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"Why geniosis live so short?
They wanna stay kids."

# Ternary numbers and algebras Reflexive numbers and Berger graphs 

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#### Abstract

The Calabi-Yau spaces with $S U(m)$ holonomy can be studied by the algebraic way through the integer lattice where one can construct the Newton reflexive polyhedra or the Berger graphs. Our conjecture is that the Berger graphs can be directly related with the $n$-ary algebras. To find such algebras we study the $n$-ary generalization of the well-known binary norm division algebras, $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, which helped to discover the most important "minimal" binary simple Lie groups, $U(1), S U(2)$ and $G(2)$. As the most important example, we consider the case $n=3$, which gives the ternary generalization of quaternions and octonions, $3^{p}, p=2,3$, respectively. The ternary generalization of quaternions is directly related to the new ternary algebra and group which are related to the natural extensions of the binary $s u(3)$ algebra and $S U(3)$ group. Using this ternary algebra we found the solution for the Berger graph: a tetrahedron.


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## 1 Introduction

Our interest in ternary algebras and symmetries started from the study of the geometry based on the holonomy principle, discovered by Berger [1]. The $C Y_{m}$ spaces with $S U(m)$ holonomy [2, 3] have a special interest for us. Our conjecture [5, 6, 7] is that $C Y_{3}\left(C Y_{m}\right)$ spaces are related to the ternary ( $n$-ary) symmetries, which are natural generalization of the binary Cartan-Killing-Lie symmetries.

There are some reasons in modern quantum physics and cosmology why we are searching for new symmetries beyond Lie algebras/groups. One of the main is related with the conjecture that the Standard Model of quarks and leptons cannot be solved in terms of binary Lie groups. This problem can be formulated as incompleteness of the Standard Model in terms of the binary Lie groups [5, 6, 7, 8]. The theory of superstrings is also based on the binary Lie groups, in particular on the D-dimensional Lorentz group, and therefore the description of the Standard Model in the superstrings approach did not bring us to success. In our opinion, the main problem with the superstrings approaches (also GUTs, SUGRA) is the inadequate external symmetry at the string scale, $M_{\text {str }} \gg M_{\mathrm{SM}}$, the D-Lorentz symmetry must be generalized. This is valid also for GUTs or SUGRA approaches. So far the construction of the quantum theory of superstrings has not been finished and it might be helpul to know the point limit of superstring theories in any dimension $4 \leq D \leq 10$. This limit is known only for $D=4$, where there exist the renormalizable quantum field theories based on the $D=1+3$ Lorentz group symmetry. The extra uncompactified dimensions make quantum field theories with Lorentz symmetry much less comfortable, since the power counting is worse. A possible way out is to suppose that the propagator is more convergent than $1 / p^{2}$, such a behaviour can be obtained if we consider, instead of binary symmetry algebra, algebras with higher order relations (That is, instead of binary operations such as addition or product of 2 elements, we start with composition laws that involve at least $n$ elements of the considered algebra, n-ary algebras). For instance, a ternary symmetry could be related with membrane dynamics. To solve the Standard Model problems we suggested to generalize their external and internal binary symmetries by addition of ternary symmetries based on the ternary algebras [6, 7, 8]. For example, ternary symmetries seem to give very good possibilities to overcome the above-mentioned problems, i.e. to make the next progress in understanding of the space-time geometry of our Universe. We suppose that the new symmetries beyond the well-known binary Lie algebras/superalgebras could allow us to build the renormalizable theories for space-time geometry with dimension $D>4$. It seems very plausible that using such ternary symmetries will offer a real possibility to overcome the problems of quantization of membranes and could be a further progress beyond string theories.

## 2 Geometry and algebra of reflexive numbers

All modern theories based on the binary Lie algebras have a common property since the algebras/symmetries are related with some invariant quadratic forms. In all these approaches, a wide class of simple classical Lie algebras was used; their whose Cartan-

Killing classification contains four infinite series $A_{r}=s l(p+1), B_{r}=s o(2 p+1), C_{p}=$ $s p(2 p), D_{p}=s o(2 p)$, and five exceptional algebras $G_{2}, F_{4}, E_{6}, E_{7}, E_{8}$. There are several ways of study such classification [14, 15, 16, 17, 18, 19, 20, 21]. For example, we can recall two of them, one way is through the Cartan matrices and Dynkin graphs, the second through the theory of numbers and Clifford algebras. Our interest in the new $n$-ary $(n>2)$ algebras and their classification started from a study of infinite series of $C Y_{m}$ spaces, $m \geq 3$, characterized by holonomy groups [1]. The $C Y_{m}$ space can be defined as the quadruple ( $M, J, g, \Omega$ ), where $(M, J)$ is a complex compact m-dimensional manifold of complex structure $J, g$ is a Käller metrics with $S U(m)$ holonomy group, and $\Omega_{m}=(m, 0)$ and $\bar{\Omega}_{m}=(0, m)$ are non-zero parallel tensors which are the holomorphic volume forms.

More exactly, a $C Y_{d i m=m-2}$ space can be realized as an algebraic variety $\mathcal{M}$ in a weighted projective space $\mathrm{CP}^{m-1}(\vec{k})$ where the weight vector reads $\vec{k}=\left(k_{1}, \ldots, k_{m}\right)$. This variety is defined by

$$
\begin{equation*}
\mathcal{M} \equiv\left(\left\{x_{1}, \ldots, x_{m}\right\} \in \mathrm{CP}^{m-1}(\vec{k}): \mathcal{P}\left(x_{1}, \ldots, x_{m}\right) \equiv \sum_{\vec{s}} c_{\vec{s}} x^{\vec{s}}=0\right) \tag{1}
\end{equation*}
$$

i.e. as the zero locus of a quasi-homogeneous polynomial of degree $d_{k}=\sum_{i=1}^{m} k_{i}$, with the monomials $x^{\vec{s}} \equiv x_{1}^{s_{1}} \cdots x_{m}^{s_{m}}$. The points in $\mathrm{CP}^{m-1}$ satisfy the property of projective invariance $\left\{x_{1}, \ldots, x_{m}\right\} \approx\left\{\lambda^{k_{1}} x_{1}, \ldots, \lambda^{k_{m}} x_{m}\right\}$ leading to the constraint $\vec{s} \cdot \vec{k}=d_{k}$.

The classification of $C Y$ - spaces can be done through the reflexivity of the weight vectors $\vec{k}$ (reflexive numbers), which can be defined in terms of the Newton reflexive polyhedra [9] or Berger graphs [6]. The Newton reflexive polyhedra are determined by the exponents of the monomials participating in the $C Y$ - equation [9]. The term "reflexive" is related with the mirror duality of Calabi-Yau spaces and the corresponding Newton polyhedra 9. The Berger graphs can be constructed directly through the reflexive weight numbers $\vec{k}=\left(k_{1}, \ldots, k_{m}\right)\left[d_{k}\right]$ by the procedure shown in [6, 7]. According to the universal algebraic approach [4] (see also [10, 11]) one can find a section in the reflexive polyhedron and, according to the $r$-arity, $2 \leq r \leq m$, of this algebraic approach, the reflexive polyhedron can be constructed from $2-, 3-, \ldots$ Berger graphs. It was conjectured that the Berger graphs might correspond to some $n$-ary Lie algebras [6, 7. For example, our analis of Berger graphs showed that the exceptional algebras could be the origin of some infinite series of $n$-ary algebras, $n=2,3,4, \ldots$ [6, 7].

In these articles we tried to decode those Berger graphs by using the method of the "simple roots" (for illustration, see table (1). In this table some general properties of the Berger graphs are presented, which correspond to the subclass of the "simply -laced" reflexive numbers (Egyptian numbers). A simply-laced number $\vec{k}=\left(k_{1}, \cdots, k_{m}\right)$ with degree $d_{k}=\sum_{i=1}^{m} k_{i}$, is defined such that

$$
\begin{equation*}
\frac{d_{k}}{k_{i}} \in \mathbb{Z}^{+} \tag{2}
\end{equation*}
$$

For these numbers there is a simple way of constructing the corresponding affine Berger graphs together with their Coxeter labels [6, 7]. The corresponding Cartan and Berger matrices of these graphs are symmetric. In the Cartan case they correspond to the $A D E$

| $\vec{k}_{3,4}^{\text {ext }}$ | Rank | $h$ | Casimir $\left(B_{i i}\right)$ | Determinant |
| :--- | :---: | :---: | :---: | :---: |
| $(0,1,1,1)[3]$ | $6\left(E_{6}\right)$ | 12 | 6 | 3 |
| $(0,1,1,2)[4]$ | $7\left(E_{7}\right)$ | 18 | 8 | 2 |
| $(0,1,2,3)[6]$ | $8\left(E_{8}\right)$ | 30 | 12 | 1 |
| $(0,0,1,1,1)[3]$ | $2_{3}+10+l$ | $18+3(l+1)$ | 9 | $3^{4}$ |
| $(0,0,1,1,2)[4]$ | $2_{3}+13+l$ | $32+4(l+1)$ | 12 | $4^{3}$ |
| $(0,0,1,2,3)[6]$ | $2_{3}+15 l$ | $60+6(l-1)$ | 18 | $6^{2}$ |
| $(0,1,1,1,1)[4]$ | $1_{3}+11$ | 28 | 12 | 16 |
| $(0,2,3,3,4)[12]$ | $1_{3}+12$ | 90 | 36 | 8 |
| $(0,1,1,2,2)[6]$ | $1_{3}+13$ | 48 | 18 | 9 |
| $(0,1,1,1,3)[6]$ | $1_{3}+15$ | 54 | 18 | 12 |
| $(0,1,1,2,4)[8]$ | $1_{3}+17$ | 80 | 24 | 8 |
| $(0,1,2,2,5)[10]$ | $1_{3}+17$ | 100 | 30 | 5 |
| $(0,1,3,4,4)[12]$ | $1_{3}+17$ | 120 | 36 | 3 |
| $(0,1,2,3,6)[12]$ | $1_{3}+19$ | 132 | 36 | 6 |
| $(0,1,4,5,10)[20]$ | $1_{3}+26$ | 290 | 60 | 2 |
| $(0,1,1,4,6)[12]$ | $1_{3}+24$ | 162 | 36 | 6 |
| $(0,1,2,6,9)[18]$ | $1_{3}+27$ | 270 | 54 | 3 |
| $(0,1,3,8,12)[24]$ | $1_{3}+32$ | 420 | 72 | 2 |
| $(0,2,3,10,15)[30]$ | $1_{3}+25$ | 420 | 90 | 4 |
| $(0,1,6,14,21)[42]$ | $1_{3}+49$ | 1092 | 126 | 1 |

Table 1: Rank, Coxeter number $h$, Casimir depending on $B_{i i}$ and determinants for the non-affine exceptional Berger graphs. The maximal Coxeter labels coincide with the degree of the corresponding reflexive simply-laced vector. The numbers $1_{3}$ and $2_{3}$ denote the number of nodes with $B_{i i}=3$.
series of simply-laced algebras. In dimensions $m=1,2,3$, the Egyptian numbers are $(1),(1,1),(1,1,1),(1,1,2),(1,2,3)$. For $m=4$, from all 95 reflexive numbers, 14 are simply-laced Egyptian numbers for which some general properties can be illustrated (see Table (1) 7:

To get further progress in the solution of the Berger graphs by method of "simple roots" we should construct the $n$-ary analogue of the $s u(2) / u(1)$ node, which was the keystone in the Killing-Cartan-Lie classification [16, 17. Therefore one should search for such "minimal" $n$-ary algebras. To do this we would like to use the ideas coming from the theory of the binary norm division algebras, $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. But to define the complete decision we must find the real examples of the "minimal" simple algebras like $s u(2)$ and $g(2)$, which are directly related with quaternions and octonions, respectively. Therefore we concentrate on searching for the $n$-ary division algebras.

## 3 Division algebras and Lie algebras

Now we should briefly recall the four norm division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ [12, 13, 14, 15, [18, 19. An algebra $A$ will be a vector space that is equipped with a bilinear map $f$ : $A \times A \rightarrow A$ called by multiplication and a nonzero element $1 \in A$ called the unit, such that $f(1, a)=f(a, 1)=a$. These algebras admit an anti-involution (or conjugation) $\left(a^{*}\right)^{*}=a$ and $(a b)^{*}=b^{*} a^{*}$. A norm division algebra is an algebra $A$ that is also a normed vector space with $N(a b)=N(a) N(b)$. Such algebras exist only for $p=1,2,4,8$ dimensions
where the following identities can be obtained:

$$
\begin{equation*}
\left(x_{1}^{2}+\ldots+x_{p}^{2}\right)\left(y_{1}^{2}+\ldots+y_{p}^{2}\right)=\left(z_{1}^{2}+\ldots+z_{p}^{2}\right) \tag{3}
\end{equation*}
$$

The doubling process, which is known as the Cayley-Dickson process, forms the sequence of divison algebras

$$
\begin{equation*}
\mathbb{R} \rightarrow \mathbb{C} \rightarrow \mathbb{H} \rightarrow \mathbb{O} \tag{4}
\end{equation*}
$$

Note that next algebra is not a division algebra. So $p=1 \mathbb{R}$ and $p=2 \mathbb{C}$ these algebras are the commutative associative normed division algebras. The quaternions, $\mathbb{H}, p=4$ form the non-commutative and associative norm division algebra. The octonion algebra $p=8, \mathbb{O}$ is an non-associative alternative algebra. If the discovery of complex numbers took a long period about some centuries years, the discovery of quaternions and octonions was made in a short time, in the middle of the XIX century by W. Hamilton [12], and by J. Graves and A.Cayley [13]. The complex numbers, quaternions and octonions can be presented in the general form:

$$
\begin{equation*}
\hat{q}=x_{0} e_{0}+x_{1} e_{1}+\ldots+x_{p-1} e_{p-1}, \quad\left\{x_{0}, x_{1}, \ldots, x_{p-1}\right\} \in \mathbb{R} \tag{5}
\end{equation*}
$$

where $p=2$ and $e_{1} \equiv \mathbf{i}$ for complex numbers $\mathbb{C}, p=4$ for quaternions $\mathbb{H}$, and $p=8$ for $\left(\mathbb{O}\right.$. The $e_{0}$ is as unit and all $e_{1}, \ldots, e_{p-1}$ are imaginary units with conjugation $\bar{e}_{1}=$ $-e_{0}, \ldots, \bar{e}_{p-1}=-e_{0}$. For quaternions we have the main relation

$$
\begin{equation*}
e_{m} e_{k}=-\delta_{m k}+f_{m k l} e_{l} \tag{6}
\end{equation*}
$$

where $\delta_{m k}$ and $f_{m k l} \equiv \epsilon_{m k l}$ are the well-known Kronecker and Levi-Cevita tensors, respectively. For octonions the completely antisymmetric tensor $f_{m k l}=1$ for the following seven triple associate cycles:

$$
\begin{equation*}
\{m k l\}=\{123\},\{145\},\{176\},\{246\},\{257\},\{347\},\{365\} . \tag{7}
\end{equation*}
$$

There are also 28 non-associate cycles. Each triple accociate cycle corresponds to a quaternionic subalgebra. These algebras have a very close link with geometry. For example, the unit elements $x^{2}=1, x \in \mathbb{R},|\hat{q}|=x_{0}^{2}+x_{1}^{2}=1$ in $\mathbb{C}_{1},|\hat{q}|=x_{0}^{2}+x_{1}^{2}+x_{2}^{2}+x_{3}^{2}=1$ in $\mathbb{H}_{1},|\hat{q}|=x_{0}^{2}+x_{1}^{2}+\ldots+x_{7}^{2}=1$ in $\mathbb{O}_{1}$, define the spheres, $S^{0}, S^{1}, S^{3}, S^{7}$, respectively. The complex and quaternionic unit elements have the $U(1)$ and $S U(2)$ group properties, respectively. The $A$ series contains the complex rotations in the unit circle, $S^{1}$, and $S^{1}$ is a Lie group. The $B$ and $C$ groups both contain the quaternion rotations on the unit sphere $S^{3}$, and $S^{3}$ is a Lie group. The $D$ series contain the Lorentz group in $D=3+1$, which consists of two copies of $S^{3}-3$-rotations and 3 -boosts. The exceptional groups do not include $S^{7}$ as a Lie group. Thus $S^{7}$ is the only unit sphere in a division algebra that is not a Lie group. The reason is that the octonions are not associative; their associator is

$$
\begin{equation*}
\left\{e_{m}, e_{k}, e_{l}\right\}=\left(e_{m} e_{k}\right) e_{l}-e_{m}\left(e_{k} e_{l}\right)=f_{m k l t} e_{t}, \tag{8}
\end{equation*}
$$

where the tensor $f_{m k l t}$ is completely antisymmetric and it is non-zero for the following seven 4-cycles:

$$
\begin{equation*}
\{m k l t\}=\{4567\},\{2367\},\{2345\},\{1357\},\{1346\},\{1256\},\{1247\} \tag{9}
\end{equation*}
$$

It is also the case when three elements $\left\{e_{a}, e_{b}, e_{c}\right\}$ are not in the same three associate cycles, for example, $\left(e_{1} e_{5}\right) e_{7}-e_{1}\left(e_{5} e_{7}\right)=2 e_{3}$. Note that the octonions form the alternative algebra since the alternative condition, $\{a, b, c\}+\{c, b, a\}=0$, is always valid. The octonions are directly linked to the five exceptional groups $G(2), F(4), E(6), E(7), E(8)$ 17, 18. The automorphism of octonions is $G(2)$.

## 4 Nambu-Filippov ternary algebras

The new n-ary symmetries can be related to algebras that are based on the generalization of the Lie binary commutation relation

$$
\begin{equation*}
[x, y]=x y-y x \tag{10}
\end{equation*}
$$

by the ternary commutations relations

$$
\begin{equation*}
\left[x_{1}, x_{2}, \ldots, x_{n}\right]=(-1)^{\tau(\sigma)}\left[x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}\right] \tag{11}
\end{equation*}
$$

where $\sigma$ runs over the symmetric group $S_{n}$ and the number $\tau(\sigma)$ equals 0 or 1 , depending on the parity of the permutation $\sigma$ (see [10, 22, 23, 24, 25, 32, 27, 28, 29, 30, 31, 33, 34, [35, 36, 11).

More exactly, a ternary Lie algebra is defined by a ternary antisymmetric operation $A \times$ $A \times A \rightarrow A$ with the Jacobi-like identity [23]. Fillipov considered the $n$-ary generalizations of Lie algebras $(n>2)$. The most simple example of this Lie algebra can be the $n$-vector product of the $(n+1)$ - dimensional vectors $\vec{x}_{1}, \ldots, \vec{x}_{n}$ which is equal to the following determinant:

$$
\begin{align*}
{\left[\vec{x}_{1}, \ldots, \vec{x}_{n}\right]_{n} } & =\vec{x}_{1} \times \vec{x}_{2} \times \ldots \times \vec{x}_{n} \\
& \left|\begin{array}{ccccc}
x_{11} & x_{12} & \ldots & x_{1 n} & \vec{e}_{1} \\
x_{21} & x_{22} & \ldots & x_{2 n} & \vec{e}_{2} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
x_{(n+1) 1} & x_{(n+1) 2} & \ldots & x_{(n+1) n} & \vec{e}_{n+1}
\end{array}\right|, \tag{12}
\end{align*}
$$

where $\left(x_{1 l}, \ldots x_{(n+1) l}\right)$ are the cordinates of the vector $\vec{x}_{l}$ and $\left\{e_{l}\right\}$ is the orthonormal basis.
The other simple example is the algebra of polynomials of $n$-variables $x_{1}, \ldots x_{n}$ where n-ary operation is the functional Jacobian:

$$
\begin{equation*}
\left[f_{1}, \ldots, f_{n}\right]_{n}=\operatorname{det}\left\|\frac{\partial\left(f_{1}, \ldots, f_{n}\right)}{\partial\left(x_{1}, \ldots x_{n}\right)}\right\| \tag{13}
\end{equation*}
$$

The $n$-ary Poisson-like structure of the determinat have been used for the generalization of the classical Hamiltonian mechanics in which the binary Poisson bracket can be replaced by ternary [22] or by n-ary brackets [23, 25], respectively.

There exist several examples of multi-Hamiltonian systems possessing dynamical or hidden symmetries, which can be realized within the generalized Nambu-Hamiltonian mechanics using the $n$-ary Nambu-Poisson brackets $(n>2)$. Among such systems one can consider the $S O(4)$ Kepler problem [26]. The well-known Kepler Hamiltonian is $H=\vec{p}^{2} / 2-1 / r$, where $r=\sqrt{x_{1}^{2}+x_{2}^{2}+x_{3}^{2}}$. Such a Hamiltonian has the $S O(3)$ rotational symmetry giving the orbital angular momentum $\vec{L}=\vec{r} \times \vec{p}$ as an integral of motion. This integral of motion with a Hamiltonian implies that the orbit lies in 2-dimensional plane, but cannot explain why it is closed. Laplace discovered the other hidden $S O(3)$ symmetry, which gives the additional integral of motion $\vec{A}=\vec{p} \times \vec{L}-\vec{r} / r$. The Kepler problem was naturally solved in Nambu Hamiltonian mechanics, in which the equations of motion are given by n-Poisson-Nambu bracket:

$$
\begin{equation*}
\frac{d f}{d t}=\left[H_{1}, . ., H_{5}, f\right] . \tag{14}
\end{equation*}
$$

Other n-ary analogues of Lie algebras have also been considered by Bremner 35, 36. Bremner found the minimal (Jacobi) identity in the totally associative case, which has degree 7. One calls algebra $A$ totally associative if it satisfies the polynomial identities

$$
\begin{equation*}
t_{i}\left(a_{1}, a_{2}, \ldots, a_{2 n-1}\right)=a_{1} \ldots\left(a_{i} \ldots a_{i+n-1}\right) \ldots a_{2 n-1}-a_{1} \ldots\left(a_{i+1} \ldots a_{i+n}\right) \ldots a_{2 n-1}=0 \tag{15}
\end{equation*}
$$

for $1 \leq i \leq n-1$. The minimal identity in the partially associative case, which has degree 5, was found by Gnedbaye [32]; $A$ is called partially associative if it satisfies the polynomial identity

$$
\begin{equation*}
p\left(a_{1}, \ldots, a_{2 n-1}\right)=\sum_{i=1}^{i=n}(-1)^{(n-1) i} a_{1} \ldots a_{i-1}\left(a_{i} \ldots a_{i+n-1}\right) a_{i+n} \ldots a_{2 n-1}=0 \tag{16}
\end{equation*}
$$

Also the ternary symmetries have been intensively discussed in quantum physics [33, 30, 31], in conformal field theories [24.

One of the best ways of studing the n-ary algebras/symmetries is through the $n$ ary generalizations of Clifford algebras. The binary Clifford algebra is an associative algebra, which contains and is generated by a vector space having a quadratic form: $e_{k} e_{l}+e_{l} e_{k}=2 g_{k l} ;\left\{e_{a}\right\}$ is the basis of the vector space and the signature of this space is determined by the metric $g_{k l}=\operatorname{diag}\left(-1, \ldots,-1_{t},+1, \ldots, 1_{s}\right)$.

The binary Clifford algebras are closely related with the study of the rotation groups of multidimensional spaces. The $1_{t}+3_{s}$ Minkowski space can be described by the quadratic form $x^{2}+y^{2}+z^{2}-c^{2} t^{2}$, which remains invariant under a general Lorentz group. The general Lorentz group $O(3,1)$ consists of a proper orthochonous Lorentz group $O_{0}(3,1)$ and three reflections (discrete transformations) $P, T, P T$, where $P$ and $T$ are space and time reversal. In the general case of the real space $R^{s, t}$, the orthogonal group $O(s, t)$ is represented by the semidirect product of a connected component $O(s, t)_{0}$ and a discrete subgroup $\{1, P, T, P T\}$. The double covering of the orthogonal group $O(s, t)$ is a

Clifford group $\operatorname{Pin}(s, t)$, which can be completely constructed within a Clifford algebra, i.e. $\operatorname{Pin}(s, t) \subset \mathrm{Cl}_{s, t}$ [37]. The discrete symmetries are represented by fundamental automorphisms of the Clifford algebras, i.e. $\quad\{1, P, T, P T\} \approx \operatorname{Aut}(\mathrm{Cl})$. In contrast with the transformations $P, T, P T$, the operation C of charge conjugation is represented by a pseudoautomorphism $A \rightarrow \bar{A}$. An extended automorphism group $\operatorname{Ext}(\mathrm{Cl})$ is isomorphic to a $C P T$ group $\{1, P, T, P T, C, C P, C T, C P T\}$. The $n$-ary Clifford algebras are related to the generalizations of orthogonal groups.

## 5 The ternary generalization of quaternions and the tripling Cayley-Dickson method

We would now like to discuss the doubling Cayley-Dickson method and generalize its to construct the ternary form division algebras. The complex numbers are 2-dimensional algebra with basis $e_{0}$ and $e_{1} \equiv \mathbf{i}$,

$$
\begin{equation*}
\mathbb{C}=\mathbb{R} \oplus \mathbb{R} e_{1} \tag{17}
\end{equation*}
$$

where $e_{0}^{2}=e_{0}, e_{1} e_{0}=e_{0} e_{1}=e_{1}$ and $e_{1}$ is the imaginary unit, $\mathbf{i}^{2}=-e_{0}$. Considering one additional basis imaginary unit element $e_{2} \equiv \mathbf{j}$ in the Cayley-Dickson doubling process, we can obtain the quaternions:

$$
\begin{equation*}
\mathbb{H}=\mathbb{C} \oplus \mathbb{C} j \tag{18}
\end{equation*}
$$

By the Cayley-Dickson method, if $a, b, c, d \in X,(a, b),(c, d) \in X^{2}$ and the product can be defined in $X^{2}$ through

$$
\begin{equation*}
(a, b)(c, d)=\left(a d-c^{*} b, b c^{*}+d a\right) \tag{19}
\end{equation*}
$$

This means that quaternions can be considered as a pair of complex numbers:

$$
\begin{equation*}
q=(a+\mathbf{i} b)+j(c+\mathbf{i} d), \tag{20}
\end{equation*}
$$

where

$$
\begin{equation*}
j(c+\mathbf{i} d)=(c+\overline{\mathbf{i}} d) j=(c-\mathbf{i} d) j . \tag{21}
\end{equation*}
$$

so, that we can see that $\mathbf{i} \mathbf{j}=-\mathbf{j} \mathbf{i}=\mathbf{k}$. The quaternions

$$
\begin{equation*}
q=x_{0} e_{0}+x_{1} e_{1}+x_{2} e_{2}+x_{3} e_{3}, \quad q \in \mathbb{H}, \tag{22}
\end{equation*}
$$

produce over $\mathbb{R}$ a 4-dimensional norm division algebra, where the fourth imaginary unit $e_{3}=e_{1} e_{2} \equiv \mathbf{k}$ appears. The main multiplication rules of all imaginary elements are the following:

$$
\begin{align*}
& \mathbf{i}^{2}=\mathbf{j}^{2}=\mathbf{k}^{2}=-1 \\
& \mathbf{i j}=\mathbf{k} \quad \mathbf{j} \mathbf{i}=-\mathbf{k} . \tag{23}
\end{align*}
$$

All other identities can be obtained from cyclic permutations of $\mathbf{i}, \mathbf{j}, \mathbf{k}$. The imaginary quaternions $\mathbf{i}, \mathbf{j}, \mathbf{k}$ produce the $s u(2)$ algebra. The matrix realization of quaternions has been done through the Pauli matrices:

$$
\begin{equation*}
\sigma_{0}, \mathbf{i} \sigma_{1}, \mathbf{i} \sigma_{2}, \mathbf{i} \sigma_{3} \tag{24}
\end{equation*}
$$

The unit quaternions $q=a \mathbf{1}+b \mathbf{i}+c \mathbf{j}+d \mathbf{k} \in H_{1}, q \bar{q}=1$, produce the $S U(2)$ group:

$$
\begin{equation*}
q \bar{q}=a^{2}+b^{2}+c^{2}+d^{2}=1, \quad\{a, b, c, d\} \in S^{3}, S^{3} \approx S U(2) \tag{25}
\end{equation*}
$$

Simililarly, continuing the Cayley-Dickson doubling process, we can build the octonions:

$$
\begin{equation*}
\mathbb{O}=\mathbb{H} \oplus \mathbb{H} \mathbf{l} \tag{26}
\end{equation*}
$$

where we introduced the new basis element $\mathbf{l} \equiv e_{4}$. As a result of this process, the basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ of $\mathbb{H}$ is complemented to a basis $\left\{\mathbf{1}=e_{0}, \mathbf{i}=e_{1}, \mathbf{j}=e_{2}, \mathbf{k}=e_{3}=e_{1} e_{2}, \mathbf{l}=\right.$ $\left.e_{4}, \mathbf{i l}=e_{5}=e_{1} e_{4}, \mathbf{j l}=e_{6}=e_{2} e_{4}, \mathbf{k l}=e_{7}=e_{3} e_{4}\right\}$ of $\mathbb{O}:$

$$
\begin{equation*}
o=x_{0} e_{0}+x_{1} e_{1}+x_{2} e_{2}+x_{3} e_{3}+x_{4} e_{4}+x_{5} e_{5}+x_{6} e_{6}+x_{7} e_{7}, \tag{27}
\end{equation*}
$$

where we can see the following seven associative cycle triples:

$$
\begin{align*}
& \left\{123: e_{1} e_{2}=e_{3}\right\},\left\{145: e_{1} e_{4}=e_{5}\right\},\left\{176: e_{1} e_{7}=e_{6}\right\},\left\{246: e_{2} e_{4}=e_{6}\right\}, \\
& \left\{257: e_{2} e_{5}=e_{7}\right\},\left\{347: e_{3} e_{4}=e_{7}\right\},\left\{365: e_{3} e_{6}=e_{5}\right\} \tag{28}
\end{align*}
$$

The doubling process can consequently produce new algebras:

$$
\begin{equation*}
\mathbb{R} \rightarrow \mathbb{C} \rightarrow \mathbb{H} \rightarrow \mathbb{O} \rightarrow \mathbb{S} \rightarrow \ldots \tag{29}
\end{equation*}
$$

but just the first four from this list are the norm division alternative algebras.
In order find new division algebras, we can just relax some constraints. To do this, let us consider two basic elements, $q_{0}, q_{1}$, with the following constraints:

$$
\begin{equation*}
q_{1} \cdot q_{0}=q_{0} \cdot q_{1}=q_{1}, \quad q_{1}^{3}=q_{0} \tag{30}
\end{equation*}
$$

In this case a new element can be introduced, as $q_{2}=q_{1}^{2}=q_{1}^{(-1)}$, i.e. $q_{2} q_{1}=q_{1} q_{2}=q_{0}$.
From these three elements a new field $\mathbb{T} C$ can be built:

$$
\begin{equation*}
\mathbb{T} C=\mathbb{R} \oplus \mathbb{R} q_{1} \oplus \mathbb{R} q_{1}^{2} \tag{31}
\end{equation*}
$$

with the new numbers

$$
\begin{equation*}
\hat{z}=x_{0} q_{0}+x_{1} q_{1}+x_{2} q_{2}, \quad x_{i} \in \mathbb{R}, \quad i=0,1,2 \tag{32}
\end{equation*}
$$

which are the ternary generalization of the complex numbers. Note that the multicomplex numbers, $\mathbb{M} C_{n}=\left\{x=\sum_{i=0}^{i=n-1} x_{i} q^{i}, q^{n}=-q_{0}, x_{i} \in \mathbb{R}\right\}$ have been suggested in articles [38, 39, 21.

Let us define two operations of the conjugations:

$$
\begin{equation*}
\bar{q}_{1}=j q_{1}, \quad \overline{\bar{q}}_{1}=j^{2} q_{1} \tag{33}
\end{equation*}
$$

where $j=\exp (2 \mathbf{i} \pi) / 3$. Since $q_{2}=q_{1}^{2}$ we can easily obtain

$$
\begin{equation*}
\bar{q}_{2}=j^{2} q_{2}, \quad \overline{\bar{q}}_{2}=j q_{2} \tag{34}
\end{equation*}
$$

These two conjugation operations can thus be applied, respectively:

$$
\begin{align*}
& \overline{\hat{z}}=x_{0} q_{0}+x_{1} j q_{1}+x_{2} j^{2} q_{2}, \\
& \overline{\overline{\hat{z}}}=x_{0} q_{0}+x_{1} j^{2} q_{1}+x_{2} j q_{2} . \tag{35}
\end{align*}
$$

We now introduce the cubic form:

$$
\begin{equation*}
\langle\hat{z}\rangle^{3}=\hat{z} \overline{\hat{z}} \overline{\hat{z}}=x_{0}^{3}+x_{1}^{3}+x_{2}^{3}-3 x_{0} x_{1} x_{3}, \tag{36}
\end{equation*}
$$

And also easily check the following relation:

$$
\begin{equation*}
\left\langle\hat{z}_{1} \hat{z}_{2}\right\rangle^{3}=\left\langle\hat{z}_{1}\right\rangle^{3}\left\langle\hat{z}_{2}\right\rangle^{3}, \tag{37}
\end{equation*}
$$

which indicates the group properties of the $\mathbb{T} C$ numbers. More exactly, the unit $\mathbb{T} C$ numbers produce the Abelian ternary group. According to the ternary analogue of the Euler formula, the following ternary complex functions [38, 39, 21] can be constructed:

$$
\begin{equation*}
\Psi=\exp \left(q_{1} \phi_{1}+q_{2} \phi_{2}\right), \quad \psi_{1}=\exp \left(q_{1} \phi_{1}\right), \quad \psi_{2}=\exp \left(q_{2} \phi_{2}\right) \tag{38}
\end{equation*}
$$

where $\phi_{i}$ are the group parameters. For the functions $\psi_{i}, i=0,1,2$, i.e. we have the following analogue of Euler, formula:

$$
\begin{align*}
\Psi & =\exp \left(q_{1} \phi+q_{2} \phi_{2}\right)=f q_{0}+g q_{1}+h q_{2} \\
\psi_{1} & =\exp \left(q_{1} \phi\right)=f_{1} q_{0}+g_{1} q_{1}+h_{1} q_{2} \\
\psi_{2} & =\exp \left(q_{2} \phi\right)=f_{2} q_{0}+h_{2} q_{1}+g_{2} q_{2} \tag{39}
\end{align*}
$$

Consequently, one can introduce the conjugation operations for these functions. For example, for $\psi_{1}$ we can get:

$$
\begin{align*}
& \bar{\psi}_{1}=\exp \left(\bar{q}_{1} \phi\right)=\exp \left(j \cdot q_{1} \phi\right)=f q_{0}+j g q_{1}+j^{2} h q_{2} \\
& \bar{\psi}_{1}=\exp \left(\overline{\bar{q}}_{1} \phi\right)=\exp \left(j^{2} \cdot q_{1} \phi\right)=f_{1} q_{0}+j^{2} g_{1} q_{1}+j h_{1} q_{2} \tag{40}
\end{align*}
$$

with the following constraints:

$$
\begin{equation*}
\psi_{1} \bar{\psi}_{1} \overline{\bar{\psi}}_{1}=\exp \left(q_{1} \phi\right) \exp \left(j \cdot q_{1} \phi\right) \exp \left(j^{2} \cdot q_{1} \phi\right)=q_{0} \tag{41}
\end{equation*}
$$

which gives us the following link between the functions, $f_{1}, g_{1}, h_{1}$ :

$$
\begin{equation*}
f_{1}^{3}+g_{1}^{3}+h_{1}^{3}-3 f_{1} g_{1} h_{1}=1 \tag{42}
\end{equation*}
$$

This surface is a ternary analogue of the $S^{1}$ circle and it is related with the ternary Abelian group, $T U(1)$.

But our goal is to find a minimal non-Abelia ternary algebra. Therefore, at the next step, we are going to construct the analogue of binary quaternions $\mathbb{H}$, based on the three elements $q_{0}, q_{1}, q_{7}$. Our new constraint $q_{1}^{3}=q_{7}^{3}=q_{0}$, we can introduce the following six new products from $q_{1}$ and $q_{7}: q_{1}^{2}, q_{7}^{2}, q_{1} q_{7}, q_{1}^{2} q_{7}, q_{7}^{2} q_{1}$ and $q_{1}^{2} q_{7}^{2}$. More exactly, to get the new $3^{2}$ ternary $\mathbb{H}$ numbers, we suggest the tripling Cayley-Dickson process starting from $\mathbb{T} C$ :

$$
\begin{equation*}
\mathbb{H}=\mathbb{T} C \oplus \mathbb{T} C q_{7} \oplus \mathbb{T} C q_{7}^{2} \tag{43}
\end{equation*}
$$

or more exactly

$$
\begin{align*}
Q & =\left(x_{0} q_{0}+x_{1} q_{1}+x_{2} q_{1}^{2}\right)+q_{7}\left(y_{0} q_{0}+y_{1} q_{1}+y_{2} q_{1}^{2}\right)+q_{7}^{2}\left(z_{0} q_{0}+z_{1} q_{1}+z_{2} q_{1}^{2}\right) \\
& =\left(x_{0} q_{0}+x_{1} q_{1}+x_{2} q_{4}\right)+\left(y_{0} q_{7}+j y_{1} q_{2}+j y_{2} q_{6}\right)+\left(z_{0} q_{8}+z_{1} q_{3}+j^{2} z_{2} q_{5}\right) \\
& =\left(x_{0} q_{0}+y_{0} q_{7}+z_{0} q_{8}\right)+\left(x_{1} q_{1}+y_{1} q_{2}+z_{1} q_{3}\right)+\left(x_{2} q_{4}+j y_{2} q_{6}+j^{2} z_{2} q_{5}\right) \tag{44}
\end{align*}
$$

where we defined the new unit elements $q_{1}^{2}=q_{4}, q_{7} q_{1}=q_{2}, q_{7}^{2}=q_{8}, q_{7}^{2} q_{1}=q_{3}, q_{7} q_{1}^{2}=j q_{6}$, $q_{7}^{2} q_{1}^{2}=j^{2} q_{5}$ with $q_{a}^{3}=q_{0}$.

To find the table of multiplication (see table (2) of all $q_{a}$ basis elements we can recall the identity between three unit imaginary elements, $e_{1}, e_{2}, e_{3}$, in binary quaternion algebra:

$$
\begin{align*}
& e_{1} e_{2} e_{3}=e_{2} e_{3} e_{1}=e_{3} e_{1} e_{2}=-e_{0}, \\
& e_{3} e_{2} e_{1}=e_{2} e_{1} e_{3}=e_{1} e_{3} e_{2}=+e_{0} \tag{45}
\end{align*}
$$

Since we suppose that the $\mathbb{H}$ numbers are the ternary generalizations of quaternions we can start from the following triple identities for $q_{1}, q_{2}, q_{3}$ :

$$
\begin{align*}
& q_{1} q_{2} q_{3}=q_{2} q_{3} q_{1}=q_{3} q_{1} q_{2}=j^{2} q_{0} \\
& q_{3} q_{2} q_{1}=q_{2} q_{1} q_{3}=q_{1} q_{3} q_{2}=j q_{0} \tag{46}
\end{align*}
$$

where $j=\exp (2 \pi \mathbf{i} / 3)$.
Introducing the new elements

$$
\begin{equation*}
q_{1}^{2}=q_{4}, \quad q_{2}^{2}=q_{5}, \quad q_{3}^{2}=q_{6}, \tag{47}
\end{equation*}
$$

Table 2: The binary multiplication relations

| $N$ | $q_{1}$ | $q_{2}$ | $q_{3}$ | $q_{4}$ | $q_{5}$ | $q_{6}$ | $q_{7}$ | $q_{8}$ | $q_{0}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $q_{1}$ | $q_{4}$ | $j^{2} q_{6}$ | $j q_{5}$ | $q_{0}$ | $q_{8}$ | $q_{7}$ | $j q_{2}$ | $j^{2} q_{3}$ | $q_{1}$ |
| $q_{2}$ | $j q_{6}$ | $q_{5}$ | $j^{2} q_{4}$ | $q_{7}$ | $q_{0}$ | $q_{8}$ | $j q_{3}$ | $j^{2} q_{1}$ | $q_{2}$ |
| $q_{3}$ | $j^{2} q_{5}$ | $j q_{4}$ | $q_{6}$ | $q_{8}$ | $q_{7}$ | $q_{0}$ | $j q_{1}$ | $j^{2} q_{2}$ | $q_{3}$ |
| $q_{4}$ | $q_{0}$ | $j^{2} q_{7}$ | $j q_{8}$ | $q_{1}$ | $j^{2} q_{3}$ | $j q_{2}$ | $q_{6}$ | $q_{5}$ | $q_{4}$ |
| $q_{5}$ | $j q_{8}$ | $q_{0}$ | $j^{2} q_{7}$ | $j q_{3}$ | $q_{2}$ | $j^{2} q_{1}$ | $q_{4}$ | $q_{6}$ | $q_{5}$ |
| $q_{6}$ | $j^{2} q_{7}$ | $j q_{8}$ | $q_{0}$ | $j^{2} q_{2}$ | $j q_{1}$ | $q_{3}$ | $q_{5}$ | $q_{4}$ | $q_{6}$ |
| $q_{7}$ | $q_{2}$ | $q_{3}$ | $q_{1}$ | $j q_{6}$ | $j q_{4}$ | $j q_{5}$ | $q_{8}$ | $q_{0}$ | $q_{7}$ |
| $q_{8}$ | $q_{3}$ | $q_{1}$ | $q_{2}$ | $j^{2} q_{5}$ | $j^{2} q_{6}$ | $j^{2} q_{4}$ | $q_{0}$ | $q_{7}$ | $q_{8}$ |
| $q_{0}$ | $q_{1}$ | $q_{2}$ | $q_{3}$ | $q_{4}$ | $q_{5}$ | $q_{6}$ | $q_{7}$ | $q_{8}$ | $q_{0}$ |

we can immediately get the following triple identities:

$$
\begin{align*}
& q_{4} q_{5} q_{6}=q_{5} q_{6} q_{4}=q_{6} q_{4} q_{5}=j^{2} q_{0} \\
& q_{6} q_{5} q_{4}=q_{5} q_{4} q_{6}=q_{4} q_{6} q_{5}=j q_{0} \tag{48}
\end{align*}
$$

We suggest the following 24 commutation relations:

$$
\begin{array}{ccc}
q_{1} q_{2}=j q_{2} q_{1}, & q_{2} q_{3}=j q_{3} q_{2}, & q_{3} q_{1}=j q_{1} q_{3} \\
q_{4} q_{5}=j q_{5} q_{4}, & q_{5} q_{6}=j q_{6} q_{5}, & q_{6} q_{4}=j q_{4} q_{6} \\
& \\
q_{5} q_{1}=j q_{1} q_{5}, & q_{6} q_{2}=j q_{2} q_{6}, & q_{4} q_{3}=j q_{3} q_{4} \\
q_{6} q_{1}=j^{2} q_{1} q_{6}, & q_{5} q_{3}=j^{2} q_{3} q_{5}, & q_{4} q_{2}=j^{2} q_{2} q_{4} \\
& \\
q_{1} q_{7}=j q_{7} q_{1}, & q_{2} q_{7}=j q_{7} q_{2}, & q_{3} q_{7}=j q_{7} q_{3} \\
q_{1} q_{8}=j^{2} q_{8} q_{1}, & q_{2} q_{8}=j^{2} q_{8} q_{2}, & q_{3} q_{8}=j^{2} q_{8} q_{3}  \tag{52}\\
& \\
q_{7} q_{4}=j q_{4} q_{7}, & q_{7} q_{5}=j q_{5} q_{7}, & q_{7} q_{6}=j q_{6} q_{7} \\
q_{8} q_{4}=j^{2} q_{4} q_{8}, & q_{8} q_{5}=j^{2} q_{5} q_{8}, & q_{8} q_{6}=j^{2} q_{6} q_{8}
\end{array}
$$

and 4 commuting pairs:

$$
\begin{equation*}
q_{1} q_{4}=q_{4} q_{1}, q_{2} q_{5}=q_{5} q_{2}, q_{3} q_{6}=q_{6} q_{3}, q_{7} q_{8}=q_{8} q_{7} \tag{53}
\end{equation*}
$$

at last, we can present the following table 2 of binary multiplication relations:
Note if we could consider the other choice for $q_{a}$, for example, $q_{a}^{3}=-q_{0}$ for $a=1, \ldots, 6$. The similar condition for $q_{7}, q_{8}$ does not play so important role and we can take them as was in the first case, i.e. $q_{7}^{3}=q_{8}^{3}=q_{0}$. As result of such a choice one can get some

$$
\mathbf{q}_{\mathbf{a}}{ }^{3}=\mathbf{q}_{0} \quad \mathbf{a}=1,2, \ldots, 8 \quad j=\exp (2 / 3 \pi i)
$$



$$
\begin{aligned}
& \mathbf{q}_{7} \mathbf{q}_{1}=\mathbf{q}_{2} \\
& \mathbf{q}_{7} \mathbf{q}_{2}=\mathbf{q}_{3} \\
& \mathbf{q}_{7} \mathbf{q}_{3}=\mathbf{q}_{1} \\
& \\
& \\
& \\
& \mathbf{q}_{1} \mathbf{q}_{7}=\mathbf{j} \mathbf{q}_{2} \\
& \mathbf{q}_{2} \mathbf{q}_{7}=\mathbf{j} \mathbf{q}_{3} \\
& \mathbf{q}_{3} \mathbf{q}_{7}=\mathbf{j} \mathbf{q}_{1}
\end{aligned}
$$

$$
\begin{aligned}
& \mathbf{q}_{1}{ }^{2}=\mathbf{q}_{4} \\
& \mathbf{q}_{2}{ }^{2}=\mathbf{q}_{5} \\
& \mathbf{q}_{3}{ }^{2}=\mathbf{q}_{6} \\
& \mathbf{q}_{7}{ }^{2}=\mathbf{q}_{8}
\end{aligned}
$$

$\mathbf{q}_{2} \mathbf{q}_{1} \mathbf{q}_{3}=\mathbf{j} \mathbf{q}_{0}$
$\mathbf{q}_{5} \mathbf{q}_{4} \mathbf{q}_{6}=\mathbf{j} \mathbf{q}_{0}$
changing in the binary relations between $q_{a}$, like as $q_{1}^{2}=-q_{4}, q_{2}^{2}=-q_{5}, q_{3}^{2}=-q_{6}$ and $q_{1}^{4}=-q_{1}, q_{5}^{2}=-q_{2}, q_{6}^{2}=-q_{3}$, respectively, but the commutation relations remain the same.

The $q_{k}$ elements that satisfy the ternary algebra are:

$$
\begin{equation*}
\{A, B, C\}_{S_{3}}=A B C+B C A+C A B-B A C-A C B-C B A \tag{54}
\end{equation*}
$$

Here $j=\exp (2 \mathbf{i} \pi / 3)$ and $S_{3}$ is the permutation group of three elements.
We can consider the $3 \times 3$ matrix realization of $q$--algebra:

$$
\begin{align*}
& q_{1}=\left(\begin{array}{lll}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{array}\right), q_{2}=\left(\begin{array}{ccc}
0 & 1 & 0 \\
0 & 0 & j \\
j^{2} & 0 & 0
\end{array}\right) q_{3}=\left(\begin{array}{lll}
0 & 1 & 0 \\
0 & 0 & j^{2} \\
j & 0 & 0
\end{array}\right) \\
& q_{4}=\left(\begin{array}{lll}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right) q_{5}=\left(\begin{array}{lll}
0 & 0 & j \\
1 & 0 & 0 \\
0 & j^{2} & 0
\end{array}\right) q_{6}=\left(\begin{array}{ccc}
0 & 0 & j^{2} \\
1 & 0 & 0 \\
0 & j & 0
\end{array}\right) \\
& q_{7}=\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & j & 0 \\
0 & 0 & j^{2}
\end{array}\right) q_{8}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & j^{2} & 0 \\
0 & 0 & j
\end{array}\right) q_{0}=\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right) \tag{55}
\end{align*}
$$

which satisfy the ternary algebra:

$$
\begin{equation*}
\left\{q_{k}, q_{l}, q_{m}\right\}_{S_{3}}=f_{k l m}^{n} q_{n} \tag{56}
\end{equation*}
$$

We can check that each triple commutator $\left\{q_{k}, q_{l}, q_{m}\right\}$ is defined by triple numbers, $\{k l m\}$, with $k, l, m=0,1,2, \ldots, 8$, that it gives just one matrix $q_{n}$ with the corresponding coefficient $f_{k l m}^{n}$ given in Table 3,

There are $C_{9}^{2}=84$ ternary commutation relations, but there are also $C_{8}^{2}=28$ commutation relations which correspond to the $s u(3)$ algebra! Therefore, it is natural to represent the $q$-numbers as a ternary generalization of quaternions. The $S_{3}$ commutation relations naturally go to the binary, $S_{2}$, Lie commutation relations:

$$
\begin{equation*}
\left\{q_{a}, q_{b}, q_{0}\right\}_{S_{3}}=q_{a} q_{b} q_{0}+q_{b} q_{0} q_{a}+q_{0} q_{a} q_{b}-q_{b} q_{a} q_{0}-q_{a} q_{0} q_{b}-q_{0} q_{b} q_{a}=q_{a} q_{b}-q_{b} q_{a} \tag{57}
\end{equation*}
$$

where $a \neq b \neq 0$. On table for those 28 cases, one can see that we have $\{k l 0\}$.
Note using matrix realization one can define a cubic form:

$$
\left\langle Q_{D e t}\right\rangle^{3}=\operatorname{Det}(Q)=\operatorname{Det}\left(\begin{array}{ccc}
z_{0} & z_{1} & \tilde{z}_{2}  \tag{58}\\
z_{2} & \tilde{z}_{0} & \tilde{z}_{1} \\
\tilde{\tilde{z}}_{1} & \tilde{z}_{2} & \tilde{z}_{0}
\end{array}\right)
$$

Table 3: The ternary commutation relations

| $N$ | $\{k l m\} \rightarrow\{n\}$ | $f_{k l m}^{n}$ | $N$ | $\{k l m\} \rightarrow\{n\}$ | $f_{k l m}^{n}$ | $N$ | $\{k l m\} \rightarrow\{n\}$ | $f_{k l m}^{n}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | $\{123\} \rightarrow\{0\}$ | $3\left(j^{2}-j\right)$ | 2 | $\{124\} \rightarrow\{2\}$ | $j(1-j)$ | 3 | $\{125\} \rightarrow\{1\}$ | $2\left(j^{2}-j\right)$ |
| 4 | $\{126\} \rightarrow\{3\}$ | $j(1-j)$ | 5 | $\{127\} \rightarrow\{5\}$ | $2(1-j)$ | 6 | $\{128\} \rightarrow\{4\}$ | $2\left(j^{2}-1\right)$ |
| 7 | $\{120\} \rightarrow\{6\}$ | $\left(j^{2}-j\right)$ | 8 | $\{134\} \rightarrow\{3\}$ | $\left(j^{2}-j\right)$ | 9 | $\{135\} \rightarrow\{2\}$ | $2\left(j-j^{2}\right)$ |
| 10 | $\{136\} \rightarrow\{1\}$ | $\left(j^{2}-j\right)$ | 11 | $\{137\} \rightarrow\{4\}$ | $2(j-1)$ | 12 | $\{138\} \rightarrow\{6\}$ | $2\left(1-j^{2}\right)$ |
| 13 | $\{130\} \rightarrow\{5\}$ | $\left(j-j^{2}\right)$ | 14 | $\{145\} \rightarrow\{5\}$ | $\left(j-j^{2}\right)$ | 15 | $\{146\} \rightarrow\{6\}$ | $\left(j^{2}-j\right)$ |
| 16 | $\{147\} \rightarrow\{7\}$ | $\left(j^{2}-j\right)$ | 17 | $\{148\} \rightarrow\{8\}$ | $\left(j-j^{2}\right)$ | 18 | $\{140\} \rightarrow \tilde{O}$ | 0 |
| 19 | $\{156\} \rightarrow\{4\}$ | $2 j(j-1)$ | 20 | $\{157\} \rightarrow\{0\}$ | $3(1-j)$ | 21 | $\{158\} \rightarrow\{7\}$ | $2(1-j)$ |
| 22 | $\{150\} \rightarrow\{8\}$ | $(1-j)$ | 23 | $\{167\} \rightarrow\{8\}$ | $2\left(1-j^{2}\right)$ | 24 | $\{168\} \rightarrow\{0\}$ | $3\left(1-j^{2}\right)$ |
| 25 | $\{160\} \rightarrow\{7\}$ | $\left(1-j^{2}\right)$ | 26 | $\{178\} \rightarrow\{1\}$ | $\left(j-j^{2}\right)$ | 27 | $\{170\} \rightarrow\{2\}$ | $(j-1)$ |
| 28 | $\{180\} \rightarrow\{3\}$ | $\left(j^{2}-1\right)$ | 29 | $\{234\} \rightarrow\{1\}$ | $2\left(j^{2}-j\right)$ | 30 | $\{235\} \rightarrow\{3\}$ | $\left(j-j^{2}\right)$ |
| 31 | $\{236\} \rightarrow\{2\}$ | $\left(j-j^{2}\right)$ | 32 | $\{237\} \rightarrow\{6\}$ | $2(1-j)$ | 33 | $\{238\} \rightarrow\{5\}$ | $2\left(j^{2}-1\right)$ |
| 34 | $\{230\} \rightarrow\{4\}$ | $\left(j^{2}-j\right)$ | 35 | $\{245\} \rightarrow\{4\}$ | $\left(j-j^{2}\right)$ | 36 | $\{246\} \rightarrow\{5\}$ | $2\left(j-j^{2}\right)$ |
| 37 | $\{247\} \rightarrow\{8\}$ | $2\left(1-j^{2}\right)$ | 38 | $\{248\} \rightarrow\{0\}$ | $3\left(1-j^{2}\right)$ | 39 | $\{240\} \rightarrow\{7\}$ | $\left(1-j^{2}\right)$ |
| 40 | $\{256\} \rightarrow\{6\}$ | $\left(j-j^{2}\right)$ | 41 | $\{257\} \rightarrow\{7\}$ | $\left(j^{2}-j\right)$ | 42 | $\{258\} \rightarrow\{8\}$ | $\left(j-j^{2}\right)$ |
| 43 | $\{250\} \rightarrow \tilde{O}$ | 0 | 44 | $\{267\} \rightarrow\{0\}$ | $3(1-j)$ | 45 | $\{268\} \rightarrow\{7\}$ | $2(1-j)$ |
| 46 | $\{260\} \rightarrow\{8\}$ | $(1-j)$ | 47 | $\{278\} \rightarrow\{2\}$ | $\left(j-j^{2}\right)$ | 48 | $\{270\} \rightarrow\{3\}$ | $(j-1)$ |
| 49 | $\{280\} \rightarrow\{1\}$ | $\left(j^{2}-1\right)$ | 50 | $\{345\} \rightarrow\{6\}$ | $2\left(j^{2}-j\right)$ | 51 | $\{346\} \rightarrow\{4\}$ | $\left(j^{2}-j\right)$ |
| 52 | $\{347\} \rightarrow\{0\}$ | $3(1-j)$ | 53 | $\{348\} \rightarrow\{7\}$ | $2(1-j)$ | 54 | $\{340\} \rightarrow\{8\}$ | $(1-j)$ |
| 55 | $\{356\} \rightarrow\{5\}$ | $j-j^{2}$ | 56 | $\{357\} \rightarrow\{8\}$ | $2\left(1-j^{2}\right)$ | 57 | $\{358\} \rightarrow\{0\}$ | $3\left(1-j^{2}\right)$ |
| 58 | $\{350\} \rightarrow\{7\}$ | $\left(1-j^{2}\right)$ | 59 | $\{367\} \rightarrow\{7\}$ | $\left(j^{2}-j\right)$ | 60 | $\{368\} \rightarrow\{8\}$ | $\left(j-j^{2}\right)$ |
| 61 | $\{360\} \rightarrow O$ | 0 | 62 | $\{378\} \rightarrow\{3\}$ | $\left(j-j^{2}\right)$ | 63 | $\{370\} \rightarrow\{1\}$ | $(j-1)$ |
| 64 | $\{380\} \rightarrow\{2\}$ | $\left(j^{2}-1\right)$ | 65 | $\{456\} \rightarrow\{0\}$ | $3\left(j^{2}-j\right)$ | 66 | $\{457\} \rightarrow\{1\}$ | $2(1-j)$ |
| 67 | $\{458\} \rightarrow\{2\}$ | $2\left(j^{2}-1\right)$ | 68 | $\{450\} \rightarrow\{3\}$ | $\left(j^{2}-j\right)$ | 69 | $\{467\} \rightarrow\{1\}$ | $2(j-1)$ |
| 70 | $\{468\} \rightarrow\{1\}$ | $2\left(1-j^{2}\right)$ | 71 | $\{460\} \rightarrow\{2\}$ | $\left(j-j^{2}\right)$ | 72 | $\{478\} \rightarrow\{4\}$ | $\left(j^{2}-j\right)$ |
| 73 | $\{470\} \rightarrow\{6\}$ | $(1-j)$ | 74 | $\{480\} \rightarrow\{5\}$ | $\left(1-j^{2}\right)$ | 75 | $\{567\} \rightarrow\{2\}$ | $2(1-j)$ |
| 76 | $\{568\} \rightarrow\{3\}$ | $2\left(j^{2}-j\right)$ | 77 | $\{560\} \rightarrow\{1\}$ | $\left(j^{2}-j\right)$ | 78 | $\{578\} \rightarrow\{5\}$ | $\left(j^{2}-j\right)$ |
| 79 | $\{570\} \rightarrow\{4\}$ | $(1-j)$ | 80 | $\{580\} \rightarrow\{6\}$ | $\left(1-j^{2}\right)$ | 81 | $\{678\} \rightarrow\{6\}$ | $\left(j^{2}-j\right)$ |
| 82 | $\{670\} \rightarrow\{5\}$ | $(1-j)$ | 83 | $\{680\} \rightarrow\{4\}$ | $\left(1-j^{2}\right)$ | 84 | $\{780\} \rightarrow \tilde{O}$ | 0 |

where

$$
\begin{array}{lll}
z_{0}=x_{0}+x_{7}+x_{8}, & \tilde{z}_{0}=x_{0}+j x_{7}+j^{2} x_{8}, & \tilde{z}_{0}=x_{0}+j^{2} x_{7}+j x_{8}, \\
z_{1}=x_{1}+x_{2}+x_{3}, & \tilde{z}_{1}=x_{1}+j x_{2}+j^{2} x_{3}, & \tilde{\tilde{z}}_{1}=x_{1}+j^{2} x_{2}+j x_{3},  \tag{59}\\
z_{2}=x_{4}+x_{5}+x_{6}, & \tilde{z}_{2}=x_{4}+j^{2} x_{4}+j x_{5}, & \tilde{z}_{2}=x_{4}+j x_{5}+j^{2} x_{6} .
\end{array}
$$

According to the permutation groups $S_{2}$ and $S_{3}$, one can consider $S_{2}$ and two $S_{3}$ transposition operations:

$$
\begin{array}{cc}
t\left(S_{2}\right): & 1 \rightarrow 2 ; 2 \rightarrow 1 \\
T_{+}\left(S_{3}\right): & 1 \rightarrow 2 ; 2 \rightarrow 3 ; 3 \rightarrow 1  \tag{60}\\
T_{-}\left(S_{3}\right): & 3 \rightarrow 2 ; 2 \rightarrow 1 ; 1 \rightarrow 3 .
\end{array}
$$

For example, in the $S_{3}$ transposition, we can obtain the following transformation (conjugations) for the $q_{i}(i=1,2,3)$ elements:

$$
\begin{array}{llll}
q_{1}^{T_{+}}=q_{1}, & q_{2}^{T_{+}}=j^{2} q_{2}, & q_{3}^{T_{+}}=j q_{3}, \\
q_{4}^{T_{+}}=q_{4}, & q_{5}^{T_{+}}=j^{2} q_{5}, & q_{6}^{T_{+}}=j q_{6}, \\
q_{0}^{T_{+}}=q_{0}, & q_{7}^{T_{+}}=j^{2} q_{7}, & q_{8}^{T_{+}}=j q_{8} . \tag{61}
\end{array}
$$

and

$$
\begin{array}{lll}
q_{1}^{T_{-}}=q_{1}, & q_{2}^{T_{-}}=j q_{2}, & q_{3}^{T_{-}}=j^{2} q_{3} \\
q_{4}^{T_{-}}=q_{4}, & q_{5}^{T_{-}}=j q_{5}, & q_{6}^{T_{+}}=j^{2} q_{6} \\
q_{0}^{T_{-}}=q_{0}, & q_{7}^{T_{-}}=j q_{7}, & q_{8}^{T_{-}}=j^{2} q_{8}, \tag{62}
\end{array}
$$

respectively.
In the two subsequent $S_{3}{ }^{+}$transpositions, the $Q$-matrices are transformed in the following way:

$$
Q^{T_{+}}=\left(\begin{array}{ccc}
x_{0}+j^{2} x_{7}+j x_{8} & x_{1}+j^{2} x_{2}+j x_{3} & x_{4}+j x_{5}+j^{2} x_{6}  \tag{63}\\
x_{4}+j^{2} x_{5}+j x_{6} & x_{0}+x_{7}+x_{8} & x_{1}+x_{2}+x_{3} \\
x_{1}+j x_{2}+j^{2} x_{3} & x_{4}+x_{5}+x_{6} & x_{0}+j x_{7}+j^{2} x_{8}
\end{array}\right)
$$

and

$$
Q^{T T_{+}}=\left(\begin{array}{ccc}
x_{0}+j x_{7}+j^{2} x_{8} & x_{1}+j x_{2}+j^{2} x_{3} & x_{4}+x_{5}+x_{6}  \tag{64}\\
x_{4}+j^{2} x_{5}+j x_{6} & x_{0}+j^{2} x_{7}+j x_{8} & x_{1}+j^{2} x_{2}+j x_{3} \\
x_{1}+x_{2}+x_{3} & x_{4}+j^{2} x_{5}+j x_{6} & x_{0}+x_{7}+x_{8}
\end{array}\right)
$$

All such ternary conjugations conserve the cubic form, $\left\langle Q_{D e t}\right\rangle^{3}$, which can be written as the following cubic polynomial:

$$
\begin{align*}
& \left(x_{0}^{3}+x_{7}^{3}+x_{8}^{3}-3 x_{0} x_{7} x_{8}\right)+\left(x_{1}^{3}+x_{2}^{3}+x_{3}^{3}-3 x_{1} x_{2} x_{3}\right)+\left(x_{4}^{3}+x_{5}^{3}+x_{6}^{3}-3 x_{4} x_{5} x_{6}\right) \\
& -3 x_{0}\left(x 1 x_{4}+x_{2} x_{5}+x_{3} x_{6}\right)-3 j^{2} x_{7}\left(x_{1} x_{5}+x_{2} x_{6}+x_{3} x_{4}\right)-3 j x_{8}\left(x_{1} x_{6}+x_{2} x_{4}+x_{3} x_{5}\right) . \tag{65}
\end{align*}
$$

This form is also symmetric under the complex conjugation and the following transformations: $x_{1} \leftrightarrow x_{4}, x_{2} \leftrightarrow x_{5}, x_{3} \leftrightarrow x_{6}, x_{7} \leftrightarrow x_{8}$. To make this form real under complex conjugation we should use all possible conjugations or automorphisms of this algebra [40]:

$$
\begin{equation*}
<Q>^{3}=\sum_{\text {automor }} Q \bar{Q} \overline{\bar{Q}} \tag{66}
\end{equation*}
$$

From the composition rule

$$
\begin{equation*}
\left\langle Q_{1} Q_{2}\right\rangle^{3}=\left\langle Q_{1}\right\rangle^{3}\left\langle Q_{2}\right\rangle^{3} \tag{67}
\end{equation*}
$$

we can see that the ternary $Q$-numbers produce a cubic form division algebra, which could be linked to the ternary group $\operatorname{TSU}(3)$, which is itself related with the observed ternary algebra.

If the binary alternative division algebras (real numbers, complex numbers, quaternions, octonions) over the real numbers have the dimensions $2^{p}, p=0,1,2,3,4, \ldots$, the ternary algebras have the following dimensions $3^{p}, p=0,1,2,3,4, \ldots$, respectively:

$$
\begin{array}{lc|cc}
\mathbb{R}: & 2^{0}=1 & \mathbb{R}: & 3^{0}=1  \tag{68}\\
\mathbb{C}: & 2^{1}=1+1 & \mathbb{T} C: & 3^{1}=1+1+1 \\
\mathbb{H}: & 2^{2}=1+2+1 & \mathbb{T} H: & 3^{2}=1+2+3+2+1 \\
\mathbb{O}: & 2^{3}=1+3+3+1 & \mathbb{T} O: & 3^{3}=1+3+6+7+6+3+1 \\
\mathbb{S}: & 2^{4}=1+4+6+4+1 & \mathbb{T} S: & 3^{4}=1+4+10+16+19+16+10+4+1
\end{array}
$$

In the last line the sedenions do not produce the division algebra.
For both cases we have the unit elements $e_{0}$ and $q_{0}$, and the $(n-1)$ basis elements

$$
\begin{array}{cc}
\left\{e_{a}: \quad e_{a}^{2}=-e_{0}\right\}, \\
\left\{q_{a}:\right. & \left.q_{a}^{3}=q_{0}\right\} \tag{69}
\end{array}
$$

respectively.

$$
\begin{equation*}
\mathbb{R} \rightarrow \mathbb{T} C \rightarrow \mathbb{T} H \rightarrow \mathbb{T} O \rightarrow \mathbb{T} S \rightarrow \ldots \tag{70}
\end{equation*}
$$

To build the ternary "octonions" from ternary "quaternion", one needs the additional basis element. For illustration we take the following three basis elements: the previous two, $q_{1}, q_{7}$ and new third element $q_{21}$. Then, applying the generalized Cayley-Dickson
method, we can get the 27-dimensional algebra with $q_{a}^{3}=q_{0}, a=1,2, \ldots, 26$ (see figure (5)

$$
\begin{align*}
T O= & =\left(x_{0} q_{0}+x_{1} q_{1}+x_{2} q_{2}+x_{3} q_{3}+x_{4} q_{4}+x_{5} q_{5}+x_{6} q_{6}+x_{7} q_{7}+x_{8} q_{8}\right) \\
& +\left(y_{0} q_{0}+y_{1} q_{1}+y_{2} q_{2}+y_{3} q_{3}+y_{4} q_{4}+y_{5} q_{5}+y_{6} q_{6}+y_{7} q_{7}+y_{8} q_{8}\right) q_{21} \\
& +\left(z_{0} q_{0}+z_{1} q_{1}+z_{2} q_{2}+z_{3} q_{3}+z_{4} q_{4}+z_{5} q_{5}+z_{6} q_{6}+z_{7} q_{7}+z_{8} q_{8}\right) q_{21}^{2} \tag{71}
\end{align*}
$$

## 6 Ternary algebras of higher rank

Now we would like to create the ternary algebra of rank 3 from the simple ternary algebra, which can be described by a tetrahedron Berger graph [6]. This is similar to the way of creating the $s u(3)(s u(r+1))$ Lie algebra from $s u(2)$-algebras. The $s u(r+1)$ algebra is the algebra of rank r having $r$-simple roots $\alpha_{a}(a=1,2, \ldots, r)$, each of them corresponding to the $s u(2) / u(1)$ algebra. This is a way to construct the $s u(r+1)$ algebra from r-su(2) algebras! We can try to do same to construct the $\operatorname{Berg}_{r}$ algebras from the minimal simple Berg-algebra.

Let us take three ternary algebras based on the $q_{A}(m, n, i), q_{B}(n, p, j), q_{C}(m, p, k)$ matrices, where $A(m, n, p), B(n, p, j), C(m, p, k)=0,1,2, \ldots, 8$. According to our conception (Figure 11) we would like to obtain the ternary algebra of higher rank defined by generators $q_{D}(i, j, k)$. For this we should consider all ternary commutation relations: $\left[Q_{a}, Q_{a}^{\prime}, Q_{a}^{\prime \prime}\right]_{S_{3}}$, where $Q_{a}$ is the complete set of all considered elements $Q_{a}=\left\{q_{A}, q_{B}, q_{C}\right\}$. For example, we can consider the following:

$$
\begin{align*}
& q_{A}=\left(\begin{array}{cccccc}
0 & m n & 0 & 0 & 0 & 0 \\
0 & 0 & n i & 0 & 0 & 0 \\
i m & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) q_{T A}=\left(\begin{array}{cccccc}
0 & 0 & 0 & m i & 0 & 0 \\
n m & 0 & 0 & 0 & 0 & 0 \\
0 & i n & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) q_{A}^{0}=\left(\begin{array}{cccccc}
m m & 0 & 0 & 0 & 0 & 0 \\
n n & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)  \tag{72}\\
& q_{B}=\left(\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & n p & \cdot 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p j \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & j n & 0 & 0 & 0 & 0
\end{array}\right) q_{B}^{T}=\left(\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & n j \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & p n & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & j p & 0 & 0
\end{array}\right) q_{B}^{0}=\left(\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & n n & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & p p & \cdot 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & j j
\end{array}\right) \tag{73}
\end{align*}
$$




Figure 1: The rank-3 Berger graph

$$
\begin{align*}
& q_{C}=\left(\begin{array}{cccccc}
0 & 0 & 0 & m p & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & p k & 0 \\
k m & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) q_{C}^{T}=\left(\begin{array}{cccccc}
0 & 0 & 0 & 0 & m k & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
p m & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & k p & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) q_{C}^{0}=\left(\begin{array}{cccccc}
m m & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & p p & 0 & 0 \\
0 & 0 & 0 & 0 & k k & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)  \tag{74}\\
& q_{D}=\left(\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i k & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & k j \\
0 & 0 & j i & 0 & 0 & 0
\end{array}\right) q_{D}^{T}=\left(\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & i j \\
0 & 0 & k i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & j k & 0
\end{array}\right) q_{D}^{0}=\left(\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & i i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & k k & 0 \\
0 & 0 & 0 & 0 & 0 & j j
\end{array}\right) \tag{75}
\end{align*}
$$

$$
t_{s u(2)}=\left(\begin{array}{c|c|cc|c|c}
m m & 0 & 0 & 0 & 0 & m j  \tag{76}\\
\hline 0 & n n & 0 & 0 & n k & 0 \\
\hline 0 & 0 & i i & p i & 0 & 0 \\
0 & 0 & i p & p p & 0 & 0 \\
\hline 0 & k n & 0 & 0 & k k & k j \\
\hline j m & 0 & 0 & 0 & 0 & j j
\end{array}\right)
$$

## 7 Appendix: Quaternary algebra

Now we would like to illustrate the quaternary algebra.

$$
\begin{equation*}
q_{m}^{4}=1, \quad m=0,1,2,3, \ldots, 14,15 \tag{77}
\end{equation*}
$$

$$
\begin{align*}
& q_{m+1}=q_{13}^{m} q_{1}, \quad q_{4-m}=q_{15}^{m} q_{4}, m=0,1,2,3  \tag{78}\\
& q_{5+m}=q_{13}^{m} q_{5}, q_{8-m}=q_{15}^{m} q_{8}, m=0,1,2,3 \\
& q_{9+m}=q_{13}^{m} q_{9}, \quad q_{12-m}=q_{15}^{m} q_{12} . m=0,1,2,3
\end{align*}
$$

$$
q_{13}^{2}=q_{14}, \quad q_{13}^{3}=q_{15}
$$

$$
q_{14}^{2}=q_{0}
$$

$$
\begin{equation*}
q_{15}^{2}=q_{14}, \quad q_{15}^{3}=q_{13} \tag{79}
\end{equation*}
$$

$$
\begin{align*}
& q_{1}^{2}=q_{5}, \quad q_{2}^{2}=j q_{7}, \quad q_{3}^{2}=j^{2} q_{5}, \quad q_{4}^{2}=j^{3} q_{7}, \\
& q_{1}^{3}=q_{9}, \quad q_{2}^{3}=j^{3} q_{12}, \quad q_{3}^{3}=j^{2} q_{11}, \quad q_{4}^{3}=j q_{10} \tag{80}
\end{align*}
$$

$$
\begin{equation*}
q_{5}^{2}=q_{0}, q_{6}^{2}=j^{2} q_{14}^{2}, q_{7}^{2}=q_{0}, q_{8}^{2}=j^{2} q_{14} \tag{81}
\end{equation*}
$$

$$
\begin{align*}
& q_{9}^{2}=q_{5}, \quad q_{10}^{2}=j^{3} q_{7}, \quad q_{11}^{2}=j^{2} q_{5}, \quad q_{12}^{2}=j q_{7}, \\
& q_{9}^{3}=q_{1}, \quad q_{10}^{3}=j q_{4}, \quad q_{11}^{3}=j^{2} q_{3}, \quad q_{12}^{3}=j^{3} q_{2} \tag{82}
\end{align*}
$$

We can consider the $4 \times 4$ matrix realization of $q$-algebra:
where $j=\exp 2 \mathbf{i} \pi / 4$.

$$
\begin{align*}
& q_{1}=\left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{array}\right) q_{2}=\left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & j & 0 \\
0 & 0 & 0 & j^{2} \\
j^{3} & 0 & 0 & 0
\end{array}\right) q_{3}=\left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & j^{2} & 0 \\
0 & 0 & 0 & 1 \\
j^{2} & 0 & 0 & 0
\end{array}\right) q_{4}=\left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & j^{3} & 0 \\
0 & 0 & 0 & j^{2} \\
j & 0 & 0 & 0
\end{array}\right) \\
& q_{5}=\left(\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{array}\right) q_{6}=\left(\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & j \\
j^{2} & 0 & 0 & 0 \\
0 & j^{3} & 0 & 0
\end{array}\right) q_{7}=\left(\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & j^{2} \\
1 & 0 & 0 & 0 \\
0 & j^{2} & 0 & 0
\end{array}\right) q_{8}=\left(\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & j^{3} \\
j^{2} & 0 & 0 & 0 \\
0 & j & 0 & 0
\end{array}\right) \\
& q_{9}=\left(\begin{array}{cccc}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right) q_{10}=\left(\begin{array}{cccc}
0 & 0 & 0 & 1 \\
j & 0 & 0 & 0 \\
0 & j^{2} & 0 & 0 \\
0 & 0 & j^{3} & 0
\end{array}\right) q_{11}=\left(\begin{array}{cccc}
0 & 0 & 0 & 1 \\
j^{2} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & j^{2} & 0
\end{array}\right) q_{12}=\left(\begin{array}{cccc}
0 & 0 & 0 & 1 \\
j^{3} & 0 & 0 & 0 \\
0 & j^{2} & 0 & 0 \\
0 & 0 & j & 0
\end{array}\right) \\
& q_{13}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & j & 0 & 0 \\
0 & 0 & j^{2} & 0 \\
0 & 0 & 0 & j^{3}
\end{array}\right) q_{14}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & j^{2} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & j^{2}
\end{array}\right) q_{15}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & j^{3} & 0 & 0 \\
0 & 0 & j^{2} & 0 \\
0 & 0 & 0 & j
\end{array}\right) q_{0}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) \tag{83}
\end{align*}
$$

## 8 Conclusions and Acknowledgements

Our interest in the search for the $n$-ary algebras has a long history and started in 1998 when we have read the pioneer article of P. Candelas and M. Font [41].

We found the "minimal" non-Abelian ternary algebra which we intend using for solving the set of Berger graphs. This could help us to find the ternary generalization of Cartan-Killing-Lie classification. Also it is very important to understand the ternary "octonions" and find a link between binary and ternary exceptional graphs and algebras. Examples of the ternary algebras, which are presented here, could help us to find some physical applications, for example, in solving of the tetrahedron Baxter equation. The other physical application is related to searching for a ternary generalization of Lorentz group. In the end we hope that the ternary symmetries could help us to make the quantium theory of membranes.

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