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Dual realizations of dynamical symmetry breaking

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ABSTRACT: We show the infrared equivalence between a recently proposed model containing a six dimensional scalar field with a four-dimensional localized Higgs type potential and the four-dimensional Nambu-Jona-Lasinio (NJL) model. In the dual NJL description, the fermions are localized at the origin of a large two-dimensional compact space. Due to a classical running effect above the compactification scale, the four-fermion coupling of the NJL model increases from the cutoff scale down to the compactification scale, providing the large Fermi coupling needed for the dynamical symmetry breaking. We also present a string theory embedding of our field-theory construction. On more general grounds, our results suggest that 4d models with dynamical symmetry breaking can be given a higher dimensional description in terms of field theories with nontrivial boundary conditions in the internal space.

KEYWORDS: Duality in Gauge Field Theories, Large Extra Dimensions, Compactification and String Models, Supersymmetry and Duality.

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1. Introduction and conclusions

Dimensional transmutation and generation of a small scale is a remarkable result common to many quantum field theories, most notably the four-dimensional QCD and the two-dimensional Gross-Neveu model. The effect is also realized in two-dimensional quantum mechanics with a deep (delta-like) attractive potential and in six-dimensional scalar models¹ with 4d localized scalar potential and a large but compact transverse space. The last example, put forward in [2], was analyzed from the point of the quantum-mechanical problem in the case of a perturbative coupling μ appearing as the (dimensionless) localized parameter interpreted as a mass term in 4d. It was shown that for a 6d scalar field with Dirichlet boundary condition on a large two-dimensional compact space taken for simplicity to be a disk, there is a phase transition with a very light (compared to the compactification scale) particle for a small critical value $\mu_c \ll 1$. The parameter μ was shown to run between the cutoff scale Λ and the compactification scale R^{-1} , such that precisely at the critical point, μ becomes large at R^{-1} . A very similar phenomenon of appearance of a light state close to a critical point where an (four-fermion) interaction becomes strong is in the 4d Nambu-Jona-Lasinio (NJL) model [3]. The purpose of the present paper is to study closer this analogy and argue that the 6d model studied in [2] and the 4d NJL model are, in a

¹For gravitational aspects of codimension two models, see e.g. [1].

sense which will be defined in detail later on, dual descriptions of the same physics. Our starting point is to provide an explicit framework in which the 4d localized potential is generated, by a Yukawa interaction of the bulk scalar field with N 4d localized fermions. In the large N limit, integrating out the fermions produces precisely the potential needed for the symmetry breaking. Alternatively, we show that integrating out the bulk scalar leads to a dual 4d NJL model with chiral symmetry breaking, where the Fermi coupling is generated at tree-level by scalar bulk exchange. We show that the critical Fermi coupling calculated by NJL methods in the large N limit agrees with the bulk 6d calculation of the critical coupling calculated as a problem with nontrivial boundary conditions. The $6d \leftrightarrow 4d$ duality we study exchanges some quantum and classical natures of the symmetry breaking phenomenon. In the bulk 6d picture, the quantum (Yukawa) interactions are completely encoded in a boundary condition, the localized scalar potential, whereas the symmetry breaking can be studied as a quantum-mechanical problem with nontrivial boundary condition and can be understood as a result of a classical running effect in the transverse 2d space. In the 4d NJL picture, the symmetry breaking is provided by the nonperturbative self-consistent gap equation [3–5], but in addition the four-fermion coupling has a classical logarithmic running between the cutoff and the compactification scale. When in the bulk picture $\mu = \mu_c$ at Λ , the four-fermion coupling in the NJL picture at the compactification scale reaches the critical value for the dynamical symmetry breaking $G(R^{-1}) = G_c$. Our main interest in this equivalence is that, whereas a consistent treatment of the NJL model involves nonperturbative techniques like the large N expansion or going below 4d and using UV nontrivial fixed points and $1/\epsilon$ techniques, the bulk analysis is essentially classical² and does not need, in principle, any nonperturbative techniques.

The structure of the paper is as follows. In section 2 we review the six dimensional model worked out in [2] and argue that, in addition to the perturbative critical coupling μ_c we found there, there are other critical points corresponding to large values $\mu_c^{(n)} > 1$. In section 2.1 we study a similar setup in which the nontrivial boundary condition in the compact space is replaced by a bulk mass and Neumann boundary condition. In section 3 we define the 6d model on an orbifold space instead of a disk, which is more suitable for a microscopic (string theory) realization. Section 4 contains the main arguments concerning the infrared equivalence between the 6d scalar model with nontrivial boundary condition and the 4d NJL model. Section 5 generalizes the previous section to a (softly broken) supersymmetric theory. Section 6 provides an explicit string theory realization of the present setup in an orientifold of type IIB strings with D-branes.

2. Six dimensional phase transition: perturbative and nonperturbative critical couplings

Recently, [2] addressed the problem of spontaneous symmetry breaking in a 6d scalar model

²Throughout the paper by “classical” we mean classical from the point of view of quantum field theory, i.e. no quantum interaction. The treatment is still quantum mechanical.

with 4d localized Higgs potential. The corresponding action reads:³

$$S = \int d^4x d^2y \left[\frac{1}{2} (\partial_M \phi)^2 - V_\delta(\phi) \right],$$

$$V_\delta(\phi) = \left(-\frac{\mu^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4 \right) \cdot \delta^2(y). \quad (2.1)$$

The scalar field ϕ has dimension two and therefore μ^2 is dimensionless. The scalar potential is localized at the origin of the compact space. We resolve the singularity at $y = 0$ by introducing a disk $r < \epsilon$ supporting the potential,

$$V(\phi) = \frac{1}{\pi \epsilon^2} \left(-\frac{\mu^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4 \right) \quad \text{for } 0 < r < \epsilon, \quad (2.2)$$

$$V(\phi) = 0 \quad \text{for } \epsilon < r < R.$$

According to [6, 7, 2], there is a "classical" running of the tachyonic mass parameter

$$\frac{1}{\mu^2(Q)} = \frac{1}{\mu^2(\Lambda)} - \frac{1}{2\pi} \ln\left(\frac{\Lambda}{Q}\right). \quad (2.3)$$

It was shown in [2] for a compact two-dimensional space, chosen to be a disk of radius R , with 4d localized Higgs potential and Dirichlet boundary condition

$$\phi|_{r=R} = 0, \quad (2.4)$$

that this model has a phase transition for a small critical value

$$\frac{\mu_c^2}{2\pi} \ln(R\Lambda) = 1, \quad (2.5)$$

where Λ is a UV cutoff defined in connection with the resolution of the delta singularity $\Lambda = 1/\epsilon$. So the phase transition happens precisely when the renormalized value $\mu_c^2(R^{-1}) \rightarrow \infty$ blows up at the compactification scale. The running interpretation breaks down close to the phase transition point. The classical running of μ induces also a running for the self-coupling λ , according to the RG equation

$$Q \frac{d\lambda}{dQ} = -\frac{2}{\pi} \mu^2 \lambda, \quad (2.6)$$

which, by using (2.3), readily integrates to

$$\lambda(Q) = \frac{\lambda(\Lambda)}{\left(1 - \frac{\mu^2}{2\pi} \ln \frac{\Lambda}{Q}\right)^4}. \quad (2.7)$$

Notice that at the phase transition point $\mu = \mu_c$,

$$\mu(R^{-1}) \rightarrow \infty, \quad \lambda(R^{-1}) \rightarrow \infty. \quad (2.8)$$

³We are using a (+, -, -, -, -, -) metric. The index M denotes bulk coordinates and runs from 0, 1, 2, 3, 4, 5, while $\mu = 0, 1, 2, 3$ denotes brane coordinates. We'll use either $x^{4,5}$ or $y_{1,2}$ to denote the two extra dimensions.

We will argue later on in section 4 that in a dual 4d theory which turns out to be a NJL theory, the conditions (2.8) have the interpretation of compositeness conditions of [5]. Close to the critical coupling, however, the running interpretation breaks down and actually the higher-dimensional 6d and also the 4d physics turn out to be perturbative.

We now review and slightly update the arguments of [2] by arguing that there are actually additional but large critical couplings $\mu_c^{(n)} \geq \sqrt{4\pi}$, defined by the presence of a 4d massless mode in the spectrum. Assuming that this exists, slightly below it $\mu \leq \mu_c$, in the background $\phi_c = 0$, the field eqs. for a 4d mode of mass $M^2 = p^2$, are

$$\begin{aligned} \Delta^{(2)}\phi + \frac{\mu^2}{\pi\epsilon^2}\phi &= 0, \quad r < \epsilon, \\ \Delta^{(2)}\phi + p^2\phi &= 0, \quad r > \epsilon, \end{aligned} \tag{2.9}$$

where we neglected the mass p^2 inside the brane (this is a very good approximation for all masses much lighter than the cutoff $\Lambda = 1/\epsilon$). The solutions of (2.9) with Dirichlet boundary condition (2.4) and for $p^2 \ll R^{-2}$ are

$$\begin{aligned} \phi(r) &= f_0 J_0\left(\frac{\mu r}{\sqrt{\pi\epsilon}}\right), \quad r < \epsilon, \\ \phi(r) &= a \left[\ln \frac{R}{r} - \frac{p^2 r^2}{4} \ln \frac{R}{r} + \frac{p^2}{4}(R^2 - r^2) \right], \quad r > \epsilon. \end{aligned} \tag{2.10}$$

The zero mass solutions $p^2 = 0$ define the whole set of critical couplings. The matching conditions of the logarithmic derivative of the wave function at $r = \epsilon$ then give

$$\frac{\mu_c J'_0\left(\frac{\mu_c}{\sqrt{\pi}}\right)}{\sqrt{\pi} J_0\left(\frac{\mu_c}{\sqrt{\pi}}\right)} \ln \frac{R}{\epsilon} = -1. \tag{2.11}$$

For small μ_c , eq. (2.11) has the unique solution (2.5), which is indeed small provided that $R^{-1} \ll \Lambda$. Equation (2.11), however, has an *infinite discrete set* of solutions, as can easily be shown by a numerical plot. The peculiarity of the perturbative solution (2.5) is that the wave function (2.10) inside the brane $r < \epsilon$, and actually also the wave functions of the massive modes below the cutoff Λ , are almost constant. On the contrary, the wavefunctions corresponding to the “nonperturbative” critical couplings (in the sense $\mu_c^2/4\pi > 1$) have substantial variation inside the brane. Slightly below the critical coupling(s) μ_c , the zero mode becomes massive, with a mass M given approximately by

$$M^2 = p_\mu p^\mu \simeq \frac{2(\mu_c^2 - \mu^2)}{\pi R^2} \left(\ln \frac{R}{\epsilon}\right)^2. \tag{2.12}$$

The mass (2.12) reduces to the one computed in [2] in the case of the perturbative critical coupling (2.5). Close to the other critical couplings $\mu \simeq \mu_c^{(n)}$ the light mode can also be described in a 4d effective field theory. The 6d field is decomposed according to $\phi(x, r) = \sigma(x)\chi(r)$, where

$$\begin{aligned} \chi(r) &= \sqrt{\frac{2}{\pi R^2}} \ln\left(\frac{R}{\epsilon}\right) \frac{J_0\left(\frac{\mu r}{\sqrt{\pi\epsilon}}\right)}{J_0\left(\frac{\mu}{\sqrt{\pi}}\right)}, \quad r < \epsilon, \\ \chi(r) &= \sqrt{\frac{2}{\pi R^2}} \ln \frac{R}{r}, \quad r > \epsilon. \end{aligned} \tag{2.13}$$

The effective 4d potential for σ is given by

$$V_{\text{eff}}(\sigma) = \frac{m_4^2}{2}\sigma^2 + \frac{\lambda_4}{4}\sigma^4, \quad (2.14)$$

where the mass parameter and the coupling are given by

$$\begin{aligned} m_4^2 &= -\frac{2\mu^2}{\epsilon^2} \int_0^\epsilon r dr \chi^2 + \frac{2\mu^2}{\epsilon^2} \int_0^\epsilon r dr (\chi')^2 + 2\pi \int_\epsilon^R r dr (\chi')^2, \\ \lambda_4 &= \frac{2\lambda}{\epsilon^2} \int_0^\epsilon r dr \chi^4(r) = \frac{8\lambda}{\pi^2 \mu^2 R^4} \frac{1}{J_0^4(\mu/\sqrt{\pi})} \left(\ln \frac{R}{\epsilon}\right)^4 \int_0^{\frac{\mu}{\sqrt{\pi}}} x dx J_0^4(x), \end{aligned} \quad (2.15)$$

where the derivative in χ' is wrt the argument of the Bessel functions. Very close to $\mu_c^{(n)}$, the resulting mass coincides with M^2 in (2.12), showing the validity of the 4d description. The presence of light 4d modes close to the large critical couplings $\mu_c^{(n)}$ is a signature of a UV-IR mixing, where the UV physics changes the masses in the IR. While in a microscopic theory in which μ is generated dynamically, large values ask presumably for nonperturbative effects, from the bulk 2d viewpoint, μ changes only the boundary conditions of the scalar field and its consequences can be treated exactly quantum-mechanically. On the other hand, the explicit values of $\mu_c^{(n)}$ depend on the way we regularize the origin of the 2d space and thus on the UV physics. Therefore, the physical consequences of the large critical couplings are probably highly sensitive on the UV physics. This is not the case for the small critical coupling (2.5), whose value is insensitive to the regularization procedure and therefore of the UV physics, as we explicitly check by using a different regularization in section 3.

2.1 Phase transition with bulk mass and boundary Higgs potential

A natural question arising in the present setup is what happens if one replaces the positive contribution to the mass coming from the Dirichlet boundary condition by a bulk mass m , keeping the 4d localized Higgs-type potential. The action describing this case is given by

$$\begin{aligned} S &= \int d^4x d^2y \left[\frac{1}{2}(\partial_M \phi)^2 - \frac{1}{2}m^2 \phi^2 - V_\delta(\phi) \right], \\ V_\delta(\phi) &= \left(-\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 \right) \cdot \delta^2(y), \end{aligned} \quad (2.16)$$

where the field ϕ has now Neumann boundary condition

$$\partial_r \phi|_{r=R} = 0. \quad (2.17)$$

We are working in the unbroken phase $\phi_c = 0$, in which case the field equations for a 4d field of mass p^2 are

$$\begin{aligned} \Delta^{(2)}\phi + \frac{\mu^2}{\pi\epsilon^2}\phi &= 0, \quad r < \epsilon, \\ \Delta^{(2)}\phi + (p^2 - m^2)\phi &= 0, \quad r > \epsilon. \end{aligned} \quad (2.18)$$

By defining $q^2 = m^2 - p^2$, we find the solutions of (2.18) with Neumann boundary conditions (2.17) and for $q^2 \ll R^{-2}$ to be

$$\begin{aligned} \phi(r) &= f_0 J_0\left(\frac{\mu r}{\sqrt{\pi\epsilon}}\right) \quad , \quad r < \epsilon, \\ \phi(r) &= a \left[1 + \frac{q^2 R^2}{2} \ln \frac{R}{r} + \frac{q^2}{4} r^2 \right] \quad , \quad r > \epsilon. \end{aligned} \quad (2.19)$$

Matching conditions at $r = \epsilon$ for zero mass solutions $p^2 = 0$ define the critical couplings in this case to be given by the solutions of

$$\left[1 - \frac{\mu_c}{\sqrt{\pi}} \frac{J_1\left(\frac{\mu_c}{\sqrt{\pi}}\right)}{J_0\left(\frac{\mu_c}{\sqrt{\pi}}\right)} \ln \frac{R}{\epsilon} \right] \frac{m^2 R^2}{2} = \frac{\mu_c}{\sqrt{\pi}} \frac{J_1\left(\frac{\mu_c}{\sqrt{\pi}}\right)}{J_0\left(\frac{\mu_c}{\sqrt{\pi}}\right)}. \quad (2.20)$$

Analogously to the case discussed in the previous section, eq. (2.20) has an infinity but discrete number of solutions $\mu_c^{(n)}$, out of which only *one* is perturbative $\mu_c \ll 1$. In this perturbative case, similarly to (2.5), there is a classical running interpretation of the critical coupling

$$\pi R^2 m^2 = \frac{\mu_c^2}{1 - \frac{\mu_c^2}{2\pi} \ln \frac{R}{\epsilon}} = \mu_c^2(R^{-1}), \quad (2.21)$$

where $\mu_c^2(R^{-1})$ is the renormalized value of the (perturbative) critical coupling at the compactification scale $Q = R^{-1}$. From (2.21) it follows that for small bulk mass $m^2 R^2 \ll 1$, μ stays perturbative at all energies above the compactification scale, whereas for large masses $m^2 R^2 \geq 1$, μ enters strong coupling regime if there is a light 4d mode in the spectrum.

Slightly below the critical couplings, the light 4d mass is given by

$$p^2 = m^2 - \frac{2}{R^2} \frac{\mu J_1\left(\frac{\mu}{\sqrt{\pi}}\right)/(\sqrt{\pi} J_0\left(\frac{\mu}{\sqrt{\pi}}\right))}{1 - \ln \frac{R}{\epsilon} \mu J_1\left(\frac{\mu}{\sqrt{\pi}}\right)/(\sqrt{\pi} J_0\left(\frac{\mu}{\sqrt{\pi}}\right))}. \quad (2.22)$$

Very close to the perturbative critical coupling (2.21), the light mass becomes

$$p^2 = m^2 - \frac{1}{\pi R^2} \frac{\mu^2}{1 - \frac{\mu^2}{2\pi} \ln \frac{R}{\epsilon}} = m^2 - \frac{1}{\pi R^2} \mu^2(R^{-1}) \quad (2.23)$$

and has again a transparent interpretation in terms of the classical running between the compactification scale and the cutoff.

There are also light 4d states for $q^2 \sim R^{-2}$ or larger. However, we are especially interested in the case of small bulk masses $m^2 R^2 \ll 1$, for reasons to be explained in the dual NJL formulation of a supersymmetric extension of this model.

3. Symmetry breaking phase transition in orbifolds

More standard and easy to handle spaces in string theory are orbifolds. We will consider in the following a compactification on the orbifold T^2/\mathbb{Z}_2 and check as a warmup the properties of the phase transition. The orbifold acts as the reflection $(y_1, y_2) \rightarrow (-y_1, -y_2)$.

This orbifold has four fixed points. The fixed points and their corresponding \mathbb{Z}_2 coordinate transformations are summarized as:

$$\begin{array}{cccc} y_1 \rightarrow -y_1 & y_1 \rightarrow -y_1 + 2\pi R_1 & y_1 \rightarrow -y_1 & y_1 \rightarrow -y_1 + 2\pi R_1 \\ y_2 \rightarrow -y_2 & y_2 \rightarrow -y_2 & y_2 \rightarrow -y_2 + 2\pi R_2 & y_2 \rightarrow -y_2 + 2\pi R_2 \\ (0, 0) & (\pi R_1, 0) & (0, \pi R_2) & (\pi R_1, \pi R_2) . \end{array} \quad (3.1)$$

In complex notation, the action of \mathbb{Z}_2 on the compact space is a two-dimensional π rotation, $Z_2(y_1 + iy_2) = e^{i\pi}(y_1 + iy_2)$.

The field equation is free in the bulk and has a delta function source at the origin, suitably replaced by a mass distribution

$$\partial_M \partial^M \Phi + \frac{\partial V}{\partial \Phi} \delta^2(\mathbf{y}) = 0 . \quad (3.2)$$

Let us now proceed to study the mass spectrum for a scalar field with antiperiodic boundary conditions in the y_1 -direction⁴

$$\Phi(y_1 + 2\pi R_1, y_2) = -\Phi(y_1, y_2) . \quad (3.3)$$

If the scalar field Φ is even under the orbifold action, it can be decomposed on a complete basis formed by the cosine functions:

$$\Phi(x, \mathbf{y}) = \sum_{(k_1, k_2) \in \mathcal{I}} \frac{1}{\sqrt{2\pi^2 R_1 R_2}} \cos\left(\frac{k_1 + 1/2}{R_1} y_1 + \frac{k_2}{R_2} y_2\right) \phi_{(k_1, k_2)}(x) . \quad (3.4)$$

The indices $k_{1,2}$ belong to the set \mathcal{I}

$$\mathcal{I} = \{(0; 0), (1 \dots \infty; 0), (0, -1; 1 \dots \infty), (1 \dots \infty; 1 \dots \infty), (1 \dots \infty; -\infty \dots -1)\} . \quad (3.5)$$

In the unbroken vacuum, the quadratic part of the scalar action takes the following form after integration over the two extra dimensions

$$\mathcal{L} = \mathcal{L}_{\text{kin}} - \frac{1}{2} \sum_{(k_1, k_2) \in \mathcal{I}} \left(\frac{(k_1 + 1/2)^2}{R_1^2} + \frac{k_2^2}{R_2^2} \right) \phi_{(k_1, k_2)}^2 + \bar{\mu}^2 \left(\sum_{(k_1, k_2) \in \mathcal{I}} \phi_{(k_1, k_2)} \right)^2 , \quad (3.6)$$

where

$$\bar{\mu}^2 \equiv \frac{\mu^2}{4\pi^2 R_1 R_2} \quad (3.7)$$

is the naive (volume suppressed) four dimensional lightest scalar mass. The mass term of the 4d action is

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2} \sum_{(k_1, k_2), (p_1, p_2) \in \mathcal{I}} \phi_{(k_1, k_2)} \mathcal{M}_{(k_1, k_2), (p_1, p_2)}^2 \phi_{(p_1, p_2)} , \quad (3.8)$$

⁴This is the analog of the Dirichlet boundary condition on the disk imposed in [2]. As will become clear from our discussion, different boundary conditions, for example $\Phi(y_1, y_2 + 2\pi R_2) = -\Phi(y_1, y_2)$ or $\Phi(y_1 + 2\pi R_1, y_2 + 2\pi R_2) = -\Phi(y_1, y_2)$ lead to the same critical coupling (2.5).

with the mass matrix given by

$$\mathcal{M}_{(k_1, k_2), (p_1, p_2)}^2 = -2\bar{\mu}^2 + \left(\frac{(k_1 + 1/2)^2}{R_1^2} + \frac{k_2^2}{R_2^2} \right) \delta_{k_1, p_1} \delta_{k_2, p_2} . \quad (3.9)$$

The diagonalization of this mass matrix defines the physical mass eigenstates.

Let us now try to find the eigenvalues and eigenvectors of the mass matrix (3.9). We use the techniques used in 5d models in [8, 9] and in 6d models in [7]. The characteristic equation is given by

$$\mathcal{M}^2 \Psi_m = m^2 \Psi_m , \quad (3.10)$$

where m^2 represents the eigenvalues and Ψ is the eigenvector in the basis $|k_1, k_2\rangle_{(k_1, k_2) \in \mathcal{I}}$, i.e $\Psi_{(k_1, k_2)} = \langle k_1, k_2 | \Psi_m \rangle$. The matrix equation (3.10) is equivalent to the infinite set of explicit equations for every $(k_1, k_2) \in \mathcal{I}$

$$2\bar{\mu}^2 \Psi' = \left(-m^2 + \frac{(k_1 + 1/2)^2}{R_1^2} + \frac{k_2^2}{R_2^2} \right) \Psi_{(k_1, k_2)} , \quad (3.11)$$

where Ψ' is independent of (k_1, k_2) . The solution of the equations (3.11) is given by

$$\Psi_{(k_1, k_2)} = \frac{\mathcal{N}}{-m^2 + (k_1 + 1/2)^2/R_1^2 + k_2^2/R_2^2} , \quad (3.12)$$

where \mathcal{N} is a normalization constant independent of (k_1, k_2) . Putting this solution back in the equation (3.11) and using the fact that

$$\sum_{(k_1, k_2) \in \mathcal{I}} = \frac{1}{2} \sum_{k_1, k_2 = -\infty}^{\infty} , \quad (3.13)$$

we obtain the eigenvalue equation

$$\frac{1}{\bar{\mu}^2} = \sum_{k_1, k_2 = -\infty}^{\infty} \frac{1}{-m^2 + (k_1 + 1/2)^2/R_1^2 + k_2^2/R_2^2} , \quad (3.14)$$

or equivalently

$$\frac{1}{\mu^2} = D(p^2 = m^2, y_1 = y_2 = 0) \quad , \quad \text{where} \\ D(p^2, y_1, y_2) = \frac{1}{4\pi^2 R_1 R_2} \sum_{k_1, k_2 = -\infty}^{\infty} \frac{\cos[(k_1 + 1/2)y_1/R_1 + k_2 y_2/R_2]}{-p^2 + (k_1 + 1/2)^2/R_1^2 + k_2^2/R_2^2} \quad (3.15)$$

is the propagator in a mixed, 4d momentum and 2d position, representation. We want to find an estimate for the lightest solution, m^2 , of the eigenvalue equation (3.14). The procedure we are using is similar to the one used in [7] and we only give the result here. It is clear from (3.14) that there is a critical coupling, defined for arbitrary radii by

$$\frac{4\pi^2 R_1 R_2}{\mu_c^2} = \sum_{|k_i| < k_i^{\max}} \frac{1}{(k_1 + 1/2)^2/R_1^2 + k_2^2/R_2^2} , \quad (3.16)$$

which signals a second order phase transition, where the lightest mass m^2 changes sign. For equal and large radii and by cutting the sums at $k_i^{\max} = R\Lambda$, we find μ_c to be exactly equal to the value (2.5) worked out in [2]. A puzzle arises however in this KK approach to the phase transition. Indeed, whereas we accurately describe the perturbative critical coupling (2.5), eq. (3.16) does not contain the nonperturbative couplings $\mu_c^{(n)}$ present in (2.11). We believe that this is due to the way the logarithmic divergence is handled in (3.16), or equivalently, to the “brane resolution” for nonperturbative values of μ^2 . Indeed, as we already mentioned, in this case wave functions oscillate significantly inside the brane and the regularization procedure becomes more subtle.

Very close and slightly below the phase transition we can linearize the mass equation (3.14) in order to find for the lightest mode

$$m^2 \simeq \frac{4\pi^2}{\alpha R_1 R_2} \frac{\mu_c^2 - \mu^2}{\mu_c^4} \simeq \frac{4\pi^2}{\alpha R_1 R_2} \left(\frac{1}{\mu^2} - \frac{1}{\mu_c^2} \right), \quad (3.17)$$

where

$$\alpha = R_1^{-2} R_2^{-2} \sum_{k_1, k_2 = -\infty}^{\infty} \frac{1}{[(k_1 + 1/2)^2/R_1^2 + k_2^2/R_2^2]^2}. \quad (3.18)$$

The mass (3.17) is not exactly the same as the one worked out in [2]. The reason is that the geometries in [2] and in the present section are different and affect the IR physics, in particular physical masses. UV physics, however is the same for both geometries ; in particular the value (at the leading order in μ^2) of the perturbative critical coupling μ_c defining the phase transition is the same as in (2.5). Since the regularizations used on the disk and in (3.16) are different, this shows the regularization independence of the perturbative critical coupling.

Above the critical value, the scalar gets a vev (more precisely, a profile in the compact space)

$$\phi_c(y_1, y_2) = \phi_0 \mathcal{N} \sum_{k_1, k_2} \frac{\cos[(k_1 + 1/2)y_1/R_1 + k_2 y_2/R_2]}{(k_1 + 1/2)^2/R_1^2 + k_2^2/R_2^2}, \quad (3.19)$$

where according to [2]

$$\phi_0^2 = \frac{\mu^2 - \mu_c^2}{\lambda} \quad (3.20)$$

and \mathcal{N} is a normalization constant such that at the origin (more precisely, at the regularized mass distribution), $\phi_c(0, 0) = \phi_0$.

4. Localized matter and dual description: the NJL model

An immediate question is how to generate in a natural way the localized scalar potential needed for the symmetry breaking. The simplest idea is to add (Weyl for definiteness, but the situation is similar for Dirac fermions) N fermions χ_i on the boundary, with Yukawa couplings

$$g\chi_i\chi_i\phi(y=0) + \text{h.c.}, \quad (4.1)$$

to the (now complex) bulk field. The model has, in addition to a global $\text{SO}(N)$ symmetry, a continuous chiral $\text{U}(1)$ symmetry under which ϕ is charged.

4.1 The bulk picture

One-loop quantum corrections generated by the fermion loops automatically generate a scalar potential of the appropriate form (2.1), plus higher-order terms. The continuous chiral symmetry will now be spontaneously broken at the phase transition.

In the large N limit, the leading induced scalar potential is

$$V_{\text{eff}}(\phi) = -N \int \frac{d^4p}{(2\pi)^4} \ln(p^2 + 4g^2|\phi|^2), \quad (4.2)$$

which can be expanded in powers of ϕ as

$$V_{\text{eff}}(\phi) = -4Ng^2|\phi|^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2} + 8Ng^4|\phi|^4 \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^4} + \dots \quad (4.3)$$

We therefore induced radiatively, to the leading order in an $1/N$ expansion, a potential of the form (2.1) with

$$\mu^2 = \frac{Ng^2}{4\pi^2} \Lambda^2. \quad (4.4)$$

As usual [10] the power expansion in ϕ has severe IR divergences, which are however resummed in the effective potential (4.2). Then the condition defining the symmetry breaking phase is

$$\frac{Ng^2}{4\pi^2} \Lambda^2 > \frac{4\pi}{\ln(R^2\Lambda^2)}, \quad (4.5)$$

whereas the perturbative expansion used in [2], for $\mu^2 \ll 1$, translates here into

$$\frac{Ng^2}{4\pi^2} \Lambda^2 \ll 1. \quad (4.6)$$

For $\langle\phi\rangle \neq 0$ the brane fermions χ_i acquire a mass and the chiral symmetry is spontaneously broken, with the imaginary part of ϕ being the Goldstone boson.

4.2 Dual picture: the NJL model

There is a dual description in which the bulk field ϕ is integrated out at tree-level and the chiral symmetry breaking is entirely described in terms of nonperturbative brane dynamics. The resulting brane lagrangian has the simple form

$$S_{\text{brane}} = -i \chi_i \sigma^\mu \partial_\mu \bar{\chi}_i + G \chi_i \chi_i \bar{\chi}_j \bar{\chi}_j, \quad (4.7)$$

with

$$G = \frac{g^2}{2\pi^2 R_1 R_2} \sum_{(k_1, k_2) \in \mathcal{I}} \frac{1}{\frac{(k_1+1/2)^2}{R_1^2} + \frac{k_2^2}{R_2^2}} \simeq \frac{g^2}{4\pi} \ln(R^2\Lambda^2), \quad (4.8)$$

where in the last equality we considered equal and large radii $R_1 = R_2 = R \gg M_*^{-1}$, where M_* is the 6d fundamental (Planck) scale. Therefore the "dual" lagrangian (4.7) is the Nambu-Jona-Lasinio model [3], in which the chiral symmetry is dynamically broken

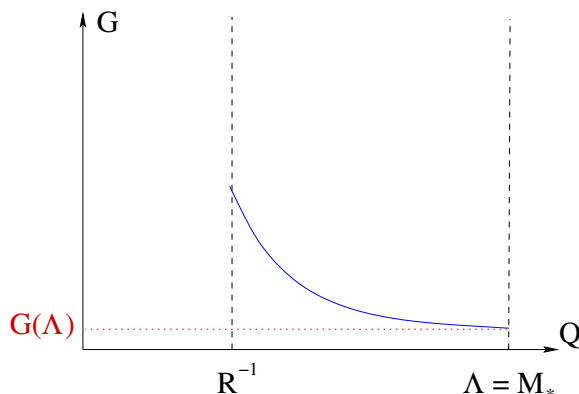


Figure 1: Classical running of the four-fermion coupling induced by the cumulative effect of the Kaluza-Klein states. The coupling becomes strong $G = G_c$ at the compactification scale and induces chiral symmetry breaking.

by the fermion condensate $\langle \chi_i \chi_i \rangle$ for values of the four-fermion coupling above the critical value

$$G > G_c \quad , \quad \text{where } G_c^{-1} = \frac{N\Lambda^2}{4\pi^2} \quad . \quad (4.9)$$

By using (4.4) and (4.8), we find that the condition (4.9) is precisely the same as the condition for the broken phase derived in the "bulk" approach (4.5).

Whereas in the deep IR the 6d bulk model is equivalent to the 4d NJL model, their UV behaviour is different.⁵ In particular, due to the cumulative effects of the KK states, the four-fermion coupling G has a logarithmic running

$$G(Q) \simeq \frac{g^2}{4\pi} \ln(\Lambda^2/Q^2) \quad (4.10)$$

from the cutoff scale Λ to the compactification scale R^{-1} , as illustrated in figure (1). So G increases in the IR and can generate dynamical chiral symmetry breaking. Even for couplings $g^2 \ll 4\pi$ such that a perturbative treatment is available, the non-decoupling of heavy KK states generates a large four-fermion coupling in the infrared which drives the symmetry breaking.

In the bulk formulation, the mass parameter was generated through quantum loops and the phase transition had a "classical" (quantum mechanical) nature. In the NJL formulation, the four-fermion coupling is generated classically while the phase transition is generated in a nonperturbative fashion through the quantum gap equation

$$1 = 4NG \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m^2} = 4 N G D(x = 0, m^2) \quad . \quad (4.11)$$

In the previous expression, $D(x, m^2)$ is the 4d propagator in the position representation of a scalar field, and m is the dynamical fermion mass.

⁵For earlier ideas of the role of extra dimensions in dynamical symmetry breaking, see [11]. For a recent extensive review on strong dynamics, see e.g. [22].

The counterpart of the NJL quantum gap equation (4.11) in the bulk formulation is the classical gap equation (3.14)-(3.15). The similarity of the bulk equation (3.15) and the NJL gap equation (4.11) is transparent.

Notice that the dual NJL description is valid in principle only below the phase transition. In this case and when the classical running interpretation is valid, the connection between the Higgs-localized scalar potential and the NJL model can also be understood in terms of the compositeness condition of [5]. There, it was argued, by introducing a composite scalar field H , that the kinetic term $Z_H |\partial H|^2$ vanishes, $Z_H \rightarrow 0$, at the compositeness scale Λ_C . By a rescaling of the kinetic term to the canonical form, this is equivalent of imposing the boundary conditions $m_H \rightarrow \infty$, $\lambda_H \rightarrow \infty$, where the parameters m_H, λ_H are defined from the effective scalar potential $V = -m_H^2 |H|^2 + \lambda_H |H|^4$. In our 6d model, the running between the UV cutoff and the compactification scale R^{-1} produces μ^2 and λ to diverge at R^{-1} precisely at the critical point. With a rescaling $\phi \rightarrow Z^{1/2} \phi$ of the scalar to go from the convention of [2] to the normalization of [5], keeping the scalar mass fixed, the wave function is

$$Z(Q) = 1 - \frac{\mu^2}{2\pi} \ln \frac{\Lambda}{Q} . \tag{4.12}$$

At the phase transition $\mu = \mu_c$, we get

$$Z(Q = R^{-1}, \mu = \mu_c) = 0 \quad , \quad \lambda(Q = R^{-1}) = \lambda(\Lambda) Z^{-2}(Q = R^{-1}, \mu = \mu_c) \rightarrow \infty . \tag{4.13}$$

The first of these conditions is similar to the one-loop (large N) induced wave function displayed in [5], whereas the one for scalar self-coupling is different, since the UV physics is different. A natural candidate for the compositeness scale in our case is therefore $\Lambda_C = 1/R$.

Analogously to the scalar parameters μ, λ , the Yukawa coupling g gets an induced running which can be easily integrated:

$$Q \frac{dg}{dQ} = -\frac{\mu^2}{2\pi} g \quad , \quad g(Q) = \frac{g(\Lambda)}{1 - \frac{\mu^2}{2\pi} \ln \frac{\Lambda}{Q}} . \tag{4.14}$$

The compositeness conditions have therefore to be supplemented with

$$g(Q = R^{-1}) \rightarrow \infty , \tag{4.15}$$

in analogy with the compositeness condition for the top quark in the top condensation scenario [5]. Whereas the first and the third conditions (4.13)-(4.15) are indeed similar to the compositeness conditions in [5], close to the critical coupling μ_c the running interpretation breaks down and the 4d physics is actually completely perturbative. Indeed, defining the 4d effective theory as in [2] by

$$\begin{aligned} \phi(x^\mu, r) &= \sqrt{\frac{2}{\pi R^2}} \ln \frac{R}{\epsilon} \sigma(x^\mu) \quad , \quad r < \epsilon , \\ \phi(x^\mu, r) &= \sqrt{\frac{2}{\pi R^2}} \ln \frac{R}{r} \sigma(x^\mu) \quad , \quad \epsilon < r < R , \end{aligned} \tag{4.16}$$

we find, at $\mu \simeq \mu_c$, the 4d scalar self-coupling and Yukawa coupling to the fermions to be

$$\lambda_4 = \frac{64\pi^2}{\mu_c^8} \frac{\lambda}{R^4} \quad , \quad g_4 = \frac{2\sqrt{2\pi}}{\mu_c^2} \frac{g}{R} . \quad (4.17)$$

Since $1/\mu_c^2 \sim \ln(\Lambda R)$, the couplings (4.17) are log enhanced compared to their naive (volume suppressed) values, but are still perturbative and under control.

Whereas the NJL 4d description and the bulk 6d one with appropriate boundary conditions are equivalent in the IR, the 6d picture can sometimes be simpler to use in order to describe the symmetry breaking pattern. For example, consider two set of fermions living in two different fixed points, $(0, 0)$ and $(0, \pi R_2)$ and interacting both with (relatively large) Yukawa coupling to the same 6d bulk scalar, which has nontrivial boundary conditions (3.3). In the NJL picture, there are three four-fermionic interactions coming from the two sets of fermions, with specific four-fermion couplings. In the bulk picture, localized loops of fermions generate localized scalar potentials at both fixed points. The scalar action in this case is

$$\begin{aligned} S &= \int d^4x d^2y \left[\frac{1}{2} (\partial_M \phi)^2 - V_{\delta,1}(\phi) - V_{\delta,2}(\phi) \right] , \\ V_{\delta,1}(\phi) &= \left(-\frac{\mu_1^2}{2} \phi^2 + \frac{\lambda_1}{4} \phi^4 \right) \cdot \delta(y_1) \delta(y_2) , \\ V_{\delta,2}(\phi) &= \left(-\frac{\mu_2^2}{2} \phi^2 + \frac{\lambda_2}{4} \phi^4 \right) \cdot \delta(y_1) \delta(y_2 - \pi R_2) . \end{aligned} \quad (4.18)$$

It is a straightforward exercise to work out the equation defining the mass eigenstates (3.10) in the unbroken phase. We find

$$\begin{aligned} &\left[\frac{1}{\bar{\mu}_1^2} - \sum_{k_i=-\infty}^{\infty} \frac{1}{m^2 + (k_1 + 1/2)^2/R_1^2 + k_2^2/R_2^2} \right] \times \left[\frac{1}{\bar{\mu}_2^2} - \sum_{k_i=-\infty}^{\infty} \frac{1}{m^2 + (k_1 + 1/2)^2/R_1^2 + k_2^2/R_2^2} \right] \\ &= \left(\sum_{k_i=-\infty}^{\infty} \frac{(-1)^{k_2}}{m^2 + (k_1 + 1/2)^2/R_1^2 + k_2^2/R_2^2} \right)^2 . \end{aligned} \quad (4.19)$$

Putting different fermions in different positions in the compact space can also provide a geometrical understanding of various values of their Yukawa couplings via the wave function profile of the bulk scalar.

5. Supersymmetric extension and coupling to localized chiral fields

The basic mechanism we used for generating a sizable running uses the logarithmic terms coming from the renormalization of the localized mass term. This implies in particular that the corrections to the mass should be forbidden in the bulk. The natural way to implement this is to have supersymmetry in the bulk. Since the potential has to be generated on the brane, we add boundary chiral fields in supersymmetric multiplets, with supersymmetry softly broken on the boundary.

There are two inequivalent ways to supersymmetrize, by using either bulk hypermultiplets or bulk vector multiplets. Let us start with the first case and consider a bulk hypermultiplet in 6d, which in 4d $\mathcal{N} = 1$ supersymmetric language has two chiral (super)fields $\Phi_{1,2} = (\phi_{1,2}, \psi_{1,2})$, and N localized matter superfields $A_i = (z_i, \chi_i)$, with $i = 1 \cdots N$, where we denoted in parenthesis the scalar and fermionic components of the multiplets. By using the $\mathcal{N} = 1$ superfield formalism of ref. [12], the action is given by

$$S = \int d^2 y d^4 \theta \left(\Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2 \right) + \left(\int d^2 y d^2 \theta \Phi_1 (\partial_5 + i\partial_6) \Phi_2 + \text{h.c.} \right) + \int d^4 \theta A_i^\dagger A_i (1 - \Sigma^2 \theta^2 \bar{\theta}^2) + \left(\int d^2 \theta g \Phi_1(y=0) A_i A_i + \text{h.c.} \right), \quad (5.1)$$

where Σ is a scalar soft mass term for the boundary fields. In order to write the component lagrangian, we first solve for the auxiliary fields

$$\begin{aligned} F_{\phi,1}^\dagger &= -(\partial_5 + i\partial_6)\phi_2 - g z_i z_i \delta^2(y), \\ F_{\phi,2}^\dagger &= (\partial_5 + i\partial_6)\phi_1, \quad F_{A_i}^\dagger = -2g\phi_1 z_i. \end{aligned} \quad (5.2)$$

After eliminating auxiliary fields, the component lagrangian is

$$\begin{aligned} S = & \int d^2 y \left[|\partial_M \phi_1|^2 + |\partial_M \phi_2|^2 - i\psi_1 \sigma^\mu \partial_\mu \bar{\psi}_1 - i\psi_2 \sigma^\mu \partial_\mu \bar{\psi}_2 + (\psi_1 (\partial_5 + i\partial_6) \psi_2 + \text{h.c.}) \right] \\ & - \left[S_{\text{kin}}(z_i, \chi_i) + g\phi_1 \chi_i \chi_i + 2g z_i \psi_1 \chi_i + 4g^2 |\phi_1|^2 |z_i|^2 \right. \\ & \left. + g \left\{ z_i z_i (\partial_5 - i\partial_6) \bar{\phi}_2 + \text{h.c.} \right\} + \Sigma^2 |z_i|^2 + g^2 z_i^2 \bar{z}_j^2 \delta^2(0) \right] \Big|_{y=0}. \end{aligned} \quad (5.3)$$

Analogously to the non supersymmetric case we impose nontrivial boundary conditions

$$\Phi_{1,2}(y_1 + 2\pi R_1, y_2) = -\Phi_{1,2}(y_1, y_2). \quad (5.4)$$

The \mathbb{Z}_2 orbifold has a nontrivial action on the bulk hypermultiplet fields. A consistent assignment is Φ_1 to be even and Φ_2 to be odd. The KK expansions in this case are

$$\begin{aligned} \Phi_1(x, \mathbf{y}) &= \sum_{(k_1, k_2) \in \mathcal{I}} \frac{1}{\sqrt{2\pi^2 R_1 R_2}} \cos \left(\frac{k_1 + 1/2}{R_1} y_1 + \frac{k_2}{R_2} y_2 \right) \Phi_1^{(k_1, k_2)}(x), \\ \Phi_2(x, \mathbf{y}) &= \sum_{(k_1, k_2) \in \mathcal{I}} \frac{1}{\sqrt{2\pi^2 R_1 R_2}} \sin \left(\frac{k_1 + 1/2}{R_1} y_1 + \frac{k_2}{R_2} y_2 \right) \Phi_2^{(k_1, k_2)}(x). \end{aligned} \quad (5.5)$$

The second possibility of supersymmetrization is to add bulk 6d vector multiplets, which in 4d $\mathcal{N} = 1$ language are described by vector V and chiral ϕ supermultiplets, both in the adjoint representation of a gauge group G . In order to be able to couple ϕ to the localized matter, we need to choose \mathbb{Z}_2 , the orbifold action, such that V is odd and therefore has no zero modes, whereas ϕ is even and can therefore couple to boundary chiral multiplets. More precisely, we can start from a nonabelian gauge group and give a nontrivial action of the orbifold on the gauge degrees of freedom

$$V(-y_1, -y_2) = P^\dagger V(y_1, y_2) P, \quad \phi(-y_1, -y_2) = -P^\dagger \phi(y_1, y_2) P, \quad (5.6)$$

where P is a matrix in the adjoint representation of the gauge group, such that $P^2 = 1$. The surviving (even) gauge group generators T_a satisfy $[T_a, P] = 0$, whereas the remaining ones T_α are projected out. The complementary states ϕ_α from the adjoint scalar ϕ survive at the fixed points and can consistently be coupled to the localized matter. By imposing nontrivial boundary conditions on ϕ_α we generate a setup where ϕ_α plays the role of the scalar field with localized Higgs potential and can trigger the phase transition. While this can be an interesting alternative, we will not pursue this possibility further on.

5.1 Bulk picture

The leading quantum corrections in $1/N$ come at one-loop with the chiral (super)fields A_i running in the loop. There is an induced effective potential for ϕ_1 which can be computed in the standard way [10]. The result is

$$V_{\text{eff}}(\phi_1) = N \int \frac{d^4 p}{(2\pi)^4} \{ \ln(p^2 + \Sigma^2 + 4g^2|\phi_1|^2) - \ln(p^2 + 4g^2|\phi_1|^2) \}, \quad (5.7)$$

which can be expanded in powers of ϕ_1 as

$$V_{\text{eff}}(\phi_1) = -4N\Sigma^2 g^2 |\phi_1|^2 \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2(p^2 + \Sigma^2)} + 8N\Sigma^2 g^4 |\phi_1|^4 \int \frac{d^4 p}{(2\pi)^4} \frac{2p^2 + \Sigma^2}{p^4(p^2 + \Sigma^2)^2} + \dots \quad (5.8)$$

We therefore induced radiatively, in the leading order in an $1/N$ expansion, a potential of the form (2.1) with

$$\mu^2 = \frac{Ng^2}{4\pi^2} \Sigma^2 \ln \frac{\Lambda^2}{\Sigma^2}. \quad (5.9)$$

In a first approximation, we can consider this as the bare coupling in the model of [2]. Then the condition defining the symmetry breaking phase, for large and equal radii $R_1 = R_2 \gg \frac{1}{\Lambda}$, is

$$\frac{Ng^2}{4\pi^2} \Sigma^2 \ln \frac{\Lambda^2}{\Sigma^2} > \frac{4\pi}{\ln(R^2\Lambda^2)}, \quad (5.10)$$

whereas the perturbativity condition translates here into

$$\frac{Ng^2}{4\pi^2} \Sigma^2 \ln \frac{\Lambda^2}{\Sigma^2} \ll 1. \quad (5.11)$$

For $\langle \phi_1 \rangle \neq 0$ the brane fermions χ_i acquire a mass and the chiral symmetry is spontaneously broken. Finally, fermion loops also induce localized operators of the form $|(\partial_5 + i\partial_6)\phi_2|^2$. Their effect is to renormalize the KK masses of the odd field ϕ_2 , but this effect has no relevance for our present discussion.

Notice that for natural values of (the dimensionful coupling) g , eq. (5.10) can be satisfied only for large values of the soft breaking parameter Σ . This is easy to interpret in the dual NJL description, to which we now turn.

5.2 Dual description: the softly supersymmetric NJL model

There is a dual description in which the bulk fields Φ_i are integrated out at tree-level and the chiral symmetry breaking is entirely described in terms of nonperturbative brane dynamics. Since the bulk plus the interaction action is supersymmetric, the integrating out procedure gives rise to a supersymmetric effective action, to add to the brane lagrangian with softly broken supersymmetry. There are some subtleties in proving that the integration out leads to a well-defined four-dimensional action without ill-defined (i.e. $\delta^2(0)$) terms. Analogously to former studies in 5d [13], it can be checked that the singular terms cancel out as they should. The resulting brane lagrangian has the simple form

$$S_{\text{brane}} = \int d^4\theta \{ A_i^\dagger A_i (1 - \Sigma^2 \theta^2 \bar{\theta}^2) + G A_i A_i A_j^\dagger A_j^\dagger \}, \quad (5.12)$$

where

$$G = \frac{g^2}{2\pi^2 R_1 R_2} \sum_{(k_1, k_2) \in \mathcal{I}} \frac{1}{\frac{(k_1+1/2)^2}{R_1^2} + \frac{k_2^2}{R_2^2}}. \quad (5.13)$$

Therefore the "dual" lagrangian is a softly-broken supersymmetric version [14, 15] of the Nambu-Jona-Lasinio model [3]. The dynamics of the softly broken supersymmetric version of the NJL model in the large N expansion was investigated in detail in [15]. It was found there that chiral symmetry is dynamically broken by the fermion condensate $\langle \chi_i \chi_i \rangle$, for values of the four-fermion coupling above the critical value

$$G > G_c \quad , \quad \text{where } G_c^{-1} = \frac{N \Sigma^2}{4\pi^2} \ln \frac{\Lambda^2}{\Sigma^2}. \quad (5.14)$$

By using (5.13), we find that the condition (5.14) is precisely the same as the condition for the broken phase derived in the previous section, which for equal and large radii is displayed in (5.10).

As in the 4d supersymmetric NJL model, there is a naturalness problem in the 6d construction. In the 4d SUSY NJL model, the symmetry breaking occurs for values of the four-fermion interaction $G \gtrsim 1/\Sigma^2 \gg 1/\Lambda^2$ much larger than its natural value. For small supersymmetry breaking $\Sigma R \ll 1$, this generates strong four-fermion interactions well below the compositeness scale $\Lambda_c = R^{-1}$. As transparent in (5.13), the natural scale of the strong four-fermion interactions for our 6d explicit realization is actually R^{-1} , unless g is much larger than its natural value. The reason is simpler to understand by rewriting (5.12) in an appropriate form to compare with the Minimal Supersymmetric Standard Model (MSSM) [14]

$$S_{\text{brane}} = \int d^4\theta \{ A_i^\dagger A_i (1 - \Sigma^2 \theta^2 \bar{\theta}^2) + H_1^\dagger H_1 \} + \left(\int d^2\theta H_2 (m H_1 - g A_i A_i) + \text{h.c.} \right). \quad (5.15)$$

Since H_2 is a Lagrange multiplier in (5.15), the two lagrangians (5.12) and (5.15) are equivalent for $G = g^2/m^2$. As explained in [14, 15], H_2 acquires a kinetic term $Z_2 H_2^\dagger H_2$ at one-loop in the large N expansion, which vanishes at the compositeness scale $Z_2(Q =$

$\Lambda_c) = 0$. Below Λ_c , both H_1 and H_2 are dynamical fields and are to be identified with the two Higgs doublets of the MSSM. In the language of (5.15), the naturalness problem is that in order to keep the Higgs mass light, the supersymmetric mass term should be of order $m \sim \Sigma$, which is nothing but the reincarnation of the so-called μ -problem of MSSM in the SUSY NJL case. In the 6d case, the analog of the action (5.15) is (5.1), the analog of H_2 is Φ_1 , whereas the analog of the supersymmetric mass m is the mixing term $\Phi_1(\partial_5 + i\partial_6)\Phi_2$ in (5.1). Upon KK expansion, we get $m \sim (1/2R_1)$. Equivalently, this can be seen from the nontrivial boundary conditions (5.4) via the KK expansion (5.5). Since the symmetry breaking only occurs for $m \sim \Sigma$, the model also requires large supersymmetry breaking scale, as already anticipated.

If we believe in the equivalence between the 6d scalar model and the 4d NJL model also close to the nonperturbative critical couplings of section 2, then the corresponding Yukawa coupling g can be large and produce a larger Fermi coupling G . However it will be hard to argue reliably for very large Yukawas. Maybe a simpler road is to use a small bulk mass $mR \ll 1$, which will generate a large Fermi coupling $G \sim g^2/R^2m^2$. Another solution in order to get symmetry breaking compatible with small supersymmetry breaking scale Σ is to start with small supersymmetry mass generated by boundary conditions. In other words, we need

$$(\Phi_1 + i\Phi_2)(y_1 + 2\pi R_1, y_2) = e^{2\pi i\omega}(\Phi_1 + i\Phi_2)(y_1, y_2), \quad (5.16)$$

with $\omega \ll 1$, in which case $m \sim (\omega/R_1)$. Whereas from first principles in string theory ω is quantized and cannot be very small, in analogy with known 5d examples [9], $\omega \ll 1$ can actually be realized by starting with periodic boundary conditions and adding small supersymmetric mass terms for bulk fields localized at the fixed points $(\pi R_1, 0)$ and/or $(\pi R_1, \pi R_2)$. Supersymmetry is broken softly with $\Sigma\Lambda \ll 1$ only at the origin $(0, 0)$. After re-diagonalization of the mass matrix, this is equivalent to starting with nontrivial boundary conditions (5.16) and no localized mass terms. This is technically natural in the sense that a small supersymmetric mass term in the fixed points is protected by supersymmetry.

6. String theory realization

It is legitimate to ask if it is possible to realize the field theory construction we did provide in [2] and in this paper from a string theory framework. The answer is positive and we provide here one possible construction for the (softly broken) supersymmetric case of section 5.⁶ The main requirement for the string construction is to provide a large and flat two-dimensional compact space, so there should be no localized source which would curve the large 2d space. Indeed, our field theory analysis was done in flat space. This situation can be realized in orientifold constructions, where orientifold planes cancel the sources provided by the branes. The basic ingredients of the construction are that the field

⁶Other realizations of dynamical symmetry breaking can be found in [16] for string realizations of a nonlocal NJL version and [17] ([18]) for a string (field-theory) realization of the chiral symmetry breaking in QCD.

ϕ arises from D5 branes wrapping our large 2d space, whereas the localized fermions arise at the intersection between the D5 branes and a different set of branes, the intersection being four dimensional and generating chiral fermions. We use orientifolds of type IIB string theory with D5-branes wrapping different coordinates of the internal space. Our example is based on the orientifold projection $\Pi' = \Pi \pi_4 \pi_5 \pi_8 \pi_9$, where Π is the left-right world-sheet interchange, π_4, π_5 are parity operations in two compact coordinates x_4 and x_5 , to be identified with the two large dimensions (y_1, y_2) in our field theoretical construction and π_8, π_9 are parity operations in two internal noncompact coordinates x_8 and x_9 . The basic building block for brane configuration we consider is then the following

coord.	0	1	2	3	4	5	6	7	8	9	
D5 ₁	x	x	x	x	x	x	0	0	0	0	
D5 ₂	x	x	x	x	0	0	x	x	0	0	
O5 ₂	x	x	x	x	0	0	x	x	0	0	(6.1)

In (6.1), crosses x denote coordinates parallel to the branes, whereas 0 denotes orthogonal coordinates. Notice that D5₂ branes and O5₂ planes are orthogonal to the large 2d space (x_4, x_5) . In order to keep the 2d space flat we need a configuration with D5₂ branes on top of the O5₂ planes, with locally zero tension and charge. We add a Wilson line $\langle W_4 \rangle$ on the D5₁ branes in the compact x_4 coordinate, which has the effect of breaking the gauge group and giving masses to fields charged under W_4 . There are in particular four charged scalar fields ϕ_i , which get a mass from this Hosotani mechanism and will be identified with the master field(s) of our field theory model. Notice first of all that this field lives indeed in six dimensions, in the bulk of our large 2d compact space and it corresponds to a hypermultiplet $\phi = (\phi_1, \phi_2)$ from the 4d viewpoint, as in section 5. The mass of ϕ is positive and it corresponds to the mass generated by boundary conditions analyzed in [2] and in sections 2, 4 and 5 of the present paper. Alternatively, using a Wilson line in one of the last four coordinates $x_6 \cdots x_9$ is equivalent to considering a bulk supersymmetric mass as in the section 3. For simplicity, we can consider the last three coordinates $x_7 \cdots x_9$ as being noncompact, whereas x_6 is a circle and will be used to break supersymmetry a la Scherk-Schwarz.

The D5₂ branes gauge fields are nondynamical in four dimensions and play the role of global symmetries. The D5₁ brane degrees of freedom, on the other hand, are dynamical and contain in particular gauge fields and the field(s) ϕ . The D5₁-D5₂ sector, after additional orbifold projections to be discussed below, contains massless $\mathcal{N} = 1$ chiral multiplets localized in four dimensions, to be identified with the 4d chiral multiplets A_i in section 5. At the effective low energy action level, the setup is similar to the one considered in section 5, with couplings of the form $A\phi A$ and one expects the arguments presented there to apply and generate dynamical symmetry breaking. Non-trivial boundary conditions in the compact coordinate x_6 a la Scherk-Schwarz break supersymmetry at tree-level in the D5₂ sector, whereas the D5₁ branes, being orthogonal to the x_6 coordinate, feel the breaking only through radiative corrections [19]. Notice that the dynamics in the large bulk coordinates x_4, x_5 is supersymmetric at that stage. At one-loop, supersymmetry breaking

propagates in the D5₁- D5₂ sector and generates the localized 4d soft terms that were used in section 5.

The setup preserve until now $\mathcal{N} = 2$ supersymmetry in 4d spontaneously broken to $\mathcal{N} = 0$ by the Scherk-Schwarz deformation, so additional ingredients are needed in order to generate chirality. The standard internal spaces used in this respect are the Calabi-Yau spaces or the orbifolds. We choose here the second possibility. We introduce additional Z_2 and Z_3 orbifold operations acting on the internal coordinates as

$$Z_2(z_1, z_2, z_3) = (-z_1, -z_2, z_3) \quad , \quad Z_3(z_1, z_2, z_3) = (e^{\frac{2\pi i}{3}} z_1, z_2, e^{-\frac{2\pi i}{3}} z_3),$$

where

$$z_1 = \frac{x_4 + ix_5}{\sqrt{2}} \quad , \quad z_2 = \frac{x_6 + ix_7}{\sqrt{2}} \quad , \quad z_3 = \frac{x_8 + ix_9}{\sqrt{2}} \quad (6.2)$$

are the three complex internal coordinates. The resulting orientifold, which is dual to the so-called $Z_3 \times Z_2$ or Z'_6 type I orbifold in the literature [20] after performing four T-dualities in x_6, x_7, x_8, x_9 , reduces supersymmetry down to $\mathcal{N} = 1$ in 4d. The 4d type I $Z_3 \times Z_2$ orbifold has D9 brane / O9 planes and one set of D5 branes / O5 planes, wrapping the third internal torus. After the four T-dualities, the D9 branes (O9 planes) become our D5₁ branes (O5₁ planes) and the D5 (O5) branes become our D5₂ branes (O5₂ planes). Our Wilson line is in the type I orbifold a Wilson line on the D9 branes.

In order to break completely supersymmetry, as already announced we are adding a Scherk-Schwarz operation in the compact coordinate x_6 , compatible with the two orbifold operations. Our Scherk-Schwarz deformation is a 2π rotation in x_6 and one 4d spacetime coordinate. The corresponding worldsheet current anticommutes with the Z_2 orbifold projection and commutes with the Z_3 one, as required by the consistency of the string construction [21]. A last subtlety, explained in detail in [19] is that due to the Scherk-Schwarz operation, the O5₁ planes, which are perpendicular to the x_6 coordinate used in the supersymmetry breaking, are actually pairs of O5₁ orientifold - $\bar{O}5_1$ antiorientifold planes, situated at $x_6 = 0$ and $x_6 = \pi R_6$, respectively. If the radius $R_6 \gg l_s$ is large enough, the closed string tachyon is massive and the timescale for the instability can be large enough. The D5₁ branes should be at (or close to) the point $x_6 = 0$, such that the strings D5₁ - $\bar{O}5_1$, which break supersymmetry, to be very massive.

A possible objection to the present setup is that the boundary conditions generated mass and/or the bulk hypermultiplet mass m are not constant but field-dependent, given by the vev of the Wilson line on the D5₁ branes. Since the setup in non-supersymmetric, the Wilson line acquires a potential and its vev will be dynamically fixed. This last point needs further investigation which is however beyond the goals of the present paper. This objection is also valid for the field-theory construction, in that we assumed that the radii of the large 2d space were stabilized by additional dynamics. To conclude, the setup presented in this section does realize the (softly) supersymmetric model work out in section 5, where the supersymmetry breaking is soft and comes from a Scherk-Schwarz deformation in an extra space coordinate.

The original non-supersymmetric setup of [2] and in sections 2,3 of the present paper can be in principle also realized for smaller values of $R_6 \gtrsim l_s$. However in this case we

expect severe tachyonic instabilities in the system, which need to be suppressed in order for the string picture to be a viable description of the field-theory construction.

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