The continuous spin limit of higher spin equations of motion

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Abstract

We show that the Wigner equations describing the continuous spin representations can be obtained as a limit of massive higher-spin equations. The limit involves a suitable scaling of the wave function, the mass going to zero and the spin to infinity with their product being fixed. The result allows to transform the Wigner equations to a gauge invariant Fronsdal-like form. We also give the generalisation of the Wigner equations to higher dimensions with fields belonging to arbitrary representations of the massless little group.

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1 Introduction

The unitary irreducible representations (UIR) of the Poincaré group $P_D = ISO(D-1,1)$, as Wigner has shown, are determined by those of the little group [1]. On the one hand, the little group for massless particles in D-dimensional Minkowski space-time $\mathbb{R}^{D-1,1}$ is the non-compact Euclidean group $E_{D-2} = ISO(D-2)$. Its UIR are infinite-dimensional[5] except for the case where all the translation-like generators vanish. The latter case characterises the "helicity" representations whose little group is effectively $SO(D-2)$. The generic case gives rise to the so called "continuous spin" [2, 3, 4, 5, 6] representations[6] with a nonvanishing value of the second Casimir operator $W = \mu^2$, where μ is a real parameter with the dimension of a mass. Wigner proposed a set of manifestly covariant equations to describe fields carrying these UIR in four space-time dimensions [2]. The wave function depends on the usual space-time coordinates and in addition on an internal four-vector. These equations are reviewed in the second section. Three of the Wigner equations allow to constrain this four-vector to a transverse angle variable which is at the origin of the "continuous spin" name.

The massive representations, on the other hand, are determined by representation of the rotation group $SO(D-1)$. From the group theoretical point of view, the UIR of the orthogonal and Euclidean groups are related by an Inönü-Wigner contraction $SO(D-1) \to E_{D-2}$ [7]. It follows that one can obtain the continuous spin representations from the massive ones in a massless limit $m \to 0$. The second Casimir operator is related to the spin s of the particle as $W = m^2 s(s + D - 3)$. In order to keep W nonvanishing, the massless limit must be such that the product $s\,m$ remains finite, $sm \to \mu$, so that the spin goes to infinity, for a group theoretical discussion see [8]. The main goal of this paper is to obtain covariant wave equations for continuous spin representations, in any space-time dimension, from massive higher-spin equations.

A massive spin-s particle can be described by a rank-s tensor field [9] or, more conveniently, as a Kaluza-Klein mode of a massless spin-s particle [10] on a higher dimensional space-time[7] with a nonvanishing momentum along the extra dimension [11, 12]. This gives rise, as reviewed in Section 3, to a collection of totally symmetric tensors having ranks less than or equal to s. If one introduces an auxiliary vector u^μ, one can interpret the tensors as the Taylor coefficients of the expansion in powers of u. We will show, in Section 4, that this is the way that the Wigner internal vector arises and the Wigner equations emerge in the contraction limit. This limit involves a proper rescaling of the wave function and the auxiliary variables in order to be well behaved. Starting from the equations of motion of the higher-spin particle in de Donder's gauge we get the Wigner equations in the aforementioned limit. This suggests that the Wigner equations correspond to a gauge-fixed version of gauge invariant equations of motion. We show that this is the case in Section 5, where we determine the new gauge symmetries. The equations can be formulated with a restricted gauge invariance, in analogy with the Fronsdal equations [10], or with an unconstrained gauge parameter. In the second case one has to introduce a "compensator" field [13]. In Sections 6 and 7 we discuss generalisations of the equations we found to arbitrary UIR of $SO(D-3)$, the little group of E_{D-2}, which we shall call the "short" little group [6]. We

[5] For space-time dimension $D \geqslant 4$.

[6] They are also called "infinite spin" representations [3].

[7] The helicity little group in $D+1$ dimensions is the group $SO(D-1)$, which is identified with the massive little group in D dimensions.

shall consider the spinorial representation in Section 6 and the “exotic” ones in Section 7.

There are many physical motivations to study higher spin fields and their limits. We refer, for example, to [14] for a comprehensive discussion. Let us mention that String theory gives rise to particles with arbitrary spins and a proper understanding of its symmetries may be approached by revealing the symmetries underlying the higher spin fields and the constraints from the consistency of their interactions. In this respect, some higher derivative string theory [15] gives rise to particles belonging to the continuous spin representation. The way that conventional string theory and this tensionless higher derivative one are related may be clarified by the relation between the continuous spin fields and the higher spin massive fields which is the subject of this paper.

2 The continuous spin representation

Consider a massless particle in D-dimensional spacetime[8]. Let its momentum have zero components except for p^+ ($V^\pm = \frac{1}{\sqrt{2}}(V^0 \pm V^{D-1}), V^+ = -V_-$). The little group leaving the momentum invariant is generated by M_{ij} and $M_{+i} = \pi_i$, they verify the Lie algebra

$$\begin{aligned} [\pi_i, \pi_j] &= 0,\ [M_{ij}, M_{kl}] = i(\delta_{jk}M_{il} - \delta_{ik}M_{jl} - \delta_{jl}M_{ik} + \delta_{il}M_{jk}) \\ [\pi_i, M_{kl}] &= i(\delta_{ik}\pi_l - \delta_{il}\pi_k), \end{aligned} \tag{2.1}$$

which is the Lie algebra of the $D-2$ dimensional Euclidean group E_{D-2}. The Casimir $\pi^i\pi_i = \mu^2$ classifies the representations of E_{D-2}. If μ vanishes then the irreducible representations are given by those of $SO(D-2)$, these are the helicity states. When μ is nonvanishing we get the continuous spin representations. They are of infinite dimension and are determined by the UIR of the subgroup leaving a given π^i invariant, the short little group, $SO(D-3)$.

In fact, the second Casimir operator of the Poincaré group, is given by

$$W = -\frac{1}{2}p^2 M_{\mu\nu}M^{\mu\nu} + M_{\mu\alpha}p^\alpha M^{\mu\beta}p_\beta. \tag{2.2}$$

It reduces to μ^2 for the massless particle and to $m^2 s(s + D - 3)$ for the massive one. Here s corresponds to the rank of the traceless completely symmetric tensor in D space-time dimensions.

2.1 Wigner’s wave equation

A wave equation whose physical content is the single valued continuous spin representation was proposed by Wigner [2]. The wave function depends on two vectors: the momentum p and the additional vector ξ which is dimensionless. There are two independent equations obeyed by the wave function $\overline{\Psi}(p\,, \xi)$ and two other which are consequences of these. They read

$$\mathcal{E}_1\overline{\Psi} \equiv p\cdot\frac{\partial\overline{\Psi}}{\partial\xi} - i\mu\,\overline{\Psi} = 0, \tag{2.3}$$

$$\mathcal{E}_2\overline{\Psi} \equiv (\xi^2 - 1)\overline{\Psi} = 0\,. \tag{2.4}$$

[8] Our conventions are as follows: Greek indices such as μ, ν, ... denote space-time indices running from 0 to $D-1$ while Latin indices such as i, j, ... denote transverse indices running from 1 to $D-2$. The Minkowski metric is mostly plus and reads in light-cone coordinates: $ds^2 = -2dx^+dx^- + dx^i dx_i$. Dots denote contraction of (implicit) space-time indices.

The first compatibility condition reads

$$[\mathcal{E}_1, \mathcal{E}_2]\overline{\Psi} \;\equiv\; 2\,\mathcal{E}_3\overline{\Psi} \;=\; 2\,p\cdot\xi\,\overline{\Psi} = 0, \tag{2.5}$$

and the second compatibility condition

$$[\mathcal{E}_1, \mathcal{E}_3]\overline{\Psi} \;\equiv\; \mathcal{E}_4\overline{\Psi} \;=\; p^2\,\overline{\Psi} = 0 \tag{2.6}$$

is the mass-shell constraint. There are no more compatibility conditions. of a higher derivative classical action [16].

The equation (2.3) reflects the fact that the couples $(p\,,\xi)$ and $(p\,,\xi+\alpha p)$ are physically equivalent for arbitrary $\alpha\in\mathbb{R}$. Indeed, one gets

$$\overline{\Psi}(p\,,\xi+\alpha p) \;=\; e^{i\alpha\mu}\,\overline{\Psi}(p\,,\xi) \tag{2.7}$$

from Equation (2.3). The equation (2.4) states that the internal vector ξ is a unit space-like vector while the mass-shell condition (2.6) states that the momentum is light-like. From the equation (2.5), one obtains that the internal vector is orthogonal to the momentum. All together, one finds that ξ lives on the unit hypersphere S^{D-3} of the transverse hyperplane $\mathbb{R}^{D-2}$. In conclusion, the "continuous spin" degrees of freedom essentially correspond to $D-3$ angular variables, whose Fourier conjugates are discrete variables analogous to the usual spin degrees of freedom.

2.2 The Fourier transformed wave equation

In fact, it is useful for later purposes to write the equations (2.3) - (2.6) in terms of w, the Fourier conjugate to ξ. The equations now read

$$(p\cdot w+\mu)\,\Psi = 0\,, \tag{2.8}$$

$$\left(\frac{\partial}{\partial w}\cdot\frac{\partial}{\partial w}+1\right)\Psi = 0\,, \tag{2.9}$$

$$\left(p\cdot\frac{\partial}{\partial w}\right)\Psi = 0\,, \tag{2.10}$$

$$p^2\,\Psi = 0\,. \tag{2.11}$$

In order to explicit the physical content of the equations, let us consider a plane wave,

$$\Psi(p,w) \;=\; \delta(p-p_0)\,\psi_{p_0}(w), \tag{2.12}$$

with $p_0^2=0$. Suppose that the only non-vanishing component of p_0^μ is p_0^+, then Equation (2.8) implies that

$$\psi_{p_0}(w) \;=\; \delta(w^-p_0^+-\mu)\;\phi\,(w^+,w^i), \tag{2.13}$$

where w^i are the transverse coordinates. Equation (2.10) implies that ϕ does not depend on w^+ and, finally, Equation (2.9) becomes the Helmholtz equation

$$\frac{\partial^2\phi}{\partial w^i\partial w_i}+\phi = 0. \tag{2.14}$$

There are several formal ways to write the solutions to the equation (2.14). A first way is to expand $\phi(w^i)$ in powers of w:

$$\phi(w^i) = \sum_{n=0}^{\infty} \frac{1}{n!}\, \phi_{i_1 \dots i_r} w^{i_1} \dots w^{i_n}, \tag{2.15}$$

with symmetric coefficients obeying

$$\phi_{i_1 \dots i_n} = -\phi^j{}_{j\, i_1 \dots i_n}. \tag{2.16}$$

A second way is to to expand in spherical harmonics as

$$\phi(w^i) = \sum_{n=0}^{\infty} f_n(r) f_{i_1 \dots i_n} \hat{w}^{i_1} \dots \hat{w}^{i_n}, \tag{2.17}$$

where $r^2 = w_i w^i$, and $\hat{w}^i = w^i/r$ are the coordinates on the sphere S^{D-3}. In equation (2.17), the constant tensors $f_{i_1 \dots i_n}$ are totally symmetric and traceless. The function f_n verifies the differential equation

$$f_n'' + \frac{D-3}{r} f_n' - \frac{n(n+D-4)}{r^2} f_n + f_n = 0, \tag{2.18}$$

which results from (2.14). The solution which is regular at $r = 0$ to equation (2.18) is given by

$$f_n(r) = r^{2-\frac{D}{2}} J_{n+\frac{D}{2}-2}(r), \tag{2.19}$$

where J_ν is the Bessel function of the first kind.

In both expansions, one gets totally symmetric tensors that one is tempted to compare with the fields appearing in the description of a massive higher-spin particle. The first expansion (2.15) turns out to be the one which will allow to make contact with the massive case. Rougly speaking, the point is that the physical components of a spin-s massive symmetric field correspond to a spin-s irreducible representation of the massive little group $SO(D-1)$, *i.e.* a rank-s traceless symmetric $D-1$ tensor $\phi_{I_1 \dots I_s}$ ($I_k = 1, \dots, D-1$), which decomposes as a tower of totally symmetric $D-2$ tensor $\phi_{i_1 \dots i_r}$ of rank r running from zero to s and satisfying precisely (2.16). The second expansion (2.17) has the merit of exhibiting the physical content of the Wigner equations: the general solution is given by a sum of plane waves with functions over the (internal) hypersphere S^{D-3} as coefficients. We stress that the continuous spin wave function Ψ has a number of "components" which is infinite but *countable*. The only *continuous* parameter is actually the label μ which is merely the analog of the mass m in the massive representation. When μ goes to zero we get an infinite sum over all helicity states represented above by the tensors $f_{i_1 \dots i_n}$.

The Hilbert space of functions on S^{D-3} carries the UIR of the massless little group E_{D-2} with a trivial representation of the short little group $SO(D-3)$. We shall consider cases with arbitrary irreducible representations of the latter in Section 6.

3 Massless and massive higher-spin fields

A convenient way of obtaining the equations of motion for the massive higher-spin fields is to start from a massless spin-s field in one extra space dimension and to compactify on a circle with a non vanishing momentum.

3.1 Massless higher-spin field

The Fronsdal equation for a massless higher-spin field described by a rank-s totally symmetric tensor $\varphi_{\mu_1 \ldots \mu_s}$ with a vanishing double trace is given by [10]

$$p^2 \varphi_{\mu_1 \ldots \mu_s} - p_{(\mu_1} p^\nu \varphi_{\mu_2 \ldots \mu_s)\nu} + p_{(\mu_1} p_{\mu_2} \varphi_{\mu_3 \ldots \mu_s)\nu}{}^\nu = 0, \tag{3.1}$$

where the curly bracket denotes complete symmetrisation by summing over all different permutations. These equations are invariant under the gauge transformations

$$\delta \varphi_{\mu_1 \ldots \mu_s} = p_{(\mu_1} \varepsilon_{\mu_2 \ldots \mu_s)}, \tag{3.2}$$

where the tensor ε is traceless. If one introduces an auxiliary vector u and defines

$$\varphi(x, u) \; = \; \frac{1}{s!} \, \varphi_{\mu_1 \ldots \, \mu_s} \, u^{\mu_1} \ldots u^{\mu_s}, \tag{3.3}$$

then the Fronsdal equation (3.1) may be rewritten as

$$\left[p^2 - (p \cdot u) \left(p \cdot \frac{\partial}{\partial u} \right) + \frac{1}{2} \, (p \cdot u)^2 \left(\frac{\partial}{\partial u} \cdot \frac{\partial}{\partial u} \right) \right] \varphi(x, u) = 0 \, . \tag{3.4}$$

The double trace constraint on the gauge field becomes

$$\left(\frac{\partial}{\partial u} \cdot \frac{\partial}{\partial u} \right)^2 \varphi = 0 \, , \tag{3.5}$$

and the homogeneity in u of the function φ defined by (3.3) implies

$$\left(u \cdot \frac{\partial}{\partial u} - s \right) \varphi = 0 \, . \tag{3.6}$$

The gauge transformations (3.2) read

$$\delta \varphi \; = \; (p \cdot u) \, \varepsilon, \tag{3.7}$$

with

$$\left(u \cdot \frac{\partial}{\partial u} - s + 1 \right) \varepsilon = 0 \tag{3.8}$$

and the tracelessness of ε leads to

$$\left(\frac{\partial}{\partial u} \cdot \frac{\partial}{\partial u} \right) \, \varepsilon = 0. \tag{3.9}$$

In de Donder's gauge

$$\left[p \cdot \frac{\partial}{\partial u} - \frac{1}{2} \, (p \cdot u) \left(\frac{\partial}{\partial u} \cdot \frac{\partial}{\partial u} \right) \right] \varphi = 0, \tag{3.10}$$

the Fronsdal equation (3.1) simplifies to $p^2 \varphi = 0$. There is a residual gauge invariance with ε subject, in addition to the tracelessness condition, to

$$p^2 \, \varepsilon = 0 \, , \quad \left(p \cdot \frac{\partial}{\partial u} \right) \, \varepsilon = 0 \, . \tag{3.11}$$

The remaining gauge invariance [17] allows to impose the tracelessness of φ

$$\left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u}\right)\varphi=0, \tag{3.12}$$

so that de Donder's gauge becomes

$$\left(p\cdot\frac{\partial}{\partial u}\right)\varphi=0, \tag{3.13}$$

which expresses the transversality of φ. We can now define a gauge invariant field by

$$\Phi=\delta(u\cdot p)\,\varphi\,. \tag{3.14}$$

The new field Φ is no more a polynomial in u but it is nevertheless a homogeneous distribution in u of degree $s-1$. It allows to write compatible and covariant equations describing the massless higher spin degrees of freedom

$$(p\cdot u)\,\Phi=0, \tag{3.15}$$
$$\left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u}\right)\Phi=0, \tag{3.16}$$
$$\left(p\cdot\frac{\partial}{\partial u}\right)\Phi=0, \tag{3.17}$$
$$p^2\,\Phi=0\,, \tag{3.18}$$
$$\left(u\cdot\frac{\partial}{\partial u}-s+1\right)\Phi=0\,. \tag{3.19}$$

In fact, Equations (3.15) - (3.19) describe by themselves a massless rank-s totally symmetric field, it is no longer necessary to assume the polynomiality in u. Notice the similarity with the Wigner equations, which have the same form except for constants in the first and third equations and the absence of an analogue of the last equation.

3.2 Massive higher-spin field

In order to get the equations for a massive particle we start with a massless higher-spin particle in dimension $D+1$ and consider a mode with $p^D=m$. We divide the $D+1$ auxiliary vector into a D vector u and a scalar v with respect to $SO(1,D-1)$.

The Fronsdal equation (3.1) becomes

$$\left[p^2+m^2-(p\cdot u+mv)\left(p\cdot\frac{\partial}{\partial u}+m\frac{\partial}{\partial v}\right)+\frac{1}{2}\,(p\cdot u+mv)^2\left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u}+\frac{\partial^2}{\partial v^2}\right)\right]\varphi=0\,. \tag{3.20}$$

The double trace constraint on the gauge field becomes

$$\left[\left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u}\right)^2+2\left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u}\right)\frac{\partial^2}{\partial v^2}+\frac{\partial^4}{\partial v^4}\right]\varphi=0\,, \tag{3.21}$$

and the homogeneity constraint reads

$$\left(u\cdot\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}-s\right)\varphi=0\,. \tag{3.22}$$

The equation (3.22) implies that field φ has the following expansion in the variables u and v

$$\varphi(x,u,v)\;=\;\sum_{r=0}^{s}\frac{1}{r!(s-r)!}\,\varphi_{\mu_1\dots\mu_r}\,u^{\mu_1}\dots u^{\mu_r}v^{s-r}\,. \tag{3.23}$$

Hence the Kaluza-Klein mechanism produces a tower of totally symmetric tensors of rank going from zero to s. The gauge transformation are

$$\delta\varphi\;=\;(p\cdot u+mv)\,\varepsilon, \tag{3.24}$$

with

$$\left(u\cdot\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}-s+1\right)\varepsilon=0 \tag{3.25}$$

and the tracelessness of ε leads to

$$\left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u}+\frac{\partial^2}{\partial v^2}\right)\,\varepsilon=0. \tag{3.26}$$

Nowadays, this description of massive fields is frequently referred to as "Stückelberg formulation". General procedures to make connection with the work of Singh and Hagen [11], by solving the constraint (3.21) and by fixing completely the gauge transformations (3.24), were presented in [12].

The "gauge-fixed" equations (3.15) - (3.19) become

$$(p\cdot u+mv)\,\Phi=0, \tag{3.27}$$

$$\left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u}+\frac{\partial^2}{\partial v^2}\right)\Phi=0, \tag{3.28}$$

$$\left(p\cdot\frac{\partial}{\partial u}+m\frac{\partial}{\partial v}\right)\Phi=0, \tag{3.29}$$

$$(p^2+m^2)\,\Phi=0. \tag{3.30}$$

$$\left(u\cdot\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}-s+1\right)\Phi=0, \tag{3.31}$$

4 Continuous spin from massive higher spin

We are now in a position to examine the limit where the mass goes to zero, the spin to infinity with their product being fixed. It is clear from equation (3.31) that this limit is ill defined on the field Φ. In order to get a well defined limit, one has also to assume a suitable scaling of the scalar v. Let us define the parameter μ and the variable α by

$$\mu=sm,\;\;\alpha=\frac{v}{s}, \tag{4.1}$$

the precise limit we are interested in is when s goes to infinity with finite μ and α. Before examining this limit, let us rewrite the massive equations with the new variables. It will be very convenient to first write the solution of equation (3.31)

$$\left(u \cdot \frac{\partial}{\partial u} + \alpha \frac{\partial}{\partial \alpha} - s + 1\right) \Phi = 0, \tag{4.2}$$

as

$$\Phi = \alpha^{s-1} \Psi \left(\frac{u}{\alpha}\right), \tag{4.3}$$

where we introduced the new field Ψ. This is the field which will remain well defined in the limit. Define $w = u/\alpha$. The equations (3.27) - (3.30) lead to

$$(p \cdot w + \mu)\, \Psi = 0\,, \tag{4.4}$$

$$\left[\frac{\partial}{\partial w} \cdot \frac{\partial}{\partial w} + \frac{1}{s^2}\left((s-1)(s-2) - (2s-3)\left(w \cdot \frac{\partial}{\partial w}\right) + \left(w \cdot \frac{\partial}{\partial w}\right)^2\right)\right] \Psi = 0\,, \tag{4.5}$$

$$\left[p \cdot \frac{\partial}{\partial w} + \frac{\mu}{s^2}\left(s - 1 - w \cdot \frac{\partial}{\partial w}\right)\right] \Psi = 0\,, \tag{4.6}$$

$$\left[p^2 + \left(\frac{\mu}{s}\right)^2\right] \Psi = 0\,, \tag{4.7}$$

using the useful relations

$$\frac{\partial}{\partial u} = \frac{1}{\alpha}\frac{\partial}{\partial w}\,, \qquad \frac{\partial}{\partial v} = \frac{1}{s}\left(-\frac{w}{\alpha} \cdot \frac{\partial}{\partial w} + \frac{\partial}{\partial \alpha}\right). \tag{4.8}$$

The limit of infinite spin is now non-singular and we get precisely the Wigner equations (2.8) - (2.11). It is now clear that although the field Φ has an ill defined limit the product $\alpha^{1-s}\Phi$ has a finite limit which is the continuous spin wave function Ψ. We stress that it was crucial in the above procedure that the auxiliary variable v grows as s in the limit. In terms of the Taylor expansion of the original field Φ this amounts to a rescaling of the various tensor fields by complicated factors depending on the spin s.

5 Continuous spin from Fronsdal equations

In the preceding section, we showed how to obtain the Wigner equations for the continuous spin particle starting from a gauge fixed version of the Stückelberg formulation. Here, we perform the contraction directly on the Fronsdal-like equations in order to find the gauge invariance leading to the continuous spin. For the spin-1 and spin-2 particles, the gauge invariance plays a crucial role in understanding various aspects of the Maxwell and Einstein theories. For higher-spin fields the gauge invariance and its deformations are also crucial in discussing possible interactions. Generally speaking, gauge invariance is an important ingredient for deriving covariant field equations from an action principle.

We solve as before the homogeneity condition (3.22) as

$$\varphi(u, v) = \alpha^s \psi(w), \tag{5.1}$$

where we used the same change of variables as before. Next, we take the limit of infinite spin from (3.20), rewritten for ψ, and the resulting equation is the Fronsdal-like continuous spin equation

$$\left[p^2 - (p\cdot w+\mu)\left(p\cdot\frac{\partial}{\partial w}\right) + \frac{1}{2}(p\cdot w+\mu)^2\left(\frac{\partial}{\partial w}\cdot\frac{\partial}{\partial w}+1\right)\right]\psi = 0\,. \tag{5.2}$$

The double tracelessness condition (3.21) becomes

$$\left(\frac{\partial}{\partial w}\cdot\frac{\partial}{\partial w}+1\right)^2\psi = 0\,. \tag{5.3}$$

The gauge transformation parameter can similarly be written as

$$\varepsilon(u,v) = \alpha^{s-1}\epsilon(w)\,. \tag{5.4}$$

The gauge transformation becomes

$$\delta\psi = (p\cdot w+\mu)\epsilon, \tag{5.5}$$

with the gauge parameter ϵ subject to the trace condition

$$\left(\frac{\partial}{\partial w}\cdot\frac{\partial}{\partial w}+1\right)\epsilon = 0. \tag{5.6}$$

So the continuous spin field is now described *à la* Stückelberg. The relation with the preceding description is obtained by first fixing the gauge with a de Donder-like condition

$$\left[p\cdot\frac{\partial}{\partial w} - \frac{1}{2}(p\cdot w+\mu)\left(\frac{\partial}{\partial w}\cdot\frac{\partial}{\partial w}+1\right)\right]\psi = 0\,. \tag{5.7}$$

The equation of motion reduces to $p^2=0$ and the residual gauge invariance allows to impose

$$\left(\frac{\partial}{\partial w}\cdot\frac{\partial}{\partial w}+1\right)\psi = 0\,. \tag{5.8}$$

If we now define

$$\Psi = \delta(p\cdot w+\mu)\,\psi \tag{5.9}$$

then the wave function Ψ is gauge invariant and we get back the Wigner equations for Ψ.

A suggestive way of writing the gauge invariant equation is to decompose ψ as a sum of homogeneous functions ψ_r of degree r in w, the coefficient of which is a rank-r tensor:

$$\psi = \sum_{r=0}^{\infty}\psi_r. \tag{5.10}$$

The equation (5.2) decomposes as

$$\begin{aligned}\left[p^2 - (p\cdot w)\left(p\cdot\frac{\partial}{\partial w}\right) + \frac{1}{2}(p\cdot w)^2\left(\frac{\partial}{\partial w}\cdot\frac{\partial}{\partial w}\right)\right]\psi_r \;=\;& \\ -\frac{1}{2}(p\cdot w)^2\,\psi_{r-2}& \\ -\mu\left\{(p\cdot w)\,\psi_{r-1} + \left[(p\cdot w)\left(\frac{\partial}{\partial w}\cdot\frac{\partial}{\partial w}\right) - \left(p\cdot\frac{\partial}{\partial w}\right)\right]\psi_{r+1}\right\}& \\ -\frac{\mu^2}{2}\left\{\psi_r + \left(\frac{\partial}{\partial w}\cdot\frac{\partial}{\partial w}\right)\psi_{r+2}\right\}\,.&\end{aligned} \tag{5.11}$$

The left-hand-side is just the Fronsdal operator acting on ψ_r and the right-hand-side contains the couplings of the rank-r field with the other fields. The constraint (5.3) yields

$$\psi_r + 2\left(\frac{\partial}{\partial w}\cdot\frac{\partial}{\partial w}\right)\psi_{r+2} + \left(\frac{\partial}{\partial w}\cdot\frac{\partial}{\partial w}\right)^2\psi_{r+4} = 0. \tag{5.12}$$

Similarily to the higher-spin case, the gauge parameter was constrained by Equation (5.6). It is possible to remove this trace constraint by introducing a compensator field χ which transforms as [13]

$$\delta\chi = \left(\frac{\partial}{\partial w}\cdot\frac{\partial}{\partial w} + 1\right)\epsilon\,. \tag{5.13}$$

The gauge invariant equations of motion are now given by

$$\left[p^2 - (p\cdot w+\mu)\left(p\cdot\frac{\partial}{\partial w}\right) + \frac{1}{2}(p\cdot w+\mu)^2\left(\frac{\partial}{\partial w}\cdot\frac{\partial}{\partial w}+1\right)\right]\psi = \frac{1}{2}\,(p\cdot w+\mu)^3\chi\,, \tag{5.14}$$

and

$$\left(\frac{\partial}{\partial w}\cdot\frac{\partial}{\partial w}+1\right)^2\psi = \left[(p\cdot w+\mu)\left(\frac{\partial}{\partial w}\cdot\frac{\partial}{\partial w}+1\right) + 2\left(p\cdot\frac{\partial}{\partial w}\right)\right]\chi\,. \tag{5.15}$$

The partial gauge fixing $\chi = 0$ gives back the previous equations (5.2) and (5.3).

6 Fermionic equations

In this section we consider the double-valued continuous spin representation, formulated with the aid of a D-dimensional spinor.

For a totally symmetric spinor-tensor

$$\varphi^\alpha = \frac{1}{s!}\,\varphi^\alpha_{\mu_1\dots\mu_s}\,u^{\mu_1}\dots u^{\mu_s}, \tag{6.1}$$

the Fang-Fronsdal equation reads [19]

$$\left[\Gamma\cdot p - (u\cdot p)\left(\Gamma\cdot\frac{\partial}{\partial u}\right)\right]\varphi = 0. \tag{6.2}$$

It is invariant under the gauge transformations

$$\delta\varphi = (p\cdot u)\,\varepsilon\,, \tag{6.3}$$

with the spinor-tensor gauge parameter ε constrained by the gamma-trace condition

$$\left(\Gamma\cdot\frac{\partial}{\partial u}\right)\varepsilon = 0\,. \tag{6.4}$$

The analog of the double trace constraint is now

$$\left(\Gamma\cdot\frac{\partial}{\partial u}\right)\left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u}\right)\varphi = 0\,. \tag{6.5}$$

The homogeneity of φ is expressed by

$$\left(u\cdot\frac{\partial}{\partial u}-s\right)\varphi=0. \tag{6.6}$$

In the infinite spin limit, following the same procedure as before and defing ψ by

$$\varphi=\alpha^s\psi, \tag{6.7}$$

we get for ψ

$$\left[\Gamma\cdot p-(w\cdot p+\mu)\left(\Gamma\cdot\frac{\partial}{\partial w}+\Gamma^{D+1}\right)\right]\psi=0\,, \tag{6.8}$$

which is invariant under the Stückelberg-like gauge transformations

$$\delta\psi=(w\cdot p+\mu)\,\epsilon\,, \tag{6.9}$$

with

$$\left(\Gamma\cdot\frac{\partial}{\partial w}+\Gamma^{D+1}\right)\epsilon=0\,. \tag{6.10}$$

The double trace constraint becomes

$$\left(\Gamma\cdot\frac{\partial}{\partial u}+\Gamma^{D+1}\right)\left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u}+1\right)\psi=0. \tag{6.11}$$

Analogously to the bosonic case, we can now partially fix the gauge to obtain

$$\left(\Gamma\cdot\frac{\partial}{\partial w}+\Gamma^{D+1}\right)\psi=0,\quad \Gamma\cdot p=0. \tag{6.12}$$

If we define, similarly to the bosonic case, the gauge invariant field Ψ by

$$\Psi=\delta(w\cdot p+\mu)\psi, \tag{6.13}$$

then we obtain the following equations for the double valued continuous spin field

$$\begin{aligned}(w\cdot p+\mu)\Psi &= 0,\quad (\Gamma\cdot p)\Psi=0,\\ (p\cdot\frac{\partial}{\partial w})\Psi &= 0,\quad \left(\Gamma\cdot\frac{\partial}{\partial w}+\Gamma^{D+1}\right)\Psi=0\end{aligned} \tag{6.14}$$

Notice that the massive spinor-tensor that we get by compactification is non chiral and so in the limit we get a non chiral double valued continuous spin field. The fourth equation in the system (6.14) is absent in the fermionic Wigner equations; it is replaced by its square (2.9) and by a chirality constraint (in $D=4$). In both cases, the same number of spinorial components is eliminated. Our formulation is valid for arbitrary dimension D.

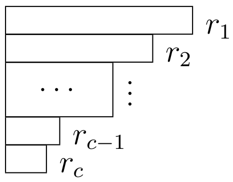
7 Exotic representations of the short little group

The representation of the short little group $SO(D-3)$ need not necessarily be the trivial one for space-time dimensions $D \geqslant 5$ [6].

The finite-dimensional representations of the pseudo-orthogonal groups $SO(p,q)$ correspond to rank-r (gamma)-traceless (spinor)-tensors with "mixed symmetries" labeled[9] by partition of the positive integer r into c integer parts $r_1\,,\,r_2\,,\,\ldots\,,\,r_c$ with

$$r_1 + r_2 + \ldots + r_c = r\,,\quad r_1 \geqslant r_2 \geqslant \ldots \geqslant r_c > 0\,, \tag{7.1}$$

and $c < p+q$. Such a partition is denoted by $(r_1\,,r_2\,,\,\ldots\,,r_c)$ and is depicted by the Young diagram



(7.2)

made of c rows, with the nth row containing r_n boxes. To describe such irreducible representations of the (pseudo) orthogonal groups, it is convenient to introduce c commuting auxiliary variable (for more details, see *e.g.* Section 3 of [20]).

7.1 Mixed symmetry gauge fields

As before we start from the helicity representations to build the continuous spin representations. During the last years, several steps have been performed towards a detailed understanding of mixed symmetry (also called "exotic") gauge fields in Minkowski space-time [21].

For the sake of simplicity, we will focus on gauge-fixed field equations generalizing (3.15) - (3.19) though one could start from the Labastida equations that generalize the Fronsdal formulation [22]. The wave function $\Phi(x, u_A)$ is a polynomial in the commuting variables u^μ_A. The subscripts will run from 0 to c and be denoted by capital Latin letters $A\,,\,B\,,\,$ *etc.* Proper wave equations are

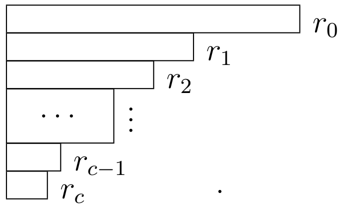
$$\begin{aligned} p^2\,\Phi &= 0\,, && (7.3)\\ (p\cdot u_A)\,\Phi &= 0\,, && (7.4)\\ \left(p\cdot\frac{\partial}{\partial u_A}\right)\Phi &= 0\,, && (7.5)\\ \left(\frac{\partial}{\partial u_A}\cdot\frac{\partial}{\partial u_B}\right)\Phi &= 0\,, && (7.6)\\ \left[\left(u_A\cdot\frac{\partial}{\partial u_B}\right) - (r_A - 1)\,\delta_{AB}\right]\Phi &= 0\,,\quad (A\leqslant B)\,. && (7.7) \end{aligned}$$

Equation (7.4) can be solved as

$$\Phi = \delta(u_A\cdot p)\,\varphi\,, \tag{7.8}$$

[9] In order to discard "dual" representations of $SO(p,q)$ one may further restrict the positive integer c to be smaller or equal to the integer part of $\frac{p+q}{2}$.

and leads together with Equation (7.5) to the fact that φ depends only on the transverse variables u^i_A. Then the last two conditions (7.6) - (7.7) express the fact that the coefficients of $\varphi(u^i_A)$ belong to an irreducible tensor representation of the helicity little group $SO(D-2)$ characterized by the Young diagram



. (7.9)

7.2 Exotic continuous spin particles

Performing the Kaluza-Klein compactification leads to a splitting of the auxiliary variables into (u^μ_A, v_A). Moreover, one makes contact with the procedure of Section 4 by identifying the couple of variables (u, v) with the couple (u_0, v_0). Let the Latin letters such as a, b, ... be subscripts running from 1 to c.

The length of the first row in (7.9) may be identified with the "spin" $s = r_0$. The infinite spin limit $s \to \infty$ of the massive equations with $A = B = 0$ leads to the Wigner equations (2.8) - (2.11), exactly as in Section 4. The novelty is that they are supplemented by new equations.

To start with, one should look at the equation

$$\left[\left(u_0 \cdot \frac{\partial}{\partial u_b}\right) + v_0 \frac{\partial}{\partial v_b}\right] \Phi = 0 \tag{7.10}$$

which implies that

$$\frac{\partial \Psi}{\partial v_a} = -\frac{1}{s}\left(w \cdot \frac{\partial \Psi}{\partial u_a}\right), \tag{7.11}$$

where we recall that $\Phi = \alpha^{s-1}\Psi$ and $v_0 = s\alpha$. Thus, the wave function does not depend on v_a in the infinite spin limit. The condition (7.11) allows to derive the last equations

$$(p \cdot u_a)\,\Psi = 0\,, \tag{7.12}$$

$$\left(p \cdot \frac{\partial}{\partial u_a}\right)\Psi = 0\,, \tag{7.13}$$

$$\left(\frac{\partial}{\partial w} \cdot \frac{\partial}{\partial u_a}\right)\Psi = 0 \tag{7.14}$$

$$\left(\frac{\partial}{\partial u_a} \cdot \frac{\partial}{\partial u_b}\right)\Psi = 0\,, \tag{7.15}$$

$$\left[\left(u_a \cdot \frac{\partial}{\partial u_b}\right) - (r_a - 1)\,\delta_{ab}\right]\Psi = 0\,, \quad (a \leqslant b)\,. \tag{7.16}$$

As before, Equations (2.8) and (7.12) can be solved as

$$\Psi = \delta(w \cdot p + \mu)\,\delta(u_a \cdot p)\,\psi\,. \tag{7.17}$$

Then Equations (2.10) and (7.13) imply in turn that ψ depends only on the transverse variables w^i and u^i_a. Equation (7.16) further eliminates one direction of the variables u_a. Eventually, Equations (7.15) - (7.16) impose that the physical components are in an irreducible representation of the short little group $SO(D-3)$ depicted by the Young diagram (7.9). This result is in complete agreement with the group-theoretical analysis of [6].

To summarize the single valued exotic case, one may say that the Wigner equations should be supplemented with the set (7.12) - (7.16). In the double valued exotic case, one should further add Equations (6.14) and the set

$$\left(\Gamma\cdot\frac{\partial}{\partial u_a}\right)\Psi = 0\,. \tag{7.18}$$

which expresses the gamma tracelessness for each index.

8 Conclusions and perspectives

We obtained covariant wave equations for the continuous spin field in various forms as an infinite spin limit of massive higher-spin equations. The first ones are identical to Wigner equations and were obtained from the gauge fixed equations. The second ones arise from the Fronsdal equations and exhibit gauge symmetries with constrained or unconstrained gauge parameters. A natural question arises: Is it possible to derive some of these equations from an action principle? In this respect, we mention that the limit of the "Einstein tensor" is singular even with the field redefinitions we performed. Thus the possibility of formulating the action principle remains open.

Higher-spin fields are known to propagate consistently in constant curvature backgrounds, therefore the infinite spin limit suggests to look for the flat space limit of some $(A)dS$ representations that would lead to the continuous spin representations. The mass-shell condition for totally symmetric representations of the anti de Sitter group $SO(D-1,2)$ is [23]

$$\left[D^2 - m^2 - \frac{1}{R^2}\Big(s^2 + (6-D)s + 2(D-3)\Big)\right]\Phi = 0\,, \tag{8.1}$$

where D is the covariant derivative and R is the radius of AdS_D. In order for the infinite spin limit of (8.1) to be non-singular it appears to be indeed necessary to require that the radius R goes to infinity at least as fast as s. This argument also holds for any massive representation of the AdS_D isometry group. In this respect,

Looking at the continuous spin equation expanded in the homogeneity degree, the parameter μ plays the role of a coupling constant of bilinear interactions between fields of different degrees. These coupling terms are responsible for the fact that the representation is irreducible. In the decoupling limit $\mu\to 0$ the single (or double) valued continuous spin representation, as we showed in Section 2.2, indeed decomposes as an infinite sum of helicity representations for all integer (or half-odd) spins. Therefore this limit provides a natural mechanism to generate an infinite tower of massless higher-spins. The latter seems to be a proper starting point to switch on interactions (see *e.g.* [24, 20]) so it would be interesting to try introducing self-interactions for the continuous spin field itself. [10]

[10] Even in flat space-time, the fact that the parameter μ has mass dimension is promising for writing consistent interaction vertices, since μ might play a role analogue to the inverse AdS-radius $1/R$ in Vasiliev's construction.

At first sight, second quantisation of the continuous spin field seems to lead either to non-locality or to a breakdown of causality [5]. Wigner himself argued against the continuous spin particle because they should lead to an infinite heat capacity of the vacuum, essentially because the number of polarizations (*i.e.* the spin in $D = 4$) is infinite [3]. It would be very satisfactory if the infinite spin limit could explain – if not resolve – the elusive properties of the continuous spin representation. Since the original higher-spin massive field is well behaved, the subtle infinite spin limit should be at the origin of these strange properties and finding a proper limit might regularise some of those unconventional characteristics.

References

[1] E. P. Wigner, Annals Math. **40** (1939) 149 [Nucl. Phys. Proc. Suppl. **6** (1989) 9].

[2] E.P. Wigner, Z. Physik **124** (1947) 665;
V. Bargmann and E. P. Wigner, Proc. Nat. Acad. Sci. **34** (1948) 211.

[3] E. P. Wigner, "Invariant quantum mechanical equations of motion," in *International Atomic Energy Agency* (Vienna, 1963) 59.

[4] J. Yngvason, Commun. Math. Phys. **18** (1970) 195;
A. Chakrabarty, J. Math. Phys. **12** (1971) 1813;
G.J. Iverson and G. Mack, Ann. Phys. **64** (1971) 211.

[5] L. F. Abbott, Phys. Rev. D **13** (1976) 2291;
K. Hirata, Prog. Theor. Phys. **58** (1977) 652;
J. Mund, B. Schroer and J. Yngvason, Phys. Lett. B **596** (2004) 156 [arXiv:math-ph/0402043].

[6] L. Brink, A. M. Khan, P. Ramond and X. Xiong, J. Math. Phys. **43** (2002) 6279 [hep-th/0205145].

[7] E. Inönü and E. P. Wigner, Proc. Nat. Acad. Sci. **39** (1953) 510.

[8] A. M. Khan and P. Ramond, J. Math. Phys. **46** (2005) 053515 [Erratum-ibid. **46** (2005) 079901] [hep-th/0410107].

[9] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. A **173**, 211 (1939).

[10] C. Fronsdal, Phys. Rev. D **18** (1978) 3624.

[11] L. P. S. Singh and C. R. Hagen, Phys. Rev. D **9** (1974) 898.

[12] C. Aragone, S. Deser and Z. Yang, Annals Phys. **179** (1987) 76;
A. I. Pashnev, Theor. Math. Phys. **78** (1989) 272 [Teor. Mat. Fiz. **78** (1989) 384].
For a review see X. Bekaert, I. L. Buchbinder, A. Pashnev and M. Tsulaia, Class. Quant. Grav. **21** (2004) S1457 [hep-th/0312252].

[13] D. Francia and A. Sagnotti, Class. Quant. Grav. **20** (2003) S473 [hep-th/0212185];
"Minimal local lagrangians for higher-spin geometry," hep-th/0507144;
A. Sagnotti and M. Tsulaia, Nucl. Phys. B **682** (2004) 83 [hep-th/0311257].

[14] D. Sorokin, "Introduction to the classical theory of higher spins," AIP Conf. Proc. **767**, 172 (2005) `hep-th/0405069` .
Proceedings of the First Solvay Workshop on Higher-Spin Gauge Theories (Brussels, May 2004), http://www.ulb.ac.be/sciences/ptm/pmif/Solvay1proc.pdf,
in particular see N. Bouatta, G. Compere and A. Sagnotti, `hep-th/0409068`; M. Bianchi and V. Didenko, `hep-th/0502220`; S. de Buyl and A. Kleinschmidt, `hep-th/0410274`; A. Sagnotti, E. Sezgin and P. Sundell, `hep-th/0501156`.

[15] G. K. Savvidy, Int. J. Mod. Phys. A **19** (2004) 3171 [`hep-th/0310085`];
J. Mourad, "Continuous spin particles from a string theory," `hep-th/0504118`.

[16] D. Zoller, Class. Quant. Grav. **11** (1994) 1423;
J. Mourad, "Continuous spin and tensionless strings," `hep-th/0410009`;
L. Edgren, R. Marnelius and P. Salomonson, "Infinite spin particles," `hep-th/0503136`.

[17] A. Mikhailov, "Notes on higher spin symmetries," `hep-th/0201019`.

[18] M. Bianchi, P. J. Heslop and F. Riccioni, "More on *la grande bouffe*," `hep-th/0504156`.

[19] J. Fang and C. Fronsdal, Phys. Rev. D **18** (1978) 3630.

[20] X. Bekaert, S. Cnockaert, C. Iazeolla and M.A. Vasiliev, "Nonlinear Higher Spin Theories in Various Dimensions," [`hep-th/0503128`].

[21] X. Bekaert and N. Boulanger, Commun. Math. Phys. **245** (2004) 27 [`hep-th/0208058`];
P. de Medeiros and C. Hull, JHEP **0305** (2003) 019 [`hep-th/0303036`].

[22] J. M. Labastida, Phys. Rev. Lett. **58** (1987) 531; Nucl. Phys. B **322** (1989) 185.

[23] R. R. Metsaev, Phys. Lett. B **590** (2004) 95 [`hep-th/0312297`].

[24] M. A. Vasiliev, Phys. Lett. B **567** (2003) 139 [`hep-th/0304049`]; Fortsch. Phys. **52** (2004) 702 [`hep-th/0401177`]; Comptes Rendus Physique **5** (2004) 1101 [`hep-th/0409260`].