Hopf algebras in renormalization theory: Locality and Dyson-Schwinger equations from Hochschild cohomology

Christoph BERGBAUER and Dirk KREIMER

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

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HOPF ALGEBRAS IN RENORMALIZATION THEORY: LOCALITY AND DYSON-SCHWINGER EQUATIONS FROM HOCHSCHILD COHOMOLOGY

C. BERGBAUER AND D. KREIMER

ABSTRACT. In this review we discuss the relevance of the Hochschild cohomology of renormalization Hopf algebras for local quantum field theories and their equations of motion.

CONTENTS

INTRODUCTION AND ACKNOWLEDGMENTS

The relevance of infinite dimensional Hopf and Lie algebras for the understanding of local quantum field theory has been established in the last couple of years. Here, we focus on the role of the 1-cocycles in the Hochschild cohomology of such renormalization Hopf algebras.

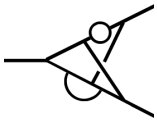
After an introductory overview which recapitulates the well-known Hopf algebra of rooted trees we exhibit once more the crucial connection between 1-cocycles in the Hochschild cohomology of the Hopf algebra, locality and the structure of the quantum equations of motion. For the latter, we introduce combinatorial Dyson-Schwinger equations and show that the perturbation series provides Hopf subalgebras indexed only by the order of the perturbation. We then discuss assorted applications of such equations which focus on the notion of self-similarity and transcendence.

This paper is based on an overview talk given by one of us (D. K.), extended by a more detailed exhibition of some useful mathematical aspects of the Hochschild cohomology of the relevant Hopf algebras. It is a pleasure to thank the organizers of the *75ème Rencontre entre Physiciens Théoriciens et Mathématiciens* for organizing that enjoyable workshop. D. K. thanks Karen Yeats for discussions on the transcendental nature of DSEs. C. B. acknowledges support by the Deutsche Forschungsgemeinschaft. He also thanks Boston University and the IHES for hospitality.

1. Rooted trees, Feynman graphs, Hochschild cohomology and local counterterms

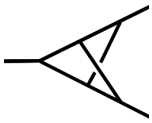
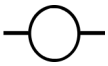
1.1. **Motivation.** Rooted trees store information about nested and disjoint subdivergences of Feynman graphs in a natural way. This has been used at least implicitly since Hepp's proof of the BPH subtraction formula [22] and Zimmermann's forest formula [37]. However it was only decades later that the algebraic structure of the Bogoliubov recursion was elucidated by showing that it is essentially given by the coproduct and the corresponding antipode of a Hopf algebra on rooted trees [24, 9]. The same result can be formulated more directly in terms of a very similar Hopf algebra on 1PI Feynman graphs [10]. We start with the description in terms of rooted trees which serves as a universal role model for all Hopf algebras of this kind.

For instance, the subdivergences of the ϕ^3 diagram in six spacetime dimensions

$\Gamma =$ 

can be represented by the decorated tree

where

$\gamma_1 =$  , $\gamma_2 =$  , $\gamma_3 =$  .

Additional labelling (which we do not care about here) would be needed to keep track of the actual insertion places. However, since one is ultimately interested in the sum of all Feynman graphs of a given order in perturbation theory, for the purpose of the Bogoliubov recursion all possible insertions of γ_2 and γ_3 into γ_1 can be considered at the same time, when due care is given to the resolution of graphs with overlapping divergences into appropriate linear combinations of trees.

In a moment we will need the trees

and


whose meaning should be clear: They represent the graph γ_1 for which γ_2 or γ_3, respectively, is suitably inserted.

Now how is Γ renormalized? According to the Bogoliubov recursion, the renormalized value is given by

$$\begin{aligned}\phi_R\left(\bullet^{\gamma_1}_{\gamma_2}\!\!\wedge\!\!\bullet_{\gamma_3}\right) \;:=\; & (id-R)\left(\phi\left(\bullet^{\gamma_1}_{\gamma_2}\!\!\wedge\!\!\bullet_{\gamma_3}\right) - R\phi(\bullet_{\gamma_2})\phi\left(\bullet^{\gamma_1}_{\bullet_{\gamma_3}}\right) - \right.\\ & -R\phi(\bullet_{\gamma_3})\phi\left(\bullet^{\gamma_1}_{\bullet_{\gamma_2}}\right) - R\left(\phi(\bullet_{\gamma_2}\bullet_{\gamma_3}) - \right.\\ & \left.\left. -\phi(\bullet_{\gamma_2})R\phi(\bullet_{\gamma_3}) - \phi(\bullet_{\gamma_3})R\phi(\bullet_{\gamma_2})\right)\phi(\bullet_{\gamma_1})\right)\end{aligned} \tag{1}$$

where ϕ denotes the unrenormalized but possibly regularized (if we do not renormalize on the level of the integrand) contribution of the graph which a given tree represents. For example, in dimensional regularization, ϕ is a map into the algebra $V := \mathbb{C}[\epsilon^{-1}, \epsilon]]$ of Laurent series with finite pole part. The map $R : V \to V$ is a renormalization scheme. For example, the minimal subtraction scheme is obtained by defining R to be the projector onto the proper pole part, $R(\epsilon^k) = \epsilon^k$ if $k < 0$ and $R(\epsilon^k) = 0$ otherwise. We emphasize though that the use of a regulator can be avoided by defining a suitable renormalization scheme on the level of integrands. Such an approach can then be directly formulated on the level of Dyson–Schwinger equations, where the choice of a renormalization scheme can be non-perturbatively given as the choice of a boundary condition for the accompanying integral equation.

Now consider the polynomial algebra $\mathcal{H}$ generated by all decorated rooted trees of this kind. There is a coproduct on it which disentangles trees into subtrees and thus divergences into subdivergences, as will be discussed in subsection 1.3. Using this coproduct Δ, the above algebra $\mathcal{H}$ becomes a Hopf algebra. Let S be its antipode. By definition, S satisfies the recursive relation (in Sweedler's notation, omitting the primitive part)

$$S(x) = -x - \tilde{\sum} S(x')x''$$

It turns out that, if one similarly defines a "twisted antipode" S_R^ϕ as a map $\mathcal{H} \to V$ by $S_R^\phi(1) = 1$ and

$$S_R^\phi(x) = -R\left(\phi(x) + \tilde{\sum} S_R^\phi(x')\phi(x'')\right),$$

then S_R^ϕ provides the counterterm and the convolution $S_R^\phi \star \phi = m_V(S_R \otimes \phi)\Delta$ solves the Bogoliubov recursion: $\phi_R = S_R^\phi \star \phi$ [24, 9]. Using this algebraic approach to the combinatorial intricacies of renormalization, many important questions in perturbative and non-perturbative quantum field theory can be treated from a convenient conceptual point of view, some of which will be reviewed in the following sections.

This picture translates rather easily to renormalization in coordinate space [1], as will be briefly discussed in subsection 1.6.

Before continuing the discussion of renormalization, we introduce some key algebraic notions. We will come back to the example of the graph Γ later on.

1.2. **Basic definitions and notation.** Let k be a field of characteristic zero. We consider k-bialgebras $(A, m, \mathbb{I}, \Delta, \epsilon)$ that are graded connected, that is

$$A = \bigoplus_{n=0}^{\infty} A_n, \quad A_0 \cong k, \quad A_m A_n \subseteq A_{m+n}, \quad \Delta(A_n) \subseteq \bigoplus_{l+m=n} A_l \otimes A_m.$$

By abuse of notation, we write $\mathbb{I}$ both for the unit and the unit map. Also, we sometimes consider ϵ as a map $A \to A_0$. We assume that $\Delta(\mathbb{I}) = \mathbb{I} \otimes \mathbb{I}$. It follows that $\epsilon(\mathbb{I}) = 1$ while $\epsilon(A_n) = 0$ for $n \neq 0$. The kernel of ϵ is called the augmentation ideal, and the map $P : A \to A$, $P = id - \epsilon$, is called the projection onto the augmentation ideal. The coproduct Δ gives rise to another coassociative map: $\tilde{\Delta}$, defined by

$$\tilde{\Delta}(x) = \Delta(x) - \mathbb{I} \otimes x - x \otimes \mathbb{I}.$$

Recall that elements in the kernel of $\tilde{\Delta}$ are called *primitive*. We will occasionally use Sweedler's notation $\Delta(x) = \sum x' \otimes x''$ and also $\tilde{\Delta}(x) = \tilde{\sum} x' \otimes x''$.

It is a well known fact that connected graded bialgebras are Hopf algebras. Indeed, the sequence defined by the recursive relation

$$S(x) = -x - \tilde{\sum} S(x')x'' \text{ for } x \notin A_0, \quad S(\mathbb{I}) = \mathbb{I} \tag{2}$$

converges in $\mathrm{End}_k(A)$.

For a coalgebra (A, Δ) and an algebra (B, m), the vector space $\mathrm{Hom}_k(A, B)$ of linear maps $A \to B$ is equipped with a convolution product $\star$ by $(f, g) \mapsto f \star g = m(f \otimes g)\Delta$. Thus $(f \star g)(x) = \sum f(x')g(x'')$. Now let Q be the linear endomorphism of $A^{\otimes 2}$ defined by $Q(\mathbb{I} \otimes \mathbb{I}) = -\mathbb{I} \otimes \mathbb{I}$ and $Q = id \otimes P$ otherwise. Using the modified product $\star_Q : (f, g) \mapsto f \star_Q g = m(f \otimes g)Q\Delta$, equations (2) can be rewritten

$$S = -S \star_Q id$$

which will be convenient later on. Note that for $x \neq \mathbb{I}$, this is the same as saying

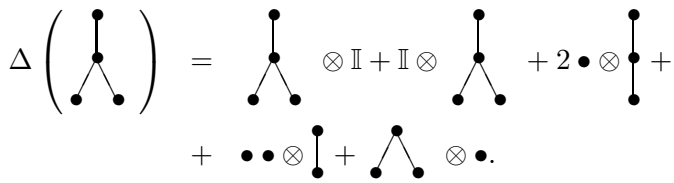
$$S(x) = -m(S \otimes P)\Delta(x).$$

1.3. **The Hopf algebra of rooted trees.** Now we give a more detailed construction of the Hopf algebra $\mathcal{H}$ of rooted trees [24, 9] that is in the center of all our considerations. An (undecorated, non-planar) rooted tree is a connected contractible compact graph with a distinguished vertex, the root. By convention, we will draw the root on top. We are only interested in isomorphism classes of rooted trees (an isomorphism of rooted trees being an isomorphism of graphs which maps the root to the root) which we, by abuse of language, simply call rooted trees again. As a graded algebra, $\mathcal{H}$ is the free commutative algebra generated by trees (including the empty tree which we consider the unit $\mathbb{I}$) with the weight grading: the weight of a tree is the number of its vertices. Remember that our trees are compact. A product of rooted trees is called a forest – obviously the weight of a forest is the sum of the weights of its trees. On $\mathcal{H}$ a coproduct Δ is introduced by

$$\Delta(\tau) = \mathbb{I} \otimes \tau + \tau \otimes \mathbb{I} + \sum_{adm.c} P_c(\tau) \otimes R_c(\tau) \tag{3}$$

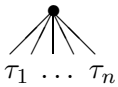
where the sum goes over all *admissible cuts* of the tree τ. By a cut of τ we mean a nonempty subset of the edges of τ that are to be removed. The product of subtrees

which "fall down" upon removal of those edges is called the *pruned part* and is denoted $P_c(\tau)$, the part which remains connected with the root is denoted $R_c(\tau)$. This makes sense only for certain "admissible" cuts: by definition, a cut $c(\tau)$ is admissible, if for each leaf l of τ it contains at most one edge on the unique path from l to the root. For instance,



The coassociativity of Δ is shown in [24]. $\mathcal{H}$ is obviously not cocommutative. Since the coproduct is compatible with the grading, $\mathcal{H}$ is a Hopf algebra. There is an important linear endomorphism of $\mathcal{H}$, the grafting operator B_+ defined as follows:

$$B_+(\mathbb{I}) = \bullet$$

(4) $B_+(\tau_1 \dots \tau_n) =$  for trees τ_i

In words: B_+ creates a new root and connects it with each root of its argument. The special importance of B_+ will become evident in subsection 1.5: B_+ is a closed but not exact Hochschild 1-cochain.

1.4. **Tree-like structures and variations on a theme.**

Tree-like structures. ¿From the Hopf algebra $\mathcal{H}$, defined in the previous subsection, several generalizations can be constructed: Hopf algebras of decorated trees, of planar trees, etc. This can be phrased most elegantly from a general point of view in terms of "tree-like structures", as for example introduced by Turaev in [36]: Consider the category of rooted trees and embeddings (an embedding $\tau' \to \tau$ is an isomorphism from τ' to a subtree of τ). A rooted tree-structure is then defined to be a contravariant functor from this category to the category of sets. For example, decorated (labelled) trees can be described by the functor ϕ which maps a tree onto a certain set its vertices and/or edges are decorated with. Being contravariant, ϕ maps embeddings of trees to the respective restrictions of decorations. Similarly, a planar structure is provided by a functor ϕ mapping a tree to the set of its topological embeddings into the real plane modulo orientation-preserving homeomorphisms of $\mathbb{R}^2$ onto itself. Now let ϕ be a rooted tree-structure. A rooted ϕ-tree is a pair (τ, s) where τ is a tree and s is an element of $\phi(\tau)$. The notions of isomorphisms and subtrees of rooted ϕ-trees are immediate.

Generalizations of $\mathcal{H}$. Using this convenient framework, we have immediately other Hopf algebras at hand: Let S be a set. The Hopf algebra $\mathcal{H}(S)$ is defined as in the previous subsection, replacing the word tree by S-decorated tree (for our purposes, we only decorate vertices, not edges). Similarly, $\mathcal{H}_{pl}$ is the (noncommutative) Hopf algebra of planar rooted trees. In particular, for these Hopf algebras, the proofs of the coassociativity of Δ are verbatim the same. The planar Hopf algebra and its decorated versions $H_{pl}(S)$ were extensively studied by Foissy [18, 19]. He showed that they are self-dual and constructed isomorphisms to several other Hopf algebras on trees that have appeared in the literature.

The Hopf algebra of Feynman graphs. While rooted trees describe nested divergences in an obvious manner, the resolution of *overlapping* divergences into trees requires some care [37, 25, 17]. This problem exists only in momentum space. By basing a Hopf algebra directly on Feynman graphs instead of trees, these issues can be avoided [25, 10]. As an algebra, let H_{CK} be the free commutative algebra on 1PI Feynman graphs (of a given theory; the case of a non-scalar theory requires to take form factors (external structures) into account which we avoid here). The empty graph serves as a unit $\mathbb{I}$. In the following, a product of graphs is identified with the disjoint union of these graphs. On a graph, a coproduct is given [10] by

$$\Delta(\Gamma) = \mathbb{I} \otimes \Gamma + \Gamma \otimes \mathbb{I} + \sum_{\gamma \subsetneq \Gamma} \gamma \otimes \Gamma/\gamma$$

where the sum is over all 1PI superficially divergent proper subgraphs γ of Γ. A few examples are given in [10].

Still, thanks to the universal property mentioned at the end of subsection 1.5, Hopf algebras of rooted trees serve as an excellent role model for various questions and, moreover, yield most interesting links to different branches of mathematics [12, 21]. In the present paper, we will be mainly concerned with Hopf algebras of trees. In many cases, it is only a matter of notation to translate these results into the Hopf algebra of Feynman graphs, easily achieved by the practitioner of QFT [4].

In view of the preceding paragraphs, the reader might wish to try to describe $\mathcal{H}_{CK}$ as a Hopf algebra of suitable tree-like structures, using the results of [25].

1.5. **Hochschild cohomology of bialgebras.**

Definition. Let A be a bialgebra. We consider linear maps $L : A \to A^{\otimes n}$ as n-cochains and define a coboundary operator b by

$$bL := (id \otimes L)\Delta + \sum_{i=1}^{n} (-1)^i \Delta_i L + (-1)^{n+1} L \otimes \mathbb{I} \tag{5}$$

where Δ denotes the coproduct and Δ_i the coproduct Δ applied to the i-th factor in $A^{\otimes n}$. The map $L \otimes \mathbb{I}$ is given by $x \mapsto L(x) \otimes \mathbb{I}$. It is essentially due to the coassociativity of Δ that b squares to zero, which gives rise to a cochain complex (C, b). Clearly (C, b) captures only information about the coalgebra structure of A. The cohomology of (C, b), denoted $HH^\bullet_\epsilon(A)$, is easily seen to be the dual (A considered as a bi*co*module rather than a bimodule over itself) notion of the Hochschild cohomology of algebras. Note that the right bicomodule action is here $(id \otimes \epsilon)\Delta$ which explains the last summand in (5) and the subscript in $HH^\bullet_\epsilon$.

The role of $HH^1_\epsilon(\mathcal{H})$. For $n = 1$, the cocycle condition $bL = 0$ reduces to, for $L : A \to A$,

$$\Delta L = (id \otimes L)\Delta + L \otimes \mathbb{I}. \tag{6}$$

Sometimes the following equivalent statement, using the map $\tilde{\Delta}$, is more convenient:

$$\tilde{\Delta} L = (id \otimes L)\tilde{\Delta} + id \otimes L(\mathbb{I}). \tag{7}$$

Let us now try to understand the space $HH^1_\epsilon(\mathcal{H})$ of "outer coderivations on $\mathcal{H}$." We first describe the 0-coboundaries ("inner coderivations"). They are of the form

$$L(\tau) = \sum \alpha_{\tau''}\tau' - \alpha_{\tau'}\mathbb{I}$$

in Sweedler's notation, where α_τ is an element of k for each forest τ. For example, $L : \tau \mapsto \sum \tau' - \mathbb{I}$ is a 0-coboundary. Note that $\mathbb{I}$ is in the kernel of any 0-coboundary.

It is a crucial fact that the grafting operator B_+, introduced in subsection 1.3 is a 1-cocycle [9]:

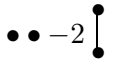
$$\Delta B_+ = (id \otimes B_+)\Delta + B_+ \otimes \mathbb{I}. \tag{8}$$

$$\tilde{\Delta} B_+ = (id \otimes B_+)\tilde{\Delta} + id \otimes \bullet. \tag{9}$$

When looking at equation (9), the statement is rather immediate: Let τ be a forest. The first summand at the right side of (9) refers to cuts of $B_+(\tau)$ which affect at most all but one of the edges connecting the new root of $B_+(\tau)$ to the roots of τ, while the second summand takes care of the cut which completely separates the root of $B_+(\tau)$ from all its children.

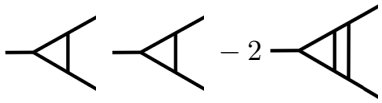
Since B_+ is a homogeneous linear endomorphism of degree 1, it is not a 0-coboundary – note that the coboundaries have no chance to increase the degree. Thus B_+ is a generator (among others) of $HH^1_\epsilon(\mathcal{H})$.

When looking for other generators L of $HH^1_\epsilon(\mathcal{H})$, the cocycle conditions (6,7) immediately yield the requirement that $L(\mathbb{I})$ be a primitive element (and zero if L is exact). While $\bullet$ is up to scalar factors obviously the only primitive element in degree 1, there are plenty of primitives in higher degrees. For example,

(10) 

is a primitive element in degree 2. Foissy [18] showed that $L \mapsto L(\mathbb{I})$ is a surjective map $HH^1_\epsilon(\mathcal{H}) \to$Prim$(\mathcal{H})$ onto the set of primitive elements of $\mathcal{H}$. In the case of Hopf algebras of decorated rooted trees $\mathcal{H}(S)$ obviously any element $s \in S$ yields a homogeneous cocycle of degree 1 denoted B^s_+ which, applied to a forest, connects its roots to a new root decorated by s.

It should be clear that each 1PI Feynman graph which is free of subdivergences is a primitive element of $\mathcal{H}_{CK}$. In general, there are primitive elements in higher degrees too, for example, cf. (10), the linear combination



in ϕ^3 theory in six dimensions.

Universal property. The category of objects (A, L) consisting of a commutative bialgebra A and a Hochschild 1-cocycle L on A with morphisms bialgebra morphisms commuting with the cocycles has the initial object $(\mathcal{H}, B_+)$. This is a result of [9]. Indeed, let (A, L) be such a pair. The map $\rho : \mathcal{H} \to A$ is simply defined by $\rho(\mathbb{I}) = \mathbb{I}$ and pushing forward along B_+ (and L) and the multiplication. The fact

that ρ is a morphism of coalgebras is an easy consequence of (8).

Also it was shown in [1] that, conversely, the coproduct Δ of $\mathcal{H}$ is determined if one requires the map B_+ to be a 1-cocycle. This may serve to find different presentations of $\mathcal{H}$.

For any $\mathcal{H}$-bicomodule B, the higher Hochschild cohomology $HH^n(\mathcal{H}, B)$, $n \geq 2$, is trivial [18], thus in particular $HH^n_\epsilon(\mathcal{H}) = 0$.

1.6. **Finiteness and locality from the Hopf algebra.** We have now accumulated enough algebraic notions to come back to the original physical application already sketched in subsection 1.1. Given a specific quantum field theory, Hopf algebras $\mathcal{H}(S)$ and $\mathcal{H}_{CK}$ are determined by its perturbative expansion into Feynman graphs. We denote this Hopf algebra generically by $\mathcal{H}$. Every divergent graph γ without subdivergences determines a Hochschild 1-cocycle B_+^γ, and any relevant tree or graph is in the range of a 1-cocycle of this kind.

Momentum space. The next step is to choose a target algebra V and regularized Feynman rules $\phi : \mathcal{H} \to V$, and a renormalization scheme $R : V \to V$. The map ϕ is supposed to be a (unital) algebra homomorphism. We stick to the example $(V = \mathbb{C}[\epsilon^{-1}, \epsilon]], \phi)$ of dimensional regularization as in subsection 1.1, but stress once more that the reader can find suitable generalizations in the literature [15]. The minimal subtraction scheme where R is the projector onto the proper pole part is only one of many choices one can make. However, in any case we require R to preserve the UV divergent structure (i. e. the pole part) and to satisfy the Rota-Baxter equation

$$R(xy) + R(x)R(y) = R(xR(y)) + R(R(x)y). \tag{11}$$

Moreover we demand that $R(1) = 1$. It is easy to check that the minimal subtraction scheme satisfies (11). The Rota-Baxter equation is the algebraic key to the link between renormalization and Birkhoff decomposition, see for example [10, 14, 15]. It also guarantees that the renormalized Feynman rules are again an algebra homomorphism [26] as are the unrenormalized rules ϕ. Now the twisted antipode is defined by

$$S_R^\phi = -R(S_R^\phi \star_Q \phi) \tag{12}$$

equivalently, in Sweedler's notation

$$S_R^\phi(\tau) = -R\left(\phi(\tau) + \tilde{\sum} S_R^\phi(\tau')\phi(\tau'')\right), \text{ for } \tau \neq \mathbb{I}, \quad S_R^\phi(\mathbb{I}) = 1$$

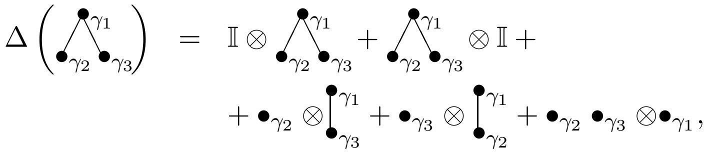
where the term "twisted antipode" should be justified by a glance at the recursive expression (2) for the regular antipode. The map S_R^ϕ, as can be inferred from the example in Figures 1–3 yields the counterterm for ϕ. The complete unrenormalized evaluation function is then given by

$$\phi_R = S_R^\phi \star \phi. \tag{13}$$

One can find a non-recursive description of ϕ_R [24, 9] which shows the equivalence with Zimmermann's forest formula [37].

Example. In order to understand the twisted antipode, we come back to the example of subsection 1.1. On the relevant trees, the coproduct acts as follows:

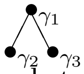
(14)



$$\Delta(\bullet_{\gamma_i}) = \mathbb{I}\otimes\bullet_{\gamma_i}+\bullet_{\gamma_i}\otimes\mathbb{I},$$

According to (12) and (13), the algorithm for ϕ_R consists of the following steps:

(F) Apply the coproduct Δ to the tree under consideration
(C_n) (for $n = 1\ldots$) apply the map $Q\Delta\otimes id^{\otimes n}$ until all summands are of the form $\mathbb{I}\otimes\ldots$.
(C'_n) (for $n = \ldots 1$) apply the map $-R\phi m\otimes id^{\otimes n}$ until we end up in $V\otimes\mathcal{H}$
(F') apply the map $m_V(id\otimes\phi)$ to get into V.

For the tree  this algorithm is performed in Figures 1–3. While in our simple example of only two disjoint subdivergences, the Bogoliubov recursion could have been performed by hand without using the Hopf algebra, when going to higher loop orders, the Hopf algebra approach provides significant computational advantage [4, 5].

Locality of counterterms from Hochschild cohomology. Moreover, the Hopf algebra can be used to give a direct proof of finiteness of renormalization and locality of counterterms from a purely algebraic point of view. For a simple toy model, this has been done in a recent paper [27]. The basic observation is that the fact that every relevant tree or graph is in the range of a homogeneous Hochschild 1-cocycle of degree 1 allows for easy and clean inductive proofs of various statements for arbitrary loop number. This also holds on the level of graphs: the sum over all primitive graphs of a given loop order n defines a 1-cocycle B_+^n such that every graph is generated in the range of these 1-cocycles. This allows, as observed in [28, 32], to prove locality in general.

Indeed, let B_+^n be a 1-cocycle, and let μ_+ be the measure defined by the n-loop integrand of $B_+^n(\mathbb{I})$. Let $\phi(B_+^n(X))$ be a Feynman amplitude defined by insertion of a collection of subdivergences X into those n-loop primitive graphs. We write $\phi(B_+^n(X)) = \int\phi(X)d\mu_+$, emphasizing that subgraphs become subintegrals under the Feynman rules.

Recall that P denotes the projection onto the augmentation ideal. Since $P(\mathbb{I}) = 0$, $PB_+^n = B_+^n$, we can write

$$m(S_R^\phi\otimes\phi P)\Delta(B_+^n(X)) = \int S_R^\phi\star\phi(X)d\mu_+.$$

This proves locality in a straightforward manner by induction over the augmentation degree, i. e. using the coradical filtration of the Hopf algebra.

Coordinate space. The language of rooted trees is especially suited for describing renormalization in coordinate space [1]. A particularly appealing approach to coordinate space renormalization is the work of Epstein and Glaser [16] (see also

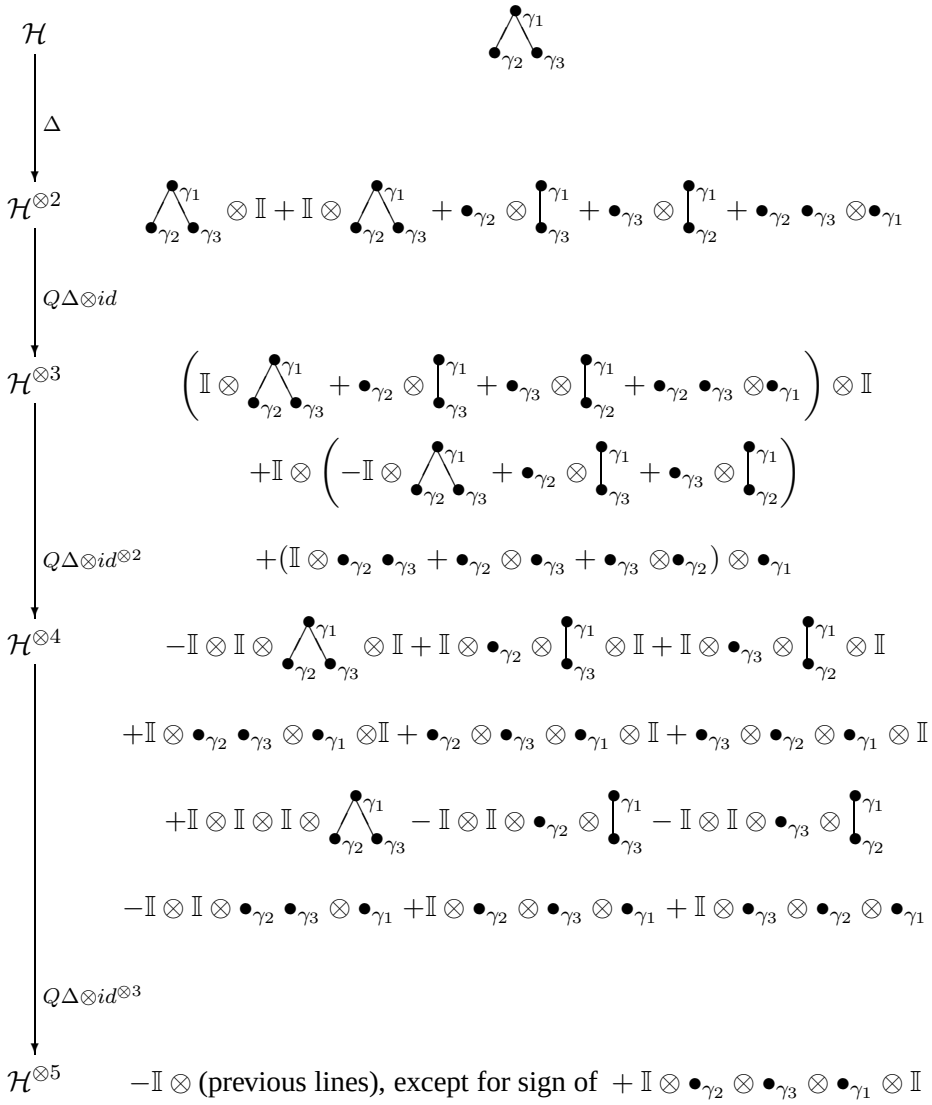


FIGURE 1. First part of the calculation of ϕ_R. Apply Δ and then $Q\Delta\otimes id^{\otimes n}$ until all summands are of the form $\mathbb{I}\otimes\ldots$.

[35, 7]) who, starting from ideas of Bogoliubov [3] and others, extracted a set of axioms for time-ordered product and constructed such time-ordered products in terms of rigorous functional analysis. The result is completely equivalent to momentum space renormalization but has conceptual (albeit not computational) advantages. It is no surprise that the Hopf algebra picture fits equally nice into this framework [1], if one takes into account the specific features of Epstein-Glaser renormalization such as the absence of overlapping divergences and regularization parameters. In view of highly interesting mathematical ramifications such as a possible analogy to the Fulton-MacPherson compactification of configuration spaces [20, 33], it seems most appropriate to attack this problem using trees [1] rather than coordinate space Feynman diagrams.

$$\xrightarrow{-R\phi m\otimes id^3}$$

$$V\otimes\mathcal{H}^{\otimes 3}\qquad -1\otimes\mathbb{I}\otimes\bullet_{\gamma_1}\!\!\begin{smallmatrix}\diagdown & \diagup\\ \bullet_{\gamma_2} & \bullet_{\gamma_3}\end{smallmatrix}\otimes\mathbb{I}+1\otimes\bullet_{\gamma_2}\otimes\bullet_{\gamma_1}\!\!\begin{smallmatrix}|\\ \bullet_{\gamma_3}\end{smallmatrix}\otimes\mathbb{I}+1\otimes\bullet_{\gamma_3}\otimes\bullet_{\gamma_1}\!\!\begin{smallmatrix}|\\ \bullet_{\gamma_2}\end{smallmatrix}\otimes\mathbb{I}$$

$$+1\otimes\bullet_{\gamma_2}\bullet_{\gamma_3}\otimes\bullet_{\gamma_1}\otimes\mathbb{I}-R\phi(\bullet_{\gamma_2})\otimes\bullet_{\gamma_3}\otimes\bullet_{\gamma_1}\otimes\mathbb{I}$$

$$\xrightarrow{-Rm_V(id\otimes\phi)\otimes id^2}\quad -R\phi(\bullet_{\gamma_3})\otimes\bullet_{\gamma_2}\otimes\bullet_{\gamma_1}\otimes\mathbb{I}+1\otimes\mathbb{I}\otimes\mathbb{I}\otimes\bullet_{\gamma_1}\!\!\begin{smallmatrix}\diagdown & \diagup\\ \bullet_{\gamma_2} & \bullet_{\gamma_3}\end{smallmatrix}$$

$$-1\otimes\mathbb{I}\otimes\bullet_{\gamma_2}\otimes\bullet_{\gamma_1}\!\!\begin{smallmatrix}|\\ \bullet_{\gamma_3}\end{smallmatrix}-1\otimes\mathbb{I}\otimes\bullet_{\gamma_3}\otimes\bullet_{\gamma_1}\!\!\begin{smallmatrix}|\\ \bullet_{\gamma_2}\end{smallmatrix}-1\otimes\mathbb{I}\otimes\bullet_{\gamma_2}\bullet_{\gamma_3}\otimes\bullet_{\gamma_1}$$

$$+1\otimes\bullet_{\gamma_2}\otimes\bullet_{\gamma_3}\otimes\bullet_{\gamma_1}+1\otimes\bullet_{\gamma_3}\otimes\bullet_{\gamma_2}\otimes\bullet_{\gamma_1}$$

$$V\otimes\mathcal{H}^{\otimes 2}\qquad 1\otimes\bullet_{\gamma_1}\!\!\begin{smallmatrix}\diagdown & \diagup\\ \bullet_{\gamma_2} & \bullet_{\gamma_3}\end{smallmatrix}\otimes\mathbb{I}-R\phi(\bullet_{\gamma_2})\otimes\bullet_{\gamma_1}\!\!\begin{smallmatrix}|\\ \bullet_{\gamma_3}\end{smallmatrix}\otimes\mathbb{I}-R\phi(\bullet_{\gamma_3})\otimes\bullet_{\gamma_1}\!\!\begin{smallmatrix}|\\ \bullet_{\gamma_2}\end{smallmatrix}\otimes\mathbb{I}$$

$$-R\phi(\bullet_{\gamma_2}\bullet_{\gamma_3})\otimes\bullet_{\gamma_1}\otimes\mathbb{I}+R(R\phi(\bullet_{\gamma_2})\phi(\bullet_{\gamma_3}))\otimes\bullet_{\gamma_1}\otimes\mathbb{I}$$

$$+R(R\phi(\bullet_{\gamma_3})\phi(\bullet_{\gamma_2}))\otimes\bullet_{\gamma_1}\otimes\mathbb{I}-1\otimes\mathbb{I}\otimes\bullet_{\gamma_1}\!\!\begin{smallmatrix}\diagdown & \diagup\\ \bullet_{\gamma_2} & \bullet_{\gamma_3}\end{smallmatrix}+1\otimes\bullet_{\gamma_2}\otimes\bullet_{\gamma_1}\!\!\begin{smallmatrix}|\\ \bullet_{\gamma_3}\end{smallmatrix}$$

$$+1\otimes\bullet_{\gamma_3}\otimes\bullet_{\gamma_1}\!\!\begin{smallmatrix}|\\ \bullet_{\gamma_2}\end{smallmatrix}+1\otimes\bullet_{\gamma_2}\bullet_{\gamma_3}\otimes\bullet_{\gamma_1}-R\phi(\bullet_{\gamma_2})\otimes\bullet_{\gamma_3}\otimes\bullet_{\gamma_1}$$

$$\xrightarrow{-Rm_V(id\otimes\phi)\otimes id}\qquad -R(\bullet_{\gamma_3})\otimes\bullet_{\gamma_2}\otimes\bullet_{\gamma_1}$$

$$V\otimes\mathcal{H}\qquad\qquad\ldots$$

$$\xrightarrow{m_V(id\otimes\phi)}$$

FIGURE 2. Second part of the calculation of ϕ_R. Apply $-R\phi m\otimes id^{\otimes n}$ until arrival in $V\otimes\mathcal{H}$. Then apply $m_V(id\otimes\phi)$.

If there are no subdivergences, Epstein-Glaser renormalization amounts to a Taylor subtraction on test functions: Let ${}^0t$ be a distribution on some $\mathbb{R}^d-\{0\}$ with singularity at 0, for example ${}^0t=x^{-(d+1)}$. In order to extend ${}^0t$ onto all of $\mathbb{R}^d$, consider

$$t: f\mapsto {}^0t\left(f-\sum_{|\alpha|\leq\rho}w_\alpha\partial^\alpha f(0)\right)\tag{15}$$

where f is a test function and the w_α are auxiliary test functions with $\partial^\beta w_\alpha(0)=\delta_\alpha^\beta$. If ρ is large enough with respect to the degree of divergence of ${}^0t$ at 0, the modified distribution t is defined on all of $\mathbb{R}^d$. It is natural to consider the first summand in (15) as the unrenormalized contribution and the second summand as the counterterm.

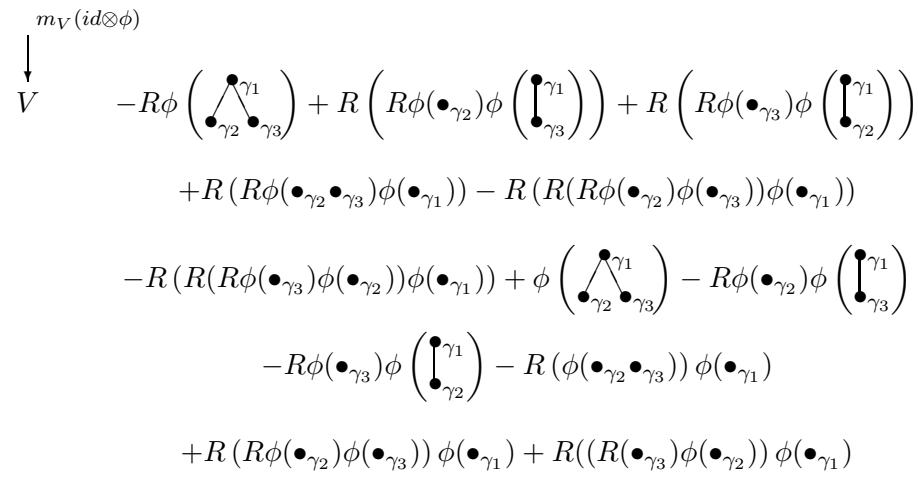
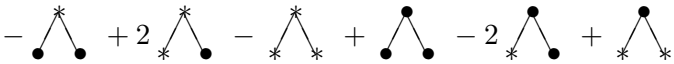


FIGURE 3. Third part of the calculation of ϕ_R. The reader should compare the result with (1). Using the fact that S_R^ϕ is an algebra homomorphism (if R is a Rota-Baxter map), the last step (C3) in Figure 1 and the first step (C3') in Figure 2 could have been avoided.

Epstein and Glaser describe how to take care of distributions which may have an overall divergence and subdivergences, i. e. distributions which are not only singular on the thin diagonal $\{x_1 = \ldots = x_n\}$ of some M^n but on the fat diagonal $\{x_i = x_j \text{ for some } i, j\}$. The algorithm is, with the above identification of unrenormalized part and counterterm, structurally very similar to the Bogoliubov recursion and can thus be describe by a twisted antipode [1].

Using some techniques of [1], notably the "cut product" $\odot$ of certain linear endomorphisms on $\mathcal{H}$ as a replacement for the convolution product, one can construct the map R as an algebra endomorphism on a Hopf algebra of trees with decorated vertices $\mathcal{H}(\{\bullet, *\})$ and consider the "twisted antipode" defined by $S_R = -R(S_R \odot_Q id)$, and the renormalization map $S_R \odot id$. Starting from the tree  , the map $S_R \odot id$ yields



which should be compared to the last line in Figure 3 – vertices of type $\bullet$ mark unrenormalized contributions, vertices of type $\star$ the corresponding counterterms. These trees are then mapped into an appropriate space of operator-valued distributions: the above example describes terms needed for the renormalization of the fourth order time-ordered product. Using this somewhat modified approach, where the combinatorics happen entirely in the Hopf algebra (as opposed to between the Hopf algebra and the target ring), checking locality simply amounts to calculating the commutator of $S_R \odot id$ and $B_{+\bullet}$:

$$(S_R \odot id)B_{+\bullet} = (id - R)B_{+\bullet}(S_R \odot id).$$

Thus once the subdivergences are taken care of, it suffices to subtract the superficial divergence.

2. Hopf subalgebras and Dyson-Schwinger equations

Hopf subalgebras of the Hopf algebras of (decorated) rooted trees or Feynman graphs are in close relation with Dyson-Schwinger equations. Indeed, any Dyson-Schwinger equation (to be defined below) gives rise to a Hopf subalgebra. This is a statement about self-similarity: a 1-cocycle like B_+ ensures that a product of trees is mapped to a tree, and this is a rather general phenomenon: the Green functions appear as functionals of themselves, the functionals being provided by the Dyson skeleton graphs which appear as the integral kernel $\phi(B_+^n(\mathbb{I}))$.

It will turn out in Theorem 2 that all Hopf subalgebras coming from a reasonably general class of Dyson-Schwinger equations are in fact isomorphic.

2.1. **Hopf subalgebras of decorated rooted trees.**

For simplicity, we start our considerations in the Hopf algebra $\mathcal{H}$ of undecorated rooted trees. A full classification of their Hopf subalgebras is far beyond reach. However, we give a few examples the last of which will be directly related to Dyson-Schwinger equations.

Bounded fertility, finite parts, primitive elements. For $n \in \mathbb{N}$ let $\mathcal{H}_n$ be the subalgebra of $\mathcal{H}$ generated by trees whose vertices have fertility bounded from above by n. A glance at the definition of the coproduct (3) suffices to see that $\mathcal{H}_n$ is a Hopf subalgebra of $\mathcal{H}$. In particular, the Hopf algebra $\mathcal{H}_1$ with one generator in each degree is known as the Hopf algebra of *ladders*. It is closely related to iterated integrals [8, 26].

Similarly, the free commutative algebra generated by trees of degree $\leq n$ forms a Hopf subalgebra for any n since the coproduct respects the grading. Another example where there is nothing to check are subalgebras generated by an arbitrary collection of primitive elements of $\mathcal{H}$.

The Connes-Moscovici Hopf subalgebra. A less trivial example of a Hopf subalgebra of $\mathcal{H}$ arose in the work of Connes and Moscovici on local index formulas for transversally hypoelliptic operators on foliations [12, 9, 13]. In the case of a foliation of codimension 1, the relevant Hopf algebra $\mathcal{H}_T$ is defined by the generators X, Y, δ_n for $n \in \mathbb{N}$, the relations

$$[X, Y] = -X, \quad [X, \delta_n] = \delta_{n+1}, \quad [Y, \delta_n] = n\delta_n, \quad [\delta_n, \delta_m] = 0,$$

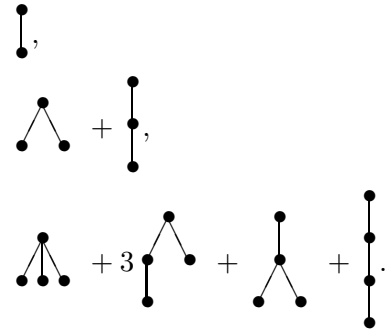
and the coproduct

$$\Delta(X) = X \otimes \mathbb{I} + \mathbb{I} \otimes X + \delta_1, \quad \Delta(Y) = Y \otimes \mathbb{I} + \mathbb{I} \otimes Y, \quad \Delta(\delta_1) = \delta_1 \otimes \mathbb{I} + \mathbb{I} \otimes \delta_1.$$

Note that the relations above and the requirement that Δ be an algebra homomorphism determine Δ on the generators δ_n for $n \geq 2$ as well. Let N be the linear operator, called *natural growth operator*, on $\mathcal{H}$, defined on a tree τ by adding a branch to each vertex of τ and summing up the resulting trees, extended as a

derivation onto all of $\mathcal{H}$. For example,

$$
\begin{aligned}
N(\mathbb{I}) &= \bullet, \\
N^2(\mathbb{I}) &= \\
N^3(\mathbb{I}) &= \\
N^4(\mathbb{I}) &=
\end{aligned}
\tag{16}
$$



Now identifying δ_1 with $\bullet$, and generally δ_n with $N^n(\mathbb{I})$, the commutative Hopf subalgebra of $\mathcal{H}_T$ generated by the δ_n can be embedded into $\mathcal{H}$ [9]. The resulting Hopf subalgebra is denoted $\mathcal{H}_{CM}$. For example,

$$
\begin{aligned}
\tilde{\Delta}(\delta_1) &= 0, \\
\tilde{\Delta}(\delta_2) &= \delta_1 \otimes \delta_1, \\
\tilde{\Delta}(\delta_3) &= 3\delta_1 \otimes \delta_2 + (\delta_2 + \delta_1^2) \otimes \delta_1.
\end{aligned}
$$

The δ_n can be specified in a non-recursive manner:

$$\delta_n = \sum_{\tau \in \mathcal{T}_n} c_\tau \tau$$

where the integers c_τ, called *Connes-Moscovici weights*, have been computed in [26, 18] using the tree factorial

$$c_\tau = \frac{n!}{\tau! \, \mathrm{Sym}(\tau)}$$

where $\mathrm{Sym}(\tau)$ is the symmetry factor (rank of the group of symmetries) of τ.

A quadratic Dyson-Schwinger equation. Now we turn to the study of another source of Hopf subalgebras, the combinatorial Dyson-Schwinger equations. As a first example, we consider the equation

$$X = \mathbb{I} + \alpha B_+(X^2) \tag{17}$$

in $\mathcal{H}[[\alpha]]$. Using the ansatz

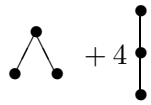
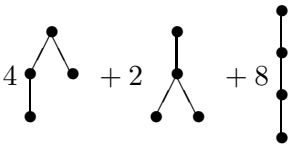
$$X = \sum_{n=0}^{\infty} \alpha^n c_n$$

one easily finds $c_0 = \mathbb{I}$ and

$$c_{n+1} = \sum_{k=0}^{n} B_+(c_k c_{n-k}) \tag{18}$$

which determine X by induction. The first couple of c_n are easily calculated:

$$c_0 = \mathbb{I},$$
$$c_1 = \bullet,$$
$$c_2 = 2$$

$$c_3 =$$

$$+4$$
$$c_4 = 4$$

$$+2$$
$$+8$$

We observe that c_n is a weighted sum of trees with vertex fertility bounded by 2 – this is due to the square of X in the Dyson-Schwinger equation (17). The reader should compare this to the Connes-Moscovici trees (16) discussed in the previous subsection. The recursive nature of (17) makes one suspect that the c_n generate a Hopf subalgebra of $\mathcal{H}$. Indeed, for each $n \geq 1$ and $k \leq 1$ there are polynomials P_k^n in the c_l for $l \leq n$ such that

$$\Delta c_n = \sum_{k=0}^{n} P_k^n \otimes c_k. \tag{19}$$

They are inductively determined by

$$P_{k+1}^{n+1} = \sum_{l=0}^{n-k} P_0^l P_k^{n-l} \tag{20}$$

and $P_0^{n+1} = c_{n+1}$. For a proof of this statement, see the more general Theorem 2 in the next subsection. For the moment, we merely display the first P_k^n in an upper triangular matrix where columns are indexed by $n = 0 \ldots 5$ and rows by $k = 0 \ldots n$.

$$\begin{bmatrix} \mathbb{I} & c_1 & c_2 & c_3 & c_4 & c_5 \\ & \mathbb{I} & 2c_1 & 2c_2 + c_1^2 & 2c_3 + 2c_1c_2 & 2c_4 + 2c_1c_3 + c_2^2 \\ & & \mathbb{I} & 3c_1 & 3c_2 + 3c_1^2 & 6c_1c_2 + c_1^3 + 3c_3 \\ & & & \mathbb{I} & 4c_1 & 6c_1^2 + 4c_2 \\ & & & & \mathbb{I} & 5c_1 \\ & & & & & \mathbb{I} \end{bmatrix}$$

The coefficients are basically polynomial coefficients as will become clear in the next subsection.

2.2. **Combinatorial Dyson-Schwinger equations.** Let A be any connected graded Hopf algebra which is free or free commutative as an algebra, and $(B_+^{d_n})_{n\in\mathbb{N}}$ a collection of Hochschild 1-cocycles on it (not necessarily pairwise distinct). The most general Dyson-Schwinger equation we wish to consider here is

$$X = \mathbb{I} + \sum_{n=1}^{\infty} \alpha^n w_n B_+^{d_n}(X^{n+1}) \tag{21}$$

in $A[[\alpha]]$. The parameter α plays the role of a coupling constant. The w_n are scalars in k. Again we decompose the solution

$$X = \sum_{n=0}^{\infty} \alpha^n c_n$$

with $c_n \in A$.

LEMMA 1. *The Dyson-Schwinger equation (21) has a unique solution described by* $c_0 = \mathbb{I}$ *and*

$$c_n = \sum_{m=1}^{n} w_m B_+^{d_m} \left(\sum_{k_1+\ldots+k_{m+1}=n-m,\, k_i \geq 0} c_{k_1} \ldots c_{k_{m+1}} \right). \tag{22}$$

Proof. Inserting the ansatz into (21) and sorting by powers of α yields the result. Uniqueness is obvious. □

THEOREM 2. *The* c_n *generate a Hopf subalgebra of* A :

$$\Delta(c_n) = \sum_{k=0}^{n} P_k^n \otimes c_k$$

where the P_k^n *are homogeneous polynomials of degree* $n-k$ *in the* c_l, $l \leq n$:

$$P_k^n = \sum_{l_1+\ldots+l_{k+1}=n-k} c_{l_1} \ldots c_{l_{k+1}}. \tag{23}$$

In particular, the P_k^n *are independent of the* w_n *and* $B_+^{d_n}$.

We emphasize that the main ingredient for the proof of this theorem is the fact that the $B_+^{d_n}$ are Hochschild 1-cocycles, the rest being a cumbersome but straightforward calculation.

Proof. We proceed by proving inductively the following statements:

- (α_n) The theorem holds up to order n.
- (β_n) For a given $m \in \{1 \ldots n\}$ let $l_1 + \ldots + l_{m+1} =: p \in \{0 \ldots n-m\}$, $l_i \geq 0$. Then the right hand sum

$$P(n-m, m, p) := \sum_{k_1+\ldots+k_{m+1}=n-m,\, k_i \geq l_i} P_{l_1}^{k_1} \ldots P_{l_{m+1}}^{k_{m+1}} \tag{24}$$

  does not depend on the single l_i but only on p, $n-m$ and m, justifying the notation $P(n-m, m, p)$.
- (γ_n) In the above notation and for any $q \in \{1 \ldots n\}$, the term $P(n-m, m, q-m)$ does not depend on $m \in \{1 \ldots q\}$.

To start the induction, we note that (α_0) is obvious. (β_1) is trivial as $m = 1$ enforces $l_1 = l_2 = 0$. Similarly, for (γ_1) only one m is in range and the statement thus trivially satisfied. We proceed to (α_n). By definition, and using (6) for the

$B_+^{d_n}$,

$$\Delta(c_n) = \sum_{m=1}^{n} w_m((id \otimes B_+^{d_m})\Delta + B_+^{d_m} \otimes \mathbb{I}) \left(\sum_{k_1+\ldots+k_{m+1}=n-m,\, k_i \geq 0} c_{k_1} \ldots c_{k_{m+1}} \right)$$

(using the induction hypothesis (α_{n-1}))

$$= c_n \otimes \mathbb{I} + \sum_{m=1}^{n} w_m (id \otimes B_+^{d_m}) \sum_{k_1+\ldots+k_{m+1}=n-m,\, k_i \geq 0} \sum_{l_1 \ldots l_{m+1}=0}^{k_1 \ldots k_{m+1}} P_{l_1}^{k_1} \ldots P_{l_{m+1}}^{k_{m+1}} \otimes c_{l_1} \ldots c_{l_{m+1}} =$$

(by rearranging indices)

$$= c_n \otimes \mathbb{I} + \sum_{m=1}^{n} w_m \sum_{p=0}^{n-m} \sum_{l_1+\ldots+l_{m+1}=p} \sum_{k_1+\ldots+k_{m+1}=n-m,\, k_i \geq l_i} P_{l_1}^{k_1} \ldots P_{l_{m+1}}^{k_{m+1}} \otimes B_+^{d_m}(c_{l_1} \ldots c_{l_{m+1}}) =$$

(by the induction hypothesis (β_n) and using the notation of (24))

$$= c_n \otimes \mathbb{I} + \sum_{m=1}^{n} w_m \sum_{p=0}^{n-m} P(n-m, m, p) \otimes \sum_{l_1+\ldots+l_{m+1}=p} B_+^{d_m}(c_{l_1} \ldots c_{l_{m+1}}) =$$

(rearranging indices (q replaces $m+p$) and using (γ_n))

$$= c_n \otimes \mathbb{I} + \sum_{q=1}^{n} \sum_{m=1}^{q} w_m P(n-m, m, q-m) \otimes \otimes \sum_{l_1+\ldots+l_{m+1}=q-m} B_+^{d_m}(c_{l_1} \ldots c_{l_{m+1}}) =$$

$$= c_n \otimes \mathbb{I} + \sum_{q=1}^{n} P(n-q, q, 0) \otimes \sum_{m=1}^{q} w_q \sum_{l_1+\ldots+l_{m+1}=q-m} B_+^{d_m}(c_{l_1} \ldots c_{l_{m+1}}).$$

Since the right hand tensor factor is c_q, a glance at (24), using that $P_0^k = c_k$, verifies (α_n).

The items (β_n) and (γ_n) follow from (α_{n-1}) :

$$\begin{aligned} P(n-m,m,p) &= \sum_{k_1+\ldots+k_{m+1}=n-m,\, k_i\geq l_i} P^{k_1}_{l_1}\ldots P^{k_{m+1}}_{l_{m+1}} = \\ &= \sum_{k_1+\ldots+k_{m+1}=n-m,\, k_i\geq l_i} \sum_{r^1_1+\ldots+r^1_{l_1+1}=k_1-l_1} \ldots \\ &\quad \ldots \sum_{r^{m+1}_1+\ldots+r^{m+1}_{l_{m+1}+1}=k_{m+1}-l_{m+1}} c_{r^1_1}\ldots c_{r^{m+1}_{l_{m+1}+1}} = \\ &= \sum_{r_1+\ldots+r_{m+p+1}=n-m-p} c_{r_1}\ldots c_{r_{m+p+1}}, \end{aligned}$$

which is independent of any l_i whence (β_n). Substituting $p=q-m$ shows (γ_n). $\square$

At first sight the fact that the coproduct on the c_i does not depend on the w_k and hence that all Dyson-Schwinger equations of this kind yield isomorphic Hopf subalgebras (provided there are no relations among the c_n) might well come as a surprise. The deeper reason for this is the recursiveness of (21) as will become more apparent in the next paragraphs.

Description in terms of trees. Now we specialize to the case $A=\mathcal{H}(S)$ where $(S=\dot\cup S_n, |\cdot|)$ is an arbitrary graded set of decorations such that $|d_n|=n$ for all n (one can even allow $d_n\subset S_n$ and define $B^{d_n}_+ := \sum_{\delta\in d_n} B^\delta_+$). Using the following lemma, which gives an explicit presentation of the c_i in terms of trees, Theorem 2 can be proven in a more comprehensive way.

LEMMA 3. *The solution of (21) can be described by* $c_0=\mathbb{I}$ *and*

$$c_n = \sum_{\tau\in\mathcal{T}(S),\,|\tau|=n} \frac{\tau}{\mathrm{Sym}(\tau)} \prod_{v\in\tau^{[0]}} \gamma_v \tag{25}$$

where

$$\gamma_v = \begin{cases} w_{|dec(v)|}\frac{(|dec(v)|+1)!}{(|dec(v)|+1-\mathrm{fert}(v))!} & \textit{if } \mathrm{fert}(v)\leq |dec(v)|+1 \\ 0 & \textit{else}. \end{cases}$$

Here $\mathcal{T}(S)$ *denotes the set of* S*-decorated trees,* $\tau^{[0]}$ *the set of vertices of* τ, $dec(v)$ *the decoration (in* S*) of* v, $|\tau|$ *the decoration weight of* τ, *i. e.* $|\tau| = \sum_{v\in\tau^{[0]}}|dec(v)|$, *and* $\mathrm{fert}(v)$ *the fertility (number of outgoing edges) of the vertex* v.

Note that only trees contribute where at each vertex the fertility does not exceed the degree of its decoration plus 1.

Proof. This is an easy induction using the following argument: Let τ be a given tree in c_n and let its root o be decorated by something in degree m. According to (22), $\tau = B^{d_m}_+(\mathbb{I}^{k_0}\tau_1\ldots\tau_{m+1-k_0})$ where the τ_i are trees different from $\mathbb{I}$. The fertility of the root is thus $m+1-k_0$. We assume $\tau_1\ldots\tau_{m+1-k_0} = \sigma_1^{k_1}\ldots\sigma_p^{k_p}$ where the σ_i are pairwise different trees. In (22), there are $C := \frac{(m+1)!}{k_0!\ldots k_p!}$ choices to

make which yield the tree τ. Since the γ_v are simply multiplied for all vertices v of a tree, it remains to see that for the only new vertex o in τ, we have

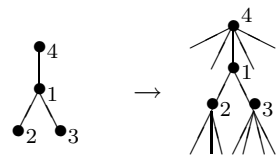
$$\gamma_o = \frac{(m+1)!}{k_0!} = C\frac{\text{Sym}(\tau)}{\text{Sym}(\tau_1)\dots\text{Sym}(\tau_{m+1-k_0})}.$$

This however follows immediately from the definition of Sym. □

As a matter of fact, the coefficients

$$\prod_v \gamma_v \tag{26}$$

can be interpreted as follows: Consider each tree as an "operadic" object with $|deg(v)| + 1 - \text{fert}(v)$ inputs at each vertex v. For example,



Clearly, the total number of inputs is $n+1$ for any tree of weight n. Now the coefficient (26) is nothing but the number of planar embeddings of this operadic tree (where the trunk, i. e. the original tree is kept fixed). In other words, (26) counts the number of ways that the input edges can sway around the original tree. Using this idea, we obtain the following

Operadic proof of Theorem 2. As a variation of (21) let us consider the operadic fixpoint equation

$$G(\alpha) = \mathbb{I} + \sum_n \alpha^n \mu_{n+1}(G(\alpha)^{\otimes(n+1)}).$$

Here, μ_j is a map $\in O^{[j]} : V^{\otimes j} \to V$ for some space V and $G(\alpha)$ is a formal series in α with coefficients in the $O^{[j]}$. We regard $\mathbb{I} : V \to V$ as the identity map. We write $G(\alpha) = \mathbb{I} + \sum_k \alpha^k \nu_k$. It follows easily by induction that $\nu_k \in O^{[k+1]}$. Clearly, $G(\alpha)$ is a sum (with unit weights) over all maps which we obtain by composition of some undecomposable maps μ_n.

The coproduct of decorated rooted trees acts on the ν_k in an obvious manner. A given monomial $\nu_{i_1}^{r_1} \cdots \nu_{i_l}^{r_l}$ (which lives in the PROP $V^{\otimes(r_1 i_1 + \dots + r_l i_r + r)} \to V^{\otimes r}$, where $r = \sum r_i$) can be composed with any element in $O^{[r-1]}$ in

$$\frac{(k+1)!}{r_1! \dots r_l!} \tag{27}$$

ways. Hence, as the ν_i sum over all maps with unit weight, this is the contribution to the term in the coproduct which has ν_k on the right hand side and the given monomial on the left hand side. Going back to the initial Dyson-Schwinger equation (21), we see that the same argument (27) also determines the coproduct on the c_k there, in agreement with (23):

$$P_k^n = \sum_{\substack{i_1 r_1 + \dots + i_l r_l = n-k \\ 0 \le i_s < i_{s+1},\, \sum r_i = k+1}} \frac{(k+1)!}{r_1! \dots r_l!} c_{i_1}^{r_1} \dots c_{i_l}^{r_l} \tag{28}$$

Indeed, the trees in c_k were weighted by a product over vertices (26), and the coproduct respects the planar structure. □

The coefficients in (26) and (28) arise thus in a completely natural way due to the transition from a noncommutative (planar) to a commutative (non-planar) setting.

Final remarks. Before we ultimately turn to the more analytical side of Dyson-Schwinger equations, let us mention that by Theorem 2, the Connes-Moscovici Hopf subalgebra presented in the preceding subsection is not generated by a Dyson-Schwinger equation of the form (21) if we restrict ourselves to one-cocycles into the linear space of generators, as they typically appear in local quantum field theory. Note that the Hopf algebras which appear as solutions of (21) are studied under the name Faa di Bruno algebras in [17], to which the Connes-Moscovcci algebra can be related through an isomorphism..

Of course one can consider much more general equations such as

$$X = \alpha B_+ \left(\frac{\mathbb{I}}{\mathbb{I} - X} \right)$$

and would not expect the resulting Hopf subalgebras (if any) to be isomorphic to the ones encountered here at all.

Finally let us emphasize that the ladder Hopf algebra $\mathcal{H}_1$ introduced in the last subsection, can be generated by the *linear* Dyson-Schwinger equation

$$X = \mathbb{I} + \alpha B_+(X)$$

The Hopf algebra $\mathcal{H}_1$ plays a special role at the fixpoint of the renormalization group flow [34], see also the next subsection.

As opposed to the above example, we call Dyson-Schwinger equations of the form (21) (where some $w_n \neq 0$ for $n \geq 2$) *nonlinear*. They necessarily generate trees with sidebranchings.

2.3. **Applications in physics and number theory.** In physics, Dyson-Schwinger equations, usually derived by formal means using functional integrals, describe the loop expansion of Green functions in a recursive way. An alternative to derive these equations is given by the very existence of a Hopf algebra underlying perturbation theory. These Hopf algebras provide Hochschild 1-cocycles, and we can obtain the Dyson–Schwinger equations for them in a straightforward manner.

In the following, we first exhibit Dyson-Schwinger equations in three different contexts: as a source for transcendental numbers, as a manner to define a generating function for the polylogarithm, and as the equations of motion for a renormalizable quantum field theory. The presentation is by no means self-contained, and we refer the reader to the growing literature for more detail [28, 32, 29, 30, 31, 23].

A simple toy model. Let us consider the equation we had before (17),

$$X_2 = \mathbb{I} + \alpha B_+(X_2^2), \tag{29}$$

and let us exhibit the difference between such an equation and the associated linear system

$$X_1 = \mathbb{I} + \alpha B_+(X_1). \tag{30}$$

We will study toy Feynman rules on these Hopf algebras, regarded as characters on the Hopf algebra. We explore that $\phi B_+(\mathbb{I})$ defines an integral kernel k, such that

$$\phi B_+(\mathbb{I})[z] = \int_0^\infty k(x,z)dx,$$

where the kernel is homogeneous: $k(ux,uz) = k(x,z)/u$. We regard the integral as the Fourier transform of the kernel with respect to the multiplicative group $\mathbb{R}_+$. To define our first set of renormalized Feynman rules, we simply set

$$\phi B_+(h)[z] = \int_0^\infty (k(x,z) - k(x,1))\phi(h)[x]dx.$$

Note that we have $\phi(h_1h_2) = \phi(h_1)\phi(h_2)$ which implies $\phi(\mathbb{I})[z] = 1$.

Let us define the transform $K(\gamma)$ of the kernel $k(x,1)$ to be

$$K(\gamma) = \int_0^\infty k(x,1)x^{-\gamma}dx.$$

This determines

$$\int_0^\infty k(x,z)x^{-\gamma}dx = z^{-\gamma}K(\gamma).$$

Let us now look at (30). Applying ϕ to both sides delivers an integral equation for the Green function

$$\phi(X_1)[z;\alpha] = 1 + \alpha\int_0^\infty \phi(X_1)[x;\alpha](k(x,z) - k(x,1))dx.$$

Note that our choice of renormalized Feynman rules corresponds to the choice of a boundary condition for the Dyson-Schwinger equation, $\phi(X_1)[1;\alpha] = 1$. Omitting the subtraction of the kernel at $z = 1$ defines the unrenormalized Feynman rule ϕ_u, which reconstructs the renormalized one, $\phi = S_R^{\phi_u} \star \phi_u$, where R is the evaluation map at $z = 1$.

Equation (30) can be solved by an Ansatz $\phi(X_1)[z] = z^{-\gamma(\alpha)}$, which leads to

$$z^{-\gamma(\alpha)} = 1 + \alpha(z^{-\gamma(\alpha)} - 1)K(\gamma(\alpha)),$$

i. e. the series $\gamma(\alpha)$ is the solution of the equation $1 = \alpha K(\gamma(\alpha))$. The non-linear case (29) can not be solved by such an *Ansatz*. Maintaining the same boundary condition we get the equation

$$\phi(X_2)[z;\alpha] = 1 + \alpha\int_0^\infty (\phi(X_2)[x;\alpha])^2(k(x,z) - k(x,1))dx.$$

At this moment, it is instructive to introduce a bit of quantum field theory wisdom. We observe that we can write this integral equation in the form

$$\phi(X_2)[z;\alpha] = 1 + \int_0^\infty \phi(X_2)[x;\alpha]\sqsupset(x;\alpha)(k(x,z) - k(x,1))dx,$$

where the running coupling $\beth(x;\alpha) = \alpha\phi(X_2)[x;\alpha]$ has been introduced. We see that we just modify the linear Dyson-Schwinger equation by this running coupling, which forces us to look for solutions not of the form

$$G(\alpha, z) = e^{-\gamma(\alpha)},$$

but instead in the more general form

$$G(\alpha, z) = e^{-\sum_{j=1}^{\infty} \gamma_j(\alpha) \ln^j z}, \tag{31}$$

where the γ_j themselves are recursively defined through γ_1 thanks to the renormalization group.

Indeed, assume now that the running coupling is constant, $\partial_{\ln z}\beth = 0$. This turns the non-linear Dyson-Schwinger equation into the linear one (30). That is a general phenomenon: the linear Dyson-Schwinger equation appears in the limit of a vanishing β-function, and signifies a possible fixpoint of the renormalization group.

This suggests a natural expansion in terms of the coefficients of the β-function, which will be presented elsewhere.

All this has a combinatorial counterpart:

$$\frac{\partial \ln X_2(\alpha)}{\partial \alpha} = S \star Y(X_2(\alpha_R)),$$

where Y is again the grading operator. Let us work this out in an example. We consider the solution $X_2(\alpha)$ of (29). Setting $X_2 = \mathbb{I} + \sum_{k=1}^{\infty} \alpha^k c_k$, we find to $O(\alpha^3)$,

$$\begin{array}{ll}
\tilde{\Delta}(c_1) = 0 & S * Y(c_1) = c_1 \\
\tilde{\Delta}(c_2) = 2c_1 \otimes c_1 & S * Y(c_2) = 2c_2 - 2c_1^2 \\
\tilde{\Delta}(c_3) = 3c_1 \otimes c_2 + (2c_2 + c_1^2) \otimes c_1 & S * Y(c_3) = 3c_3 - 8c_1c_2 + 5c_1^3.
\end{array}$$

Furthermore,

$$\alpha\partial_\alpha \ln X_2(\alpha) = \alpha c_1 + 2\alpha^2(c_2 - \frac{1}{2}c_1^2) + 3\alpha^3 \left(c_3 - c_1c_2 + \frac{1}{3}c_1^3 \right).$$

Setting

$$\alpha = \frac{\alpha_R}{X(\alpha)},$$

and recursively replacing α by α_R,

$$\begin{aligned}
\alpha(\alpha_R) &= \frac{\alpha_R}{1 + \alpha(\alpha_R)c_1 + \alpha^2(\alpha_R)c_2 + \cdots} \\
&= \frac{\alpha_R}{1 + \frac{\alpha_R c_1}{1+\alpha_R c_1 + \cdots} + \alpha_R^2 c_2 + \cdots} \\
&= \frac{\alpha_R}{1 + \alpha_R c_1 + \alpha_R^2(c_2 - c_1^2) + \cdots},
\end{aligned}$$

confirms the result to that order. The general proof is a straightforward application of the results in [11].

Furthermore, the reader can check that

$$F_{c_m}(c_k) = (k - m + 1)c_{k-m}$$

for all $k \geq m$, which is at the heart of a recursive determination of the above coefficients $\gamma_j(\alpha)$ in (31). Here, F_{c_m} is the befooting operator

$$F_{c_m}(c_k) = \langle Z_{c_m} \otimes \mathrm{id}, \Delta(c_k) \rangle$$

of [6]. As a final remark, we mention that it is not the non-linearity which provides the major challenge in solving a non-linear Dyson-Schwinger equation, but the fact that the one-variable Fourier calculus presented above has to be replaced by a multi-variable calculus which leads to transcendental extensions [31] which is a fascinating topic in its own right.

Indeed, consider once more the linear equation (30), now with Feynman rules defined by a two-variable kernel $k(x,y,z) = 1/(x+y+z)^2$ with $k(xz,yz,z) = k(x,y,1)/z^2$

$$\phi(B_+(h))(z) = \int dxdy \frac{(\phi(h)(x))^{q_2}(\phi(h)(y))^{q_1}}{(x+y+z)^2} - |_{z=1},$$

and q_1, q_2 are two positive rational numbers which add to one. Comparing this system with the degenerate system where one of the q_i vanishes (and the other thus is unity) shows that the perturbative expansion in α provides coefficients which are transcendental extensions of the ones obtained in the degenerate case. This rather general phenomenon leads deeply into the transcendental structure of Green functions, currently under investigation.

Dyson-Schwinger equation for the polylog. Quantum field theory is concerned with the determination of correlators which we can regard as generating functions for a perturbative expansion of amplitudes. These correlators are solutions of our Dyson-Schwinger equations, the latter being typical fixpoint equations: the correlator equals a functional of the correlator. Such self-similarities appear in many branches of mathematics. Here, we want to exhibit one such appearance which we find particularly fascinating: the generating function for the polylog [30].

Following [30] consider the following $N \times N$ matrix once more borrowed from Spencer Bloch's function theory of the polylogarithm [2]:

$$\begin{array}{c} \alpha^0 \\ \alpha^1 \\ \alpha^2 \\ \alpha^3 \\ \cdots \end{array} \underbrace{\left(\begin{array}{rcrcrcrcr} +1 & | & 0 & | & 0 & | & 0 & | & \cdots \\ -\mathrm{Li}_1(z) & | & 2\pi i & | & 0 & | & 0 & | & \cdots \\ -\mathrm{Li}_2(z) & | & 2\pi i \ln z & | & [2\pi i]^2 & | & 0 & | & \cdots \\ -\mathrm{Li}_3(z) & | & 2\pi i \frac{\ln^2 z}{2!} & | & [2\pi i]^2 \ln z & | & [2\pi i]^3 & | & \cdots \\ \cdots & | & \cdots & | & \cdots & | & \cdots & | & \cdots \end{array} \right)}_{\begin{array}{ccccc} u^0 & u^1 & u^2 & u^3 & \ldots \end{array}},$$

given up to $N = 4$. We assign an order in a small parameter α to each row, counting rows $0, 1, \ldots$ from top to bottom, similarly we count columns $0, 1, \ldots$ from left to right by a parameter u, and assign an order u^i to the i-th column. The polylog is defined by

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

inside the unit circle and analytically continued with a branch cut along the real axis from one to plus infinity.

As the n-th polylog appears as the integral over the $(n-1)$-th polylog, we expect to be able to find a straightforward integral equation for its generating function which resembles a Dyson-Schwinger equation. Consider

$$\begin{aligned} F(\alpha,u;z) &= 1-\frac{1}{1-z}+\frac{2\pi iu\alpha}{1-2\pi iu\alpha}+\alpha\left[\int_0^z\frac{F(\alpha,0;x)}{x}dx\right. \\ &\quad \left.+\int_1^z\frac{F(\alpha,u;x)-F(\alpha,0;x)}{x}dx\right], \end{aligned}$$

where we call $F(\alpha,u;z)$ a renormalized Green function, α the coupling (a small parameter, $0<\alpha<1$) and consider the perturbative expansion

$$F(\alpha,u;z)=1-\frac{1}{1-z}+\sum_{k=1}^{\infty}\alpha^k f_k(u;z).$$

We distinguished the lowest order term $f_0(z)=z/(z-1)$ (which corresponds to the term without quantum corrections in QFT) at order α^0 which here equals $-\mathrm{Li}_0(z)$. The limit $u\to 1$ can be taken in the above Dyson-Schwinger equation. We note that upon introducing a counterterm $Z(\alpha,u;\ln\rho)$, the above equation is the renormalized solution at $\rho\to 0$ of the equation

$$F_\rho(\alpha,u;z)=Z(\alpha,u;\ln\rho)-\frac{1}{1-z}+\frac{2\pi iu\alpha}{1-2\pi iu\alpha}+\alpha\int_0^z\frac{F_\rho(\alpha,u;x)}{x}dx.$$

We immediately confirm that, for $k>0$, the term of order $\alpha^k u^i$ in the renormalized solution of this Dyson-Schwinger equation is the entry (k,i) in the above matrix: the above matrix provides in its non-trivial entries the solution of the Dyson-Schwinger equation so constructed.

We now work with the cocommutative Hopf algebra $\mathcal{H}_1$ determined by the Dyson-Schwinger equation $X=\mathbb{I}+\alpha B_+(X)$, so $X=\sum_{k=0}^{\infty}\alpha^k t_k$ where the t_k are k-fold application of B_+ to $\mathbb{I}$, and let $Li\equiv Li(z)$ and $L\equiv L(z)$ be characters on the Hopf algebra defined by

$$-\phi(t_n)(z,0)\equiv Li(t_n)(z)=\mathrm{Li}_n(z),\ L(t_n)(z)=\frac{\ln^n(z)}{n!}.$$

We can regard the character Li as a Feynman rule and the transition $Li\to L$ as a renormalization map which leaves the behavior at infinity unchanged.
We know [2] that the elimination of all ambiguities due to a choice of branch lies in the construction of functions $a_p(z)=(2\pi i)^{-p}\,\widetilde{a}_p(z)$ where

$$\widetilde{a}_p(z):=\mathrm{Li}_p(z)-\cdots+(-1)^j\mathrm{Li}_{p-j}(z)\frac{\ln^j(z)}{j!}+\cdots+(-1)^{p-1}\mathrm{Li}_1(z)\frac{\ln^{p-1}(z)}{(p-1)!}.$$

This is now a very familiar equation:

PROPOSITION 4. *For $z\in\mathbb{C}$,*

$$\tilde{a}_p(z)=m(L^{-1}\otimes Li)(\mathrm{id}\otimes P)\Delta(t_p),$$

where $L^{-1}=LS$, with S the antipode in $\mathcal{H}_1$, and P the projection onto the augmentation ideal.

Proof: elementary combinatorics confirming that

$$LS(t_n/n!)(z) = (-\ln(z))^n/n!.$$

□

There is a strong analogy here to the Bogoliubov R operation in renormalization theory [28, 30], thanks to the fact that Li and L have matching asymptotic behavior for $\mid z \mid \rightarrow \infty$. Indeed, if we let R be defined to map the character Li to the character L, $R(Li) = L$, and P the projector onto the augmentation ideal of $\mathcal{H}_1$, then

$$LS = S_R^{Li} = -Rm(S_R^{Li} \otimes Li)(\mathrm{id} \otimes P)\Delta \equiv -R\left(\overline{Li}\right),$$

for example

$$S_R^{Li}(t_2) = -R(Li(t_2) + S_R^{Li}(t_1)Li(t_1)) = -L(t_2) + L(t_1)L(t_1) = \frac{\ln^2(z)}{2!},$$

where $\overline{Li}(t_2) = Li(t_2) - L(t_1)Li(t_1)$. Thus, a_p is the result of the Bogoliubov map

$$\overline{Li} = m(S_R^{Li} \otimes Li)(\mathrm{id} \otimes P)\Delta$$

acting on t_n. We have two completely equivalent mechanism for the removal of ambiguities at this moment:

$$L^{-1} \star Li \text{ vs } S_R^{\phi} \star \phi.$$

This points towards an analogy between the structure of the polylog and QFT Green functions which very much suggests to explore QFT from the viewpoint of mixed Hodge structures in the future.

Dyson-Schwinger equations for full QFT. The quantum equations of motion, the Dyson-Schwinger equations of a full fledged quantum field theory, can be obtained in precisely the same manner as discussed above. They typically are of the form

$$\Gamma^{\underline{r}} = \mathbb{I} + \sum_{\substack{\gamma \in H_L^{[1]} \\ \mathrm{res}(\gamma) = \underline{r}}} \frac{\alpha^{|\gamma|}}{\mathrm{Sym}(\gamma)} B_+^{\gamma}(X_{\mathcal{R}}^{\gamma}) = \mathbb{I} + \sum_{\substack{\Gamma \in H_L \\ \mathrm{res}(\Gamma) = \underline{r}}} \frac{\alpha^{|\Gamma|}\Gamma}{\mathrm{Sym}(\Gamma)}, \tag{32}$$

where the first sum is over a countable set of Hopf algebra primitives γ, $\mathrm{res}(\gamma) = \underline{r}$,

$$\Delta(\gamma) = \gamma \otimes \mathbb{I} + \mathbb{I} \otimes \gamma,$$

indexing the Hochschild 1-cocycles B_+^{γ} above, while the second sum is over all one-particle irreducible graphs contributing to the desired Green function, all weighted by their symmetry factors. In more traditional terms, the primitive graphs γ correspond to skeletons into which vertex and propagator corrections are to be inserted.

Here, $\Gamma^{\underline{r}}$ is to be regarded as a formal series

$$\Gamma^{\underline{r}} = \mathbb{I} + \sum_{k \geq 1} c_k^{\underline{r}} \alpha^k, \; c_k^{\underline{r}} \in H.$$

These coefficients of the perturbative expansion deliver Hopf subalgebras in their own right, cf. Theorem 2. Indeed, the maps

$$B_+^{\underline{r},n} = \sum_{\substack{\gamma \in H_L^{[1]} \\ \mathrm{res}(\gamma) = \underline{r},\, |\gamma| = n}} B_+^{\gamma},$$

where the sum is over all primitive 1PI n-loop graphs γ with external leg structure $\underline{r}$, are 1-cocycles. They are implicitly defined by the second equality in (32), the remarkable feature is the fact that these maps can be shown to be Hochschild closed and hence ensure locality. A detailed account of this fact, which illuminates in particular the structure of gauge theories, is upcoming [23].

In (32), $X^{\gamma}_{\mathcal{R}}$ is of the form

$$X^{\gamma}_{\mathcal{R}} = \Gamma^{\mathrm{res}(\gamma)} \left(X_{\mathrm{coupl}}\right)^{|\gamma|},$$

where X_{coupl} is the vertex function divided by the square roots of the inverse propagator functions. Under the Feynman rules X_{coupl} hence maps to the invariant charge.

As an example, consider QED. We have a set of residues (external leg structures)

$$\mathrm{Res} = \Big\{ \ , \ , \ \Big\}.$$

We finish our paper by exhibiting the action of the Hochschild 1-cocycle B_{+} on the order α expansion of

$$X \ = \frac{\left(\Gamma \ \right)^{3}}{\left(\Gamma \ \right)^{2}\left(\Gamma \ \right)} = \Gamma \ \left(X_{\mathrm{coupl}}\right)^{2},$$

with

$$X_{\mathrm{coupl}} = \Gamma \ \left(\Gamma \ \sqrt{\Gamma \ }\right)^{-1}.$$

To order α, one finds

$$X \ = 1 + \alpha \left(3 \ + 2 \ + \ .\right)$$

Hence, the non-primitive two-loop vertex graphs of QED are obtained as

$$B_{+} \ \left(3 \ + 2 \ + \ \right).$$

Hochschild closedness demands that this equals

$$+ \ + \ + \ + \ + \ ,$$

as then

$$\tilde{\Delta}(\ldots) = \left(3 \ + 2 \ + \ \right) \otimes \ .$$

In this manner one determines the Hochschild 1-cocycles for a renormalizable quantum field theory. This works particularly nice for gauge theories, as will be exhibited in [23].

2.4. **Final remarks.** There is a very powerful structure behind the above decomposition into Hopf algebra primitives – the fact that the sum over all Green functions $G^{\underline{n}}$ is indeed the sum over all 1PI graphs, and this sum, the effective action, can be written nicely as $\prod \frac{1}{1-\gamma}$, a product over "prime" graphs – graphs which are primitive elements of the Hopf algebra and which index the Hochschild 1-cocycles, delivering a complete factorization of the action. A single such Euler factor with its corresponding Dyson-Schwinger equation and Feynman rules was evaluated in [6], a calculation which was entirely in accordance with our study: an understanding of the weight of contributions $\sim \ln(z)$ from a knowledge of the weight of such contributions of smaller degree in α, dubbed propagator-coupling duality in [6]. Altogether, this allows to summarize the structure in QFT as a vast generalization of results summarized here. It turns out that even the quantum structure of gauge theories can be understood along these lines [27]. A full discussion is upcoming [23].

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FREIE UNIVERSITÄT BERLIN, INSTITUT FÜR MATHEMATIK II, ARNIMALLEE 2-6, D-14195 BERLIN AND
INSTITUT DES HAUTES ÉTUDES SCIENTIFIQUES, 35 ROUTE DE CHARTRES, F-91440 BURES-SUR-YVETTE
E-mail address: bergbau@math.fu-berlin.de

CNRS AT INSTITUT DES HAUTES ÉTUDES SCIENTIFIQUES, 35 ROUTE DE CHARTRES, F-91440 BURES-SUR-YVETTE AND CENTER FOR MATHEMATICAL PHYSICS, BOSTON UNIVERSITY
E-mail address: kreimer@ihes.fr