Collapsed 5-manifolds with pinched sectional curvature

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COLLAPSED 5-MANIFOLDS WITH PINCHED POSITIVE SECTIONAL CURVATURE

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Abstract. Let M be a closed 5-manifold of pinched curvature $0 < \delta \le \sec_M \le 1$. We prove that M is homeomorphic to a spherical space form if M satisfies one of the following conditions: (i) $\delta = 1/4$ and the fundamental group is a non-cyclic group of order $\ge C$, a constant. (ii) The center of the fundamental group has index $\ge w(\delta)$, a constant depending on δ. (iii) The ratio of the volume and the maximal injectivity radius is $< \epsilon(\delta)$. (iv) The volume is less than $\epsilon(\delta)$ and the fundamental group $\pi_1(M)$ has a center of index at least w, a universal constant, and $\pi_1(M)$ is either isomorphic to a spherical 5-space group or has an odd order.

0. Introduction

The sphere theorem asserts that if a manifold M admits a quarter pinched metric, $\frac{1}{4} < \sec_M \le 1$, then the universal covering space of M is homeomorphic to a sphere. It is natural to ask whether M is homeomorphic to a spherical space form, S^n/Γ, $\Gamma \subset O(n+1)$ ([GKR], [IHR]). We will call Γ a *spherical n-space group.* Clearly, a positive answer implies that the fundamental group $\pi_1(M)$ is isomorphic to a spherical space group and the $\pi_1(M)$-action on the universal covering is conjugate to a linear action. A subtlety is that neither of these holds without a positive curvature condition: in every odd dimension $n \ge 5$ ([Ha]),

(0.1) There are infinitely many non-spherical space groups acting freely on an n-sphere.

(0.2) There are infinitely many distinct free actions on an n-sphere by a spherical space group which do not conjugate to any linear action.

Obviously, the quotient manifolds in (0.1) and (0.2) are not homeomorphic to any spherical space form.

In this paper, as a first step we investigate the case of dimension 5. We obtain several rigidity results (Theorems A-D) on a δ-pinched 5-manifold M whose fundamental group is not small (equivalently, whose volume is small). In particular, we rule out (0.1) and (0.2) in our circumstances via studying certain symmetry structure on M discovered by Cheeger-Fukaya-Gromov ([CFG], [Ro1]).

We now begin to state the main results in this paper.

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Theorem A.

Let M be a closed 5-manifold with $\frac{1}{4} < sec_M \leq 1$. If the fundamental group $\pi_1(M)$ is a non-cyclic group of order $\geq C$ (a constant), then M is homeomorphic to a spherical space form.

With an arbitrary δ-pinching, we obtain the following weak generalization of Theorem A (which is slightly stronger than the case of $\delta = 1/4$ in Theorem B).

Theorem B.

Let M be a closed 5-manifold with $0 < \delta \leq sec_M \leq 1$. If the center of $\pi_1(M)$ has index $\geq w(\delta)$ (a constant depending on δ), then M is homeomorphic to a spherical space form.

Observe that M in Theorem B has diameter $\leq \pi/\sqrt{\delta}$ (Bonnet theorem) and small volume (volume comparison). According to [CFG] (cf. [Ro1]), M admits local compatible isometric T^k-actions with $k \geq 1$ of some nearby metric (details will be given shortly). In our circumstance, we show that $k \geq 2$, and thus the universal covering of M is diffeomorphic to a sphere ([Ro2]). The main work is to show, using the local symmetry structure, that $\pi_1(M)$ is isomorphic to a spherical space group and the $\pi_1(M)$-action is conjugate to a linear one (see Theorem E, compare to (0.1) and (0.2)).

Any δ-pinched 5-manifold satisfies $\text{vol}(M)/\max \text{injrad}(M, x) \leq d(\delta)$ (by the Cheeger's lemma). We find if the ratio is actually small, then $k \geq 2$ as in the above.

Theorem C.

For $0 < \delta \leq 1$, there exists a small number, $\epsilon(\delta) > 0$, such that if a closed 5-manifold M satisfies

$$0 < \delta \leq sec_M \leq 1, \qquad \frac{vol(M)}{\max injrad\,(M, x)} < \epsilon(\delta),$$

then M is homeomorphic to a spherical space form.

Note that Theorems B and C do not hold if one relaxes the condition "$\delta > 0$" to "$\delta \geq 0$" (see Example 2.7). We intend to discuss the classification with "$\delta \geq 0$" elsewhere.

Consider a collapsed δ-pinched 5-manifold M close in the Gromov-Hausdorff distance to a metric space X of dimension 4 (equivalently, $k = 1$). A new trouble is to determine the topology of the universal covering space, or equivalently the topology of X. This looks quite difficult; it seems to require a classification of positively curved 4-manifolds in the case when X is smooth (and thus the fundamental group of M is cyclic).

Theorem D.

For $0 < \delta \leq 1$, there exists $\epsilon(\delta) > 0$ such that if a closed 5-manifold M satisfies

$$0 < \delta \leq sec \leq 1, \qquad vol(M) < \epsilon(\delta),$$

then M is homeomorphic to a spherical space form, provided $\pi_1(M)$ has a center of index at least $w > 0$, a constant (independent of δ), and $\pi_1(M)$ is a spherical 5-space group or $|\pi_1(M)|$ is odd.

A spherical 5-space group is either cyclic or has a normal cyclic subgroup of index three ([Wo]). In either case, there are infinitely many spherical 5-space forms satisfying the conditions of Theorems A-D (see Example 2.6).

A natural question is when M in Theorems A-C is diffeomorphic to a spherical 5-space form? We mention the following: a spherical 5-space form S^5/Γ admits exactly one or two different smooth structures depending on $|\Gamma|$ odd or even ([KS], [Wa]). Moreover, both smooth structures may allow a non-negatively curved metric (e.g., there are exactly four smooth manifolds homotopy equivalent to $\mathbb{R}P^5$, all of them admit metrics of non-negative sectional curvature ([GZ]), and two of them are not homeomorphic to each other).

A question of Yau ([Yau]) asks if in a given homotopy type contains at most finitely many different diffeomorphism types that can support a metric of positive sectional curvature. A positive answer is known only in dimensions 2 and 3 ([Ha]). When restricting to the class of pinched metrics, positive answers are known in even dimensions and the class of manifolds (odd-dimensions) with finite second homotopy groups ([FR1], [PT]).

Since there are at most finitely many spherical space forms (up to diffeomorphism) with a given homotopy type, Theorem C immediately implies:

Corollary 0.3.

Let M be a close δ-pinched 5-manifold. Then the homotopy type of M contains at most $c(\delta)$ many diffeomorphism types that support a δ-pinched metric, provided $\pi_1(M)$ has a center with index $\geq w(\delta)$.

As mentioned earlier, our approach to Theorems A-D is based on the fibration theorem of Cheeger-Fukaya-Gromov on collapsed manifolds with bounded sectional curvature and diameter ([CFG], [CG1,2]). In our circumstances, the fibration theorem asserts that there is a constant $v(n,\delta)>0$ such that if a δ-pinched n-manifold M has a volume less than $v(n,\delta)$, then M admits a pure F-structure all whose orbits are of positive dimensions. By the Ricci flows technique, one can show that there is invariant metric which is at least $\delta/2$-pinched ([Ro1]).

In the case of a finite fundamental group, the notion of a pure F-structure is equivalent to that of a π_1*-invariant torus* T^k*-action* on a manifold M, which is defined by an effective T^k-action on the universal covering space $\tilde M$ of M such that it extends to a $T^k \rtimes_\rho \pi_1(M)$-action, where $\rho : \pi_1(M) \to \text{Aut}(T^k)$ is a homomorphism from the fundamental group to the automorphism group of T^k. Clearly, the T^k-action on $\tilde M$ is the lifting of a T^k-action on M if and only if ρ is trivial or equivalently, the T^k-action and the $\pi_1(M)$-action commute. Hence, the notion of a π_1-invariant torus action generalizes that of a global torus action and the T^k-orbit structure on $\tilde M$ projects onto M so that each orbit is a flat submanifold.

Consider M as in Theorems A-D. In view of the above, we may assume that M admits a π_1-invariant isometric T^k-action ($k\geq 1$).

Theorem E.

Let M be a closed 5-manifold of positive sectional curvature. If M admits a π_1-invariant isometric T^k-action with $k>1$, then M is homeomorphic to a spherical space form.

Theorem E is known in the following special cases: M (itself) admits an isometric T^3-action ([GS]) or M is simply connected ([Ro2]). In particular, the universal covering space of M is diffeomorphic to a sphere.

We show that under the assumptions of Theorems B and C, $k>1$, and thus Theorem E implies Theorems B and C. In the case $k=1$, the T^1-action on the universal covering $\tilde M$ of M is free if $\tilde M$ is a sphere and if $\pi_1(M)$ is not cyclic. We then complete the proof of Theorem A by proving the following topological result: Let a finite group Γ act freely on S^5. If S^5 admits a free Γ-invariant T^1-action such that the induce Γ-action on S^5/T^1 is pseudo-free, then S^5/Γ is homeomorphic to a spherical space form (see Proposition 1.3).

In view of the above, Theorem D follows from

Theorem F.

Let M be a closed 5-manifold of positive sectional curvature which admits a π_1-invariant fixed point free isometric T^1-action. Then the universal covering $\tilde M$ is diffeomorphic to S^5, provided $\pi_1(M)$ has a center of index $\geq w$, and $\pi_1(M)$ has odd order or is a spherical 5-space group.

It is worth to point it out that *every* spherical 5-space form admits a π_1-invariant isometric T^3-action and a free isometric T^1-action and many admit $\pi_1(M)$-invariant isometric T^2-actions ([Wo], p.225).

We would like to put Theorems D and E in a little perspective. In the study of positive sectional curvature, due to the obvious ambiguity the class of positively curved manifolds with (large) symmetry has frequently been a focus of the investigations. According to K. Grove, this also serves as a strategy of searching for new examples and obstructions. There has been significant progress in the last decade on classification of simply connected manifolds with large symmetry rank (the rank of the isometry group), cf. [GS], [FR1,2], [HK], [Ro1-3], [Wi1,2]. However, not much is known for non-simply connected manifolds with large symmetry rank. These results may be treated as an attempt in this direction.

We now give an outline of the proof of Theorems E and F.

By the compact transformation group theory ([Bre]), the topology of a T^k-space M is closely related to that of the singular set (the union of all non-singular orbits) and the orbit space M/T^k. In the presence of an invariant metric of positive sectional curvature, the singular set and the orbit space are very restricted, and this is the ultimate reason for a possible classification. In our proofs, we will thoroughly investigate of the structure of the singularity and the orbit space.

The proof of Theorem E divides into two situations: $k=3$ (Theorem 3.1) and $k=2$, and the main work is in the case of $k=2$. When $k=2$, we first prove Theorem E at the level of fundamental groups (Theorem 4.1). Then we divide the proof into two cases: the π_1-invariant T^2-action is pseudo-free (section 5) and not pseudo-free (section 6). In the former case, we study the T^2-action on the universal covering space, $\tilde M\simeq S^5$, which has a singular set, $\mathcal S$, consisting of three isolated circle orbits. It suffices to show that the $(\pi_1(M),T^2)$-bundle, $\tilde M-\mathcal S\to(\tilde M-\mathcal S)/T^2$, is conjugate to a standard linear model from spherical space forms. This can be done following [FR2] if one can assume that the orbit space, $X=\tilde M/T^2$, is a homeomorphic sphere. The problem is that we only know that X is a homotopy 3-sphere. We overcome this difficulty by combining the above with tools from the s-cobordism theory in dimension 5. In the non-pseudo-free case, the singular set has dimension 3, and by analyzing the singular structure, we are able to view M as a gluing of standard pieces.

In the proof of Theorem F, by studying the induced $\pi_1(M)$-action on $\tilde M/T^1$ (which is not trivial because $\pi_1(M)$ is not cyclic), we bound from above the Euler characteristic of $\tilde M/T^1$ via the technique of q-extent (Proposition 7.5, cf. [Gr], [Ya]). With this constraint, we show that the condition on the fundamental groups implies that $\tilde M$ is a homology sphere, the T^1-action is free and the $\pi_1(M)$-action on $\tilde M/T^1$ is pseudo-free (note that the standard free linear T^1-action on S^5 is preserved by any spherical 5-space group, [Wo]). By employing results in [HL] and [Wi1,2] on pseudo-free actions by finite groups on a homeomorphic complex projective plane, we show that the $\pi_1(M)$-action is homeomorphically conjugate to a linear action.

Remark 0.5. By the improved sphere theorem due to Abresch-Meyer [AM], the $\frac{1}{4}$-pinching in Theorem A may be replaced by a slightly weaker pinching constant $\frac{1}{4}-\varepsilon$.

Remark 0.6. Theorems C and E, and the Corollaries 0.1 and 0.2 hold in a much less restrictions on $\pi_1(M)$ (see Section 3).

Remark 0.7. Our approach seems not able to apply to higher dimensions to obtain analogies of Theorems A, B and C. One difficulty is the analog of Theorem 3.2 in higher dimensions seems not true. An analog of Theorem A for cyclic group is plausible if one could establish a generalized version of Theorem 3.2 for 4-dimensional orbifold, and for non-pseudofree actions. However, there are examples of non-linear pseudofree circle actions on S^5 (cf. [FS]).

The rest of the paper is organized as follows: In Section 1 we give a proof of Theorem A by assuming Theorem E. In Section 2, we prove Theorem B by assuming Theorem E and construct examples mentioned following Theorem B. In Section 3, we prove Theorem C by assuming Theorems D, E. In Section 4 we provide the main tools that will be used in the proof of Theorems D and E. In Section 5, we prove the case of Theorem E for $k=3$. In Section 6, we prove Theorem E at the level of fundamental groups. In Sections 7 and 8, we complete the proof of Theorem E for $k=2$. In Section 9, we prove Theorem F.

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1. Proof of Theorem A by Assuming Theorem E

Consider M as in Theorem A; whose universal covering space $\tilde M$ is homeomorphic to S^5 (the sphere theorem). Because the volume of M is small, $\tilde M$ admits a π_1-invariant isometric T^k-action (Theorems 1.1 and 1.2). We will use this structure to show that M is homeomorphic to a spherical space form (compare to (0.1) and (0.2)). By Theorem E, we may assume that $k=1$. Because $\pi_1(M)$ is non-cyclic, we show that the isometric T^1-action must be free and commute with the $\pi_1(M)$-action such that the induced $\pi_1(M)$-action on $\tilde M/T^1$ is pseudo-free (Lemma 1.9). The key is to show, based on a result (Lemma 1.4) in [HL] and [Wil1,2], that these properties imply that $\pi_1(M)$ is isomorphic to a spherical 5-space group (Lemma 1.6) and the $\pi_1(M)$-action is conjugate to a linear action (Proposition 1.3).

a. π_1-invariant isometric T^k-actions on collapsed manifolds.

According to [CFG], if a closed n-manifold M with bounded curvature and diameter has a small volume, then M admits a pure nilpotent Killing structure whose orbits are infra-nilmanifolds. When the fundamental group of M is finite, the infra-nilmanifolds are actually flat and thus the nilpotent Killing structure is equivalent to a π_1-invariant almost isometric T^k-action ([Ro1]). The π_1-invariance implies that the T^k-orbits on the universal covering descend to M, also denoted by $T^k(x), x \in M$. A T^k-orbit on M is called regular, if it has a tubular neighborhood in which T^k-orbits form a fiber bundle. Let $\mathcal{S}$ denote the set of all non-regular orbits. Then $M - \mathcal{S}$ is an open dense subset. For a small number $\eta > 0$, let $U_{-\eta} = \{x \in U : d(x, \mathcal{S}) > \eta\}$.

Theorem 1.1 ([CFG]).
Given $n, d > 0$, there exist constants, $\epsilon(n,d), c(n) > 0$, such that if a closed n-manifold M of finite fundamental group satisfying

$$|sec_M| \leq 1, \qquad diam\ (M) \leq d, \qquad vol\ (M) < \epsilon(n,d),$$

then M admits a π_1-invariant T^k-action satisfying
(1.1.1) Every T^k-orbit has a positive dimension.
(1.1.2) Any orbit in $U_{-\eta}$ has a second fundamental form, $|II| \leq c(n)\eta^{-1}$.
(1.1.3) For any $\epsilon > 0$, there is a T^k-invariant metric of bounded sectional curvature by one which is ϵ-close to the original metric in C^1-norm.

Note that one can always assume a small constant $\eta = \eta(n) > 0$ such that $U_{-\eta} \neq \emptyset$ (if $U_{-\eta} = \emptyset$, then diam$(M) < \eta$ and thus M is almost flat ([Gr1]). Then $\mathcal{S} = \emptyset$ and therefore $U_{-\eta} = M$, a contradiction). Using the Ricci flows in [Ha], one can obtain a T^k-invariant metric with additional properties.

Theorem 1.2 ([Ro1]).
Let (M, g) satisfy the conditions of Theorem 1.1. For $\epsilon > 0$, there is a T^k-invariant metric g_ϵ such that

$$|g - g_\epsilon|_{C^1} < \epsilon, \qquad \min sec_g - \epsilon \leq sec_{g_\epsilon} \leq \max sec_g + \epsilon.$$

b. A criterion of a pseudo-free linear action on 5-spheres.

Consider a finite group Γ acting freely on S^5. As mentioned in (0.1) and (0.2), Γ may not be isomorphic to any spherical 5-space group, nor, even assuming Γ isomorphic to a spherical 5-space group, the Γ-action on S^5 may not be conjugate to any linear action (see (0.1) and (0.2)). Obviously, additional conditions are required for the Γ-action to conjugate a linear action.

We will give a criterion, Proposition 1.3, and use it to prove Theorem A. We point it out that this criterion will be also used in the proofs of Theorems D and E.

Spherical space forms have been completely classified, see [Wo]. From p225. [Wo], we observe that if a finite group $\Gamma \subset SO(6)$ acts freely on S^5, then Γ commutes with a standard free linear T^1-action on S^5. If, in addition, Γ is not cyclic, then the induced Γ-action on S^5/T^1 is pseudo-free i.e., any non-trivial element has only isolated fixed point.

The above properties are indeed sufficient conditions for a free Γ-action on S^5 to conjugate to a linear action.

Proposition 1.3.

Let a finite group Γ act freely on S^5. If Γ commutes with a free T^1-action on S^5 such that the induced Γ-action on S^5/T^1 is pseudo-free, then the Γ-action is topologically conjugate to a linear action.

A G-action is called locally linear, if each singular point has an invariant neighborhood which is equivariantly homeomorphic to a neighborhood of 0 in a real representation space. In particular, smooth actions are locally linear.

The following result (cf. [HL], [Wi1-2]) plays a crucial role in the proof of Proposition 1.3.

Theorem 1.4 ([Wi1]).

Any pseudo-free locally linear action by a finite group on a 4-manifold homeomorphic to $\mathbb{C}P^2$ is topologically conjugate to the linear action of a subgroup of $PSU(3)$ on $\mathbb{C}P^2$.

Note that $PSU(3) = SU(3)/\mathbb{Z}_3$, where $\mathbb{Z}_3$ is the center of $SU(3)$. It is perhaps useful to recall some details of which finite subgroups of $PSU(3)$ may act linearly and pseudo-freely on $\mathbb{C}P^2$. By [Wi2] (also [HL]) the group is either a cyclic group $\mathbb{Z}_n = \langle x \rangle$, or noncyclic with a presentation

$$(1.5) \qquad \{x, y : yxy^{-1} = x^r, x^n = y^3 = 1, \text{ where } r^2 + r + 1 = 0 \mod (n)\}$$

The linear action of the group on $\mathbb{C}P^2$ is given by

$$x[z_0, z_1, z_2] = [\omega z_0, \omega^{-r} z_1, z_2]; \; y[z_0, z_1, z_2] = [z_1, z_2, z_0]$$

where $\omega = e^{\frac{2\pi i}{n}}$ is the n-th root of the unit.

Observe that the group in (1.5) is $\mathbb{Z}_3 \oplus \mathbb{Z}_3 = \langle x, y \rangle$ if $n = 3$, and n can not be an integral multiple of 9. Therefore, it can never be a 5-dimensional spherical space form group if $n > 1$ (cf. [Wo] page 225). However, Petrie [Pe] constructed a free action of (1.5) on S^5 if $n = 7$ and $r = 2$.

Lemma 1.6.

Let Γ be as in Proposition 1.3. Let $\Gamma_0 \subset \Gamma$ be the principal isotropy group of the Γ-action on S^5/T^1. Then $\Gamma_0 \subset C(\Gamma)$, the center of Γ. Moreover, Γ is isomorphic to a subgroup of $SU(3)$, acting pseudo-freely on S^5.

Proof. For any nontrivial $\gamma \in \Gamma/\Gamma_0$, since γ acts on $S^5/T^1 \approx \mathbb{C}P^2$ with isolated fixed points, the subgroup of Γ generated by γ and Γ_0 acts freely on the circle orbit over a fixed point of γ, and so it is cyclic. This implies the first assertion.

Obviously, Γ_0 is isomorphic to a cyclic group $\mathbb{Z}_\ell$. By the above discussion, Γ/Γ_0 is either cyclic, or a noncyclic group (1.5). The desired result follows in the former case because Γ itself must be cyclic.

Let Γ/Γ_0 be the group (1.5). From the presentation Γ/Γ_0 has trivial center. Moreover, n must be coprime to 3 because otherwise it contains $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ as a subgroup, and so Γ contains a non-cyclic abelian subgroup, this is absurd since Γ acts freely on S^5. Similarly, one concludes that Γ must be a center extension of $\mathbb{Z}_\ell$ by Γ/Γ_0 such that its restrictions on $\langle x\rangle$, $\langle y\rangle$ are cyclic groups of order ℓn and 3ℓ respectively. Therefore Γ contains $\mathbb{Z}_{\ell n}$ as a cyclic subgroup of index 3, and the only 3-Sylow group is cyclic. Therefore, writing $|\Gamma| = ks = 3\ell n$ with $(k,3) = 1$ (e.g., we may take $k = n(\ell, n)$ and $s = 3\ell n/(\ell, n)$), by the Burnside Theorem (cf. [Wo] Theorem 5.4.1, p.163), Γ is generated by two elements A and B with relations

$$A^k = B^s = 1,\ BAB^{-1} = A^r$$

where $((r-1)s, k) = 1$ and $r^s \equiv 1(\text{mod } k)$. Since $\{A, B^3\}$ generates a cyclic index 3 subgroup, $r^3 \equiv 1(\text{mod } k)$. Therefore, Γ may be realized as the subgroup of $SU(3)$ generated by the matrices

$$\begin{pmatrix} R(1/k) & 0 & 0 \\ 0 & R(r/k) & 0 \\ 0 & 0 & R(r^2/k) \end{pmatrix}, \qquad \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ R(3/s) & 0 & 0 \end{pmatrix},$$

where $R(\theta)$ denote the standard 2×2 rotation matrix with rotation angle $2\pi\theta$, and I the 2×2 identity matrix. □

Proof of Proposition 1.3.

By Freedman [Fr], S^5/T^1 is homeomorphic to $\mathbb{C}P^2$. Consider the induced Γ-action on S^5/T^1, let Γ_0 be the principal isotropy group, and let $\bar\Gamma = \Gamma/\Gamma_0$. Clearly, $\Gamma_0 \cong \mathbb{Z}_\ell$ is a subgroup of T^1. By Theorem 2.4, the $\bar\Gamma$-action is conjugate to a linear $\bar\Gamma$-action on $\mathbb{C}P^2$ (and thus $\bar\Gamma$ is identified with a subgroup of $PSU(3)$, denoted by $\bar\Gamma_\ell$) by an equivariant homeomorphism $f : (S^5/T^1, \bar\Gamma) \to (\mathbb{C}P^2, \bar\Gamma_\ell)$.

By Lemma 1.6, $\Gamma \cong \Gamma_\ell \subset SU(3)$. And Γ_ℓ acts linearly on S^5 lifting the $\bar\Gamma = \bar\Gamma_\ell$-action on $\mathbb{C}P^2$ (but we should note that the Γ_ℓ-action may not be free a priorly). For the sake of convenience, let us identify Γ_ℓ with Γ. It remains to prove that, the free Γ-action on S^5 is conjugate to the linear Γ_ℓ-action.

Consider the (Γ, T^1)- (resp. (Γ_ℓ, T^1)) principal bundle $S^5 \to S^5/T^1$.

By Theorem (??) it suffices to prove that the induced principal T^1-bundle

$$T^1 \to E\Gamma \times_\Gamma S^5 \to E\Gamma \times_\Gamma \mathbb{C}P^2 \tag{1.7}$$

is equivalent to the corresponding principal T^1-bundle of (Γ_ℓ, T^1)-bundle on S^5. It is easy to see that the fundamental group of $E\Gamma\times_\Gamma\mathbb{C}P^2$ is Γ, and $\pi_2(E\Gamma\times_\Gamma\mathbb{C}P^2) \cong \mathbb{Z}$. Therefore, $H^2(E\Gamma \times_\Gamma \mathbb{C}P^2; \mathbb{Z}) \cong \mathbb{Z} \oplus H_1(\Gamma)$, where the free part may be regarded as $\text{Hom}(\pi_2(E\Gamma \times_\Gamma \mathbb{C}P^2); \mathbb{Z})$.

Let e_Γ denote the Euler class of the principal T^1-bundle. By the homotopy exact sequence one sees that e_Γ is a primitive element of $H^2(E\Gamma \times_\Gamma \mathbb{C}P^2; \mathbb{Z})$, i.e. modulo the torsion group $H_1(\Gamma)$ it generates the group. Moreover, with the notions in the proof of Lemma 3.4, $H_1(\Gamma) \cong \mathbb{Z}_s$.

Let $\mathbb{Z}_s$ be the subgroup of Γ generated by B, which acts on $\mathbb{C}P^2$ with three isolated fixed points. Let $[p] \in \mathbb{C}P^2$ be such a fixed point with isotropy group $\mathbb{Z}_s$. Consider the orbit $\Gamma[p] \subset \mathbb{C}P^2$. The restriction of the fiber bundle (1.7) on $E\Gamma \times_\Gamma \Gamma[p] = E\mathbb{Z}_s \times_{\mathbb{Z}_s} [p]$ is equivalent to the principal bundle

$$T^1 \to E\mathbb{Z}_s \times T^1 \to E\mathbb{Z}_s \times_{\mathbb{Z}_s} [p] \simeq B\mathbb{Z}_s \tag{1.8}$$

whose total space is homotopy equivalent to $T^1/\mathbb{Z}_s = T^1$. Therefore, the Euler class e_Γ, restricts to the Euler class of (1.7), which is clearly a generator of $H^2(E\mathbb{Z}_s \times_{\mathbb{Z}_s} [p]) = \mathbb{Z}_s$. On the other hand, the generator of $\mathrm{Hom}(\pi_2(E\Gamma \times_\Gamma \mathbb{C}P^2), \mathbb{Z})$, considered as a subgroup of $H^2(E\Gamma \times_\Gamma \mathbb{C}P^2)$, restricts to zero in $H^2(E\mathbb{Z}_s \times_{\mathbb{Z}_s} [p])$, since $\pi_2(E\mathbb{Z}_s \times_{\mathbb{Z}_s} [p]) = 0$. Therefore, we may write e_Γ in the following form

$$(1, \alpha) \in \mathrm{Hom}(\pi_2(E\Gamma \times_\Gamma \mathbb{C}P^2), \mathbb{Z}) \oplus H_1(\Gamma) \cong \mathbb{Z} \oplus H_1(\Gamma)$$

where $\alpha \in H_1(\Gamma) \cong \mathbb{Z}_s$ is a generator.

Fix a generator $1 \in H_1(\Gamma) \cong \mathbb{Z}_s$. By [Wo] Theorem 5.5.6 (cf. page 168, where $d = 3$ for our case) there always exists an automorphism $\psi = \psi_{1,t,u} : \Gamma \to \Gamma$ such that the induced automorphism $[\psi] \in \mathrm{Aut}(H_1(\Gamma))$ satisfies that $[\psi](\alpha) = \pm 1$ (depending $\mod (3)$ type of t). Therefore, by composing the Γ-action with an automorphism ψ of Γ, we may assume that $e_\Gamma = (1, \pm 1)$.

The same goes through for the linear Γ_ℓ-action on S^5. And the Euler class $e_{\Gamma_\ell} = (1, \pm 1) \in H^2(E\Gamma_\ell \times \mathbb{C}P^2, \mathbb{Z})$. It is easy to see that, for the complex conjugated linear action of Γ_ℓ on S^5, the Euler class is $(1, -1)$ (resp. $(1, 1)$), if $e_{\Gamma_\ell} = (1, 1)$ (resp. resp. $(1, -1)$). Therefore the T^1-principal bundle (1.7) is equivalent to the T^1-principal bundle associated to some linear Γ_ℓ-action. In particular, Γ acts freely and linearly on S^5, and so Γ is isomorphic to a 5-dimensional spherical space form group. The desired result follows. □

c. Proof of Theorem A by assuming Theorem E.

We need the following lemma to apply the criterion in Proposition 1.3.

Lemma 1.9.
Let M be a closed 5-manifold of positive curvature whose universal covering space is a sphere. Assume that M admits a π_1-invariant isometric T^1-action. If $\pi_1(M)$ is not cyclic, then the T^1-action is free and commutes with the $\pi_1(M)$-action such that the induced $\pi_1(M)$-action on $\tilde M/T^1$ is pseudo-free.

Proof. Consider the holonomy representation, $\rho : \pi_1(M) \to \mathrm{Aut}(S^1) = \{\pm 1\}$, and let $\Gamma = \ker(\rho)$, a normal subgroup of index at most 2 which commutes with the T^1-action. It is easy to see that if the T^1-action is not free, then $\Gamma =< \gamma >$ is cyclic and the induced γ-action on $\tilde M/T^1$ is pseudo-free. Otherwise, $\tilde M$ has a T^1-invariant totally geodesic 3-submanifold N. Because $\pi_1(M)$ preserves N, $\pi_1(M)$ is cyclic, a contradiction.

Consider the γ-action on $\tilde M/T^1 = \mathbb{C}P^2$. Then γ has three isolated fixed point. Assume that $\beta \notin \Gamma$.

By [RW] the T^1-action on $\tilde M$ is free. If the induced $\pi_1(M)$-action on $\tilde M/T^1$ is not pseudo-free, then there is $\gamma \in \pi_1(M)$ such that the induced $\bar\gamma$-action on $\tilde M/T^1$ has a fixed point component $\bar F_0$ of dimension 2. For any $\eta \in \pi_1(M)$, we then have that $\bar\eta(\bar F_0) \cap \bar F_0 \neq \emptyset$ i.e. there is $\bar x \in \bar F_0$ such that $\bar\gamma(\bar\eta(\bar x)) = \bar\eta(\bar x)$. This implies that $\bar\eta^{-1}\bar\gamma\bar\eta(\bar x) = \bar x$ and thus $\eta^{-1}\gamma\eta$ and γ generalize a cyclic subgroup of $\pi_1(M)$. Consequently, $\eta^{-1}\gamma\eta$ is in the normalizer of $< \gamma >$, and thus the normalizer is a normal subgroup of $\pi_1(M)$. It is easy to see that the normalizer is cyclic, and thus $< \gamma >$ is a normal subgroup of $\pi_1(M)$, and therefore $\pi_1(M)$ is cyclic because it coincides with the normalizer, a contradiction. □

Proof of Theorem A by assuming Theorem E.

Consider M as in Theorem A, whose Riemannian universal covering space $\tilde{M}$ is homeomorphic to S^5 (the sphere theorem). Let S^5_δ denote a sphere of constant curvature δ. By the volume comparison,

$$\mathrm{vol}(M) = \frac{\mathrm{vol}(\tilde{M})}{|\pi_1(M)|} \le \frac{\mathrm{vol}(S^5_{1/4})}{C} < \epsilon$$

is small, and thus without loss of generality we may assume $\tilde{M}$ admits a $\pi_1(M)$-invariant isometric T^k-action (Theorems 1.1 and 1.2). By Theorem E, we may further assume that $k = 1$. By Lemma 1.9, we can apply Proposition 1.3 to conclude the desired result. □

2. Proof of Theorems B and C by Assuming Theorem E

d. Proof of Theorem B by assuming Theorem E.

Consider M be as in Theorem B. As in the proof of Theorem A, the volume of M is small and thus $\tilde{M}$ admits a $\pi_1(M)$-invariant isometric T^k-action. By Theorem E, it suffices to show that the condition on the fundamental groups implies that $k > 1$.

Consider a π_1-invariant T^1-action on $\tilde{M}$. The kernel of the holonomy representation, $\rho : \pi_1(M) \to \mathrm{Aut}(T^1) \cong \mathbb{Z}_2$, is a normal subgroup of index at most two. Then the T^1-action on $\tilde{M}$ descends to a T^1-action on $\tilde{M}/\ker(\rho)$, which is either M or a double covering of M. In particular, there is a local T^1-action on M. We will call isotropy groups of the local T^1-action isotropy groups of the π_1-invariant T^1-action.

Lemma 2.1.

Let M_i be a sequence of closed n-manifolds of $|sec_{M_i}| \le 1$ which converges in the Gromov-Hausdorff distance to a compact metric space X of dimension $(n-1)$. Then there is a uniform upper bound on the order of isotropy group of the π_1-invariant T^1-action on M_i.

Proof. We argue by contradiction, assuming that $x_i \in M_i$ such that the isotropy group $T^1_{x_i} \cong \mathbb{Z}_{h_i}$ with $h_i \to \infty$ (see (1.1.1)). Passing to a subsequence if necessary, we may assume that $x_i \to x \in X$. Note that an open neighborhood of x is homeomorphic to a cone over the limit of $S^\perp_{x_i}/T^1_{x_i}$, where $S^\perp_{x_i}$ is the unit sphere in the normal space to $T^1(x_i)$, and $T^1_{x_i}$ is the acts on $S^\perp_{x_i}$ via the isotropy representation. Because $h_i \to \infty$, the limit of $S^\perp_{x_i}/T^1_{x_i}$ has dimension $\le n-3$, and thus the cone has dimension $\le n-2$, a contradiction to $\dim(X) = n-1$. □

Consider an exceptional T^1-orbit $T^1(x)$ in M, with isotropy group $\mathbb{Z}_h$. Then a lower bound for h is related to the fundamental group in the following way. Let γ denote the homotopy class of $T^1(x)$ with order r, and let σ be the homotopy class of a principal T^1-orbit with order s. Then $h \ge r/s$.

In the proof of Theorem B, we will use the following result on fundamental groups of positively curved manifolds ([Ro3]). A cyclic subgroup of $\pi_1(M)$ is called maximal, if it is not properly contained in any cyclic subgroup of $\pi_1(M)$.

Theorem 2.2.

Let M be a closed n-manifold of positive sectional curvature. If M admits a π_1-invariant isometric T^k-action, then any maximal normal subgroup of $\pi_1(M)$ has index $\le w(n)$.

Proof of Theorem B by assuming Theorem E.

By the volume comparison, $\text{vol}(M) = \text{vol}(\tilde{M})/|\pi_1(M)| \le \text{vol}(S^5_\delta)/w(\delta)$. We may assume that $w(\delta)$ is large so that $\text{vol}(M) < \epsilon$. By Theorems 1.1 and 1.2, without loss of generality we may assume that M admits a π_1-invariant isometric T^k-action. By Theorem E, it suffices to show that $k > 1$.

We argue by contradiction: assuming a sequence, M_i, satisfying the above with $w_i \to \infty$, and $k = 1$. By the Gromov's compactness, we may assume that $M_i \xrightarrow{d_{GH}} X$. Because $k = 1$, it follows that $\dim(X) = 4$, and thus any isotropy group of the π_1-invariant T^1-action on M_i has order $\le c$ (Lemma 2.1).

To get a contradiction, we will find an isotropy group of order $> c$. Take any maximal normal cyclic subgroup $H_i =< \gamma_i >\subset \pi_1(M_i)$. By Theorem 2.2, we obtain

$$\begin{aligned} w_i &\le [\pi_1(M_i) : \text{cent}(\pi_1(M_i))] \\ &\le [\pi_1(M_i) : H_i \cap \text{cent}(\pi_1(M_i))] \\ &= [\pi_1(M_i) : H_i] \cdot [H_i : H_i \cap \text{cent}(\pi_1(M_i))] \\ &\le w(5) \cdot [H_i : H_i \cap \text{cent}(\pi_1(M_i))]. \end{aligned}$$

(note that the above implies that H_i is not trivial) Clearly, for i large we may assume that $[H_i : H_i \cap \text{cent}(\pi_1(M_i))] > c$. Let σ_i denote the homotopy class of a principal T^1-orbit on M_i. Then σ_i is in the center of $\pi_1(M_i)$. Assume that γ_i preserves some T^1-orbit $T^1(\tilde{x})$ ([Ro3]). Because σ preserves all T^1-orbits, γ_i and σ_i generate a normal cyclic subgroup, and the maximality of H_i implies that $\sigma_i \in H_i$. Note that γ_i is a multiple of the homotopy class of the projection of $T^1(\tilde{x})$ in M. Then the isotropy group of the projection has order at least $[H_i :< \sigma_i >] \ge [H_i : H_i \cap \text{cent}(\pi_1(M_i))] > c$, a contradiction. □

e. Proof of Theorem C by assuming Theorem E.

Similar to the proof of Theorem B, Theorem C is a consequence of the following proposition and Theorem E.

Proposition 2.3.

Let M be a closed n-manifold of finite fundamental group satisfying

$$|sec_M| \le 1, \qquad diam\ (M) \le d, \qquad \frac{vol\ (M)}{\sup injrad\ (M, x)} < \epsilon_1(n, d).$$

Then M admits a π_1-invariant isometric T^k-action with $k > 1$.

For a motivation of Proposition 2.3, consider the metric product of a unit sphere and a flat ϵ-torus, $M_\epsilon = S^n \times \epsilon^2 T^k$. Then vol $(M_\epsilon)/\sup \text{injrad}\ (M_\epsilon) \to 0$ (resp. is proportional to vol (S^n)) if $k > 1$ (resp. $k = 1$), and the π_1-invariant structure in Theorem 1.1 is the multiplication on the T^k-factor.

Proof of Proposition 2.3.

We may assume that $\epsilon_1(n,d)$ is small so that vol $(M) < \epsilon(n,d)$, and thus M admits a π_1-invariant almost isometric T^k-action (Theorem 1.1). Without the loss of generality, we may assume that the metric is T^k-invariant (Theorem 1.2).

We argue by contradiction; assuming a sequence, M_i, as in the above such that $\mathrm{vol}(M_i)/\max\mathrm{injrad}(M_i) \to 0$ and $k=1$. Without loss of generality, we may assume that $M_i \xrightarrow{d_{GH}} X$. Let $x_i \in M_i$ such that $\mathrm{injrad}(M_i,x_i) = \max\mathrm{injrad}\ (M_i,x)$. We claim that there is a constant, $\eta > 0$, such that (for all i) $T^1(x_i)$ is contained in the η-tube U_i of some T^1-orbit, $T^1(y_i)$. Assuming the claim (whose proof is given at the end of the proof), we will bound $\frac{\mathrm{vol}(U_i)}{\mathrm{injrad}\ (M_i,x_i)}$ from below by a positive constant (depending on η), a contradiction.

Because T^1 acts isometrically on U_i,

$$\mathrm{vol}(U_i) = \mathrm{length}(T^1(y_i)) \cdot \mathrm{area}(D_i^\perp), \tag{2.3.1}$$

where $D_i^\perp$ denotes a normal slice of U_i. We shall bound $\mathrm{area}(D_i^\perp)$ from below, and bound $\mathrm{length}(T^1(y_i))$ in terms of $\mathrm{length}(T^1(x_i))$.

Let $T^1_{y_i}$ be the isotropy group of $T^1(y_i)$, and let $p_i : \tilde U_i \to U_i$ denote the Riemannian $|T^1_{y_i}|$-covering map, where $\tilde U_i = T^1 \times D_i^\perp$, $U_i = T^1 \times_{T^1_{y_i}} D_i^\perp$ (the Slice lemma) and the lifting T^1-action acts on $\tilde U_i$ by the rotation of the T^1-factor. By the Gray-O"Neill Riemannian submersion formula, the sectional curvature on $\tilde U_i/T^1$ is upper bounded by a constant $c_1(n)$. By the volume comparison, we conclude that $\mathrm{vol}(D_i^\perp) = \mathrm{area}(\tilde U_i/T^1) \ge \mathrm{vol}(B_\eta)$, where B_η is a η-ball in the $(n-1)$-space form of constant curvature $c_1(n)$. From (2.3.1), we get

$$\mathrm{vol}(U_i) \ge \mathrm{length}(T^1(y_i)) \cdot \mathrm{vol}(B_\eta). \tag{2.3.2}$$

By (1.1.2) we may assume that the second fundamental group of all T^1-orbits on $\tilde U_i$ are uniformly bounded by a constant $c(n)\eta^{-1}$. Then we may assume a constant $c(n,r)$ such that $\mathrm{length}(T^1(\tilde x_i)) \le c(n,\eta)\cdot \mathrm{length}(T^1(\tilde y_i))$, where $p_i(\tilde x_i) = x_i$ and $p_i(\tilde y_i) = y_i$. Then

$$\mathrm{length}(T^1(x_i)) \le c(n,\eta)\cdot |T^1_{y_i}| \cdot \mathrm{length}(T^1(y_i)). \tag{2.3.3}$$

Recall that the T^1-orbit at any point represents all the collapsed directions of the metric (cf. [CFG]). In particular, we may assume that

$$\mathrm{injrad}(M_i,x_i) \le \frac{1}{2}\mathrm{length}(T^1(x_i)). \tag{2.3.4}$$

Using (2.3.2)-(2.3.4), we derive

$$\begin{aligned}\frac{\mathrm{vol}\ (M_i)}{\max\mathrm{injrad}\ (M_i)} &\ge \frac{\mathrm{vol}\ (U_i)}{\frac{1}{2}\mathrm{length}(T^1(x_i))} \\ &\ge \frac{\mathrm{length}(T^1(y_i))\cdot \mathrm{vol}\ (B_\eta)}{\frac{1}{2}\mathrm{length}(T^1(x_i))} \\ &\ge \frac{2\cdot\ \mathrm{vol}\ (B_\eta)}{c(n,\eta)\cdot |T^1_{y_i}|}.\end{aligned} \tag{2.3.5}$$

By Lemma 2.1, we may assume that $|T_x^1| \le h(n,d)$ and thus see a contradiction in (2.3.5) because the left hand side converges to zero.

We now verify the claim. Recall that M_i/T^1 is homeomorphic and ϵ_i-isometric X, $\epsilon_i \to 0$, and the projection of the singular set on M_i into X converging to the singular set of X (with respect to the Hausdorff distance). On the other hand, X is the metric quotient, $X = Y/O(n)$, where Y is a Riemannian manifold on which $O(n)$ acts isometrically (cf. [CFG]). We can now pick up η from the stratification structure on $(Y, O(n)$ i.e. there are $O(n)$-invariant subsets,

$$Y = S_0 \supset S_1 \supset \cdots \supset S_r, \qquad \bar S_i = \bigcup_{j \ge i} S_j$$

such that each component of S_i has a unique isotropy group, and S_r is a closed totally geodesic submanifold. We can choose a sequence of numbers, $1 > \eta_r >> \eta_{r-1} >> \cdots > \eta_1$ such that each $x \in S_i = \bigcup_{j>i} T_{\eta_j}(S_j)$ satisfies that $O(n)(x)$ has a η_i-tube. We then choose $\eta = \eta_1/2$. $\square$

Remark 2.4.

Observe that the above proof goes through if one replaces the assumption, "$\mathrm{vol}(M)/\sup \mathrm{injrad}(M,x) < \epsilon_1(n,d)$" by "$\mathrm{vol}(M)/\mathrm{injrad}(M) < \epsilon_1(n,d)$". However, any spherical 5-space form satisfies $\mathrm{vol}(S^5/\Gamma)/\mathrm{injrad}(S^5/\Gamma) \ge \pi/3$ while given $\epsilon > 0$, there are many spherical 5-spaces satisfying $\mathrm{vol}(S^5/\Gamma)/\max \mathrm{injrad}(S^5/\Gamma) < \epsilon$ (see Example 2.6).

The following may be viewed as a converse to Proposition 2.3.

Lemma 2.5.
Let $M_i \xrightarrow{d_{GH}} X$ such that $|sec_{M_i}| \le 1$ and diam$(M_i) \le d$. If $\dim(X) \le n-2$, then vol$(M_i)/\max$ injrad$(M_i, x) \to 0$.

Proof. We argue by contradiction; without loss of generality we may assume that $\mathrm{vol}(M_i)/\max \mathrm{injrad}(M_i, z) \ge c > 0$ for all i. By the Cheeger's lemma, the ratio,

$$c \le \frac{\mathrm{vol}(M_i)}{\mathrm{injrad}(M)} \cdot \frac{\mathrm{injrad}(M_i)}{\max \mathrm{injrad}(M_i, z)} = \frac{\mathrm{vol}(M_i)}{\max \mathrm{injrad}(M, z)} \le c(n,d),$$

and thus $1 \le \frac{\max \mathrm{injrad}(M_i,z)}{\mathrm{injrad}(M_i)} \le c(n,d)/c$.

Let S_i denote the singular set of the $\pi_1(M_i)$-invariant isometric T^k-action on $\overline{M_i}$, and let U_i denote the ϵ-tube of S_i. Then the orbit projection, $p_i : M_i - U_i \to \overline{M_i - U_i} = (M_i - U_i)/T^k$ $(\bar x = p_i(x))$ is a Riemannian submersion with fiber a flat manifold F_i. We may choose ϵ small so that

$$\frac{c}{2} \le \frac{\mathrm{vol}(M_i - U_i)}{\max \mathrm{injrad}(M_i, z)} = \frac{\int_{\overline{M_i - U_i}} \mathrm{vol}(p_i^{-1}(\bar x)) d\overline{vol}}{\max \mathrm{injrad}(M_i, z)}.$$

Because $\mathrm{diam}(p_i^{-1}(\bar x)) \to 0$ uniformly as $i \to \infty$, again by the Cheeger's lemma we may assume that

$$\mathrm{vol}(p_i^{-1}(\bar x)) \le \mathrm{injrad}(p_i^{-1}(\bar x))\epsilon_i,$$

where $\epsilon_i \to 0$ as $i \to \infty$.

Because the fiber $p_i^{-1}(\bar{x})$ points all collapsed directions ([CFG]), we may assume that $\text{injrad}(M_i, x) \simeq \text{injrad}(p_i^{-1}(\bar{x}))$. Then

$$\begin{aligned}\frac{c}{2} &\le \frac{\text{vol}(M_i - U_i)}{\max \text{injrad}(M_i, z)} \\ &\le \int_{\overline{M_i - U_i}} \frac{2\text{injrad}(M_i, x)}{\max \text{injrad}(M_i, z)} \epsilon_i d\overline{vol} \\ &\le \frac{2c(n,d)}{c} \text{vol}(\overline{M_i - U_i})\epsilon_i \to 0,\end{aligned}$$

a contradiction. □

We now apply Lemma 2.5 to construct spherical 5-space forms satisfying Theorem C.

Example 2.6.

We first construct lens spaces, $S^5/\mathbb{Z}_p$. Consider a semi-free linear T^2-action on S^5. Let $T_i^1 \subset T^2$ such that $\text{diam}(T^2/T_i^1) \le i^{-1}$ and T_i^1 has no fixed point. We then choose a large prime p_i so that $\mathbb{Z}_{p_i}(\subset T_i^1)$ acts freely on S^5 and $\text{length}(T^1/\mathbb{Z}_{p_i}) \le i^{-1}$ (because the T_i^1-action has only finitely many isotropy groups). Clearly, the Gromov-Hausdorff distance, $d_{GH}(S^5/\mathbb{Z}_{p_i}, S^5/T^2) \to 0$. By Lemma 5.2, $S^5/\mathbb{Z}_{p_i}$ satisfies the conditions of Theorem C.

We now construct non-lens spaces. Recall from [Wo] (p.225) that if a spherical 5-space group, Γ, is not cyclic, then Γ is generated by two elements,

$$\gamma_1 = \begin{pmatrix} R(1/m) & 0 & 0 \\ 0 & R(r/m) & 0 \\ 0 & 0 & R(r^2/m) \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ R(3l/n) & 0 & 0 \end{pmatrix},$$

satisfying that $\gamma_1^m = \gamma_2^n = 1, \gamma_2\gamma_1\gamma_2^{-1} = \gamma_1^r$ with $n = 0 \mod (3), (n(r-1), m) = 1$, $r \ne r^3 = 1 \mod (m)$ and $(l, n/3) = 1$, where $R(\theta)$ denote the standard 2×2 rotation matrix with rotation angle θ. Clearly, the center of Γ is generated by γ_2^3. Hence, the index of $[\Gamma : \gamma_2^3] \ge m$. Then S^5/Γ satisfies conditions of Theorems B and C when m large.

Example 2.7.

We will construct examples showing that Theorems B and C are false if relaxing "$\delta > 0$" to "$\delta \ge 0$".

According to [Wo], there is a sequence of non-cyclic spherical 3-space groups, Γ_i, such that S^3/Γ_i converges to a closed interval I and $[\Gamma_i : \text{cent}(\Gamma_i)] \to \infty$. Then $M_i = S^2/\Gamma_i \times S^2$ converges to $I \times S^2$ with $0 \le \sec_{M_i} \le 1$. However, Γ_i cannot act freely on S^5 (any finite group acting freely both on S^3 and S^5 must be cyclic, cf. [Bro]). Note that by Lemma 2.5, $\text{vol}(M_i)/\max \text{injrad}(M_i) \to 0$.

3. Proof of Theorem D by Assuming Theorems E and F

In this section we give a generalization of Theorems E and F.

To exclude such an ambiguity, we introduce the following notion. We call an abelian subgroup c_s of a finite group Γ a *semi-center*, if its centralizer has index at most two in Γ and if $|c_s|$ is 'maximal' among such abelian subgroups. Obviously, a semi-center contains the center, and coincides with the center when $|\Gamma|$ is odd. However, a semi-center may not be unique when $|\Gamma|$ is even.

The following theorem is a generalization of Theorem D.

Theorem 3.1.

Given $0 < \delta \le 1$, there exists $\epsilon(\delta) > 0$ such that if a closed 5-manifold M satisfies

$$0 < \delta \le sec \le 1, \qquad vol(M) < \epsilon(\delta),$$

then M is homeomorphic to a spherical space form, provided

(3.1.1) a semi-center of the fundamental group $\pi_1(M)$ has index at least $w > 0$, a universal constant (independent of δ);

(3.1.2) $\pi_1(M)$ does not contain any index ≤ 2 subgroup isomorphic to a spherical 3-space group.

By combining Theorem 1.1, Theorem E, Proposition 1.3 and Lemma 1.9, the following generalized version of Theorem F implies Theorem 3.1.

Theorem 3.2.

Let M be a closed 5-manifold of positive sectional curvature which admits a π_1-invariant fixed point free isometric T^1-action. Then the universal covering $\tilde{M}$ is diffeomorphic to S^5, provided $\pi_1(M)$ satisfies (3.1.1) and (3.1.2).

Proof of Theorem D (resp. E) by assuming Theorem 3.1 (resp. 3.2).

It suffices to verify the conditions (3.1.1) and (3.1.2) in the circumstance of Theorem C. Note that a spherical 3-space group of odd order is cyclic, therefore, (3.1.1) and (3.1.2) hold if $\pi_1(M)$ has odd order, since $\pi_1(M)$ is not cyclic. If $\pi_1(M)$ is a non-cyclic spherical 5-space group, by [Wo] p225 it contains an index 3 normal cyclic subgroup, and therefore a semi-center coincides again with its center, and $\pi_1(M)$ contains no spherical 3-space group of index at most 2. The desired result follows. $\square$

4. Preparations

In this section, we supply materials that will be used in the proofs of Theorems D and E in the rest of this paper.

a. Fixed point set of abelian groups.

Consider a compact Lie group G acting isometrically on a closed manifold M. Let $F(G, M)$ denote the set of G-fixed points. Then each component of $F(G, M)$ is a closed totally geodesic submanifold. If $G = T^k$, then F has even codimension. For a generic compact Lie group, the topology of M may not be well related to the topology of $F(G, M)$. However, the opposite situation occurs when G is abelian (cf. [Bre], p.163).

Theorem 4.1.

Let M be a compact $\mathbb{Z}_h$-space. If $h = p$ is a prime, then

$$\chi(M) = \chi(F(\mathbb{Z}_p, M)) \mod p.$$

If $\mathbb{Z}_p$ (p is a prime) acts trivially on the homology group $H^*(M;\mathbb{Z})$, then

$$\chi(M) = \chi(F(\mathbb{Z}_p, M)).$$

The last assertion of the above lemma is from [Bre] III exercise 13 (p.169).

Theorem 4.2.
Let a compact abelian Lie group G act effectively on a closed manifold M, and let N denote an invariant subset. Then

$$\mathrm{rank}(H^*(F(G,M), F(G,N);\ell)) \leq \mathrm{rank}(H^*(M,N;\ell)),$$

where $G = T^k$ and $\ell = \mathbb{Q}$ or $G = \mathbb{Z}_p^k$ and $\ell = \mathbb{Z}_p$.

A consequence of Theorem 4.2 is

Theorem 4.3 (Smith).
Let a torus T^k act effectively on a closed manifold M. If M is a rational homology sphere, then $F(T^k, M)$ is a rational homology sphere.

The T^k-action on a sphere without fixed points is well understood.

Theorem 4.4 ([Bre], P. 164).
Let M be a n-dimensional homology sphere and admit a T^k action with no fixed point. If $H \subset T^k$ is a subtorus of dimension $k-1$, let $r(H)$ denote that integer, for which $F(H,M)$ is a homology $r(H)$-sphere. Then with H ranging over all subtori of dimension $k-1$ and $r(H) \geqslant 0$, we have

$$n + 1 = \sum_H (r(H) + 1).$$

A basic relation between the fundamental groups of M and that of its orbit space is the following homotopy lifting property.

Lemma 4.5 ([Bre]).
Let M be a manifold which admits a compact Lie group G-action. If either G is connected or G has a fixed point, then the orbit projection, $p : M \to M/G$, induces an onto map on the fundamental groups.

b. A generalized Lashof-May-Segal theorem.

Let G denote a compact Lie group (G can be finite). For two G-spaces, Y and Z, a map, $f : Y \to Z$, is called an *G-map* if $f(g \cdot y) = g \cdot f(y)$ for all $y \in Y$ and $g \in G$.

A *principal (G, T^k)-bundle* is a principal T^k-bundle, $T^k \to E \xrightarrow{p} Y$, such that E and Y are G-spaces, p is a G-map and the G-action on E preserves the structural group of the bundle. Note that the G-action and T^k-action may not commute. Two principal (G, T^k)-bundles are called *equivalent*, if there is a G-equivariant bundle equivalent map.

Let $\mathcal{B}(G,T^k)(B)$ be the set of equivalence classes of principal (G,T^k)-bundles over B. When $G=\{1\}$, we will skip G from the notation.

Let EG be the infinite join of G, a contractible free G-CW complex (cf. [Hu]). Put $B_G = EG \times_G B$ and $E_G = EG \times_G E$. There is a natural transformation,

$$\Phi : \mathcal{B}(G,T^k)(B) \to \mathcal{B}(T^k)(B_G)$$

by sending a principal (G,T^k)-bundle $p : E \to B$ to the principal T^k-bundle $p_G : E_G \to B_G$. The following theorem is a special case of [FR2] Theorem 3.3 which generalizes the Lashof-May-Segal theorem.

Theorem 4.6.
Let B, G be as in the above. Then $\Phi : \mathcal{B}(G,T^k)(B) \to \mathcal{B}(T^k)(B)$ is a bijection.

Theorem 4.6 can be used in the following situation (see Section 7): Let M be a closed manifold of finite fundamental group which admits a pseudo-free T^k-action. Let $\tilde M_0 = \tilde M - \mathcal{S}$, and let $X = (\tilde M - \mathcal{S})/T^k$. Then $\tilde M_0 \to X$ is a $(\pi_1(M), T^k)$-bundle.

c. Positive curvature and isometric torus actions.

In the rest of this section, we will consider an isometric T^k-action on a closed manifold M of positive sectional curvature. As seen in Theorems 4.1 and 4.2 and Lemma 4.5, the topology of M is closely related to the singular structure and the orbit space of the T^k-action. In the presence of a positive curvature, the singular structure and the orbit space is very restricted. This is the ultimate reason for many results in this field.

A basic constraint on the singular structure is given by following Berger's vanishing theorem ([Ro1], also [GS], [Su]).

Theorem 4.7.
Let a torus T^k act isometrically on a closed manifold M of positive sectional curvature. Then there is a T^k-orbit which is a circle. Moreover, the fixed point set is not empty when $\dim(M)$ *is even.*

Theorem 4.7 implies, via the isotropy representation at a circle orbit, that large k yields closed totally geodesic submanifolds of small codimension.

In the study of the fundamental group of a positively curved manifold on which T^k acts isometrically, the following result is a basic tool.

Theorem 4.8 ([Ro4]).
Let M be a closed manifold of positive sectional curvature on which T^k acts isometrically, and let ϕ be an isometry on M which commutes with the T^k-action. Then ϕ preserves some T^k-orbit which is a circle.

We now illustrate a situation where Theorem 2.8 may be applied. Let T^k act isometrically on a closed manifold M of finite fundamental group, and let $\pi : \tilde M \to M$ denote the Riemannian universal covering. Let $p : \tilde M \to \tilde M/\tilde T^k$ be the orbit projection, where $\tilde T^k$ denotes the covering torus of T^k acting on $\tilde M$. For any $\gamma \in \pi_1(M)$, because the γ-action commutes with the $\tilde T^k$-action, γ induces an isometry of $\tilde M/\tilde T^k$, denoted by $\bar\gamma$.

(4.9.1) $\bar{\gamma}$ is trivial if and only if $\gamma \in H$, the subgroup generated by loops in a principal T^k-orbit.
(4.9.2) γ preserves an orbit, say $\tilde{T}^k(\tilde{x})$, if and only $\bar{\gamma}$ fixes $\bar{x} = p(\tilde{T}^k(\tilde{x}))$.
(4.9.3) If $k = 1$ and γ preserves $\tilde{T}^1(\tilde{x})$, then $T^1(x)$ is an exceptional orbit, $x = \pi(\tilde{x})$ whose isotropy group contains a subgroup $\mathbb{Z}_h$ with h the exponent of γ.

Corollary 4.10.

Let M be a closed manifold of positive sectional curvature. If M admits an isometric T^k-action, then the subgroup generated by loops in a principal T^k-orbit is cyclic, say $< \alpha >$. If $\alpha \neq 1$, then for all $\gamma \in \pi_1(M)$, α and γ generate a cyclic subgroup.

Theorem 4.11 ([GS]).

Let M be a closed n-manifold of positive sectional curvature. If M admits an isometric T^k-action, then $k \leq [\frac{n+1}{2}]$ and "$=$" implies that M is diffeomorphic to a sphere, or a lens space, or a complex projective space.

Theorem 4.12 ([Ro2]).

Let M be a closed 5-manifold of positive sectional curvature. If M admits an isometric T^2-action, then M is diffeomorphic to a sphere.

Consider an isometric T^k-action on a closed manifold of positive sectional curvature. A consequence of Lemma 2.5 is that large k implies a closed totally geodesic submanifold of small codimension. The following connectedness theorem of Wilking provides a useful tool to contract information on homotopy groups from the existence of a closed totally geodesic submanifold of small codimension (see [FMR] for a further development).

A map from N to M is called $(i+1)$-*connected*, if it induces an isomorphism up to the i-th homotopy groups and a surjective homomorphism on the $(i+1)$-th homotopy groups.

Theorem 4.13 ([Wi]).

Let M be a closed n-manifold of positive sectional curvature, and let N be a closed totally geodesic k-submanifold. If there is a Lie group G that acts isometrically on M and fixes N pointwisely, then the inclusion map is $(2k-n+1+C(G))$-connected, where $C(G)$ is the dimension of a principal orbit of G.

We close this subsection by giving the following obstructions on a closed manifold of non-negative sectional curvature from ([Gr2]).

Theorem 4.14.

Let M be a closed n-manifold of non-negative sectional curvature.
(4.14.1) $\pi_1(M)$ can be generated by at most $\ell(n)$-elements.
(4.14.2) rank$(H_*(M;\ell)) \leq b(n)$, where ℓ is any coefficient field.

5. Proof of Theorem E for $k = 3$

Consider the case $k = 3$ in Theorem E. By Theorem 4.11, we may assume that the T^3-action on $\tilde{M}$ does not commute with the $\pi_1(M)$-action (equivalently,

$\rho : \pi_1(M) \to \mathrm{Aut}(T^3) = \mathrm{GL}(3, \mathbb{Z})$ is not trivial). For examples of such spherical 5-space forms, see [Wo] p.225.

The goal of this section is to prove the following:

Theorem 5.1.

Let M be a closed 5-manifold of positive sectional curvature. If M admits a π_1-invariant isometric T^3-action, then M is homeomorphic to a spherical space form.

By Proposition 1.3, the following two lemmas imply Theorem 5.1.

Lemma 5.2.

Let M be a closed 5-manifold of positive sectional curvature. Suppose that M admits a π_1-invariant isometric T^3-action. Then T^3 has a circle subgroup T^1 which acts freely on $\tilde{M}$ and which commutes with the $\pi_1(M)$-action.

Proof. Let $\rho : \pi_1(M) \to \mathrm{Aut}(T^3) = \mathrm{SL}(\mathbb{Z}, 3)$ be the holonomy representation. Without the loss of generality, we may assume that ρ is not trivial (see Theorem 4.11). We claim that $\ker \rho$ has index 3 in $\pi_1(M)$. First, $\tilde{M}$ is diffeomorphic to S^5 (Theorem 4.11) and $\tilde{M}/T^3$ is homeomorphic to a simplex $\triangle^2$ as stratified set; the three vertices are the projection of isolated three circle orbits in $\tilde{M}$ and the three edges are the projection of three components of T^2-orbits (cf. [FR3]). Because $\pi_1(M)/\ker \rho$ acts effectively on $\tilde{M}/T^3 \simeq \triangle^2$ which preserves the three vertices and the three edges, it is clear that $\pi_1(M)/\ker \rho \cong \mathbb{Z}_3$.

Consider the standard $\mathrm{SL}(\mathbb{Z}, 3)$-action on a three torus. Note that $SL(\mathbb{Z}, 3)$ has a unique subgroup isomorphic to $\mathbb{Z}_3$, generated by the permutation of three factors, and the diagonal circle subgroup of T^3 is $\mathbb{Z}_3$-invariant. (cf. [Wo], p.225). This implies that T^3 has a circle subgroup, T^1, on which $\rho(\pi_1(M)) \cong \mathbb{Z}_3$ acts trivially. Therefore T^1-action and $\rho(\pi_1(M))$-action commute, since $t\gamma \cdot x = \gamma\rho(\gamma)(t) \cdot x$, for any $\gamma \in \pi_1(M)$, $t \in T^3$, and $x \in \tilde{M}$.

It remains to show that the T^1-action is free. If $1 \neq H \subset T^1$ such that $F(H, \tilde{M}) \neq \emptyset$, then $F(H, \tilde{M})$ is either a circle or a totally geodesic three sphere (Theorems 2.3 and 2.13). Because $\pi_1(M)$ preserves $F(H, \tilde{M})$ and $\pi_1(M)/\ker \rho$ acts effectively on Δ^2, we may assume that $F(H, \tilde{M})$ is a totally geodesic three sphere. In particular, $F(H, \tilde{M})$ contains two isolated circle orbits (of the T^3-action). This implies that $\pi_1(M)$ must fix a vertex of $\tilde{M}/T^3 \simeq \triangle^2$, and this implies that $\pi_1(M)$ acts trivially on $\tilde{M}/T^3$, a contradiction. □

Lemma 5.3.

Let T^1 be as in Lemma 5.2. If $\rho : \pi_1(M) \to GL(\mathbb{Z}, 3)$ is non-trivial. Then the induced $\pi_1(M)$-action on $\tilde{M}/T^1$ is pseudo-free.

Proof. Suppose not, let $\gamma \in \pi_1(M)$ so that it acts on $\tilde{M}/T^1$ ($\approx \mathbb{C}P^2$ up to orientation by Freedman's result) with a 2-dimensional fixed point set. Let $\pi : \tilde{M}/T^1 \to \tilde{M}/T^3 \approx \Delta^2$ be the orbit projection. By the proof of Lemma 5.2 we know that $\pi_1(M)/\ker(\rho) \cong \mathbb{Z}_3$.

If $\rho(\gamma)$ is non-trivial, clearly the induced action of γ on $\tilde{M}/T^3$ has only an isolated fixed point in the interior of the disc Δ^2. Thus its preimage in $\tilde{M}/T^1$ is a 2-torus.

This preimage contains the fixed point set of the γ-action on $\tilde{M}/T^1$, which is a totally geodesic 2-dimensional submanifold (a sphere or $\mathbb{R}P^2$). A contradiction.

It remains to consider the case when $\rho(\gamma)$ is trivial. If γ has a 2-dimensional fixed point set in $\tilde{M}/T^1$, there is an element $t_0 \in T^1$ so that the fixed point set of $t_0\gamma \in T^3 \rtimes \pi_1(M)$ contains a 3-dimensional totally geodesic submanifold $F \subset \tilde{M}$. Observe that F is T^3-invariant (since $\gamma \in \ker(\rho)$) and its principal isotropy group (of the T^3-action) is a circle subgroup $C \subset T^3$. Note that $\ker(\rho)$ is a normal cyclic subgroup of $\pi_1(M)$ of index 3. Therefore, $\langle\gamma\rangle$ is also a normal subgroup in $\pi_1(M)$, and so $\pi_1(M)$ acts on the fixed point set F. On the other hand, for an element $\alpha \in \pi_1(M)$ so that $\rho(\alpha)$ is non-trivial, the principal isotropy group of $\alpha(F) = F$ is $\rho(\alpha)(C)$. This proves that $\rho(\alpha)(C) = C$. Thereby $C = T^1$, since the diagonal is the unique fixed point circle for the holonomy representation ρ. This is absurd by Lemma 5.2, since T^1 acts freely on $\tilde{M}$. □

6. Proof of Theorem E at the level of fundamental Group

In this section, we will prove Theorem C at the level of fundamental groups. The main result in this section is the following:

Theorem 6.1.
Let M be a closed 5-manifold of positive sectional curvature. If M admits a π_1-invariant isometric T^k-action ($k > 1$), then the fundamental group of M is isomorphic to that of a spherical 5-space form.

By Theorem 5.1, it suffices to prove Theorem 6.1 for $k = 2$.

According to [Wo], the fundamental group of a spherical 5-space form, Γ, is either cyclic or is generated by two elements satisfying

$$A^m = B^n = 1, \qquad BAB^{-1} = A^r,$$

such that $n = 0(\text{mod } 9), (n(r-1), m) = 1, r \neq r^3 = 1 \mod (m)$. We first give the following criterion of a spherical 5-space form group.

Lemma 6.2.
A finite non-cyclic group Γ is isomorphic to the fundamental group of a spherical 5-space form, if Γ satisfies the following conditions:
(6.2.1) Every subgroup of order $3p$ is cyclic, for any prime p.
(6.2.2) Γ has a normal cyclic subgroup of index 3.

Proof. Writing $|\Gamma| = mn$ with $(m, 3) = 1$, by the Burnside Theorem (cf. [Wo] Theorem 5.4.1, p.163) and [Wo] Theorem 5.3.2 on page 161, Γ is metacyclic, i.e., it is generated by two elements A and B with relations

$$A^m = B^n = 1, \; BAB^{-1} = A^r$$

where $((r-1)n, m) = 1$ and $r^n \equiv 1(\text{mod } m)$. In particular, any Sylow 3-subgroup of Γ is cyclic. By (6.2.2), $\{A, B^3\}$ generates a cyclic subgroup of index 3, and thus $r^3 \equiv 1(\text{mod } m)$, but $r \neq 1(\text{mod } m)$, otherwise Γ is cyclic. By [Wo] page 225, it only remains to prove that n is divisible by 9. If not, i.e., $(\frac{n}{3}, 3) = 1$. For any prime factor p of m, let p^i be the largest p-factor of m. Since the automorphism group $\mathrm{Aut}(\mathbb{Z}_{p^i})$ is cyclic if p is odd, and an abelian 2-group if $p = 2$, its only order 3 subgroup is contained in $\mathrm{Aut}(\mathbb{Z}_p)$. By (6.2.1), B commutes with the order p elements, i.e., corresponds to zero in $\mathrm{Aut}(\mathbb{Z}_p)$. Therefore, B commutes with every element of order p^i. This implies that Γ is an abelian group, and so it is cyclic. A contradiction. $\square$

In the proof of Theorem 6.1, we will establish (6.2.1) and (6.2.2).

Lemma 6.3.
Let M be a closed 5-manifold of positive sectional curvature. Suppose that M admits a π_1-invariant isometric T^k-action ($k > 1$). If the T^k-fixed point set is not empty, then $\pi_1(M)$ is cyclic.

Proof. Because $\pi_1(M)$ preserves $F(T^k, \tilde M)$, it suffices to show that $F(T^k, \tilde M)$ is a circle. By Theorem 4.12, the universal covering space $\tilde M$ is diffeomorphic to S^5. By Theorems 4.3 and 4.8, $F(T^k, \tilde M)$ is connected. Because T^k acts effectively on the normal space of $F(T^k, \tilde M)$, $F(T^k, \tilde M)$ is a circle. $\square$

Recall that a T^k-action ($k > 1$) is *pseudo-free*, if all singular orbits are isolated and outside of which the T^k-action is free.

Lemma 6.4.
Let M be a closed 5-manifold of positive sectional curvature. Suppose that M admits a π_1-invariant isometric T^2-action without fixed point. If the T^2-action on $\tilde M$ is not pseudo-free, then $\pi_1(M)$ is cyclic.

Proof. First, $\tilde M$ is diffeomorphic to S^5 (Theorem 4.12). Hence, if $H \subset T^2$ is a circle or $\mathbb{Z}_p$ (p is a prime), then $F(H, \tilde M)$ is connected (Theorem 4.3).

Consider *all* circle subgroups of T^2 with nonempty fixed point sets: According to Theorem 4.4, there are two possibilities: (1) There are two distinct circle subgroups: T^1_1, T^1_2 such that $\dim(F(T^1_1, \tilde M)) = 3$ and $F(T^1_2, \tilde M)$ is a circle. (2) There are three distinct circle subgroups with fixed points set of dimension one.

Because $F(\rho(\gamma)(T^1_1), \tilde M) = \gamma(F(T^1_1, \tilde M))$, $\rho(\gamma)(T^1_1) = T^1_1$ (otherwise, there two distinct circle subgroups with fixed point sets of dimension 3). Consequently, $\rho(\gamma)(T^1_2) = T^1_2$ and thus $\pi_1(M)$ preserves $F(T^1_2, \tilde M)$ (this implies that $\pi_1(M)$ is cyclic).

We then consider (2) such that there is a $\mathbb{Z}_p \subset T^2$ (p is a prime) with $\dim(F(\mathbb{Z}_p, \tilde M)) = 3$. If $\rho(\gamma)(\mathbb{Z}_p) = \mathbb{Z}_p$ for all $\gamma \in \pi_1(M)$, then $\pi_1(M)$ preserves $F(\mathbb{Z}_p, \tilde M)$. Because $F(\mathbb{Z}_p, \tilde M)$ contains exactly two of the three circle orbits in (2), $\pi_1(M)$ must preserve the unique circle orbit outside $F(\mathbb{Z}_p, \tilde M)$ and thus $\pi_1(M)$ is cyclic.

If there is $\gamma \in \pi_1(M)$ such that $\rho(\gamma)(\mathbb{Z}_p) \neq \mathbb{Z}_p$, then $\gamma(F(\mathbb{Z}_p, \tilde M)) = F(\rho(\gamma)(\mathbb{Z}_p), \tilde M)$, and $F_0 = F(\mathbb{Z}_p, \tilde M) \cap F(\rho(\gamma)(\mathbb{Z}_p), \tilde M)$ is a circle which is the fixed point set of $\mathbb{Z}_p^2 \subset T^2$ (Theorem 4.4). Because $\rho(\gamma)(\mathbb{Z}_p^2) = \mathbb{Z}_p^2$ for all $\gamma \in \pi_1(M)$, $\pi_1(M)$ preserves F_0, and thus $\pi_1(M)$ is cyclic. $\square$

Lemma 6.5.
Let M be a closed 5-manifold of positive sectional curvature. Suppose that M admits a π_1-invariant isometric T^2-action with an empty fixed point set. If the T^2-action on $\tilde M$ is pseudo-free, then either $\pi_1(M)$ is cyclic or satisfies

(6.5.1) $\pi_1(M)$ has a normal cyclic subgroup of index 3.
(6.5.2) Any subgroup of $\pi_1(M)$ with order $3q$ is cyclic, where q is a prime.

Combining Lemmas 6.2-6.5, we conclude the case of Theorem 6.1 for $k = 2$, and therefore the proof of Theorem 6.1, by Theorem 5.1.

From the proof of Lemma 6.4, there are exactly three isolated circle orbits.

The pseudo-free condition implies that $\tilde{M}/T^2$ is a topological manifold and thus a homotopy 3-sphere because $\tilde{M}/T^2$ is simply connected. The $\pi_1(M)$-action on $\tilde{M}$ induces a $\pi_1(M)$-action on $\tilde{M}/T^2$. Because $\gamma \in \pi_1(M)$ maps a circle orbit to a circle orbit, we may view $\pi_1(M)$ acting on three points by permutations i.e. there is a homomorphism, $\phi : \pi_1(M) \to S_3$, the permutation group of three letters. The kernel of ϕ is a normal subgroup, which acts trivially on the three points (or equivalently, which preserving each of the circle orbits). Thus $\ker(\phi)$ is cyclic.

In the proof of Lemma 6.5 we need

Lemma 6.6.
Let M be as in Lemma 6.5. Then ϕ is trivial if and only if the holonomy representation $\rho : \pi_1(M) \to Aut(T^2)$ is trivial.

Proof. Let H_i, $i = 1, 2, 3$, denote the three isotropy groups of the isolated single orbits of the pseudofree T^2-action on $\tilde{M}$. Note that, for any $\gamma \in \pi_1(M)$, and $x \in \tilde{M}$ with isotropy group I_x, the isotropy group of $\gamma(x)$, $I_{\gamma(x)} = \rho(\gamma)(I_x)$. Therefore, if ρ is trivial, then $\pi_1(M)$ preserves the isotropy groups, and so preserves every singular orbits, i.e., ϕ is trivial.

Conversely, if ϕ is trivial, $\pi_1(M)$ preserves the three singular orbits. In particular, $\pi_1(M)$ is cyclic. Therefore, $\rho(\gamma)(H_i) = H_i$ for a generator $\gamma \in \pi_1(M)$. It is easy to see that H_i, $i = 1, 2, 3$, generate T^2. Therefore, in the Lie algebra of T^2, $\mathbb{R}^2$, the automorphism $\rho(\gamma) \in GL(\mathbb{Z}, 2)$ has three different eigenvectors whose eigenvalues are 1 or -1. This implies that $\rho(\gamma)$ is the identity. The desired result follows. $\square$

Our proof of Lemma 6.5 involves a homotopy invariant, 'the first k-invariant'. This invariant can be used to distinguish two connected spaces whose first and second homotopy groups are the same. Let's now briefly recall its definition ([Wh]).

Let X be a connected space, and $K(\pi_i(X), \ell)$ denote the Eilenberg-Maclane space. Corresponding to each map, $k_1 : K(\pi_1(X), 1) \to K(\pi_2(X), 3)$, there is a unique fibration,

$$\begin{array}{ccccc} K(\pi_2(X),2) & \longrightarrow & E_{k_1} & & \\ & & \big\downarrow f & & \\ & & K(\pi_1(X),1) & \xrightarrow{k_1} & K(\pi_2(X),3) \end{array}$$

with fiber $K(\pi_2(X), 2)$. Moreover, there is a unique E_{k_1} such that the classifying map f has a lifting, $\tilde{f} : X \to E_{k_1}$,

$$\begin{array}{ccc} & & E_{k_1} \\ & & \big\downarrow \\ X & \xrightarrow{f} & K(\pi_1(X),1) \end{array}$$

satisfying that $\tilde{f}_* : \pi_i(X) \to \pi_i(E_{k_1})$ is an isomorphism for $i = 1, 2$. The corresponding cohomology class $k_1 \in H^3(K(\pi_1(X); \pi_2(X))$ is called the first k-invariant of X. Clearly, the first k-invariant is a homotopy invariant.

Let $L_\ell = S^3/\mathbb{Z}_\ell$ denote a lens space. It is well-known that the punctured lens space has non-trivial first k-invariant, i.e., for $p \in L_\ell$, $k_1(L_\ell - \{p\}) \neq 0$ (cf. [EM]).

Proof of Lemma 6.5.

(6.5.1) By the discussion after Lemma 6.4, we have a homomorphism $\phi : \pi_1(M) \to S_3$. A priori, $\text{Im}(\phi)$ could be $\{1\}$, $\mathbb{Z}_3$, or S_3 or $\mathbb{Z}_2$.

If $\text{Im}(\phi) \cong \{1\}$, then $\pi_1(M)$ fixes all isolated circle orbits, therefore $\pi_1(M)$ acts on every circle orbit freely. This implies that $\pi_1(M)$ is cyclic.

We will now rule out the latter two cases.

If $\text{Im}(\phi) \cong \mathbb{Z}_2$, there is an element $\gamma \in \pi_1(M)$ so that $\phi(\gamma)$ is nonzero. By definition γ preserves a unique singular circle orbit, with isotropy group $H \cong S^1$, and permutes the rest two circle orbits. Since $\tilde{M}/T^2$ is a homotopy 3-sphere, the induced action of γ on $\tilde{M}/T^2$ has at least a fixed point which is not the isolated singular points (the singular orbits). This implies that γ preserves a principal orbit $T^2 \cdot x$ and acts freely on. By Lemma 6.6 $\rho(\gamma)$ is also nonzero, of order 2. It is easy to show that the γ-action and the transitive T^2-action on $T^2 \cdot x$ do not commute. Therefore, the free γ-action on $T^2 \cdot x$ has a quotient space the Klein bottle. This implies that, up to conjugation, the action of γ on the orbit is given by the composition of the multiplication $\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \in T^2$ with the rotation $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \rho(\gamma)$ on T^2. Thus, for $t_0 = \begin{bmatrix} \alpha_1^{-1} \\ \alpha_2^{-1} \end{bmatrix} \in T^2$, the fixed point set of $t_0\gamma$ on $T^2 \cdot x$ contains two disjoint circles, $S^1 \times \{\pm 1\}$. Therefore, the fixed point set F of $t_0\gamma$ on $\tilde{M}$ has dimension 3, a homology 3-sphere, which intersects with every principal orbit either empty, or two disjoint circles.

Observe that F projects to the fixed point set γ on $\tilde{M}/T^2$. Therefore, γ acts on $\tilde{M}/T^2$ with fixed point set a 2-dimensional homology sphere. If F does not intersect with the circle orbit with isotropy group H (preserved by γ), the fixed point set of γ on $\tilde{M}/T^2$ is not connected, a contradiction. Since H is preserved by $\rho(\gamma)$, i.e. $\rho(\gamma)(H) = H$, H is either $S^1 \times \{1\}$, or $\{1\} \times S^1$ in the standard coordinate for T^2. Now the intersection of F with the singular circle orbit consists of either two points, or the whole singular orbit, depending on whether the reduced automorphism $\rho(\gamma) \in \text{Aut}(T^2/H)$ is trivial or not. In the former case, H acts F semifreely, with two isolated fixed points. A contradiction, since F is a homotopy 3-sphere by Theorem 4.13, because otherwise H acts on two points punctured 3-sphere freely, absurd by Euler characteristic reasoning. For the latter case, the quotient F/H is a 2-disk, with an action of $\{\pm 1\} \subset 1 \times S^1$ freely in the interior of the disk. A contradiction again by the Brower fixed point theorem.

If $\phi(\pi_1(M)) = S_3$, we may use $\phi^{-1}(\mathbb{Z}_2)$ instead of $\pi_1(M)$ to get the contradiction.

It remains to prove (6.5.2). By (6.5.1), it suffices to prove $\pi_1(M)$ satisfies the pq-condition for $q = 3$. If $\pi_1(M)$ contains a non-cyclic subgroup of order $3q$, where q is a prime, then the subgroup contains $\mathbb{Z}_3$ as a subgroup which acts pseudo-freely on $\tilde{M}/T^2$ (denoted by Σ), a homotopy 3-sphere. Let $\tilde{M}_0$ denote the complement of the three isolated circle orbits on $\tilde{M}$, and let $\Sigma_0 = \tilde{M}_0/T^2$. Consider the T^2-bundle,

$$T^2 \to \tilde{M}_0/\mathbb{Z}_3 \to \Sigma_0/\mathbb{Z}_3,$$

and its classifying map, $f : \Sigma_0/\mathbb{Z}_3 \to B(T^2 \rtimes \mathbb{Z}_3)$. Because $\tilde{M}_0$ is 2-connected, by the transversality f is a 3-equivalence. This implies that the first k-invariant of $\Sigma_0/\mathbb{Z}_3$ is zero, a contradiction, because $\Sigma_0/\mathbb{Z}_3$ is homeomorphic to the punctured lens space $L_3 - \{p\}$ which has a non-zero first k-invariant. □

7. Proof of Theorem E for Pseudo-free T^2-actions

The goal of this section is to prove Theorem E where the π_1-invariant isometric T^2-action is pseudo-free (Theorem 7.1). In the complementary situation, there is a totally geodesic submanifold of codimension 2 that requires a different argument (see Section 8). In the proof we need to use the topology of the orbit space $\tilde{M}/T^2$, which is a homotopy 3-sphere. A serious problem is that the Poincaré conjecture is open and we can not conclude that it is homeomorphic to S^3. This can be solved because everything can go through by employing s-cobordism theory in differential topology, and using the well-known s-cobordism theorem (due to Smale) in dimension 5. In this section we will give our proof by assuming the Poincare conjecture and leave the proof in the general case to an Appendix, since it is probably easier to follow the geometric ideas, and also partly since the work of Perelman.

Theorem 7.1.
Let the assumptions be as in Theorem E. If $k = 2$ and the T^2-action is pseudo-free, then M is homeomorphic to a spherical space form.

To start with, it is very helpful to look at the linear model of a locally pseudofree T^2-action on a spherical space form S^5/Γ. There is a linear pseudofree T^2-action on S^5 defined as follows:

$$(e^{i\theta}, e^{i\phi})(z_0, z_1, z_2) = (e^{i(\theta+\phi)}z_0, e^{i(\theta-2\phi)}z_1, e^{i(-2\theta+\phi)}z_2)$$

with principal isotropy group $\mathbb{Z}_3$ generated by $(e^{\frac{2}{3}\pi i}, e^{\frac{4}{3}\pi i})$.

A linear action of a cyclic group $\mathbb{Z}_{k\ell}$ on S^5 commuting with the linear pseudofree T^2-action reduces to a linear action on the orbit space $S^5/T^2 = S^3$. Let $\mathbb{Z}_k$ denote the principal isotropy group of the reduced linear action on S^3. It is equivalent to say that $\mathbb{Z}_k$ acts along the T^2-orbits. The reduced $\mathbb{Z}_\ell = \mathbb{Z}_{k\ell}/\mathbb{Z}_k$ action on S^3 has a fixed point set S^1, which spans a linear plane of $\mathbb{R}^4$. The condition of the linear $\mathbb{Z}_k$ action (defined by multiplying $(e^{\frac{a}{k}\pi i}, e^{\frac{b}{k}\pi i}, e^{\frac{c}{k}\pi i})$) along the T^2-orbits can be written as $a + b + c = 0(\text{mod } 2k)$.

Let Γ be a non-cyclic spherical 5-group acting freely and linearly on S^5, $\Gamma \subset SO(6)$ is generated by two matrices

$$A = \begin{pmatrix} R(1/m) & 0 & 0 \\ 0 & R(r/m) & 0 \\ 0 & 0 & R(r^2/m) \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ R(3/n) & 0 & 0 \end{pmatrix},$$

where $R(\theta)$ denote the standard 2×2 rotation matrix with rotation angle $2\pi\theta$, and I the 2×2 identity matrix, $r, n \in \mathbb{Z}$ satisfying $r^2 + r + 1 = 0(\text{mod } m)$. Note that the action by A may not be along the T^2-orbits.

In the following context we will continue to use Γ_0 (resp. T_0^2) to mean a linear Γ-action (T^2-action) on S^5 so that it extends to a linear action of $T_0^2 \rtimes \Gamma_0$.

Consider the pseudofree linear T^2-action on S^5 defined above. Let S^5_0 denote the complement of small open tubes ($\simeq D^4 \times T^1$) around the three isolated circle orbits on S^5. Let M be as in Theorem 7.1. By Theorem 4.4 the pseudofree T^2-action on $\tilde{M} \approx S^5$ has exactly three isolated circle orbits. Let $\tilde{M}_0$ denote the complement of small open tubes of the three isolated circle orbits. By Theorem 6.1 $\Gamma = \pi_1(M)$ is a spherical 5-space group. Our main effort is to show that $M_0 = \tilde{M}/\Gamma$ is homeomorphic to S^5_0/Γ_0. By [SW] the gluing of a handle $D^4 \times T^1$ is unique up to homeomorphism, and therefore M is homeomorphic to S^5/Γ_0.

Let $\rho : \pi_1(M) = \Gamma \to \mathrm{Aut}(T^2)$ denote the holonomy representation of the π_1-invariant action. By Lemma 6.6 we know that $\ker(\rho)$ is cyclic, and the image $\rho(\Gamma)$ is either trivial or isomorphic to $\mathbb{Z}_3$. Let $\mathbb{Z}_k$ denote the principal isotropy group of the reduced Γ-action on $\tilde{M}/T^2 := \Sigma$. By definition one sees that $\mathbb{Z}_k$ acts on $\tilde{M}$ through the T^2-orbits.

Now let us consider the principal T^2-bundle $T^2 \to \tilde{M}_0 \to \tilde{M}_0/T^2 = \Sigma_0$. Assuming the Poincare conjecture, $\Sigma_0 = S^3_0$ is the complement of S^3 by removing three small 3-disks. Note that Γ acts on this principal bundle, and moreover, the sub-action of $\ker(\rho)$ commutes with the T^2-action. The following lemma is immediate:

Lemma 7.2. *The principal T^2-bundle is unique up to weak equivalence. Therefore, every pseudofree T^2-action on S^5 is conjugately linear.*

Proof. Note that the bundle is uniquely determined by its Euler class, an element in $H^2(\Sigma_0; \mathbb{Z}^2) \cong \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z}^2)$. Considering the Euler class as a 2×2 matrix (given by its classifying map to BT^2), its determinant is ± 1 since the total space $\tilde{M}_0$ is 2-connected by the transversality theorem. Therefore, up to the left action by $GL(\mathbb{Z}, 2)$, i.e. up to an automorphism of T^2, the bundle is unique. The desired result follows. □

By Lemma 7.2 the sub-action by $\mathbb{Z}_k$ on $\tilde{M} \approx S^5$ is conjugately linear. Therefore $\tilde{M}/\mathbb{Z}_k$ is diffeomorphic to a lens space $S^5/\mathbb{Z}_k$. Of course one should note that there are possibly many different ways to embed a $\mathbb{Z}_k$ in T^2 which acts freely on S^5, and consequently the lens space may not be unique.

Let us consider the reduced principal T^2-bundle $T^2/\mathbb{Z}_k \to \tilde{M}_0/\mathbb{Z}_k \to \Sigma_0$, regarded as a $\Gamma/\mathbb{Z}_k$-equivariant bundle. By Lemma 7.2 $\tilde{M}/\mathbb{Z}_k$ is a lens space.

Lemma 7.3. *If the Γ-action and T^2-action on $\tilde{M}$ commute, then the above Γ-equivariant principal T^2-bundle is Γ-equivariantly equivalent to a linear T^2-bundle $T^2/\mathbb{Z}_k \to S^5_0 \to S^3_0$.*

Proof. Note that Γ is cyclic. Let us write $\Gamma = \mathbb{Z}_{k\ell}$ where $\mathbb{Z}_k$ is as above. The effective action on $\mathbb{Z}_\ell$ on $\Sigma \approx S^3$ is has fixed point. By a deep theorem of [BLP] this action of $\mathbb{Z}_\ell$ on Σ is conjugate to a linear action on S^3. Therefore, by Theorem 4.6 it suffices to prove that the associated principal T^2-bundle

$$T^2/\mathbb{Z}_k \to E\mathbb{Z}_\ell \times_{\mathbb{Z}_\ell} \tilde{M}_0/\mathbb{Z}_k \to E\mathbb{Z}_\ell \times_{\mathbb{Z}_\ell} \Sigma_0$$

is unique up to weak equivalence. We need only to show its Euler class $e(\gamma) \in H^2(E\mathbb{Z}_\ell \times_{\mathbb{Z}_\ell} \Sigma_0; \mathbb{Z}^2)$ can be realized by the Euler class of a linear Γ_0-equivariant principal T^2-bundle over S^3_0 with total space $S^5_0/\mathbb{Z}_k$, where $\mathbb{Z}_k$ acts linearly on S^5 along the T^2-orbits. By some standard calculation we get that $H^2(E\mathbb{Z}_\ell \times_{\mathbb{Z}_\ell} \Sigma_0; \mathbb{Z}^2) \cong \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z}^2) \oplus \mathbb{Z}^2_\ell$. By restricting the bundle to $E\mathbb{Z}_\ell \times_{\mathbb{Z}_\ell} [p]$ where

$[p] \in \Sigma_0$ is a fixed point of the $\mathbb{Z}_\ell$-action, one gets immediately that $e(\gamma)$ restricts to a generator of $H^2(B\mathbb{Z}_\ell; \mathbb{Z}^2) \cong \mathbb{Z}_\ell^2$ (an element of order ℓ).

By comparing with the linear model discussed at the beginning of this section, it is straightforward to check that every pair $(a, b) \in \mathbb{Z}_\ell^2$ generating an order ℓ element can be realized as the torsion component of the Euler class of a linear $\mathbb{Z}_\ell$-equivariant principal T^2-bundle on S_0^3 with total space $\tilde{M}_0/\mathbb{Z}_{k\ell}$. The torsion free part of $e(\gamma)$ is uniquely determined by its lifting to $H^2(E\mathbb{Z}_\ell \times \Sigma_0; \mathbb{Z}^2) \cong \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z}^2)$, which is the Euler class of the forgetful principal T^2-bundle on Σ_0, regarded as a non-equivariant bundle. By Lemma 7.2 this Euler class is uniquely determined by the total space $\tilde{M}_0/\mathbb{Z}_k$, or equivalently, by the conjugacy class of the embedding of $\mathbb{Z}_k$ in T^2. This proves the desired result. □

Next let us consider the case where the holonomy $\rho : \pi_1(M) = \Gamma \to \mathrm{Aut}(T^2)$ is non-trivial.

Lemma 7.4. *Let M be as in Theorem 7.1. If the holonomy $\rho : \Gamma \to Aut(T^2)$ is non-trivial, then the Γ-equivariant principal T^2-bundle $\tilde{M}_0/\mathbb{Z}_k \to \Sigma_0$ is uniquely determined by the total space $\tilde{M}_0/\mathbb{Z}_k$ up to weak equivalent.*

Proof. By Lemma 6.6 the image $\rho(\Gamma) \cong \mathbb{Z}_3 \subset \mathrm{GL}(2, \mathbb{Z}) = \mathrm{Aut}(T^2)$. Recall that $\mathrm{SL}(2, \mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ has a subgroup of order 3, unique up to conjugation, which is generated by

$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}.$$

The action of $\rho(\Gamma)$ (regard an element of order 3 in Γ) on Σ_0 is free. The quotient group $\Gamma/\mathbb{Z}_k$ acts effectively on $\Sigma \approx S^3$, and which is not free, unless $\Gamma/\mathbb{Z}_k \cong \mathbb{Z}_3$. By Theorem 6.1 it is easy to see that $\Gamma/\mathbb{Z}_k \cong \mathbb{Z}_3$ only if Γ is cyclic. Therefore, by [BLP] once again the reduced action on $\Gamma/\mathbb{Z}_k$ on $\Sigma \approx S^3$ is conjugate to a linear action on S^3, unless $\Gamma/\mathbb{Z}_k = \mathbb{Z}_3$. In the latter case (if $\Gamma/\mathbb{Z}_k = \mathbb{Z}_3$) we may replace the "conjugation" by "s-cobordism" (cf. Appendix), and everything goes through. For the sake of simplicity we now assume that $\Gamma/\mathbb{Z}_k$ acts on $\Sigma \approx S^3$ is a linear action. It is easy to see that $\Gamma/\mathbb{Z}_k$ is cyclic, since Γ is a spherical 5-group (Theorem 6.1), and the action on Σ is linear. Let us write $\Gamma/\mathbb{Z}_k = \mathbb{Z}_{3n}$ (in the linear model this is generated by the matrix B).

By Theorem 4.6 the affine $(\mathbb{Z}_{3n}, T^2)$-bundle $\tilde{M}_0/\mathbb{Z}_k \to \Sigma_0$, is uniquely determined by the associated affine T^2-bundle with the above holonomy $\rho : \mathbb{Z}_{3n} \to GL(\mathbb{Z}, 2)$. The Euler class for this affine bundle sits in the local cohomology group $H^2(E\mathbb{Z}_{3n} \times_{\mathbb{Z}_{3n}} \Sigma_0; \mathbb{Z}_\rho^2)$. By the short exact sequence $1 \to \mathbb{Z}_\rho^2 \to \mathbb{Z}[\mathbb{Z}_3] \to \mathbb{Z} \to 1$ we can calculate the local cohomology group

$$1 \to H^2(E\mathbb{Z}_{3n} \times_{\mathbb{Z}_{3n}} \Sigma_0; \mathbb{Z}_\rho^2) \to H^2(E\mathbb{Z}_n \times_{\mathbb{Z}_n} \Sigma_0; \mathbb{Z}) \to H^2(E\mathbb{Z}_{3n} \times_{\mathbb{Z}_{3n}} \Sigma_0; \mathbb{Z})$$

where the middle space $E\mathbb{Z}_n \times_{\mathbb{Z}_n} \Sigma_0$ is the three fold covering of $E\mathbb{Z}_{3n} \times_{\mathbb{Z}_{3n}} \Sigma_0$. In the above exact sequence, the middle term is isomorphic to $\mathbb{Z}^2 \oplus \mathbb{Z}_n$, and the last term is isomorphic to $\mathbb{Z}_{3n}$. By transgression we see readily that the torsion part of the middle term goes injectively into the last term. Therefore, the local cohomology group $H^2(E\mathbb{Z}_{3n} \times_{\mathbb{Z}_{3n}} \Sigma_0; \mathbb{Z}_\rho^2)$ is torsion free, and of rank 2. By the universal coefficients theorem, $H^2(E\mathbb{Z}_{3n} \times_{\mathbb{Z}_{3n}} \Sigma_0; \mathbb{Z}_\rho^2)$ is given by $\mathrm{Hom}^\rho(H_2(\Sigma_0), \mathbb{Z}^2) \cong \mathrm{Hom}^\rho(\mathbb{Z}^2, \mathbb{Z}^2)$, where Hom^ρ denotes the ρ-invariant homomorphisms. Therefore the forgetful homomorphism $\mathrm{Hom}^\rho(\mathbb{Z}^2, \mathbb{Z}^2) \to \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z}^2)$ corresponds to the forgetful map from a $\mathbb{Z}_{3n}$-equivariant T^2-bundle on Σ_0 to a principal T^2-bundle. This is clearly injective since the torsion freeness. The desired result follows. $\square$

Proof of Theorem 7.1.

Let M be as in Theorem 7.1. By Theorem 6.1 $\pi_1(M) = \Gamma$ is a spherical 5-space group. By Lemmas 7.3 and 7.4 we know that $\tilde{M}_0/\Gamma := M_0$ is diffeomorphic to S_0^5/Γ_0. Since M is obtained by gluing three handles $S^1 \times D^4$ along the boundary components $S^1 \times S^3$. Because every self diffeomorphism of $S^1 \times S^3$ extends to a self homeomorphism of $S^1 \times D^4$ (cf. [SW]), the homeomorphism type does not depend on the gluing. Therefore, M is homeomorphic to S^5/Γ_0. The desired result follows. $\square$

8. Completion of the proof of Theorem E

After the works in Sections 6-7, we are ready to finish the remaining case in the proof of Theorem E.

Proof of Theorem E.

First, by Theorems 7.1 we only need to consider a non-pseudo-free T^2-action, i.e. the T^2-action has a non-empty fixed point, a 3-dimensional stratum with circle isotropy group, or a nontrivial finite isotropy group but without fixed point. In all cases, $\pi_1(M) := \Gamma$ is cyclic (Lemmas 6.3 and 6.4). Recall that $\tilde{M} = S^5$ with an action of $T^2 \rtimes_\rho \Gamma$.

Case (i). The T^2-action has a non-empty fixed point set;

Note that the fixed point set must be a circle (Theorem 4.2). By local isotropy representation of T^2 at the fixed point set, there are two circle isotropy groups with three dimensional fixed point sets, two totally geodesic S^3. Observe that the T^2-action on $\tilde{M}$ is free outside the union of the two 3-dimensional strata, and the quotient space $\tilde{M}/T^2$ is homeomorphic to the 3-ball D^3, whose boundary $S^2 = D^2_+ \cup D^2_-$, where $D^2_\pm$ is the image of the two 3-dimensional strata and $D^2_+ \cap D^2_-$ is the image of the fixed point set of T^2. We claim that *the T^2-action and Γ-action commute* (equivalently, ρ is trivial). By Theorem 4.11 this implies that M is diffeomorphic to a lens space.

Identify $\tilde{M}/T^2$ with D^3. Observe that Γ acts isometrically on $\tilde{M}/T^2$ and preserves the boundary $\partial(\tilde{M}/T^2) = \partial D^3$. If the commutativity fails, Γ acts nontrivially on $\tilde{M}/T^2$. By the well-known Brouwer fixed point theorem, Γ has at least a fixed point in the interior of D^3, which represents a principal orbit, say $T^2 \cdot x$. As in the proof of Lemma 6.5, Γ acts on $T^2 \cdot x$ with quotient a Klein bottle, since $\rho(\Gamma)$ is not trivial. Therefore, the same argument of Lemma 6.5 implies an element $t_0 \in T^2$ so that $t_0\gamma$ has a 3-dimensional fixed point set F in $\tilde{M}$. By the Frankel's theorem, F intersects with the two 3-dimensional strata, of circle isotropy groups. Clearly, F projects to the fixed point set of Γ in D^3. Recall that F intersects with a principal orbit in two circles. This together shows that the fixed point set of Γ in D^3 is a 2-dimensional, and so a disk, with non-empty intersections with both D^2_+ and D^2_-. Therefore, the fixed point set $F(\Gamma, D^3)$ contains at least a point of $D^2_+ \cap D^2_-$, the fixed point set of the T^2-action. For any such a point $[x]$, its preimage $x \in \tilde{M}$ satisfies $\gamma x = x$. A contradiction, since Γ acts freely on $\tilde{M}$.

Case (ii). The T^2-action has no fixed point, but it has a 3-dimensional stratum with circle isotropy group;

Let $T^1 \subset T^2$ denote the unique circle isotropy group with 3-dimensional fixed point set. Since Γ preserves the strata, Γ preserves the isotropy group T^1, that is, for any $g \in \Gamma$, $\rho(g)(T^1) = T^1$. Therefore, $T^1 \rtimes \Gamma$ acts on $\tilde{M}$. We claim that the *T^1-action and Γ-action commute.* Then M admits a T^1-action with three dimensional fixed point set. By Theorem 4.11 again this implies that M is diffeomorphic to a lens space.

It is clear that the T^1-action on $\tilde{M}$ is semi-free with fixed point set a totally geodesic S^3, and the orbit space $\tilde{M}/T^1$ is homeomorphic to D^4. Note that Γ acts freely on $\partial(\tilde{M}/T^1) = S^3$. Therefore Γ acts on D^4 with a unique fixed point in the interior, saying $0 \in D^4$, which is a principal orbit for the T^1-action on $\tilde{M}$. If the commutativity fails, then there is a generator $\gamma \in \Gamma$ such that $\rho(\gamma) \in \mathrm{Aut}(T^1)$ is given by the inverse automorphism. Let $T^1 \cdot x_0$ denote the principal T^1-orbit over $0 \in D^4$. Since $\gamma(x_0) \neq x_0$, let $t \in T^1$ satisfy the equation $t^2 x_0 = \gamma x_0$. Then $x = t x_0$ satisfies the equation $\gamma x = \gamma t x_0 = \rho(\gamma)(t)\gamma x_0 = t^{-1}\gamma x_0 = x$. A contradiction, since γ acts freely on $T^1 \cdot x_0$.

Case (iii). The T^2-action has only isolated singular orbits but has finite order isotropy groups;

Assume that $\mathbb{Z}_p \subset T^2$ is an isotropy group of order p, whose fixed point set $\tilde{M}^{\mathbb{Z}_p}$ is a totally geodesic 3-sphere. Note that Γ preserves $\mathbb{Z}_p$ and it acts freely on $\tilde{M}^{\mathbb{Z}_p}$. By Theorem 4.4 the T^2-action has three isolated singular orbits, and the orbit space $\tilde{M}/T^2$ is a homotopy 3-sphere. We first claim that *the Γ-action and the T^2-action commute* and therefore T^2 acts on M. We argue by contradiction. Suppose not, there is a nontrivial Γ-action on the orbit space $\tilde{M}/T^2$ acting transitively on the set of three singular orbits (by Lemma 6.6). Therefore, there is an element $\gamma \in \Gamma$ which moves the three points transitively. Note that the image of $\tilde{M}^{\mathbb{Z}_p}/T^2$ is an interval, $[0, 1]$, connecting two singular orbits. Let $\rho : \Gamma \to \mathrm{Aut}(T^2)$ denote the holonomy. By Lemma 6.6 $\rho(\gamma)$ is non-trivial. Note that $\mathbb{Z}_p$, $\rho(\gamma)(\mathbb{Z}_p)$, $\rho(\gamma^2)(\mathbb{Z}_p)$ are all isotropy groups of the T^2-action on $\tilde{M}$ such that they have pairwisely different fixed point set. However, since any two of $\mathbb{Z}_p$, $\rho(\gamma)(\mathbb{Z}_p)$,$\rho(\gamma^2)(\mathbb{Z}_p)$ generate the same subgroup isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$, of rank 2 in T^2, so the fixed point set of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ is the union of three isolated singular circle orbits in $\tilde{M} \approx S^5$. A contradiction, by Theorem 4.3.

By quotient away the finite order isotropy group $\mathbb{Z}_p$, we obtain a $T^2/\mathbb{Z}_p$-action on the quotient manifold $M/\mathbb{Z}_p$, and $\tilde{M}/\mathbb{Z}_p$. Without loss of the generality we may assume that $T^2/\mathbb{Z}_p$ is pseudofree on $\tilde{M}/\mathbb{Z}_p$ (indeed, there are at most two finite isotropy groups). Applying Theorem 7.1 to $M/\mathbb{Z}_p$ we conclude that $M/\mathbb{Z}_p$ is homeomorphic to a lens space. Note that $M^{\mathbb{Z}_p}$ is also a lens space whose compliment is homotopy equivalent to the unique singular orbit outside $M^{\mathbb{Z}_p}$. Because the branched covering of a 5-dimensional lens space along a 3-dimensional lens space with compliment a homotopy circle is again a lens space (using the same fact about the gluing along $S^1 \times S^3$ we discussed in the proof of Theorem 7.1), the desired result follows. □

As we noticed in Section 3, Theorem 3.2 implies Theorem F. The goal of this section is to prove Theorem 3.2. Recall that the algebraic conditions on the fundamental group $\pi_1(M)$ in Theorem 3.2 is essentially as follows:

(9.0.1) Any index ≤ 2 normal subgroup $\Gamma \lhd \pi_1(M)$ has a center $C(\Gamma)$ of index at least w.

(9.0.2) Any index ≤ 2 normal subgroup $\Gamma \lhd \pi_1(M)$ is not a spherical 3-space group.

In fact (9.0.2) may be replaced by Γ is neither cyclic, nor generalized quaternionic group, and binary dihedral group.

Lemma 9.1.

Let the assumptions be as in Theorem 3.2. Then
(9.1.1) Each singular stratum of the T^1-action is a circle and $\tilde X = \tilde M/T^1$ is a simply connected orbifold with only isolated singularities.
(9.1.2) $H_2(\tilde M;\mathbb{Z})$ is torsion free and has rank equal to $b_2(\tilde X) - 1$.

Proof of Lemma 9.1.

(9.1.1) If there is a nontrivial subgroup $\mathbb{Z}_p \subset T^1$ with a fixed point component $\mathrm{Fix}(\mathbb{Z}_p, \tilde M)$ of dimension 3, note that the fixed point set $\mathrm{Fix}(\mathbb{Z}_p, \tilde M)$ is invariant by the free $\pi_1(M)$-action. Thus $\pi_1(M)$ is isomorphic to the fundamental group of a 3-dimensional spherical space form, by Theorem 4.13, and therefore, it has a cyclic subgroup of index at most two by [Wo]. A contradiction to the algebraic condition (9.0.1).

(9.1.2) First, $H_2(\tilde M;\mathbb{Z}) \cong \pi_2(\tilde M)$ (by the Hurewicz theorem). Let $\tilde M_0$ denote the union of all principal T^1-orbits. By (7.3.1) and the transversality, $\pi_i(\tilde M) \cong \pi_i(\tilde M_0)$, $i = 1, 2$. Because $\tilde X_0 = \tilde M_0/T^1$ is obtained by possibly removing some isolated points, the exceptional T^1-orbits, by Lemma 2.5 $\tilde X_0$ is simply connected. Since every singularity in $\tilde X$ is a conical point whose neighborhood in $\tilde X$ is a cone over a lens space $S^3/\mathbb{Z}_p$, it is easy to see that $H_2(\tilde X;\mathbb{Z}) = H_2(\tilde X_0;\mathbb{Z}) \cong \pi_2(\tilde X_0)$ (the last isomorphism is from Hurewicz theorem). From the homotopy exact sequence of the fibration,

$$1 \to \pi_2(\tilde M_0) \to \pi_2(\tilde X_0) \to \mathbb{Z} \to 1,$$

it suffices to show that $H_2(\tilde X_0;\mathbb{Z})$ is torsion free. This is true because $H_2(\tilde X_0;\mathbb{Z}) \cong H^2(\tilde X_0, \partial \tilde X_0;\mathbb{Z})$ (the Lefschetz duality) whose torsion is isomorphic to the torsion of $H_1(\tilde X_0, \partial \tilde X_0;\mathbb{Z}) = 0$ (the universal coefficient theorem). □

b. Estimate the Euler characteristic of $\tilde X$.

To determine the topology of $\tilde M$, we will first estimate the Euler characteristic of $\tilde X$ using the constraint on the fundamental groups.

Proposition 9.2.

Let the assumptions be as in Theorem 3.2. Then $\chi(\tilde X) = 2 + b_2(\tilde X) \le 5$ or equivalently, $b_2(\tilde X) = b_2(\tilde M) + 1 \le 3$.

Let $p : \tilde{M} \to \tilde{X} = \tilde{M}/T^1$ denote the orbit projection. Then $\pi_1(M)$ acts on $\tilde{X}$ by isometries. For $\gamma \in \pi_1(M)$, we will use $\bar{\gamma}$ to denote the isometry on $\tilde{X}$ induced by γ (cf. §2). By Theorem 4.8, $F(\bar{\gamma}, \tilde{X}) \neq \emptyset$, if γ commutes with the T^1-action. To estimate the characteristic of $\tilde{X}$, we will first estimate the number of isolated $\bar{\gamma}$-fixed points (see Theorem 4.1) via the technique of q-extent estimate ([GM], [Ya]).

The q-extent $xt_q(X)$, $q \geq 2$, of a compact metric space (X, d) is, by definition, given by the following formula:

$$xt_q(X) = \binom{q}{2}^{-1} \max\Big\{ \sum_{1\leq i<j\leq q} d(x_i, x_j) : \{x_i\}_{i=1}^q \subset X \Big\}$$

Given a positive integer n and integers $k, l \in \mathbb{Z}$ coprime to n, let $L(n; k, l)$ be the 3-dimensional lens space, the quotient space of a free isometric $\mathbb{Z}_n$-action on S^3 defined by

$$\psi_{k,l} : \mathbb{Z}_n \times S^3 \to S^3;\ g(z_1, z_2) = (\omega^k z_1, \omega^l z_2)$$

with $g \in \mathbb{Z}_n$ a generator, $\omega = e^{i\frac{2\pi}{n}}$ and $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$.

Note that $L(n; k, l)$ and $L(n; -k, l)$ (resp. $L(n; l, k)$) are isometric (cf. [Ya] p.536). Obviously $L(n; -k, l)$ and $L(n; n - k, l)$ are isometric. Therefore, up to isometry we may *always* assume $k, l \in (0, n/2)$ without loss of generality. The proof of Lemma 7.3 in [Ya] works identically for $L(n; k, l)$ with $0 < k, l < n/2$ to prove

Lemma 9.3 ([Ya]).
Let $L(n; k, l)$ be a 3-dimensional lens space of constant sectional curvature one. Then

$$xt_q(L(n; k, l)) \leq arccos\Big\{ cos(\alpha_q) cos\ \pi n^{-\frac{1}{2}} - \frac{1}{2}\{(cos\ \pi n^{-\frac{1}{2}} - cos\ \pi/n)^2 + sin^2(\alpha_q)(n^{\frac{1}{2}} sin\ \pi/n - sin\ \pi n^{-\frac{1}{2}})^2\}^{\frac{1}{2}} \Big\}$$

where $\alpha_q = \pi/(2(2 - [(q + 1)/2]^{-1}))$.

Corollary 9.4.
Let $L(n; k, l)$ be a 3-dimensional lens space of constant sectional curvature one. If $n \geq 61$, then $xt_5(L(n; k, l)) < \pi/3$.

Corollary 9.5.
If the exponent of $\bar{\gamma}$ is at least 61, then $F(\bar{\gamma}, \tilde{X})$ contains at most five isolated fixed points.

Proof. We argue by contradiction, assuming $\bar{x}_1, ..., \bar{x}_6$ are six isolated $\bar{\gamma}$-fixed points. Let $\bar{X} = \tilde{X}/\langle\bar{\gamma}\rangle$. Connecting each pair of points by a minimal geodesic in $\bar{X}$, we obtain a configuration consisting of twenty geodesic triangles. Because $\bar{X}$ has positive curvature in the comparison sense ([Pe]), the sum of the interior angles of each triangle is $> \pi$ and thus the sum of total angles of the twenty triangles, $\sum \theta_i > 20\pi$. We then estimate the sum of the total angles in the following way, first estimate from above of the ten angles around each $\bar{x}_i$ and then sum up over the six points. We claim that the sum of angles at $\bar{x}_i$ is bounded above by $10 \cdot xt_5(\bar{x}_i) \leq 10\frac{\pi}{3}$ and thus $\sum \theta_i \leq 6(10 \cdot \frac{\pi}{3}) = 20\pi$, a contradiction.

Let $\tilde{x}_i \in \tilde{M}$ such that $p(\tilde{x}_i) = \bar{x}_i$, and let $\tilde{t} \in T^1$ such that $\tilde{t} \cdot \gamma$ fixes $T^1(\tilde{x}_i)$. Let $S^{\perp}_{\tilde{x}_i}$ denote the unit 3-sphere in the normal space of $T^1(\tilde{x}_i)$. If the isotropy group at $\tilde{x}_i$ is trivial, then the space of directions at $\bar{x}_i$ in $\bar{X}$ is isometric $S^{\perp}_{\tilde{x}_i}/\langle \tilde{t} \cdot \gamma \rangle$ which is a lens space with a fundamental group of order $|\bar{\gamma}|$. By Corollary 9.4, we conclude that the sum of the ten angles is bounded above by

$$\binom{5}{2} xt_5(L) = 10 \cdot \frac{\pi}{3}.$$

If the isotropy group at $\tilde{x}_i$ is not trivial, then the above estimate still holds because the 5-extent only gets smaller when passing to the quotient of $S^{\perp}_{\tilde{x}_i}$ by the isotropy group. □

Lemma 9.6.

Let the assumptions be as in Theorem E. Then the subgroup of $\pi_1(M)$ acting trivially on $H^*(\tilde{X};\mathbb{Z})$ has order $\geq |\pi_1(M)|/k_0$, where k_0 is a universal constant depending only on the Gromov constant $b = b(4)$ (cf. 4.14).

Proof. Let $\rho : \pi_1(M) \to \mathrm{Aut}(H^*(\tilde{X};\mathbb{Z}))$ denote the homomorphism induced by the $\pi_1(M)$-action on $H^*(\tilde{X};\mathbb{Z})$. As seen in the proof of Lemma 9.1, $H^*(\tilde{X};\mathbb{Z}) \cong \mathbb{Z}^{\ell}$, where $\ell = b_2(\tilde{X}) + 2 = b_2(\tilde{M}) + 3 \leq b$. Because the order of the torsion subgroup of $\mathrm{Aut}(H^*(\tilde{X};\mathbb{Z}))$ is bounded above by a constant depending only on b ([Th]), say k_0, the conclusion follows. □

Proof of Proposition 9.2.

Because $\tilde{X}$ is a simply connected orbifold with isolated singularities (Lemma 9.1), we can apply the Poincaré duality on homology groups with rational coefficients to conclude that $\chi(\tilde{X}) = 2 + b_2(\tilde{X})$. Let Γ_0 be the principal isotropy group of $\pi_1(M)$ on $\tilde{X}$. By 4.10 Γ_0 is cyclic and belongs to the center of a certain index at most two normal subgroup (the subgroup of orientation preserving isometries in $\pi_1(M)$). Under the assumption of Theorem 3.2, i.e., (9.0.1), the index $[\pi_1(M) : \Gamma_0] \geq w$. By Lemma 9.6, the normal subgroup $G \subset \pi_1(M)/\Gamma_0$, acting trivially on $H^*(\tilde{X};\mathbb{Z})$ has order $\geq w/k_0$.

For any $\bar{\beta} \in G$, by Theorem 4.8 the fixed point set $F(\bar{\beta}, \tilde{X}) \neq \emptyset$ and by Theorem 4.13 we may assume that $F(\bar{\beta}, \tilde{X})$ is a finite set (otherwise $\tilde{M}$ contains a 3-dimensional totally geodesic submanifold, and so Theorem 4.13 implies that it is a homotopy sphere). If $\bar{\beta}$ is of prime order and $|\bar{\beta}| \geq 61$, by Theorem 4.1

$$\chi(\tilde{X}) = \chi(F(\bar{\beta}, \tilde{X})).$$

Therefore, by Corollary 9.5 we conclude $\chi(\tilde{X}) \leq 5$.

Now we assume that $|G|$ has all prime factors ≤ 60. Since there are at most 17 primes less than 60, there is a prime $p \leq 60$ so that G has a p-Sylow subgroup G_p of order $\geq \frac{w}{17k_0}$. Let $\bar{\beta}_0 \in G_p$ be an element of order p generating a normal subgroup of G_p (by group theory it always exists), we may assume that $\bar{\beta}_0$ has only isolated fixed points for the same reasoning as above, say $p_1, \cdots, p_n$. By Theorem 4.1, $n = \chi(F(\bar{\beta}_0, \tilde{X})) = \chi(\tilde{X})$. Observe that $n \leq b = b(4)$. Now G_p acts on the fixed point set $F(\bar{\beta}_0, \tilde{X})$, thus we get a homomorphism $h : G_p \to S_n$, where S_n is the permutation group of n-words. Therefore, the kernel of h has order at least $\frac{w}{17k_0 \cdot n!} \geq \frac{w}{17k_0 \cdot b!}$. By Gromov $\pi_1(M)$ may be generated by a bounded number of generators, so is G_p, generated by c elements. Hence $\ker(h)$ contains an element, say $\bar{\beta}_1$, of order at least 61, if $\frac{w}{17k_0 \cdot b!}$ is sufficiently large. Since $\bar{\beta}_1$ fixes the set $\{p_1, \cdots, p_n\}$ pointwisely, by Corollary 9.5 $n \leq 5$. Therefore, $\chi(\tilde{X}) = n \leq 5$. $\square$

c. The completion of the proof of Theorem 3.2.

Lemma 9.7.

Let M be as in Theorem 3.2. Then the T^1-action on $\tilde{M}$ is free.

Proof. We argue by contradiction, assuming the T^1-action is not free. Then there is at least a finite isotropy group $\mathbb{Z}_p \subset T^1$. By Theorem 4.13 we may assume that $\dim(F(\mathbb{Z}_p, \tilde{M})) = 1$, since otherwise, $\tilde{M}$ contains a totally geodesic 3-manifold. Then $F(\mathbb{Z}_p, \tilde{M})$ consists of at most two components (circles), if $b_2(\tilde{M}) \leq 1$, or three components if $b_2(\tilde{M}) \leq 2$ (Theorem 4.2).

Let H denote the subgroup of $\pi_1(M)$ preserving all components of $F(\mathbb{Z}_p, \tilde{M})$. Then H is a cyclic normal subgroup such that the quotient $\pi_1(M)/H$ acting effectively on the set of exceptional orbits. If $b_2(\tilde{M}) \leq 1$, by counting the number of components $\pi_1(M)/H$ has order at most 2. A contradiction to (9.0.1).

If $b_2(\tilde{M}) = 2$, and $F(\mathbb{Z}_p, \tilde{M})$ has exactly three components, it is easy to see that H acts on $\tilde{X} = \tilde{M}/T^1$ has at most five isolated fixed points, three of them are the exceptional T^1-orbits with isotropy group $\mathbb{Z}_p$. Because H is normal in $\pi_1(M)$, thus $\pi_1(M)$ acts on the fixed point set and it sends an exceptional orbit to an exceptional orbit. Therefore, $\pi_1(M)$ acts on the union of the rest at most two isolated T^1-orbits fixed by H. This implies once again that $\pi_1(M)$ has an index ≤ 2 cyclic normal subgroup. The desired result follows. $\square$

Lemma 9.8.

Let M be a closed 5-manifold of positive sectional curvature which admits a π_1-invariant isometric T^1-action. If $\pi_1(M)$ satisfies (9.0.1), then the induced $\pi_1(M)$-action on $\tilde{M}/T^1$ is pseudo-free.

Proof. We need only to show that every singular point of the $\pi_1(M)$-action is isolated. We argue by contradiction, assuming an element $\gamma \in \pi_1(M)$ with a fixed point component F of dimension two in $\tilde{M}/T^2$. By Lemma 9.7, the T^1-action is free. Then its pre-image $\tilde{F} = p^{-1}(F) \subset \tilde{M}$, is a fixed point component of $t\gamma$ of dimension three, for some $t \in T^1$. Because $\tilde{F} \hookrightarrow \tilde{M}$ is 2-connected (Theorem 4.13), $\pi_2(\tilde{M}) \cong \pi_2(\tilde{F}) = 0$ and thus $\tilde{M}$ is a homotopy sphere. From the homotopy exact sequence of the fibration, $T^1 \to \tilde{M} \to \tilde{M}/T^1$, it is clear that $\tilde{M}/T^1$ is a homotopy complex projective plane and thus homeomorphic to $\mathbb{C}P^2$ ([Fr]).

Note that γ has a unique isolated fixed point $\bar{x}$, because $\tilde{M}/T^1$. We claim that
(9.8.1) The normalizer of $\langle\gamma\rangle$ is cyclic, say $\langle\beta\rangle$.
(9.8.2) For any $\eta \in \pi_1(M)$, $\eta^{-1}\gamma\eta$ commutes with γ and thus $\eta^{-1}\gamma\eta \in \langle\beta\rangle$.

Because $\langle\beta\rangle$ has a unique subgroup of order $|\gamma|$, (9.8.2) implies that $\langle\gamma\rangle$ is a normal subgroup in $\pi_1(M)$, and thus by (9.8.1) $\pi_1(M)$ is cyclic since then $\pi_1(M)$ acts freely on the unique circle orbit, a contradiction. Because the normalizer of $\langle\gamma\rangle$

fixes $\bar{x}$, this proves (9.8.1). Because $\eta(\bar{F}) \cap \bar{F} \neq \emptyset$ (Theorem 4.13), there is $\bar{y} \in \bar{F}$ such that $\gamma(\eta(y)) = \eta(\bar{y})$ and thus $\eta^{-1}\gamma\eta(\bar{y}) = \bar{y}$, and this proves (9.8.2). $\square$

Lemma 9.9.

Let M be as in Theorem 3.2. Then $\tilde{M}$ is a homotopy sphere.

Proof. Because the T^1-action on $\tilde{M}$ is free (Lemma 9.7), $\tilde{X}$ is a closed simply connected smooth 4-manifold of positive sectional curvature. Because $b_2(\tilde{X}) = b_2(\tilde{M}) + 1 \geq 1$, by Proposition 9.2, $3 \leq 2 + b_2(\tilde{X}) = \chi(\tilde{X}) \leq 5$.

If $\chi(\tilde{X}) = 3$, then $b_2(\tilde{M}) = b_2(\tilde{X}) - 1 = 0$. This together with (9.1.2) implies that $\tilde{M} \approx S^5$.

It suffices to rule out the cases $\chi(\tilde{X}) = 4$ and $\chi(\tilde{X}) = 5$.

Case (a) If $\chi(\tilde{X}) = 4$;

Note that $\tilde{X}$ is homeomorphic to $S^2 \times S^2$, $\mathbb{C}P^2\#\mathbb{C}P^2$ or $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, up to a possible orientation reversing ([Fr]). By the classification of simply connected 5-manifolds (cf. [Ba]) $\tilde{M} = S^2 \times S^3$ or $S^2\tilde{\times}S^3$. Because $\pi_1(M)$ preserves the Euler class of the principal circle bundle $T^1 \to \tilde{M} \to \tilde{X}$, the natural homomorphism given by the $\pi_1(M)$-action on the second homology group, $\alpha : \pi_1(M) \to \mathrm{Aut}(H^2(\tilde{X};\mathbb{Z});I)$, where $\mathrm{Aut}(H^2(\tilde{X};\mathbb{Z});I)$ is the automorphism group preserving the intersection form.

Let Γ_0 (resp. Γ) be the principal isotropy group of $\pi_1(M)$-action on $\tilde{X}$ (resp. orientation preserving subgroup of $\pi_1(M)$). Recall that $\Gamma \subset \pi_1(M)$ is a normal subgroup of index at most 2, and Γ_0 is in the center of Γ which generates a cyclic subgroup with any element of Γ (cf. 4.10) Let $G = \Gamma/\Gamma_0$. The homomorphism α reduces to a homomorphism $\bar{\alpha} : G \to \mathrm{Aut}(H^2(\tilde{X};\mathbb{Z});I)$. It is easy to see that the image of $\bar{\alpha}$ has order at most 2 (cf. [Mc2]).

Subcase (a1) If $\ker(\bar{\alpha})$ has odd order;

Consider the action of $\ker(\bar{\alpha})$ on $\tilde{X}$. Observe that the action is pseudo-free, and every isotropy group is cyclic. By [Mc2] Lemma 7.5 and the first paragraph in the proof of Lemma 4.7 [Mc2] (which identically extends to our case) it follows that, $\ker(\bar{\alpha})$ is cyclic of odd order. If $Im(\bar{\alpha})$ is not trivial and $\tilde{X} \approx S^2 \times S^2$, by [Bre] VII Corollary 7.5 there is an involution on X with a 2-dimensional fixed point set. A contradiction by Lemma 9.8. If $\tilde{X} \approx \mathbb{C}P^2\#\mathbb{C}P^2$ or $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$, in the former case every self homeomorphism of $\tilde{X}$ is orientation preserving, and for the latter $Im(\bar{\alpha}) = 0$. Therefore, in either cases, $\pi_1(M)$ has an image in $\mathrm{Aut}(H^2(\tilde{X};\mathbb{Z});I)$ of order at most 2, and by Corollary 4.10 it contains a normal cyclic subgroup of index at most 2, a contradiction to (9.0.1).

Subcase (a2) If $\ker(\bar{\alpha})$ has even order;

By [Bre] VII Lemma 7.4 any involution in $\ker(\bar{\alpha})$ has 2-dimensional fixed point set on $\mathbb{C}P^2\#\mathbb{C}P^2$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$. Therefore, by Lemma 9.8 we may assume that $\tilde{X} = S^2 \times S^2$. By [Mc2] Theorem 3.3 $\ker(\bar{\alpha})$ is polyhedral. Therefore $\ker(\bar{\alpha})$ is either cyclic, dihedral, or is a non-abelian group of order of order 12 (Tec. and two others) or $\ker(\bar{\alpha})$ is Oct. (of order 24), or Icos. (of order 60). Thus, $\ker(\bar{\alpha})$ is cyclic or dihedral if the order $|G| > 120$. We may require our constant w in Theorem E is larger than 120. By [Mc2] Theorems 3.9 and 4.10 G is listed as follows:

(9.9.1) a cyclic, or dihedral group;

(9.9.2) $Q_{2^km} \times \mathbb{Z}_n$; where Q_{2^km} the generalized quaternionic group, m, n are coprime odd integers;

(9.9.3) $D_{2^k m} \times \mathbb{Z}_n$, where $D_{2^k m} = \mathbb{Z}_m \rtimes \mathbb{Z}_{2^k}$ and $\mathbb{Z}_{2^k}$ acts on $\mathbb{Z}_m$ by inverse automorphism, m, n are coprime odd integers, and $k \geq 2$;

(9.9.4) A non-splitting extension of a dihedral group by $\mathbb{Z}_2$.

Since Γ is a center extension of a cyclic group by G, by Lemma 9.10 below the group Γ satisfies $2p$-condition, i.e., for any prime p, a subgroup of order $2p$ is cyclic (cf. [Mi]). By group extension theory, it is not hard to verify, the dihedral groups in (9.9.1), (9.9.2) and (9.9.4) must be reduced from a quaternionic subgroup of Γ, and the group $D_{2^k m}$ in (9.9.3) is reduced from $D_{2^{k'} m}$, where $k' \geq 2$. Moreover, for G in (9.9.1), (9.9.2) and (9.9.3), Γ is isomorphic to a 3-dimensional spherical space form group (compare [Mi] Theorem 2), and so $\pi_1(M)$ contains an index at most 2 subgroup isomorphic to a 3-dimensional spherical space form group, a contradiction by the assumption. If G is a group in (9.9.4), and $[\pi_1(M) : \Gamma] = 2$, there is a *maximal* cyclic subgroup $\langle\gamma\rangle$ of $\pi_1(M)$ of index 8. Therefore, $\pi_1(M)$ acts on the 4 fixed points of γ on $S^2 \times S^2$ without any even order isotropy group by the maximality of $\langle\gamma\rangle$. A contradiction, since the permutation group S_4 does not contain cyclic subgroup of order 8. This shows that $\Gamma = \pi_1(M)$, and it contains a quaternionic subgroup of index 2. A contradiction again to the assumption.

Case (b) If $b_2(\tilde{X}) = 3$;

Since the Euler characteristic $\chi(\tilde{X})$ is odd, by [Bre] VII Corollary 7.6 any involution on X has a 2-dimensional fixed point set. Therefore, by Lemma 9.8 we may assume that $\pi_1(M)/\Gamma_0$ has odd order. In particular, $\pi_1(M) = \Gamma$, i.e. any element of $\pi_1(M)$ preserves the orientation of $\tilde{X}$. For the homomorphism $\bar{\alpha} : G \to \mathrm{Aut}(H_2(\tilde{X}; \mathbb{Z}))$, by [Mc1] the kernel $G_0 = \ker(\bar{\alpha})$ is an abelian subgroup of T^2 with non-empty fixed point set. Therefore, G_0 must be cyclic by combining Lemma 9.8, otherwise, there exists a cyclic subgroup of G_0 with a 2-dimensional fixed point set. Therefore $\pi_1(M)/G_0$ is isomorphic to an odd order subgroup of the automorphism group $\mathrm{Aut}(H_2(\tilde{X}; \mathbb{Z})) = \mathrm{GL}(\mathbb{Z}, 3)$. By [Th] the finite subgroup of $\mathrm{GL}(\mathbb{Z}, 3)$, up to possible 2-torsion, is a subgroup of $\mathrm{GL}(\mathbb{Z}_2, 3)$ which has order $(2^3 - 1)(2^3 - 2)(2^3 - 4)$, which is coprime to 5. Note that $\pi_1(M)$ preserves the fixed point set $\mathrm{Fix}(G_0, \tilde{X})$, which consists of 5 isolated points. Therefore $\pi_1(M)/G_0$ is an odd order subgroup of S_5 and $\pi_1(M)/G_0 \neq \mathbb{Z}_5$. This implies that $\pi_1(M)$ fixes at least 2 points among $\mathrm{Fix}(G_0, \tilde{X})$ and so $\pi_1(M)$ is cyclic. A contradiction. □

Lemma 9.10.
Let $S^2 \times S^3 \to S^2 \times S^2$ be a G-equivariant principal T^1-bundle. If G acts freely on $S^2 \times S^3$, pseudofreely on $S^2 \times S^2$. Then there is no order 2 element of G acting non-trivially on $S^2 \times S^2$ inducing the identity on homology groups.

Proof. If not, for an order 2 element β, its fixed point set F in $S^2 \times S^2$ consists of 4 points (by Theorem 4.1). For such a fixed point $[x] \in S^2 \times S^2$ with $x \in S^2 \times S^3$, there is an element $t \in T^1$ so that $\beta x = tx$. The freeness of T^1-action and β-action implies that the order of t is the same as β, of order 2. Since T^1 has only one element of order 2, say t_0, this proves that βt_0^{-1} has fixed point set the union of 4 circles. A contradiction by Theorem 4.2. □

Appendix: s-cobordism theory

In the proof of Theorem E for pseudofree T^2-action (Section 7) we assumed the Poincaré conjecture holds. The goal of this appendix is to give an account how this assumption can be removed by using the s-cobordism theory.

We call two closed n-manifolds M_1 and M_2 are *s-cobordant* if there is an $(n+1)$-manifold W with boundary $M_1 \sqcup M_2$ so that the inclusions $i_1 : M_1 \to W$ and $i_2 : M_2 \to W$ are both simple homotopy equivalences (cf. [Ke]). The manifold W is called an *s-cobordism.* The deep s-cobordism theorem (originally due to Smale for simply connected manifold, and extended to non-simply connected case by Barden-Mazur-Stallings, cf. [Ke] for a detailed account) asserts that an s-cobordism is diffeomorphic (or homeomorphic if manifolds are TOP.) to the product $M_1 \times [0,1]$, provided $n \geq 5$. Hence, M_1 and M_2 are diffeomorphic (resp. homeomorphic). The dimension assumption is crucial. In fact, counterexamples exist for $n = 3, 4$, by Cappell-Shaneson, and Donaldson's theory. An s-cobordism theorem for simply connected 4-manifold (i.e. $n = 3$) would imply the 3-dimensional Poincaré conjecture.

It is a well-known fact that every simply connected 3-manifold is s-cobordant (by terminology should be called h-cobordant) to S^3. The key issue of our solution in the proof Theorem E without assuming the Poincaré conjecture is to replace the homeomorphism (equivalently diffeomorphism) by s-cobordism. If we could obtain an s-cobordism between our 5-manifold M with a spherical space form, the s-cobordism theorem applies to imply

our desired result. Note that the dimension shifting is important and this can be obtained by using the additional π_1-invariant T^2-action on the manifold.

Definition A.1.

Let M_1 (resp. M_2) be a closed 5-manifold with a smooth action by $T^2 \rtimes G$ where the T^2-action is pseudofree and the G-action is free. We call that W is an s-cobordism between $(M_1, T^2 \rtimes G)$ and $(M_2, T^2 \rtimes G)$ if

(A.1.1) W is an s-cobordism between M_1 and M_2;

(A.1.2) There is an action by $T^2 \rtimes G$ on W whose restriction on M_1 (resp. M_2) gives $(M_1, T^2 \rtimes G)$(resp. $(M_1, T^2 \rtimes G)$), and the T^2-action is pseudofree and the G-action is free.

By the s-cobordism theory (cf. [Ke]), W is homeomorphic to the product $M_i \times [0,1]$ when $\dim(M_i) \geq 5$. However, the induced T^2-action on $M_i \times [0,1]$ may not preserve the slice $M_i \times \{t\}$ $(i = 1, 2)$, $t \in (0,1)$, so that one may not conclude that (M_1, T^2) is conjugate to (M_2, T^2).

As we have seen before, if T^2 acts pseudofree on S^5, the orbit space S^5/T^2 is a manifold homotopy equivalent to S^3. Hence, if W is an s-cobordism between two pseudofree T^2-actions on S^5, obviously, the orbit space W/T^2 gives an s-cobordism between the two orbit spaces (two homotopy 3-spheres) of the actions on S^5. We will show below the converse also holds.

Lemma A.2.

A pseudo-free T^2-action on S^5 is s-cobordant to a pseudo-free linear T^2-action.

Proof. As in Section 7 we make use the convention that T_0^2 indicates a linear T^2-action on S^5. Let V be an s-cobordism of the homotopy 3-sphere $S^5/T^2 := \Sigma$

and $S^5/T_0^2 = S^3$. By Theorem 4.4 T^2 (resp. T_0^2) has three isolated circle orbits, denoted by $T_i^1 \times D^4$ (resp. $\hat{T}_i^1 \times D^4$) respectively. Let p_i (resp. q_i) denote the projections of these circle orbits in S^5/T^2 (resp. S^5/T_0^2), $i = 1, 2, 3$, respectively. Connecting p_i and q_i by a simple disjoint paths in V, $i = 1, 2, 3$, let V_0 denote the complement of the three simple paths. Observe that V_0 is homotopy equivalent to $S_0^3 = S^3 - q_1 \cup q_2 \cup q_3$ (resp. $\Sigma_0 = \Sigma_0 - p_1 \cup p_2 \cup p_3$). It is easy to see that $H^2(S_0^3; \mathbb{Z}^2) \cong \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z}^2)$. Consider the Euler classes of principal T^2-bundles on Σ_0 and S_0^3 as 2×2 matrices in the above group. Hence, the determinants of the matrices are ± 1 because the 2-connectedness of the complement of the singular orbits. Therefore, modifying by an automorphism of T^2 if necessary (in other words, up to weak equivalence), we may assume that the two Euler classes are homotopic, regarded as maps to the classifying space BT^2. The homotopy gives a map $h : V_0 \to BT^2$ whose restrictions on S_0^3 and Σ_0 correspond to the classifying maps of the principal T^2-bundles. Let W_0 denote the total space of the T^2-bundle over V_0, which is a 6-manifold with boundary.

Finally, we can attach equivariantly three copies of $T^1 \times D^4 \times [0, 1]$ to W_0 along the boundaries, so that the three copies of $T^1 \times D^4 \times \{0\}$ fill the three singular orbits in S^5. This gives a T^2-manifold W with boundary $S^5 \sqcup S^5$, which yields the desired s-cobordism between (S^5, T^2) and (S^5, T_0^2). $\square$

In Lemmas 7.3 and 7.4 the induced Γ-action on the homotopy 3-sphere Σ (the orbit space) may not be trivial. We need to find an equivariant s-cobordism between (Σ, Γ) with a standard linear action on (S^3, Γ). Recently, a deep result (cf. [BLP]) on 3-orbifold implies that every finite group acting non-freely on S^3 is conjugate to a linear action. Let Γ be a finite group. For a smooth or locally linear nonfree

Γ-action on a homotopy 3-sphere (possibly reducible) Σ, the result of [BLP] implies that (Σ, Γ) is Γ-equivariantly diffeomorphic to $(S^3, \Gamma) \# (|\Gamma| \cdot \Sigma_0, \Gamma)$, where Γ acts linearly on S^3, and acts freely on $|\Gamma| \cdot \Sigma_0$ and Σ_0 is a homotopy 3-sphere (we thank Porti for pointing out this fact to us). Therefore, it is easy to get a 4-dimensional s-cobordism (V, Γ) between (Σ, Γ) and (S^3, Γ). Indeed, we may take $(V, \Gamma) = (S^3 \times [0, 1], \Gamma) \natural (|\Gamma| \cdot B_0, \Gamma)$, where B_0 is a contractible 4-manifold with boundary Σ_0, and $\natural$ is the boundary connected sum along the boundary piece $S^3 \times \{1\}$. An exceptional case is $\Gamma = \mathbb{Z}_3$ which acts freely on the homotopy 3-sphere Σ. In this case the main result of [BLP] does not apply. However, it is easy to see that $\Sigma/\mathbb{Z}_3$ is simple homotopy equivalent to the lens space $S^3/\mathbb{Z}_3$ (unique), because that the Whitehead torsion $Wh(\mathbb{Z}_3) = 0$ (cf. [Coh]). Hence, there is an s-cobordism between $S^3/\mathbb{Z}_3$ and $\Sigma/\mathbb{Z}_3$, which gives exactly a $\mathbb{Z}_3$-equivariant s-cobordism between $(S^3, \mathbb{Z}_3)$ and $(\Sigma, \mathbb{Z}_3)$.

Lemma A.3.

The Γ-equivariant principal T^2-bundle over Σ_0 in Lemmas 7.3 and 7.4 is Γ-equivariantly s-cobordant to a linear principal T^2-bundle over S_0^3. Hence, M_0 is diffeomorphic to the linear model S_0^5/Γ_0.

Proof. Because the proofs of Lemmas 7.3 and 7.4 involve only homotopy theory, using the Γ-equivariant s-cobordism V defined above, there is a well-defined Γ-equivariant principal T^2-bundle over V_0, where V_0 is obtained by removing certain Γ-equivariant three simple paths, e.g., in the case Γ acts non-freely on Σ, just take the product $(p_1 \cup p_2 \cup p_3) \times [0, 1] \subset V$; in the case Γ acts freely on Σ, take the preimage of any simple path joining the singular point of $S^3/\mathbb{Z}_3$ and $\Sigma/\mathbb{Z}_3$. The total space of the principal T^2-bundle over V_0 gives a Γ-equivariant s-cobordism between $(\tilde{M}_0, \Gamma)$ and (S^5_0, Γ_0), which implies the diffeomorphism between S^5_0/Γ_0 and M_0. $\square$

References

[AM] U. Abresch; W. T. Meyers, *Pinching below* 1/4*, injectivity radius estimate, and sphere theorems*, J. Diff. Geom **40** (1994), 643-691.

[AW] S. Aloff; N. R. Wallach, *An infinite family of* 7*-manifolds admitting positive curved Riemannian structures*, Bull. Amer. Math. Soc. **81** (1975), 93-97.

[Bre] G. Bredon, *Introduction to compact transformation groups*, Academic Press **48** (1972).

[Bro] K. S. Brown, *Cohomology of groups*, Springer-Verlag New York (1982).

[BGP] Y. Burago; M. Gromov; Perel'man, *A.D. Alexandov spaces with curvature bounded below*, Uspekhi Mat. Nauk **47:2** (1992), 3-51.

[BLP] M. Boileau; B. Leeb; J. Porti, *Geometrization of* 3*-dimensional orbifolds*, Annals of Math. **162** (2005), 195-290.

[CFG] J. Cheeger; K. Fukaya; M. Gromov, *Nilpotent structures and invariant metrics on collapsed manifolds*, J. A.M.S **5** (1992), 327-372.

[EM] S. Eilenberg; S. MacLane, *On the groups* $H(\pi, n)$*, III*, Ann. of Math. **60** (1954), 513-557.

[FMR] F. Fang; S. Mendonca; X. Rong, *A connectedness principle in the geometry of positive curvature*, Comm. Anal. Geom. **13** (2005), 671-696.

[FR1] F. Fang; X. Rong, *Positive curvature, volume and second Betti number*, Geom. Funct. Anal. **9** (1999), 641-674.

[FR2] F. Fang; X. Rong, *The second twisted Betti numbers and the convergence of collapsing Riemannian manifolds*, Invent. Math. **150** (2002), 61-109.

[FR3] F. Fang; X. Rong, *Homeomorphic classification of positively curved manifolds with almost maximal symmetry rank*, Math. Ann **332** (2005), 81-101.

[Fr1] T. Frankel, *Manifolds of positive curvature*, Pacific J. Math. **11** (1961), 165-174.

[Fr] M. Freedman, *Topology of Four Manifolds*, J. of Diff. Geom. **28** (1982), 357-453.

[Gr2] M. Gromov, *Curvature, diameter and Betti numbers*, Commentarii Math. helvetici **56** (1981), 179-195.

[GKR] K. Grove; H. Karcher; E. Ruh, *Group actions and curvature*, Invent. Math. **23** (1974), 31-48.

[GM] K. Grove, S. Markvosen, *New extremal problems for the Riemannian recognition program via Alexandrov geometry*, J. Amer. Math. Soc. **8** (1995), 1-28.

[GS2] K. Grove, C. Searle, *Positively curved manifolds with maximal symmetry-rank*, J. Pure Appl. Alg **91** (1994), 137-142.

[GZ] K. Grove, W. Ziller, *Curvature and symmetry of Milnor spheres*, Ann. of Math. **152** (2000), 331-367.

[HL] Hambleton, I, Lee, R., *Finite group actions on* $\mathbb{C}P^2$, J. Algebra **116** (1988), 227-242.

[Ha] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. **17** (1982), 255-306.

[Hs] W. Hsiang, *Cohomology theory of topological transformation groups*, Ergebnisse der Mathematik und inere Grenzgebiete **85** (1975).

[HK] W. Hsiang, B. Kleiner, *On the topology of positively curved* 4*-manifolds with symmetry*, J. Diff. Geom **30** (1989), 615-621.

[IHR] Im Hof; E. Ruh, *An equivariant pinching theorem*, Comment. math. Helv. **50** (1975), 389-401.

[KPT] V. Kapovitch; A. Petrunin; W. Tuschmann, *Nonnegative pinching, moduli spaces and bundles with infinitely many souls*, Preprint (2005).

[Ke] M. Kervaire, *Le théorème de Barden-Mazur-Stallings*, Comment. Math. Helv. **40** (1965), 31-42.

[KS] R. Kirby; L. Siebenmann, *Foundational essays on topological manifolds, smoothings and triangulations, with notes by J. Milnor an M. Atiyah,*, Ann. Math. Studies **88** (1977, Princeton Univ. Press).

[KlS] W. Klingenberg; T. Sakai, *Remarks on the injectivity radius estimate for almost* 1/4-*pinched manifolds*, Lecture Notes in Math. Springer **1201** (1986), 156-164.

[Ko] S. Kobayashi, *Transformation groups in differential geometry*, Springer-Verlag Berlin Heidelberg New York (1972).

[Mc1] Mccooey, M, *Symmetry groups of four manifolds*, Topology **41** (2002), 835-851.

[Mc2] Mccooey, M, *Groups which acts pseudo-freely on* $S^2 \times S^2$, preprint arXiv:math.GT/9906159 (1999).

[Mi] J. Milnor, *Groups which acts on* S^n *without fixed points*, Amer. J. Math. **79** (1957), 623-630.

[Pe] T. Petrie, *Free metacyclic group actions on spheres*, Ann. of Math. **94** (1971), 108-124.

[PT] A. Petrunin; W. Tuschmann, *Diffeomorphism finiteness, positive pinching and second homotopy*, Geom. Funct. Anal. **9** (1999), 736-774.

[Ro1] X. Rong, *On the fundamental group of manifolds of positive sectional curvature*, Ann. of Math **143** (1996), 397-411.

[Ro2] X. Rong, *Positively curved manifolds with almost maximal symmetry rank*, Geom. Dedicata **95** (2002), 157-182.

[Ro3] X. Rong, *On fundamental groups of positively curved manifolds with local torus actions*, Preprint (2003).

[Se] J.P. Serre, *Linear representations of finite groups*, GTM (Springer-Verlag).

[Sh] K. Shankar, *Isometry groups of homogeneous spaces with positive sectional curvature*, Differential Geom. Appl. **14** (2001), 57-78.

[SW] R. Stong; Z. Wang, *Self-homeomorphisms of* 4*-manifolds with fundamental group* $\mathbb{Z}$, Topology. Appl. **106** (2000), 49-56.

[Sy] J. Synge, *On the connectivity of spaces of positive curvature*, Quarterly J.Math. Oxford **7** (1936), 316-320.

[Wi1] Wilczynski, D, *Periodic maps on simply connected four manifolds*, Topology **30** (1991), 55-65.

[Wi2] Wilczynski, D, *Group actions on complex projective planes*, Trans. Amer. Math. Soc. **303** (1987), 707-731.

[Wi] B. Wilking, *Torus actions on manifolds of positive sectional curvature*, Acta Math (2003), 259-297.

[Wo] J. A. Wolf, *The spaces of constant curvature*, McGraw-Hill series in higher mathematics (1976).

[Ya] D. Yang, *On the topology of non-negatively curved simply connected* 4*-manifolds with discrete symmetry*, Duke Math. J. (1994), 531-545.

[Yau] S.-T. Yau, *Open problems in differential geometry*, Proc. Sympo. in pure Math. **54** (1993), 1-28.

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