Jordan structures in harmonic functions and Fourier algebras on homogeneous spaces

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Abstract. We study the Jordan structures and geometry of bounded matrix-valued harmonic functions on a homogeneous space and their analogue, the harmonic functionals, in the setting of Fourier algebras of homogeneous spaces.

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1. Introduction

In this paper, we study two classes of complex Banach spaces, related to harmonic functions on Riemannian symmetric spaces, namely, the Banach space of bounded matrix-valued σ-harmonic functions on a homogeneous space, and its analogue, the space of harmonic functionals on the Fourier algebra of a homogeneous space. They are shown to be JB*-triples and therefore their open unit balls are symmetric manifolds. This gives rise to interesting classes of Jordan triple systems for harmonic analysis and motivates our investigation of the detailed algebraic analytic structures of these spaces.

We begin with some background. Let X be a Riemannian symmetric space of non-compact type, represented as the right coset space $H\backslash G$ of a connected semisimple Lie group G by a maximal compact subgroup H. Furstenberg [20] has shown that the bounded harmonic functions on X can be characterized by an integral equation as follows. Let Δ be the Laplace-Beltrami operator on X. Then there is an H-invariant absolutely continuous probability measure σ on G such that a bounded C^2 function f on X satisfies $\Delta f = 0$ if, and only if,

$$\widetilde{f}(Ha) = (\widetilde{f} * \sigma)(Ha) := \int_G \widetilde{f}(Hay^{-1})d\sigma(y) \qquad (Ha \in X = H\backslash G)$$

where we define $\widetilde{f}(Ha) = f(Ha^{-1})$. This motivates the definition of a harmonic function with respect to a measure in a wider context of coset spaces of locally compact groups without differential structures.

In what follows, G denotes a locally compact group with a right-invariant Haar measure λ in which case the convolution of two Borel functions f and h on G is defined

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by

$$f * h(x) = \int_G f(xy^{-1})h(y)d\lambda(y) \qquad (x \in G)$$

whenever it exits.

Let $\Omega = H\backslash G$ be a homogeneous space of G, represented as a right coset space $H\backslash G$ by a closed subgroup H. Let $x \in \Omega \mapsto x.g \in \Omega$ be the action of G on Ω. Fix a quasi-invariant measure ν on Ω (cf. [15]). We take $\nu = \lambda$ if $H = \{e\}$ is the identity subgroup. Let M_n be the C^*-algebra of $n \times n$ complex matrices, with identity I. Given an M_n-valued measure σ on G, an M_n-valued Borel function $f : \Omega \to M_n$ is called *σ-harmonic* if the convolution $f * \sigma$ exists locally ν-almost everywhere and $f(x) = f * \sigma(x)$ whenever the convolution

$$(f * \sigma)(x) = \int_G f(x.y^{-1})d\sigma(y) \quad (x \in \Omega)$$

exists. We recall that a Borel set $S \subset \Omega$ is *locally ν-null* if $\nu(S \cap K) = 0$ for every compact set $K \subset \Omega$. Let $L^\infty(\Omega, M_n)$ be the Lebesgue space of essentially bounded M_n-valued functions on Ω, with respect to the measure ν. Our first main object of study is the following Banach space of (essentially) bounded M_n-valued σ-harmonic functions on Ω:

$$H_\sigma(\Omega, M_n) = \{f \in L^\infty(\Omega, M_n) : f = f * \sigma\}.$$

In the special case of $\Omega = G$ and a complex measure σ, the space $H_\sigma(G, \mathbb{C})$ of complex σ-harmonic functions on G has been studied in [8], and the space $H_\sigma(G, M_n)$, for matrix-valued σ, was considered in [6]. Our results subsume some of those in [6,8] and are applicable to symmetric spaces. The intention of studying the matrix equation $f = f * \sigma$ is to view it as a system of convolution equations.

Our second object of study is an analogue of $H_\sigma(\Omega, \mathbb{C})$ in the setting of Fourier algebras $A(\Omega)$ of homogeneous spaces $\Omega = H\backslash G$. This is the Banach space $A(\Omega)^*_\sigma$ of *harmonic functionals* on $A(\Omega)$. The Fourier algebra $A(\Omega)$ is an ideal in the Fourier-Stieltjes algebra $B(\Omega)$ of the homogeneous space Ω and the dual $A(\Omega)^*$ identifies with a one-sided ideal of the von Neumann envelope of the group C*-algebra $C^*(G)$. There is a natural action $(\sigma, T) \in B(\Omega) \times A(\Omega)^* \mapsto \sigma \cdot T \in A(\Omega)^*$. Given $\sigma \in B(\Omega)$, the space $A(\Omega)^*_\sigma$ is defined by

$$A(\Omega)^*_\sigma = \{T \in A(\Omega)^* : T = \sigma \cdot T\}.$$

If G is abelian, the right coset space $\Omega = H\backslash G$ is a group. Given a complex measure σ on the dual group $\widehat{\Omega}$, its Fourier transform $\widehat{\sigma}$ belongs to $B(\Omega)$ and the space $H_{\widetilde{\sigma}}(\widehat{\Omega}, \mathbb{C})$ is isometrically isomorphic to $A(\Omega)^*_{\widehat{\sigma}}$, where $d\widetilde{\sigma}(x) = d\sigma(x^{-1})$. Therefore one can view the harmonic functionals $A(\Omega)^*_\sigma$, for arbitrary Ω, as a 'non-commutative' analogue of harmonic functions on Ω.

We now give a brief review of the paper. We first prove in Section 2 some relevant basic results on matrix-valued functions and measures. We give a brief introduction to Jordan triple systems in Section 3 and prove a spectral result for a later application to matrix-valued harmonic functions. In Section 4, we study matrix-valued harmonic functions on homogeneous spaces. We first show that, as in the case when Ω is a group [6], there is a contractive projection from $L^\infty(\Omega, M_n)$ onto $H_\sigma(\Omega, M_n)$, for $\|\sigma\| = 1$. Consequently, $H_\sigma(\Omega, M_n)$ is a Jordan triple system and is, moreover, isometric to a finite ℓ^∞-sum $\oplus_k L^\infty(\Omega_k) \otimes C_k$, where C_k is a finite-dimensional Cartan factor of type $1, 2, 3$ or 4. It follows from the spectral result in Section 3 that the spectrum of $H_\sigma(\Omega, M_n)$, regarded as the *Poisson space* of $H_\sigma(\Omega, M_n)$, is a finite disjoint union of Stonean spaces.

We give an explicit formula for the Jordan triple product in $H_\sigma(G, M_n)$. The Jordan structures of $H_\sigma(\Omega, M_n)$ generally differ from that of $L^\infty(\Omega, M_n)$. We give necessary and sufficient conditions for the coincidence of these two structures. We also show that $H_\sigma(\Omega, M_n)$ is always a subalgebra of $L^\infty(\Omega, M_n)$ whenever Ω is a homogeneous space of a nilpotent group and σ is symmetric.

To conclude the section, we discuss the connection to Riemannian symmetric spaces. Given a homogeneous space $\Omega = H\backslash G$ with H compact, and given an H-invariant measure $\sigma \in M(G, M_n)$ with norm one, we show that, if $H_\sigma(\Omega, M_n)$ contains only constant functions, then G acts amenably on Ω, in the sense of [21]. If Ω is a Riemannian symmetric space of non-compact type, with the Laplace-Beltrami operator Δ, then the Δ-harmonic $L^\infty(\Omega, \mathbb{C})$-functions are exactly the functions in $\mathcal{H}(\Omega, \mathbb{C}) = \{\widetilde{f} : f \in H_\sigma(\Omega, \mathbb{C})\}$ for some H-invariant probability measure σ on G. Hence $\mathcal{H}(\Omega, \mathbb{C})$ is identified with $H_\sigma(\Omega, \mathbb{C})$ which cannot be trivial by the above result, and in this case, the Jordan triple structure in $\mathcal{H}(\Omega, \mathbb{C})$ is associative, that is, $\mathcal{H}(\Omega, \mathbb{C})$ becomes a C*-algebra whose pure state space then provides the Poisson space and Poisson representation of $\mathcal{H}(\Omega, \mathbb{C})$.

We study the space $A(\Omega)^*_\sigma$ in Section 5. We show that $A(\Omega)^*_\sigma$ is also a Jordan triple system. The non-commutative analogue of the Liouville theorem for harmonic functions is the triviality of $A(\Omega)^*_\sigma$. We give some necessary and sufficient conditions for this. For instance, if H is a compact subgroup of a second countable group G, then one can find some $\sigma \in B(H\backslash G)$ such that $A(H\backslash G)^*_\sigma$ is trivial. Conversely, for a locally compact group G with a compact subgroup H, the triviality of $A(H\backslash G)^*_\sigma$ implies that $H\backslash G$ is first countable. The Liouville theorem is closely related to amenability of the underlying groups as shown in Section 4 (see also [8, Proposition 2.1.3]). Given a compact subgroup H of a second countable group G, we show that G is amenable if, and only if, $I_\sigma(H\backslash G)$ has a bounded approximate identity for every norm-one $\sigma \in B(H\backslash G)$, where $A(H\backslash G)^*_\sigma$ is the annihilator of $I_\sigma(H\backslash G)$. We also study the related questions of the existence of a bounded approximate identity in $A(H\backslash G)$ and the existence of a topological invariant mean on the dual $A(H\backslash G)^*$.

2. Matrix-valued functions and measures

We begin by proving some basic Banach algebraic results for matrix-valued measures and functions on groups. Let $\mathrm{Tr} : M_n \to \mathbb{C}$ be the canonical trace of the C^*-algebra M_n. Every continuous linear functional $\varphi : M_n \to \mathbb{C}$ is of the form $\varphi(\cdot) = \mathrm{Tr}(\cdot A_\varphi)$ where the matrix $A_\varphi \in M_n$ is unique and $\|\varphi\| = \mathrm{Tr}(|A_\varphi|)$ which is the trace-norm $\|A_\varphi\|_1$ of A_φ. We will identify the dual M_n^*, via the map $\varphi \in M_n^* \mapsto A_\varphi \in M_n$, with the vector space M_n equipped with the trace-norm $\|\cdot\|_1$.

By an M_n-*valued measure* μ on a locally compact space Ω, we mean a (norm) countably additive function $\mu : \mathcal{B} \to M_n$ where $\mathcal{B}$ is the σ-algebra of Borel sets in Ω. Since the trace-norm $\|\cdot\|_1$ is equivalent to the C^*-algebra norm on M_n and $M_n^* = (M_n, \|\cdot\|_1)$, we can regard an M_n^*-valued measure on Ω as an M_n-valued measure on Ω, and *vice versa*. We can denote an M_n-valued measure μ on Ω by an $n \times n$ matrix $\mu = (\mu_{ij})$ of complex-valued measures μ_{ij} on Ω. Since each μ_{ij} is of bounded variation [34, Theorem 6.4], μ is also of bounded variation, that is, $|\mu|(\Omega) < \infty$, where the variation $|\mu|$ of μ is a positive real measure on Ω defined by

$$|\mu|(E) = \sup_{\mathcal{P}} \Big\{ \sum_{E_i \in \mathcal{P}} \|\mu(E_i)\| \Big\} \qquad (E \in \mathcal{B})$$

with the supremum taken over all partitions $\mathcal{P}$ of E into a finite number of pairwise disjoint Borel sets. We define the norm of μ to be $\|\mu\| = |\mu|(\Omega)$. As shown in [6, p. 21], μ has a polar representation $\mu = \omega \cdot |\mu|$ where $\omega : \Omega \to M_n$ is a Bochner integrable function with $\|\omega(\cdot)\| = 1$. Likewise, if μ is an M_n^*-valued measure, we define its norm by $\|\mu\|_1(\Omega) = |\mu|_1(\Omega) = \sup_{\mathcal{P}}\{\sum_{E_i \in \mathcal{P}} \|\mu(E_i)\|_1\}$.

Let $M(\Omega, M_n^*)$ be the space of all M_n^*-valued measures on Ω, equipped with the total variation norm $\|\cdot\|_1$. It is linearly isomorphic to the space $M(\Omega, M_n)$ of M_n-valued measures on Ω, equipped with the total variation norm $\|\cdot\|$. Let $C_0(\Omega, M_n)$ be the Banach space of continuous M_n-valued functions on Ω vanishing at infinity, equipped with the supremum norm. It has been shown in [6, Lemma 5] that $M(\Omega, M_n^*)$ is linearly isometric order-isomorphic to the dual of $C_0(\Omega, M_n)$, where a measure $\mu \in M(\Omega, M_n^*)$ and a function $f \in C_0(\Omega, M_n)$ are *positive* if $\mu(E)$ and $f(x)$ are positive matrices for all $E \in \mathcal{B}$ and $x \in \Omega$. The above duality is given by

$$\langle\,,\,\rangle : C_0(\Omega, M_n) \times M(\Omega, M_n^*) \to \mathbb{C}$$

$$\langle f, \mu\rangle = \mathrm{Tr}\left(\int_\Omega f d\mu\right) = \sum_{i,k} \int_\Omega f_{ik} d\mu_{ki}$$

where $f = (f_{ij}) \in C_0(\Omega, M_n)$ and $\mu = (\mu_{ij}) \in M(\Omega, M_n^*)$ (cf. [6, Lemma 5]). A function $f = (f_{ij}) : \Omega \longrightarrow M_n$ is said to be *μ-integrable* if each f_{ij} is a Borel function and the integrals $\int_\Omega f_{ij} d\mu_{k\ell}$ exist and we define, for any $E \in \mathcal{B}$,

$$\int_E f d\mu = \left(\sum_k \int_E f_{ik} d\mu_{kj}\right) \in M_n.$$

Let ν be a regular Borel real measure on Ω. Let $L^1(\Omega, M_n^*)$ be the complex Banach space of (equivalence classes of) M_n^*-valued Bochner ν-integrable functions on Ω. The dual space of $L^1(G, M_n^*)$ identifies with the space $L^\infty(\Omega, M_n)$ of M_n-valued essentially bounded weakly locally ν-measurable functions on Ω (cf. [36, 1.22.13]). It is a von Neumann algebra under the pointwise product and involution:

$$(fg)(x) = f(x)g(x), \quad f^*(x) = f(x)^* \qquad (f, g \in L^\infty(\Omega, M_n),\ x \in \Omega).$$

If Ω is a locally compact group G, we will always take $\nu = \lambda$ to be the right invariant Haar measure on G in which case, we will use the duality between $C_0(G, M_n)$ and $M(G, M_n^*)$ to show that $M(G, M_n^*)$ is a Banach algebra in the Arens product.

Given $a \in G$, we define left and right translations on $C_0(G, M_n)$ by

$$(\ell_a f)(x) = f(ax), \quad (r_a f)(x) = f(xa) \quad (x \in G)$$

for $f \in C_0(G, M_n)$.

Lemma 2.1 *Given $f \in C_0(G, M_n)$, the maps $a \in G \mapsto \ell_a f \in C_0(G, M_n)$ and $a \in G \mapsto r_a f \in C_0(G, M_n)$ are continuous.*

Proof. Let $C_c(G, M_n)$ be the subspace of $C_0(G, M_n)$, consisting of functions with compact support. Since $C_c(G, M_n)$ is dense in $C_0(G, M_n)$, it suffices to prove the result for $C_c(G, M_n)$. Let $f \in C_c(G, M_n)$ have compact support K and suppose, for contradiction, that the map $a \in G \mapsto \ell_a f$ is not continuous at the identity $e \in G$. Then there exists

$\varepsilon > 0$ and a net (a_α) in G converging to e such that $\|\ell_{a_\alpha} f - f\| \geq \varepsilon$ for each α. Let V be a compact neighbourhood of e such that $a_\alpha \in V^{-1}$ from some α_0 onwards. Then the support of $\ell_{a_\alpha} f - f$ is contained in VK from α_0 onwards. By compactness again, there exist $v_\alpha \in V$ and $x_\alpha \in K$ such that

$$\|\ell_{a_\alpha} f - f\| = \|f(a_\alpha v_\alpha x_\alpha) - f(v_\alpha x_\alpha)\|_{M_n} \quad \text{for} \quad \alpha \geq \alpha_0.$$

Passing to a subnet if necessary, we may assume $(v_\alpha x_\alpha)$ converges to some $x \in VK$. Then we have

$$\varepsilon \leq \|\ell_{a_\alpha} f - f\| = \|f(a_\alpha v_\alpha x_\alpha) - f(v_\alpha x_\alpha)\|_{M_n} \to \|f(x) - f(x)\|_{M_n} = 0$$

which is impossible. So the map $a \in G \mapsto \ell_a f$ is continuous. Likewise, the map $a \mapsto r_a f$ is continuous.

□

Lemma 2.2 *For $f \in C_0(G, M_n)$ and $\mu \in M(G, M_n^*)$, the functions $a \in G \mapsto \int_G (\ell_a f) d\mu \in M_n$ and $a \in G \mapsto \int_G (r_a f) d\mu \in M_n$ are in $C_0(G, M_n)$.*

Proof. This follows from Lemma 2.1 and the fact that $\| \int_G h d\mu\| \leq \|h\| \|\mu\|$ for every $h \in C_0(G, M_n)$, by [6, Lemma 4].

□

Given any $u \in C_0(G)$, the orbit of the left translates of u is relatively weakly compact in $C_0(G)$. Apply this fact entry-wise to a matrix-valued function on G, we obtain the following analogous result.

Lemma 2.3 *Let $f \in C_0(G, M_n)$. Then the orbits $\mathcal{L}O(f) = \{\ell_a f : a \in G\}$ and $\mathcal{R}O(f) = \{r_a f : a \in G\}$ are relatively weakly compact in $C_0(G, M_n)$.*

Now we construct the Arens products on $M(G, M_n^*)$. Let $\mu, \nu \in M(G, M_n^*)$. We define the Arens products $\mu \Box \nu$ and $\mu \circ \nu$ by

$$\begin{aligned} \langle f, \mu \Box \nu \rangle &= \langle \nu_\ell(f), \mu \rangle, \\ \langle f, \mu \circ \nu \rangle &= \langle \mu_r(f), \nu \rangle \end{aligned} \qquad \big(f \in C_0(G, M_n)\big)$$

where $\nu_\ell(f)$, $\mu_r(f) \in C_0(G, M_n)$ are defined by

$$\begin{aligned} \nu_\ell(f)(a) &= \textstyle\int_G (\ell_a f) d\nu, \\ \mu_r(f)(a) &= \textstyle\int_G (r_a f) d\mu \end{aligned} \qquad (a \in G).$$

We remark that if G is discrete, then $M(G, M_n^*) = \ell^1(G, M_n^*) \subset \ell^1(G, M_n^*)^{**}$ and the above products are the restrictions of the Arens products on $\ell^1(G, M_n^*)^{**}$.

By a matrix-valued Fubini Theorem, the above two Arens products actually coincide on $M(G, M_n^*)$. Indeed, we have, by [6, Proposition 7], that

$$\mu \Box \nu = \mu \circ \nu = \mu * \nu \tag{2.1}$$

where $\mu * \nu$ is the *convolution* defined by

$$(\mu * \nu)(E) = (\mu \times \nu)\{(x, y) \in G \times G : xy \in E\} \quad (E \in \mathcal{B}).$$

The M_n-valued product measure $\mu \times \nu$ is defined as in the scalar case (cf. [6, p. 24]).

Proposition 2.4 *The Banach space* $\big(M(G, M_n^*), \|\cdot\|_1\big)$ *is a Banach algebra under the convolution product in* (2.1).

Proof. It suffices to show that $\|\mu * \nu\|_1 \leq \|\mu\|_1\|\nu\|_1$. Let $\mu = \omega^\mu \cdot |\mu|_1$ and $\nu = \omega^\nu \cdot |\nu|_1$ be the polar representations shown in [6, p. 21], where $\omega^\mu, \omega^\nu : G \to M_n^*$ are Bochner integrable functions with $\|\omega^\mu(\cdot)\|_1 = \|\omega^\nu(\cdot)\|_1 = 1$, and $|\mu|_1$, $|\nu|_1$ are positive real measures on G. We have

$$\begin{aligned}
\|\mu * \nu\|_1 &= \sup\Big\{\Big|\mathrm{Tr}\Big(\int_G f d(\mu * \nu)\Big)\Big| : f \in C_0(G, M_n), \|f\| \leq 1\Big\}\\
&= \sup\Big\{\Big|\mathrm{Tr}\Big(\int_G\int_G f(x,y) d\mu(x) d\nu(y)\Big)\Big| : f \in C_0(G, M_n), \|f\| \leq 1\Big\}\\
&= \sup\Big\{\Big|\mathrm{Tr}\Big(\int_G\int_G f(x,y)\omega^\mu(x)\omega^\nu(y) d|\mu|_1(x) d|\nu|_1(y)\Big)\Big| :\\
&\qquad\qquad f \in C_0(G, M_n), \|f\| \leq 1\Big\}\\
&\leq \sup\Big\{\int_G\int_G \Big|\mathrm{Tr}\Big(f(x,y)\omega^\mu(x)\omega^\nu(y)\Big)\Big| d|\mu|_1(x) d|\nu_1|(y) :\\
&\qquad\qquad f \in C_0(G, M_n), \|f\| \leq 1\Big\}\\
&\leq \sup\Big\{\int_G\int_G \|f(x,y)\|\,\|\omega^\mu(x)\|_1\|\omega^\nu(y)\|_1 d|\mu|_1(x) d|\nu|_1(y) :\\
&\qquad\qquad f \in C_0(G, M_n), \|f\| \leq 1\Big\}\\
&\leq |\mu|_1(G)|\nu|_1(G) = \|\mu\|_1\|\nu\|_1.
\end{aligned}$$

□

Under the convolution product, the Banach space $L^1(G, M_n^*)$ is also a Banach algebra. The second dual $L^1(G, M_n^*)^{**} = L^\infty(G, M_n)^*$ is a Banach algebra with respect to the two Arens products which are different in general. We recall that a Banach algebra $\mathcal{A}$ is called *Arens regular* if the Arens products on the second dual $\mathcal{A}^{**}$ coincide. It has been shown in [44] that $L^1(G)$ is Arens regular if, and only if, G is finite. This result remains true for $L^1(G, M_n^*)$ by considering $L^1(G)$ as a subalgebra of $L^1(G, M_n^*)$ via the embedding

$$f \mapsto \begin{pmatrix} f & & 0 \\ & \ddots & \\ 0 & & f \end{pmatrix}.$$

Let $C_b(G, M_n)$ be the Banach space of bounded M_n-valued continuous functions on G, equipped with the supremum norm. Let $C_{ru}(G, M_n)$ (respectively, $C_{\ell u}(G, M_n)$) be the subspace of $C_b(G, M_n)$, consisting of right uniformly continuous (respectively, left uniformly continuous) functions, where a function $f \in C_b(G, M_n)$ is *right uniformly continuous* if the map $a \in G \mapsto \ell_a f \in C_b(G, M_n)$ is continuous, and the *left uniform continuity* is defined by the continuity of the map $a \in G \mapsto r_a f \in C_b(G, M_n)$.

For $f \in C_{ru}(G, M_n)$ and $m \in L^\infty(G, M_n)^*$, we can define their product $m \cdot f \in L^\infty(G, M_n)$ by the duality $\langle\cdot,\cdot\rangle : L^1(G, M_n^*) \times L^\infty(G, M_n) \to \mathbb{C}$:

$$\langle \varphi, m \cdot f\rangle = \langle f \cdot \varphi, m\rangle \qquad \big(\varphi \in L^1(G, M_n^*)\big)$$

where $f \cdot \varphi \in L^\infty(G, M_n)$ is defined by

$$\langle \psi, f \cdot \varphi\rangle = \langle \varphi * \psi, f\rangle \qquad \big(\psi \in L^1(G, M_n)\big).$$

Lemma 2.5 *Let G be a locally compact group. Then we have*

(i) $C_{ru}(G, M_n) = \{f \in L^\infty(G, M_n) : a \in G \mapsto \ell_a f \in L^\infty(G, M_n) \text{ is continuous}\}$;
(ii) $C_{\ell u}(G, M_n) = \{f \in L^\infty(G, M_n) : a \in G \mapsto r_a f \in L^\infty(G, M_n) \text{ is continuous}\}$.

Further, for $f \in C_{ru}(G, M_n)$ and $m \in L^\infty(G, M_n)^*$, we have $m \cdot f \in C_{ru}(G, M_n)$ and $(m \cdot f)(a) = m(\ell_a f)$ for $a \in G$. Analogous result also holds for $f \in C_{\ell u}(G, M_n)$.

Proof. This follows by applying entrywise the result for the scalar case [29].

□

3. Jordan triple systems

For later applications, we introduce Jordan triple systems in this section as well as the concept of a spectrum for such a system.

A *Jordan triple system* is a complex vector space V with a *Jordan triple product*

$$\{\cdot, \cdot, \cdot\} : V \times V \times V \to V$$

which is symmetric and linear in the outer variables, conjugate linear in the middle variable and satisfies the *Jordan triple identity*

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}.$$

A complex Banach space Z is called a *JB^*-triple* if it is a Jordan triple system such that for each $z \in Z$, the linear map

$$D(z, z) : v \in Z \mapsto \{z, z, v\} \in Z$$

is Hermitian, that is, $\|\exp\big(itD(z, z)\big)\| = 1$ for all $t \in \mathbb{R}$, with non-negative spectrum and $\|D(z, z)\| = \|z\|^2$.

JB*-triples play an important role in the theory of symmetric Banach manifolds. Indeed, generalizing the Riemann Mapping Theorem, Kaup [27] has shown that the bounded symmetric domains in complex Banach spaces are exactly the open unit balls of JB*-triples, up to biholomorphic equivalence. It is therefore interesting that, as we will show, certain harmonic function spaces form JB*-triples.

A JB^*-triple Z is called a *JBW^*-triple* if it is a dual Banach space, in which case its predual is unique, denoted by Z_*, and the Jordan triple product on Z is separately weak*-continuous. The second dual Z^{**} of a JB^*-triple Z is naturally a JBW^*-triple. A subspace of a JB^*-triple is called a *subtriple* if it is closed with respect to the Jordan triple product. The JB^*-triples form a large class of Banach spaces. They include, for instance, C^*-algebras, Hilbert spaces and spaces of rectangular matrices. The triple product in a C^*-algebra $\mathcal{A}$ is given by

$$\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x).$$

JB^*-triples arise as tangent spaces to complex symmetric Banach manifolds which are infinite-dimensional generalization of the Hermitian symmetric spaces classified by E. Cartan [5] using Lie groups. The non-compact irreducible Hermitian symmetric spaces are, up to biholomorphic equivalence, the open unit balls of six types of finite-dimensional spaces of matrices. The infinite-dimensional generalization of these spaces are the following six types of JBW^*-triples, known as the *Cartan factors*:

type 1 $\mathcal{L}(H,K)$,
type 2 $\{z \in \mathcal{L}(H,H) : z^t = -z\}$,
type 3 $\{z \in \mathcal{L}(H,H) : z^t = z\}$,
type 4 spin factor,
type 5 $M_{1,2}(\mathcal{O}) = 1 \times 2$ matrices over the Cayly algebra $\mathcal{O}$,
type 6 $M_3(\mathcal{O}) = 3 \times 3$ hermitian matrices over $\mathcal{O}$,

where $\mathcal{L}(H,K)$ is the Banach space of bounded linear operators between complex Hilbert space H and K, with Jordan triple product

$$\{x,y,z\} = \frac{1}{2}(xy^*z + zy^*x)$$

and z^t is the transpose of z induced by a conjugation on H. Cartan factors of type 2 and 3 are subtriples of $\mathcal{L}(H,H)$, the latter notation is shortened to $\mathcal{L}(H)$. A *spin factor* is a Banach space that is equipped with a complete inner produce $\langle \cdot, \cdot \rangle$ and a conjugation j on the resulting Hilbert space, with triple product

$$\{x,y,z\} = \frac{1}{2}\big(\langle x,y\rangle z + \langle z,y\rangle x - \langle x,jz\rangle jy\big)$$

such that the given norm and the Hilbert space norm are equivalent. The Cayley algebra $\mathcal{O}$ is a non-associative complex algebra with a basis $\{e_0,\ldots,e_7\}$. Given $a = \alpha_0 e_0 + \cdots + \alpha_7 e_7 \in \mathcal{O}$, we define $\overline{a} = \overline{\alpha_0}e_0 + \cdots + \overline{\alpha_7}e_7$ and $a^* = \overline{\alpha_0}e_0 - (\overline{\alpha_1}e_1 + \cdots + \overline{\alpha_7}e_7)$. The Jordan triple product in $M_{1,2}(\mathcal{O})$ is given by $\{x,y,z\} = \frac{1}{2}\big(x(y^*z) + z(y^*z)\big)$ where $y^* = \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix}$ for $y = (y_1, y_2)$. Matrices in $M_3(\mathcal{O})$ are hermitian with respect to the involution $\left(\sum \alpha_k e_k\right)^i := \alpha_0 e_0 - \sum_1^7 \alpha_k e_k$ in $\mathcal{O}$. The triple product in $M_3(\mathcal{O})$ is defined by

$$\{x,y,z\} = x \circ (y^* \circ z) - y^* \circ (x \circ z) + z \circ (x \circ y^*)$$

where $x \circ y = \frac{1}{2}(xy + yx)$ and $(y_{ij})^* = (\overline{y_{ij}})$. The Cartan factors $M_{1,2}(\mathcal{O})$ and $M_3(\mathcal{O})$ are *exceptional* which means that they cannot be embedded as a subtriple of $\mathcal{L}(H)$. A JBW^*-triple which can be embedded as a subtriple of $\mathcal{L}(H)$ is called a JW^**-triple*. We refer to [9,26,35,40,41] for more details of JB*-triples and symmetric manifolds.

Let $Z \subset \mathcal{L}(H)$ and $W \subset \mathcal{L}(K)$ be JW^*-triples. Then their algebraic tensor product $Z \odot W$ identifies naturally as a subtriple of $\mathcal{L}(H \otimes K)$, where $H \otimes K$ is the usual Hilbert space tensor product. The ultraweak closure $Z \overline{\otimes} W$ of $Z \odot W$ in $\mathcal{L}(H \otimes K)$ is a JW^*-triple.

Let Z be a JB^*-triple. An element $e \in Z$ is called a *tripotent* if $e = \{e,e,e\}$. If Z is a C^*-algebra, the tripotents are exactly the partial isometries. A tripotent $e \in Z$ is said to be *minimal* if $\{e,Z,e\} = \mathbb{C}e$, to be *unitary* if $\{e,Z,e\} = Z$, and *complete* if $\{e,e,x\} = 0$ implies $x = 0$. The complete tripotents are exactly the extreme points of the closed unit ball of Z. We note that a JB^*-triple may not have any tripotent, but JBW^*-triples contain an abundance of tripotents. Given any extreme point ρ in the closed unit ball Z_1^* of Z^*, there is a unique minimal tripotent $s(\rho) \in Z^{**}$, called the *support* of ρ, such that $\rho\big(s(\rho)\big) = 1$ [18].

A closed subspace J of Z is called an *ideal* if $\{Z,J,Z\}+\{Z,Z,J\} \subset J$. If Z is a C^*-algebra, this definition is equivalent to saying that J is a closed two-sided ideal. If Z is a JBW^*-triple, then every weak*-closed ideal J in Z is 1-complemented, that is, there is a weak*-continuous contractive projection $P: Z \to Z$ with $J = P(Z)$ [24].

The Murray-von Neumann classification of von Neumann algebras into types I, II and III can be extended to JBW*-triples. These types are preserved by weak* continuous contractive projections as follows [11].

Proposition 3.1 *Let Z be a JBW^*-triple of type j, where $j =$ I, II or III. Let $P : Z \longrightarrow Z$ be a weak* continuous contractive projection. Then each direct summand of $P(Z)$ is a JBW^*-triple of type k with $k \leq j$.*

We now define the spectrum of a JB^*-triple. The structure space of primitive M-ideals of a Banach space was introduced and investigated in [1]. In JB^*-triples, the primitive M-ideals are the kernels of Cartan factor representations [3]. Let Z be a JB^*-triple. A *Cartan factor representation* of Z is a (nonzero) Jordan triple homomorphism $\pi : Z \to M$ where M is a Cartan factor and $\pi(Z)$ is weak*-dense in M. Two Cartan factor representations $\pi : Z \to M$ and $\tau : Z \to N$ are *equivalent* if there is a Jordan triple isomorphism $\psi : M \to N$ such that $\tau = \psi\pi$. Given an extreme point $\rho \in Z_1^*$, let Z_ρ^{**} be the weak*-closed ideal in Z^{**} generated by the support $s(\rho)$ of ρ. Then Z_ρ^{**} is a Cartan factor and the composite map

$$\pi_\rho : Z \hookrightarrow Z^{**} \xrightarrow{P_\rho} Z_\rho^{**}$$

is a Cartan factor representation, where $P_\rho : Z^{**} \to Z_\rho^{**}$ is the natural weak*-continuous projection mentioned above. In fact, as shown in [3], all Cartan factor representations are of this form, and two representations π_ρ and π_η are equivalent if, and only if, $s(\rho) = s(\eta)$. Hence the set Prim Z of primitive M-ideals of Z is given by

$$\text{Prim } Z = \{\ker \pi_\rho : \rho \in \partial_e Z_1^*\}$$

where $\partial_e Z_1^*$ denotes the set of extreme points in Z_1^*. We equip Prim Z with the hull-kernel topology in which the closure of a set $\mathcal{S} \subset$ Prim Z is the hull-kernel $hk(\mathcal{S})$ defined by

$$hk(\mathcal{S}) = \{J \in \text{ Prim } Z : J \supset k(\mathcal{S})\}, \quad k(\mathcal{S}) = \cap\{J : J \in \mathcal{S}\}.$$

The *spectrum* of a JB^*-triple Z is the set $\widehat{Z}$ of equivalence classes of Cartan factor representations of Z, equipped with the topology induced by the hull-kernel topology of Prim Z via the following surjective map:

$$\pi \in \widehat{Z} \mapsto \ker \pi \in \text{ Prim } Z$$

where, following the usual convention, we use the same symbol π for its equivalence class.

Given a Cartan factor C, its spectrum $\widehat{C}$ reduces to a singleton since C contains no proper weak*-closed ideal and hence every Cartan factor representation of C is equivalent to the identity representation.

If $Z = I \oplus J$ is an ℓ_∞-sum of two closed ideals I and J, then as in [3, p. 22], we have $\widehat{Z} = \widehat{I} \cup \widehat{J}$.

Proposition 3.2 *Let Z be a finite-dimensional JB^*-triple and let $C(S,Z)$ be the JB^*-triple of continuous Z-valued functions on a compact Hausdorff space S, with pointwise triple product. Then the spectrum $\widehat{C(S,Z)}$ is homeomorphic to $S \times \widehat{Z}$.*

Proof. For each $s \in S$, let $\delta_s \in C(S)^*$ be the unit mass at s. We have $\partial_e C(S)_1^* = \{\alpha\delta_s : s \in S, |\alpha| = 1\}$. The support tripotent $s(\delta_s)$ of δ_s is a minimal projection $p \in C(S)^{**}$ whereas the support tripotent of $\alpha\delta_s$ is αp. We have $C(S)^{**}_{\delta_s} = C(S)^{**}p = \mathbb{C}p$ and the Cartan factor representation $\pi_{\delta_s} : C(S) \to C(S)^{**}_{\delta_s}$ is one-dimensional with $\pi_{\delta_s}(f) = f(s)p$, which is equivalent to $\pi_{\alpha\delta_s} : C(S) \to C(S)^{**}_{\alpha\delta_s} = \mathbb{C}p$, given by $\pi_{\alpha\delta_s}(f) = \alpha f(s)p$. Hence $\widehat{C(S)}$ is homeomorphic to S, via $s \in S \mapsto \pi_{\delta_s} \in \widehat{C(S)}$. Identify $C(S, Z)$ with the injective tensor product $C(S)\check{\otimes}Z$. Since Z is finite-dimensional, we have $\big(C(S)\check{\otimes}Z\big)^{**} \cong C(S)^{**}\check{\otimes}Z$, by a simple application of [12], say.

Let $(\pi_{\delta_s}, \pi_\rho) \in \widehat{C(S)} \times \widehat{Z}$. Then we can define a natural Cartan factor representation

$$\pi_{\delta_s} \otimes \pi_\rho : C(S)\check{\otimes}Z \to \mathbb{C}p \otimes Z_\rho^{**}.$$

Two such representations $\pi_{\delta_s} \otimes \pi_\rho$ and $\pi_{\delta_t} \otimes \pi_\eta$ are equivalent if, and only if, $s = t$ and $s(\rho) = s(\eta) \in Z^{**}$. Thus we can define an injective map

$$(\pi_{\delta_s}, \pi_\rho) \in \widehat{C(S)} \times \widehat{Z} \mapsto \pi_{\delta_s} \otimes \pi_\rho \in \widehat{C(S,Z)}. \tag{3.1}$$

To see the map is surjective, consider a Cartan factor representation

$$\pi_\varphi : C(S)\check{\otimes}Z \to \big(C(S)^{**}\check{\otimes}Z\big)_\varphi$$

where $\varphi \in \partial_e C(S,Z)_1^*$ and $\big(C(S)^{**}\check{\otimes}Z\big)_\varphi$ is the weak*-closed ideal generated by the support tripotent $s(\varphi)$ in $C(S)^{**}\check{\otimes}Z$. It is well-known [37] that

$$\varphi = \alpha\delta_s \otimes \rho$$

for some $s \in S$ and $\rho \in \partial_e Z_1^*$. It follows that π_φ is equivalent to $\pi_{\delta_s} \otimes \pi_\rho$ and the map in (3.1) is surjective. The above arguments also show that the following map is bijective:

$$\begin{aligned}\Phi : (\ker\pi_{\delta_s}, \ker\pi_\rho) \in \text{ Prim } C(S) \times \text{ Prim } Z \mapsto \text{ } & \ker(\pi_{\delta_s} \otimes \pi_\rho) \\ \in \text{ } & \text{Prim}\,\big(C(S)\check{\otimes}Z\big).\end{aligned}$$

Further, Φ is a homeomorphism as it is straightforward to verify that $\Phi\big(hk(\mathcal{S})\big) = hk\Phi(\mathcal{S})$ for $\mathcal{S} \subset$ Prim $C(S) \times$ Prim Z.

Since Z is finite-dimensional, the map $\pi \in \widehat{Z} \mapsto \ker\pi \in$ Prim Z is a homeomorphism. Hence $\widehat{C(S)} \times \widehat{Z}$ is homeomorphic to Prim $C(S) \times$ Prim Z and the following diagram implies that $\widehat{C(S)} \times \widehat{Z}$ is homeomorphic to $\widehat{C(S,Z)}$:

$$\begin{array}{ccc} \widehat{C(S)} \times \widehat{Z} & \longrightarrow & \text{Prim } C(S) \times \text{ Prim } Z \\ \downarrow & & \downarrow \\ \widehat{C(S,Z)} & \longrightarrow & \text{Prim } C(S,Z). \end{array}$$

□

4. Harmonic functions on homogeneous spaces

In this section, we study the Jordan algebraic analytic structures of bounded matrix-valued harmonic functions on homogeneous spaces. Let G be a locally compact group acting on a locally compact Hausdorff space Ω by a *right* action $(g,\xi) \in G\times\Omega \mapsto \xi\cdot g \in \Omega$. If G acts transitively, then there is a closed subgroup $H \subset G$ and a continuous bijection from the right coset space $H\backslash G$ onto Ω. If this bijection is a homeomorphism, for example, this happens when G is a countable union of compact sets, then we call $\Omega \approx H\backslash G$ a *homogeneous space* of G, where the action $(g,\xi) \mapsto \xi\cdot g$ is equivalent to the natural action of G on $H\backslash G$.

There is a quasi-invariant measure ν on the homogeneous space $\Omega = H\backslash G$ (cf. [15]). Here *quasi-invariance* means that all translates of ν are mutually absolutely continuous. We fix such a measure ν on Ω. The Lebesgue spaces of ν on Ω are denoted by $L^p(\Omega)$ for $1 \le p \le \infty$. Likewise $L^p(\Omega, M_n)$ denotes the space (of equivalence classes) of M_n-valued L^p-functions on Ω.

Given $\sigma \in M(G, M_n)$ and a Borel function $f : \Omega \to M_n$, we define their convolution $f * \sigma$ by

$$(f*\sigma)(\xi) = \int_G f(\xi\cdot y^{-1})d\sigma(y) \qquad (\xi \in \Omega)$$

whenever the integral exists. As before, we let

$$H_\sigma(\Omega, M_n) = \{f \in L^\infty(\Omega, M_n) : f = f*\sigma\}$$

be the space of (essentially) bounded σ-harmonic functions on Ω.

In the scalar case $\sigma \in M(G,\mathbb{C})$, we have

$$\langle h*\widetilde{\sigma}, f\rangle = \langle h, f*\sigma\rangle \qquad \big(h \in L^1(\Omega),\, f \in L^\infty(\Omega)\big) \tag{4.1}$$

and it follows that the map $f \in L^\infty(\Omega) \mapsto f*\sigma \in L^\infty(\Omega)$ is weak* continuous. Moreover, if we let J be the norm closure in $L^1(\Omega)$ of the subspace

$$\{h*\widetilde{\sigma} - h : h \in L^1(\Omega)\},$$

then $H_\sigma(\Omega,\mathbb{C}) = \big(L^1(\Omega)/J\big)^*$.

In the matrix-valued case $\sigma \in M(G, M_n)$, however, (4.1) need not hold for M_n-valued functions, but the weak* continuity of the map $f \in L^\infty(\Omega, M_n) \mapsto f*\sigma \in L^\infty(\Omega, M_n)$ follows from entrywise computation. Indeed, let (f_α) be a net weak* convergent to f in $L^\infty(\Omega, M_n)$. Then the net $(f_\alpha)_{ij}$ of each entry weak* converges to f_{ij} in $L^\infty(\Omega)$. For any $h \in L^1(\Omega, M_n^*)$, we have

$$\begin{aligned}
\langle h, f_\alpha * \sigma\rangle &= \operatorname{Tr}\left(\int_\Omega h(x)(f_\alpha*\sigma)(x)d\nu(x)\right)\\
&= \sum_{ijk}\int_\Omega\int_G h_{ik}(x)(f_\alpha)_{kj}(x\cdot y^{-1})d\sigma_{ji}(y)d\nu(x)\\
&= \sum_{ijk}\int_\Omega (h_{ik}*\widetilde{\sigma}_{ji})(x)(f_\alpha)_{kj}(x)d\nu(x)\\
&\longrightarrow \sum_{ijk}\int_\Omega (h_{ik}*\widetilde{\sigma}_{ji})(x)(f_{kj})(x)d\nu(x) = \langle h, f*\sigma\rangle.
\end{aligned}$$

Therefore $H_\sigma(\Omega, M_n)$ is weak*-closed in $L^\infty(\Omega, M_n)$ and if we let

$$J = \{h \in L^1(\Omega, M_n^*) : \langle h, f\rangle = 0, \forall f \in H_\sigma(\Omega, M_n)\} \tag{4.2}$$

then $H_\sigma(\Omega, M_n) = \big(L^1(\Omega, M_n^*)/J\big)^*$.

We now show that, for $\|\sigma\| = 1$, there exists a contractive projection from $L^\infty(\Omega, M_n)$ onto $H_\sigma(\Omega, M_n)$. The construction of such a projection is similar to that in [6,8] for groups. We outline the main steps below.

Proposition 4.1 *Let Ω be a homogeneous space and let $\sigma \in M(G, M_n)$ with $\|\sigma\| = 1$. Then there is a contractive projection $P_\sigma : L^\infty(\Omega, M_n) \to L^\infty(\Omega, M_n)$ with range $H_\sigma(\Omega, M_n)$. Further, given any weak* continuous map $T : L^\infty(\Omega, M_n) \to L^\infty(\Omega, M_n)$ satisfying $T(f * \sigma) = (Tf) * \sigma$, then $PT = TP$.*

Proof. Define the convolution operator $\Lambda : L^\infty(\Omega, M_n) \to L^\infty(\Omega, M_n)$ by

$$\Lambda(f) = f * \sigma \qquad \big(f \in L^\infty(\Omega, M_n)\big).$$

Then $\|\Lambda\| \leq 1$ since $\|\sigma\| = 1$. By the above remark, Λ is weak* continuous. Let $L^\infty(\Omega, M_n)^{L^\infty(\Omega, M_n)}$ be equipped with the product weak* topology $\mathcal{T}$. Let K be the $\mathcal{T}$-closed convex hull of $\{\Lambda^n : n = 1, 2, \dots\}$ where $\Lambda^n = \Lambda \circ \cdots \circ \Lambda$ (n-times). Define $\Phi : K \to K$ by

$$\Phi(\Gamma)(f) = \Gamma(f) * \sigma \qquad \big(\Gamma \in K, f \in L^\infty(\Omega, M_n)\big).$$

Then Φ is well-defined, affine and $\mathcal{T}$-continuous. Therefore there exists $P_\sigma \in K$ such that $\Phi(P_\sigma) = P_\sigma$, by the Markov-Kakutani fixed point theorem, and P_σ is the required contractive projection.

The last assertion follows from $T\Lambda = \Lambda T$.

□

Corollary 4.2 *Let Ω be a homogeneous space and let $\sigma \in M(G, M_n)$ with $\|\sigma\| = 1$. Then $H_\sigma(\Omega, M_n)$ forms a JW*-triple and is either $\{0\}$ or linearly isometric to a finite ℓ^∞-sum $\bigoplus_k L^\infty(\Omega_k) \otimes C_k$ where C_k is a finite-dimensional Cartan factor of type $1, 2, 3$ or 4.*

Proof. Since $H_\sigma(\Omega, M_n)$ is the range of a contractive projection P_σ on $L^\infty(\Omega, M_n)$, by [19], $H_\sigma(\Omega, M_n)$ is a JW*-triple with the Jordan triple product defined by

$$\{f, g, h\} = \frac{1}{2}\, P_\sigma(fg^*h + hg^*f)$$

for $f, g, h \in H_\sigma(\Omega, M_n)$. As $L^\infty(\Omega, M_n)$ is a finite type I von Neumann algebra, by [6, Proposition 11], $H_\sigma(\Omega, M_n)$ is either $\{0\}$ or a finite direct sum $\bigoplus_k L^\infty(\Omega_k) \otimes C_k$ where C_k is a finite-dimensional Cartan factor which cannot be exceptional, that is, of type $1, 2, 3$ or 4.

□

Remark 4.3 The proof of Proposition 4.1 implies that there is a net $\{\mu_\alpha\}$ in the convex hull of $\{\sigma^n : n = 1, 2, \dots\}$ such that

$$P_\sigma(f) = w^* - \lim_\alpha f * \mu_\alpha$$

for every $f \in L^\infty(\Omega, M_n)$ which gives

$$\{f,g,h\} = \frac{1}{2}\, P_\sigma(fg^*h + hg^*f)$$
$$= w^* - \lim_\alpha \frac{1}{2}\,(fg^*h + hg^*f) * \mu_\alpha$$

for $f,g,h \in H_\sigma(\Omega, M_n)$. We note that, in general, the range of a contractive projection on a JB^*-triple V need not be a *subtriple* of V, although it is a JB^*-triple in its own right. We will discuss when $H_\sigma(\Omega, M_n)$ is a subtriple of $L^\infty(\Omega, M_n)$.

Example 4.4. Let G be a discrete group with at least two elements. Let $\sigma \in M(G, M_2)$ be supported on $\{a,b\} \subset G$ and defined by

$$\sigma\{a\} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma\{b\} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Then $\|\sigma\| = 1$ and for any $f = (f_{ij}) \in H_\sigma(G, M_2)$, we have $f_{i1} * \sigma_{11} = f_{i1}$ and $f_{i2} * \sigma_{22} = f_{i2}$, where $\|f_{ij} * \sigma_{jj}\| \le \|f_{ij}\|\|\sigma_{jj}\|$ and $\|\sigma_{11}\| = \|\sigma_{22}\| < 1$ imply $f_{ij} = 0$ for all i,j. So $H_\sigma(G, M_2) = \{0\}$.

If $\sigma \in M(G, \mathbb{C})$ with $\|\sigma\| = 1$, then $H_\sigma(G, \mathbb{C})$ is an abelian von Neumann algebra [8, Corollary 2.2.9] and its spectrum is the natural Poisson boundary for the Poisson representation of $H_\sigma(G, \mathbb{C})$ [8, Proposition 2.2.13]. In the matrix-valued case, we have the following description of the spectrum of $H_\sigma(\Omega, M_n)$.

Corollary 4.5 *Let $\sigma \in M(G, M_n)$ with $\|\sigma\| = 1$. Then the spectrum $H_\sigma\widehat{(\Omega, M_n)}$ is compact Hausdorff and homeomorphic to a finite disjoint union of Stonean spaces.*

Proof. We can ignore the trivial case $H_\sigma(\Omega, M_n) = \{0\}$. By Corollary 4.2, we have $H_\sigma(\Omega, M_n) = \ell_\infty \bigoplus_k L^\infty(\Omega_k) \otimes C_k$ where the direct sum is finite and C_k's are Cartan factors. Therefore we have the disjoint union

$$H_\sigma\widehat{(\Omega, M_n)} = \bigcup_k L^\infty\widehat{(\Omega_k)\otimes} C_k$$
$$\cong \bigcup_k \widehat{L^\infty(\Omega_k)} \times \widehat{C_k}$$
$$\simeq \bigcup_k \widehat{L^\infty(\Omega_k)}$$

by Proposition 3.2.

□

The pointwise product of two harmonic functions need not be harmonic. We now determine when $H_\sigma(\Omega, M_n)$ is a subalgebra of $L^\infty(\Omega, M_n)$.

Theorem 4.6. *Let Ω be a homogeneous space and let $\sigma \in M_n(G, M_n)$ be positive with $\sigma(G) = I$. The following conditions are equivalent:*

(i) $H_\sigma(\Omega, M_n)$ *is a subtriple of* $L^\infty(\Omega, M_n)$*;*
(ii) $H_\sigma(\Omega, M_n)$ *is a subalgebra of* $L^\infty(\Omega, M_n)$*;*
(iii) $H_\sigma(\Omega, M_n)$ *is a von Neumann subalgebra of* $L^\infty(\Omega, M_n)$*;*
(iv) $H_\sigma(\Omega, M_n) = \{f \in L^\infty(\Omega, M_n) : \forall\, \xi \in \Omega, f(\xi \cdot y^{-1}) = f(\xi) \text{ for } |\sigma| - \text{a.e. } y \in G\}$.

Proof. We first note that, since $\sigma(G) = I$, the constant functions $f(\cdot) = A \in M_n$ are σ-harmonic.
(i)$\Longrightarrow$(ii). Let $1(\cdot) = I$ be the constant function with value $I \in M_n$. Then for any $f, g \in H_\sigma(\Omega, M_n)$, we have

$$fg = \{f, 1, g\} \in H_\sigma(\Omega, M_n).$$

(ii)$\Longrightarrow$(iii). It suffices to show that $f \in H_\sigma(\Omega, M_n)$ implies $f^* \in H_\sigma(\Omega, M_n)$, that is, $f^* * \sigma = f^*$.

Let $\varphi(\cdot) = \operatorname{Tr}(A\cdot)$ be a state of M_n where A is a positive matrix in M_n. Let $g : \Omega \to M_n$ be the constant function $g(\cdot) = A$. By condition (ii), we have $fg \in H_\sigma(\Omega, M_n)$ and therefore, for $\xi \in \Omega$,

$$\begin{aligned}
\varphi\big(f^* * \sigma(\xi)\big) &= \operatorname{Tr}\Big(A \int_G f^*(\xi \cdot y^{-1}) d\sigma(y)\Big) \\
&= \operatorname{Tr}\Big(\int_G \big(f(\xi \cdot y^{-1})A\big)^* d\sigma(y)\Big) \\
&= \operatorname{Tr}\Big(\Big(\int_G f(\xi \cdot y^{-1}) g(\xi \cdot y^{-1}) d\sigma(y)\Big)^*\Big) \\
&= \operatorname{Tr}\big(g(\xi)^* f(\xi)^*\big) = \operatorname{Tr}\big(Af^*(\xi)\big) = \varphi\big(f^*(\xi)\big).
\end{aligned}$$

As φ was arbitrary, we have $f^* * \sigma = f^*$.
(iii)$\Longrightarrow$(iv). Clearly, if $f \in L^\infty(\Omega, M_n)$ satisfies

$$f(\xi \cdot y^{-1}) = f(\xi) \quad (|\sigma| - \text{a.e. } y \in G)$$

then $f \in H_\sigma(\Omega, M_n)$. We need to prove the converse. Let $\sigma = \omega \cdot |\sigma|$ be the polar representation of σ as in [6, p.21]. Let $p \in M_n$ be a minimal projection. Then for every $y \in G$, we have $p\omega(y)p = \lambda(y)p$ for some $\lambda(y) \in [0, 1]$. Let $f \in H_\sigma(\Omega, M_n)$ and $\xi \in \Omega$. We may assume that $f = f^*$. By condition (iii), we have

$$\big(f - f(\xi)\big)^2 p \in H_\sigma(\Omega, M_n).$$

Therefore

$$\begin{aligned}
\int_G \big(f(\xi \cdot y^{-1}) - f(\xi)\big)^2 p d\sigma(y) &= \big(f - f(\xi)\big)^2 p * \sigma(\xi) \\
&= \big(f(\xi) - f(\xi)\big)^2 p = 0
\end{aligned}$$

which gives

$$\int_G \big(f(\xi \cdot y^{-1}) - f(\xi)\big)^2 p\omega(y) p d|\sigma|(y) = 0$$

and hence

$$\int_G \operatorname{Tr}\Big(\big(f(\xi \cdot y^{-1}) - f(\xi)\big)^2 p\Big)\lambda(y) d|\sigma|(y) = 0.$$

It follows that $\operatorname{Tr}\Big(p\big(f(\xi \cdot y^{-1}) - f(\xi)\big)^2 p\Big) = 0$ for $|\sigma|$-almost every $y \in G$. This implies that $p\big(f(\xi \cdot y^{-1}) - f(\xi)\big)^2 p = 0$ for $|\sigma|$-almost every $y \in G$. Since p was arbitrary, we conclude that $f(\xi \cdot y^{-1}) = f(\xi)$ for $|\sigma|$-almost every $y \in G$.
(iv)$\Longrightarrow$(iii)$\Longrightarrow$(i). Obvious.

□

We call a measure $\sigma \in M(G, M_n)$ *symmetric* if $d|\sigma|(x) = d|\sigma|(x^{-1})$. The support of the measure $|\sigma|$ is denoted by $\operatorname{supp}|\sigma|$.

Proposition 4.7 *Let Ω be a homogeneous space of a nilpotent group G and let $\sigma \in M(G, M_n)$ be symmetric, positive and $\|\sigma\| = 1$. Let $f \in H_\sigma(\Omega, M_n)$. Then we have*

$$f(\xi \cdot a) = f(\xi)$$

for every $\xi \in \Omega$ and $a \in \operatorname{supp}|\sigma|$. In particular, $H_\sigma(\Omega, M_n)$ is a subalgebra of $L^\infty(\Omega, M_n)$.

Proof. Let $q : G \to \Omega$ be the natural map onto the right coset space Ω. Then $f \circ q$ is σ-harmonic on the nilpotent group G. Using this device, we only need to prove the result for the special case of $\Omega = G$. Let G_σ be the closed subgroup of G generated by $\operatorname{supp}|\sigma|$. Since σ is symmetric, $|\sigma|$ is a *non-degenerate* probability measure on G_σ, that is, G_σ is the closed semi-group generated by $\operatorname{supp}|\sigma|$.

We first prove the result for a bounded left uniformly continuous σ-harmonic function f on G. By restricting f and σ to G_σ, we may regard f as a bounded left uniformly continuous σ-harmonic function on G_σ. By [10, Theorem 4], f is constant on G_σ and in particular, we have

$$f(xa) = f(x) \qquad (a \in \operatorname{supp}|\sigma|)$$

for all $x \in G_\sigma$, and hence for all $x \in G$, by applying the above formula to the left translate $\ell_x f$ of f at $e \in G_\sigma$.

Now for any $f \in H_\sigma(G, M_n)$ and $\psi \in L^1(G, M_n)$, the function $\widetilde{\psi} * f$ is bounded, left uniformly continuous and σ-harmonic where $\widetilde{\psi}(x) = \psi(x^{-1})$. Therefore the above arguments imply, for $a \in \operatorname{supp}|\sigma|$,

$$\langle f - f_a, \psi \rangle = \operatorname{Tr}(\widetilde{\psi} * f)(e) - \operatorname{Tr}(\widetilde{\psi} * f)(a) = 0$$

where $f_a(\cdot) = f(\cdot a)$. Hence $f = f_a$ in $H_\sigma(G, M_n)$.

The last assertion follows from direct verification.

□

We now derive further properties of $H_\sigma(\Omega, M_n)$ for $\Omega = G$. As before, given any M_n-valued function f on G, we define $\widetilde{f}(x) = f(x^{-1})$ for $x \in G$.

Lemma 4.8 *Given $\sigma \in M(G, M_n)$, the space $H_\sigma(G, M_n) \cap C_{ru}(G, M_n)$ is weak* dense in $H_\sigma(G, M_n)$.*

Proof. Let $\{h_\alpha\}$ be a bounded approximate identity in $L^1(G)$ where h_α has compact support and $\widetilde{h_\alpha} = h_\alpha$. Then the M_n-valued functions

$$\varphi_\alpha = \begin{pmatrix} h_\alpha & & 0 \\ & \ddots & \\ 0 & & h_\alpha \end{pmatrix}$$

form a bounded approximate identity in $L^1(G, M_n)$ and $\widetilde{\varphi_\alpha} = \varphi_\alpha$.

Given any $f \in H_\sigma(G, M_n)$, we have $\varphi_\alpha * f \in H_\sigma(G, M_n) \cap C_{ru}(G, M_n)$ and for every $\psi \in L^1(G, M_n)$, $\varphi_\alpha \psi = \psi \varphi_\alpha$ implies that

$$\langle \psi, \varphi_\alpha * f \rangle = \langle \varphi_\alpha * \psi, f \rangle.$$

Hence $(\varphi_\alpha * f)$ weak* converges to f.

□

We describe below the Jordan triple product of matrix-valued harmonic functions on G in more detail which extends [8, Theorem 3.3.12].

Theorem 4.9. *Let $\sigma \in M(G, M_n)$ with $\|\sigma\| = 1$. Then $H_\sigma(G, M_n) \cap C_{ru}(G, M_n)$ is a subtriple of $H_\sigma(G, M_n)$ and there is a net $\{\mu_\alpha\}$ in the convex hull of $\{\sigma^n : n \in \mathbb{N}\}$ such that the triple product of $f, g, h \in H_\sigma(G, M_n)$ is given by*

$$2\{f, g, h\} = \lim_\alpha (fg^*h + hg^*f) * \mu_\alpha$$

where the convergence is uniform on compact subsets of $L^1(G, M_n^*)$. Further, if $f, g, h \in H_\sigma(G, M_n) \cap C_{ru}(G, M_n)$, then the convergence is uniform on compact subsets of G.

Proof. Let $f, g, h \in H_\sigma(G, M_n) \cap C_{ru}(G, M_n)$. Then

$$2\{f, g, h\} = P_\sigma(fg^*h + hg^*f)$$

where $P_\sigma : L^\infty(G, M_n) \to H_\sigma(G, M_n)$ is the contractive projection in Proposition 4.1 and P_σ commutes with any weak* continuous map $T : L^\infty(G, M_n) \to L^\infty(G, M_n)$ satisfying $T(f * \sigma) = (Tf) * \sigma$ for all $f \in L^\infty(G, M_n)$. In particular, P_σ commutes with the left translations ℓ_a $(a \in G)$. Hence by Lemma 2.5, we have $\{f, g, h\} \in C_{ru}(G, M_n)$ which proves that $H_\sigma(G, M_n) \cap C_{ru}(G, M_n)$ is a subtriple of $H_\sigma(G, M_n)$.

The second assertion follows from Remark 4.3 and the Mackey-Arens Theorem.

Now let $f, g, h \in H_\sigma(G, M_n) \cap C_{ru}(G, M_n)$. To show that the above limit converges uniformly on compact sets in G, it suffices to prove it entrywise. Let

$$\varphi = \frac{1}{2}(fg^*h + hg^*f).$$

We show that, for each $i, j, k \in \{1, \ldots, n\}$, there is a subset $\{\mu_\beta\}$ of $\{\mu_\alpha\}$ such that $\{\sum_k \varphi_{ik} * (\mu_\beta)_{kj}\}_\beta$ converges to $\{f, g, h\}_{ij}$ uniformly on compact sets in G.

Write $C_{ru}(G)$ for $C_{ru}(G, \mathbb{C})$. For $m \in C_{ru}(G)^*$, we define $m \cdot \varphi_{ik} \in C_{ru}(G)$ by

$$(m \cdot \varphi_{ik})(x) = m(\ell_x \varphi_{ik}) \qquad (x \in G).$$

Let $S = \{m \cdot \varphi_{ik} : m \in C_{ru}(G)^*,\ \|m\| \leq 1\}$. Then S is a pointwise compact subset of $C_{ru}(G)$. Moreover, S is bounded and equicontinuous. Indeed, the inequality $\|m \cdot \varphi_{ik}\| \leq \|m\|\ \|\varphi_{ik}\|$ implies that S is bounded. Given $\varepsilon > 0$, there is a neighbourhood V of the identity $e \in G$ such that

$$uv^{-1} \in V \quad \text{implies} \quad \|\ell_u \varphi_{ik} - \ell_v \varphi_{ik}\| < \varepsilon.$$

Hence $|(m \cdot \varphi_{ik})(u) - (m \cdot \varphi_{ik})(v)| = |m(\ell_u \varphi_{ik}) - m(\ell_v \varphi_{ik})| < \varepsilon$.

We have

$$\varphi * \mu_\alpha(x) = \int_G \varphi(xy^{-1}) d\mu_\alpha(y) = \int_G \varphi(xy) d\widetilde{\mu}_\alpha(y)$$

where $d\widetilde{\mu}_\alpha(y) = d\mu_\alpha(y^{-1})$. We may regard $(\widetilde{\mu}_\alpha)_{kj}$ as an element in $C_{ru}(G)^*$ by defining

$$(\widetilde{\mu}_\alpha)_{kj}(w) = \int_G w d(\widetilde{\mu}_\alpha)_{kj} \qquad \big(w \in C_{ru}(G)\big)$$

in which case we have

$$\int_G \varphi_{ik}(xy)d(\widetilde{\mu}_\alpha)_{kj}(y) = (\widetilde{\mu}_\alpha)_{kj} \cdot \varphi_{ik}(x)$$

and $(\varphi * \mu_\alpha)_{ij} = \sum_k (\widetilde{\mu}_\alpha)_{kj} \cdot \varphi_{ik}$. By Ascoli's theorem, there is a subnet $\{(\widetilde{\mu}_\gamma)_{kj} \cdot \varphi_{ik}\}$ which converges uniformly on compact subsets of G, and therefore converges in the weak* topology of $L^\infty(G)$.

Running through the index $k \in \{1, \dots, n\}$, we can therefore find a subnet $\{\mu_\beta\}$ of $\{\mu_\alpha\}$ such that $\{\sum_k (\widetilde{\mu}_\beta)_{kj} \cdot \varphi_{ik}\}_\beta$ converges uniformly on compact subsets of G, and in the weak* topology of $L^\infty(G)$. It follows that

$$\begin{aligned}\{f, g, h\}_{ij} &= \lim_\beta (\varphi * \mu_\beta)_{ij} \\ &= \lim_\beta \sum_k (\widetilde{\mu}_\beta)_{kj} \cdot \varphi_{ik}\end{aligned}$$

uniformly on compact sets in G. Finally, running through the ij-entries, we can find a subnet $\{\mu_{\alpha'}\}$ such that

$$\{f, g, h\} = \lim_{\alpha'} \varphi * \mu_{\alpha'}$$

uniformly on compact subsets of G.

□

We now consider the case when the dual space $H_\sigma(G, M_n)^*$ is a Banach algebra. Let $H_\sigma(G, M_n) = \big(L^1(G, M_n^*)/J\big)^*$ where J is defined in (4.2). We note that $H_\sigma(G, M_n)$ is a left translation invariant subspace of $L^\infty(G, M_n)$ and the identity

$$\langle \ell_a h, f \rangle = \Delta(a^{-1})\langle h, \ell_{a^{-1}} f \rangle$$

implies that J is left-translation invariant in $L^1(G, M_n)$, where Δ is the modular function of G. If $H_\sigma(G, M_n)$ is also right-translation invariant, then $L^1(G, M_n^*)/J$ is a Banach algebra by the following lemma. It follows that $H_\sigma(G, M_n)^* = \big(L^1(G, M_n^*)/J\big)^{**}$ is a Banach algebra with respect to the two Arens products.

Given $h \in L^1(G, M_n^*)$ and $f \in L^\infty(G, M_n)$, we can define two functions $h \cdot f$, $f \cdot h \in L^\infty(G, M_n)$ by

$$\begin{aligned}(h \cdot f)(x) &= \int_G h(y)(r_x f)(y) d\lambda(y) \\ (f \cdot h)(x) &= \int_G h(y)(\ell_x f)(y) d\lambda(y)\end{aligned}$$

for $x \in G$.

Proposition 4.10 *Let $\sigma \in M(G, M_n)$. Then the following conditions are equivalent :*

(i) *$H_\sigma(G, M_n)$ is translation invariant;*
(ii) *J is translation invariant;*
(iii) *J is an ideal in $L^1(G, M_n^*)$;*
(iv) *$\int_G f(xy^{-1})d\sigma(y) = \int_G f(y^{-1}x)d\sigma(y)$ for $f \in H_\sigma(G, M_n) \cap C_{ru}(G, M_n)$;*
(v) *if $h \in L^1(G, M_n^*)$ and $f \in H_\sigma(G, M_n)$, then $h \cdot f$, $f \cdot h \in H_\sigma(G, M_n)$.*

Proof. (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$ (iii). Given $h \in L^1(G, M_n^*)$ and $f \in L^\infty(G, M_n)$, we have

$$\langle h, r_a f\rangle = \Delta(a^{-1})\langle r_{a^{-1}}h, f\rangle$$

for every $a \in G$. Hence $H_\sigma(G, M_n)$ is right-translation invariant if, and only if, J is. The equivalence of (ii) and (iii) is standard.

(i)$\Longrightarrow$(iv). Let $f \in H_\sigma(G, M_n) \cap C_{ru}(G, M_n)$. The right invariance of $H_\sigma(G, M_n)$ gives

$$\begin{aligned}\int_G f(xy^{-1})d\sigma(y) &= (f * \sigma)(x) \\ &= r_x(f * \sigma)(e) \\ &= (r_x f)(e) = (r_x f * \sigma)(e) \\ &= \int_G f(y^{-1}x)d\sigma(y).\end{aligned}$$

(iv)$\Longrightarrow$(i). Let $f \in H_\sigma(G, M_n) \cap C_{ru}(G, M_n)$. Then $\ell_x f \in H_\sigma(G, M_n) \cap C_{ru}(G, M_n)$ and

$$\begin{aligned}(r_a f)(x) = r_a(f * \sigma)(x) &= \int_G f(xay^{-1})d\sigma(y) \\ &= \int_G (\ell_x f)(ay^{-1})d\sigma(y) \\ &= \int_G (\ell_x f)(y^{-1}a)d\sigma(y) \\ &= \int_G f(xy^{-1}a)d\sigma(y) \\ &= (r_a f) * \sigma(x)\end{aligned}$$

which yields the right invariance of $H_\sigma(G, M_n) \cap C_{ru}(G, M_n)$, and hence, of $H_\sigma(G, M_n)$ by weak* density from Lemma 4.8.

(i)$\Longrightarrow$(v). Since $h \cdot f = \widetilde{h} * f$, we have $h \cdot f \in H_\sigma(G, M_n)$ whenever $f \in H_\sigma(G, M_n)$. By (i), we also have $r_z f \in H_\sigma(G, M_n)$ and

$$\begin{aligned}(f \cdot h) * \sigma(x) &= \int_G (f \cdot h)(xy^{-1})d\sigma(y) \\ &= \int_G \int_G h(z) f(xy^{-1}z)d\sigma(y)d\lambda(z) \\ &= \int_G \int_G h(z)(r_z f)(xy^{-1})d\sigma(y)d\lambda(z) \\ &= \int_G h(z)(r_z f)(x)d\lambda(z) = (f \cdot h)(x).\end{aligned}$$

(v)$\Longrightarrow$(iii). We show J is a right ideal. Let $h \in J$ and $k \in L^1(G, M_n^*)$. For any $f \in H_\sigma(G, M_n)$, we have $f \cdot k \in H_\sigma(G, M_n)$ and

$$\begin{aligned}\langle h * k, f\rangle &= \operatorname{Tr}\left(\int_G (h*k)(x) f(x) d\lambda(x)\right) \\ &= \operatorname{Tr}\left(\int_G \int_G h(xy^{-1})k(y)f(x)d\lambda(y)d\lambda(x)\right) \\ &= \langle h, f \cdot k\rangle = 0.\end{aligned}$$

Hence $h * k \in J$.

□

To conclude the section, we discuss bounded σ-harmonic functions in the context of Riemannian symmetric spaces. Let X be a Riemannian symmetric space and let Δ be the Laplace-Beltrami operator on X. A C^2 function $f : X \longrightarrow \mathbb{R}$ is *harmonic* if $\Delta f = 0$. A complex-valued function on X is *harmonic* if its real and imaginary parts are harmonic.

Let X be a simply connected Riemannian symmetric space. Then X is a product

$$X = X_0 \times X_+ \times X_-$$

where X_0 is Euclidean, X_+ is of compact type and X_- is of non-compact type [22, p.244]. Since X_0 and X_+ have non-negative sectional curvatures, the bounded harmonic functions on these manifolds are constant by a result of Yau [43], and we need only consider symmetric spaces of non-compact type.

Let $X = H\backslash G$ be a Riemannian symmetric space of non-compact type, represented as a homogeneous space $H\backslash G$ of a semisimple Lie group G by a maximal compact subgroup H. An example is the symmetric space $S0(n)\backslash SL(n, \mathbb{R})$. Furstenberg [20] has shown that there is an absolutely continuous H-invariant probability measure σ on G such that a bounded function f on $X = H\backslash G$ is harmonic if, and only if, the *inverted function* $\widetilde{f}$ is σ-harmonic, where we define

$$\widetilde{f}(Ha) = f(Ha^{-1}) \qquad (Ha \in H\backslash G).$$

Let $\mathcal{H}(X, \mathbb{C}) = \{f \in L^\infty(X, \mathbb{C}) : \Delta f = 0\}$ be the space of bounded harmonic functions on X. Then the map $f \in \mathcal{H}(X, \mathbb{C}) \mapsto \widetilde{f} \in H_\sigma(X, \mathbb{C}) = \{h \in L^\infty(X, \mathbb{C}) : h * \sigma = h\}$ is a surjective linear isometry and we have readily the following result.

Corollary 4.11 $\mathcal{H}(X, \mathbb{C})$ *is a unital abelian* C^**-algebra.*

Proof. Since σ is a probability measure, $H_\sigma(X, \mathbb{C})$ contains constant functions. Since it is the range of a contraction projection $P : L^\infty(X, \mathbb{C}) \longrightarrow H_\sigma(X, \mathbb{C})$, by Proposition 4.1 and Corollary 4.2, $H_\sigma(X, \mathbb{C})$ is an abelian C*-algebra with product $f \cdot g = P(fg)$.

□

Remark 4.12 Let M be the pure state space of the C*-algebra $\mathcal{H}(X, \mathbb{C}) \approx H_\sigma(X, \mathbb{C})$. Then $\mathcal{H}(X, \mathbb{C})$ is isometric to the space $C(M)$ of continuous functions on M which gives a Poisson representation of $\mathcal{H}(X, \mathbb{C})$, with M as the Poisson boundary. The construction is similar to [8, Proposition 2.2.13].

Let $\Omega = H\backslash G$ be a homogeneous space. A measure $\sigma \in M(G, M_n)$ is said to be *H-invariant* if it satisfies $d\sigma(hk) = d\sigma(h)$ for all $k \in H$. Following [21], we say that G *acts amenably* on Ω if there is a right invariant mean on $L^\infty(\Omega, \mathbb{C})$, in which case, we also say that Ω is *G-amenable*. We now show when Ω must be G-amenable if it admits the Liouville theorem. We note that Example 4.4 shows that $H_\sigma(G, M_n) = \{0\}$ can occur for non-amenable groups G. However, if σ is a probability measure on G, then the constant function $f(\cdot) = I \in M_n$ is σ-harmonic on Ω.

Theorem 4.13 *Let $\Omega = H\backslash G$ be a homogeneous space where H is compact. Let $\sigma \in M(G, M_n)$ be a positive H-invariant measure with $\|\sigma\| = 1$. If all σ-harmonic $L^\infty(\Omega, M_n)$-function on Ω are constant, not all 0, then Ω is G-amenable.*

Proof. Given $f \in L^\infty(\Omega, M_n)$, we have $f \in H_\sigma(\Omega, M_n)$ if, and only if, $f \circ q \in H_\sigma(G, M_n)$ where $q : G \longrightarrow H\backslash G$ is the quotient map. Hence $H_\sigma(G, M_n)$ is non-zero; but we show it contains only constant functions. Let $F \in H_\sigma(G, M_n)$ and define a function $f : \Omega \longrightarrow M_n$ by

$$f(Ha) = \int_H F(ha)dh$$

where dh is the normalized Haar measure on the compact group H. Then f is σ-harmonic on Ω:

$$\begin{aligned}(f * \sigma)(Ha) &= \int_G f(Hay^{-1})d\sigma(y)\\ &= \int_H \int_G F(hay^{-1})d\sigma(y)dh\\ &= \int_H F(ha)dh = f(Ha).\end{aligned}$$

Hence f is constant by hypothesis and in particular,

$$\int_H F(ha)dh = \int_H F(h)dh \qquad (a \in G).$$

By H-invariance of σ, we have

$$\begin{aligned}F(e) &= \int_G F(y^{-1})d\sigma(y)\\ &= \int_G \int_H F(hy^{-1})dhd\sigma(y)\\ &= \int_G \int_H F(h)dhd\sigma(y)\\ &= \left(\int_H F(h)dh\right)\sigma(G).\end{aligned}$$

Applying the above result to the left translate $F(a\cdot)$ of F, we obtain

$$F(a) = \left(\int_H F(ah)dh\right)\sigma(G) \qquad (a \in G)$$

which gives $F(ah) = F(a)$ for all $h \in H$. Therefore we can define a function $\varphi : \Omega \longrightarrow M_n$ by

$$\varphi(Ha) = F(a^{-1}) \qquad (a \in G).$$

Then $\widetilde{\varphi}(Ha) = F(a)$ is σ-harmonic on Ω and hence constant. Therefore F is constant.

Now, by [6, Corollary 19], G is amenable and hence H is G-amenable.

□

As noted above, on a Riemannian symmetric space $X = H\backslash G$ of non-compact type, the bounded harmonic functions are the functions $\{\widetilde{f} : f \in H_\sigma(X, \mathbb{C})\}$, for some H-invariant probability measure σ on G. We see from Theorem 4.13 that $\mathcal{H}(X, \mathbb{C})$ contains non-constant functions.

5. Fourier algebras of homogeneous spaces

In this section, we study harmonic functionals on the Fourier algebras of homogeneous spaces and extend the results for groups in [8, Chapter 3] to this setting. Let H be a closed subgroup of a locally compact group G. As in [16], one can define the Fourier algebra $A(H\backslash G)$ and the Fourier-Stieltjes algebra $B(H\backslash G)$ associated to the homogeneous space $H\backslash G$. This extends the notion of the Fourier algebra $A(G)$ and Fourier-Stieltjes algebra $B(G)$ of a group G. The harmonic functionals on $A(G)$ have been studied in [8]. We define harmonic functionals on $A(H\backslash G)$ and extend the results in [8] to $A(H\backslash G)$.

Let G be a locally compact group. We recall that the *Fourier-Stieltjes algebra* $B(G)$ is a commutative Banach algebra consisting of complex-valued coefficient functions $\langle \pi(\cdot)\xi, \eta\rangle$ on G, where π is a continuous unitary representation of G on a Hilbert space H_π and $\xi, \eta \in H_\pi$, and the multiplication in $B(G)$ is pointwise. As a Banach space, $B(G)$ can be identified with the dual of the group C^*-algebra $C^*(G)$ of G. The *Fourier algebra* $A(G)$ is the subspace of $B(G)$, consisting of coefficient functions $\langle \rho(\cdot)\xi, \eta\rangle$ where ρ is the right regular representation of G. In fact, $A(G)$ is a closed ideal of $B(G)$ and its dual $A(G)^*$ is isometrically isomorphic to the group von Neumann algebra $VN(G)$ which is the ultra weak closure of the linear span of $\rho(G)$ in the algebra $\mathcal{L}\big(L^2(G)\big)$ of bounded operators on $L^2(G)$.

Let H be a closed subgroup of G and let $q : G \to H\backslash G$ be the canonical right coset map. As in [16], we define the following closed subalgebras of $B(G)$:

$$B(H\backslash G) = \{u \in B(G) : u(hx) = u(x), \forall x \in G, h \in H\};$$
$$A(H\backslash G) = \{u \in B(H\backslash G) : q(\text{supp } u) \text{ is compact in } H\backslash G\}^-$$

where '$-$' denotes the closure in the norm topology of $B(G)$. These algebras are called respectively the *Fourier-Stieltjes* and *Fourier algebras* of the homogeneous space $H\backslash G$. If H is normal, then $B(H\backslash G)$ and $A(H\backslash G)$ are just the Fourier-Stieltjes and Fourier algebras of the quotient group $H\backslash G$.

Evidently, both $B(H\backslash G)$ and $A(H\backslash G)$ identify naturally as subalgebras, but not closed unless H has finite index, of the C^*-algebra $C_b(H\backslash G)$ of bounded complex continuous functions on the homogeneous space $H\backslash G$. We will use this identification whenever it is convenient.

To introduce harmonic functionals on $A(H\backslash G)$, our first task is to identify the dual space $A(H\backslash G)^*$. We begin by recalling the following basic results which have been proved in [16]. We note that, in [16], the algebras $A(H\backslash G)$ and $B(H\backslash G)$ are defined for the left coset space G/H, but the results carry to our setting of the right coset space $H\backslash G$.

Lemma 5.1 *Let H be a closed subgroup of G. Then $A(H\backslash G)$ is a closed ideal of $B(H\backslash G)$ and we have*

(i) $A(H\backslash G) = B(H\backslash G)$ *if, and only if,* $H\backslash G$ *is compact.*
(ii) $A(H\backslash G) \cap A(G) \neq \{0\}$ *if, and only if,* H *is compact.*

Given that H *is compact, then* $A(H\backslash G)$ *is a regular Banach algebra whose spectrum is homeomorphic to* $H\backslash G$*. Further,* G *is amenable if, and only if,* $A(H\backslash G)$ *has an approximate identity* $\{u_\alpha\}$ *with* $\|u_\alpha\| \leq 1$.

Let $C^*(G)$ be the group C^*-algebra of G. We can identify the dual $B(G)^*$ with the enveloping von Neumann algebra $W^*(G)$ of $C^*(G)$. We denote the dual pairing between $B(G)$ and $W^*(G)$ by $\langle \cdot, \cdot \rangle : B(G) \times W^*(G) \to \mathbb{C}$. There is a continuous monomorphism $\omega : G \to W^*(G)$ such that

$$\langle u, \omega(a) \rangle = u(a) \quad \big(u \in B(G), a \in G\big)$$

and $\omega(G)$ is w^*-dense in $W^*(G)$ (cf. [13,14]). For each $u \in B(G)$ and $T \in W^*(G)$, we define $T \cdot u \in B(G)$ by

$$\langle T \cdot u, S \rangle = \langle u, ST \rangle \quad \big(S \in W^*(G)\big).$$

Then $\omega(a) \cdot u$ is just the right translate u_a of u for $a \in G$. Since $B(H\backslash G)$ and $A(H\backslash G)$ are both right translation invariant in $B(G)$, and since $\omega(G)$ is w^*-dense in $W^*(G)$, we see that $B(H\backslash G)$ and $A(H\backslash G)$ are left invariant subspaces of $B(G)$ with respect to the above product $T \cdot u$:

$$T \cdot B(H\backslash G) \subset B(H\backslash G); \quad T \cdot A(H\backslash G) \subset A(H\backslash G) \quad \big(T \in W^*(G)\big).$$

Hence, by [38, p. 124], there are projections $p_H,\ q_H \in W^*(G)$ such that $B(H\backslash G) = B(G) \circ q_H$ and $A(H\backslash G) = B(G) \circ p_H$ where $u \circ p_H$ is defined by

$$\langle u \circ p_H, S \rangle = \langle u, p_H S \rangle \qquad (S \in W^*(G))$$

for $u \in B(G)$. Taking duals, we have the following description of the dual spaces $B(H\backslash G)^*$ and $A(H\backslash G)^*$.

Lemma 5.2 *Let* H *be a closed subgroup of* G*. Then there are projections* p_H *and* q_H *in* $W^*(G)$ *such that* $A(H\backslash G)^* = p_H W^*(G)$ *and* $B(H\backslash G)^* = q_H W^*(G)$.

We note that, by the work of Takesaki and Tatsuuma [39], the closed right translation invariant $*$-subalgebras of $A(G)$ are exactly the algebras $A(H\backslash G)$ with compact subgroups $H \subset G$ in which case we have

$$A(H\backslash G) = p_H^c A(G) \tag{5.1}$$

for some projection $p_H^c \in VN(G)$ and

$$A(H\backslash G)^* = p_H^c VN(G) \tag{5.2}$$

(see also [25, Theorem 5.3.5] for a similar result for the left coset space G/H).

Motivated by the fact that the dual $A(G)^*$ identifies with the group von Neumann algebra $VN(G)$, we will denote $A(H\backslash G)^*$ by $VN(H\backslash G)$ which is a right ideal of $W^*(G)$ and hence a JW*-subtriple of $W^*(G)$. If H is a normal subgroup of G, then both $B(H\backslash G)$ and $A(H\backslash G)$ are also right invariant subspaces of $B(G)$, and therefore p_H and q_H are central projections in $W^*(G)$. Likewise we define $C^*(H\backslash G) := q_H C^*(G)$. Then we have $C^*(H\backslash G)^* = \big(q_H C^*(G)\big)^* = B(G) \circ q_H = B(H\backslash G)$.

Remark 5.3 We have $A(G) = B(G) \circ p_{\{e\}}$ where $p_{\{e\}}$ is a central projection in $W^*(G)$. Also $(B(G) \circ p_{\{e\}}) \cap (B(G) \circ p_H) = A(G) \cap A(H\backslash G) = \{0\}$ if H is non-compact in which case $p_{\{e\}} p_H = 0$ and $A(G) \circ p_H = B(G) \circ (p_{\{e\}} p_H) = \{0\}$. Likewise $A(G) \circ q_H = \{0\}$ if H is non-compact as $A(G) \cap B(H\backslash G) = \{0\}$.

We are now ready to introduce the harmonic functionals of $A(H\backslash G)$. Let $\sigma \in B(H\backslash G)$ and let

$$I_\sigma(H\backslash G) = \{\sigma\varphi - \varphi : \varphi \in A(H\backslash G)\}^-$$

where '$^-$' denotes the norm closure. Evidently $I_\sigma(H\backslash G)$ is a closed ideal of $A(H\backslash G)$. Let $\langle \cdot, \cdot \rangle : A(H\backslash G) \times VN(H\backslash G) \to \mathbb{C}$ be the dual pairing of $A(H\backslash G)^* = VN(H\backslash G)$. Since $A(H\backslash G)$ is an ideal of $B(H\backslash G)$, we can define an action of $B(H\backslash G)$ on $VN(H\backslash G)$ as follows. For $\sigma \in B(H\backslash G)$ and $T \in VN(H\backslash G)$, we define $\sigma \cdot T \in VN(H\backslash G)$ by

$$\langle \varphi, \sigma \cdot T \rangle = \langle \sigma\varphi, T \rangle \qquad \big(\varphi \in A(H\backslash G)\big). \tag{5.3}$$

It is readily verified that the annihilator of $I_\sigma(H\backslash G)$ is given by

$$\begin{aligned} I_\sigma(H\backslash G)^\perp &= \{T \in VN(H\backslash G) : \langle I_\sigma(H\backslash G), T \rangle = 0\} \\ &= \{T \in VN(H\backslash G) : \sigma \cdot T = T\}. \end{aligned}$$

We call the elements in $I_\sigma(H\backslash G)^\perp$ the *σ-harmonic functionals* of $A(H\backslash G)$ and we denote $A(H\backslash G)^*_\sigma := I_\sigma(H\backslash G)^\perp$.

We first show that $A(H\backslash G)^*_\sigma$ is a JW^*-triple.

Proposition 5.4 *Let $\sigma \in B(H\backslash G)$ with $\|\sigma\| = 1$. Then there exists a contractive projection $P_\sigma : VN(H\backslash G) \to A(H\backslash G)^*_\sigma$ such that $P_\sigma(\varphi \cdot T) = \varphi \cdot P_\sigma(T)$ for $\varphi \in A(H\backslash G)$ and $T \in VN(H\backslash G)$. In particular, $A(H\backslash G)^*_\sigma$ is a JW^*-triple.*

Proof. The arguments are similar to those for the case of $H = \{e\}$ given in [8, Proposition 3.3.1]. We sketch the construction of P_σ. Let $\mathcal{L}\big(VN(H\backslash G)\big)$ be the locally convex space of bounded linear maps from $VN(H\backslash G)$ to itself, equipped with the weak*-operator topology. For $n = 1, 2, \dots$, define $\Lambda_n : VN(H\backslash G) \to VN(H\backslash G)$ by

$$\Lambda_n(T) = \sigma^n \cdot T.$$

Let K be the closed convex hull of $\{\Lambda_n : n = 1, 2, \dots\}$ in $\mathcal{L}\big(VN(H\backslash G)\big)$ and define a continuous affine map $\Phi : K \to K$ by $\Phi(\Lambda) = \sigma \cdot \Lambda$. By compactness of K and the Markov-Kakutani fixed point theorem, there exists $P_\sigma \in K$ such that $\Phi(P_\sigma) = P_\sigma$ which is the required projection.

Finally $VN(H\backslash G) = p_H W^*(G)$ is a JW^*-triple as it is a right ideal of a von Neumann algebra and it follows that $A(H\backslash G)^*_\sigma$, being the range of a contractive projection on JW^*-triple, is also a JW^*-triple, by [28].

□

We refer to [8, 3.3] for examples in which the above projection P_σ is weak* continuous.

Corollary 5.5 *Let $\sigma \in B(H\backslash G)$ with $\|\sigma\| = 1$. If $VN(H\backslash G)$ is of type j for $j =$* I, II *or* III*, and if the projection $P_\sigma : VN(H\backslash G) \longrightarrow A(H\backslash G)^*_\sigma$ is weak* continuous, then each direct summand of $A(H\backslash G)^*_\sigma$ is a JW^*-triple of type k with $k \leq j$.*

Proof. By Proposition 3.1.

□

Remark 5.6 If $H = \{e\}$, then $A(H\backslash G)^*_\sigma$ is always a right ideal of a von Neumann algebra. To see this, let $\sigma \in B(G)$ with $\|\sigma\| = 1$ and let

$$Z(I_\sigma) = \{x \in G : \sigma(x) = 1\}$$

be the zero set of $I_\sigma := I_\sigma(\{e\}\backslash G) \subset A(G)$. If $Z(I_\sigma) = \emptyset$, then $I_\sigma = A(G)$ and $A(\{e\}\backslash G)^*_\sigma = I_\sigma^\perp = VN(G)$. If $Z(I_\sigma) \neq \emptyset$, pick $a \in Z(I_\sigma)$ and let $\gamma = \ell_{a^{-1}}\sigma$ be the left translate of σ. We have $\|\gamma\| = \gamma(e) = 1$ and $K = \{x \in G : \gamma(x) = 1\}$ is a closed subgroup of G by [23, 32.7].

Write $I_\gamma = I_\gamma(\{e\}\backslash G)$. Its zero set $Z(I_\gamma)$ is the group K. By [8, 3.2.10], $I_\gamma^\perp$ is the von Neumann subalgebra $VN_K(G)$ of $VN(G)$, generated by $\{\rho(x) : x \in Z(I_\gamma)\} = \{\rho(x) : x \in K\}$, where ρ is the right regular representation of G. We have $Z(I_\sigma) = a^{-1}K$ and $I_\sigma^\perp = \{\rho(x) : x \in a^{-1}K\}'' = \rho(a^{-1})\{\rho(x) : x \in K\}'' = \rho(a^{-1})VN_K(G)$. We make this observation following a suggestion of V. Losert and M. Leinert. More generally, for any compact subgroup H, we have

$$I_\gamma(H\backslash G)^\perp = \{T \in p_H^c VN(G) : \sigma \cdot T = T\} \subset I_\gamma^\perp = VN_K(G)$$

where p_H^c is the projection in (5.2). Therefore

$$I_\gamma(H\backslash G)^\perp = p_H^c VN(G) \cap VN_K(G) = p_H^c VN_K(G).$$

Likewise

$$I_\sigma(H\backslash G)^\perp = p_H^c VN(G)$$

if $Z(I_\sigma) = \emptyset$ while

$$I_\sigma(H\backslash G)^\perp = p_H^c(\rho(a^{-1})VN_K(G))$$

if there exists $a \in Z(I_\sigma)$ as above.

We note that the case $Z(I_\sigma) = \emptyset$ can occur as the following simple example shows.

Example 5.7 Let $G = \{a, e\}$ and consider the complex measure $\mu = \frac{1}{2}\delta_a + \frac{i}{2}\delta_e$, where δ_x denotes the point mass at $x \in G$. It is easily seen that there is no nonzero function $f : G \longrightarrow \mathbb{C}$ satisfying the convolution equation $f * \mu = f$. Let $\sigma \in A(\widehat{G}) = B(\widehat{G})$ be the Fourier transform of μ. Then $I_\sigma^\perp = \{0\}$, in other words, $I_\sigma = A(\widehat{G})$ and $Z(I_\sigma) = \emptyset$.

We now study the ideal $I_\sigma(H\backslash G)$ whose annihilator in $VN(H\backslash G)$ is the space of σ-harmonic functionals. As a non-commutative analogue of the Liouville theorem for harmonic functions, we are interested in the question of triviality of $A(H\backslash G)^*_\sigma = I_\sigma(H\backslash G)^\perp$. In the special case of $H = \{e\}$, we have shown in [8, Proposition 3.2.7] that one has $A(\{e\}\backslash G)^*_\sigma = \mathbb{C}I$ for some positive definite $\sigma \in B(G)$ with $\sigma(e) = 1$ only if G is first countable. We will prove similar results for compact subgroups $H \subset G$. For this, we first need to develop some relevant results on topological means on $VN(H\backslash G)$ which are closely related to the amenability of the homogeneous space $H\backslash G$.

We note that, in Lemma 3.2.4 and hence the proof of $(ii) \Rightarrow (i)$ in Proposition 3.2.7 of [8], the group G is required to be *second countable.*

Let H be a closed subgroup of G and let

$$A(H\backslash G)_0 = \{\varphi \in A(H\backslash G) : \varphi(e) = 0\}$$
$$S(H\backslash G) = \{\varphi \in A(H\backslash G) : \|\varphi\| = \langle \varphi, p_H \rangle = 1\}$$

where $VN(H\backslash G) = p_H W^*(G)$ as in Lemma 5.2, and $\langle \varphi, p_H\rangle = \langle \varphi, p_H\omega(e)\rangle = \langle \varphi \circ p_H, \omega(e)\rangle = \langle \varphi, \omega(e)\rangle = \varphi(e)$. We note that $S(H\backslash G)$ is a commutative semigroup with pointwise multiplication.

A functional $m \in VN(H\backslash G)^*$ is called a *mean* if $m(p_H) = \|m\| = 1$. A mean m is called a *topological invariant mean* if

$$\langle \varphi \cdot T, m\rangle = \langle T, m\rangle$$

for all $\varphi \in A(H\backslash G)$ and $T \in VN(H\backslash G)$, where $\varphi \cdot T \in VN(H\backslash G)$ is defined in (5.3).

Lemma 5.8 *Let G be a locally compact group and H be a closed subgroup of G. Let m be a linear functional on $p_H W^*(G)$ such that $m(p_H) = 1 = \|m\|$. Then there exists a net (ψ_α) in $S(H\backslash G)$ such that (ψ_α) converges to m in the weak*-topology.*

Proof. Write $p = p_H$. Define $\widetilde{m} : W^*(G) \to \mathbb{C}$ by $\widetilde{m}(T) = m(pT)$. Then $\widetilde{m}(I) = \widetilde{m}(p) = 1 = \|\widetilde{m}\|$, so $\widetilde{m}$ is a state of $W^*(G)$. By the Cauchy-Schwarz inequality, $\widetilde{m}\big(T(I-p)\big) = \widetilde{m}\big((I-p)T\big) = 0$ for each $T \in W^*(G)$ because $\widetilde{m}(I-p) = 0$. Hence $\widetilde{m}(T) = \widetilde{m}(Tp) = \widetilde{m}(pT) = \widetilde{m}(pTp)$. Consequently $\widetilde{m}$ is supported by p and we may regard $\widetilde{m} : pW^*(G)p \to \mathbb{C}$ as a state on $pW^*(G)p$. Let $\varphi_\alpha : pW^*(G)p \to \mathbb{C}$ be normal states such that $\varphi_\alpha \overset{w^*}{\to} \widetilde{m}$ on pW^*p. Define $\Phi : W^*(G) \to pW^*(G)p$ by $\Phi(T) = pTp$. We have $\varphi_\alpha \circ \Phi : W^*(G) \to \mathbb{C}$ and

$$(\varphi_\alpha \circ \Phi)(T) = \varphi_\alpha(pTp) \to \widetilde{m}(pTp) = \widetilde{m}(T) = m(pT).$$

Define $\psi_\alpha : pW^*(G) \to \mathbb{C}$ by

$$\psi_\alpha(pT) = (\varphi_\alpha \circ \Phi)(pT).$$

Then $\psi_\alpha(p) = (\varphi_\alpha \circ \Phi)(p) = \varphi_\alpha(p) = 1$ and

$$\|\psi_\alpha\| \le \|\varphi_\alpha \circ \Phi\| \le 1.$$

Hence $\|\psi_\alpha\| = 1$, and $\psi_\alpha \in S(H\backslash G)$. Also

$$\psi_\alpha(pT) = (\varphi_\alpha \circ \Phi)(pT) \to m(pT)$$

so $\psi_\alpha \overset{w^*}{\to} m$.

□

We now extend Renauld's result in [32] to homogeneous spaces.

Lemma 5.9 *Let H be a closed subgroup of G. Then there exists a topological invariant mean on $VN(H\backslash G)$. Also, there exists a net (ψ_α) in $S(H\backslash G)$ such that $\|\psi_\alpha\varphi - \psi_\alpha\| \to 0$ for each $\varphi \in S(H\backslash G)$.*

Proof. Let $K = \{m \in VN(H\backslash G)^* : m(p_H) = \|m\| = 1\}$ which is a w^*-compact convex set in $VN(H\backslash G)^*$. For each $\varphi \in S(H\backslash G)$, we can define an affine map $\pi_\varphi : K \to K$ by

$$\langle T, \pi_\varphi(m)\rangle = \langle \varphi \cdot T, m\rangle \quad \big(T \in VN(H\backslash G)\big).$$

This map is well-defined since $\|\pi_\varphi(m)\| \le \|m\| \le 1$ and $\pi_\varphi(m)(p_H) = \langle \varphi \cdot p_H, m\rangle = \langle p_H, m\rangle = 1$ where $\varphi \in S(H\backslash G)$ implies $\varphi \cdot p_H = p_H$ as $\langle \psi, \varphi \cdot p_H\rangle = \langle \varphi\psi, p_H\rangle = \psi(e) = \langle \psi, p_H\rangle$ for all $\psi \in A(H\backslash G)$. By the Markov-Kakutani fixed-point theorem, the commuting family $\{\pi_\varphi : \varphi \in S(H\backslash G)\}$ has a fixed-point $m \in K$ which is the required topological invariant mean.

Now fix a topological invariant mean m on $VN(H\backslash G) = p_H W^*(G)$. Then $m(p_H\cdot)$ is a state on the von Neumann algebra $W^*(G)$. By Lemma 5.8, we can find a net (φ_α) in $S(H\backslash G)$ w^*-converging to $m(p_H\cdot)$.

We now use an idea of I. Namioka. Let $\mathcal{E} = A(H\backslash G)^{S(H\backslash G)}$ be the product space with the product topology τ. Define a map $\pi : S(H\backslash G) \to \mathcal{E}$ by

$$\pi(\psi)(\varphi) = \varphi\psi - \psi \quad \big(\varphi, \psi \in S(H\backslash G)\big).$$

Since (φ_α) w^*-converges to m and m is a topological invariant mean, we have $\pi(\varphi_\alpha) \to 0$ in the weak topology of $\mathcal{E}$, that is, 0 is in the weak closure of $\pi\big(S(H\backslash G)\big)$ and hence in the τ-closure of $\pi\big(S(H\backslash G)\big)$. Therefore there is a net (ψ_α) in $S(H\backslash G)$ such that $\pi(\psi_\alpha) \xrightarrow{\tau} 0$, that is, $\|\varphi\psi_\alpha - \psi_\alpha\| \to 0$ for all $\varphi \in S(H\backslash G)$.

□

A closed subgroup H of G is called *neutral* if for every neighbourhood U of the identity $e \in G$, there exists a neighbourhood V of e such that $VH \subset HU$ (cf. [33]). We note that normal, open or compact subgroups are neutral, as well as all closed subgroups of a SIN-group.

Lemma 5.10 *Let H be a neutral subgroup of G. Then there is a net (σ_α) in $S(H\backslash G)$ such that* supp $\sigma_\alpha \downarrow H$.

Proof. Let $\mathcal{U}$ be a neighbourhood base of e consisting of compact sets. Let $U \in \mathcal{U}$. Then one can find a compact symmetric neighbourhood V of e such that $VH = HV$ and $V^2 \subset U$. It suffices to show that there exists $\sigma_U \in S(H\backslash G)$ with supp $\sigma_U \subset HU$ which gives the required net $(\sigma_U)_{U\in\mathcal{U}}$. The existence follows directly from an argument in the proof of Proposition 2.2 in [17].

□

The following generalizes Proposition 3.5 and Corollary 3.6 in [16].

Corollary 5.11 *Let H_1 and H_2 be neutral subgroups of G. Then $A(H_1\backslash G) = A(H_2\backslash G)$ if, and only if, $H_1 = H_2$. Also, $B(H_1\backslash G) = B(H_2\backslash G)$ if, and only if, $H_1 = H_2$.*

Given $T \in VN(G)$, the *support* of T, supp$\,(T)$, is defined to be the set of those elements $x \in G$ such that $\rho(x)$ is the w*-limit of a net $(\varphi_\alpha \cdot T)$ with $\varphi_\alpha \in A(G)$, where ρ is the right regular representation of G.

Lemma 5.12 *Let H be a compact subgroup of G. Then $A(H\backslash G)_0$ is the closed linear span of $\bigcup\{I_\sigma : \sigma \in S(H\backslash G)\}$, where $I_\sigma = I_\sigma(H\backslash G)$.*

Proof. Clearly $A(H\backslash G)_0$ contains the closed linear span I of $\bigcup\{I_\sigma : \sigma \in S(H\backslash G)\}$. It suffices to show that $I^\perp$ has dimension 1 in $A(H\backslash G)^* = p_H^c VN(G)$, that is, $I^\perp = \mathbb{C}p_H^c$, where p_H^c is the projection in (5.2).

Let $T \in I^\perp$. Then $\sigma \cdot T = T$ for all $\sigma \in S(H\backslash G)$. By Lemma 5.10, there is a net (σ_α) in $S(H\backslash G)$ such that supp $\sigma_\alpha \downarrow H$. By [14, Proposition 4.8], we have supp$\,(T) =$ supp$\,(\sigma_\alpha \cdot T) \subset$ supp σ_α and it follows that supp $T \subset H$. By [39, Theorem 3], T belongs to the w^*-closed linear span of $\{\rho(h) : h \in H\}$. Take a linear combination $\sum_\alpha \lambda_\alpha \rho(h_\alpha)$.

For any $\varphi \in A(H\backslash G)$, we have

$$\begin{aligned}\left\langle \varphi, \sum \lambda_\alpha \rho(h_\alpha)\right\rangle &= \sum \lambda_\alpha \langle \varphi, \rho(h_\alpha)\rangle \\ &= \sum \lambda_\alpha \varphi(h_\alpha) \\ &= \sum \lambda_\alpha \varphi(e) \\ &= \sum \lambda_\alpha \langle \varphi, p_H^c\rangle \\ &= \left\langle \varphi, \sum \lambda_\alpha p_H^c\right\rangle.\end{aligned}$$

It follows that $T = \lambda p_H^c$ for some $\lambda \in \mathbb{C}$ which completes the proof.

□

We now prove results on the triviality of $A(H\backslash G)^*_\sigma$.

Proposition 5.13 *Let H be a compact subgroup of a second countable group G. Then there exists some $\sigma \in S(H\backslash G)$ such that $I_\sigma(H\backslash G) = A(H\backslash G)_0$ and hence $A(H\backslash G)^*_\sigma = \mathbb{C}p_H^c$.*

Proof. It suffices to prove the first assertion since it implies that $A(H\backslash G)^*_\sigma = I_\sigma(H\backslash G)^\perp = A(H\backslash G)_0^\perp = \mathbb{C}p_H^c$. By Lemma 5.9, there is a net (φ_α) in $S(H\backslash G)$ such that $\|\sigma\varphi_\alpha - \varphi_a\| \to 0$ for all $\sigma \in S(H\backslash G)$. So $\|(\sigma\varphi - \varphi)\varphi_\alpha\| = \|\varphi(\sigma\varphi_\alpha - \varphi_\alpha)\| \to 0$ for $\sigma \in S(H\backslash G)$ and $\varphi \in A(H\backslash G)$. Let $\varepsilon > 0$ and $\varphi_1, \ldots, \varphi_n \in A(H\backslash G)_0$. From what we have just obtained and Lemma 5.12, one can find $\varphi_\beta \in S(H\backslash G)$ such that $\|\varphi_i\varphi_\beta\| < \varepsilon$ for $i = 1, \ldots, n$. It follows from $\varphi_\beta\varphi_i - \varphi_i \in I_{\varphi_\beta}(H\backslash G)$ that

$$d\big(\varphi_i, I_{\varphi_\beta}(H\backslash G)\big) = \inf\big\{\|\varphi_i - \psi\| : \psi \in I_{\varphi_\beta}(H\backslash G)\big\} < \varepsilon$$

for $i = 1, 2, \ldots, n$.

Since G is second countable, $A(G)$ is norm separable and so is $A(H\backslash G)$. So the conditions in Lemma 1.1 in [42] are satisfied and by Remark 3 of [42], we have $A(H\backslash G)_0 = I_\sigma(H\backslash G)$ for some $\sigma \in S(H\backslash G)$.

□

Lemma 5.14 *Let H be a compact subgroup of G and let $\sigma \in B(H\backslash G)$ be positive definite with $\sigma(e) = 1$. The following conditions are equivalent:*

(i) $A(H\backslash G)^*_\sigma = \mathbb{C}p_H^c$;
(ii) $x \in H$ *whenever* $\sigma(x) = 1$.

Proof. (i) $\Longrightarrow$ (ii). Let $\sigma(x) = 1$. Then for any $\varphi \in A(H\backslash G)$, we have

$$\langle \varphi, \sigma \cdot \rho(x)\rangle = \langle \sigma\varphi, \rho(x)\rangle = \varphi(x) = \langle \varphi, \rho(x)\rangle.$$

Therefore $\sigma \cdot \rho(x) = \rho(x)$ and $\rho(x)p_H^c \in A(H\backslash G)^*_\sigma$ gives $\rho(x)p_H^c = \lambda p_H^c$ for some $\lambda \in \mathbb{C}$.

By [14, Lemma 3.2], we can pick $\psi \in A(G)$ such that $\psi(H) = \psi(Hx) = \{1\}$. Define

$$\widetilde{\psi}(\cdot) = \int_H \psi(h\,\cdot)dh.$$

Then $\widetilde{\psi} \in A(H\backslash G)$ and $\widetilde{\psi}(x) = \widetilde{\psi}(e) = 1$. Hence $1 = \widetilde{\psi}(x) = \langle\widetilde{\psi}, \rho(x)\rangle = \langle\widetilde{\psi}, \rho(x)p_H^c\rangle = \lambda\langle\widetilde{\psi}, p_H^c\rangle = \lambda\widetilde{\psi}(e)$ gives $\lambda = 1$. Consequently $x \in H$ for otherwise one can find $\varphi \in A(H\backslash G)$ such that $\varphi(x) \neq \varphi(e)$ [16, Theorem 4.1] which leads to the contradiction $\langle\rho(x)p_H^c, \varphi\rangle \neq \langle p_H^c, \varphi\rangle$.

(ii) $\Longrightarrow$ (i). The proof is similar to that of Lemma 3.2.6 in [8].

□

Proposition 5.15 *Let H be a compact subgroup of G. If $A(H\backslash G)^*_\sigma = \mathbb{C}p^c_H$ for some positive definite $\sigma \in B(H\backslash G)$ with $\sigma(e) = 1$, then $H\backslash G$ is first countable.*

Proof. By Lemma 5.14, $\sigma(x) = 1$ implies $x \in H$. We note that, for any net (Hx_α) in a compact neighbourhood of H in $H\backslash G$, the net (Hx_α) converges to H in the homogeneous space $H\backslash G$ if, and only if, $\sigma(x_\alpha) \to \sigma(e) = 1$. Indeed, if $Hx_\alpha \nrightarrow H$, then there is a subnet $Hx_\beta \to Hx$ for some $x \notin H$. Therefore $\lim_\beta \sigma(x_\beta) = \sigma(x) \neq 1$.

Now let $\mathcal{C}$ be a compact neighbourhood of H in $H\backslash G$. Let $\mathcal{K} = \{r_a\sigma : Ha \in \mathcal{C}\}$ be the right translates of σ by $\mathcal{C}$. Then $\mathcal{K}$ is a norm compact subset of $A(H\backslash G)$ and therefore has a norm dense sequence (ψ_n) such that a net (Hx_α) converges to Hx in $\mathcal{C}$ if and only if $\psi_n(x_\alpha) \to \psi_n(x)$ for all n. So $\mathcal{C}$ is metrizable. This proves the first countability of $H\backslash G$.

□

The above result generalizes [8, Proposition 3.2.7]. We note that, in [8, Proposition 3.2.7], the proof of $(i) \Rightarrow (ii)$ requires that the net (x_α) there to be taken from a compact neighbourhood of the identity e, as in the above arguments.

We now discuss the existence of bounded approximate identity in $I_\sigma(H\backslash G)$ and $A(H\backslash G)$. We recall the first Arens multiplication on $A(H\backslash G)^{**} = VN(H\backslash G)^*$ as follows. Let $m, n \in A(H\backslash G)^{**}$ and $T \in VN(H\backslash G)$. We define $m \circ n \in A(H\backslash G)^{**}$ by

$$\langle T, m \circ n\rangle = \langle n \cdot T, m\rangle \quad \big(T \in VN(H\backslash G)\big)$$

where $n \cdot T \in VN(H\backslash G)$ is defined by

$$\langle \varphi, n \cdot T\rangle = \langle T, \varphi \cdot n\rangle \quad \big(\varphi \in A(H\backslash G)\big)$$

and $\varphi \cdot n \in A(H\backslash G)^{**}$ is given by

$$\langle S, \varphi \cdot n\rangle = \langle \varphi \cdot S, n\rangle \quad \big(S \in VN(H\backslash G)\big)$$

with $\varphi \cdot S \in VN(H\backslash G)$ as defined in (5.3).

We note that $\langle p_H, m \circ n\rangle = m(p_H)n(p_H)$ where $VN(H\backslash G) = p_H W^*(G)$ as in Lemma 5.2.

Lemma 5.16 *Let H be a closed subgroup of G and let $m \in A(H\backslash G)^{**}$ be a topological invariant mean on $VN(H\backslash G)$. Then we have $n \circ m = m$ for each $n \in A(H\backslash G)^{**}$ satisfying $\|n\| = n(p_H) = 1$.*

Proof. By Lemma 5.8, we can find a net (φ_α) in $A(H\backslash G)$ with $\|\varphi_\alpha\| = \varphi_\alpha(p_H) = 1$ and w^*-converging to n. Hence $n \circ m = w^*\text{-}\lim_\alpha \varphi_\alpha \circ m = m$ by topological invariance of m.

□

Theorem 5.17 *Let H be a closed subgroup of G. The following conditions are equivalent:*

(i) *$A(H\backslash G)$ has a bounded approximate identity;*
(ii) *$A(H\backslash G)_0$ has a bounded approximate identity.*

Proof. (i) $\Longrightarrow$ (ii). Let $\{\varphi_\alpha\}$ be a bounded approximate identity in $A(H\backslash G)$ and let $\theta \in A(H\backslash G)^{**}$ be a weak* cluster point of $\{\varphi_\alpha\}$. Pick $\psi \in A(H\backslash G)^{**}$ such that $\|\psi\| = \psi(p_H) = 1$. Then

$$(\varphi_\alpha \circ \psi)(p_H) = \varphi_\alpha(p_H)\psi(p_H) = \varphi_\alpha(p_H)$$

implies $\theta(p_H) = 1$. Let m be a topological invariant mean on $VN(H\backslash G)$ and let $\delta = \theta - m$. Let

$$J_0 = \{n \in A(H\backslash G)^{**} : n(p_H) = 0\}.$$

We have $A(H\backslash G)_0 \subset J_0$ and it suffices to show that $A(H\backslash G)_0^{**} \approx J_0$ has a right identity (cf. [4, p.146]). We first show that

$$n \circ m = 0$$

for all $n \in J_0$.

Recall that $A(H\backslash G)^* = p_H W^*(G)$ and write $p = p_H$. For any $n \in W^*(G)^*$, we define a linear functional $L_p n$ on $W^*(G)$ by

$$L_p n(\cdot) = n(p\cdot).$$

If n is positive, the Cauchy-Schwarz inequality implies that $\|L_p n\| = n(p)$. Now let $n \in A(H\backslash G)^{**}$ and define $\widetilde{n} \in W^*(G)^*$ by $\widetilde{n}(\cdot) = n(p\cdot)$. Then $\widetilde{n}$ is a linear combination of positive linear functionals on $W^*(G)$ and $L_p\widetilde{n} = \widetilde{n}$. It follows that $\widetilde{n}$ is a linear combination $\sum_i \alpha_i L_p n_i$ where, after normalizing, each positive functional n_i satisfies $\|L_p n_i\| = 1 = n_i(p)$. By Lemma 5.16, we have $(L_p n_i) \circ m = m$ for all i. Consequently, for each $n \in J_0$, we have $\widetilde{n}(p) = \sum_i \alpha_i$ and $n \circ m = \sum_i \alpha_i (L_p n_i \circ m) = \sum_i \alpha_i m = 0$. Therefore $n \circ \delta = n \circ (\theta - m) = n \circ \theta = n$ since θ is a right identity of $A(H\backslash G)^{**}$. As $\delta \in J_0$, we have shown that J_0 has a right identity δ.

(ii) $\Longrightarrow$ (i). Since the closed ideal $A(H\backslash G)_0$ has co-dimension 1 in $A(H\backslash G)$, the result follows from the fact that a Banach algebra $\mathcal{A}$ has a bounded approximate identity if it has a closed ideal $\mathcal{I}$ having one, as well as the quotient $\mathcal{A}/\mathcal{I}$.

□

Remark 5.18 If H is compact, condition (i) above is equivalent to the amenability of the homogeneous space $H\backslash G$ (see [16, Theorem 4.2]). If $H = \{e\}$, condition (ii) is equivalent to the amenability of G as shown in [30].

Lemma 5.19 *Let H be a closed subgroup of G such that $A(H\backslash G)$ has a bounded approximate identity. Then $I_\sigma(H\backslash G)$ also has a bounded approximate identity for $\sigma \in B(H\backslash G)$ with $\|\sigma\| = 1$.*

Proof. Let $\{\varphi_\alpha\}$ be a bounded approximate identity in $A(H\backslash G)$. Let $\sigma_n = \frac{1}{n}\sum_{k=1}^{n} \sigma^k$ for $n = 1, 2, \ldots$. Then $\|\sigma_n\| \leq 1$ and $\{\varphi_\alpha - \sigma_n\varphi_\alpha\}_{\alpha,n}$ is a bounded approximate identity in $I_\sigma(H\backslash G)$.

□

Theorem 5.20 *Let H be a compact subgroup of a second countable group G. Then G is amenable if and only if $I_\sigma(H\backslash G)$ has a bounded approximate identity for each $\sigma \in B(H\backslash G)$ with $\|\sigma\| = 1$.*

Proof. Since H is compact, amenability of G is equivalent to that of $H\backslash G$, which is in turn equivalent to the existence of a bounded approximate identity in $A(H\backslash G)$ (cf. [16]). Therefore necessity follows from Lemma 5.19. Conversely, by the proof of Proposition 5.13, we have $A(H\backslash G)_0 = I_\sigma(H\backslash G)$ for some $\sigma \in B(H\backslash G)$ with $\|\sigma\| = 1$. So $A(H\backslash G)_0$ has a bounded approximate identity. Hence $A(H\backslash G)$ has a bounded approximate identity by Theorem 5.17.

□

We conclude with further results on topological invariant means on $VN(H\backslash G)$.

Lemma 5.21 *Let H be an open subgroup of G. Then $VN(H\backslash G)$ has a topological invariant mean in $A(H\backslash G)$.*

Proof. Since H is open, the characteristic function 1_H belongs to $B(G)$. We have $q(\operatorname{supp} 1_H) = H$ and $\|1_H\| = 1 = 1_H(e)$ and therefore $1_H \in S(H\backslash G)$ and 1_H is a mean on $VN(H\backslash G)$. Moreover 1_H is a topological invariant mean since $\varphi 1_H = 1_H$ for all $\varphi \in S(H\backslash G)$.

□

Proposition 5.22 *Let H be a neutral subgroup of G. Then $VN(H\backslash G)$ has a topological invariant mean in $A(H\backslash G)$ if, and only if, H is open.*

Proof. We only need to show the necessity. Let $m \in A(H\backslash G)$ be a topological invariant mean on $VN(H\backslash G)$. Suppose H is not open. Then there exists $x \notin H$ such that $m(x) \neq 0$. We can find a compact neighbourhood U of e such that $x \notin HU$. As in the proof of Lemma 5.10, there exists $\sigma \in S(H\backslash G)$ with supp $\sigma \subset HU$. As $\sigma m = m$, we have $m(x) = \sigma(x)m(x) = 0$ which is a contradiction. So H is open.

□

Our final result concerns the uniqueness of a topological invariant mean on $VN(H\backslash G)$ when H is compact in which case, we first show that the net (ψ_α) in Lemma 5.9 has a more explicit construction.

Lemma 5.23 *Let H be a compact subgroup of G and let $\mathcal{U}$ be a neighbourhood base of e consisting of compact sets. For each $U \in \mathcal{U}$, let $\sigma_U \in S(H\backslash G)$ satisfy* $\operatorname{supp} \sigma_U \subset UH$. *Then we have $\|\psi\sigma_U - \sigma_U\| \to 0$ as $U \to \{e\}$, for each $\psi \in S(H\backslash G)$.*

Proof. We first show that $S(H\backslash G) \cap C_c(H\backslash G)$ is norm dense in $S(H\backslash G)$. Let $P_H : A(G) \to A(H\backslash G)$ be the contractive projection

$$(P_H\varphi)(x) = \int_H \varphi(hx)dh$$

as defined in [16, Theorem 3.3]. We note that $P_H\big(A(G) \cap P^1(G)\big) \subset S(H\backslash G)$ and $\operatorname{supp}(P_H\varphi) \subset H(\operatorname{supp}\varphi)$, where $P^1(G)$ is the set of all positive definite functions on G having value 1 at e. Now let $\varphi \in S(H\backslash G)$ and let (φ_n) be a sequence in $A(G) \cap P^1(G) \cap C_c(G)$ such that $\|\varphi_n - \varphi\|_{A(G)} \to 0$. Then $P_H\varphi_n \in S(H\backslash G) \cap C_c(H\backslash G)$ and $\|P_H\varphi_n - \varphi\|_{A(H\backslash G)} \to 0$.

Let $\psi \in S(H\backslash G)$. We show $\|\psi\sigma_U - \sigma_U\| \to 0$. Let $\varepsilon > 0$. Pick $\psi' \in S(H\backslash G) \cap C_c(H\backslash G)$ such that $\|\psi - \psi'\| < \frac{\varepsilon}{2}$. Let $K =$ supp $\psi' \subset G$. By regularity of $A(H\backslash G)$ [16, Theorem 4.1], we can find $\varphi \in A(H\backslash G)$ such that $\varphi^{-1}\{1\} = K$. Since $\psi'(e) = 1$ and $e \in K$, we have $(\psi' - \varphi)(e) = 0$. By [14, p.229], there exists $\eta \in A(G)$ such that $\|\psi' - \varphi - \eta\| < \frac{\varepsilon}{2}$

and $\eta(V) = \{0\}$ for some neighbourhood V of e. We have $\|\psi' - \varphi - P_H\eta\| < \frac{\varepsilon}{2}$. Hence for every $U \in \mathcal{U}$ satisfying $U \subset V \cap K$, we have $\sigma_U\varphi = \sigma_U$ and $\sigma_U\eta = 0$ which gives

$$\begin{aligned}\|\psi\sigma_U - \sigma_U\| &\leq \|(\psi - \psi')\sigma_U\| + \|\psi'\sigma_U - \sigma_U\| \\ &< \frac{\varepsilon}{2} + \|(\psi' - \eta - \varphi)\sigma_U\| < \varepsilon.\end{aligned}$$

□

We end with the following result which extends a result in [32] for groups.

Proposition 5.24 *Let H be a compact subgroup of a second countable group G. If $VN(H\backslash G)$ has a unique topological invariant mean, then H is open.*

Proof. Since G is second countable, $A(H\backslash G)$ is norm separable and from Lemma 5.23, we can choose a sequence (σ_n) in $S(H\backslash G)$ such that

$$\|\psi\sigma_n - \sigma_n\| \to 0$$

for each $\psi \in S(H\backslash G)$. Let m be a topological invariant mean. As every weak cluster point of (σ_n) in $A(H\backslash G)^{**}$ is a topological invariant mean on $VN(H\backslash G)$, m must be the only cluster point by uniqueness. We note that $A(G)$ and hence $A(H\backslash G)$ is weakly sequentially complete. It follows that (σ_n) converges weakly to m and $m \in A(H\backslash G)$. Hence H is open by Proposition 5.22.

□

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