Quantum Cosmology of the Brane Universe

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We canonically quantize the dynamics of the brane universe embedded into the five-dimensional Schwarzschild–anti-de Sitter bulk space-time. We show that in the brane-world settings the formulation of the quantum cosmology, including the problem of initial conditions, is conceptually more simple than in the (3 + 1)-dimensional case. The Wheeler-DeWitt equation is a finite-difference equation. It is exactly solvable in the case of a flat universe and we find the ground state of the system. The closed brane universe can be created as a result of decay of the bulk black hole.

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Introduction.—Quantum effects almost certainly played a crucial role in the early universe evolution and in the process of universe creation. Understanding and study of quantum cosmology is important not only from the conceptual point of view, but, hopefully, may provide us with constraints on possible topology of the universe and initial conditions for the inflationary stage [1–3]. Appropriate theoretical frameworks that would incorporate all quantum gravitational effects are yet to be constructed, however.

String theory, eventually, may provide the consistent approach to the quantum cosmology realm, but the formulation of the string theory on a nontrivial and significantly Lorentzian space-time is a very complicated and unsolved task (see, for example, [4] and references therein). That is why the approaches based on canonical quantization of the Einstein gravity [5] still prove to be more successful in addressing the problems of quantum cosmology. Here one has to adopt a modest approach and restrict consideration to quantum phenomena below the Plank energy scale. Quantizing the universe as a whole, one has to resort further to the "mini-superspace" modeling [1–3,6] in order to get to definite final results (for a recent interesting development see, however, Ref. [7] where effective action for the scale factor was derived integrating out other gravitational degrees of freedom using numerical simulations).

Even then, within the mini-superspace approach, many conceptual and technical problems remain, such as the problem of ascribing physical meaning to the wave function of the universe [6]. Other important issues are the choice of the boundary conditions that one imposes at the big-bang point (e.g., "no-boundary" [6], "tunneling" [2], etc.) and the problem of unboundedness of the gravitational action (see, e.g., [8]).

In this Letter we pursue the viewpoint that the presence of extra dimensions can resolve or relax some of these problems. Indeed, in the brane-world scenario [9], the problem of quantum cosmology is replaced by a much better defined problem of quantum mechanics of the brane, which moves in the bulk space-time. (Quantization of a matter on the classical brane has been considered in [10] using anti-de Sitter/conformal field theories (CFT) correspondence.) This has several important consequences. First, one may hope that probabilistic interpretation, initial and boundary conditions, tunneling, "scattering," and "ground" states of the universe become better defined (for discussion see also [11]). Second, one can escape, to some extent, solving the problems of quantum gravity. Indeed, the big-bang point, i.e., the point of vanishing brane size, can be unreachable due to quantum uncertainty. Thus, quantization of matter in a self-consistently calculated "external" gravitational field can be sufficient.

The conceptual simplicity of the brane quantum cosmology does not imply its "technical" simplicity: one has to take into account self-consistently the interaction of the brane with the bulk on both classical and quantum levels. Here we can benefit capitalizing on the fact that the dynamics of (3 + 1)-dimensional brane embedded in (4 + 1)-dimensional bulk is very similar to the dynamics of self-gravitating shells in conventional (3 + 1)-dimensional general relativity, which was studied extensively both at classical [12,13] and quantum levels [14–16]. In this Letter we generalize the formalism developed in [16] to the case of the (3 + 1)-dimensional brane universe embedded into (4 + 1)-dimensional bulk.

We may hope that some results found in frameworks of brane quantum cosmology may hold even if the universe is (3 + 1) dimensional. In particular, the distinctive feature of quantum mechanics of branes is that the differential Schrödinger (or "Wheeler-DeWitt") equation for the wave function is replaced by a finite-difference equation [14,16]. This may be a general property of "true" quantum cosmology. Note in this respect that finite-difference equations for the wave function of the universe appear also in the frameworks of loop-quantum gravity [17].

Hamiltonian description of the classical motion of a gravitating brane.—We construct the Hamiltonian formalism that describes the motion of a self-gravitating thin shell of matter starting from the action of (4 + 1)-dimensional Einstein gravity with bulk cosmological constant. The brane part of the action contains the term proportional to the brane tension λ and the term that describes (in the simplest case) dustlike matter on the brane with the mass μ per unit comoving volume. The total action of the system is

$$S = \frac{1}{4l_{\rm Pl}^3}\int_{\rm bulk}\sqrt{g}[\Lambda +^{(4)}\mathcal{R} + ({\rm Tr}\mathcal{K})^2 - {\rm Tr}\mathcal{K}^2] - 8\pi\mu\int_{\rm brane} d\tau - \lambda\int_{\rm brane}\sqrt{-\hat{g}}d\tau d^3\hat{x}, \qquad (1)$$

where $l_{\rm Pl}^{-1}$ is the $(4+1)$-dimensional Planck mass, $\hat{g}$ is the induced metric on the brane, τ is the proper time of comoving observers in the brane universe, Λ is the bulk cosmological constant, and ${}^{(4)}\mathcal{R}$, $\mathcal{K}_{AB}$ are the four-dimensional Ricci scalar and the external curvature of the spatial section of $(4+1)$-dimensional space-time. We restrict ourselves to the case of homogeneous and isotropic brane, which may describe the open, flat, or closed brane universe.

For generally covariant systems the Hamiltonian dynamics is encoded in a system of constraints [5]. For a spherically symmetric space without matter, and in any space-time dimensions, these constraints can be solved explicitly classically as well as quantum mechanically; see Ref. [18]. This result can be understood noticing that in this case gravity has only global degrees of freedom. The most convenient way to parametrize these global degrees of freedom is to use the Schwarzschild-like representation of the metric

$$ds^2 = -F(t,r)dT^2 + \frac{dR^2}{F(t,r)} + R^2 d\Omega_3^2, \qquad (2)$$

where $T = T(t,r)$ and $R(t,r)$ are arbitrary functions of time and radial coordinates (t,r), while the function $F(t,r)$ has the form $F(t,r) = k - l_{\rm Pl}^3 M(t,r)R^{-2} - \Lambda R^2$, where $k = 0, \pm 1$ for the cases of flat, closed, and open spatial sections, respectively.

In the Hamiltonian formalism the canonical variables describing the bulk gravitational field are $(R, M; P_R, P_M)$. It turns out that $T' = \partial T/\partial r$ is the momentum conjugate to M [18]. The conventional constraints of canonical formalism reduce to the set of equations, $P_R = 0$ and $M' = \partial M/\partial r = 0$. One can see that if $M = \text{const}$, the metric (2) coincides with the metric of five-dimensional Schwarzschild–anti-de Sitter black hole of mass M.

The canonical constraint on the brane is

$$\hat{H} = \frac{3\hat{R}^2}{l_{\rm Pl}^3}\sigma\sqrt{|\hat{F}|}\cosh\left\{\frac{l_{\rm Pl}^3\hat{P}_{\hat{R}}}{3\hat{R}^2}\right\} - (\mu + \lambda\hat{R}^3) = 0, \qquad (3)$$

where a hat denotes the values of corresponding variables on the brane, e.g., $\hat{R} = R(t,r)|_{\rm brane}$ and $\sigma = \pm 1$. For the geometrical meaning of the sign function σ see Ref. [12]. At the classical level σ is integral of the motion, but the change of sign is possible at the quantum level [15,16]. Note that the Hamiltonian constraint Eq. (3) does not describe the most general case (e.g., the Schwarzschild parameter M can be different on both sides of the brane in general situations); rather, the Z_2 symmetry was assumed following Ref. [9]. Positive (negative) sign of σ corresponds to the positive (negative) brane tension in the case of the classical regime of Randall-Sundrum cosmology. For the discussion of general brane Hamiltonian in the quantum case see Refs. [15,16].

The equation of motion for $\hat{R}$ found from the Hamiltonian (3) is $d\hat{R}/d\tau = \sigma\sqrt{|F|}\sinh(3l_{\rm Pl}^3\hat{P}_R/\hat{R}^2)$, which, upon substitution into (3), gives

$$\frac{(d\hat{R}/d\tau)^2}{\hat{R}^2} + \frac{k}{\hat{R}^2} = \frac{l_{\rm Pl}^6(\mu + \lambda\hat{R}^3)^2}{9\hat{R}^6} + \frac{l_{\rm Pl}^3 M}{\hat{R}^4} + \Lambda. \qquad (4)$$

Being written in this form, the equation of motion of the brane resembles closely the Friedmann equation [19], in which the density of matter on the brane $\rho_m = \mu/\hat{R}^3$ enters quadratically at small $\hat{R}$, the presence of nonzero bulk black hole mass M results in the effective "dark radiation" contribution $\rho_{dr} = M/\hat{R}^4$, and the effective cosmological constant on the brane is a certain combination of the bulk cosmological constant and the brane tension $\Lambda_{(3+1)} = l_{\rm Pl}^6\lambda^2/9 + \Lambda$. Note, however, that Eq. (4) is a "square" of the true dynamical equation, and important information encoded in σ is lost. Therefore, its use can be inappropriate in some situations, especially in the quantum regime.

Quantum dynamics of the brane universe.—In canonically quantized theory the Hamiltonian constraint (3) is replaced by an equation on the wave function of the universe, $\hat{H}\Psi = 0$, which in the present case is a differential equation of infinite order. This infinite-order equation takes a simple form after the canonical transformation $v = \hat{R}^3$; $P_v = \hat{P}_{\hat{R}}/(3\hat{R}^2)$, which brings the Hamiltonian $\hat{H}$ into the form

$$\hat{H} = \frac{3v^{2/3}}{l_{\rm Pl}^3}\sigma\sqrt{|\hat{F}|}\cosh\{l_{\rm Pl}^3 P_v\} - (\mu + \lambda v). \qquad (5)$$

In the new variables, after quantization $P_v \to -i\partial/\partial v$, the hyperbolic cosine that enters $\hat{H}$ becomes an operator of finite shift along the imaginary axis, $\exp(l_{\rm Pl}^3 P_v)\Psi(v) = \Psi(v - il_{\rm Pl}^3)$. Substituting this into $\hat{H}\Psi = 0$ we find the following finite-difference equation, which determines the quantum dynamics of a self-gravitating brane universe:

$$v^{2/3}F^{1/2}\{\Psi(v + il_{\rm Pl}^3) + \Psi(v - il_{\rm Pl}^3)\} - \tfrac{2}{3}l_{\rm Pl}^3(\mu + \lambda v)\Psi(v) = 0. \qquad (6)$$

We have chosen a particular operator ordering in the above equation. The general case can be studied along the lines of Ref. [16]. A different choice of operator ordering changes some details but does not change qualitatively the results presented below.

Since the shift of the argument of the wave function is along the imaginary axis, one has to consider the above equation in the complex plane, or, more precisely, on the corresponding Riemanian surface. Indeed, the function

$F^{1/2}$ is a branching function on the complex plane. The two branches, $F^{1/2} = \pm\sqrt{F}$, correspond to the two possible choices of σ. Therefore, if one finds the solutions of the above equation on the Riemann surface, the wave function Ψ is defined simultaneously in $\sigma = +1$ and $\sigma = -1$ domains.

In order to understand qualitatively the behavior of the solutions of Eq. (6), we start with an analysis of the distances much larger than $l_{\rm Pl}$. In this limit we can expand $\Psi(v \pm i l_{\rm Pl}^3)$ in powers of the shift parameter, $\Psi(v \pm i l_{\rm Pl}^3) \simeq \Psi(v) \pm i l_{\rm Pl}^3 \Psi'(v) - \frac{1}{2} l_{\rm Pl}^6 \Psi''(v) + \cdots$. In the first nontrivial order Eq. (6) reduces to (we restrict ourselves to the case $\sigma = 1$ here)

$$\Psi'' + \frac{2}{l_{\rm Pl}^6}\left(1 - \frac{l_{\rm Pl}^3(\mu + \lambda v)}{3v^{2/3}F^{1/2}}\right)\Psi = 0, \quad (7)$$

which is a Schrödinger-like equation for particle motion in a potential

$$U = 1 - \frac{l_{\rm Pl}^3(\mu + \lambda v)}{3(kv^{4/3} - 2GMv^{2/3} + |\Lambda|v^2)^{1/2}}. \quad (8)$$

For large $v = \hat{R}^3$ the potential approaches a constant, $U \to 1 - l_{\rm Pl}^3\lambda/(3\sqrt{|\Lambda|})$. If $l_{\rm Pl}^3\lambda > 3\sqrt{|\Lambda|}$ the wave function behaves in the limit of large R^3 as a flat wave that describes an expanding or contracting universe.

Exactly solvable case of the flat universe.—To make more detailed analysis of the quantum mechanics of the brane, e.g., to study its spectrum, one needs to impose boundary conditions at the origin. At first sight the issue of boundary conditions at the big-bang point $v = 0$ looks conceptually simpler for the brane universe. Indeed, since the scale factor of the universe is now just a position of the brane moving in the external space (rather than purely gravitational degree of freedom), this is just the question of boundary conditions on the wave function at the origin of spherical coordinates. However, in the region $v \sim l_{\rm Pl}^3$ one cannot expand Eq. (6) in powers of $l_{\rm Pl}$, and the intuition based on Eq. (7) is not applicable anymore. Instead, one has to deal with the exact finite difference Eq. (6). (We assume that the mini-superspace model based on the thin-wall approximation is still valid in the limit of small v.)

The finite-difference equations and, in particular, Eq. (6) possess a number of interesting general properties. Being understood as infinite-order differential equations, they have to be supplemented with an infinite set of boundary conditions. At the same time, starting from a single particular solution Ψ_0 one can generate an infinite set of solutions simply by multiplying $\Psi_0(z)$ by a function $C(z)$, which is periodic with respect to the finite shift parameter [i.e., $C(z + i l_{\rm Pl}^3) = C(z)$ in the case of Eq. (6)]. The appropriate methods of analysis of finite-difference equations are discussed in Refs. [16,20,21].

In order to illustrate these methods, it is convenient to consider the special case when Eq. (6) is exactly solvable, namely, the case of the flat universe $k = 0$ and zero bulk Schwarzschild mass $M = 0$. For this choice of parameters, Eq. (6) takes the form

$$\Psi(v + i l_{\rm Pl}^3) + \Psi(v - i l_{\rm Pl}^3) - \frac{2 l_{\rm Pl}^3}{3\sqrt{|\Lambda|}}\left(\lambda + \frac{\mu}{v}\right)\Psi(v) = 0, \quad (9)$$

which coincides with the finite-difference analog of the quantum-mechanical problem of motion in the Coulomb potential [20]. A general solution of Eq. (9) is given by (up to multiplication by an arbitrary $i l_{\rm Pl}^3$—periodic function) $\Psi(S) = v e^{-\alpha v}\mathcal{F}(1 - iv, 1 - \beta{:}2{:}1 - e^{-2i\alpha})$, where $\mathcal{F}$ is the hypergeometric function. Parameters α and β are defined by relations $\cos\alpha = l_{\rm Pl}^3\lambda/(3\sqrt{|\Lambda|})$ and $\beta\sin\alpha = l_{\rm Pl}^3\mu/(3\sqrt{|\Lambda|})$.

As is usual in quantum mechanics, the single solution can be selected only when the proper set of boundary conditions is chosen. The correct boundary conditions can be determined from the requirement of vanishing of the probability flow $J = i(\Psi^\dagger \hat{H}\Psi - \Psi\hat{H}\Psi^\dagger)$ at $v = 0$. In the case of Eq. (9) this reduces to the set of conditions [21] $\Psi^{(2n)}(0) = 0$, $n = 0, 1, \ldots$.

Similar to the conventional quantum mechanics with the Coulomb potential, there are bound states and continuous spectrum. Using the above boundary conditions as well as appropriate conditions at infinity, one can see that bound states exist when the quantization condition

$$\Lambda_{(3+1)} = \frac{l_{\rm Pl}^6\lambda^2}{9} + \Lambda = -\frac{4\mu^2}{9n^2}, \qquad n = 1, 2, \ldots,$$

is satisfied. It relates the effective brane cosmological constant $\Lambda_{(3+1)}$ to the matter density on the brane. In particular, the ground state of the universe corresponds to $n = 1$. The wave functions of continuous spectrum ($l_{\rm Pl}^3\lambda > 3\sqrt{|\Lambda|}$, which corresponds to the positive effective cosmological constant on the brane) contain both the collapsing (ingoing wave) and expanding (outgoing) branes. Thus, in the case of continuous spectrum, the wave function of the universe corresponds to the so-called "big bounce" situation. One can consider also transitions between the bound states and the states from continuous spectrum (e.g., an expanding brane universe can result from the excitation of the ground state). However, the analysis of perturbations of the spherically symmetric system considered above goes beyond the mini-superspace approximation.

Tunneling from the bound states.—In order to study qualitatively the more general cases when the bulk Schwarzschild mass is not zero, let us come back to the analysis of the truncated Eq. (7). The behavior of the potential U for the cases $k \neq 0$ and/or $M \neq 0$ is shown in Fig. 1. One can see that if $l_{\rm Pl}^3 M \geq (\Lambda_{(3+1)})^{-1}$ there is a potential barrier, which separates the regions of bound and unbound motion of the brane. This means that the spectrum of quantum states of the brane can contain, apart from the discrete and continuous part, also "resonances." In this case the expanding brane universe is the result of decay (or

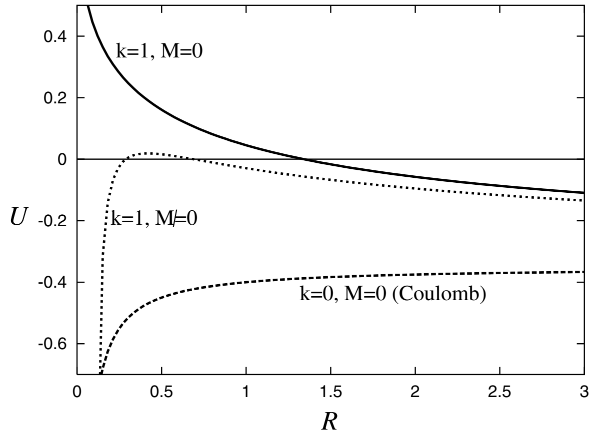

FIG. 1. The potential U (8) for different choices of parameters. The potential is singular at the gravitational radius of the bulk Schwarzschild–anti-de Sitter black hole when $M \neq 0$.

tunneling) of an almost stable state localized near the origin. The main difference of the tunneling states considered here from the $(3 + 1)$-dimensional ones is that in the $(3 + 1)$-dimensional case the choice of the boundary conditions at $\hat{R} = 0$ is ambiguous and the existence of the tunneling state is, in fact, just postulated [2].

Discussion.—In this Letter we have constructed quantum cosmology of the brane universe and have shown that it has several distinctive features. In particular, one can avoid the conceptual problems related to the interpretation of the wave function of the universe. Indeed, in the brane-world setup one does not quantize pure gravity, but rather deals with quantum mechanics of a matter source (brane) moving through higher-dimensional space-time. The problem of the choice of boundary conditions on the wave function of the universe is also free from ambiguities: one simply has to impose the usual quantum-mechanical conditions on the wave function at the origin of coordinates. This allows for the detailed analysis of bound states, continuous spectrum, and tunneling states, where creation of the universe from "nothing" can be interpreted as a decay of a bound state resonance.

When gravitational self-interaction of the brane universe is important, as, for example, in the setup of the Randall-Sundrum cosmology studied here, one has to correctly account for the bulk-brane interaction not only classically, but also on the quantum level. As a result, the classical brane Hamiltonian constraint (3) becomes after quantization a finite-difference equation (6).

Although the appearance of finite-difference equations is a novel feature of the quantum brane cosmology, the analysis of the boundary conditions and of the wave functions of discrete and continuous spectra can be carried in a way similar to the one used in conventional quantum mechanics. From the point of view of quantization of gravitating systems, the appearance of a nonlocal equation (with nonlocality at the Plank scale) is natural to expect. Such equations appear in several other models (see, e.g., [22,23]). It implies a deformation of the Lorentz symmetry and generalized uncertainty principle [24,25].

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