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Optimal Machine Maintenance and Replacement by Linear Programming

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ABSTRACT

The problem of an optimal strategy of machine maintenance and replacement over an infinite horizon is formulated as a linear programming problem. Computational experience on real data is given including sensitivity of the LP solution to the maximum life of the machine and to various parameters of the model including discount factor, the method of computing depreciation, effect of rebuilding, maintenance and rebuild costs.

1. Introduction

A number of papers on when to optimally replace an aging or obsolescent machine have appeared in the management science literature. For example, there are the papers of Derman [2], Eppen [4], Klein [9] and Kolesar [10], [11] containing Markovian models of machine deterioration or failure. Other papers include the dynamic programming model of Bellman [1] and the papers by Eisen and Liebowitz [3] and Kamien and Schwartz [8]. A survey of the literature to 1965 is given by McCall [12] and the book edited by Jardine [7] contains some recent research on the machine replacement problem. Wagner's book [15] also contains a fairly extensive discussion of machine replacement models.

The majority of these papers and books contain quantitative models that are intended primarily to provide qualitative insights into the machine replacement decision process. By contrast, this paper reports on a machine replacement model that allows computation of optimal maintenance and replacement policies from the appropriate cost data. Specifically, we give an infinite horizon deterministic dynamic programming formulation of the maintenance and replacement problem which can be solved as a linear programming problem. Computational experience using the linear programming algorithm MPSX on the IBM 360 system is given for a realistic problem.

It is important at the onset to comment upon the simplification inherent in our deterministic model of a decision process studied as a stochastic process by most of the authors mentioned above. First, our model reflects the increasing probability of failure of a machine by maintenance costs which increase with time. Moreover, our model can be easily generalized to incorporate stochastic failure if the appropriate probability

distributions are known or estimated. It is important to emphasize, however, that knowledge of these probability distributions requires in most cases difficult and costly statistical data analysis.

The plan of this paper is the following. In section 2, we give the dynamic programming model which underlies our equipment replacement problem and the equivalent linear programming model used for computation. Although a number of papers have analyzed the theoretical relationship between infinite horizon dynamic programming and linear programming, we have not located a published paper in which actual computation using a linear programming algorithm was done for a specific dynamic programming application.

Section 3 discusses the specialization of the dynamic programming model to the equipment replacement problem. Computational experience on data pertaining to a continuous miner is given in section 4. A continuous miner is a piece of heavy equipment which automatically mines coal from the face of a mine. Section 5 contains a few suggestions for further study.

2. Overview of General Dynamic Programming Model

The equipment model we are studying is a special case of a general dynamic programming model. Since much of the methodology of this paper is appropriate to the general model, it is appropriate to begin our discussion with this model. The general model is best described using network terminology [13]. The network is $G = [\bar{N}, \bar{A}]$ where $\bar{N} = \{1,\ldots,s,\ldots,S\}$ is a collection of nodes or states, and $\bar{A}$ is a collection of directed arcs connecting pairs of nodes. Associated with each arc is an immediate return c_{st}.

At the start of any time period, the system can be in one of the states $s \in \bar{N}$. An arc (s,t) corresponds to the decision to make the transition to state t during the current period with return c_{st}. Let

$$A(s) = \{t: (s,t) \in \bar{A}\}, \quad s = 1,\ldots,S$$

and

$$B(s) = \{r: (r,s) \in \bar{A}\}.$$

We assume $A(s)$ is not empty for each s.

If the planning horizon is T time periods and the system begins in state s_0, the optimization problem is to find a path (sequence of decisions) $\mu = (s_0,s_1),(s_1,s_2),\ldots,(s_{T-1},s_T)$ consisting of T arcs in G such that $\sum_{i=0}^{T-1} \alpha^i c_{s_i,s_{i+1}}$ is a maximum, where α is a discount factor, $0 \leq \alpha < 1$.

This problem can be solved by backward iteration with the recursions

$$v_s^n = \max_{t \in A(s)} \{c_{st} + \alpha v_t^{n-1}\} \qquad s = 1,\ldots,S; \quad n = 1,\ldots,T$$

$$v_s^0 = 0 \qquad s = 1,\ldots,S.$$

A relevant mathematical construct is the infinite horizon problem which results when we let $T \to \infty$. The recursions become the functional equations

$$v_s^\infty = \max_{t \in A(s)} \{c_{st} + \alpha v_t^\infty\} \qquad s = 1,\ldots,S.$$

The relationship between the finite and infinite horizon models is two-fold. First, we have convergence of the return values v_s^T; namely $\lim_{T\to\infty} v_s^T = v_s^\infty$, $s = 1,\ldots,S$. This is the principle of successive approximation. Equally important, however, is the following result. For each state s, define

$$A^T(s) = \{t\colon v_s^T = c_{st} + \alpha v_t^{T-1}\}$$

and

$$A^\infty(s) = \{t\colon v_s^\infty = c_{st} + \alpha v_t^\infty\}.$$

Then the following obtains [13]: There exists a T^* such that for all $T \geq T^*$ and any state s

$$A^T(s) \subseteq A^\infty(s);$$

that is, the optimal immediate decision when there are $T \geq T^*$ periods remaining is to make a decision that is optimal if the planning horizon were infinite.

Since machines are maintained or rebuilt and new machines bought over a long indefinite (but finite) planning horizon T, it seems reasonable to assume that $T \geq T^*$ and we wish to make optimal steady state or infinite horizon decisions. Finding optimal infinite horizon decisions can be done

with less computational effort than doing backward iteration through many periods. In particular, the infinite horizon decision problem can be solved by solving one of the following two dual linear programs; see Hadley [16, sections 11-16, 11-17, 11-18], Wagner [34, section 12-6]. The parameters q_s below are arbitrary positive numbers.

Primal

$$\max \sum_{(s,t)\varepsilon\bar{A}} c_{st}x_{st} \qquad \text{subject to}$$

$$\sum_{(s,t)\varepsilon A(s)} x_{st} - \alpha\sum_{(r,s)\varepsilon B(s)} x_{rs} = q_s \quad \text{for all } s$$

$$x_{st} \geq 0 \text{ for } (s,t) \; \varepsilon \; \bar{A}$$

Dual

$$\min \sum_{s=1}^{S} q_s y_s \qquad \text{subject to}$$

$$y_s - \alpha y_t \geq c_{st} \text{ for } (s,t) \; \varepsilon \; \bar{A}$$

$$y_s \text{ unrestricted in sign, } s = 1,\ldots,S.$$

It can be shown that the optimal values y_s^* to the linear programming dual satisfy the infinite horizon functional equations; namely,

$$y_s^* = \max_{t\varepsilon A(s)} \{c_{st} + y_t^*\},\; s = 1,\ldots,S.$$

3. Equipment Replacement Model

We now discuss how the general dynamic programming model of section 2 can be applied to the machine replacement problem. At each decision point or state, there are three possible alternatives: maintain the present machine, rebuild the present machine, or purchase a new machine. The immediate decision results in an immediate profit which is a function of the decision and of the state of the machine.

1) Maintain: This decision includes normal repairs, preventive maintenance, routine breakdown maintenance, and more extensive repairs which may be of an overhaul nature but which do not constitute a complete rebuild.

2) Rebuild: This decision entails removing the machine from production for a period of time and completely rebuilding it.

3) Buy: The old machine is replaced with a new one.

Each node (state of the machine) is labeled with three indices, IJN. The first index equals the number of times the machine has been rebuilt; the second index equals the period in which the machine was last rebuilt (J = 0 if the machine has never been rebuilt); and the third index equals the age of the machine. We assume there is some absolute upper bound on the age of the machine before which a new machine will be bought. This assumption ensures that the state space is finite.

In the infinite horizon dynamic programming model, we maximize the infinite stream of discounted profits generated over the infinite planning horizon. The purchase of a new machine

is a regeneration point from which the process begins anew each time it is reached. As we have noted, the regeneration point must be reached after no more than a fixed number of years.

The advantages of the infinite horizon model include:

1) The optimal solution it provides represents a steady state solution without the transient effect of going out of business after a finite planning horizon.

2) Once the immediate profits of going from one state to another have been computed, the model can be solved as a linear programming problem by a standard linear programming package.

3) The sensitivity of the optimal solution to given parameters can be tested by the use of linear programming sensitivity analysis routines which are available on standard linear programming packages.

The model does not allow for the possibility of making decisions between periods. The necessity of making such decisions might result from some major accident or from stochastic machine failure. We deal with this shortcoming by employing the expected values of the various profits. Nevertheless, when compared with a model which would more fully take into account the probabilistic considerations, the present model is conservative with respect to the purchase of a new machine, i.e., the present model will tend to keep an old machine longer than would the probabilistic model.

3.1. Number of States and Number of Arcs in the Network

Let s(n) equal the number of states whose third index is n. Index j equals the period in which the machine was last rebuilt and therefore

varies from zero to n - 1. Index I equals the number of times the machine has been rebuilt and therefore varies from zero to J. Thus,

$$s(n) = 1 + \sum_{J=1}^{n-1} \sum_{I=1}^{J} 1$$

$$s(n) = 1 + n(n - 1)/2$$

where the left term of the right hand side counts state (0,0,n). Let us assume that the machine will not be kept longer than some absolute upper bound, say L. Then, if S(L) equals the total number of states in the network, we have

$$S(L) = \sum_{n=1}^{L} s(n) = L[1 + (L + 1)(L - 1)/6].$$

The number of arcs in the network is now easily derived. There are three arcs emanating from each state whose third index is less than L. (Each arc corresponds to one of the three decisions.) If the third index is equal to L, then there exists only one possible decision (buy) and therefore only one corresponding arc. Thus if A(L) equals the number of arcs in the network

$$A(L) = 3S(L - 1) + s(L)$$

$$A(L) = 3(L - 1)[1 + L(L - 2)/6] + 1 + L(L - 1)/2$$

3.2. Matrix Generation

Beginning with state (0,0,1), the state space is generated recursively for N less than L according to the following rules:

Maintain	(IJN) → (I,J,N+1)
Rebuild	(IJN) → (I+1,N,N+1)
Buy	(IJN) → (0,0,1)

When N = L, only the third decision is available. The states are then numbered in some fashion. Suppose state (IJN) is numbered Q and state (I'J'N') is numbered P. We define C(Q,P) as the cost of traversing the arc from state Q to state P. Then

C(Q,P) = CM(IJN) if (I'J'N') = (I,J,N+1)

C(Q,P) = CR(IJN) if (I'J'N') = (I+1,N,N+1)

C(Q,P) = CB(IJN) if (I'J'N') = (0,0,1)

Here, CM(IJN), CR(IJN) and CB(IJN) equal respectively the immediate profit of maintaining, rebuilding or replacing a machine in state (IJN). Having the state space and the arc costs, it is a straightforward matter to generate either the primal or the dual.

3.3. Mathematical Programming System Extended

Our matrix generator computer program produces the linear programming problem in standard linear programming format appropriate as input for IBM's Mathematical Programming System Extended (MPSX). MPSX in turn outputs the optimal solution to the linear program. The solution gives the optimal strategy out of any state whatever, and the value of that strategy. The optimal basis is stored for the purpose of sensitivity analysis. A modified matrix generator computer program passes to MPSX a matrix which has been altered in such a way as to test the sensitivity of the solution to a particular parameter. MPSX retrieves the optimal basis from the

last computer run and uses this old basis as the new initial basis. This procedure drastically reduces the number of iterations required to reach optimality. For example, for a maximum machine life of fifteen years, the reduction for the case of the primal is from approximately 1000 iterations to approximately 100 iterations. Sometimes, as few as about ten iterations are needed to reach optimality. It is more economical to solve the primal than the dual since the primal has fewer rows than the dual.

4. Computational Experience with Equipment Replacement Model

We now consider the specific problem of when to rebuild or replace continuous miners, machines employed in the coal industry, by specializing the equipment model.

4.1. Assumptions

·Costs and profits are subject to inflation.

·There is a cost associated with operating and maintaining a new machine and this cost increases as the machine ages.

·There is a cost associated with rebuilding a machine for the first time and this cost increases for each period that the machine is not rebuilt.

·Rebuilding reduces the maintenance cost to some level after which the cost increases from that level with age.

·Technological improvement affects the production capacity of a new machine.

·The initial and present capacity of a machine is quantifiable.

·Production capacity decays with age.

·Rebuilding restores a certain amount of production potential.

The present model does not consider every conceivable cost, but it is a straightforward matter to include any expense which can be identified and quantified. It is a virtue of the model that the costs need not be expressed in analytic form but may be expressed in an arbitrary manner, e.g., they may be tabulated in tables.

4.2. Input Variables

Following are the parameters which are needed in computing the various profits:

•Discount Rate (DR).

•Inflation Rate (IR).

•Effective Tax Rate (ETR).

•Depreciation Inflow [DI(J,N)] -- The term "Depreciation Inflow" is, strictly speaking, a misnomer. In the case of a capitalized rebuild, this term is equal to the difference between the cash outflow and the resulting depreciation inflow (the net result being a cash outflow). If a new machine is being purchased, the term equals the trade-in or salvage value of the old machine. The term is therefore calculated separately for each decision. The double declining balance method of depreciation is used, switching to the straight line method when appropriate.

•Purchase Price of a new machine (PP).

•Maintenance Cost (MC) -- The cost associated with operating and maintaining a machine.

•Maintenance Cost Increase Rate (MCIR) -- The rate at which the cost of maintaining a machine increases.

•Rebuild Cost (RC) -- Cost of rebuilding a machine.

•Rebuild Cost Increase Rate (RCIR) -- The rate at which the rebuilding cost increases for each period that the machine is not rebuilt.

•Effect of Technology on the Capacity of new machines (ETC) -- The increase in production capacity resulting from technological improvements.

•Production Decay (PD) -- The annual rate at which production decreases.

•Base Capacity (BC) -- The capacity of a new machine.

•Effect of Rebuilding on Machine capacity (ERM).

•After Tax Profit of a ton of coal (ATP).

•Equipment Life (EL) -- The Internal Revenue Service guideline life. It is used in computing the depreciation charges.

•Write-off Period for a Capitalized Rebuild (WPCR) -- The number of years over which the Internal Revenue Service allows writing off the cost of rebuilding a machine which is older than its guideline life.

4.3. Profit Functions

In the following formulas, positive terms represent cash inflows and negative terms represent cash outflows. Normally, a tax credit (25%) based upon the tax shield generated by the depreciation, maintenance and rebuild charges is considered a cash inflow. For the purpose of simplifing the equations, instead of taking 25% of the tax credit, we will express the cash effect of the cash credit in a slightly unorthodox manner. Instead of taking 25% of the tax credit as a cash inflow, we consider 75% of the maintenance cost and rebuild cost as a cash outflow and 25% of depreciation as a cash inflow.

Example:

Depreciation	=	$12,000
Maintenance	=	$60,000
Rebuild	=	$20,000

a) Tax shield = 12 + 60 + 20 = 92

25% tax credit (cash inflow) = 23

Net cash outflow = 80 - 23 = 57

b) Cash outflow = 3/4 (60) + 3/4 (20) = 60

Cash inflow = 1/4 (12) = 3

Net cash outflow = 57

Rebuild Profit Function: CR(IJN) -- Equation 1 states that the profit of rebuilding a machine in state (IJN) is equal to minus the increased rebuild cost minus the maintenance cost plus the profit from the output of an N year old machine whose production capability is summarized by the two indices I and J plus the inflow from depreciation. The double declining balance method of depreciation is used (eq. 1a), switching to the straight line method when appropriate (eq. 1b). When a machine is older than the guideline life, the cost of rebuilding is capitalized (a cash outflow) and a new depreciation charge is computed based upon its new life (eq. 1c). If the machine has already experienced a capitalized rebuild, then there is an existing depreciation charge which must be continued (eq. 1d).

Maintain Profit Function: CM(IJN) -- The profit of maintaining a machine in state (IJN) is equal to minus the increased

maintenance expense plus the profit from production plus the inflow from depreciation (eq. 2). If a machine is older than its guideline life, there is no depreciation charge unless there was a previous capitalized rebuild (eqs. 2c, 2d, 2e).

Buy Profit Function: CB(IJN) -- Equation 3 states that the profit of replacing an old machine with a new one is equal to minus the purchase price and the maintenance cost of the new machine plus the inflow from the trade-in, which is assumed to be equal to the remaining tax depreciation on the old machine (eqs. 3 and 3a). If the old machine is older than the guideline life and has had a capitalized rebuild, then there is some remaining depreciation which must be considered (eq. 3b).

4.4. Effective Discount Factor

We combine the discount, inflation and technological improvement rates into a single effective discount factor given by the formula

$$(1 + \text{inflation rate})/[(1 + \text{discount rate})(1 + \text{rate of technological improvement})].$$

The rationale behind the addition of the technological improvement rate to the ostensible discount rate is as follows: One dollar invested today at a discount rate of 15% is worth \$1.15 next year, and if technology is better next year by 1%, then we can buy \$1.16 of productivity next year with one dollar invested this year.

Rebuild Profit Function: CR(IJN)

1) $$CR(IJN) = -(1-ETR)(RC)[1+(N-J-1)(RCIR)] - (1-ETR)(MC) + (ATP)[(BC)(ERM)^{I+1} - (PD)(N-I-1)] + DI(JN)$$

Let

$$x(N) = (PP)[1-\sum_{k=0}^{N}(DDB)(1-DDB)^k]/[(EL)-N] - (PP)(DDB)(1-DDB)^N$$

and let N_o be the smallest value of N for which $x(N) \geq 0$.

1a) If $x(N) < 0$, then $DI(JN) = (ETR)(PP)(DDB)(1-DDB)^N$

1b) If $x(N) \geq 0$, then $DI(JN) = (ETR)(PP)[1-\sum_{k=0}^{N_o}(DDB)(1-DDB)^k]/[(EL)-N_o]$

1c) If $EL < N$, then $DI(JN) = (ETR)(RC)[1+(N-J-1)(RCIR)]/(WPCR) - (RC)[1+(N-J-1)(RCIR)]$

1d) If $EL + 1 < N$ and $J = N-1$, then

$$DI(JN) = (ETR)(RC)[1+(N-J-1)(RCIR)]/(WPCR) - (RC)[1+(N-J-1)(RCIR)] + (ETR)(RC)/(WPCR)$$

Maintain Profit Function: CM(IJN)

2) $$CM(IJN) = -(1-ETR)[(MC)+(MCIR)(N-J)] + (ATP)[(BC)(ERM)^{I} - (PD)(N-J)] + DI(JN)$$

2a) If $x(N) < 0$, then $$DI(JN) = (ETR)(PP)(DDB)(1-DDB)^{N}$$

2b) If $x(N) \geq 0$, then $$DI(JN) = (ETR)(PP)[1- \sum_{k=0}^{N_o} (DDB)(1-DDB)^{k}] / [(EL)-N_o]$$

2c) If $EL + 1 = N$, then $DI(JN) = 0$

2d) If $EL < N$ and $J \neq N - 1$, then $DI(\_N) = 0$

2e) If $EL + 1 < N$ and $J = N - 1$, then $DI(JN) = (ETR)(RC)/(WPCR)$

B

Buy Profit Function: CB(IJN)

3) $$CB(IJN) = -(PP) - (1-ETR)(MC) + (ETR)(DDB)(PP) + DI(JN)$$

3a) If $J \leq EL$, then $$DI(JN) = (PP)[1- \sum_{k=0}^{N} (DDB)(1-DDB)^{k}]$$

3b) If $J > EL$ and $J = N - 1$, then

$$DI(JN) = (PP)[1- \sum_{k=0}^{N} (DDB)(1-DDB)^{k}] + (ETR)(RC)[(WPCR)-N+J]/(WPCR)$$

Table 1

Parameters for the Continuous Miner

DR = Discount Rate = 15%

RI = Inflation Rate = 5%

ETC = Effect of Technology on Capacity of New Machine = 1%

$$\alpha = (1 + RI)/((1 + DR)\ (1 + ETC)) = 0.904$$

ETR = Effective Tax Rate = 25%

PP = Purchase Price of New Machine = $180,000

MC = Maintenance Cost = $15,000

MCIR = Maintenance Cost Increase Rate = $10,000 Per Year

RC = Rebuild Cost = $20,000

RCIR = Rebuild Cost Increase Rate = 10%

ATP = After Tax Profit of Ton of Coal = $1.50

EL = Equipment Life = 10 Years

PD = Production Decay = 7,000 Tons Per Year

ERM = Effect of Rebuilding on Machine Capacity = 95%

BC = Base Capacity = 150,000 Tons Per Year

WPCR = Write-Off Period for a Capitalized Rebuild = 4 Years

DDB = A Constant Used in the Depreciation Equations = 0.2

4.5. Computational Experience

Tables 2 through 12 report on computational experience using the data of Table 1. Table 2 gives the size of the primal linear programming problem as a function of maximum life L and the number of iterations to solve them (without starting bases) on an IBM 360/65 using MPSX. The optimal strategies as a function of L are given in Table 3. In the tables, we abbreviate the decisions to rebuild, maintain or buy as R, M and B, respectively.

It is easy to show that the optimal value in Table 3 is monotonically increasing as a function of L. Approximate upper bound calculations indicate that the error introduced by setting L equal to 15 is small. Thus, for practical purposes, the optimal strategy and the value of the optimal strategy for all larger L is obtained for L = 15.

Table 2

Maximum Life L	Rows (states)	Columns (arcs)	Simplex Iterations
5	25	53	40
10	175	433	304
15	575	1573	1134
16	696	1846	1302

Tables 4 through 12 report on the sensitivity of the optimal solution to other parameters. In all cases the maximum life was taken to be 15 years, and the number of iterations of the simplex algorithm are greatly reduced because the optimal basis from Table 3 (L = 15) was used as a starting basis. Notice from Table 9 that the regeneration cycle is quite sensitive to the effect of rebuilding on machine capacity. Notice also

from Table 11 that the two methods of depreciation give almost identical results.

Table 3

Sensitivity Analysis: Maximum Life (Years)

	5	10	15	16
Year	Decision	Decision	Decision	Decision
1	M	M	M	M
2	M	M	M	M
3	R	R	R	R
4	M	M	M	M
5	B	M	M	M
6		R	R	R
7		M	M	M
8		M	M	M
9		M	M	M
10		B	R	R
11			M	M
12			M	M
13			M	M
14			B	B
	Value	Value	Value	Value
	$1,405,346	$1,587,561	$1,600,849	$1,600,849

Table 4

Sensitivity Analysis: Effective Discount Factor

	0.500	0.800	0.904	0.950	0.990
Year	Decision	Decision	Decision	Decision	Decision
1	M	M	M	M	M
2	M	M	M	M	M
3	R	R	R	R	M
4	M	M	M	M	R
5	R	M	M	M	M
6	M	R	R	R	M
7	M	M	M	M	R
8	M	M	M	M	M
9	R	M	M	R	M
10	M	R	R	M	M
11	M	M	M	M	B
12	M	M	M	B	
13	M	M	M		
14	M	M	B		
15	B	B			
	Value	Value	Value	Value	Value
	$381,245	$849,126	$1,600,849	$2,902,923	$13,738,916
LP Iterations	66	58	--	128	320

Table 5

Sensitivity Analysis: Rebuild Cost Increase Rate

	0.1	1.0	3.0	5.0
Year	Decision	Decision	Decision	Decision
1	M	M	R	R
2	M	M	R	R
3	R	R	R	R
4	M	M	R	R
5	M	M	R	R
6	R	R	R	R
7	M	M	R	R
8	M	M	R	R
9	M	R	B	B
10	R	M		
11	M	M		
12	M	M		
13	M	B		
14	B			
	Value	Value	Value	Value
	$1,600,849	$1,531,827	$1,457,898	$1,457,898
LP Iterations	--	1085*	74	93

*no starting basis

Table 5

Sensitivity Analysis: Rebuild Cost Increase Rate

	0.1	1.0	3.0	5.0
Year	Decision	Decision	Decision	Decision
1	M	M	R	R
2	M	M	R	R
3	R	R	R	R
4	M	M	R	R
5	M	M	R	R
6	R	R	R	R
7	M	M	R	R
8	M	M	R	R
9	M	R	B	B
10	R	M		
11	M	M		
12	M	M		
13	M	B		
14	B			
	Value	Value	Value	Value
	$1,600,849	$1,531,827	$1,457,898	$1,457,898
LP Iterations	--	1085*	74	93

*no starting basis

Table 6

Sensitivity Analysis: Maintenance Cost Increase Rate

	$10,000	$20,000	$30,000	$40,000
Year	Decision	Decision	Decision	Decision
1	M	M	M	R
2	M	M	R	R
3	R	R	M	R
4	M	M	R	R
5	M	M	M	R
6	R	R	R	R
7	M	M	M	R
8	M	M	R	R
9	M	R	M	R
10	R	M	B	R
11	M	M		B
12	M	B		
13	M			
14	B			
	Value	Value	Value	Value
	$1,600,849	$1,518,553	$1,473,455	$1,456,837
LP Iterations	--	81	113	134

Table 7

Sensitivity Analysis: Maintenance Cost

	$15,000	$45,000	$90,000
Year	Decision	Decision	Decision
1	M	M	M
2	M	M	M
3	R	R	R
4	M	M	M
5	M	M	M
6	R	R	R
7	M	M	M
8	M	M	M
9	M	M	M
10	R	R	R
11	M	M	M
12	M	M	M
13	M	M	M
14	B	B	B
	Value	Value	Value
	$1,600,849	$1,366,464	$1,014,887
LP Iterations	--	57	57

Table 8

Sensitivity Analysis: Rebuild Cost

	$20,000	$60,000	$100,000	$150,000
Year	Decision	Decision	Decision	Decision
1	M	M	M	M
2	M	M	M	M
3	R	M	M	M
4	M	R	R	M
5	M	M	M	R
6	R	M	M	M
7	M	M	M	M
8	M	R	R	M
9	M	M	M	M
10	R	M	M	M
11	M	M	M	B
12	M	M	M	
13	M	B	B	
14	B			
	Value	Value	Value	Value
	$1,600,849	$1,525,168	$1,459,404	$1,398,442
LP Iterations	--	44	49	84

Table 9

Sensitivity Analysis: Effect of Rebuilding on Machine Capacity

Year	75% Decision	95% Decision	100% Decision
1	M	M	R
2	M	M	R
3	M	R	R
4	M	M	R
5	M	M	R
6	R	R	R
7	M	M	R
8	M	M	R
9	B	M	R
10		R	R
11		M	R
12		M	M
13		M	R
14		B	M
15			B
	Value	Value	Value
	$1,361,157	$1,600,849	$1,964,784
LP Iterations	188		255

Table 10

Sensitivity Analysis: After Tax Profit of Ton of Coal

Year	$0.50 Decision	$1.00 Decision	$1.50 Decision	$3.00 Decision
1	M	M	M	M
2	R	M	M	M
3	M	R	R	M
4	R	M	M	R
5	M	M	M	M
6	M	R	R	M
7	R	M	M	M
8	M	M	M	R
9	M	M	M	M
10	R	R	R	M
11	M	M	M	M
12	M	M	M	B
13	M	M	M	
14	B	B	B	
	Value	Value	Value	Value
	$347,099	$973,727	$1,600,849	$3,492,839
LP Iterations	120	68		59

Table 11

Sensitivity Analysis: Depreciation

Double Declining Balance vs. Straight Line

Year	Double Declining Balance Decision	Straight Line Decision
1	M	M
2	M	M
3	R	R
4	M	M
5	M	M
6	R	R
7	M	M
8	M	M
9	M	M
10	R	R
11	M	M
12	M	M
13	M	M
14	B	B
	Value	Value
	$1,600,849	$1,602,617
LP Iterations		62

Table 12

Sensitivity Analysis: Purchase Price

Year	$100,000 Decision	$180,000 Decision
1	M	M
2	M	M
3	R	R
4	M	M
5	M	M
6	R	R
7	M	M
8	M	M
9	R	M
10	M	R
11	M	M
12	B	M
13		M
14		B
	Value	Value
	$1,615,845	$1,600,849
LP Iterations	58	

5. Suggestions for Further Study

The deterministic infinite horizon dynamic programming model of this paper can be readily extended to a stochastic formulation by allowing, for example, a random transition to a new machine due to unexpected failure. The state space can also be expanded to allow states of the machine in various stages of aging which are probabilistically attained. If the appropriate transition probabilities are known, the linear programming formulations are basically the same.

Other areas of future interest are:

·Streamlining of the matrix generation, e.g., elimination of states of obvious low value.

·Proof (if possible) that the value of the optimal strategy is a concave function of maximum machine life.

·Development of tight upper bounds on the value of the optimal strategy as a function of maximum machine life.

·Consideration of lead time if physical replacement occurs well after the decision to replace is made.

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