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Non-topological gravitating defects in five-dimensional anti-de Sitter space

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Abstract

A class of five-dimensional warped solutions is presented. The geometry is everywhere regular and tends to five-dimensional anti-de Sitter space for large absolute values of the bulk coordinate. The physical features of the solutions change depending on the value of an integer parameter. In particular, a set of solutions describes generalized gravitating kinks where the scalar field interpolates between two different minima of the potential. The other category of solutions describes instead gravitating defects where the scalar profile is always finite and reaches the same constant asymptote both for positive and negative values of the bulk coordinate. In this sense the profiles are non-topological. The physical features of the zero modes are discussed.

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In the presence of infinite extra-dimensions [1] (see also [2, 3]) fields of various spin are localized around higher dimensional gravitating defects whose properties determine, at least partially, the features of the localized interactions. Consider, therefore, one of the simplest incarnations of this idea and suppose that there is only one infinite extra dimension that will be denoted, in what follows, by w. The five-dimensional line element can then be written as

$$ds^2 = g_{AB} d\,x^A d\,x^B = a^2(w)[\eta_{\mu\nu} d\,x^{\mu} d\,x^{\nu} - dw^2], \tag{1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric with signature mostly minus; the Latin (uppercase) indices run over all the five dimensions while the Greek indices run over the (3+1) observable dimensions. The coordinate w runs continuously from $-\infty$ to $+\infty$. In the situation described by Eq. (1), five-dimensional domain-wall solutions are known to exist [4, 5, 6, 7] (see also [8, 9, 10]) and they have the structure of gravitating kinks whose associated geometry is rather similar to the one generated by 3-brane sources supplemented by a negative cosmological constant [11]. Five-dimensional gravitating kinks can arise both in the case of Einstein-Hilbert gravity and in the case of quadratic gravity theories of Euler-Gauss-Bonnet type (see, for instance, [12, 13, 14, 15] and references therein).

The scalar-tensor action adopted for illustrating the present considerations will then be given by

$$S = \int d^5x \sqrt{|g|}\biggl[-\frac{R}{2\kappa} + \frac{1}{2} g^{AB}\partial_A\phi\partial_B\phi - U(\phi)\biggr], \qquad \kappa = 8\pi G_5 = 8\pi M_5^{-3}, \tag{2}$$

leading to the equations [2]

$$\mathcal{F}^2 = \frac{\kappa}{6}\biggl[\frac{\phi'^2}{2} - U a^2\biggr], \qquad \mathcal{F}' = -\frac{\kappa}{4}\biggl[\phi'^2 + \frac{2}{3} U a^2\biggr], \tag{3}$$

$$\phi'' + 3\mathcal{F}\phi' - \frac{\partial U}{\partial \phi} a^2 = 0, \tag{4}$$

where $\mathcal{F} = a'/a$. A consistent solution of Eqs. (3) and (4) can be obtained in the form

$$a(w) = [(bw)^{2\nu} + 1]^{-\frac{1}{2\nu}}, \qquad \nu \geq 1, \tag{5}$$

$$\phi(w) = v \pm \frac{1}{\beta} \arctan[(bw)^{\nu}], \tag{6}$$

$$U(\phi) = U_0\, \Lambda(\phi)^{\frac{\nu-1}{\nu}} [(2\nu - 1) - (3 + 2\nu)\Lambda(\phi)] \tag{7}$$

where b is a parameter (with dimensions of inverse length) related with the thickness of the scalar profile and where:

$$\Lambda(\phi) = \sin^2(\phi - v), \qquad \beta = \sqrt{\frac{\kappa}{3}\biggl(\frac{\nu^2}{2\nu - 1}\biggr)}, \qquad U_0 = \frac{3b^2}{2\kappa}. \tag{8}$$

[2]In the following, the prime will always denote a derivation with respect to the bulk coordinate w.

In Eq. (6), v arises as an integration constant with the same dimensions of β^{-1}. In Eqs. (5), (6) and (7) ν is a positive integer (i.e. $\nu \geq 1$). Since the bulk coordinate w may take both positive and negative values, if ν would be rational or even real, the functions defining the solution will may become imaginary [3]. The curvature invariants pertaining to the solution defined by Eqs. (5), (6) and (7) can be simply computed and they are

$$R^2 = 16\, b^4 \frac{(bw)^{4(\nu-1)}[(2-4\nu)+5(bw)^{2\nu}]^2}{[1+(bw)^{2\nu}]^{4-\frac{2}{\nu}}}, \tag{9}$$

$$R_{AB}R^{AB} = 4\, b^4 \frac{(bw)^{2(\nu-2)}[20(bw)^{6\nu}+5(1-2\nu)^2(bw)^{2\nu}+16(1-2\nu)(bw)^{4\nu}]}{[1+(bw)^{2\nu}]^{4-\frac{2}{\nu}}}, \tag{10}$$

$$R_{ABCD}R^{ABCD} = 8b^4 \frac{(bw)^{4(\nu-1)}[5(bw)^{4\nu}+2(1-2\nu)^2+4(bw)^{2\nu}(1-2\nu)]}{[1+(bw)^{2\nu}]^{4-\frac{2}{\nu}}}, \tag{11}$$

where R_{ABCD}, R_{AB} and R are, respectively, the Riemann tensor, the Ricci tensor and the Ricci scalar. In the case of the metric (1), the Weyl invariant vanishes. Since $\nu \geq 1$, Eq. (5) implies that $a(w) \to |bw|^{-1}$ in the limit $|bw| \to \infty$, i.e. for values of the bulk coordinate much larger than the thickness of the configuration. Since $\nu \geq 1$ is a positive integer, the curvature invariants do not have poles for any finite value of w. The quantity $w_0 = b^{-1}$ is the radius of the (asymptotic) AdS_5 space. Consistently with this behaviour, the explicit form of the curvature invariants goes to a constant for $|bw| \to \infty$.

If ν is *odd*, i.e. $\nu = 2\,m + 1$ with $m = 0, 1, 2, 3, ...$, Eq. (6) implies that $\beta(\phi - v)$ varies between $-\pi/2$ and $\pi/2$. The plus sign in Eq. (6) corresponds to the kink solution while the minus sign corresponds to the anti-kink solution. In the case of one spatial dimension, spatial infinity consists of two points, i.e. $\pm\infty$; a topological charge is then customarily defined for the characterization of $(1+1)$-dimensional defects such as the ones arising in the case of sine-Gordon system [16]. When ν is odd, therefore, we will have that the topological charge does not vanish and is given, in particular, by

$$Q_m = \frac{1}{2\pi}\int_{-\infty}^{\infty} dw \frac{\partial\phi}{\partial w} = \pm\frac{\sqrt{4m+1}}{2m+1}\lambda, \qquad \lambda = \sqrt{\frac{3}{4\kappa}}, \tag{12}$$

where the plus and the minus signs correspond, respectively, to a kink and to an anti-kink solution.

If ν is *even*, i.e. $\nu = 2n$ with $n = 1, 2, 3, ...$, from Eq. (6), $\beta(\phi - v)$ goes, asymptotically, to the same value both for $w \to -\infty$ and for $w \to +\infty$. Therefore, applying the definition reported in the first equality of Eq. (12), we will have, in this case that $Q_n = 0$ in spite of the sign appearing in Eq. (6). This second class of solutions seems then to describe more non-topological rather than topological defects. It should be clear that five-dimensional gravity

[3] From a swift inspection of Eqs. (5), (6) and (7) it may seem that the solution can be continued also for negative values of ν. This is not correct since, if $\nu < 0$, ${\phi'}^2$ becomes negative, or, equivalently, the parameter β defined in Eq. (8) becomes imaginary.

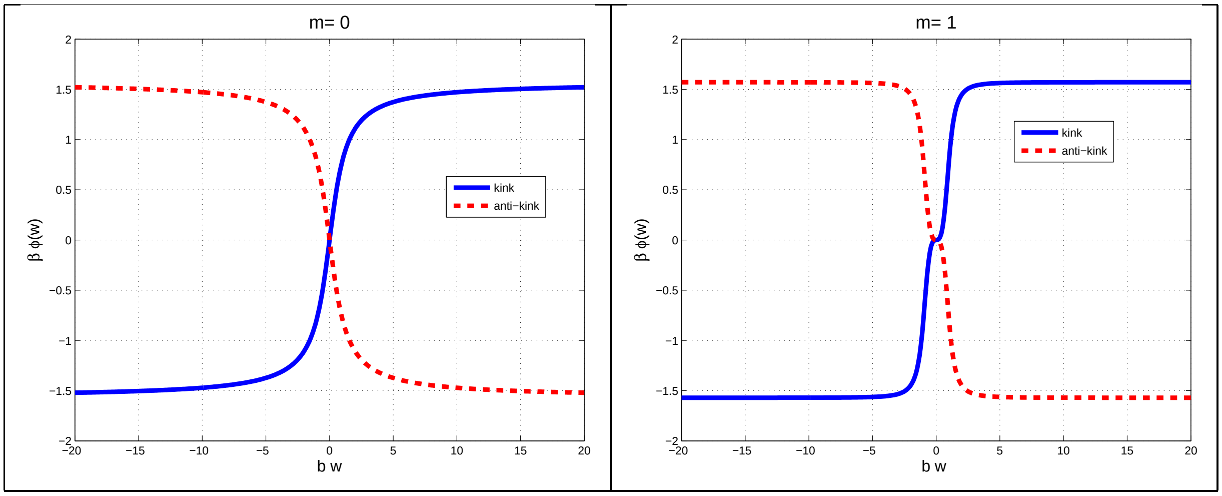

Figure 1: Two cases of kink and anti-kink solutions are illustrated. They both arise for odd ν and, in particular, for $\nu = 1$ (plot at the left) and $\nu = 3$ (plot at the right). Recall that, when ν is odd, it is conventionally parametrized as $\nu = 2m + 1$.

is essential in order to have this type of profiles. In the absence of gravity, non-topological defects in $(1 + 1)$ dimensions are connected with an additive conservation law, so that one should demand that the system contains, at least, a complex scalar field with global $U(1)$ symmetry [16] or, equivalently, two real scalar fields [17]. Here, however, because of the presence of gravity, bell-like profiles can arise even if ϕ is not complex as the solution (6) demonstrates explicitly when ν is even.

For graphical illustration, it is practical to rescale ϕ through β in such a way that the rescaled field, i.e. $\overline{\phi}(w) = \beta\phi(w)$ is dimensionless. In the following, when not otherwise stated, we will also fix, without loss of generality, $v = 0$. The cases $m = 0$ and $m = 1$ (i.e. $\nu = 1$ and $\nu = 3$) are illustrated, respectively, in the left and in the right plot of Fig. 1. As m increases an intermediate plateau develops close to $w = 0$ (see the right plot in Fig. 1 and the left plot in Fig. 2). In the case $m = 0$ (full line in the right plot of Fig. 2) the potential is of sine-Gordon type and it is, according to Eq. (7), $U(\phi) = U_0(5\cos 2\beta\phi - 3)/2$. The minima of the potential are located, for $\nu = 1$, in $-\pi/2$ and in $+\pi/2$ (see right plot of Fig. 2). By looking simultaneously at Fig. 1 (full line in the left plot) and at Fig. 2 (full line in the right plot) it appears that the kink solution connects the minimum in $-\pi/2$ to the the minimum in $\pi/2$ and $\beta\phi$ correctly interpolates between these two values. This situation reminds a bit the sine-Gordon system in $(1 + 1)$ dimensions [16] where, however, the potential vanishes at the minima while here it is negative due to the gravitating nature of the solution. As m increases the potential develops, at the centre of the interval of periodicity, a second (local) minimum which is located, for the interval chosen in Fig. 2, in $\phi = 0$. Since the minimum is only local (and not global) the field does not settle down and finally reaches the true global

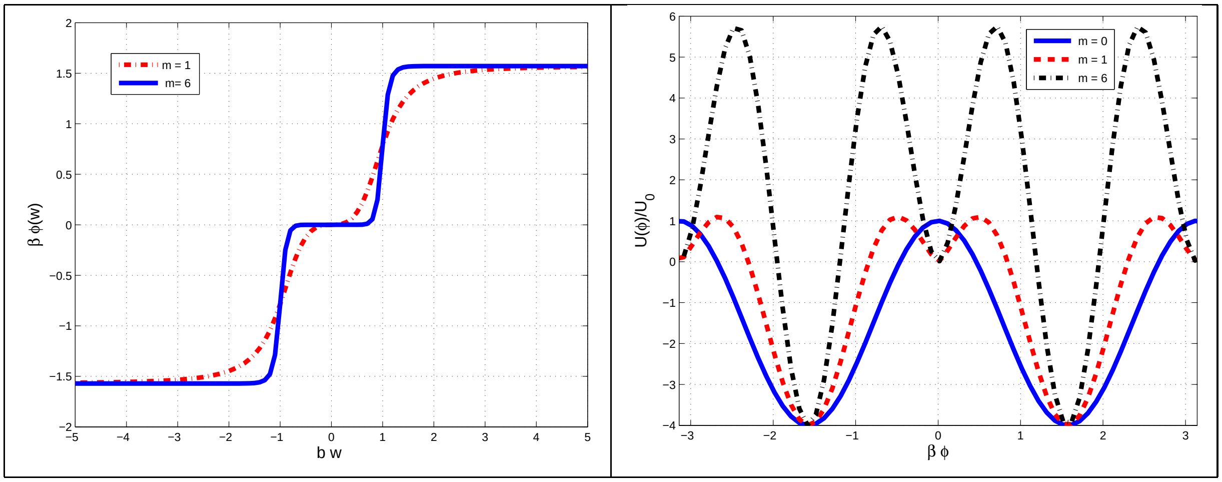

Figure 2: In the left plot the behaviour of ϕ is illustrated for large (odd) ν (as in Fig. 1, $\nu = 2m+1$). In the plot at the right the potential $U(\phi)$ is reported for three different values of m as specified in the legend.

minimum in $\pi/2$. As m increases further (dot-dashed lines in both plots of Fig. 2), the local minimum becomes more and more pronounced and the length of the intermediate plateau in ϕ gets larger (see Fig. 2, left plot). According to Eq. (6), both ϕ' and ϕ'^2 are always finite and regular for every value of the bulk radius. In Figs. 1 and 2 $\beta v = 0$ has been assumed.

If ν is even the scalar profile goes, asymptotically, to the same value for $w \to \pm\infty$. In Fig. 3 we report the profile of $\beta\phi$ and its related potential for few values of even ν and for two different values of βv (i.e. $\beta v = 0$ and $\beta v = \pi/2$). In the left plot of Fig. 3 the scalar field is illustrated as a function of the bulk radius for two different values of βv (i.e. $\beta v = 0$ and $\beta v = \pi/2$). By increasing the value of n the width of $\beta\phi$ increases (dashed line in the left plot of Fig. 3). Given the properties of this second class of solutions the case of even ν resembles the one of a non-topological defect.

As already mentioned the geometry is AdS$_5$ for $|bw| \to \infty$. This aspect can be clearly appreciated from Fig. 4 (left plot) where the warp factor is illustrated for different values of ν: for $|bw| \to \infty$, $a(w) \simeq 1/|bw|$. The curvature invariants, in the same limit, reach a constant value. In Fig. 4 (right plot) the Riemann invariant is illustrated. The other curvature invariants are qualitatively similar. As a consequence of the features of the geometry the four-dimensional Planck mass is finite since it is simply given by

$$M_P^2 \simeq M_5^3 \int_{-\infty}^{\infty} dw a^3(w) = 2bM_5^3 \frac{\Gamma(1+\frac{1}{2\nu})\Gamma(\frac{1}{\nu})}{\Gamma(\frac{3}{2\nu})}, \qquad \nu \geq 1 \tag{13}$$

where the second equality follows by performing explicitly the integral when $a(w)$ is given by Eq. (5) and when, as assumed throughout, $\nu \geq 1$. Since the four-dimensional Planck mass is finite, the tensor fluctuations of the geometry are localized on the profile both for even and

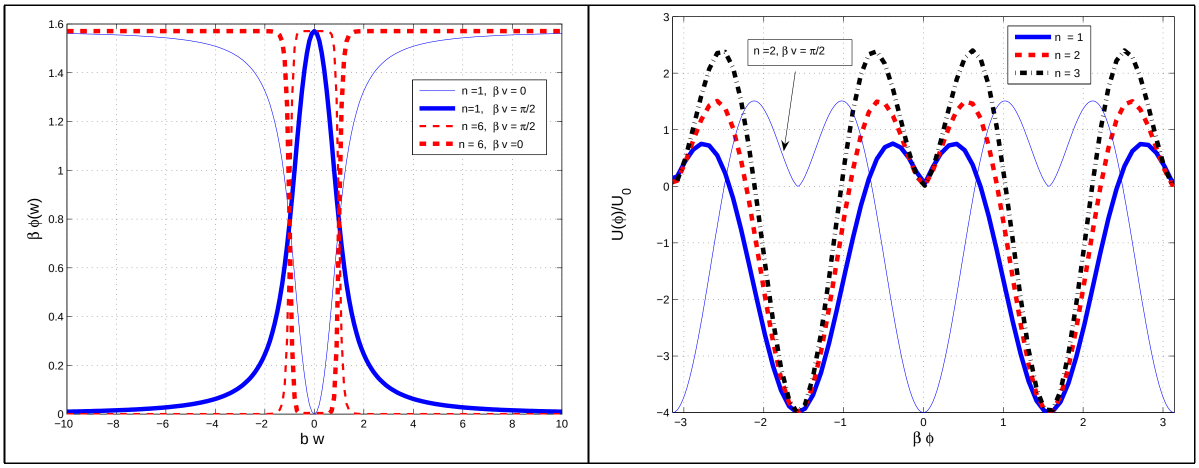

Figure 3: The profiles of the scalar field (plot at the left) and the potential (plot at the right) are illustrated for even values of ν. In this case we parametrize $\nu = 2n$ with n positive integer. The cases $n = 1$ and $n = 6$ are reported (plot at the left), respectively, with the full and thin lines. The thick lines (plot at the left) refer to different boundary conditions as specified in the legend.

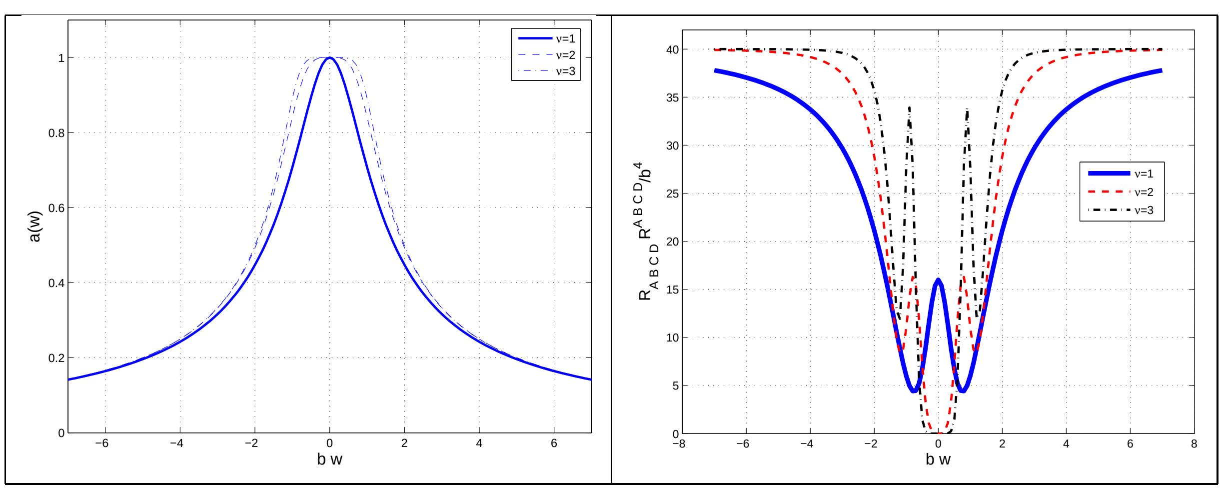

Figure 4: The profile of the warp factor (left plot) and the Riemann invariant (right plot) for different values of ν. For the other two curvature invariants reported in Eqs. (9) and (10), the plots are qualitatively similar since they tend to a constant value for $|bw| \to \infty$ and they never get singular for finite w.

odd ν. This occurrence is common also to the case when the defect is modeled by a 3-brane [11]. Less obvious is the fact that the scalar fluctuations of the sources are not localized. These results stem from the analysis of the zero modes of the configurations defined by Eqs. (5), (6) and (7) and will now be swiftly addressed. To discuss this problem one can then adopt the formalism developed in [6]. In five dimensions the perturbed geometry leads to 15 independent degrees of freedom which can be classified according to the way they transform under four-dimensional Poincaré transformations. To the fluctuations of the geometry one has also to add the fluctuation of the ϕ, i.e. the fluctuation of the profile of the defect. Therefore, the fluctuations of the geometry and of the scalar profile can be written as

$$g_{AB}(x^{\mu}, w) = \overline{g}_{AB}(w) + \delta g_{AB}(x^{\mu}, w), \qquad \phi(x^{\mu}, w) = \phi(w) + \chi(x^{\mu}, w). \tag{14}$$

where

$$\delta g_{AB} = a^2(w)\begin{pmatrix} 2h_{\mu\nu} + (\partial_{\mu} f_{\nu} + \partial_{\nu} f_{\mu}) + 2\eta_{\mu\nu}\psi + 2\partial_{\mu}\partial_{\nu} E & D_{\mu} + \partial_{\mu} C \\ D_{\mu} + \partial_{\mu} C & 2\xi \end{pmatrix}. \tag{15}$$

On top of $h_{\mu\nu}$ which is divergence-less and trace-less (i.e. $\partial_{\mu} h^{\mu}_{\nu}$=0, $h_{\mu}^{\mu} = 0$) there are four scalars (i.e. E, ψ, ξ and C) and two divergence-less vectors (D_{μ} and f_{μ}).

The analysis can be conducted in gauge-invariant terms without assuming any specific form of the background geometry. Neglecting the vector modes of the geometry [4] the relevant zero modes are the ones associated with the graviton and with the scalar fluctuations. The decoupled evolution equation of the tensor modes can be written as [6]

$$\mu_{\mu\nu}'' - \partial_{\alpha}\partial^{\alpha}\mu_{\mu\nu} - \frac{(a^{3/2})''}{a^{3/2}}\mu_{\mu\nu} = 0. \tag{16}$$

where $\mu_{\mu\nu} = a^{3/2} h_{\mu\nu}$ is the canonical normal mode of the of the action (2) perturbed to second order in the amplitude of tensor fluctuations [6]. The lowest mass eigenstate of Eq. (16) is $\mu(w) = \mu_0 a^{3/2}(w)$. Hence, the normalization condition of the tensor zero mode implies

$$|\mu_0|^2 \int_{-\infty}^{\infty} a^3 \, dw = 2|\mu_0|^2 \int_{0}^{\infty} a^3(w) \, dw = 1. \tag{17}$$

The integral appearing in Eq. (17) is always convergent if, as assumed throughout the paper, ν is a positive integer. Therefore, the graviton zero mode is always localized on the configurations discussed here. The scalar normal mode of the action is a linear combination χ (defined in Eq. (14)) and and of ψ (defined in Eq. (15). The canonical variable is then [6] $\mathcal{G} = a^{3/2}\psi - z\chi$ and it obeys the equation

$$\mathcal{G}'' - \partial_{\alpha}\partial^{\alpha}\mathcal{G} - \frac{z''}{z}\mathcal{G} = 0, \qquad z(w) = \frac{a^{3/2}\phi'}{\mathcal{F}}. \tag{18}$$

[4]The vector modes are not localized since their corresponding zero mode is not normalizable as it follows from the evolution equations of D_{μ}; the other vector, i.e. f_{μ} can be gauged away by using the freedom of fixing the coordinate system.

The lowest mass eigenstate of Eq. (18) is given by $\mathcal{G}(w) = \mathcal{G}_0 z(w)$ which is normalizable iff

$$\int_{-\infty}^{\infty} z^2(w)\, dw = \frac{3}{\kappa}(2\nu - 1)\int_{-\infty}^{+\infty} \frac{dw}{(bw)^{2\nu}[1 + (bw)^{2\nu}]^{\frac{3}{2\nu}}}, \qquad \nu \geq 1 \tag{19}$$

where the right hand side follows from the definition off $z(w)$ and from the explicit form of the solution. But the integrand in Eq. (19) is divergent for $|bw| \to 0$ as $|bw|^{-2\nu}$. As ν increases, the divergence becomes always more severe. We then conclude that the scalar modes of the geometry are not localized on the defect.

In conclusion a new class of solutions of five-dimensional warped geometries has been presented and discussed. This class of solution contains, simultaneously, kink-like profiles and bell-like scalar profiles. The regular geometry of the configuration allows the localization of the tensor modes of the geometry. Neither the scalar nor the vector modes are localized. The present findings seem to suggest that not only domain walls but also gravitating non-topological defects if five dimensions may be used to localize gravitational interactions. Furthermore, it is intriguing that these two rather different physical situations may arise in the same class of solutions.

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