Exponentially Small Splitting of Invariant Manifolds near a Hamiltonian-Hopf Bifurcation

by

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Declarations

I declare that, to the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise indicated, cited, or commonly known.

The material in this thesis is submitted to the University of Warwick for the degree of Doctor of Philosophy and has not been submitted to any other university or for any other degree.

The results in chapter 5 have been submitted for publication in Nonlinearity and are currently under consideration.

Abstract

Consider an analytic two-degrees of freedom Hamiltonian system with an equilibrium point that undergoes a Hamiltonian-Hopf bifurcation, i.e., the eigenvalues of the linearized system at the equilibrium change from complex $\pm\beta \pm i\alpha$ $(\alpha, \beta > 0)$ for $\epsilon > 0$ to pure imaginary $\pm i\omega_1$ and $\pm i\omega_2$ $(\omega_1 \neq \omega_2 \neq 0)$ for $\epsilon < 0$. At $\epsilon = 0$ the equilibrium has a pair of doubled pure imaginary eigenvalues. Depending on the sign of a certain coefficient of the normal form there are two main bifurcation scenarios. In one of these (the stable case), two dimensional stable and unstable manifolds of the equilibrium shrink and disappear as $\epsilon \to 0^+$. At any order of the normal form the stable and unstable manifolds coincide and the invariant manifolds are indistinguishable using classical perturbation theory. In particular, Melnikov's method is not capable to evaluate the splitting.

In this thesis we have addressed the problem of measuring the splitting of these manifolds for small values of the bifurcation parameter ϵ. We have estimated the size of the splitting which depends on a singular way from the bifurcation parameter. In order to measure the splitting we have introduced an *homoclinic invariant* ω_ϵ which extends the Lazutkin's homoclinic invariant defined for area-preserving maps. The main result of this thesis is an asymptotic formula for the homoclinic invariant. Assuming reversibility, we have proved that there is a symmetric homoclinic orbit such that its homoclinic invariant can be estimated as follows,

$$\omega_\epsilon = \pm 2e^{-\frac{\pi\alpha}{2\beta}}(\omega_0 + O(\epsilon^{1-\mu})).$$

where $\mu > 0$ is arbitrarily small and ω_0 is known as the *Stokes constant*. This asymptotic formula implies that the splitting is exponentially small (with respect to ϵ). When $\omega_0 \neq 0$ then the invariant manifolds intersect transversely. The Stokes constant ω_0 is defined for the Hamiltonian at the moment of bifurcation only. We also prove that it does not vanish identically. Finally, we apply our methods to study homoclinic solutions in the Swift-Hohenberg equation. Our results show the existence of multi-pulse homoclinic solutions and a small scale chaos.

Chapter 1

Introduction

The subject of this thesis is related to a phenomenon first observed by the French mathematician Henri Poincaré around 1890, when investigating the question of the stability of the solar system. Poincaré considered the system formed by three bodies Sun-Earth-Moon, under the action of Newton's laws of gravity. In an attempt to prove the stability of the three body system, Poincaré used perturbation series and realized its divergence character due to the presence of a transverse homoclinic orbit [63]. He also realized that the evolution of such system was often chaotic in the sense that a small perturbation in the initial positions or velocities of one of the bodies would lead to a radically different state when compared to the unperturbed system, uncovering for the first time what is now commonly known as chaos in deterministic systems. Poincaré decided to send his results to an international competition created in 1885 by King Oscar II of Sweden on the occasion of his 60th birthday, to award the best mathematical research in four different areas, one of which was the question of stability of the solar system. The jury, consisting of Mittag-Leffler, Weierstrass and Hermite decided to award the prize to Poincaré and noted that although his paper [63] couldn't be regarded as a solution to the original problem it would mark the beginning of a new era in celestial mechanics.

However, when his work was about to be published in Acta Mathematica, the editor of the journal found an error in Poincaré's arguments and Mittag-Leffler prevented the respective publication. The situation was very embarrassing for everyone and in particular to Poincaré who spent the time between March 1887 and July 1890 working on the correction of that major error. The outcome of this work was impressive. Poincaré invented a series of methods endowed with a geometric flavour, which laid the grounds for the development of the field up to the present day. Methods of which included the first-return (Poincaré) maps, stability theory for fixed points and periodic orbits, stable and unstable manifolds, the Poincaré recurrence theorem, integral invariants, etc. which can be found in his three volume treatise [64].

Inspired by the work of Poincaré, Jacques Hadamard published in 1898 an article where he studied geodesics on surfaces of negative curvature [38]. Hadamard introduced a method of symbolic description to study the dynamics of the geodesic flow which originated what is now known as symbolic dynamics. Poincaré appreciated Hadamard's results although he believed that the trajectories of the three body problem were rather comparable to geodesics on convex surfaces [65].

From a historical point of view, a more detailed account of Poincaré's work on the three body problem can be found in this excellent book [6].

In order to better understand what Poincaré observed we consider the following model,

$$\ddot{x} = \sin x + \mu \cos x \cos \frac{t}{\epsilon}, \tag{1.1}$$

which he derived when studying periodic orbits of two degrees of freedom Hamiltonian systems. System (1.1) describes a pendulum with an oscillating suspension point. Of course, the simple pendulum $\ddot{x} = \sin x$ is integrable and at the points $x = 0\,(mod\ 2\pi)$ we have saddle equilibria and centers for $x = \pi\,(mod\ 2\pi)$. Using the 2π-periodicity in x we can restrict our analysis to the interval $[0, 2\pi]$ and the conservation of energy allow us to fully understand the dynamics of the pendulum and obtain a phase portrait similar to Figure 1.1. The curves that connect the points 0 and 2π were initially referred by Poincaré as *bi-asymptotic* orbits and later in his book [64] he named them heteroclinic

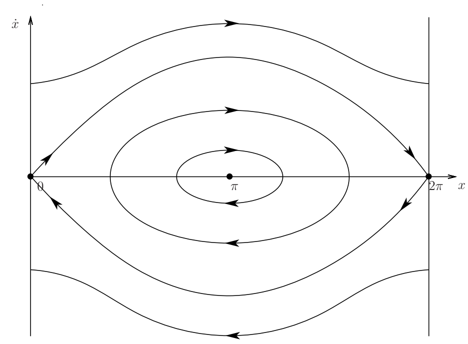

Figure 1.1: Phase portrait of the pendulum.

orbits (resp. homoclinic). Because these curves separate different types of motion they are also known as separatrices. So now we can ask he following question: how different is the phase portrait of system (1.1) from the one in Figure 1.1. Following Poincaré, to better understand the dynamics of (1.1) we construct hyperbolic periodic orbits by taking the system,

$$\ddot{x} = \sin x, \quad \dot{\theta} = 1,$$

where $\theta \in \mathbb{S}^1$. The phase space of this system is $\mathbb{R}^2 \times \mathbb{S}^1$ and $x = 0$ is now a hyperbolic periodic orbit. The study of the system reduces to the study of the map $P_0 : \{\theta = 0\} \to \{\theta = 2\pi\}$ (Poincaré map) which is defined in the obvious way using the orbits of the system. The phase portrait of this map looks similar to Figure 1.1 except that the orbits are now discrete sets. Now system (1.1) is equivalent to,

$$\ddot{x} = \sin x + \mu \cos x \cos \frac{\theta}{\epsilon}, \quad \dot{\theta} = 1,$$

and its Poincaré map P_μ has a hyperbolic fixed point x_μ close to $x = 0$ for μ and ϵ sufficiently small. Moreover, the separatrices split in the way shown in Figure 1.2. After discovering this splitting, Poincaré wrote in [64] the following,

> "If one attempts to imagine the figure formed by these two curves and their infinitely many intersections, each of which corresponds to a bi-asymptotic

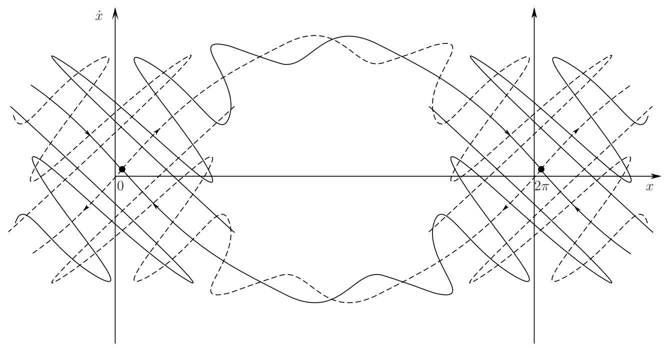

Figure 1.2: Splitting of the separatrices of the perturbed pendulum.

> solution, these intersections form something like a lattice or fabric or a net with infinitely tight loops. None of these loops can intersect itself, but it must wind around itself in a very complicated fashion in order to intersect all the other loops of the net infinitely many times. One is struck by the complexity of this figure, which I shall not even attempt to draw. Nothing gives us a better idea of the complicated nature of the three-body problem and the problems of dynamics in general, in which there is no unique integral and in which the Bohlin series diverge."

Poincaré was aware of the complexity of motion near a transverse homoclinic orbit and he also knew that in some cases the splitting of the separatrices is exponentially small. In fact, for the present example (1.1) the splitting is of order $O(\mu\epsilon^{-1}e^{-\frac{\pi}{2\epsilon}})$ (see [30]).

1.1 Homoclinic Chaos

It was not until the work of Birkhoff [10] in 1935 that more light was shed into the dynamical consequences near a transverse homoclinic orbit. In that paper, Birkhoff proved that given a two dimensional area-preserving analytic diffeomorphism T having a saddle fixed point p with a transverse homoclinic orbit Γ, then in every neighbourhood of the closure of a homoclinic orbit there exists a countable set of periodic orbits having all periods greater than equal to some natural number. Years later, around 1960, Smale found his horseshoe while strolling the beaches of Rio de Janeiro. He then used it as a model basis for finding chaotic dynamics near transverse homoclinic orbits. In his paper [70] in 1965 he proved a result which became known as Smale-Birkhoff Theorem which says that given a diffeomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ having a hyperbolic fixed point p and a homoclinic point $q \neq p$ such that the stable and unstable manifolds of p intersect transversely, there exists a hyperbolic invariant set Λ on which T is topologically conjugated to a shift on two symbols. In that same year, Shilnikov proved in [68] that given a three-dimensional system having an equilibrium of saddle-focus type, where its eigenvalues are of the form $\{\rho \pm i\omega, \lambda\}$ where $\omega > 0$, $\rho < 0$, $\lambda > 0$ such that $\rho + \lambda > 0$ and the equilibrium has a homoclinic orbit, then one can define a Poincaré map P in a transversal neighbourhood of the homoclinic orbit such that P has a countable set of "Smale horseshoes". A couple of years later, Shilnikov gave a complete description of all orbits in a neighbourhood of the closure of a homoclinic orbit (see [69]). Subsequently, the foundations of the general theory were laid by Alekseev in [1, 2, 3].

One important corollary of the results mentioned above is for two degrees of freedom Hamiltonian systems having a saddle-focus equilibrium with stable and unstable manifolds intersecting transversely. In [21] Devaney extended the previous results to this case and proved that in any neighbourhood of a transverse homoclinic orbit, the system admits a suspended horseshoe as an invariant set.

Further results and generalizations have been obtained by many people and therefore, the literature on this subject is vast. As a last remark, let us just mention that for three or more degrees of freedom near integrable Hamiltonian systems, the splitting of invariant manifolds is an important ingredient in the so called Arnold diffusion [4]. It is clear that for more than two degrees of freedom the invariant tori of KAM theory are no longer obstructions for diffusion since their co-dimension is at least 2. In this case the stable and unstable manifolds of the invariant tori work as paths for diffusion provided the invariant manifolds split and the size of the splitting is sufficiently large to allow the transition from one torus to another. It is believed that for *a priori* stable systems (which is the case of the Arnold example [4]) the Arnold diffusion is a generic phenomenon. In this setting the splitting of invariant manifolds is expected to be exponentially small and the diffusion time is exponentially long [61].

1.2 Poincaré-Arnold-Melnikov Method

The theory of splitting of invariant manifolds (or separatrices) has evolved in parallel both for maps and for flows. At present, the standard method for determining the transversality of invariant manifolds is the Poincaré-Arnold-Melnikov method [57]. In the following, we shall restrict our explanations to the case of time-periodic perturbations of one degree of freedom Hamiltonian systems, although one can apply the method in more general situations, see for instance [43] where Melnikov's method is applied in Hamiltonian systems of higher degrees of freedom or [36] where the method is developed for systems with arbitrary dimensions or even more recently [53] where Melnikov's method is developed for diffeomorphisms.

Consider the following Hamiltonian,

$$H(q,p,t,\mu) = H_0(q,p) + \mu H_1(q,p,t), \tag{1.2}$$

where μ is a small parameter. Suppose that the Hamiltonian H_0 has a saddle equilibrium, say at the origin, and a corresponding homoclinic orbit $\Gamma_0(t) = (q_0(t), p_0(t))$, i.e. $\lim_{t\to\pm\infty} \Gamma_0(t) = 0$. The implicit function theorem can be applied to obtain a periodic hyperbolic orbit γ for the full system (1.2) such that $\gamma = O(\mu)$. Moreover the corresponding stable and unstable manifolds of the periodic orbit γ are μ-close to the unperturbed homoclinic orbit Γ_0. Using classical perturbation theory one can write parametrisations of the stable (resp. unstable) manifold as powers series in the parameter μ and by properly choosing a transverse section to a certain homoclinic point $\Gamma_0(t_0)$ it is possible, to compute the difference $d(t_0)$ between the points of first intersection of the section with the stable and unstable manifolds of the periodic orbit γ. Using H_0 as one of the coordinates, V. Arnold derived the following simple formula,

$$d(t_0) = M(t_0)\mu + O(\mu^2), \quad M(t_0) = \int_{-\infty}^{+\infty} \{H_0, H_1\}|_{\Gamma_0(t), t+t_0} \, dt, \tag{1.3}$$

where $M(t_0)$ is known as the *Melnikov's function*. Notice the brackets inside the integral of the previous formula are the Poisson brackets. It immediately follows from the previous formula that simple zeros of the Melnikov function yield transverse homoclinic orbits for the full system (1.2). Note that Melnikov method is a first order perturbation method as it compares stable and unstable manifolds at the order $O(\mu)$. Additionally, when H_1 depends on an extra parameter ϵ, for instance as in example (1.1), then the Melnikov function may also depend on that parameter. In the example above, where the frequency of the perturbation is ϵ^{-1}, we have that,

$$M(t_0) = -\frac{2\pi}{\epsilon^2 \cosh(\frac{\pi}{2\epsilon})} \cos \frac{t_0}{\epsilon},$$

and the Melnikov function is exponentially small with respect to ϵ. Recall from (1.3) that the error term is of order $O(\mu^2)$ which becomes greater than the leading term $M(t_0)\mu$ when ϵ is very small. Thus, in systems where exponentially small phenomena occur, Melnikov's method does not directly apply and further study is needed to justify the method and prove the correctness of the prediction. In the case of example (1.1), Gelfreich [27] and Treshchev [74] have independently obtained an asymptotic formula for the splitting which differs from the one predicted by Melnikov's theory. There are numerous examples where Melnikov's method requires further justification and obtaining the correct leading order for the splitting distance is in general a very non-trivial problem due to the presence of exponentially small phenomena. On the same line of research, let us just mention the articles [26] and [20] on the rapidly forced pendulum,

$$\ddot{x} + \sin x = \mu \epsilon^p \sin \frac{t}{\epsilon},$$

which justified Melnikov's method for $p > 0$ and $\epsilon > 0$. For a detailed survey of these results and much more the reader is referred to [30].

Fast and slow dynamics are a common theme in high frequency periodic perturbations. As they fall into the class of singular perturbation problems then this is the main reason for the failure of Melnikov's method as exponentially small phenomena is predominant in this class. Thus a new approach is required to deal with exponentially small splitting and in the following section we will briefly discuss a set of problems where estimating the size of the splitting has been done successfully.

1.3 Exponentially Small Splitting

Exponentially small splitting can be found in many systems such as high frequency periodic perturbations of autonomous systems (as previously discussed), in close to identity area preserving maps, bifurcations of resonant periodic orbits in two degrees of freedom Hamiltonian systems and as a result of this thesis in bifurcations of total elliptic equilibria in two degrees of freedom Hamiltonian systems. As explained before, detecting the exponentially small splitting of invariant manifolds is very important due to its profound consequences in the dynamics of the phase space of the system. Moreover, in many interesting cases Melnikov's method is not applicable to detect the splitting.

In the case of systems with slow-fast motions, Neishtadt's theorem [60] can be used to obtain an exponentially small upper bound for the splitting and for close to identity area preserving maps Fontich and Simó, [25] also derived an exponentially small upper bound for the splitting of separatrices. However, getting a lower bound is generally very difficult and strongly depends on the form of the equations of the system. Hence, very few results are known for generic families of systems and most cases treated in the literature are for particular systems only.

In addition to high frequency perturbations of pendula the most paradigmatic example in the exponentially small splitting is given by the *Chirikov standard map* which is defined by the following relation,

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y + \epsilon \sin x \\ y + \epsilon \sin x \end{pmatrix}.$$

This map is a diffeomorphism on a two dimensional torus $\mathbb{T}^2$ which is area-preserving and reversible. For $\epsilon = 0$ the standard map is integrable and the torus is foliated into invariant circles where y is an integral of motion. When $\epsilon > 0$ the map has a hyperbolic fixed point $(0, 0)$ and it is well known that it has stable and unstable curves (separatrices) intersecting at the primary homolinic point (π, y_ϵ) which corresponds to the first intersection of the curves with the symmetric line $x = \pi$. Note that the standard map is a ϵ-step discretization of the pendulum $\ddot{x} = \sin x$ (modulus a proper scaling of variables). Hence its phase portrait looks like the pendulum (see figure 1.1) for ϵ small but the separatrices are expected to split. In 1984, in the pioneering article [47], V.F. Lazutkin obtained an asymptotic formula for the splitting angle α_ϵ, defined by the separatrices at the primary homolinic point,

$$\alpha_\epsilon = \frac{\pi e^{-\frac{\pi^2}{\sqrt{\epsilon}}}}{\epsilon} \left(\omega_0 + O(\epsilon^{\frac{1}{8}-\delta})\right), \tag{1.4}$$

where the constant δ is an arbitrarily small positive constant and ω_0 is a positive constant defined for an ϵ-independent problem. It is not known if is possible to write ω_0 in terms of elementarily constants (e.g. π, e) and at the present, the only known way to estimate ω_0 is through numerical computations. A numerical procedure can be designed to the effect [30] and several digits of ω_0 have been computed,

$$\omega_0 = 1118.827706...$$

The proof of the asymptotic formula for the splitting angle given by Lazutkin was incomplete and only in 1999, V. Gelfreich presented in [28] a complete proof inspired by the original ideas of Lazutkin.

As the splitting angle α_ϵ depends on the homoclinic point and coordinate system, in a subsequent paper [33] the *Lazutkin homoclinic invariant* was introduced to measure the splitting of separatrices in area-preserving maps. The idea was to parametrize the stable (resp. unstable) curve $\Gamma^\pm(t) = (x^\pm(t), y^\pm(t))$ by solutions of the finite-difference system,

$$x(t+h) = x(t) + y(t+h), \quad y(t+h) = y(t) + \epsilon \sin x(t),$$

where h is a conveniently defined parameter which depends on ϵ (in fact $\epsilon \approx h^2$, see [28] for more details). Assuming that $\Gamma^{\pm}(0)$ is equal to the primary homoclinic point, then the Lazutkin homoclinic invariant could be defined as follows,

$$\omega = \det \begin{pmatrix} \dot{x}^-(0) & \dot{x}^+(0) \\ \dot{y}^-(0) & \dot{y}^+(0) \end{pmatrix}.$$

Clearly the Lazutkin homoclinic invariant is equal to the signed area formed by the tangent vectors $\Gamma^{\pm}(0)$ at the primary homolinic point and its definition is independent from any symplectic coordinate system. Moreover, it can be shown that it takes the same value for all points of the homoclinic orbit. These remarkable properties make the Lazutkin homoclinic invariant the natural quantity for detecting the splitting of separatrices in area-preserving maps. In the case of the standard map, an asymptotic expansion for ω was obtained in [33] which reads,

$$\omega \asymp \frac{4\pi}{h^2} e^{-\frac{\pi^2}{h}} \sum_{n \geq 0} \omega_n h^{2n},$$

where the symbol $\asymp$ means that if we truncate the series in the right hand side at some order then the error will be of the order of the first missing term. From the asymptotic expansion of ω one can obtain a refinement of the splitting angle.

Note that in the case of the standard map, an application of Melnikov's method gives an incorrect estimate for the splitting of the separatrices. In fact, Melnikov method is a finite order perturbation method, in the sense that it expands the separatrices in powers of the perturbation parameter ϵ and compares stable and unstable curves at the order $O(\epsilon^p)$ for some $p > 0$. However it can be shown (see Proposition 3.1 of [28]) that for every $p \in \mathbb{N}$ there is a $C > 0$ and $\epsilon_0 > 0$ such that,

$$\left|x^+(t) - x^-(t)\right| + \left|y^+(t) - y^-(t)\right| \leq C\epsilon^p, \quad t \in (-\sqrt{\epsilon}, \sqrt{\epsilon}), \quad \epsilon < \epsilon_0. \tag{1.5}$$

Since the error in Melnikov method is always polynomial in ϵ (see (1.3)) it is clear from (1.5) that it exceeds the magnitude of the splitting of separatrices, thus not giving a correct estimate for the size of the splitting. Consequently, a new method for estimating the size of the splitting had to be invented.

In relation to the splitting of separatrices of the standard map, let us also mention the work of Hakim and Mallick [39] which used Borel summation methods to study the exponential pre-factor of the asymptotic formula (1.4). Their work established a relation between Écalle's resurgence theory of functions [13] and the problems of splitting of separatrices which later inspired the work of D. Sauzin and many other people (see [31] and [66] and references therein). More recently, P. Martín, D. Sauzin and T. M. Seara have studied the splitting of separatrices in perturbations of the McMillan map (see [55] and [54]). Their approach is based on Lazutkin's original ideas and resurgent theory.

Many other maps where exponentially small splitting of separatrices is present, have been studied and an asymptotic formula measuring the splitting has been obtained (see the survey [30] for several examples and references therein), in most cases using only formal arguments. Moreover, most rigorous results in the area concern particular maps or systems and very few general results are known. As a matter of fact, in the case of maps, only very recently a preprint [29] of Gelfreich and N. Brännström appear on arxiv where an asymptotic formula for Lazutkin's homolinic invariant is formally derived which describes the exponentially small splitting of separatrices in a generic analytic family of area preserving maps near a Hamiltonian saddle-center bifurcation.

Lazutkin's approach has become standard and most rigorous proofs use more or less Lazutkin's original ideas. Roughly speaking, the approach consists in studying the analytic continuation of parametrizations of stable and unstable manifolds into the complex domain. Although the phenomena we want to study lives in a real domain, a careful analytic study of the parametrizations near a certain complex singularity is able to detect the exponentially small phenomena. Then a local rectification of the map and standard Fourier arguments are able to return to the reals and obtain the asymptotic formula describing the splitting. At the heart of the method is a "complex matching technique" which allows the passage from the analytic study of the invariant manifolds in a neighbourhood of the fixed point to the analytic study near the complex singularity.

This technique can be found in the Physics literature where problems of exponentially small splitting of invariant manifolds are also studied but use a different mathematical framework from the one used in Dynamical Systems. There the common approach is known as "asymptotics beyond all orders" [67] which is related to matched asymptotic expansions [23] that capture the exponentially small terms. Most notably, the work of Kruskal and Segur [45] in the 80's where they considered a model of crystal growth and using matched asymptotic expansions they were able to prove that a certain heteroclinic connection breaks. This work has influenced many others in the field and the same technique has been applied (at the formal level) to prove the non-persistence of homoclinic or heteroclinic solutions to certain singularly perturbed systems (for instance [35, 78, 17]). It is worth mentioning that most arguments used in the "asymptotics beyond all orders" approach are heuristics and although may produce satisfactory solutions are not rigorous mathematical proofs. More recently, the asymptotic beyond all orders approach has been applied in [73, 19, 18, 75].

In his book [52], Eric Lombardi undertook efforts to put the matched asymptotic expansions technique into rigorous arguments that could be used to solve many problems in the class of exponentially small phenomena. He realized that most problems in this class could be reduced to the study of certain oscillatory integrals which capture the exponentially small terms. He then applied his methods to study homoclinic connections of periodic orbits in reversible analytic vector fields near resonances. Let us emphasise that his results apply not only for particular examples but for one parameter families of reversible vector fields admitting some sort of resonance (in particular for a $0^2 i\omega$ or a $(i\omega_0)^2 i\omega_1$ resonance). However, we should mention that his methods do not apply to a $(i\omega)^2$ resonance, which is considered in this thesis. The reader is referred to his book [52] for more details.

As a final remark, let us refer the reader to the survey of A.R. Champneys [14] where several applications of exponentially small splitting to mechanics, fluids and optics are considered.

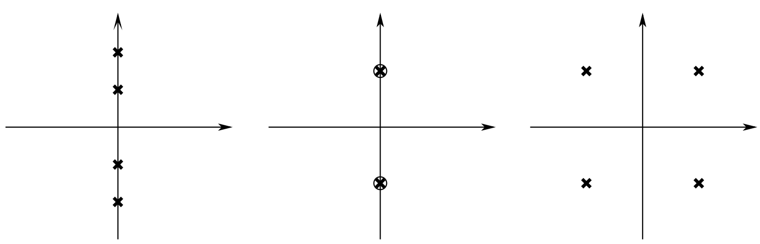

Figure 1.3: Eigenvalues of $DX_{H_\epsilon}(p)$.

1.4 Main Contributions of this Thesis

Consider an analytic one parameter family of two degrees of freedom Hamiltonian systems X_{H_ϵ} with a common equilibrium point p, i.e., $X_{H_\epsilon}(p) = 0$. We say that p undergoes a Hamiltonian-Hopf bifurcation if the eigenvalues of the linearized system at the equilibrium point change from complex $\pm\beta \pm i\alpha$ $(\alpha, \beta > 0)$ for $\epsilon > 0$ to pure imaginary $\pm i\alpha_1$ and $\pm i\alpha_2$ $(\alpha_1 \neq \alpha_2 \neq 0)$ for $\epsilon < 0$, as is shown schematically in Figure 1.3. When $\epsilon = 0$ the equilibrium has a pair of pure imaginary eigenvalues $\pm i\alpha_0$ with multiplicity two. In other words, the equilibrium p changes from hyperbolic to elliptic. This bifurcation has been extensively studied [76] and a normal form theory for the bifurcation has been developed. It is known that depending on the sign of a certain coefficient η of the normal form there are two main bifurcation scenarios (see section 2.2 of chapter 2). In one of these scenarios, which corresponds to $\eta > 0$ (the stable case) it is known that for $\epsilon > 0$ there are two dimensional stable W^s_ϵ and unstable W^u_ϵ manifolds within a three dimensional energy level set, that shrink to the equilibrium as the bifurcation parameter ϵ approaches the critical value.

At the level of the normal form the stable and unstable manifolds coincide and for the original Hamiltonian, in general, it is expected a completely different situation: stable and unstable manifolds will not coincide any longer and intersect transversely, forming a countable set of homoclinic orbits as initially described by Poincaré and all the chaos that it implies.

The question of transversality in an Hamiltonian-Hopf bifurcation has been considered by many people and finds applications in many different problems. For instance, in the study of stationary localized solutions for the Swift-Hohenberg equation [48, 18] or in the restricted three body problem where numerical evidence have shown the existence of homoclinic orbits to the Lagrange equilateral equilibrium point which are the limit of periodic orbits with long periods (blue sky catastrophe) [42]. For more applications the reader is referred to [14].

In this thesis, we have addressed the problem of determining if stable and unstable manifolds of the equilibrium intersect transversely. We have estimated the size of the splitting of the invariant manifolds which depend on a singular way from the bifurcation parameter. For $\epsilon = 0$ the equilibrium is elliptic, thus the problem of determining the transversality belongs to the class of analytic singular perturbation problems.

The most significant effort towards solving the question of transversality occurred in 2003 when P. D. McSwiggen and K. R. Meyer proved in [56] that for small positive ϵ the stable and unstable manifolds are either identical or have a transverse intersection, i.e. a transverse homoclinic orbit. However, their arguments did not show a transverse intersection and the main question remained open.

When the Hamiltonian vector field X_{H_ϵ} is reversible, Glebsky and Lerman proved in [34] the existence of two symmetric homoclinic orbits using an implicit function theorem argument. They also pointed out that stable and unstable manifolds could split and that this splitting was exponentially small. The existence of two symmetric homoclinic orbits follows from a more general result of G. Iooss and M. C. Pérouème in [44] where it is considered a four dimensional reversible vector field near a 1:1 resonance (or $(i\omega)^2$ resonance). See also [15] where the existence of symmetric homoclinic orbits is studied by considering ω and β as independent parameters.

More recently, Lombardi [52] developed several methods that allowed him to study homoclinic connections of periodic orbits in reversible analytic vector fields near certain resonances. The resonance considered in this thesis $(i\omega)^2$ is not treated in his book and in page 12 we find:

> "Observe from Figure 1.3 that for the $(i\omega)^2$ resonance such a coexistence of *slow hyperbolic part with a rapid oscillatory one* does not exist. Thus they can be studied with classical tools (see [44])."

Our results have shown that exponentially small phenomena is generic near a Hamiltonian-Hopf bifurcation, thus contradicting Lombardi's observation. More precisely, we have proved that generically stable and unstable manifolds of the equilibrium split and that the size of the splitting is exponentially small with respect to ϵ.

In order to measure the splitting of the invariant manifolds we have extended the definition of the Lazutkin's homoclinic invariant (which is defined for area-preserving maps) for our case of two degrees of freedom Hamiltonian systems. Given a homoclinic point $p_\epsilon \in W^s_\epsilon \cap W^u_\epsilon$, we have found a natural way to normalize vectors $v^{u,s}_\epsilon$ tangent to W^s_ϵ and W^u_ϵ at the homoclinic point p_ϵ and defined the following homoclinic invariant,

$$\omega_\epsilon = \Omega(v^s_\epsilon, v^u_\epsilon),$$

where Ω is the standard symplectic form in $\mathbb{R}^4$. Moreover, we have shown that it satisfies the following properties:

1. It is invariant under symplectic change of coordinates,
2. It takes the same value along the homolinic orbit defined by p_ϵ, i.e., is independent of a particular homoclinic point,
3. If $\hat{v}^{u,s}_\epsilon$ is a different pair of tangent vectors such that the homoclinic invariant $\hat{\omega}_\epsilon$ defined by those vectors satisfy the above properties then $\hat{\omega}_\epsilon$ is not independent of ω_ϵ, i.e., there exists a relation between the homoclinic invariants,
4. If $\omega_\epsilon \neq 0$ then W^s_ϵ and W^u_ϵ have a transverse intersection.

To effectively measure an exponentially small splitting we have constructed approximations of stable and unstable manifolds in complex domains and measure the splitting in places where it is detectable, that is, near singularities in complex domains. This has involved several steps, such as the construction of asymptotic expansions for the invariant manifolds in different complex domains and a complex matching technique that captures the exponentially small phenomena, as mentioned earlier in this Chapter. Assuming that the family X_{H_ϵ} is reversible with respect to the involution,

$$\mathcal{S}(q_1, q_2, p_1, p_2) = (-q_1, q_2, p_1, -p_2),$$

we have measured the splitting of the invariant manifolds at a symmetric homoclinic point p_ϵ, i.e. $p_\epsilon \in \mathrm{Fix}(\mathcal{S})$ (the set of fixed points of $\mathcal{S}$). We can now state the main result of this thesis,

Theorem 1.4.1. *If $\epsilon > 0$ and $\eta > 0$ (the stable case) then there exists a symmetric homoclinic point $p_\epsilon \in W^s_\epsilon \cap W^u_\epsilon$ such that the corresponding homoclinic invariant has the following asymptotic formula,*

$$\omega_\epsilon = \pm 2e^{-\frac{\pi\alpha}{2\beta}}(\omega_0 + O(\epsilon^{1-\mu})). \tag{1.6}$$

where $\mu > 0$ is arbitrarily small.

Recall that α and β are the imaginary and real part of the eigenvalues of the linearized system at the equilibrium point. Moreover, $\beta \to 0$ as $\epsilon \to 0^+$ (see Figure 1.3, in fact we know that $\beta = O(\sqrt{\epsilon})$). Consequently ω_ϵ is exponentially small with respect to ϵ. In addition, when ω_0 is different from zero, the previous asymptotic formula implies that the invariant manifolds have a transverse intersection.

Similar to many other problems in the class of exponentially small splitting (compare with the standard map (1.4)) the constant ω_0 is defined for an ϵ-independent problem and in our case it only depends on the Hamiltonian H_0 (at the exact moment of bifurcation). It is a remarkable fact that the leading coefficient ω_0 which determines the transversality of the family of invariant manifolds W^s_ϵ and W^u_ϵ does not depend on the unfolding of H_0. To be more precise, let $\mathcal{U} \subseteq \mathbb{C}^4$ be a neighbourhood of the origin and $\mathfrak{H}_0$ be the space of analytic Hamiltonian functions $H : \mathcal{U} \to \mathbb{C}^4$ that have the same properties as H_0. Then we have the following,

Theorem 1.4.2. *There exists a non-zero functional $\mathcal{K}_0 : \mathfrak{H}_0 \to \mathbb{R}_0^+$ satisfying the following properties:*

1. $\omega_0 = \sqrt{\mathcal{K}_0(H_0)}$ *(Stokes constant),*

2. *Given $H \in \mathfrak{H}_0$ such that $\mathcal{K}_0(H) \neq 0$ then H is non-integrable and the normal form transformation diverges,*

3. *$\mathcal{K}_0$ is independent of the symplectic coordinate system, i.e., if $\hat{H} = H \circ \Psi$ for some analytic symplectic map Ψ fixing the equilibrium p then $\mathcal{K}_0(H) = \mathcal{K}_0(\hat{H})$,*

4. *If H_ν is an analytic curve in $\mathfrak{H}_0$ then $\mathcal{K}_0$ is an analytic function of ν.*

5. *Given any analytic curve H_ν in $\mathfrak{H}_0$ where ν is defined in an open disc $\mathbb{D} \subset \mathbb{C}$, then for every $\epsilon > 0$ there is an ϵ-close analytic curve $F_\nu \in \mathfrak{H}_0$ to H_ν, i.e.*

$$\sup_{\mathbf{x} \in \mathcal{U}, \nu \in \mathbb{D}} |H_\nu(\mathbf{x}) - F_\nu(\mathbf{x})| < \epsilon,$$

   such that $\mathcal{K}_0(F_\nu)$ does not vanish on an open and dense subset of $\mathbb{D}$.

The definition of $\mathcal{K}_0$ is related to a phenomenon observed in solutions of certain differential equations known as *Stokes phenomenon* (see [62] and references therein). From the last property of Theorem 1.4.2 we conclude that the splitting of invariant manifolds near a Hamiltonian-Hopf bifurcation is a generic phenomenon.

The reversibility assumption is not necessary in most parts of this thesis. In fact, it is only used to ensure the existence of a certain primary homoclinic orbit. We believe arguments based on the preservation of the symplectic form yield the existence of a homoclinic orbit such that the asymptotic formula (1.6) holds in the non-reversible case.

Taking the Swift-Hohenberg equation as an example, we have performed high-precision numerical experiments to support validity of the asymptotic expansion (1.6) and evaluated a Stokes constant numerically using two independent methods. In particular, this study implies the existence of countably many reversible homoclinic orbits for the Swift-Hohenberg equation, which are known as *multisolitons*. The Swift-Hohenberg equation is also considered as a paradigmatic model in pattern formation theory [51, 50, 18]. Recently, similar computations to ours have been performed by S. J. Chapman and G. Kozyreff in [18] where they study localised patterns in the Swift-Hohenberg equation emerging from a subcritical modulation instability using the multiple-scales analysis beyond all orders. Although arguments in [18] are not completely rigorous they were still able to capture the exponentially small phenomena by means of analysing certain formal expansions using optimal truncation and studied their difference in the vicinity of the Stokes lines.

Our results extend those in [18] as we have developed a theory to study transversal homoclinic orbits in Hamiltonian system near a Hamiltonian-Hopf bifurcation, for which the Swift-Hohenberg is a particular example of this type of bifurcation.

Chapter 2

Preliminaries

In this chapter we review some well known results about Hamiltonian systems and describe the Hamiltonian-Hopf bifurcation in detail. In the end we shall define certain linear operators and obtain inverse theorems that will be used in subsequent chapters.

2.1 Hamiltonian Systems

The goal of this section is to present a brief introduction to Hamiltonian systems and introduce some of the notation that will be used throughout this thesis. The material of this section can be found in [5, 58].

The Hamiltonian formalism is the natural mathematical framework in which is possible to develop the theory of conservative mechanical systems since the equations of motion of a mechanical system can be transformed into a Hamiltonian system,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{2.1}$$

where H is a C^2 function defined in the even-dimensional space $\mathbb{R}^{2n}$ with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ where the configuration variable q_i is conjugated to the momentum variable p_i. In this case we say that the Hamiltonian system (2.1) has n degrees of freedom and the function H is known as the Hamiltonian.

More generally, in Hamiltonian mechanics there is a one-to-one correspondence between Hamiltonian vector fields and Hamiltonian functions which is defined by the symplectic structure. In the simplest case, the standard symplectic structure in $\mathbb{R}^{2n}$ is given by the *canonical symplectic form*,

$$\Omega(x,y) = x^T J y, \quad \text{where} \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

For a given Hamiltonian function H one can define the associated (Hamiltonian) vector field X_H in a coordinate independent way as follows,

$$\mathrm{d}H = \Omega(X_H, \cdot).$$

Moreover, we can compute the derivative of a given function F along the vector field X_H which we denote by $\{F, H\}$ where,

$$\{F, H\} = \Omega(X_F, X_H).$$

The operation $\{\cdot,\cdot\}$ is called the *Poisson bracket*. The integral curves of the Hamiltonian vector field X_H satisfy the Hamilton equations (2.1) which can be written as follows,

$$\dot{q}_i = \{q_i, H\}, \qquad \dot{p}_i = \{p_i, H\}, \tag{2.2}$$

or using the shorter notation $\dot{\mathbf{x}} = X_H(\mathbf{x})$ where $\mathbf{x} = (q,p) \in \mathbb{R}^{2n}$. The flow of this ODE is denoted by Φ_H^t. Using the Poisson bracket we can see that the derivative of the function H along the vector field X_H vanishes, since $\{H, H\} = 0$. Thus H is constant along the flow lines of the Hamiltonian vector field X_H. This property is known as conservation of energy.

Another well known fact in Hamiltonian mechanics is that the flow Φ_H^t preserves the symplectic form Ω and in particular, it preserves the volume form Ω^n given by the nth exterior product of Ω. Moreover, the transformations that preserve the symplectic form are known as canonical or symplectic transformations. From the definition of X_H it is clear that if Ψ is a canonical transformation such that $F = H \circ \Psi$ then

$X_F = (D\Psi)^{-1} X_H \circ \Psi$. Consequently the Hamiltonian structure is preserved under canonical changes of coordinates.

The ultimate goal in Hamiltonian dynamics is to understand the asymptotic behavior of most trajectories of the Hamiltonian system (2.2). A class of Hamiltonian systems where the dynamics are significantly simple to understand is the class of integrable Hamiltonian systems. Roughly speaking, an n degrees of freedom Hamiltonian system with Hamiltonian H is said to be *integrable* (in the sense of Liouville-Arnold [5]) if there exist n functions $H = F_1, \ldots, F_n$ which are independent (their differentials are pointwise linearly independent) and in involution $\{F_i, F_j\} = 0$ for all $i, j = 1, \ldots, n$. In this case, the equations of motion can be solved by "quadratures", obtaining a complete description of the structure of the orbits in the phase space. A more precise statement is given by Liouville-Arnold Theorem which says that if,

$$M_z := \left\{\mathbf{x} \in \mathbb{R}^{2n} \mid F_i(\mathbf{x}) = z_i,\ i = 1, \ldots, n\right\},$$

is connected and compact then M_z is diffeomorphic to the n-torus $\mathbb{T}^n$ and moreover in a neighbourhood of M_z there exist a canonical change of coordinates such that in the new coordinates $(I_1, \ldots, I_n, \varphi_1, \ldots, \varphi_n)$ the Hamiltonian depends only on I_i. These new coordinates are called *action-angle* coordinates. An example of an integrable system is given by the pendulum as discussed in the introduction of this thesis. Additional examples will come later when studying the normal forms.

In fact, the most interesting phenomena in Hamiltonian dynamics is given by non-integrable systems. There, one can start by studying its invariant objects (equilibrium points, periodic orbits, tori, etc) and the corresponding attracting and repelling sets. A particular case is when p is an equilibrium point of X_H, i.e., $X_H(p) = 0$, then one can define its stable and unstable set as follows,

$$\begin{aligned} W^u(p) &= \left\{\mathbf{x} \in \mathbb{R}^{2n} \mid \lim_{t \to -\infty} \Phi_H^t(\mathbf{x}) = 0\right\}, \\ W^s(p) &= \left\{\mathbf{x} \in \mathbb{R}^{2n} \mid \lim_{t \to +\infty} \Phi_H^t(\mathbf{x}) = 0\right\}. \end{aligned} \tag{2.3}$$

and assuming that $DX_H(p)$ contains no eigenvalues on the imaginary axis (hyperbolic equilibrium) then the spectrum of $DX_H(p)$ will contain n eigenvalues $\lambda_1, \ldots, \lambda_n$ with negative real part and n eigenvalues $\mu_1, \ldots, \mu_n$ with positive real part (since the spectrum of a Hamiltonian matrix is invariant under complex conjugation and symmetric with respect to the imaginary axis). Now the well known Stable Manifold theorem [37] implies that $W^{u,s}(p)$ are locally n dimensional smooth manifolds having the same degree of regularity as the vector field X_H. Thus we usually denote by $W^{u,s}_{loc}(p)$ the stable and unstable manifolds in a neighbourhood of the equilibrium p. Moreover the local stable manifold $W^s_{loc}(p)$ is tangent at p to the eigenspace of the λ_i's and the local unstable manifold $W^u_{loc}(p)$ is tangent at p to the eigenspace of the μ_i's. In general, the stable and unstable sets (2.3) are immersed manifolds and their global structure can be very complicated as Figure 1.2 shows.

Particularly interesting are the *homoclinic points* which belong to the intersection $W^u(p) \cap W^s(p)$. For a homoclinic point p_h we have the corresponding *homoclinic orbit* $\gamma_h(t) = \Phi^t_H(p_h)$ which is also in the intersection of stable and unstable manifolds. Thus $W^u(p) \cap W^s(p)$ is at least one dimensional. Recall that conservation of energy implies that both stable and unstable manifolds are contained inside the energy level $\{H = H(p)\}$ which is $2n-1$ dimensional.

A fundamental question is whether stable and unstable manifolds intersect transversally at the homoclinic orbit γ_h. That is, if for every homoclinic point q of the homoclinic orbit γ_h the tangent space of stable and unstable manifolds at q generated the space $\mathbb{R}^{2n-1}$,

$$T_q W^u(p) + T_q W^s(p) = \mathbb{R}^{2n-1}.$$

In this case we say that γ_h is a transverse homoclinic orbit. This question is of great importance as it provides a route to very complicated dynamics in a neighbourhood of the transverse homoclinic orbit as was described in the introductory chapter of this thesis.

2.1.1 Normal forms

The idea of the normal form procedure is to simplify as much as possible a given Hamiltonian H by producing suitable near identity canonical change of coordinates that kill most terms in the original Hamiltonian. The transformed Hamiltonian H^{NF} is expected to have some type of additional symmetry, such as $\mathbb{S}^1$ symmetry induced by some integral of motion.

In the following we shall restrict our explanations to normal forms around equilibria. So we suppose that H can be written as follows,

$$H = H_2 + H_3 + H_4 \cdots , \tag{2.4}$$

where $H_i \in \mathcal{H}_i$ and $\mathcal{H}_i$ is the space of homogeneous polynomials of degree i. The first step in the normalization is to bring the quadratic part H_2 into a canonical normal form. The study of normal forms for linear Hamiltonian systems is important as it is not always possible to put a linear Hamiltonian matrix into a Jordan normal form by a linear canonical change of coordinates. Thus, the classification of normal forms for linear Hamiltonian matrices is more refined then the usual Jordan normal form and for more details the reader is referred to [59]. So let us suppose that H_2 is in some canonical normal form and explain how one proceeds to normalize H_3. Given $F_3 \in \mathcal{H}_3$ we produce a near identity canonical change of coordinates Φ_3 by considering the time one Hamiltonian flow generated by F_3, i.e., $\Phi^1_{F_3}$, and compose it with H to get,

$$H \circ \Phi^1_{F_3} = H_2 + H_3 - \mathrm{adj}_{H_2}(F_3) + \text{higher order terms},$$

where $\mathrm{adj}_{H_2}(\cdot) = \{\cdot, H_2\}$ is called the *adjoint operator* or also known as the *homological operator*. Note that this change of coordinates did not affect the quadratic part. Now we will try to eliminate the order 3 terms or in other words solve the equation $H_3 - \mathrm{adj}_{H_2}(F_3) = 0$ with respect to F_3. In general, it is not always possible to solve that equation as $\mathrm{adj}_{H_2} : \mathcal{H}_3 \to \mathcal{H}_3$ may have non-trivial kernel and consequently H_3 may not belong to $\mathrm{im}(\mathrm{adj}_{H_2})$. Thus, the image of adj_{H_2} describes to a great extend the normal form to which H can be transformed. Moreover, it may have different *styles* [59] depending on the choice of complement of $\operatorname{im}\operatorname{adj}_{H_2}$. Repeating these arguments recursively we obtain the following,

Theorem 2.1.1. *Let $\mathcal{G}_i$ be linear subspaces of $\mathcal{H}_i$ such that $\mathcal{G}_i + \operatorname{im}\operatorname{adj}_{H_2} = \mathcal{H}_i$, then there exists a formal near identity canonical change of coordinates Φ such that,*

$$H^{NF} = H \circ \Phi = H_2 + \tilde{H}_3 + \tilde{H}_4 + \cdots,$$

where $\tilde{H}_i \in \mathcal{G}_i$.

So the question of computing a normal form for H reduces to computing the complements $\mathcal{G}_i$. Choosing a normal form style (or complements $\mathcal{G}_i$) depends whether X_{H_2} is semisimple or not. In the first case the $\ker\operatorname{adj}_{H_2}$ complements $\operatorname{im}\operatorname{adj}_{H_2}$ and the polynomials F_i can be chosen properly so that H^{NF} belongs to $\ker\operatorname{adj}_{H_2}$. This implies that H^{NF} is constant along the Hamiltonian flow of H_2. Thus H_2 is an integral of H^{NF}. When X_{H_2} is not semisimple then it is possible to choose a particular inner product in the linear spaces $\mathcal{H}_i$ such that the adjoint operator of $\operatorname{adj}_{H_2} : \mathcal{H}_i \to \mathcal{H}_i$ with respect to that inner product is $\operatorname{adj}_{H_2^T}$ where H_2^T is the Hamiltonian of the transposed Hamiltonian matrix $(DX_{H_2}(0))^T$. Now Fredholm alternative implies that $\ker\operatorname{adj}_{H_2^T}$ complements $\operatorname{im}\operatorname{adj}_{H_2}$ and as before one can choose polynomials F_i such that $\left\{\tilde{H}_i, H_2^T\right\} = 0$ for all $i \geq 3$.

There is also an $\mathrm{sl}(2,\mathbb{R})$ normal form style that is a ring of invariants under a modified linear flow (see [59]). This approach and the inner product often yield the same normal form but the $\mathrm{sl}(2,\mathbb{R})$ is less known due to its representation theory apparatus.

Note that as adj_{H_2} has kernel then the normal form transformation Φ is non unique.

This technique of simplifying the form of a given Hamiltonian goes back to Birkhoff [9] who studied a semisimple Hamiltonian with multiple centers,

$$H = \sum_{i=1}^{n} \omega_i L_i + \text{higher order terms}, \quad \text{where} \quad L_i = \frac{q_i^2 + p_i^2}{2}$$

and proved that in the absence of resonances in the frequencies, i.e., $(k, \omega) \neq 0$ for all $k \in \mathbb{Z}^n$, then the original Hamiltonian could be formally transformed into a Hamiltonian depending only on the L_i's.

Normal form for the nonsemisimple Hamiltonian $1 : -1$ resonance

The nonsemisimple Hamiltonian $1 : -1$ resonance is a two degrees of freedom Hamiltonian system having the following Hamiltonian function,

$$H = q_1 p_2 - q_2 p_1 + \frac{q_1^2 + q_2^2}{2} + \text{higher order terms},$$

where the higher order terms are at least cubic in the variables q_1, q_2, p_1 and p_2. We want to derive a normal form for the Hamiltonian H and for our purposes it is sufficient to consider H as a formal series. Let us denote the quadratic part of H by H_2. Note that $DX_{H_2}(0)$ is not semisimple. We have the following,

Theorem 2.1.2 (Sokol'skiĭ [71]). *There is a formal near identity canonical change of coordinates Φ such that,*

$$H^{NF} = H \circ \Phi = q_1 p_2 - q_2 p_1 + \frac{q_1^2 + q_2^2}{2} + K(q_2 p_1 - q_1 p_2, p_1^2 + p_2^2),$$

where K is a formal series in two variables starting with quadratic terms. Moreover the coefficients of K are uniquely defined, forming an infinite set of invariants for the Hamiltonian H.

Proof. In normal form theory often formulae look simpler if one considers complex coordinates given by,

$$z = q_1 + iq_2, \; w = p_1 + ip_2, \; \bar{z} = q_1 - iq_2, \; \bar{w} = p_1 - ip_2.$$

This change of variables in an automorphism of $\mathbb{C}^4$ and it deforms the canonical symplectic form Ω according to the relation,

$$dq_1 \wedge dp_1 + dq_2 \wedge dp_2 = \frac{1}{2} \left(dz \wedge d\bar{w} + d\bar{z} \wedge dw \right).$$

Thus in the new variables we multiply the Hamiltonian H by 2 and use the symplectic form $dz \wedge d\bar{w} + d\bar{z} \wedge dw$ to derive its Hamilton equations. Now, as shown in [58], on the linear space $\mathcal{H}_n$ of homogeneous polynomials of degree n in the variables $\mathbf{x} = (z, w, \bar{z}, \bar{w})$ we can introduce an inner product such that the adjoint of the linear operator $\mathrm{adj}_{H_2} : \mathcal{H}_n \to \mathcal{H}_n$ with respect to that inner product is $\mathrm{adj}_{H_2^T}$ where H_2^T is the Hamiltonian of the transposed Hamiltonian matrix $(DX_{H_2}(0))^T$,

$$H_2^T = i(z\bar{w} - \bar{z}w) - w\bar{w}.$$

The Fredholm alternative gives the splitting $\mathcal{H}_n = \ker \mathrm{adj}_{H_2^T} \oplus \mathrm{im}\,\mathrm{adj}_{H_2}$ and according to Theorem 2.1.1 there is a formal near identity canonical change of coordinates Φ such that,

$$H^{NF} = H \circ \Phi = H_2 + \tilde{H}_3 + \tilde{H}_4 + \cdots ,$$

where $\tilde{H}_n \in \ker(\mathrm{adj}_{H_2^T} : \mathcal{H}_n \to \mathcal{H}_n)$ for all $n \geq 3$. So in order to get the form of the polynomials $\tilde{H}_n$ we only need to determine a basis for $\ker \mathrm{adj}_{H_2^T}$. Recall that $\mathrm{adj}_{H_2^T}(\cdot) = \left\{\cdot, H_2^T\right\}$ where the Poisson bracket $\{\cdot,\cdot\}$ is defined by the formula,

$$\{P, Q\} = \frac{\partial P}{\partial z}\frac{\partial Q}{\partial \bar{w}} - \frac{\partial P}{\partial \bar{w}}\frac{\partial Q}{\partial z} + \frac{\partial P}{\partial \bar{z}}\frac{\partial Q}{\partial w} - \frac{\partial P}{\partial w}\frac{\partial Q}{\partial \bar{z}}.$$

To determine the kernel of $\mathrm{adj}_{H_2^T}$ we see how $\{\cdot,\cdot\}$ acts on terms of the form $z^{i_1} w^{i_2} \bar{z}^{j_1} \bar{z}^{j_2}$ where $i_1 + i_2 + j_1 + j_2 = n$ for some $n \geq 3$. Note that $\mathrm{adj}_{H_2^T}$ also splits into a semisimple part plus a nilpotent part, namely,

$$\mathrm{adj}_{H_2^T}(\cdot) = \{\cdot, i(z\bar{w} - \bar{z}w)\} + \{\cdot, -w\bar{w}\}.$$

Thus we compute,

$$\begin{aligned} \left\{z^{i_1} w^{i_2} \bar{z}^{j_1} \bar{w}^{j_2}, i(z\bar{w} - \bar{z}w)\right\} &= i(i_1 + i_2 - j_1 - j_2) z^{i_1} w^{i_2} \bar{z}^{j_1} \bar{w}^{j_2}, \\ \left\{z^{i_1} w^{i_2} \bar{z}^{j_1} \bar{w}^{j_2}, -w\bar{w}\right\} &= -i_1 z^{i_1 - 1} w^{i_2 + 1} \bar{z}^{j_1} \bar{w}^{j_2} - j_1 z^{i_1} w^{i_2} \bar{z}^{j_1 - 1} \bar{w}^{j_2 + 1}. \end{aligned} \tag{2.5}$$

From the first equation we see that the normalized Hamiltonian H^{NF} contains only homogeneous polynomials of even degree in n and moreover,

$$i_1 + i_2 = \frac{n}{2} \quad \text{and} \quad j_1 + j_2 = \frac{n}{2}.$$

Now taking into account the second equation of (2.5) it is not difficult to conclude that $\dim\ker(\mathrm{adj}_{H_2^T} : \mathcal{H}_n \to \mathcal{H}_n) = \frac{n}{2} + 1$. Moreover, we can explicitly compute the following basis

$$\ker(\mathrm{adj}_{H_2^T} : \mathcal{H}_n \to \mathcal{H}_n) = \mathrm{span}\left\{(z\bar{w} - \bar{z}w)^k (w\bar{w})^m \mid m, k \geq 0, \quad m + k = \frac{n}{2}\right\}.$$

Thus, the homogeneous polynomials $\tilde{H}_n$ can be written uniquely in terms of that basis and this concludes the proof of the Theorem. □

Remark 2.1.2.1. It is clear that H^{NF} is in involution with $q_2p_1 - q_1p_2$. Thus H^{NF} is symmetric with respect to the one-parameter group of rotations induced by the Hamiltonian flow of $q_2p_1 - q_1p_2$. Hence any truncation of H^{NF} is integrable and consequently H can be approximated by an integrable Hamiltonian at every order.

2.2 Hamiltonian-Hopf bifurcation

Let H_ϵ be an analytic family of two degrees of freedom Hamiltonians defined in a neighbourhood of the origin in $\mathbb{R}^4$. Suppose that the family of vector fields X_{H_ϵ} (with respect to the canonical symplectic form in $\mathbb{R}^4$) has a common equilibrium point which we assume to be at the origin ($X_{H_\epsilon}(0) = 0$ for every ϵ) and that as $\epsilon \to 0^+$ the equilibrium point of the family X_{H_ϵ} goes through a Hamiltonian-Hopf bifurcation as described in the introduction of this thesis: for $\epsilon > 0$ the linear system $DX_{H_\epsilon}(0)$ has two pairs of complex conjugate eigenvalues $\pm\beta_\epsilon \pm i\alpha_\epsilon$, $\alpha_\epsilon \neq 0, \beta_\epsilon \neq 0$ which approach the imaginary axis as $\epsilon \to 0^+$ yielding a single pair of pure imaginary eigenvalues $\pm\alpha_0 i$, $\alpha_0 \neq 0$ with multiplicity two for the linear system $DX_{H_0}(0)$. Therefore, in the general case the matrix $DX_{H_0}(0)$ is nonsemisimple and according to the normal form theory for Hamiltonian matrices [11] one can assume that,

$$DX_{H_0}(0) = \begin{pmatrix} 0 & -\alpha_0 & 0 & 0 \\ \alpha_0 & 0 & 0 & 0 \\ -\iota & 0 & 0 & -\alpha_0 \\ 0 & -\iota & \alpha_0 & 0 \end{pmatrix},$$

where $\iota = \pm 1$. For simplification purposes one can assume without lost of generality that $\alpha_0 = 1$ and $\iota = 1$. Indeed, by a reparametrization of time or equivalently by multiplying the Hamiltonian H_ϵ by $\iota \left|\alpha_0\right|^{-1}$ and considering the canonical linear change of variables,

$$(q_1, q_2, p_1, p_2) \to \left(\iota \frac{\alpha_0}{\sqrt{|\alpha_0|}} q_1, \sqrt{|\alpha_0|} q_2, \iota \frac{\sqrt{|\alpha_0|}}{\alpha_0} p_1, \frac{1}{\sqrt{|\alpha_0|}} p_2 \right), \tag{2.6}$$

we obtain the desired normalization of α_0 and ι. Thus we can write H_ϵ in the following form,

$$H_\epsilon(q, p) = q_1 p_2 - q_2 p_1 + \frac{q_1^2 + q_2^2}{2} + F_\epsilon(q, p), \tag{2.7}$$

where $F_\epsilon(q, p) = O((|q|+|p|+|\epsilon|)^3)$ such that $q = (q_1, q_2)$, $p = (p_1, p_2)$ and $F_\epsilon(0, 0) = 0$ and $\partial_{q,p} F_\epsilon(0, 0) = 0$.

The Hamiltonian-Hopf bifurcation corresponds to the unfolding of a nonsemisimple Hamiltonian with a $1 : -1$ resonance. This resonance has been studied by Sokol'skiĭ in [71] who investigated the stability of the equilibrium point. With the help of normal form of Theorem 2.1.2 he established its formal stability.

The definitive study of the Hamiltonian-Hopf bifurcation is attributed to van der Meer in [76] who derived the following normal form for the bifurcation,

$$H_\epsilon^{NF} = H_\epsilon^0 + \sum_{3m+2j+2l \geq 5} a_{m,j,l} I_1^m I_3^j \epsilon^l, \qquad H_\epsilon^0 = -I_1 + I_2 - \epsilon I_3 + \eta I_3^2, \tag{2.8}$$

such that,

$$I_1 = q_2 p_1 - q_1 p_2, \; I_2 = \frac{q_1^2 + q_2^2}{2}, \; I_3 = \frac{p_1^2 + p_2^2}{2}, \tag{2.9}$$

where η and the coefficients $a_{m,j,l}$ are real numbers. Note that I_1 is an integral of H^{NF}, i.e. $\left\{H^{NF}, I_1\right\} = 0$, and that any truncation of the normal form is integrable. Moreover, by an analytic near identity canonical change of coordinates Φ_n we can normalize H up to some fixed order whereas the transformation that carries H into H^{NF} is expected to diverge in general.

Also note that the normal form H_ϵ^{NF} is reversible with respect to the involution,

$$\mathcal{S}(q_1, q_2, p_1, p_2) = (-q_1, q_2, p_1, -p_2). \tag{2.10}$$

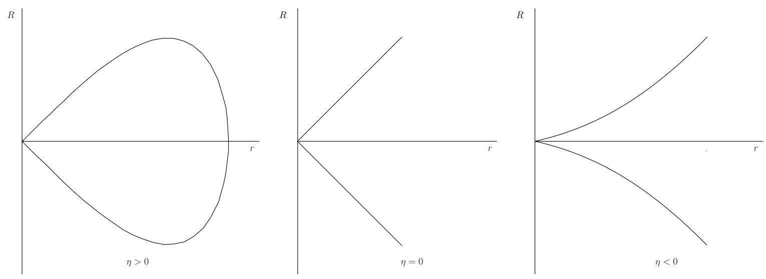

Figure 2.1: Different scenarios in the Hamiltonian-Hopf bifurcation.

That is $\mathcal{S}X_{H_\epsilon^{NF}}(\mathbf{x}) = -X_{H_\epsilon^{NF}}(\mathcal{S}\mathbf{x})$.

Now, there is a convenient scaling of variables which groups terms of the same order in the normal form (2.8). We start by scaling the bifurcation parameter by $\epsilon = \delta^2$ and change variables according to,

$$q_1 = \delta^2 Q_1,\; q_2 = \delta^2 Q_2,\; p_1 = \delta P_1,\; p_2 = \delta P_2. \tag{2.11}$$

We call this change the *standard scaling*. It is not difficult to see that the standard scaling is symplectic with multiplier δ^3. Hence we multiply the new Hamiltonian by δ^{-3} and use the canonical symplectic form Ω to derive the Hamilton equations. In these new variables the leading order Hamiltonian H_ϵ^0 becomes,

$$h^0 = -\mathcal{I}_1 + \left\{\mathcal{I}_2 - \mathcal{I}_3 + \eta\mathcal{I}_3^2\right\}\delta,$$

where the $\mathcal{I}_i$'s are defined in the same way as the I_i's but in the new variables Q and P. As h^0 is integrable a detailed bifurcation analysis of the Hamiltonian system can be performed. For that end, it is convenient to change to the following polar coordinates,

$$\begin{aligned} Q_1 &= R\cos\theta - \tfrac{\Theta}{r}\sin\theta, \quad P_1 = r\cos\theta, \\ Q_2 &= R\sin\theta + \tfrac{\Theta}{r}\cos\theta, \quad P_2 = r\sin\theta. \end{aligned} \tag{2.12}$$

In these new coordinates the Hamiltonian h^0 takes the form,

$$h^0 = -\Theta + \left\{\frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{1}{2}r^2 + \frac{\eta}{4}r^4\right\}\delta,$$

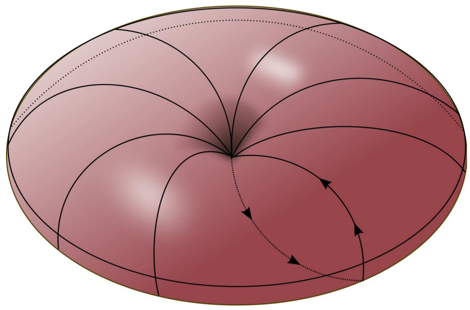

Figure 2.2: The invariant manifold of H^0_ϵ.

and $\Theta = \mathcal{I}_1$ is a first integral of h^0. Now we look for the stable and unstable manifolds of the equilibrium of h^0 which are contained inside the set $\{h^0 = 0, \Theta = 0\}$. Stable and unstable manifolds coincide and are defined by the equation,

$$R^2 = r^2 - \frac{\eta}{2} r^4, \quad \theta \in \mathbb{S}^1. \tag{2.13}$$

Due to the $\mathbb{S}^1$ invariance we can take a section $\theta = 0$ and plot the curve defined by the equation above. According to the sign of η there are three distinct cases (see Figure 2.1).

Recently, in [49] Lerman and Markova proved that when $\eta > 0$ the equilibrium of H_0 is Lyapunov stable and unstable when $\eta < 0$. Thus the stable case is when $\eta > 0$ and unstable when $\eta < 0$. The case $\eta = 0$ is called degenerate.

When $\eta > 0$ we have a single loop in the (r, R) plane as Figure 2.1 demonstrates. Taking into account the rotation $\theta \in \mathbb{S}^1$ we obtain a manifold which is homeomorphic to a 2-sphere where its north and south poles are glued together (see Figure 2.2). We can cut this invariant manifold along a transverse section $R = 0$ and obtain a circle of homoclinic points as illustrated in Figure 2.3.

In the polar coordinates (2.12) the set of fixed points $\mathrm{Fix}(\mathcal{S})$ of the involution (2.10) is given by $R = 0$ and $\theta = 0$ or $\theta = \pi$. Thus there are exactly two symmetric homoclinic points that correspond to $\theta = 0, \pi$. For the full system (2.7), the circle of homoclinic points is expected to split in two circles, stable and unstable, that intersect

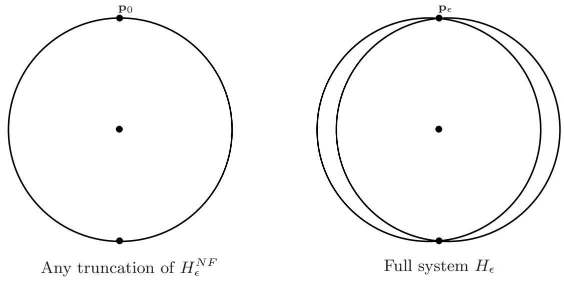

Figure 2.3: Section $R = 0$.

at two symmetric homoclinic points. In fact, the existence of such symmetric homoclinic orbits for the full system follows from an application of the implicit function theorem.

Of course, this analysis works for any truncation of the normal form H_ϵ^{NF} and therefore, when $\eta > 0$ the stable and unstable manifolds $W_\epsilon^{s,u}$ of the equilibrium of H_ϵ can be approximated at any order by a single manifold having the properties previously described. In general, W_ϵ^s and W_ϵ^u are expected to split and due to the integrability of the normal form at every orders we conclude that the invariant manifolds are extremely close. In fact, we will show that it is impossible to distinguish them using classical perturbation theory, i.e. their difference is beyond all orders, and the size of the splitting is exponentially small with respect to ϵ.

2.3 Natural Parametrizations

In a study of homoclinic trajectories it is important to have a convenient basis in the tangent space to the stable and unstable manifolds. The tangent space is given by natural parametrizations of the invariant manifolds. Below we provide a definition adapted to our problem. This definition can be of independent interest as it can be easily extended onto hyperbolic equilibria of higher dimensional systems (not necessarily Hamiltonian).

Suppose that the origin is an equilibrium of a Hamiltonian vector field X_H and that $\pm\beta \pm i\alpha$ are the eigenvalues of $DX_H(0)$. Then the origin has a two dimensional unstable manifold. According to Hartman [41] the restriction of the vector field on W^u_{loc} can be linearised by a C^1 change of variables. In the polar coordinates the linearised dynamics on W^u_{loc} takes the form:

$$\dot{r} = \beta r \qquad \dot{\varphi} = \alpha\,.$$

It is convenient to introduce $z = \log r$ so that $\dot{z} = \beta$. Then the local unstable manifold is the image of a function

$$\mathbf{\Gamma}^u : \{(\varphi, z) : \varphi \in S^1, z < \log r_0\} \to \mathbb{R}^4$$

where r_0 is the radius of the linearisation domain and S^1 is the unit circle. Since $\mathbf{\Gamma}^u$ maps trajectories into trajectories we can propagate it uniquely along the trajectories of the Hamiltonian system using the property

$$\mathbf{\Gamma}^u(\varphi + \alpha t, z + \beta t) = \Phi^t_H \circ \mathbf{\Gamma}^u(\varphi, z) \tag{2.14}$$

where Φ^t_H is the Hamiltonian flow. Note that

$$\mathbf{\Gamma}^u(\varphi + 2\pi, z) = \mathbf{\Gamma}^u(\varphi, z)$$

since φ is the angle component of the polar coordinates. Moreover,

$$\lim_{z \to -\infty} \mathbf{\Gamma}^u(\varphi, z) = 0\,.$$

Differentiating $\mathbf{\Gamma}^u$ along a trajectory we see that it satisfies the non-linear PDE:

$$\alpha \partial_\varphi \mathbf{\Gamma}^u + \beta \partial_z \mathbf{\Gamma}^u = X_H(\mathbf{\Gamma}^u)\,. \tag{2.15}$$

Note that each of the derivatives $\partial_z \mathbf{\Gamma}^u$ and $\partial_\varphi \mathbf{\Gamma}^u$ defines a vector field on W^u and equation (2.14) implies that both vector fields are invariant under the restriction of the flow $\Phi^t_H\Big|_{W^u}$.

Equation (2.15) is very important in the study of the invariant manifolds and in the subsequent chapters we will develop a theory to solve this PDE subject to certain conditions. The parametrization $\mathbf{\Gamma}^u$ is C^1 but in fact, using directly equation (2.15) we will show that when the Hamiltonian is analytic the parametrization is also analytic.

We can define $\mathbf{\Gamma}^s$ applying the same arguments to the Hamiltonian $-H$. In this case it is convenient to set $z = -\log r$ to ensure that $\mathbf{\Gamma}^s$ satisfies the same PDE as $\mathbf{\Gamma}^u$. In a reversible system with a reversing involution S, it is convenient to set

$$\mathbf{\Gamma}^s(\varphi, z) = S \circ \mathbf{\Gamma}^u(-\varphi, -z). \tag{2.16}$$

Now let us present an example. Following the previous discussion, we will parametrise the invariant manifold defined by equation (2.13) by a real analytic map $X_0 : \mathbb{R}^2 \to \mathbb{R}^4$ which is a solution of the following linear PDE,

$$\partial_\varphi X_0 + \partial_z X_0 = X_{h^0}(X_0).$$

Due to integrability of the Hamiltonian h^0 it is possible to compute explicitly a parametrisation X_0 (see Theorem 4.2.1),

$$X_0(\varphi, z) = \sqrt{\frac{2}{\eta}} \left(\frac{\cos\varphi \sinh z}{\cosh^2 z}, \frac{\sin\varphi \sinh z}{\cosh^2 z}, \frac{\cos\varphi}{\cosh z}, \frac{\sin\varphi}{\cosh z} \right)^T. \tag{2.17}$$

The curves defined by $x(t; \varphi, z) := X_0(\varphi + t, z + t)$ are integral curves of the vector field X_{h^0} and foliate the invariant manifold. Notice that X_0 is periodic in φ (due to the rotational invariance of X_{h^0}) and $\lim_{z\to\pm\infty} X_0(\cdot, z) = 0$.

We will see that X_0 in the unscaled variables can be regarded as the zeroth order approximation of the stable and unstable manifolds of H_ϵ near the equilibrium point. Note that the parametrisation X_0 has complex singularities for values of $z = i\frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$ and is $i\pi$-antiperiodic in z, i.e. $X_0(\varphi, z + i\pi) = -X_0(\varphi, z)$.

An anaytic study in a neighbourhood of the singularities of X_0 will provide a method for detecting the exponentially small splitting of the stable and unstable manifolds. Periodicity of X_0 in z allow us to restrict our analysis to the singular point $z = i\frac{\pi}{2}$. More concretely, we will study the solutions of equation (compare with (2.15)),

$$\alpha_\epsilon \partial_\varphi \mathbf{\Gamma} + \beta_\epsilon \partial_z \mathbf{\Gamma} = X_{H_\epsilon}(\mathbf{\Gamma}), \tag{2.18}$$

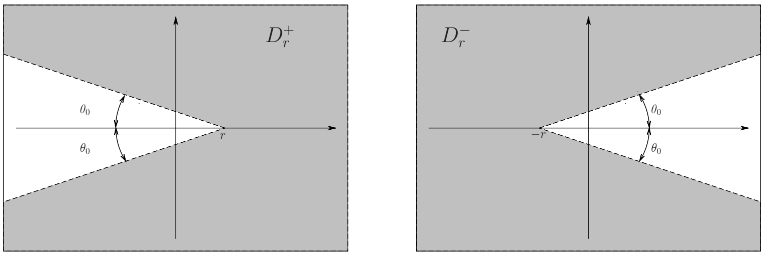

Figure 2.4: Domains $D_r^{\pm}$.

and the corresponding analytic continuation up to the singular point $z = i\frac{\pi}{2}$. For points close to the singularity, it is convenient to use the following change of variables,

$$z = \frac{\beta_\epsilon}{\alpha_\epsilon}\tau + i\frac{\pi}{2},$$

to center the singularity at the origin. The scale $\frac{\beta_\epsilon}{\alpha_\epsilon} \approx \sqrt{\epsilon}$ is used due to technical reasons which will become more apparent when performing the complex matching technique developed in chapter 4. Thus, in the new variable τ, equation (2.18) becomes,

$$\partial_\varphi \mathbf{\Gamma} + \partial_\tau \mathbf{\Gamma} = \alpha_\epsilon^{-1} X_{H_\epsilon}(\mathbf{\Gamma}) \tag{2.19}$$

This equation and equation (2.18) will be studied in detail in the subsequent chapters.

2.4 Linear Operators

In this section we define and study certain complex Banach spaces and some linear operators acting on them. The linear operators and motivated by the study of the PDE (2.19). These technical results are at the core of the proofs of the Theorems in the next chapters.

2.4.1 Solutions of first order linear differential equations

Let $0 < \theta_0 < \frac{\pi}{4}$, $h > 0$. We shall leave these parameters fixed throughout this section. Let $S_h = \{\varphi \in \mathbb{C} \mid |\mathrm{Im}\varphi| < h\}$ and for $r > 0$ consider the following domains in the complex plane,

$$D_r^- = \{\tau \in \mathbb{C} \mid |\arg(\tau + r)| > \theta_0\}, \quad D_r^+ = \left\{\tau \in \mathbb{C} \mid -\tau \in D_r^-\right\}. \tag{2.20}$$

In this section we consider the problem of solving the following linear PDE,

$$\mathcal{D}x = f, \tag{2.21}$$

where $\mathcal{D} = \partial_\varphi + \partial_\tau$ is a first order linear differential operator and f is some analytic function defined in an open subset of $\mathbb{C}^2$. We will also suppose that all functions are 2π-periodic in φ.

The simplest case is when $f = 0$. As one would expect, by using the method of characteristics, a solution of the homogeneous equation $\mathcal{D}x = 0$ must be a function which is constant along the characteristics $\dot{\varphi} = 1$ and $\dot{\tau} = 1$. Thus, is a function depending on a single variable, say $\tau - \varphi$. The next Proposition determines such function and its domain of definition,

Proposition 2.4.1. *Let $x : S_h \times B \to \mathbb{C}$ be an analytic function, 2π-periodic in φ where B is an open domain of $\mathbb{C}$. Suppose that $\mathcal{D}x = 0$, then there is a unique 2π-periodic analytic function,*

$$x_0 : \bigcup_{\tau_0 \in B} \tau_0 + S_h \to \mathbb{C}$$

such that $x(\varphi, \tau) = x_0(\tau - \varphi)$.

Proof. Given $\tau_0 \in B$ let

$$\Omega_{\tau_0} = \{(\varphi, \tau) \in S_h \times B \mid \varphi - \tau + \tau_0 \in S_h\}.$$

Note that Ω_{τ_0} is an open domain of $\mathbb{C}^2$. Now the initial value problem,

$$\mathcal{D}\xi = 0, \quad \xi(\varphi, \tau_0) = x(\varphi, \tau_0), \tag{2.22}$$

has a solution $\xi(\varphi, \tau) = x(\varphi - \tau + \tau_0, \tau_0)$. Hence ξ is an analytic function of a single variable $\tau - \varphi$ and is defined in the translated horizontal strip $\tau_0 + S_h$. By the main local existence and uniqueness theorem for analytic partial differential equations (see [24] for instance) we conclude that $x = \xi$ on Ω_{τ_0}. Thus $x(\varphi, \tau) = x(\varphi - \tau + \tau_0, \tau_0)$. Observe that for $\tau_0, \tau_1 \in B$ such that $(\tau_0 + S) \cap (\tau_1 + S) \neq \emptyset$ then $\Omega_{\tau_0} \cap \Omega_{\tau_1} \neq \emptyset$. Taking into account $S_h \times B = \bigcup_{\tau_0 \in B} \Omega_{\tau_0}$ and the uniqueness of analytic continuation we get the desired result. $\square$

When f is non-zero and for instance defined in $S_h \times D_r^{\pm}$ then equation (2.21) has two solutions,

$$x^-(\varphi, \tau) = \int_{-\infty}^{0} f(\varphi + s, \tau + s) ds \quad \text{and} \quad x^+(\varphi, \tau) = -\int_{0}^{+\infty} f(\varphi + s, \tau + s) ds,$$

provided the integrand in both functions is well defined in the domain of f and the corresponding integral converges.

Proposition 2.4.2. *Let $f : S_h \times D_r^- \to \mathbb{C}$ be an analytic function, 2π-periodic in φ and continuous on the closure of its domain. Moreover, suppose that $|f(\varphi, \tau)| \leq \frac{K_f}{|\tau|^p}$ for some $K_f > 0$ and $p \geq 2$. Then the formula,*

$$x^-(\varphi, \tau) = \int_{-\infty}^{0} f(\varphi + s, \tau + s) ds,$$

defines an analytic function in $S_h \times D_r^-$, continuous on the closure of its domain, 2π-periodic in φ. Moreover,

$$\left| x^-(\varphi, \tau) \right| \leq \frac{K_{p-1} K_f}{|\tau|^{p-1}}, \tag{2.23}$$

for some $K_p > 0$ independent of r.

In order to prove this Proposition we need the following Lemmas,

Lemma 2.4.1. *Let $p \geq 1$, $\tau \in D_r^+$. Then there exists a constant $K_p > 0$ such that,*

$$\int_{-\infty}^{0} \frac{1}{|\tau + s|^{p+1}} ds \leq \frac{K_p}{|\tau|^p}. \tag{2.24}$$

Proof. The proof of this lemma follows from easy estimates. First note that,

$$\int_{-\infty}^{0} \frac{ds}{|\tau + s|^{p+1}} \underbrace{=}_{t = \frac{s}{|\tau|}} \frac{1}{|\tau|^p} \int_{-\infty}^{0} \frac{dt}{\left|1 + e^{-i \arg(\tau)} t\right|^{p+1}}.$$

It is not difficult to get the following upper bounds,

$$\sup_{t\in(-\infty,0]} \frac{1}{\left|1+e^{-i\arg(\tau)}t\right|^{p+1}} \leq \frac{1}{(\sin\theta_0)^{p+1}} \; \forall\tau\in D_r^-$$

and,

$$\frac{1}{\left|1+e^{-i\arg(\tau)}t\right|} \leq \frac{-1}{t+\cos\arg(\tau)}, \; \forall t\leq -1 \;, \; \forall\tau\in D_r^-.$$

Using these estimates we conclude that,

$$\begin{aligned}
\int_{-\infty}^{0} \frac{dt}{\left|1+e^{-i\arg(\tau)}t\right|^{p+1}} &= \int_{-1}^{0} \frac{dt}{\left|1+e^{-i\arg(\tau)}t\right|^{p+1}} + \int_{-\infty}^{-1} \frac{dt}{\left|1+e^{-i\arg(\tau)}t\right|^{p+1}} \\
&\leq \frac{1}{(\sin\theta_0)^{p+1}} + \frac{1}{p\,(1-\cos\arg(\tau))^{p}} \\
&\leq \frac{1}{(\sin\theta_0)^{p+1}} + \frac{1}{p\,(1-\cos\theta_0)^{p}},
\end{aligned}$$

yielding the desired estimate (2.24). □

Lemma 2.4.2. *Let Ω be an open subset of $\mathbb{C}^2$, f a continuous function from $(-\infty,0)\times\Omega$ into $\mathbb{C}$. Suppose that for each $t\in(-\infty,0)$ the function $(z_1,z_2)\to f(t,z_1,z_2)$ is analytic in Ω and that both $\frac{\partial f}{\partial z_1}(t,z_1,z_2)$ and $\frac{\partial f}{\partial z_2}(t,z_1,z_2)$ are continuous functions in $(-\infty,0)\times\Omega$. Moreover, assume that for every $(z_1,z_2)\in\Omega$,*

$$F(z_1,z_2)=\int_{-\infty}^{0} f(t,z_1,z_2)dt<\infty,$$

and that $\int_{-N}^{0} f(t,z_1,z_2)dt$ converges uniformly as $N\to+\infty$ to $F(z_1,z_2)$ for (z_1,z_2) in compact subsets of Ω. Under these conditions the function F is analytic in Ω.

Proof. This result is standard in classical analysis and can be found in some text books, for instance [22]. □

Proof of Proposition 2.4.2. Let $f: S_h\times D_r^-\to\mathbb{C}$ be an analytic function as defined in the statement of the proposition. Moreover we know that $|f(\varphi,\tau)|\leq\frac{K_f}{|\tau|^p}$ for some

$K_f > 0$ and $p \geq 2$. For $N \geq 0$ we have $(\varphi - N, \tau - N) \in S_h \times D_r^-$, then,

$$\begin{aligned} \int_{-\infty}^{-N} |f(\varphi + s, \tau + s)| \, ds &\leq \int_{-\infty}^{0} |f(\varphi - N + s, \tau - N + s)| \, ds \\ &\leq \int_{-\infty}^{0} \frac{K_f}{|\tau - N + s|^p} ds \\ &\leq \frac{K_{p-1} K_f}{|\tau - N|^{p-1}}, \end{aligned} \tag{2.25}$$

by the Lemma 2.4.1. Thus, the integral $\int_{-N}^{0} f(\varphi + s, \tau + s) ds$ converges uniformly in $S_h \times D_r^-$ and we can apply Lemma 2.4.2 and deduce that,

$$x^-(\varphi, \tau) = \int_{-\infty}^{0} f(\varphi + s, \tau + s) ds,$$

defines an analytic function in $S_h \times D_r^-$. The continuity on the closure of its domain also follows from the continuity of f and the uniform convergence of the integral (2.25). The periodicity is trivial and the upper bound for x^- follows from (2.25) with $N = 0$. This concludes the proof. □

Remark 2.4.0.2. An analogous Proposition holds for the function,

$$x^+(\varphi, \tau) = -\int_{0}^{+\infty} f(\varphi + s, \tau + s) ds,$$

which is defined in $S_h \times D_r^+$.

Now we consider the problem of solving equation (2.21), which we recall for convenience,

$$\mathcal{D}x = f, \tag{2.26}$$

but for functions f defined in $S_h \times D_r^1$ where,

$$D_r^1 = D_r^+ \cap D_r^- \cap \{\tau \in \mathbb{C} \mid \mathrm{Im}\tau < -r\}.$$

Regarding this new domain D_r^1 we can not repeat the same arguments of Proposition 2.4.2 since D_r^1 does not contain an infinite horizontal segment. In order to overcome this difficulty, we construct an analytic solution of (2.26) using a technique similar to partition of unity, originally developed by V. F. Lazutkin in [46]. Following the ideas of [28] we consider the following domains,

$$\tilde{D}_r^- = \{\tau \in \mathbb{C} \mid |\arg(\tau + r)| > \theta_0 \quad \text{and} \quad \operatorname{Im}(\tau) < -r\},$$
$$\tilde{D}_r^+ = \left\{\tau \in \mathbb{C} \mid -\tau \in \tilde{D}_r^-\right\}.$$

Note that $D_r^1 = \tilde{D}_r^+ \cap \tilde{D}_r^-$. The method consists of representing in a suitable way a function f analytic in $S_h \times D_r^1$ as a sum of two functions $f^\pm$ analytic in $S_h^1 \times \tilde{D}_r^\pm$ respectively. For that purpose we need to define a partition of unity for the set ∂D_r^1 as follows. Let $\lambda_0 : \mathbb{R} \to [0,1]$ be a smooth function such that,

$$\lambda_0(t) = 0 \quad t \leq -\pi, \quad \lambda_0(t) = 1 \quad t \geq \pi, \quad \left|\lambda_0'(t)\right| \leq 1 \ \forall t \in \mathbb{R}$$

and define the following functions $\lambda^\pm : \partial D_r^1 \to [0,1]$ by,

$$\lambda^+(\tau) = \lambda_0\left(\operatorname{Re}(\tau)\right), \quad \lambda^-(\tau) = 1 - \lambda^+(\tau).$$

Lemma 2.4.3 (On the Cauchy integral). *Let $r > \frac{\pi \tan\theta_0}{1 - \tan\theta_0}$ and $f : S_h \times D_r^1 \to \mathbb{C}$ an analytic function, 2π-periodic in φ and continuous on the closure of its domain. Moreover suppose that there exists $K_f > 0$ such that*

$$|f(\varphi, \tau)| \leq \frac{K_f}{|\tau|^2} \quad \text{in} \quad S_h \times D_r^1.$$

Then the integral,

$$f^\pm(\varphi, \tau) = \frac{1}{2\pi i} \int_{\partial D_r^1} \frac{\lambda^\pm(\xi) f(\varphi, \xi)}{\xi - \tau} d\xi, \tag{2.27}$$

defines an analytic function in $S_h \times \tilde{D}_r^\pm$, 2π-periodic in φ, continuous in $\overline{S_h \times \tilde{D}_r^\pm}$ and

$$\left|f^\pm(\varphi, \tau)\right| \leq \frac{2K_f}{r^2} \quad \text{in} \quad S_h \times \tilde{D}_r^\pm.$$

Moreover,

$$f(\varphi, \tau) = f^+(\varphi, \tau) + f^-(\varphi, \tau).$$

Proof. This lemma is a special case of Lemma 9.2 in [28]. ∎

Using this lemma we are able to prove,

Proposition 2.4.3. *Let* $\epsilon \geq 0$, $r > \frac{\pi \tan\theta_0}{1-\tan\theta_0}$ *and* $p \geq 4$. *If* $f : S_h \times D_r^1 \to \mathbb{C}$ *is analytic,* 2π*-periodic in* φ*, continuous on the closure of its domain and there exists* $K_f > 0$ *such that,*

$$|f(\varphi,\tau)| \leq \frac{K_f}{\left|\tau^p e^{i\epsilon(\tau-\varphi)}\right|} \quad \text{in} \quad S_h \times D_r^1.$$

Then equation (2.26) *has an analytic solution* $x : S_h \times D_r^1 \to \mathbb{C}$, 2π*-periodic in* φ *and continuous on the closure of its domain such that,*

$$|x(\varphi,\tau)| \leq \frac{4K_f K_{p-3}}{r^2} \frac{1}{\left|\tau^{p-3} e^{i\epsilon(\tau-\varphi)}\right|}. \tag{2.28}$$

Proof. Let $\mu(\varphi,\tau) = \tau^{p-2} e^{i\epsilon(\tau-\varphi)}$ and $\tilde{f}(\varphi,\tau) = \mu(\varphi,\tau) f(\varphi,\tau)$. Now we apply lemma 2.4.3 to $\tilde{f}$ with $K_{\tilde{f}} = K_f$ to get,

$$f(\varphi,\tau) = \frac{1}{\mu(\varphi,\tau)} \left(\tilde{f}^-(\varphi,\tau) + \tilde{f}^+(\varphi,\tau) \right). \tag{2.29}$$

Note that periodicity in φ is preserved since by (2.27) the function $\tilde{f}^{\pm}$ is 2π periodic in φ as well as the function μ. Let,

$$x(\varphi,\tau) = \int_{-\infty}^{0} \frac{\tilde{f}^-(\varphi+s,\tau+s)}{\mu(\varphi+s,\tau+s)} ds - \int_{0}^{+\infty} \frac{\tilde{f}^+(\varphi+s,\tau+s)}{\mu(\varphi+s,\tau+s)} ds. \tag{2.30}$$

If formula (2.30) defines an analytic function in $S_h \times D_r^1$, then it is the desired solution of equation (2.26). Let us prove that x is analytic. Applying Lemma 2.4.1 and the upper bound from Lemma 2.4.3 to the first term of (2.30) we get,

$$\left| \int_{-\infty}^{0} \frac{\tilde{f}^-(\varphi+s,\tau+s)}{\mu(\varphi+s,\tau+s)} ds \right| \leq \frac{2K_f}{r^2 \left|e^{i\epsilon(\tau-\varphi)}\right|} \int_{-\infty}^{0} \frac{1}{|\tau+s|^{p-2}} ds \leq \frac{2K_f K_{p-3}}{r^2 \left|e^{i\epsilon(\tau-\varphi)}\right| |\tau|^{p-3}}. \tag{2.31}$$

Thus for $p \geq 4$ the integral converges uniformly in $S_r^1 \times \tilde{D}_r^-$ and by Lemma 2.4.2 it defines an analytic function in $S_h \times \tilde{D}_r^-$. The continuity on the closure of $S_h \times \tilde{D}_r^-$ also follows from uniform convergence and continuity of $\tilde{f}^-$. In an analogous way we conclude that $\int_0^{+\infty} \frac{\tilde{f}^-(\varphi+s,\tau+s)}{\mu(\varphi+s,\tau+s)} ds$ defines an analytic function in $S_h \times \tilde{D}_r^+$, continuous on the closure of its domain and having the same upper bound (2.31). Thus,

$$|x(\varphi,\tau)| \leq \frac{4K_f K_{p-3}}{r^2 \left|e^{i\epsilon(\tau-\varphi)}\right| |\tau|^{p-3}},$$

and the proof is complete. $\square$

2.4.2 Linear operators and their inverses

Let $B \subset \mathbb{C}$ be an open domain. We denote $\mathfrak{X}_p(S_h \times B)$ for $p \in \mathbb{Z}$ the space of analytic functions $f = (f_1, \dots, f_4) : S_h \times B \to \mathbb{C}^4$ continuous on the closure of $S_h \times B$, 2π-periodic in $\varphi \in S_h$ and satisfying,

$$\|f\|_p = \sup_{(\varphi,\tau)\in S_h\times B} \left(\left|\tau^{p+1} f_1(\varphi,\tau)\right| + \left|\tau^{p+1} f_2(\varphi,\tau)\right|\right.$$
$$\left. + \left|\tau^p f_3(\varphi,\tau)\right| + \left|\tau^p f_4(\varphi,\tau)\right|\right) < \infty.$$

The space $\mathfrak{X}_p(S_h \times B)$ with the norm $\|\cdot\|_p$ as defined above is a complex Banach space. When $f \in \mathfrak{X}_p(S_h \times B)$ we occasionally write

$$f(\varphi,\tau) = (\tau^{-p-1} f_1(\varphi,\tau), \tau^{-p-1} f_2(\varphi,\tau), \tau^{-p} f_3(\varphi,\tau), \tau^{-p} f_4(\varphi,\tau)),$$

where the norm of f is now $\|f\|_p = \sup_{(\varphi,\tau)\in S_h\times B} \sum_{i=1}^{4} |f_i(\varphi,\tau)|$.

For $0 < \mu < 2$ let $\mathfrak{Y}_\mu(S_h \times B)$ be the space of analytic functions $\xi = (\xi_1, \dots, \xi_4) : S_h \times B \to \mathbb{C}^4$ continuous on the closure of $S_h \times B$, 2π-periodic in $\varphi \in S_h$ and satisfying,

$$\|\xi\|_\mu = \sup_{(\varphi,\tau)\in S_h\times B} \sum_{i=1}^{4} \left|e^{(2-\mu)i(\tau-\varphi)} \xi_i(\varphi,\tau)\right| < \infty.$$

Given two Banach spaces $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ and $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ we define the usual norm on the space of linear operators $\mathcal{L} : \mathfrak{X} \to \mathfrak{Y}$ as follows,

$$\|\mathcal{L}\|_{\mathfrak{Y},\mathfrak{X}} = \sup_{\xi\in\mathfrak{X}\setminus\{0\}} \frac{\|\mathcal{L}(\xi)\|_{\mathfrak{Y}}}{\|\xi\|_{\mathfrak{X}}}.$$

When it is clear from the text we shall omit the dependence of the Banach spaces $\mathfrak{X}_p(S_h \times B)$ and $\mathfrak{Y}_\mu(S_h \times B)$ from the set $S_h \times B$. Moreover, in order to simplify the notation we shall write the norm of a linear operator $\mathcal{L} : \mathfrak{X}_p(S_h \times B) \to \mathfrak{X}_q(S_h \times B)$ as $\|\mathcal{L}\|_{q,p}$ and the norm of a linear operator $\mathcal{L} : \mathfrak{Y}_\mu(S_h \times B) \to \mathfrak{Y}_{\mu'}(S_h \times B)$ as $\|\mathcal{L}\|_{\mu',\mu}$.

The following inclusions are not difficult to prove and we shall use them when appropriate,

- $\mathfrak{X}_p\left(S_h \times D_r^-\right) \subseteq \mathfrak{X}_q\left(S_h \times D_r^-\right)$ for $p \geq q$;
- $\mathfrak{Y}_\mu(S_h \times D_r^1) \subset \mathfrak{Y}_{\mu'}(S_h \times D_r^1)$ for $\mu < \mu'$;
- $\mathfrak{X}_p(S_h \times D_{\tilde{r}}^-) \subset \mathfrak{X}_p(S_h \times D_r^-)$ for $\tilde{r} < r$;
- $\mathfrak{Y}_\mu(S_h \times D_{\tilde{r}}^1) \subset \mathfrak{Y}_\mu(S_h \times D_r^1)$ for $\tilde{r} < r$;
- $\mathfrak{Y}_\mu(S_h \times D_r^1) \subset \mathfrak{X}_p(S_h \times D_r^1)$.

Now let $A : S_h \times B \to \mathbb{C}^{4\times 4}$ be an analytic matrix-valued function which is 2π-periodic in φ. Define,

$$\mathcal{L}(\xi)(\varphi,\tau) = \mathcal{D}\xi(\varphi,\tau) - A(\varphi,\tau)\xi(\varphi,\tau), \tag{2.32}$$

where $\mathcal{D} = \partial_\varphi + \partial_\tau$ is the same differential operator defined in the previous section and $\xi : S_h \times B \to \mathbb{C}^4$ is an analytic function which is 2π-periodic in φ. In the following we shall be interested in solving the equation $\mathcal{L}(\xi) = f$ for a given f. The functions u and f will be defined later in this section. The reason why we look at this equation is because we want to solve the PDE (2.19) when $\epsilon = 0$.

We say that a 4 by 4 matrix-valued function $\mathbf{U} : S_h \times B \to \mathbb{C}^{4\times 4}$ is a *fundamental matrix* of $\mathcal{L}$ if $\mathcal{L}(\mathbf{U}) = 0$, $\det(\mathbf{U}) = 1$ and moreover,

$$\mathbf{U} = \begin{pmatrix} \tau^{-2}u_{1,1} & \tau^2 u_{1,2} & \tau u_{1,3} & \tau^{-3}u_{1,4} \\ \tau^{-2}u_{2,1} & \tau^2 u_{2,2} & \tau u_{2,3} & \tau^{-3}u_{2,4} \\ \tau^{-1}u_{3,1} & \tau^3 u_{3,2} & \tau^2 u_{3,3} & \tau^{-2}u_{3,4} \\ \tau^{-1}u_{4,1} & \tau^3 u_{4,2} & \tau^2 u_{4,3} & \tau^{-2}u_{4,4} \end{pmatrix}, \tag{2.33}$$

where $u_{i,j}$ are analytic functions in $S_h \times B$, continuous on the closure of its domain, 2π-periodic in φ and $\sup_{(\varphi,\tau)\in S_h\times B} |u_{i,j}(\varphi,\tau)| < \infty$ for every $i,j = 1,\ldots,4$. Thus, we can define,

$$K_{\mathbf{U}} := \max_{i,j} \left\{ \sup_{(\varphi,\tau)\in S_h\times B} |u_{i,j}(\varphi,\tau)| \right\}. \tag{2.34}$$

Note that the columns of $\mathbf{U}$ belong to $\mathfrak{X}_1\left(S_h \times B\right)$, $\mathfrak{X}_{-3}\left(S_h \times B\right)$, $\mathfrak{X}_{-2}\left(S_h \times B\right)$ and $\mathfrak{X}_2\left(S_h \times B\right)$ respectively.

An example: the operator $\mathcal{L}_0$

Here we define an operator $\mathcal{L}_0$ in the form of (2.32) which depends on a certain matrix A_0 and has a fundamental matrix $\mathbf{U}_0$ that will be defined below. This operator will play an important role in the perturbation theory developed in the subsequent chapters. Let us consider the following PDE,

$$\mathcal{D}\mathbf{x} = X_{H^0}(\mathbf{x}), \tag{2.35}$$

where H^0 denotes the leading order H_0^0 (we omit its subscript to simplify the notation) of H_ϵ for $\epsilon = 0$ (see (2.8)) which we recall $H^0 = -I_1 + I_2 + \eta I_3^2$. It is not difficult to check that,

$$\mathbf{\Gamma}_0(\varphi, \tau) = \left(\kappa\tau^{-2}\cos\varphi, \kappa\tau^{-2}\sin\varphi, \kappa\tau^{-1}\cos\varphi, \kappa\tau^{-1}\sin\varphi\right)^T, \tag{2.36}$$

solves equation (2.35) where $\kappa^2 = -\frac{2}{\eta}$. Indeed, using the polar coordinates,

$$q_1 = R\cos\theta, \quad p_1 = r\cos\theta, \quad q_2 = R\sin\theta, \quad p_2 = r\sin\theta.$$

we see that equation (2.35) reduces to the following equations,

$$\mathcal{D}\theta = 1, \quad \mathcal{D}r = -R, \quad \mathcal{D}R = \eta r^3.$$

The last two equations define a second order differential equation $\mathcal{D}^2 r = -\eta r^3$ which has a solution $r(\varphi, \tau) = \frac{\kappa}{\tau}$. Thus $R(\varphi, \tau) = \frac{\kappa}{\tau^2}$. Now using $\theta(\varphi, \tau) = \varphi$ as a solution of the first equation we get the desired solution $\mathbf{\Gamma}_0$.

The linearized Hamiltonian vector field $A_0 := DX_{H^0}(\mathbf{\Gamma}_0)$ evaluated at $\mathbf{\Gamma}_0$ reads,

$$A_0(\varphi, \tau) = \begin{pmatrix} 0 & -1 & -\frac{1+2\cos^2\varphi}{\tau^2} & -\frac{\sin(2\varphi)}{\tau^2} \\ 1 & 0 & -\frac{\sin(2\varphi)}{\tau^2} & -\frac{1+2\sin^2\varphi}{\tau^2} \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}. \tag{2.37}$$

Note that A_0 does not depend on the choice of κ. Moreover it is 2π-periodic in φ, analytic in $\mathbb{C} \times \mathbb{C}^*$ and bounded in $S_h \times D_r^-$. Now we define $\mathcal{L}_0$ as in (2.32) to be,

$$\mathcal{L}_0(\xi)(\varphi, \tau) = \mathcal{D}\xi(\varphi, \tau) - A_0(\varphi, \tau)\xi(\varphi, \tau), \tag{2.38}$$

where $\xi : S_h \times B \to \mathbb{C}^4$ is an analytic function which is 2π-periodic in φ. It can be checked directly (using the polar coordinates as before) that the following matrix

$$\mathbf{U}_0(\varphi,\tau) = \begin{pmatrix} -\frac{\kappa \sin\varphi}{\tau^2} & -\frac{3\tau^2\cos\varphi}{5\kappa} & \frac{2\tau\sin\varphi}{3\kappa} & -\frac{2\kappa\cos\varphi}{\tau^3} \\ \frac{\kappa\cos\varphi}{\tau^2} & -\frac{3\tau^2\sin\varphi}{5\kappa} & -\frac{2\tau\cos\varphi}{3\kappa} & -\frac{2\kappa\sin\varphi}{\tau^3} \\ -\frac{\kappa\sin\varphi}{\tau} & \frac{\tau^3\cos\varphi}{5\kappa} & -\frac{\tau^2\sin\varphi}{3\kappa} & -\frac{\kappa\cos\varphi}{\tau^2} \\ \frac{\kappa\cos\varphi}{\tau} & \frac{\tau^3\sin\varphi}{5\kappa} & \frac{\tau^2\cos\varphi}{3\kappa} & -\frac{\kappa\sin\varphi}{\tau^2} \end{pmatrix}, \tag{2.39}$$

is a fundamental matrix for the linear operator $\mathcal{L}_0$. In fact, a direct substitution of $\mathbf{U}_0$ into (2.38) yields $\mathcal{L}_0(\mathbf{U}_0) = 0$ and all entries of $\mathbf{U}_0$ are analytic functions in $\mathbb{C}\times\mathbb{C}^*$ and 2π-periodic in φ. Moreover, for any $h, r > 0$ it is clear that the columns of $\mathbf{U}_0$ belong to the spaces $\mathfrak{X}_1(S_h \times D_r^-)$, $\mathfrak{X}_{-3}(S_h \times D_r^-)$, $\mathfrak{X}_{-2}(S_h \times D_r^-)$ and $\mathfrak{X}_2(S_h \times D_r^-)$ respectively. Finally, $\mathbf{U}_0(\varphi,\tau)$ is symplectic for all $(\varphi,\tau) \in \mathbb{C}\times\mathbb{C}^*$. In particular $\det(\mathbf{U}_0) = 1$ and hence $\mathbf{U}_0$ is a fundamental matrix of $\mathcal{L}_0$.

Inverse Theorems for the linear operator $\mathcal{L}$ in certain Banach spaces

In this subsection we are interested in the question of invertibility of $\mathcal{L}$ in different Banach spaces. We state and prove several Theorems that will be used in the perturbation theory developed in the subsequent chapters.

Theorem 2.4.1. *Let $p \geq 3$ and suppose that the linear operator $\mathcal{L} : \mathfrak{X}_p(S_h \times D_r^-) \to \mathfrak{X}_p(S_h \times D_r^-)$ acting by the formula* (2.32) *has a fundamental matrix $\mathbf{U}$. Then $\mathcal{L}$ has trivial kernel. Moreover there exists an unique bounded linear operator $\mathcal{L}^{-1} : \mathfrak{X}_{p+1}(S_h \times D_r^-) \to \mathfrak{X}_p(S_h \times D_r^-)$ such that $\mathcal{L}\mathcal{L}^{-1} = \mathrm{Id}$.*

Proof. Let us prove the first assertion of the Proposition: kernel of $\mathcal{L}$ is trivial. In fact, let $\xi \in \mathfrak{X}_p(S_h \times D_r^-)$ such that $\mathcal{L}(\xi) = 0$. Then, by the definition of the operator $\mathcal{L}$, the function ξ must satisfy,

$$D\xi = A\xi.$$

Now we use the method of variation of constants and write $\xi = \mathbf{U}\mathbf{c}$ for some vector function $\mathbf{c} : S_h \times D_r^- \to \mathbb{C}^4$. Hence, in the virtue of $\det(\mathbf{U}) = 1$, it is not difficult to show that $\mathbf{c}$ must satisfy $D\mathbf{c} = 0$. Applying Proposition 2.4.1 to each component of the vector function $\mathbf{c}$ we conclude that $\mathbf{c}(\varphi, \tau) = \mathbf{c}_0(\tau - \varphi)$ where $\mathbf{c}_0 : \mathbb{C} \to \mathbb{C}^4$ is an entire, 2π-periodic vector function. Moreover, since $\mathbf{c}_0 = \mathbf{U}^{-1}\xi$ we can bound $\mathbf{c}_0$ as follows. Due to (2.33), the inverse $\mathbf{U}^{-1}$ has the following form,

$$\mathbf{U}^{-1} = \begin{pmatrix} \tau^2 \breve{u}_{1,1} & \tau^2 \breve{u}_{1,2} & \tau \breve{u}_{1,3} & \tau \breve{u}_{1,4} \\ \tau^{-2} \breve{u}_{2,1} & \tau^{-2} \breve{u}_{2,2} & \tau^{-3} \breve{u}_{2,3} & \tau^{-3} \breve{u}_{2,4} \\ \tau^{-1} \breve{u}_{3,1} & \tau^{-1} \breve{u}_{3,2} & \tau^{-2} \breve{u}_{3,3} & \tau^{-2} \breve{u}_{3,4} \\ \tau^3 \breve{u}_{4,1} & \tau^3 \breve{u}_{4,2} & \tau^2 \breve{u}_{4,3} & \tau^2 \breve{u}_{4,4} \end{pmatrix} \tag{2.40}$$

where $\breve{u}_{i,j}$ are analytic functions in $S_h \times D_r^-$, 2π-periodic in φ and

$$K_{\mathbf{U}^{-1}} := \max_{i,j} \left\{ \sup_{(\varphi,\tau) \in S_h \times D_r^-} |\breve{u}_{i,j}(\varphi, \tau)| \right\} < \infty, \tag{2.41}$$

which follows from (2.34). Thus, if $\xi = (\tau^{-p-1}\xi_1, \tau^{-p-1}\xi_2, \tau^{-p}\xi_3, \tau^{-p}\xi_4)$ then

$$\mathbf{c}_0 = \left(\tau^{-p+1} \sum_{i=1}^{4} \breve{u}_{1,j}\xi_i, \tau^{-p-3} \sum_{i=1}^{4} \breve{u}_{2,j}\xi_i, \tau^{-p-2} \sum_{i=1}^{4} \breve{u}_{3,j}\xi_i, \tau^{-p+2} \sum_{i=1}^{4} \breve{u}_{4,j}\xi_i \right). \tag{2.42}$$

It is not difficult to see that (2.42) and (2.41) imply that $\mathbf{c}_0$ is bounded in $\mathbb{C}$ for $p \geq 3$. Thus, an entire bounded function must be a constant, by Liouville's theorem. Moreover, (2.42) implies that

$$\lim_{\mathrm{Im}(s) \to \pm\infty} \mathbf{c}_0(s) = 0.$$

Thus $\xi = 0$ and the kernel of $\mathcal{L}$ is trivial.

Now let us construct an inverse of $\mathcal{L}$, i.e., let us solve the following equation,

$$\mathcal{L}(\xi) = f. \tag{2.43}$$

where $f \in \mathfrak{X}_{p+1}(S_h \times D_r^-)$. Again, we look for a solution of (2.43) using the method of variation of constants. Let $\xi = \mathbf{U}\mathbf{c}$. Then equation (2.43) is equivalent to,

$$D\mathbf{c} = \mathbf{U}^{-1} f. \tag{2.44}$$

Writing $f = (\tau^{-p-2} f_1, \tau^{-p-2} f_2, \tau^{-p-1} f_3, \tau^{-p-1} f_4)$ and taking into account (2.40) we can write the right hand side of equation (2.44) as follows $\mathbf{U}^{-1} f = (g_1, g_2, g_3, g_4)^T$ where,

$$g_1 = \tau^{-p} \sum_{i=1}^{4} \breve{u}_{1,j} f_i, \qquad g_2 = \tau^{-p-4} \sum_{i=1}^{4} \breve{u}_{2,j} f_i,$$
$$g_3 = \tau^{-p-3} \sum_{i=1}^{4} \breve{u}_{3,j} f_i, \qquad g_4 = \tau^{-p+1} \sum_{i=1}^{4} \breve{u}_{4,j} f_i.$$

Now bearing in mind that $\|f\|_{p+1} < \infty$ and (2.41) we can estimate the previous functions as follows,

$$|g_1(\varphi,\tau)| \leq \frac{K_{\mathbf{U}^{-1}} \|f\|_{p+1}}{|\tau|^{p}}, \qquad |g_2(\varphi,\tau)| \leq \frac{K_{\mathbf{U}^{-1}} \|f\|_{p+1}}{|\tau|^{p+4}},$$
$$|g_3(\varphi,\tau)| \leq \frac{K_{\mathbf{U}^{-1}} \|f\|_{p+1}}{|\tau|^{p+3}}, \qquad |g_4(\varphi,\tau)| \leq \frac{K_{\mathbf{U}^{-1}} \|f\|_{p+1}}{|\tau|^{p-1}}.$$

where the upper bounds are valid in $S_h \times D_r^-$. For integers $p \geq 3$ we can apply Proposition 2.4.2 to each component of equation (2.44) and conclude that there is a vector function $\mathbf{c} = (c_1, c_2, c_3, c_4) : S_h \times D_r^- \to \mathbf{C}^4$ such that each c_i is an analytic function in $S_h \times D_r^-$, continuous on the closure of its domain and 2π-periodic in φ. Moreover (2.23) yields,

$$|c_1(\varphi,\tau)| \leq \frac{K_{p-1} K_{\mathbf{U}^{-1}} \|f\|_{p+1}}{|\tau|^{p-1}}, \qquad |c_2(\varphi,\tau)| \leq \frac{K_{p+3} K_{\mathbf{U}^{-1}} \|f\|_{p+1}}{|\tau|^{p+3}},$$
$$|c_3(\varphi,\tau)| \leq \frac{K_{p+2} K_{\mathbf{U}^{-1}} \|f\|_{p+1}}{|\tau|^{p+2}}, \qquad |c_4(\varphi,\tau)| \leq \frac{K_{p-2} K_{\mathbf{U}^{-1}} \|f\|_{p+1}}{|\tau|^{p-2}}.$$

Finally, we define the linear operator $\mathcal{L}^{-1}$ as follows

$$\mathcal{L}^{-1}(f) = \xi,$$

where the vector function ξ is obtain through the relation $\xi = \mathbf{U}\mathbf{c}$. If ξ_i denote the components of ξ then the ξ_i can be bounded in $S_h \times D_r^-$ in the following way,

$$|\xi_1(\varphi,\tau)| \leq \frac{\bar{K}}{|\tau|^{p+1}} \|f\|_{p+1}, \qquad |\xi_2(\varphi,\tau)| \leq \frac{\bar{K}}{|\tau|^{p+1}} \|f\|_{p+1},$$
$$|\xi_3(\varphi,\tau)| \leq \frac{\bar{K}}{|\tau|^{p}} \|f\|_{p+1}, \qquad |\xi_4(\varphi,\tau)| \leq \frac{\bar{K}}{|\tau|^{p}} \|f\|_{p+1},$$

where $\bar{K} = (K_{p-1} + K_{p+3} + K_{p+2} + K_{p-2})K_{\mathbf{U}}K_{\mathbf{U}^{-1}}$. Consequently $\|\xi\|_n \leq \bar{K}\,\|f\|_{p+1}$ yielding $\|\mathcal{L}^{-1}\|_{n,n+1} \leq \bar{K}$. Thus $\mathcal{L}^{-1}$ is bounded and the uniqueness follows from the triviality of the kernel of $\mathcal{L}$. This completes the proof of the Theorem. □

Remark 2.4.1.1. It is clear that we can repeat the same arguments of the previous proof *mutatis mutandis* to the case when all the functions (including $\mathbf{U}$ and A) are analytic in $S_h \times D_r^+$. As the proof is completely equivalent we omit the details.

Theorem 2.4.2. *Let $p \geq 3$, $r > \frac{\pi \tan\theta_0}{1-\tan\theta_0}$ and suppose that the linear operator $\mathcal{L}$: $\mathfrak{X}_p(S_h \times D_r^1) \to \mathfrak{X}_p(S_h \times D_r^1)$ acting by the formula* (2.32) *has a fundamental matrix* $\mathbf{U}$. *Then the kernel of $\mathcal{L}$ consists of functions of the form*

$$\mathbf{U}(\varphi,\tau)\mathbf{c}(\tau-\varphi)$$

where $\mathbf{c} : \{s \in \mathbb{C} : \mathrm{Im}(s) < -r + h\} \to \mathbb{C}^4$ is an analytic vector function which is 2π-periodic, continuous on the closure of its domain and,

$$\lim_{\mathrm{Im}s \to -\infty} \mathbf{c}(s) = 0.$$

Moreover, there exists a bounded linear operator $\mathcal{L}^{-1} : \mathfrak{X}_{p+3}(S_h \times D_r^1) \to \mathfrak{X}_p(S_h \times D_r^1)$ such that $\mathcal{L}\mathcal{L}^{-1} = \mathrm{Id}$.

Proof. The proof of this theorem is almost identical to the previous one except that the functions are now defined in a different domain $S_h \times D_r^1$. As before, if $\xi \in \mathfrak{X}_p(S_h \times D_r^1)$ such that $\mathcal{L}(\xi) = 0$ then, by the definition of the operator $\mathcal{L}$, the function ξ must satisfy,

$$D\xi = A\xi.$$

Again, we use the method of variation of constants and write $\xi = \mathbf{U}\mathbf{c}$ for some vector function $\mathbf{c} : S_h \times D_r^1 \to \mathbb{C}^4$. Hence, $\mathbf{c}$ must satisfy the equation $D\mathbf{c} = 0$. Applying Proposition 2.4.1 to each component of the vector function $\mathbf{c}$, we conclude that $\mathbf{c}(\varphi,\tau) = \mathbf{c}_0(\tau-\varphi)$ where $\mathbf{c}_0 : \bigcup_{\tau_0 \in D_r^1} \tau_0 + S_h \to \mathbb{C}^4$ is an analytic, 2π-periodic vector function. Note that $\bigcup_{\tau_0 \in D_r^1} \tau_0 + S_h$ is equal to the half plane $\{s \in \mathbb{C} : \operatorname{Im}(s) < -r + h\}$. Moreover, since $\mathbf{c}_0 = \mathbf{U}^{-1}\xi$ we conclude as in the proof of the previous Theorem that

$$\mathbf{c}_0 = \left(\tau^{-p+1} \sum_{i=1}^{4} \breve{u}_{1,j}\xi_i, \tau^{-p-3} \sum_{i=1}^{4} \breve{u}_{2,j}\xi_i, \tau^{-p-2} \sum_{i=1}^{4} \breve{u}_{3,j}\xi_i, \tau^{-p+2} \sum_{i=1}^{4} \breve{u}_{4,j}\xi_i \right), \quad (2.45)$$

where $\breve{u}_{i,j}$ are the entries of the inverse matrix $\mathbf{U}^{-1}$ (see (2.40)) and

$$\xi = (\tau^{-p-1}\xi_1, \tau^{-p-1}\xi_2, \tau^{-p}\xi_3, \tau^{-p}\xi_4),$$

such that

$$\max_{i=1,\ldots,4} \sup_{(\varphi,\tau) \in S_h \times D_r^1} |\xi_i(\varphi, \tau)| < \infty.$$

Taking into account these observations and (2.41) we conclude that,

$$\lim_{\operatorname{Im}(s) \to -\infty} \mathbf{c}_0(s) = 0,$$

which proves the first part of the Theorem. For the second part, let us construct an inverse of $\mathcal{L}$ by solving the following equation,

$$\mathcal{L}(\xi) = f, \quad (2.46)$$

where $f \in \mathfrak{X}_{p+3}(S_h \times D_r^1)$. Again, we look for a solution of (2.46) using the method of variation of constants. Let $\xi = \mathbf{U}\mathbf{c}$. As in the proof of the previous Theorem, the equation (2.46) is equivalent to,

$$D\mathbf{c} = \mathbf{U}^{-1} f. \quad (2.47)$$

Writing $f = (\tau^{-p-4} f_1, \tau^{-p-4} f_2, \tau^{-p-3} f_3, \tau^{-p-3} f_4)$ and taking into account (2.40) we can write the right hand side of equation (2.47) as follows $\mathbf{U}^{-1} f = (g_1, g_2, g_3, g_4)^T$ where,

$$g_1 = \tau^{-p-2} \sum_{i=1}^{4} \breve{u}_{1,j} f_i, \qquad g_2 = \tau^{-p-6} \sum_{i=1}^{4} \breve{u}_{2,j} f_i,$$
$$g_3 = \tau^{-p-5} \sum_{i=1}^{4} \breve{u}_{3,j} f_i, \qquad g_4 = \tau^{-p-1} \sum_{i=1}^{4} \breve{u}_{4,j} f_i.$$

Now bearing in mind that $\|f\|_{p+3} < \infty$ and (2.41) we can estimate the previous functions as follows,

$$|g_1(\varphi,\tau)| \le \frac{K_{\mathbf{U}^{-1}} \|f\|_{p+3}}{|\tau|^{p+2}}, \qquad |g_2(\varphi,\tau)| \le \frac{K_{\mathbf{U}^{-1}} \|f\|_{p+3}}{|\tau|^{p+6}},$$
$$|g_3(\varphi,\tau)| \le \frac{K_{\mathbf{U}^{-1}} \|f\|_{p+3}}{|\tau|^{p+5}}, \qquad |g_4(\varphi,\tau)| \le \frac{K_{\mathbf{U}^{-1}} \|f\|_{p+3}}{|\tau|^{p+1}}.$$

where the upper bounds are now valid in $S_h \times D_r^1$. Since $r > \frac{\pi \tan\theta_0}{1-\tan\theta_0}$ then for $p \ge 3$ we can apply Proposition 2.4.3 with $\epsilon = 0$ to each component of equation (2.47) and conclude that there is a vector function $\mathbf{c} = (c_1, c_2, c_3, c_4) : S_h \times D_r^1 \to \mathbf{C}^4$ such that each c_i is an analytic function in $S_h \times D_r^1$, continuous on the closure of its domain and 2π-periodic in φ. Moreover (2.28) yields,

$$|c_1(\varphi,\tau)| \le \frac{4K_{p-1}K_{\mathbf{U}^{-1}} \|f\|_{p+3}}{r^2 |\tau|^{p-1}}, \qquad |c_2(\varphi,\tau)| \le \frac{4K_{p+3}K_{\mathbf{U}^{-1}} \|f\|_{p+3}}{r^2 |\tau|^{p+3}},$$
$$|c_3(\varphi,\tau)| \le \frac{4K_{p+2}K_{\mathbf{U}^{-1}} \|f\|_{p+3}}{r^2 |\tau|^{p+2}}, \qquad |c_4(\varphi,\tau)| \le \frac{4K_{p-2}K_{\mathbf{U}^{-1}} \|f\|_{p+3}}{r^2 |\tau|^{p-2}}.$$

Finally, as in the proof of the previous Theorem, we define the linear operator $\mathcal{L}^{-1}$ as follows

$$\mathcal{L}^{-1}(f) = \xi,$$

where the vector function ξ is obtain through the relation $\xi = \mathbf{U}\mathbf{c}$. If ξ_i denote the components of ξ then the ξ_i can be bounded in $S_h \times D_r^1$ in the following way,

$$|\xi_1(\varphi,\tau)| \le \frac{\bar{K}}{|\tau|^{p+1}} \|f\|_{p+3}, \qquad |\xi_2(\varphi,\tau)| \le \frac{\bar{K}}{|\tau|^{p+1}} \|f\|_{p+3},$$
$$|\xi_3(\varphi,\tau)| \le \frac{\bar{K}}{|\tau|^{p}} \|f\|_{p+3}, \qquad |\xi_4(\varphi,\tau)| \le \frac{\bar{K}}{|\tau|^{p}} \|f\|_{p+3},$$

where $\bar{K} = \frac{4}{r^2}(K_{p-1}+K_{p+3}+K_{p+2}+K_{p-2})K_{\mathbf{U}}K_{\mathbf{U}^{-1}}$. Consequently $\|\xi\|_n \le \bar{K}\|f\|_{p+3}$ yielding $\|\mathcal{L}^{-1}\|_{n,n+3} \le \bar{K}$. Thus $\mathcal{L}^{-1}$ is bounded. This completes the proof of the Theorem. $\square$

Theorem 2.4.3. *Let $p \in \mathbb{Z}$, $0 < \mu < 2$ and $r > \max\left\{1, \frac{\pi\tan\theta_0}{1-\tan\theta_0}\right\}$. Suppose that the linear operator $\mathcal{L} : \mathfrak{X}_p(S_h \times D_r^1) \to \mathfrak{X}_p(S_h \times D_r^1)$ acting by the formula* (2.32) *has a* *fundamental matrix* $\mathbf{U}$*. Then for any* $\mu' > 0$ *such that* $\mu' < \mu$ *there exists a bounded linear operator* $\mathcal{L}_{\mu'}^{-1} : \mathfrak{Y}_{\mu'}(S_h \times D_r^1) \to \mathfrak{Y}_{\mu}(S_h \times D_r^1)$ *such that* $\mathcal{L}\mathcal{L}_{\mu'}^{-1} = \mathrm{Id}$*.*

Proof. Let $\mu' > 0$ such that $\mu' < \mu$ and let us obtain an inverse of $\mathcal{L}$ by solving the following equation,

$$\mathcal{L}(\xi) = f, \tag{2.48}$$

where $f \in \mathfrak{Y}_{\mu'}(S_h \times D_r^1) \subset \mathfrak{X}_p(S_h \times D_r^1)$ for any $p \in \mathbb{Z}$. Again, we look for a solution of (2.48) using the method of variation of constants. Let $\xi = \mathbf{U}\mathbf{c}$. As before, the equation (2.48) is equivalent to,

$$D\mathbf{c} = \mathbf{U}^{-1} f. \tag{2.49}$$

Let $f(\varphi, \tau) = e^{-(2-\mu')i(\tau-\varphi)} \tilde{f}(\varphi, \tau)$ where $\tilde{f}$ is a bounded function. Taking into account that $r > 1$ and (2.40) we can bounded the components of $\mathbf{U}^{-1} f = (g_1, g_2, g_3, g_4)^T$ as follows,

$$|g_i(\varphi, \tau)| \leq \sup_{(\varphi,\tau) \in S_h \times D_r^1} \left| \tau^9 e^{-(\mu-\mu')i(\tau-\varphi)} \right| \frac{\|f\|_{\mu'} K_{\mathbf{U}^{-1}}}{\left| \tau^6 e^{(2-\mu)i(\tau-\varphi)} \right|}, \quad i = 1, \ldots, 4,$$

valid in $S_h \times D_r^1$. Note that the supremum in the previous estimate is finite since $\mu - \mu' > 0$. Now bearing in mind that $r > \frac{\pi \tan \theta_0}{1 - \tan \theta_0}$ we can apply Proposition 2.4.3 with $\epsilon = 2 - \mu$ and $p = 6$ to each component of equation (2.49) and conclude that there is a vector function $\mathbf{c} = (c_1, c_2, c_3, c_4) : S_h \times D_r^1 \to \mathbb{C}^4$ such that each c_i is an analytic function in $S_h \times D_r^1$, continuous on the closure of its domain and 2π-periodic in φ. Moreover (2.28) yields,

$$|c_i(\varphi, \tau)| \leq \frac{K_{\mathbf{c}}}{\left| \tau^3 e^{(2-\mu)i(\tau-\varphi)} \right|} \|f\|_{\mu'}, \quad i = 1, \ldots, 4, \tag{2.50}$$

where,

$$K_{\mathbf{c}} = \frac{4 \sup_{(\varphi,\tau) \in S_h \times D_r^1} \left| \tau^9 e^{-(\mu-\mu')i(\tau-\varphi)} \right| K_{\mathbf{U}^{-1}} K_3}{r^2}.$$

Finally, we define the linear operator $\mathcal{L}_{\mu'}^{-1}$ as follows

$$\mathcal{L}_{\mu'}^{-1}(f) = \xi,$$

where the vector function ξ is obtained through the relation $\xi = \mathbf{Uc}$. If ξ_i denote the components of ξ then taking into account (2.50) the ξ_i can be bounded in $S_h \times D_r^1$ in the following way,

$$|\xi_i(\varphi, \tau)| \leq \frac{4K_{\mathbf{U}}K_{\mathbf{c}}}{\left|e^{(2-\mu)i(\tau-\varphi)}\right|} \left\|f\right\|_{\mu'}, \quad i = 1, \ldots, 4,$$

where $K_{\mathbf{U}}$ is defined in (2.34). Consequently $\left\|\xi\right\|_{\mu} \leq 16K_{\mathbf{U}}K_{\mathbf{c}} \left\|f\right\|_{\mu'}$ yielding $\left\|\mathcal{L}_{\mu'}^{-1}\right\|_{\mu,\mu'} \leq 16K_{\mathbf{U}}K_{\mathbf{c}}$. Thus $\mathcal{L}_{\mu'}^{-1}$ is bounded which completes the proof of the Theorem. $\square$

Chapter 3

Inner Problem

In this chapter we study the Hamiltonian H_ϵ at the exact moment of bifurcation, i.e., for $\epsilon = 0$. We will show that the equilibrium point has a stable (resp. unstable) analytic complex manifold W_0^s (resp. W_0^u) which are obtained using a parametrisation method. Their parametrisations are defined in certain domains of $\mathbb{C}^2$ and have the same asymptotic expansion valid in a common domain of intersection. Hence their distance is beyond all algebraic orders. We prove an exponentially small upper bound for their distance. In the four dimensional space $\mathbb{C}^4$ the distance of these manifolds can be locally described by two quantities. Furthermore, since the manifolds lie inside the zero energy level of the Hamiltonian it implies that their distance can be described by a single number, which we call the Stokes constant. This is closely related to the Stokes phenomena, where the same asymptotic expansion describes two different solutions in a common region.

3.1 Introduction

Consider a two degrees of freedom Hamiltonian system where the Hamiltonian H is supposed to be analytic in a complex neighbourhood $\mathcal{U} \subseteq \mathbb{C}^4$ of the origin and continuous on its closure. We suppose that the Hamiltonian vector field X_H has an equilibrium point which we assume to be at the origin. Moreover, we assume that the linear part of the Hamiltonian vector field is not diagonalizable and has a pair of pure imaginary eigenvalues $\pm\alpha_0 i$ ($\alpha_0 > 0$) having multiplicity two. The well known normal form theory for quadratic Hamiltonians [11] implies that exists a linear symplectic change of variables that transforms the Hamiltonian H to the form,

$$H = -\alpha_0 \left(q_2 p_1 - q_1 p_2\right) + \frac{\iota}{2}\left(q_1^2 + q_2^2\right) + \text{high order terms},$$

where $\iota = \pm 1$. Without lost of generality we can assume that $\alpha_0 = 1$ and $\iota = 1$ (see (2.6)) and by a canonical change of coordinates we can suppose that H is in the general form,

$$\begin{aligned} H = H^0 + F, \quad &\text{where} \quad H^0 = -I_1 + I_2 + \eta I_3^2, \\ \text{and} \quad I_1 = q_2 p_1 - q_1 p_2, \quad &I_2 = \frac{q_1^2 + q_2^2}{2}, \quad I_3 = \frac{p_1^2 + p_2^2}{2}, \end{aligned} \tag{3.1}$$

where $\eta \in \mathbb{C}$ and $F : \mathcal{U} \to \mathbb{C}^4$ is an analytic function such that $F(q,p) = O((|q|^{\frac{1}{2}} + |p|)^5)$ where $q = (q_1, q_2)$ and $p = (p_1, p_2)$. In the following, we will consider the non-degenerate case which corresponds to,

$$\eta \neq 0. \tag{3.2}$$

It is well known that Hamiltonian (3.1) can be normalized up to a given order (see chapter 2 on the normal form). There is a formal near identity canonical change of coordinates Φ that transforms H into the following,

$$H^{NF} = H \circ \Phi = -I_1 + I_2 + \eta I_3^2 + \sum_{3l+2k \geq 5} a_{l,k} I_1^l I_3^k, \tag{3.3}$$

where the coefficients $a_{l,k} \in \mathbb{C}$. Note that, if the series (3.3) converge then since I_1 is an integral of motion it would imply that H^{NF} is integrable. The results of this chapter imply that generically the normal form transformation diverge, hence in general the Hamiltonian H is non-integrable.

Although the equilibrium point is of elliptic type, we will show the existence of a stable (resp. unstable) analytic invariant manifold immersed in $\mathbb{C}^4$ such that points on this invariant manifold converge to the equilibrium forward (resp. backward) in time under the flow. Moreover, as one might expect, the rate of convergence is of polynomial type.

Let $\mathbf{x} = (q, p) \in \mathbb{C}^4$. In the study of the invariant manifolds, we shall look for natural parametrizations (see section 2.3 of chapter 2) as solutions of the following PDE,

$$\mathcal{D}\mathbf{x} = X_H(\mathbf{x}), \qquad \text{where} \quad \mathcal{D} = \partial_\varphi + \partial_\tau. \tag{3.4}$$

Note that equation (3.4) is obtained from equation (2.19) by setting $\epsilon = 0$.

We will show that there is a stable parametrisation $\mathbf{\Gamma}^-$ and an unstable parametrisation $\mathbf{\Gamma}^+$ satisfying equation (3.4) which are defined in certain domains of $\mathbb{C}^2$ having the same asymptotic expansion valid in a common domain of intersection. Therefore their distance is beyond all algebraic orders. In addition, we will prove an exponentially small upper bound for their distance.

3.2 Formal Series

The results in this section are of formal character, therefore we do not care about the convergence of the power series involved. Let $\mathsf{T}_\mathbb{K}$ denote the space of trigonometric polynomials where $\mathbb{K} = \mathbb{C}, \mathbb{R}$, i.e., the space of functions of the form,

$$a_0 + \sum_{k=1}^{n} a_k \cos(k\varphi) + \sum_{k=1}^{n} b_k \sin(k\varphi), \qquad a_k, b_k \in \mathbb{K}, \qquad n \in \mathbb{N}_0.$$

The function $\deg_{\mathsf{T}_\mathbb{K}} : \mathsf{T}_\mathbb{K} \to \mathbb{N}_0$ stands for the usual definition of the degree.

In this section, we will look for formal solutions of equation (3.4) in the class of formal power series in the variable τ^{-1} with coefficients in $\mathsf{T}_\mathbb{C}$. It is convenient to transform H into its normal form and compute a formal solution in the normal form coordinates. Then using the normal form transformation we return to the original coordinates.

Note that the normal form (3.3) is rotationally symmetric, i.e., it commutes with the one parameter group of rotations R_φ,

$$R_\varphi = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 & 0 \\ 0 & 0 & \cos(\varphi) & -\sin(\varphi) \\ 0 & 0 & \sin(\varphi) & \cos(\varphi) \end{pmatrix}.$$

In the following we look for formal solutions of the PDE,

$$\mathcal{D}\mathbf{z} = X_{H^{NF}}(\mathbf{z}), \tag{3.5}$$

in the class of formal power series $\tau^{-1}\mathsf{T}^4_{\mathbb{C}}[[\tau^{-1}]]$. We have the following,

Theorem 3.2.1. *Equation* (3.5) *has a formal solution* $\hat{\mathbf{Z}} \in \tau^{-1}\mathsf{T}^4_{\mathbb{C}}[[\tau^{-1}]]$ *having the form,*

$$\hat{\mathbf{Z}}(\varphi,\tau) = R_\varphi \left(\psi_1(\tau), \phi_1(\tau), \phi_2(\tau), \psi_2(\tau)\right)^T,$$

where for $i = 1, 2$, $\psi_i, \phi_i \in \tau^{-1}\mathbb{C}[[\tau^{-1}]]$ *and* ψ_i *are even formal series and* ϕ_i *are odd formal series and having the leading orders,*

$$\begin{aligned} \psi_1(\tau) &= \kappa\tau^{-2} + \cdots, \quad & \phi_1(\tau) &= \frac{\kappa^3 a_{1,1}}{2}\tau^{-3} + \cdots, \\ \phi_2(\tau) &= \kappa\tau^{-1} + \cdots, \quad & \psi_2(\tau) &= \frac{\kappa^3 a_{1,1}}{2}\tau^{-2} + \cdots. \end{aligned}$$

where $\kappa^2 = -\frac{2}{\eta}$. *The formal solution* $\hat{\mathbf{Z}}$ *is unique modulus a rotation* R_π, *i.e.,* $\hat{\mathbf{Z}}$ *and* $R_\pi\hat{\mathbf{Z}}$ *are the only formal solutions satisfying the properties stated above. Moreover, for any other formal solution* $\hat{\hat{\mathbf{Z}}} \in \tau^{-1}\mathsf{T}^4_{\mathbb{C}}[[\tau^{-1}]]$ *there exist* $(\varphi_0, \tau_0) \in \mathbb{C}^2$ *such that* $\hat{\hat{\mathbf{Z}}}(\varphi,\tau) = \hat{\mathbf{Z}}(\varphi+\varphi_0, \tau+\tau_0)$.

Proof. Let us look for a formal solution of equation (3.5) in the form $\hat{\mathbf{Z}}(\varphi,\tau) = R_\varphi\hat{\xi}(\tau)$ where $\hat{\xi} \in \tau^{-1}\mathbb{C}^4[[\tau^{-1}]]$. Taking into account that the Hamiltonian vector field commutes with R_φ which has infinitesimal generator $-X_{I_1}$, then we get the following equivalent equation,

$$\partial_\tau\hat{\xi} = X_{H^{NF}+I_1}(\hat{\xi}). \tag{3.6}$$

Now, it is convenient to change to polar coordinates given by,

$$\xi^1 = R\cos\theta - \tfrac{\Theta}{r}\sin\theta, \quad \xi^3 = r\cos\theta,$$
$$\xi^2 = R\sin\theta + \tfrac{\Theta}{r}\cos\theta, \quad \xi^4 = r\sin\theta, \tag{3.7}$$

where $\hat{\xi} = (\xi^1, \xi^2, \xi^3, \xi^4)$. Note that the integral I_1 is equal to Θ. In these new variables the equation (3.6) takes the form,

$$\partial_\tau \theta = -\frac{\Theta}{r^2} - \sum_{3i+2j\geq 5} \frac{i a_{i,j}}{2^j}\Theta^{i-1} r^{2j}, \quad \partial_\tau r = -R, \quad \partial_\tau \Theta = 0, \tag{3.8}$$

$$\partial_\tau R = \left(-\frac{\Theta^2}{r^3} + \eta r^3\right) + \sum_{3i+2j\geq 5} \frac{2j a_{i,j}}{2^j}\Theta^i r^{2j-1}. \tag{3.9}$$

Let us start with the third equation of (3.8). It follows that $\Theta(\tau) = \Theta_0$ where $\Theta_0 \in \mathbb{C}$. Taking into account that $\Theta \in \tau^{-2}\mathbb{C}[[\tau^{-1}]]$ we conclude that $\Theta_0 = 0$. Hence $\Theta = 0$.

We move on and consider now the second equation of (3.8) and equation (3.9). Taking into account that $\Theta = 0$, these two equations are equivalent to the following single equation,

$$\partial_\tau^2 r = -\eta r^3 - \sum_{j\geq 2} \frac{2(j+1)a_{0,j+1}}{2^{j+1}} r^{2j+1}. \tag{3.10}$$

In the following we construct a formal solution of (3.10) belonging to $\tau^{-1}\mathbb{C}[[\tau^{-1}]]$.

Claim 3.2.1.1. Equation (3.10) has a non-zero formal solution $r \in \tau^{-1}\mathbb{C}[[\tau^{-1}]]$ having only odd powers of τ^{-1}. Moreover,

$$r(\tau) = \kappa\tau^{-1} - \frac{1}{8}a_{0,3}\kappa^5\tau^{-3} + \cdots . \tag{3.11}$$

where $\kappa^2 = -\frac{2}{\eta}$. This solution is unique if we fix one of the two values for κ. Moreover, for any other non-zero formal solution $\tilde{r} \in \tau^{-1}\mathbb{C}[[\tau^{-1}]]$ of equation (3.10) there exists τ_0 such that $\tilde{r}(\tau) = \pm r(\tau + \tau_0)$.

Proof. Let us take a formal series $r(\tau) = \sum_{k\geq 1} r_k \tau^{-k}$ and substitute into equation (3.10). After collecting terms of the same order in τ^{-k-2} we obtain an equation which we can solve for the coefficient r_k. Let us present the details. At order τ^{-3} we get the following equation for r_1,

$$2r_1 = -\eta r_1^3,$$

which implies that $r_1^2 = -\frac{2}{\eta}$ (the other solution is trivialy r_1 which leads to the zero formal solution $r = 0$). Hence we let $r_1 := \kappa$ where κ is defined by the relation $\kappa^2 = -\frac{2}{\eta}$. Note that κ can take to distinct values, i.e., $-\sqrt{-\frac{2}{\eta}}$ and $\sqrt{-\frac{2}{\eta}}$. Let us move to the next order. At order τ^{-4} we obtain,

$$6r_2 = -3\eta r_1^2 r_2.$$

Note that this equation is linear with respect to r_2. Taking into account that $r_1 = \kappa$ we can simplify the previous equation and conclude that it holds for every $r_2 \in \mathbb{C}$. Hence r_2 is a free coefficient. Since we are considering only odd powers of r we set this coefficient to zero.

At this stage, we have determined $r_1 = \kappa$ and $r_2 = 0$. Now we proceed by induction on k. First let us determine r_3. It is not difficult to write the equation for r_3 which reads,

$$6r_3 = -\frac{6}{8}a_{0,3}r_1^5.$$

Thus $r_3 = -\frac{1}{8}a_{0,3}\kappa^5$. Now suppose that all coefficients r_l, $3 \leq l \leq k$ have been defined uniquely such that for l even we have $r_l = 0$ and for l odd we have $r_l = p(\kappa)$ where $p \in \mathbb{C}[\kappa]$ and contains only odd powers in κ. Due to the induction hypothesis, at the order τ^{-k-3} we have the following equation for r_{k+1},

$$((k+1)(k+2) - 6)r_{k+1} = f_{k+1}(r_1, \ldots, r_k)$$

where f_{k+1} is a polynomial depending on a finite number of coefficients $a_{0,j+1}$ for $j \geq 2$. Note that it is always possible to solve the previous equation with respect to r_{k+1} for $k \geq 2$ since $(k+1)(k+2) - 6 = 0$ only if $k = 1$ or $k = -4$. Now we have to distinguish two cases. First consider the case when $k+1$ is even. Since the right hand side of equation (3.10) has only odd powers of r and according to the induction hypothesis

$r_l = 0$ for even l then $f_{k+1} = 0$. Thus $r_{k+1} = 0$. On the other hand, when $k+1$ is odd then by the same reasoning as above it is not difficult to see that f_{k+1} is a polynomial in $\mathbb{C}[\kappa]$, having only odd powers of κ, and r_{k+1} is determined uniquely by the formula $r_{k+1} = ((k+1)(k+2)-6)^{-1} f_{k+1}$. This completes the induction. Finally let $\tilde{r} \in \tau^{-1}\mathbb{C}[[\tau^{-1}]]$ be a another non zero formal solution of equation (3.10). We can write $\tilde{r} = \sum_{k\geq 1} \tilde{r}_k \tau^{-k}$. As before, we conclude that $\tilde{r}_1^2 = \kappa^2$ thus, $\tilde{r}_1 = \pm\kappa$. Now for $\tau_0 \in \mathbb{C}$ we have that,

$$r(\tau+\tau_0) = \frac{\kappa}{\tau+\tau_0} + \cdots = \frac{\kappa}{\tau} - \frac{\tau_0\kappa}{\tau^2} + \cdots .$$

is also a formal solution of equation (3.10). Comparing the second order coefficient $-\tau_0\kappa$ with the coefficient $\tilde{r}_2$ we conclude by the uniqueness of r that if $\tau_0 = -\frac{\tilde{r}_2}{\kappa}$ then $\tilde{r}(\tau) = \pm r(\tau+\tau_0)$ and the claim is proved. $\square$

As a direct consequence of the previous Claim and taking into account the second equation of (3.8) we conclude that $R = -\partial_\tau r$, hence $R \in \tau^{-2}\mathbb{C}[[\tau^{-1}]]$ containing only even powers in τ^{-1}. Moreover,

$$R(\tau) = \kappa\tau^{-2} + \cdots$$

Finally, using the known formal solutions Θ and r we simplify the first equation of (3.8) and obtain,

$$\partial_\tau\theta = -\sum_{j\geq 1} \frac{a_{1,j}}{2^j} \left(\sum_{k\geq 1} r_k\tau^{-k}\right)^{2j} . \tag{3.12}$$

Note that $\left(\sum_{k\geq 1} r_k\tau^{-k}\right)^{2j} \in \tau^{-2j}\mathbb{C}[[\tau^{-1}]]$ and contains only even powers in τ^{-1}. Since $\kappa^2 = -\frac{2}{\eta}$ then the right hand side is independent of the choice of κ. Hence, equation (3.12) can be simplified to give,

$$\partial_\tau\theta = \sum_{k\geq 1} b_k\tau^{-2k},$$

where $b_k \in \mathbb{C}$ and depend on a finite number of coefficients of r and $a_{1,j}$ for $j \geq 1$. For this equation a general formal solution has the form,

$$\theta(\tau) = \theta_0 + \sum_{k\geq 1} \theta_k \tau^{-2k+1},$$

where $\theta_k \in \mathbb{C}$. Let $b(\tau) := \sum_{k\geq 1} \theta_k \tau^{-2k+1}$. Note that,

$$b(\tau) = -\frac{a_{1,1}}{\eta}\tau^{-1} + \cdots .$$

At this point let us rewrite the formal solutions in the following form,

$$\begin{aligned} &\theta(\tau) = \theta_0 + b(\tau), \quad \Theta(\tau) = 0, \\ &r(\tau) = \sum_{k\geq 1} r_k \tau^{-2k+1}, \quad R(\tau) = \sum_{k\geq 1} R_k \tau^{-2k}. \end{aligned} \tag{3.13}$$

In order to conclude the proof of the Theorem, let us come back to the variable $\hat{\xi}$. First observe that,

$$\cos b(\tau) = \sum_{i\geq 0} \frac{(-1)^i}{(2i)!} \left(\sum_{k\geq 1} \theta_k \tau^{-2k+1} \right)^{2i},$$

and taking into account that the formal series inside the parenthesis of the right hand side of the previous formula is an even formal series in τ^{-1} starting with the term τ^{-2i} we conclude that,

$$\cos b(\tau) = \sum_{k\geq 0} w_k \tau^{-2k}, \tag{3.14}$$

for some $w_k \in \mathbb{C}$ depending on a finite number of coefficients θ_k for $k \geq 1$. A similar formula holds for the sine function which reads,

$$\sin b(\tau) = \sum_{k\geq 0} z_k \tau^{-2k+1}, \tag{3.15}$$

for some $z_k \in \mathbb{C}$ depending on a finite number of coefficients θ_k for $k \geq 1$. Now according to the change of variables (3.7) let us define,

$$\begin{aligned} \phi^1(\tau) &= R(\tau)\cos b(\tau), & \psi^2(\tau) &= r(\tau)\cos b(\tau), \\ \psi^1(\tau) &= R(\tau)\sin b(\tau), & \phi^2(\tau) &= r(\tau)\sin b(\tau). \end{aligned}$$

Thus,

$$\hat{\xi}(\tau) = R_{\theta_0}\left(\phi^1(\tau), \psi^1(\tau), \psi^2(\tau), \phi^2(\tau)\right)^T,$$

is a formal solution of equation (3.6). Taking into account the formulae (3.13), (3.14) and (3.15) we conclude that the formal series ψ^1, ϕ^1, ϕ^2 and ψ^2 satisfy the required properties stated in the Theorem. Thus, θ_0 must be equal to 0 or π and from the definition of κ we conclude that $\hat{\xi}$ is uniquely defined up to a rotation R_π. This completes the proof of the Theorem. □

Remark 3.2.1.1. If the original Hamiltonian H is real analytic then its normal form H^{NF} is a formal series with real coefficients, i.e., $\overline{H^{NF}(\mathbf{z})} = H^{NF}(\overline{\mathbf{z}})$ and in particular, the coefficient η is real.

Depending on the sign of η we can say more about the coefficients of the formal solutions. If $\eta < 0$ (which corresponds to the unstable case) then one can trace the proofs of the previous Theorem (in particular the proof of Claim 3.2.1.1) and conclude that the coefficients of $\hat{\mathbf{Z}}$ are real, i.e., $\hat{\mathbf{Z}} = R_\varphi \xi$ where $\xi \in \tau^{-1}\mathbb{R}^4[[\tau^{-1}]]$. Thus, $\hat{\mathbf{Z}}(\varphi,\tau) = \overline{\hat{\mathbf{Z}}(\bar{\varphi},\bar{\tau})}$ when $\eta < 0$. On the other hand, when $\eta > 0$ (which is the stable case) then the coefficients of $\hat{\mathbf{Z}}$ are pure imaginary numbers, i.e., $\hat{\mathbf{Z}} = iR_\varphi\xi$ where $\xi \in \tau^{-1}\mathbb{R}^4[[\tau^{-1}]]$. Thus, $\hat{\mathbf{Z}}(\varphi,\tau) = \overline{\hat{\mathbf{Z}}(\bar{\varphi}+\pi,\bar{\tau})}$ when $\eta > 0$.

Remark 3.2.1.2. If the original Hamiltonian H is real analytic then taking into account that the normal form vector field $X_{H^{NF}}$ is reversible with respect to the linear involution,

$$\mathcal{S}(q_1, q_2, p_1, p_2) = (-q_1, q_2, p_1, -p_2), \tag{3.16}$$

it is not difficult to see that the conditions of the previous Theorem on the formal series ψ_i and ϕ_i are equivalent to the following condition,

$$\hat{\mathbf{Z}}(\varphi,\tau) = \overline{\mathcal{S}(\hat{\mathbf{Z}}(-\bar{\varphi},-\bar{\tau}))}.$$

This condition defines the formal solution $\hat{\mathbf{Z}}$ uniquely (up to a rotation R_π) and independently from any coordinate system.

Theorem 3.2.2. *Equation* (3.4) *has a non zero formal solution* $\hat{\mathbf{\Gamma}}$ *belonging to the class* $\tau^{-1}\mathsf{T}^4_{\mathbb{C}}\left[\left[\tau^{-1}\right]\right]$ *and having the form,*

$$\hat{\mathbf{\Gamma}} = \left(\tau^{-2}\hat{\Gamma}_1, \tau^{-2}\hat{\Gamma}_2, \tau^{-1}\hat{\Gamma}_3, \tau^{-1}\hat{\Gamma}_4\right)^T,$$

where $\hat{\Gamma}_i = \sum_{k\geq 0}\Gamma^i_k\tau^{-k},\ i = 1,\dots,4,$ *such that* $\Gamma^i_k \in \mathsf{T}_{\mathbb{C}}$ *with* $\deg_{\mathsf{T}_{\mathbb{C}}}(\Gamma^i_k) = k+2$ *for* $i = 1,2$ *and* $\deg_{\mathsf{T}_{\mathbb{C}}}(\Gamma^i_k) = k+1$ *for* $i = 3,4$. *Moreover, for any other non zero formal solution* $\hat{\hat{\mathbf{\Gamma}}}$ *of* (3.4) *belonging to the same class there exist* $(\varphi_0, \tau_0) \in \mathbb{C}^2$ *such that* $\hat{\hat{\mathbf{\Gamma}}}(\varphi,\tau) = \hat{\mathbf{\Gamma}}(\varphi+\varphi_0, \tau+\tau_0)$.

Proof. By the normal form theory there exists (non unique) a near identity formal canonical change of variables $\mathbf{x} = \Phi(\mathbf{z})$ which transforms the Hamiltonian H into its normal form H^{NF} by the relation $H^{NF} = H \circ \Phi$. For our purposes, we can suppose that the transformation Φ is a formal power series having the general form,

$$\begin{aligned} q &= Q + \sum_{2|i|+|j|\geq 3}\phi^1_{i,j}Q^iP^j, \\ p &= P + \sum_{2|i|+|j|\geq 4}\phi^2_{i,j}Q^iP^j, \end{aligned} \tag{3.17}$$

written in multi-index notation, where $\phi^1_{i,j}, \phi^2_{i,j} \in \mathbb{C}^2$. Now the previous Theorem provides a formal solution $\hat{\mathbf{Z}}$ for the normal form equation (3.5) having the form $\hat{\mathbf{Z}} = R_\varphi\hat{\xi}(\tau)$ where $\hat{\xi} = (\psi_1, \phi_1, \phi_2, \psi_2)^T$ such that $\psi_1, \psi_2 \in \tau^{-2}\mathbb{C}[[\tau^{-1}]]$, $\phi_1 \in \tau^{-3}\mathbb{C}[[\tau^{-1}]]$ and $\phi_2 \in \tau^{-1}\mathbb{C}[[\tau^{-1}]]$. Substituting this formal solution into the formal series (3.17) we obtain a formal solution $\hat{\mathbf{\Gamma}} := \Phi \circ \hat{\mathbf{Z}}$ for equation (3.4). Now, since composition of formal series is again a formal series, the Theorem follows and we just need to check the degree of the trigonometric polynomials. As this should not present any difficulty we conclude the proof of the Theorem. □

Several remarks are in order,

Remark 3.2.2.1. The freedom in the definition of the formal solution $\hat{\mathbf{\Gamma}}$ can be eliminated if we fix the first two orders of the formal series $\hat{\Gamma}_i$, $i = 1,\dots,4$. In general, we cannot eliminate this freedom in a coordinate independent way.

Remark 3.2.2.2. If the original Hamiltonian H is real analytic then for any solution $\mathbf{\Gamma}$ of equation (3.4) we have that $\overline{\mathbf{\Gamma}(\overline{\varphi},\overline{\tau})}$ is also a solution of the same equation. Indeed,

$$\mathcal{D}\overline{\mathbf{\Gamma}(\bar{\varphi},\bar{\tau})} = \overline{\overline{\mathcal{D}}\mathbf{\Gamma}(\bar{\varphi},\bar{\tau})} = \overline{X_H(\mathbf{\Gamma}(\bar{\varphi},\bar{\tau}))} = X_H(\overline{\mathbf{\Gamma}(\bar{\varphi},\bar{\tau})}),$$

where $\overline{\mathcal{D}} = \partial_{\bar{\varphi}} + \partial_{\bar{\tau}}$. Moreover, as $\hat{\mathbf{\Gamma}} = \Phi \circ \hat{\mathbf{Z}}$ where Φ is a normal form transformation (3.17) which is a formal series with real coefficients, we have according to Remark 3.2.1.1 that,

$$\begin{aligned}
\overline{\hat{\mathbf{\Gamma}}(\bar{\varphi},\bar{\tau})} &= \Phi(\overline{\hat{\mathbf{Z}}(\bar{\varphi},\bar{\tau})}) = \Phi(\hat{\mathbf{Z}}(\varphi+\pi,\tau)) = \hat{\mathbf{\Gamma}}(\varphi+\pi,\tau), && \text{for} \quad \eta > 0,\\
\overline{\hat{\mathbf{\Gamma}}(\bar{\varphi},\bar{\tau})} &= \Phi(\overline{\hat{\mathbf{Z}}(\bar{\varphi},\bar{\tau})}) = \Phi(\hat{\mathbf{Z}}(\varphi,\tau)) = \hat{\mathbf{\Gamma}}(\varphi,\tau), && \text{for} \quad \eta < 0.
\end{aligned}$$

Remark 3.2.2.3. If the original Hamiltonian H is real analytic and X_H is reversible with respect to the involution (3.16) then the normal form preserves the reversibility. Thus, it follows from Remark 3.2.1.2 that one can define the formal solution $\hat{\mathbf{\Gamma}}$ in a coordinate independent way using the reversibility as follows,

$$\hat{\mathbf{\Gamma}}(\varphi,\tau) = \overline{\mathcal{S}(\hat{\mathbf{\Gamma}}(-\bar{\varphi},-\bar{\tau}))}.$$

This formal solution is unique up to a translation $\hat{\mathbf{\Gamma}}(\varphi+\pi,\tau)$.

Remark 3.2.2.4. Let $j \in \mathbb{Z}$ and $\hat{\mathbf{u}}$ be a formal series in the class $\tau^j \mathsf{T}^4_{\mathbb{C}}[[\tau^{-1}]]$ such that $\hat{\mathbf{u}} = \left(\tau^{j-1}u^1, \tau^{j-1}u^2, \tau^j u^3, \tau^j u^4\right)^T$, where $u^i = \sum_{k\geq 0} u^i_k \tau^{-k} \in \mathsf{T}_{\mathbb{C}}[[\tau^{-1}]]$. Now define,

$$\langle \hat{\mathbf{u}} \rangle_n := \left(\tau^{j-1}\sum_{k=0}^{n+j} u^1_k \tau^{-k}, \tau^{j-1}\sum_{k=0}^{n+j} u^2_k \tau^{-k}, \tau^{j}\sum_{k=0}^{n+j} u^3_k \tau^{-k}, \tau^{j}\sum_{k=0}^{n+j} u^4_k \tau^{-k}\right)^T,$$

which is just a partial sum of the formal series $\hat{\mathbf{u}}$ up to order $\tau^{-(n+1)}$ in the first two components and up to order τ^{-n} in the last two.

For $n \geq 1$ let $\mathbf{\Gamma}_n := \left\langle \hat{\mathbf{\Gamma}} \right\rangle_n$. Then we have the following important property,

$$\mathcal{D}\mathbf{\Gamma}_n - X_H(\mathbf{\Gamma}_n) = \left(\tau^{-(n+2)}R^1_n, \tau^{-(n+2)}R^2_n, \tau^{-(n+1)}R^3_n, \tau^{-(n+1)}R^4_n\right)^T, \tag{3.18}$$

for some $R^i_n \in \mathsf{T}^4_{\mathbb{C}}[[\tau^{-1}]]$, $i = 1,\ldots,4$. Indeed, for a formal series $\hat{\mathbf{\Gamma}} = \sum_{k\geq 1} \Gamma^k \tau^{-k}$ with $\Gamma^k \in \mathsf{T}^4_{\mathbb{C}}$ to solve formally the equation (3.4), then the coefficients Γ^k must solve the infinite system of equations,

$$\partial_\varphi \Gamma^k - X_{-I_1+I_2}(\Gamma^k) = (k-1)\Gamma^{k-1} + G_k(\Gamma^1, \ldots, \Gamma^{k-2}), \quad k = 1, 2, \ldots \qquad (3.19)$$

obtained from substituting the formal series into equation (3.4) and collecting terms of the same order in τ^{-k}. The function G_k can be defined in a recursive way.

Now since the first n coefficients of the sum $\mathbf{\Gamma}_n$ solve the equations (3.19) for $k = 1, \ldots, n$ then in order to get (3.18) we consider the equation (3.19) for $k = n+1$. Note that the left hand side of equation (3.19) depends only on the kth coefficient of the formal series $\hat{\mathbf{\Gamma}}$. Moreover, due to the form of the vector field $X_{-I_1+I_2}$ we can see that the first two components of the expression in the left hand side of (3.19) only depend on the first two components of Γ^k. These observations allow us to conclude (3.18).

3.2.1 Formal variational equation

In this subsection we consider the first variational equation of X_H around the formal solution $\hat{\mathbf{\Gamma}}$,

$$\mathcal{D}\mathbf{u} = DX_H(\hat{\mathbf{\Gamma}})\mathbf{u}. \qquad (3.20)$$

Our goal in this section is to construct a convenient basis for the space of formal solutions of equation (3.20). These formal solutions provide asymptotic series for certain analytic solutions of a modified equation of (3.20) that will be at the core of the perturbation theory developed in the subsequent sections.

We know already two formal solutions of the previous equation. They are $\partial_\varphi \hat{\mathbf{\Gamma}}$ and $\partial_\tau \hat{\mathbf{\Gamma}}$. Note that these formal solutions are linearly independent (as formal series in $\mathsf{T}^4_{\mathbb{C}}[[\tau^{-1}]]$). We can regard these as formal invariant tangent vectors fields of the formal invariant manifold parametrised by $\hat{\mathbf{\Gamma}}$. If the series were convergent, then we could drop the adjective "formal" and the tangent vector fields and the invariant manifold would be analytic. Moreover,

$$\Omega(\partial_\varphi \hat{\mathbf{\Gamma}}, \partial_\tau \hat{\mathbf{\Gamma}}) = 0, \qquad (3.21)$$

where Ω is the canonical symplectic form in $\mathbb{C}^4$. In other words the tangent vector fields $\partial_\varphi \hat{\mathbf{\Gamma}}$ and $\partial_\tau \hat{\mathbf{\Gamma}}$ form a Lagrangian plane. In general, these series are expected to diverge. Nevertheless, at the formal level we still have (3.21). In fact, a simple computation shows that, if $\mathbf{u}_1$ and $\mathbf{u}_2$ are two formal solutions of (3.20), then

$$\Omega(\mathbf{u}_1, \mathbf{u}_2) \in \mathbb{C}.$$

In fact, for $i \in \{1,2\}$, let $u_i \in \tau^{n_i}\mathsf{T}^4_{\mathbb{C}}[[\tau^{-1}]]$ for some $n_i \in \mathbb{Z}$ and suppose that $\mathcal{D}\mathbf{u}_i = DX_H(\hat{\mathbf{\Gamma}})\mathbf{u}_i$. Then,

$$\begin{aligned}\mathcal{D}\Omega(\mathbf{u}_1, \mathbf{u}_2) &= \Omega(\mathcal{D}\mathbf{u}_1, \mathbf{u}_2) + \Omega(\mathbf{u}_1, \mathcal{D}\mathbf{u}_2) \\ &= \Omega(DX_H(\hat{\mathbf{\Gamma}})\mathbf{u}_1, \mathbf{u}_2) + \Omega(\mathbf{u}_1, DX_H(\hat{\mathbf{\Gamma}})\mathbf{u}_2) \\ &= 0.\end{aligned} \tag{3.22}$$

In particular, $\mathcal{D}\Omega(\partial_\varphi \hat{\mathbf{\Gamma}}, \partial_\tau \hat{\mathbf{\Gamma}}) = 0$. Now the next Lemma provides the desired answer.

Lemma 3.2.1. *Let $g \in \tau^j\mathsf{T}^4_{\mathbb{C}}[[\tau^{-1}]]$ for some $j \in \mathbb{Z}$ and suppose that $\mathcal{D}g = 0$. Then $g = g_0 \in \mathbb{C}$. In addition, if $j \leq -1$ then $g = 0$.*

Proof. Let $g = \sum_{k \leq j} g_k \tau^k$ where $g_k \in \mathsf{T}^4_{\mathbb{C}}$. Substituting g into the equation $\mathcal{D}g = 0$ and collecting terms of the same order in τ^k we get,

$$\begin{aligned}&\partial_\varphi g_j = 0, \\ &\partial_\varphi g_k + (k+1)g_{k+1} = 0, \quad k \leq j-1.\end{aligned} \tag{3.23}$$

The first equation of (3.23) implies that $g_j \in \mathbb{C}$. Now using the second equation we can solve for g_k. Taking into account that $g_k \in \mathsf{T}^4_{\mathbb{C}}$ we conclude that $(k+1)g_{k+1} = 0$ for all $k \leq j-1$. Note that when $k = -1$ we have no restriction on g_0 and the Lemma follows. ☐

In the following, we will construct a matrix of formal solutions $\hat{\mathbf{U}} = (\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3, \hat{\mathbf{u}}_4)$ for the linear equation (3.20), satisfying the following properties,

1. For every $i = 1, \ldots, 4$, $\hat{\mathbf{u}}_i \in \tau^j\mathsf{T}^4_{\mathbb{C}}[[\tau^{-1}]]$ for some $j \in \mathbb{Z}$,

2. The formal series $\hat{\mathbf{u}}_i$ are linearly independent,

3. The first and fourth columns of the matrix $\hat{\mathbf{U}}$ are the known formal solutions $\hat{\mathbf{u}}_1 = \partial_\varphi \hat{\mathbf{\Gamma}}$ and $\hat{\mathbf{u}}_4 = \partial_\tau \hat{\mathbf{\Gamma}}$,

4. The columns of $\hat{\mathbf{U}}$ form a formal "symplectic basis", i.e.,

$$\Omega(\partial_\varphi \hat{\mathbf{\Gamma}}, \hat{\mathbf{u}}_2) = 0, \quad \Omega(\hat{\mathbf{u}}_2, \partial_\tau \hat{\mathbf{\Gamma}}) = 1, \quad \Omega(\partial_\varphi \hat{\mathbf{\Gamma}}, \partial_\tau \hat{\mathbf{\Gamma}}) = 0$$
$$\Omega(\partial_\varphi \hat{\mathbf{\Gamma}}, \hat{\mathbf{u}}_3) = 1, \quad \Omega(\hat{\mathbf{u}}_3, \partial_\tau \hat{\mathbf{\Gamma}}) = 0, \quad \Omega(\hat{\mathbf{u}}_3, \hat{\mathbf{u}}_2) = 0.$$

where Ω is the canonical symplectic form in $\mathbb{C}^4$. The last property of $\hat{\mathbf{U}}$ implies that,

$$\Omega(\hat{\mathbf{U}}v, \hat{\mathbf{U}}w) = \Omega(v, w), \quad \forall\, v, w \in \mathbb{C}^4.$$

Thus, $\hat{\mathbf{U}}$ as defined by the properties above is a symplectic matrix and moreover $\det(\hat{\mathbf{U}}) = 1$. A matrix $\hat{\mathbf{U}}$ satisfying the properties stated above is called a *formal normalized fundamental matrix* for equation (3.20).

Theorem 3.2.3. *The linear equation* (3.20) *has a formal normalized fundamental matrix* $\hat{\mathbf{U}}$ *such that,*

$$\hat{\mathbf{U}} = \begin{pmatrix} \tau^{-2}\hat{u}_{1,1} & \tau^{2}\hat{u}_{1,2} & \tau\hat{u}_{1,3} & \tau^{-3}\hat{u}_{1,4} \\ \tau^{-2}\hat{u}_{2,1} & \tau^{2}\hat{u}_{2,2} & \tau\hat{u}_{2,3} & \tau^{-3}\hat{u}_{2,4} \\ \tau^{-1}\hat{u}_{3,1} & \tau^{3}\hat{u}_{3,2} & \tau^{2}\hat{u}_{3,3} & \tau^{-2}\hat{u}_{3,4} \\ \tau^{-1}\hat{u}_{4,1} & \tau^{3}\hat{u}_{4,2} & \tau^{2}\hat{u}_{4,3} & \tau^{-2}\hat{u}_{4,4} \end{pmatrix},$$

where $\hat{u}_{i,j} = \sum_{k\geq 0} u_k^{i,j} \tau^{-k} \in \mathsf{T}_{\mathbb{C}}[[\tau^{-1}]]$, *for* $i, j = 1, \ldots, 4$. *Moreover for any other formal normalized fundamental matrix* $\hat{\tilde{\mathbf{U}}}$ *there is a* $c = (c_1, c_2, c_3) \in \mathbb{C}^3$ *such that* $\hat{\tilde{\mathbf{U}}} = \hat{\mathbf{U}} E_c$ *where,*

$$E_c = \begin{pmatrix} 1 & -c_1 & c_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & c_3 & c_1 & 1 \end{pmatrix}. \tag{3.24}$$

Proof. In the proof of Theorem 3.2.2 we have obtained the formal solution $\hat{\mathbf{\Gamma}}$ through the normal form Hamiltonian H^{NF} by defining $\hat{\mathbf{\Gamma}} = \Phi \circ \hat{\mathbf{Z}}$, where Φ is the normal form transformation, which is a formal series of the form (3.17), and $\hat{\mathbf{Z}}$ is the formal solution of Theorem 3.2.1. Also from the same Theorem we know that $\hat{\mathbf{Z}} = R_\varphi \hat{\xi}$ where $\hat{\xi}$ is a formal series in the class $\tau^{-1}\mathbb{C}^4[[\tau^{-1}]]$ and using the polar coordinates (3.7) we can write it as follows,

$$\hat{\xi}(\tau) = \left(R(\tau)\cos\theta(\tau), R(\tau)\sin\theta(\tau), r(\tau)\cos\theta(\tau), r(\tau)\sin\theta(\tau)\right)^T, \tag{3.25}$$

where r, R and θ are the formal series (3.13). Now using Φ, the equation (3.20) is equivalent to,

$$\mathcal{D}\mathbf{v} = DX_{H^{NF}}(\hat{\mathbf{Z}})\mathbf{v}, \tag{3.26}$$

where $\mathbf{v}$ and $\mathbf{u}$ are related by,

$$\mathbf{u} = D\Psi(\hat{\mathbf{Z}})\mathbf{v}.$$

We seek for formal solutions of (3.26) in the form $\mathbf{v} = R_\varphi \zeta$ where $\zeta \in \tau^j\mathbb{C}^4[[\tau^{-1}]]$ for some $j \in \mathbb{Z}$. Similar to the proof of Theorem 3.2.1 the ζ must satisfy the linear PDE,

$$\partial_\tau \zeta = DX_{H^{NF}+I_1}(\hat{\xi})\zeta.$$

Bearing in mind (3.25), we now rewrite the previous equation in polar coordinates,

$$\begin{aligned}
&\partial_\tau w_1 = -\sum_{l\geq 1} \frac{l a_{1,l}}{2^{l-1}} r^{2l-1} w_2 - \left(\frac{1}{r^2} + \sum_{l\geq 0} \frac{a_{2,l}}{2^{l-1}} r^{2l}\right) w_3, \quad \partial_\tau w_2 = -w_4, \\
&\partial_\tau w_3 = 0, \quad \partial_\tau w_4 = \left(3\eta r^2 + \sum_{l\geq 3} \frac{l(2l-1)a_{0,l}}{2^{l-1}} r^{2l-2}\right) w_2 + \sum_{l\geq 1} \frac{l a_{1,l}}{2^{l-1}} r^{2l-1} w_3,
\end{aligned} \tag{3.27}$$

where the relation between the variables is the following,

$$\zeta = D\Lambda(\theta, r, \Theta, R)\mathbf{w}, \tag{3.28}$$

where Λ denotes the change of variables (3.7), θ, r, Θ and R are the formal series (3.13) and $\mathbf{w} = (w_1, w_2, w_3, w_4)^T$. Recall that $\Theta = 0$. Note that Λ is symplectic with multiplier -1. Two formal solutions of (3.27) immediately follow from the formal series r, θ and R,

$$\hat{\mathbf{w}}_1 = (1,0,0,0)^T \quad \text{and} \quad \hat{\mathbf{w}}_4 = (\partial_\tau\theta, \partial_\tau r, 0, \partial_\tau R)^T. \tag{3.29}$$

We now construct two more formal solutions which are independent of (3.29). We shall look for these formal solutions of (3.27) in the class of formal series $\tau^j\mathbb{C}[[\tau^{-1}]]$. Let us consider the second and fourth equations of (3.27). They are equivalent to the single second order linear equation,

$$\partial_\tau^2 w_2 = -\left(3\eta r^2 + \sum_{l\geq 3}\frac{l(2l-1)a_{0,l}}{2^{l-1}}r^{2l-2}\right)w_2 - \sum_{l\geq 1}\frac{la_{1,l}}{2^{l-1}}r^{2l-1}w_3. \tag{3.30}$$

In order to solve the previous equation, we first consider the homogeneous part.

Claim 3.2.3.1. The linear homogeneous equation,

$$\partial_\tau^2 w_2 = -\left(3\eta r^2 + \sum_{l\geq 3}\frac{l(2l-1)a_{0,l}}{2^{l-1}}r^{2l-2}\right)w_2, \tag{3.31}$$

has two linearly independent formal solutions,

$$w_{2,1} \in \tau^{-2}\mathbb{C}[[\tau^{-1}]] \quad \text{and} \quad w_{2,2} \in \tau^3\mathbb{C}[[\tau^{-1}]]$$

such that $w_{2,1}$ is an even formal series and $w_{2,2}$ an odd formal series. Moreover $w_{2,1} = \partial_\tau r$, $w_{2,2} = \frac{\tau^3}{5\kappa} + \frac{7}{40}a_{0,3}\kappa^3\tau + \cdots$ and,

$$w_{2,2}\partial_\tau w_{2,1} - w_{2,1}\partial_\tau w_{2,2} = 1. \tag{3.32}$$

Proof. That $\partial_\tau r$ is a formal solution of the homogeneous equation is obvious. Moreover its properties follow from the properties of the formal series r. Now let us determine the second formal solution. It follows from the fact that the formal series $r \in \tau^{-1}\mathbb{C}[[\tau^{-1}]]$ is odd that the right hand side of the homogeneous equation (3.31) is a formal series of the form $b = \sum_{k\leq -1} b_k\tau^{2k}$ where b_k depend on a finite number of coefficients of r and $a_{0,l}$ for $l \geq 3$. Moreover, according to (3.11) we have,

$$r(\tau) = \kappa\tau^{-1} - \frac{1}{8}a_{0,3}\kappa^5\tau^{-3} + \cdots,$$

where $\kappa^2 = -\frac{2}{\eta}$. Using the leading orders of r, we compute the first few orders of the formal series b for further reference,

$$b_{-1} = 6 \quad \text{and} \quad b_{-2} = -\frac{21 a_{0,3}}{\eta^2}. \tag{3.33}$$

Now we are ready to solve equation (3.31) in the class of formal series. Thus, substituting the formal series $w_{2,2} = \sum_{k\leq 1} w_{2,2,k}\tau^{2k+1}$ into equation (3.31) and collecting terms of the same order in τ^k we obtain the following infinite system of linear equations,

$$(2k(2k+1)-6)\, w_{2,2,k} = \sum_{j=k-2}^{-2} w_{2,2,k-j-1} b_j, \quad k = 1, 0, -1, \ldots$$

For $k = 1$ we get no condition on the first coefficient, thus $w_{2,2,1} \in \mathbb{C}$. For $k = 0$ we obtain $w_{2,2,0} = -\frac{1}{6} w_{2,2,1} b_{-2}$. When $k \leq -1$, a simple induction argument shows that we can determine the coefficients $w_{2,2,k}$ (which depend linearly on the coefficient $w_{2,2,1}$) in a recursive way by using the previous formula since $(2k(2k+1)-6) = 0$ only if $k = 1$ or $k = -\frac{3}{2}$. Finally let us derive the equality (3.32). Since,

$$\partial_\tau \left(w_{2,2}\partial_\tau w_{2,1} - w_{2,1}\partial_\tau w_{2,2} \right) = 0,$$

due to the fact that both $w_{2,1}$ and $w_{2,2}$ solve the homogeneous equation (3.31) we have that $w_{2,2}\partial_\tau w_{2,1} - w_{2,1}\partial_\tau w_{2,2}$ is equal to some constant. Taking into account the leading orders of the formal solutions $w_{2,1}$ and $w_{2,2}$ we conclude that $w_{2,2}\partial_\tau w_{2,1} - w_{2,1}\partial_\tau w_{2,2} = 5\kappa w_{2,2,1}$. As $w_{2,2,1}$ is a free coefficient we can define $w_{2,2,1} := \frac{1}{5\kappa}$ and obtain the desired equality. This concludes the proof of the Claim. $\square$

Returning to the non-homogeneous equation (3.30), we can see that the last term of the right hand side of the equation depends on w_3 from which we know that $\partial_\tau w_3 =$. Thus $w_3 = w_{3,0} \in \mathbb{C}$ is a constant. Now, taking into account the form of the formal series r,

$$g(\tau) := \sum_{l\geq 1} \frac{l a_{1,l}}{2^{l-1}} r^{2l-1} \in \tau^{-1}\mathbb{C}[[\tau^{-1}]],$$

is an odd formal series whose coefficients depend on a finite number of coefficients of r and $a_{1,l}$ for $l \geq 1$. Using the well known method of variation of constants we can write the general formal solution of (3.30) as follows,

$$w_2 = c_1 w_{2,1} + c_2 w_{2,2} + w_{2,2} \int^\tau w_{2,1} g w_{3,0} - w_{2,1} \int^\tau w_{2,2} g w_{3,0}, \tag{3.34}$$

where $w_{3,0}, c_1, c_2 \in \mathbb{C}$. Note that the integration in the previous formula is well defined in the class of formal series $\mathbb{C}[[\tau^{-1}]][[\tau]]$. Indeed, it can be easily checked that $w_{2,1} g \in \tau^{-3}\mathbb{C}[[\tau^{-1}]]$ is an odd formal series and $w_{2,2} g \in \tau^2 \mathbb{C}[[\tau^{-1}]]$ is an even formal series. Hence both integrands do not contain the term τ^{-1}. Next we define two particular formal solutions of (3.30),

$$w_2^0 := w_{2,2} \quad \text{and} \quad w_2^{-1} := -w_{2,2} \int^\tau w_{2,1} g + w_{2,1} \int^\tau w_{2,2} g. \tag{3.35}$$

The first formal solution corresponds to setting $c_1 = w_{3,0} = 0$ and $c_2 = 1$ in the general solution (3.34) and the second corresponds to $c_1 = c_2 = 0$ and $w_{3,0} = -1$. Note that $w_2^0 \in \tau^3 \mathbb{C}[[\tau^{-1}]]$ is an odd formal series and $w_2^{-1} \in \tau \mathbb{C}[[\tau^{-1}]]$ is also odd formal series.

Now coming back to the first equation of (3.27), we can rewrite it as follows,

$$\partial_\tau w_1 = -g w_2 + f w_{3,0},$$

where,

$$f = -\frac{1}{r^2} - \sum_{l \geq 0} \frac{a_{2,l}}{2^{l-1}} r^{2l}.$$

It is not difficult to see that $f \in \tau^2 \mathbb{C}[[\tau^{-1}]]$ is an even formal series. Moreover both $g w_2^0 \in \tau^2 \mathbb{C}[[\tau^{-1}]]$ and $g w_2^{-1} \in \mathbb{C}[[\tau^{-1}]]$ are even formal series. These observations allow us to conclude that the following are formal solutions of (3.27),

$$w_1^0 = -\int^\tau g w_2^0 \quad \text{and} \quad w_1^{-1} = -\int^\tau g w_2^{-1} - \int^\tau f, \tag{3.36}$$

which are well defined in the class of formal series $\mathbb{C}[[\tau^{-1}]][[\tau]]$ and moreover $w_1^0, w_1^{-1} \in \tau^3 \mathbb{C}[[\tau^{-1}]]$ are both odd formal series. Thus we obtain two formal solutions of (3.27) defined as follows,

$$\hat{\mathbf{w}}_2 := \left(w_1^0, w_2^0, 0, -\partial_\tau w_2^0\right)^T \quad \text{and} \quad \hat{\mathbf{w}}_3 := \left(w_1^{-1}, w_2^{-1}, -1, -\partial_\tau w_2^{-1}\right)^T.$$

Note that $\{\hat{\mathbf{w}}_i\}_{i=1,\dots,4}$ is a set of linearly independent formal solutions of equation (3.27) and that,

$$\begin{aligned} \Omega(\hat{\mathbf{w}}_1, \hat{\mathbf{w}}_2) = 0, \quad \Omega(\hat{\mathbf{w}}_1, \hat{\mathbf{w}}_3) = -1, \quad \Omega(\hat{\mathbf{w}}_1, \hat{\mathbf{w}}_4) = 0, \\ \Omega(\hat{\mathbf{w}}_2, \hat{\mathbf{w}}_3) = 0, \quad \Omega(\hat{\mathbf{w}}_2, \hat{\mathbf{w}}_4) = -1, \quad \Omega(\hat{\mathbf{w}}_3, \hat{\mathbf{w}}_4) = 0. \end{aligned} \tag{3.37}$$

where Ω is the canonical symplectic form in the polar coordinates, i.e., $\Omega = d\theta \wedge \Theta + dr \wedge dR$. The top identities of (3.37) are straightforward to prove just by using the definition of $\hat{\mathbf{w}}_i$. The ones on the bottom are a bit more trickier and let us prove them. First note that similar arguments as in (3.22) show that $\partial_\tau \Omega(\hat{\mathbf{w}}_i, \hat{\mathbf{w}}_j) = 0$ for $i, j = 1, \dots, 4$. Secondly, it follows from the previous claim and from (3.11) that,

$$w_{2,2}(\tau) = \frac{\tau^3}{5\kappa} + \frac{7}{40} a_{0,3} \kappa^3 \tau + \cdots \quad \text{and} \quad r(\tau) = \kappa \tau^{-1} - \frac{1}{8} a_{0,3} \kappa^5 \tau^{-3} + \cdots. \tag{3.38}$$

Now we compute $\Omega(\hat{\mathbf{w}}_2, \hat{\mathbf{w}}_3)$. Using the definition of both $\hat{\mathbf{w}}_2$ and $\hat{\mathbf{w}}_3$ we get

$$\Omega(\hat{\mathbf{w}}_2, \hat{\mathbf{w}}_3) = -w_0^1 - w_2^0 \partial_\tau w_2^{-1} + \partial_\tau w_0^2 w_2^{-1}.$$

Bearing in mind (3.35) and (3.36) we can simplify the previous expression and rewrite it as follows,

$$\Omega(\hat{\mathbf{w}}_2, \hat{\mathbf{w}}_3) = \left(1 - w_{2,2} \partial_\tau^2 r + \partial_\tau w_{2,2} \partial_\tau r\right) \int^\tau g w_{2,2}.$$

Now using the leading orders (3.38) we conclude that the expression inside the parenthesis in the previous formula belongs to $\tau^{-4}\mathbb{C}[[\tau^{-1}]]$. Moreover $\int^\tau g w_{2,2} \in \tau^3 \mathbb{C}[[\tau^{-1}]]$ and consequently $\Omega(\hat{\mathbf{w}}_2, \hat{\mathbf{w}}_3) \in \tau^{-1}\mathbb{C}[[\tau^{-1}]]$. Applying Lemma 3.2.1 we get $\Omega(\hat{\mathbf{w}}_2, \hat{\mathbf{w}}_3) = 0$ as we wanted to show.

Now we handle $\Omega(\hat{\mathbf{w}}_2, \hat{\mathbf{w}}_4)$. Again, making use of the definitions (3.35) and recalling that $\partial_\tau R = -\partial_\tau^2 r$ and $w_{2,1} = \partial_\tau r$ we obtain,

$$\Omega(\hat{\mathbf{w}}_2, \hat{\mathbf{w}}_4) = w_{2,1} \partial_\tau w_{2,2} - w_{2,2} \partial_\tau w_{2,1}.$$

The identity now follows from (3.32).

At last, let us compute $\Omega(\hat{\mathbf{w}}_3, \hat{\mathbf{w}}_4)$. Again using the definitions of the functions involved we get,

$$\Omega(\hat{\mathbf{w}}_3, \hat{\mathbf{w}}_4) = \partial_\tau \theta + w_2^{-1} \partial_\tau R + \partial_\tau r \partial_\tau w_2^{-1}.$$

This last expression belongs to $\tau^{-2}\mathbb{C}[[\tau^{-2}]]$ and applying Lemma 3.2.1 we obtain the desired result.

Now, taking into account the change (3.28) and $\mathbf{v} = R_\varphi \zeta$ we define,

$$\hat{\mathbf{v}}_i = R_\varphi D\Lambda(\theta, r, \Theta, R)\hat{\mathbf{w}}_i.$$

Clearly $\{\hat{\mathbf{v}}_i\}_{i=1,\dots,4}$ is a set of linearly independent formal solutions of equation (3.26) and moreover $\hat{\mathbf{v}}_1 = \partial_\varphi \hat{\mathbf{Z}}$ and $\hat{\mathbf{v}}_4 = \partial_\tau \hat{\mathbf{Z}}$. Taking into account the formulae (3.14) and (3.15) for the $\cos$ and $\sin$ respectively and that $r^{-1} \in \tau\mathbb{C}[[\tau^{-1}]]$, a closer look at the Jacobian of the polar coordinates transformation Λ reveals that,

$$D\Lambda(\theta, r, \Theta, R) = \begin{pmatrix} \tau^{-3}\Lambda_1 & 0 & \Lambda_2 & \Lambda_3 \\ \tau^{-2}\Lambda_4 & 0 & \tau\Lambda_5 & \tau^{-1}\Lambda_6 \\ \tau^{-2}\Lambda_7 & \Lambda_8 & 0 & 0 \\ \tau^{-1}\Lambda_9 & \tau^{-1}\Lambda_{10} & 0 & 0 \end{pmatrix},$$

for some $\Lambda_i \in \mathbb{C}[[\tau^{-1}]]$ for $i = 1, \dots, 10$. Thus

$$\hat{\mathbf{v}}_2 = \left(\tau^2\hat{v}_{1,2}, \tau^2\hat{v}_{2,2}, \tau^3\hat{v}_{3,2}, \tau^3\hat{v}_{4,2}\right)^T \quad \text{and} \quad \hat{\mathbf{v}}_3 = \left(\tau\hat{v}_{1,3}, \tau\hat{v}_{2,3}, \tau^2\hat{v}_{3,3}, \tau^2\hat{v}_{4,3}\right)^T,$$

for some $\hat{v}_{i,1}, \hat{v}_{i,2} \in \mathsf{T}_{\mathbb{C}}[[\tau^{-1}]]$ for $i = 1, \dots, 4$. As previously observed Λ is symplectic with multiplier -1 and the identities (3.37) in the new variables read,

$$\begin{aligned} &\Omega(\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) = 0, \quad \Omega(\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_3) = 1, \quad \Omega(\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_4) = 0, \\ &\Omega(\hat{\mathbf{v}}_2, \hat{\mathbf{v}}_3) = 0, \quad \Omega(\hat{\mathbf{v}}_2, \hat{\mathbf{v}}_4) = 1, \quad \Omega(\hat{\mathbf{v}}_3, \hat{\mathbf{v}}_4) = 0. \end{aligned} \tag{3.39}$$

Finally, composing the formal solutions $\hat{\mathbf{v}}_i$ with the normal form transformation Φ we obtain the desired matrix $\hat{\mathbf{U}} = (\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3, \hat{\mathbf{u}}_4)^T$ where the $\hat{\mathbf{u}}_i$ are defined in the following way,

$$\hat{\mathbf{u}}_i := D\Phi(\hat{\mathbf{Z}})\hat{\mathbf{v}}_i.$$

It is clear that $\hat{\mathbf{U}}$ satisfies all properties of a formal normalized fundamental matrix for equation (3.20). Moreover its leading orders follow easily from the leading orders of $\hat{\mathbf{v}}_i$ and the fact that Φ is near identity. In order to conclude the proof of the Theorem, note that by the method of variation of constants a general formal solution of equation (3.20) is of the form $\hat{\mathbf{U}}c$ where c is any formal series in $\tau^j \mathsf{T}^4_{\mathbb{C}}[[\tau^{-1}]]$ for some $j \in \mathbb{Z}$, such that $\mathcal{D}c = 0$. Lemma 3.2.1 implies that $c \in \mathbb{C}^4$. Thus, if $\hat{\hat{\mathbf{U}}}$ is another formal normalized fundamental matrix of (3.20) then there exist a matrix $E \in \mathbb{C}^{4\times 4}$ such that $\hat{\hat{\mathbf{U}}} = \hat{\mathbf{U}}E$. From the third property of a formal normalized fundamental matrix we also conclude that,

$$E = \begin{pmatrix} 1 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 1 \end{pmatrix}.$$

Moreover, since $\hat{\hat{\mathbf{U}}}$ and $\hat{\mathbf{U}}$ are symplectic it also follows that E is symplectic and a simple computation shows that one can reduce the number of entries of E to obtain the form (3.24). This concludes the proof of the Theorem. $\square$

Remark 3.2.3.1. For $n \geq 1$ let,

$$\mathbf{U}_n := \left(\langle\hat{\mathbf{u}}_1\rangle_n, \langle\hat{\mathbf{u}}_2\rangle_n, \langle\hat{\mathbf{u}}_3\rangle_n, \langle\hat{\mathbf{u}}_4\rangle_n\right)^T.$$

where $\langle\cdot\rangle_n$ was defined in Remark 3.2.2.4 and $\hat{\mathbf{u}}_i$ are the columns of a *formal normalized fundamental matrix* $\hat{\mathbf{U}}$. As in Remark 3.2.2.4 it is not difficult to show that each column of,

$$\mathcal{D}\mathbf{U}_n - DX_H(\mathbf{\Gamma}_{n+3})\mathbf{U}_n$$

starts with terms of order $\tau^{-(n+2)}$ in the first two components and with terms of order $\tau^{-(n+1)}$ in the last two.

3.3 Solutions of a Variational Equation

Let $n \geq 3$ and $\xi \in \mathfrak{X}_{n+4}(S_h \times D_r^-)$ and consider the following variational equation,

$$\mathcal{D}\mathbf{u} = DX_H(\mathbf{\Gamma}_{n+3} + \xi)\mathbf{u}. \tag{3.40}$$

where $\mathbf{\Gamma}_{n+3}$ is the function defined in Remark 3.2.2.4. In this section we construct a 4 by 4 matrix function $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4) : S_h \times D_r^- :\to \mathbb{C}^{4\times 4}$ such that $\mathcal{D}\mathbf{U} = DX_H(\mathbf{\Gamma}_{n+3} + \xi)\mathbf{U}$. The vector functions $\mathbf{u}_i : S_h \times D_r^- :\to \mathbb{C}^4$ are the columns of $\mathbf{U}$ and satisfy the following properties,

1. $\mathbf{u}_1 \in \mathfrak{X}_1(S_h \times D_r^-)$, $\mathbf{u}_2 \in \mathfrak{X}_{-3}(S_h \times D_r^-)$, $\mathbf{u}_3 \in \mathfrak{X}_{-2}(S_h \times D_r^-)$ and $\mathbf{u}_4 \in \mathfrak{X}_2(S_h \times D_r^-)$.

2. $\{\mathbf{u}_i\}_{i=1,\ldots,4}$ form a "symplectic basis" in $\mathbb{C}^4$, i.e.,

$$\begin{aligned} &\Omega(\mathbf{u}_1, \mathbf{u}_2) = 0, \quad \Omega(\mathbf{u}_2, \mathbf{u}_4) = 1, \quad \Omega(\mathbf{u}_1, \mathbf{u}_4) = 0 \\ &\Omega(\mathbf{u}_1, \mathbf{u}_3) = 1, \quad \Omega(\mathbf{u}_3, \mathbf{u}_4) = 0, \quad \Omega(\mathbf{u}_3, \mathbf{u}_2) = 0. \end{aligned} \tag{3.41}$$

where Ω is the canonical symplectic form in $\mathbb{C}^4$. The last property implies that $\mathbf{U}$ is a symplectic matrix for all $(\varphi, \tau) \in S_h \times D_r^-$ and $\det(\mathbf{U}) = 1$. A matrix $\mathbf{U}$ that satisfies the properties above is called a *canonical fundamental matrix* for the variation equation (3.40).

Theorem 3.3.1. *Let $n \geq 3$ and let $\mathbf{U}_n$ be a piece of a formal normalized fundamental matrix $\hat{\mathbf{U}}$ as defined in Remark 3.2.3.1. Then there is $r_0 > 0$ sufficiently large such that for every $r > r_0$ the variational equation* (3.40) *has an unique canonical fundamental matrix $\mathbf{U} : S_h \times D_r^- :\to \mathbb{C}^{4\times 4}$ such that,*

$$\mathbf{U} - \mathbf{U}_n \in \mathfrak{X}^4_{n+1}(S_h \times D_r^-). \tag{3.42}$$

Proof. Let $n \geq 3$. We look for a canonical fundamental matrix of (3.40) in the form,

$$\mathbf{U} = \mathbf{U}_n + \mathbf{V}, \tag{3.43}$$

where $\mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) : S_h \times D_r^- \to \mathbb{C}^{4\times 4}$ is a 4 by 4 matrix function such that each vector column $\mathbf{v}_i$ belong to the space $\mathfrak{X}_n\left(S_h \times D_r^-\right)$ for some $r > 0$ (to be chosen later in the proof). Substituting (3.43) into the equation (3.40) we obtain,

$$\mathcal{D}\mathbf{V} = DX_H(\mathbf{\Gamma}_{n+3} + \xi)\mathbf{V} + DX_H(\mathbf{\Gamma}_{n+3} + \xi)\mathbf{U}_n - \mathcal{D}\mathbf{U}_n.$$

This last equation can be rewritten in the following form,

$$\mathcal{L}_0(\mathbf{V}) = \mathbf{B}\mathbf{V} + \mathbf{R}_n, \tag{3.44}$$

where the linear operator $\mathcal{L}_0$ is defined by formula (2.38) which we recall $\mathcal{L}_0(\mathbf{u}) = \mathcal{D}\mathbf{u} - A_0\mathbf{u}$ where A_0 is the matrix given by (2.37) and

$$\mathbf{B} = DX_H(\mathbf{\Gamma}_{n+3} + \xi) - A_0 \quad \text{and} \quad \mathbf{R}_n = DX_H(\mathbf{\Gamma}_{n+3} + \xi)\mathbf{U}_n - \mathcal{D}\mathbf{U}_n.$$

Note that $\mathbf{B}(\varphi, \tau) = O(\tau^{-3})$. Moreover due to Remark 3.2.3.1 each column of $\mathbf{R}_n$ belongs to $\mathfrak{X}_{n+1}\left(S_h \times D_r^-\right)$. Thus, $\mathbf{B}\mathbf{V} + \mathbf{R}_n \in \mathfrak{X}_{n+1}^4\left(S_h \times D_r^-\right)$. Now since $\mathcal{L}_0$ has a fundamental matrix $\mathbf{U}_0$ given by (2.39) then we can apply Theorem 2.4.1 and obtain an unique bounded inverse $\mathcal{L}_0^{-1} : \mathfrak{X}_{n+1}\left(S_h \times D_r^-\right) \to \mathfrak{X}_n\left(S_h \times D_r^-\right)$ for $n \geq 3$. Thus, in order to solve (3.44) for $\mathbf{V}$, it is sufficient to find a fixed point of the following operator,

$$\mathbf{V} \mapsto \mathcal{L}_0^{-1}\left(\mathbf{B}\mathbf{V}\right) + \mathcal{L}_0^{-1}\left(\mathbf{R}_n\right). \tag{3.45}$$

First note that the matrix $\mathbf{B}$ induces a linear operator $\mathcal{B} : \mathfrak{X}_n\left(S_h \times D_r^-\right) \to \mathfrak{X}_{n+1}\left(S_h \times D_r^-\right)$ such that $\mathcal{B}(\mathbf{v}) = \mathbf{B}\mathbf{v}$ for a given $\mathbf{v} \in \mathfrak{X}_n\left(S_h \times D_r^-\right)$. Thus, in order to prove the existence of a fixed point for (3.45) it is enough to show that,

$$\left\|\mathcal{L}_0^{-1} \circ \mathcal{B}\right\|_{n,n} \leq \frac{1}{2}. \tag{3.46}$$

We now show the inequality (3.46). Given $\mathbf{v} \in \mathfrak{X}_n\left(S_h \times D_r^-\right)$ we write

$$\mathbf{v} = (\tau^{-n-1}v_1, \tau^{-n-1}v_2, \tau^{-n}v_3, \tau^{-n}v_4),$$

and as $\mathbf{B} = O(\tau^{-3})$ we can write $\mathbf{B}\mathbf{v}$ as follows,

$$\mathbf{B}\mathbf{v} = \begin{pmatrix} \tau^{-n-4}\left(B_{1,1}v_1 + B_{1,2}v_2\right) + \tau^{-n-3}\left(B_{1,3}v_3 + B_{1,4}v_4\right) \\ \tau^{-n-4}\left(B_{2,1}v_1 + B_{2,2}v_2\right) + \tau^{-n-3}\left(B_{2,3}v_3 + B_{2,4}v_4\right) \\ \tau^{-n-4}\left(B_{3,1}v_1 + B_{3,2}v_2\right) + \tau^{-n-3}\left(B_{3,3}v_3 + B_{3,4}v_4\right) \\ \tau^{-n-4}\left(B_{4,1}v_1 + B_{4,2}v_2\right) + \tau^{-n-3}\left(B_{4,3}v_3 + B_{4,4}v_4\right) \end{pmatrix}, \tag{3.47}$$

for some analytic functions $B_{i,j} : S_h \times D_r^- \to \mathbb{C}$ which are 2π-periodic in φ and continuous on the closure on their domains. Moreover,

$$K_{\mathbf{B}} = \max_{i,j=1,\ldots,4} \left\{ \sup_{(\varphi,\tau)\in S_h\times D_r^-} |B_{i,j}(\varphi,\tau)| \right\} < \infty.$$

Now given $r_0 > \frac{1}{\sin\theta_0}$ then for every $r > r_0$ the following chain of inequalities hold,

$$|\tau|^{-k} \leq |\tau|^{-1} \leq \frac{1}{r_0 \sin\theta_0} \quad \text{in} \quad D_r^-.$$

The previous observation and (3.47) give the following estimate,

$$\|\mathbf{B}\mathbf{v}\|_{n+1} \leq \frac{K_{\mathbf{B}}}{r_0 \sin\theta_0} \|\mathbf{v}\|_n .$$

Thus the linear operator $\mathcal{B}$ is bounded and $\|\mathcal{B}\|_{n+1,n} \leq \frac{K_{\mathbf{B}}}{r_0\sin\theta_0}$. Now taking into account that $\mathcal{L}_0^{-1}$ is also bounded by Theorem 2.4.1 we get,

$$\left\|\mathcal{L}_0^{-1} \circ \mathcal{B}\right\|_{n,n} \leq \left\|\mathcal{L}_0^{-1}\right\|_{n,n+1} \|\mathcal{B}\|_{n+1,n} \leq \frac{K_{\mathbf{B}} \left\|\mathcal{L}_0^{-1}\right\|_{n,n+1}}{r_0 \sin\theta_0}.$$

Therefore if $r_0 > \max\left\{\frac{1}{\sin\theta_0}, \frac{2K_{\mathbf{B}}\left\|\mathcal{L}_0^{-1}\right\|_{n,n+1}}{\sin\theta_0}\right\}$ then for every $r > r_0$ we get the desired inequality (3.46) and consequently we can apply the contraction mapping theorem and obtain an unique fixed point $\mathbf{V} \in \mathfrak{X}_n^4 \left(S_h \times D_r^-\right)$ of equation (3.45). Finally, note that if we repeat the previous arguments with $n+1$ instead of n then for $\tilde{r} > 0$ sufficiently large there exists an unique $\tilde{\mathbf{V}} \in \mathfrak{X}_{n+1}^4 \left(S_h \times D_{\tilde{r}}^-\right)$ such that $\tilde{\mathbf{U}} = \mathbf{U}_{n+1} + \tilde{\mathbf{V}}$ solves equation (3.40). Now it follows that $\tilde{\mathbf{U}} - \mathbf{U}_n \in \mathfrak{X}_{n+1}^4 \left(S_h \times D_{\tilde{r}}^-\right)$ and due to the uniqueness of the fixed point we conclude that $\tilde{\mathbf{U}} - \mathbf{U}_n = \mathbf{V}$. Hence $\mathbf{V} \in \mathfrak{X}_{n+1}^4 \left(S_h \times D_r^-\right)$ for every r sufficiently large. Thus inclusion (3.42) is proved. In order to conclude the proof of the Theorem we just need to show the equalities (3.41). They follow from the fact that Ω is bilinear, $\mathcal{D}\Omega(\mathbf{u}_i, \mathbf{u}_j) = 0$ for $i, j = 1, \ldots, 4$ and Proposition 2.4.1 and the fact that $\hat{\mathbf{U}}$ is formal symplectic. This concludes the proof of the Theorem. □

Remark 3.3.1.1. As before, it is clear that the arguments of the proof of the previous Theorem work equally well when all the functions are analytic in $S_h \times D_r^+$.

3.4 Analytic Invariant Manifolds

In this section we prove the existence of an unstable (resp. stable) analytic manifold immersed in $\mathbb{C}^4$. We also provide an asymptotic expansion for both manifolds. More concretely, following (3.4) we look for parametrisations as solutions of the following PDE,

$$\mathcal{D}\mathbf{x} = X_H(\mathbf{x}). \tag{3.48}$$

Now, given a formal solution $\hat{\mathbf{\Gamma}}$ in the class $\tau^{-1}\mathsf{T}^4_{\mathbb{C}}[[\tau^{-1}]]$ of equation (3.48), which exists due to Theorem 3.2.2, we prove the existence of an unique solution $\mathbf{\Gamma}^-$ (resp. $\mathbf{\Gamma}^+$) of equation (3.48) belonging to the space $\mathfrak{X}_1\left(S_h \times D_r^-\right)$ (resp. $\mathcal{X}_1\left(S_h \times D_r^+\right)$) such that $\mathbf{\Gamma}^\pm \asymp \hat{\mathbf{\Gamma}}$, i.e.

$$\forall\, n \in \mathbb{N}, \quad \exists C > 0, \quad \left\|\mathbf{\Gamma}^\pm(\varphi,\tau) - \mathbf{\Gamma}_n(\varphi,\tau)\right\| \leq C\tau^{-n-1}, \quad \text{in} \quad S_h \times D_r^\pm, \tag{3.49}$$

where $\mathbf{\Gamma}_n$ denotes a truncation of $\hat{\mathbf{\Gamma}}$ as defined in Remark 3.2.2.4. We will prove the existence of such solution for the $-$ case only as the $+$ case is completely analogous modulus minor modifications in the definitions of the sets where the functions are analytic. Then we have the following,

Theorem 3.4.1 (Analytic unstable parametrisation)**.** *Given a formal solution $\hat{\mathbf{\Gamma}} \in \tau^{-1}\mathsf{T}^4_{\mathbb{C}}[[\tau^{-1}]]$ of equation* (3.48) *there is an $r_0 > 0$ sufficiently large such that for every $r > r_0$ the equation* (3.48) *has an unique analytic solution $\mathbf{\Gamma}^- \in \mathfrak{X}_1\left(S_h \times D_r^-\right)$ such that $\mathbf{\Gamma}^- - \mathbf{\Gamma}_n \in \mathfrak{X}_{n+1}\left(S_h \times D_r^-\right)$ for all $n \geq 6$.*

Proof. Let $n \geq 6$ and $r > 0$ (to be chosen later in the proof). Let us look for a solution of equation (3.48) of the form,

$$\mathbf{\Gamma}^- = \mathbf{\Gamma}_n + \xi, \tag{3.50}$$

where $\xi \in \mathfrak{X}_n\left(S_h \times D_r^-\right)$ and $\mathbf{\Gamma}_n$ is defined as in Remark 3.2.2.4. Substituting (3.50) into equation (3.48) we obtain,

$$\mathcal{D}\xi = X_H(\mathbf{\Gamma}_n + \xi) - \mathcal{D}\mathbf{\Gamma}_n.$$

Now we rewrite the previous equation as follows,

$$\mathcal{L}(\xi) = \mathbf{Q}(\xi) + \mathbf{R}_n, \tag{3.51}$$

where $\mathcal{L}$ is a linear operator acting according to the formula $\mathcal{L}(\xi) = \mathcal{D}\xi - DX_H(\mathbf{\Gamma}_n)\xi$ and

$$\mathbf{Q}(\xi) = X_H(\mathbf{\Gamma}_n + \xi) - X_H(\mathbf{\Gamma}_n) - DX_H(\mathbf{\Gamma}_n)\xi, \qquad \mathbf{R}_n = X_H(\mathbf{\Gamma}_n) - \mathcal{D}\mathbf{\Gamma}_n.$$

Note that it follows from Remark 3.2.2.4 that $\mathbf{R}_n \in \mathfrak{X}_{n+1}\left(S_h \times D_r^-\right)$. We focus our attention in solving equation (3.51) with respect to ξ. For that purpose we want to invert the linear operator $\mathcal{L}$ and obtain a new equation from which we can apply a fixed point argument to get the desired solution.

According to Theorem 2.4.1 we can invert the linear operator $\mathcal{L}$ as long as it has a fundamental matrix $\mathbf{U}$ and $\mathbf{Q}(\xi) \in \mathfrak{X}_{n+1}\left(S_h \times D_r^-\right)$ given $\xi \in \mathfrak{X}_n\left(S_h \times D_r^-\right)$. Due to Theorem 3.3.1 there exist an $r_0 > 0$ such that for every $r > r_0$ the linear operator $\mathcal{L}$ has a fundamental matrix $\mathbf{U}$ such that $\mathbf{U} - \mathbf{U}_{n-3} \in \mathfrak{X}^4_{n-2}(S_h \times D_r^-)$.

Now let us show that $\mathbf{Q}(\xi) \in \mathfrak{X}_{n+1}\left(S_h \times D_r^-\right)$. Denote the components of the vector field X_H by (v_1, v_2, v_3, v_4) and consider the following auxiliary functions,

$$\gamma_i(t) = v_i(\mathbf{\Gamma}_n + t\xi) - v_i(\mathbf{\Gamma}_n) - t\nabla v_i(\mathbf{\Gamma}_n)\xi, \quad i = 1, \ldots, 4.$$

Note that $\gamma_i(0) = 0$ for $i = 1, \ldots, 4$ and $\mathbf{Q}(\xi) = (\gamma_1(1), \gamma_2(1), \gamma_3(1), \gamma_4(1))^T$. Now we can integrate by parts each function γ_i to obtain,

$$\gamma_i(1) = \int_0^1 (1-s)\gamma_i''(s)ds, \quad i = 1, \ldots, 4.$$

Then by the intermediate value theorem there exist $t_i \in [0,1]$ for $i = 1, \ldots, 4$ such that $\gamma_i(1) = (1 - t_i)\gamma_i''(t_i)$ for $i = 1, \ldots, 4$ where the second derivative of γ_i can be easily computed

$$\gamma_i''(s) = \xi^T \operatorname{Hess}(v_i)|_{\mathbf{\Gamma}_n + s\xi}\,\xi. \tag{3.52}$$

Now taking into account that $\xi \in \mathfrak{X}_n\left(S_h \times D_r^-\right)$ and the analyticity of X_H it is not difficult to get the following estimate,

$$|\gamma_i(1)| \leq 2\,\|H\|_{C^3}\,|\tau|^{-2n}\,\|\xi\|_n^2,$$

where $\|\cdot\|_{C^3}$ is the usual C^3 norm of a smooth function. Using this upper bound and the fact that given $r_1 > \max\left\{r_0, \frac{1}{\sin\theta_0}\right\}$ then for every $r > r_1$ we have $|\tau|^{-2} \leq |\tau|^{-1}$ for $\tau \in D_r^-$, then we can estimate $\|\mathbf{Q}(\xi)\|_{n+1}$ in the following way,

$$\|\mathbf{Q}(\xi)\|_{n+1} \leq 8\,\|H\|_{C^3}\,\|\xi\|_n^2 \sup_{\tau\in D_r^-} |\tau|^{-n+2} \leq \frac{8\,\|H\|_{C^3}\,\|\xi\|_n^2}{(r_1\sin\theta_0)^{n-2}}, \tag{3.53}$$

where this last estimate holds since $n \geq 6$. Thus $\mathbf{Q}(\xi) \in \mathcal{X}_{n+1}\left(S_h \times D_r^-\right)$.

Thus, it follows from Theorem 2.4.1 that there is an unique bounded linear operator $\mathcal{L}^{-1}$ such that $\mathcal{L}\mathcal{L}^{-1} = \mathrm{Id}$. Thus, in order to solve equation (3.51), it is sufficient to find a fixed point in $\mathfrak{X}_n\left(S_h \times D_r^-\right)$ of the following non-linear operator,

$$\xi \mapsto \mathcal{L}^{-1}(\mathbf{Q}(\xi)) + \mathcal{L}^{-1}(\mathbf{R}_n).$$

Let us denote this non-linear operator by $\mathcal{G}$. So in order to apply the contraction mapping theorem we have to check that $\mathcal{G}$ is contracting in some invariant ball

$$\mathfrak{B}_\rho = \left\{\xi \in \mathfrak{X}_n(S_h \times D_r^-) \mid \|\xi\|_n \leq \rho\right\},$$

where $\rho > 0$. First we prove that $\mathcal{G}(\mathfrak{B}_\rho) \subseteq \mathfrak{B}_\rho$ for some $\rho > 0$. Indeed, let $\rho = 2\left\|\mathcal{L}^{-1}\right\|_{n,n+1}\|\mathbf{R}_n\|_{n+1}$ and $\xi \in \mathfrak{B}_\rho$, then (3.53) implies,

$$\left\|\mathcal{L}^{-1}(\mathbf{Q}(\xi)) - \mathcal{L}^{-1}(\mathbf{R}_n)\right\|_n \leq \left\|\mathcal{L}^{-1}\right\|_{n,n+1}\left(\frac{8\,\|H\|_{C^3}\,\|\xi\|_n^2}{(r_1\sin\theta_0)^{n-2}} + \|\mathbf{R}_n\|_{n+1}\right) \leq \rho,$$

provided r_1 is sufficiently large,

$$r_1 \geq \frac{(16\,\|H\|_{C^3}\left\|\mathcal{L}^{-1}\right\|_{n,n+1}\rho)^{\frac{1}{n-2}}}{\sin\theta_0}. \tag{3.54}$$

Thus $\mathcal{G}$ leaves invariant a closed ball $\mathfrak{B}_\rho$.

To check the contraction we let $\xi_1, \xi_2 \in \mathfrak{B}_\rho$ and consider a line connecting both points, i.e., $\theta_t = (1-t)\xi_1 + t\xi_2$. Clearly $\theta_t \in \mathfrak{B}_\rho$ for all $t \in [0,1]$. Similar as before we define the following auxiliary functions,

$$\psi_i(t) = v_i(\mathbf{\Gamma}_n + \theta_t) - v_i(\mathbf{\Gamma}_n) - \nabla v_i(\mathbf{\Gamma}_n)\theta_t, \quad i = 1,\ldots,4.$$

Note that,

$$\mathbf{Q}(\xi_1) = (\psi_1(0), \psi_2(0), \psi_3(0), \psi_4(0))^T \quad \text{and} \quad \mathbf{Q}(\xi_2) = (\psi_1(1), \psi_2(1), \psi_3(1), \psi_4(1))^T.$$

By the mean value theorem there exist $t_i \in [0,1]$ for $i = 1, \ldots, 4$ such that $\psi_i(1) - \psi_i(0) = \psi_i'(t_i)$. Differentiating the functions ψ_i we get,

$$\psi_i(1) - \psi_i(0) = (\nabla v_i (\mathbf{\Gamma}_n + \theta_{t_i}) - \nabla v_i (\mathbf{\Gamma}_n)) \cdot (\xi_2 - \xi_1), \quad i = 1, \ldots, 4. \tag{3.55}$$

Now we can easily get the following upper bounds for the differences (3.55),

$$|\psi_i(1) - \psi_i(0)| \leq 2 \|H\|_{C^3} \rho |\tau|^{-2n} \|\xi_2 - \xi_1\|_n .$$

Thus,

$$\|\mathbf{Q}(\xi_2) - \mathbf{Q}(\xi_1)\|_{n+1} \leq \frac{8\rho \|H\|_{C^3}}{(r_0 \sin\theta_0)^{n-2}} \|\xi_2 - \xi_1\|_n .$$

Applying the linear operator $\mathcal{L}^{-1}$ and taking into account (3.54) we get,

$$\begin{aligned} \left\|\mathcal{L}^{-1}(\mathbf{Q}(\xi_2) - \mathbf{Q}(\xi_1))\right\|_n &\leq \left\|\mathcal{L}^{-1}\right\|_{n,n+1} \frac{8\rho \|H\|_{C^3}}{(r_0 \sin\theta_0)^{n-2}} \|\xi_2 - \xi_1\|_n \\ &\leq \frac{1}{2} \|\xi_2 - \xi_1\|_n , \end{aligned}$$

which proves that $\|\mathcal{G}(\xi_2) - \mathcal{G}(\xi_1)\|_n \leq \frac{1}{2} \|\xi_2 - \xi_1\|_n$. Thus $\mathcal{G}$ is contracting in the ball $\mathfrak{B}_\rho$ provided $r > r_1$ where,

$$r_1 > \max\left\{ r_0, \frac{1}{\sin\theta_0}, \frac{(16 \|H\|_{C^3} \left\|\mathcal{L}^{-1}\right\|_{n,n+1} \rho)^{\frac{1}{n-2}}}{\sin\theta_0} \right\}.$$

Now let us check that the unique function $\mathbf{\Gamma}^-$ obtained with $n \geq 6$ is in fact independent of n. Increasing r, if necessary, the distance $\|\mathbf{\Gamma}^- - \mathbf{\Gamma}_6\|_6$ can be made as small as we want in order to apply the contraction mapping theorem for $n = 6$. Hence it is independent of n. Finally,

$$\mathbf{\Gamma}^- - \mathbf{\Gamma}_n = \mathbf{\Gamma}^- - \mathbf{\Gamma}_{n+1} + \mathbf{\Gamma}_{n+1} - \mathbf{\Gamma}_n \in \mathfrak{X}_{n+1}(S_h \times D_r^-).$$

This completes the proof of the Theorem. $\square$

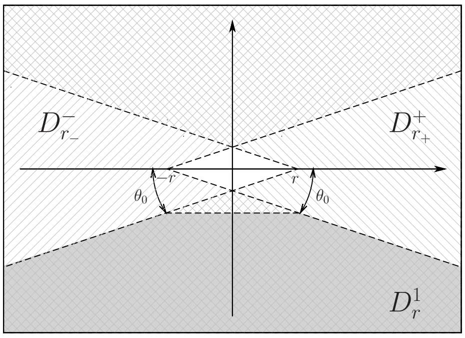

Figure 3.1: The intersection of the domains $D^{\pm}_{r_\pm}$.

As previously observed we can repeat the same arguments of the previous Theorem but now considering the functions defined on the domains $S_h \times D^+_r$. We obtain the following,

Theorem 3.4.2 (Analytic stable parametrisation). *Given a formal solution* $\hat{\mathbf{\Gamma}} \in \tau^{-1}\mathsf{T}^4_{\mathbb{C}}[[\tau^{-1}]]$ *of equation* (3.48) *there is an* $r_0 > 0$ *sufficiently large such that for every* $r > r_0$ *the equation* (3.48) *has an unique analytic solution* $\mathbf{\Gamma}^+ \in \mathfrak{X}_1\left(S_h \times D^+_r\right)$ *such that* $\mathbf{\Gamma}^+ - \mathbf{\Gamma}_n \in \mathfrak{X}_{n+1}\left(S_h \times D^+_r\right)$ *for all* $n \geq 6$.

3.5 Stokes phenomenon

Given a formal solution $\hat{\mathbf{\Gamma}}$ of (3.48) in the class $\tau^{-1}\mathsf{T}^4_{\mathbb{C}}[[\tau^{-1}]]$, Theorem 3.4.1 establishes the existence of an unique analytic vector function $\mathbf{\Gamma}^- : S_h \times D^-_{r_-} \to \mathbb{C}^4$ which parametrises an unstable analytic invariant manifold such that $\mathbf{\Gamma}^- \asymp \hat{\mathbf{\Gamma}}$ (see (3.49) for the definition of $\asymp$). Analogously, Theorem 3.4.2 yields the existence of an analytic vector function $\mathbf{\Gamma}^+ : S_h \times D^+_{r_+} \to \mathbb{C}^4$ which parametrises a stable analytic invariant manifold and having the same asymptotic expansion as $\mathbf{\Gamma}^-$ valid in its domain of definition. Both parametrisations have the same asymptotic expansion valid in the intersections of the domains $S_h \times D^{\pm}_{r_\pm}$ (see Figure 3.1). It is clear that the intersection of the domains has two connected components and the difference $\mathbf{\Gamma}^+ - \mathbf{\Gamma}^-$ is asymptotic to zero, i.e., beyond all algebraic orders. In the following we shall obtain a more precise estimate for the difference of the parametrisations on the lower component of the intersection set, i.e. $S_h \times D^1_{r_1}$ where $r_1 = \max\{r_-, r_+\}$. Similar considerations work for the upper connected component. In order to obtain such estimate we will use the fact that $\mathbf{\Gamma}^+ - \mathbf{\Gamma}^-$ is approximately a solution of the variational equation of X_H along the unstable solution $\mathbf{\Gamma}^-$. So in the following we study the analytic solutions of the variational equation,

$$\mathcal{D}\mathbf{u} = DX_H(\mathbf{\Gamma}^-)\mathbf{u}. \tag{3.56}$$

It is clear that both $\partial_\varphi \mathbf{\Gamma}^-$ and $\partial_\tau \mathbf{\Gamma}^-$ solve equation (3.56). Now using the theory of Section 3.3 we can construct two other independent analytic solutions such that together form a 4 by 4 matrix function $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4) : S_h \times D^-_r :\to \mathbb{C}^{4\times 4}$ which solves equation (3.56) where the vector functions $\mathbf{u}_i : S_h \times D^-_r :\to \mathbb{C}^4$ are the columns of $\mathbf{U}$ and satisfy the following properties,

1. $\mathbf{u}_1 \in \mathfrak{X}_1(S_h \times D^-_r)$, $\mathbf{u}_2 \in \mathfrak{X}_{-3}(S_h \times D^-_r)$, $\mathbf{u}_3 \in \mathfrak{X}_{-2}(S_h \times D^-_r)$ and $\mathbf{u}_4 \in \mathfrak{X}_2(S_h \times D^-_r)$.

2. The first and fourth columns of the matrix $\mathbf{U}$ are the known solutions $\mathbf{u}_1 = \partial_\varphi \mathbf{\Gamma}^-$ and $\mathbf{u}_4 = \partial_\tau \mathbf{\Gamma}^-$,

3. $\{\mathbf{u}_i\}_{i=1,\ldots,4}$ form a symplectic basis in $\mathbb{C}^4$, i.e.,

$$\begin{aligned} &\Omega(\mathbf{u}_1, \mathbf{u}_2) = 0, \quad \Omega(\mathbf{u}_2, \mathbf{u}_4) = 1, \quad \Omega(\mathbf{u}_1, \mathbf{u}_4) = 0 \\ &\Omega(\mathbf{u}_1, \mathbf{u}_3) = 1, \quad \Omega(\mathbf{u}_3, \mathbf{u}_4) = 0, \quad \Omega(\mathbf{u}_3, \mathbf{u}_2) = 0. \end{aligned} \tag{3.57}$$

where Ω is the canonical symplectic form in $\mathbb{C}^4$. The last property implies that $\mathbf{U}$ is a symplectic matrix for all $(\varphi, \tau) \in S_h \times D^-_r$ and $\det(\mathbf{U}) = 1$. A matrix $\mathbf{U}$ satisfying the above properties is called a *normalized fundamental matrix* for the variation equation (3.56).

Corollary 3.5.0.1. *Given an analytic unstable parametrisation $\mathbf{\Gamma}^- \asymp \hat{\mathbf{\Gamma}}$ and a formal normalized fundamental matrix $\hat{\mathbf{U}}$ then there is $r_0 > 0$ such that for every $r > r_0$*

the variational equation (3.56) *has an unique normalized fundamental matrix* $\mathbf{U} : S_h \times D_r^- \to \mathbb{C}^{4\times 4}$ *such that,*

$$\mathbf{U} - \mathbf{U}_n \in \mathfrak{X}_{n+1}^4(S_h \times D_r^-), \quad \forall n \geq 3,$$

where $\mathbf{U}_n$ *is a partial sum of the formal series* $\hat{\mathbf{U}}$ *as defined in Remark 3.2.3.1.*

Proof. From Theorem 3.3.1 we know that for every $n \geq 3$ there exists $r_0 > 0$ such that for every $r > r_0$ there exists an unique canonical fundamental matrix $\mathbf{U}$ such that $\mathbf{U} - \mathbf{U}_n \in \mathfrak{X}_{n+1}(S_h \times D_r^-)$. Thus we only need to prove that $\mathbf{U}$ is in fact independent of n. Indeed for all $n \geq 3$, we can trace the proof of Theorem 3.3.1 and see that, by increasing r if necessary, we can make $\|\mathbf{U} - \mathbf{U}_3\|_3$ as small as we want in order apply the contraction mapping theorem. Now due to the uniqueness of the fixed point we get independence from n. □

Theorem 3.5.1. *Let* $\mu_0 > 0$ *be very small, then there exist a vector* $\boldsymbol{\Theta}_* \in \mathbb{C}^4$ *and an* $r_0 > 0$ *such that for* $r > r_0$ *we have the following asymptotic formula for the difference,*

$$\boldsymbol{\Gamma}^+(\varphi, \tau) - \boldsymbol{\Gamma}^-(\varphi, \tau) = e^{-i(\tau-\varphi)}\mathbf{U}(\varphi, \tau)\boldsymbol{\Theta}_* + O(e^{-(2-\mu_0)i(\tau-\varphi)}), \tag{3.58}$$

valid in $S_h \times D_r^1$ *where* $\mathbf{U}$ *is a normalized fundamental matrix of the variational equation* (3.56).

Proof. Let $\xi_* = \boldsymbol{\Gamma}^+ - \boldsymbol{\Gamma}^-$. Notice that $\xi_* \in \mathfrak{X}_n(S_h \times D_{r_1}^1)$ for all $n \geq 6$. Let us prove that ξ_* admits an exponentially small upper bound. Using the fact that both $\boldsymbol{\Gamma}^-$ and $\boldsymbol{\Gamma}^+$ are solutions of (3.48) we can write,

$$\begin{aligned} &\mathcal{D}\xi_* + \mathcal{D}\boldsymbol{\Gamma}^- = X_H(\boldsymbol{\Gamma}^- + \xi_*) \\ &\Leftrightarrow \mathcal{D}\xi_* - DX_H(\boldsymbol{\Gamma}^-)\xi_* = X_H(\boldsymbol{\Gamma}^- + \xi_*) - X_H(\boldsymbol{\Gamma}^-) - DX_H(\boldsymbol{\Gamma}^-)\xi_*. \end{aligned}$$

Now we rewrite the previous equation as follows,

$$\mathcal{L}(\xi_*) = \mathbf{Q}(\xi_*), \tag{3.59}$$

where $\mathcal{L}(\xi_*) = \mathcal{D}\xi_* - DX_H(\mathbf{\Gamma}^-)\xi_*$ and

$$\mathbf{Q}(\xi_*) = X_H(\mathbf{\Gamma}^- + \xi_*) - X_H(\mathbf{\Gamma}^-) - DX_H(\mathbf{\Gamma}^-)\xi_*.$$

Similar estimates as in the proof of theorem 3.4.1 (in particular the estimate (3.53)) show that given $r_2 > \max\left\{r_1, \frac{1}{\sin\theta_0}\right\}$ we have,

$$\|\mathbf{Q}(\xi_*)\|_{n+3} \leq \frac{8\,\|H\|_{C^3}\,\|\xi_*\|_n^2}{(r\sin\theta_0)^n}, \tag{3.60}$$

valid in $S_h \times D_r^1$ for every $r > r_2$ and every $n \geq 6$. Therefore, $\mathbf{Q}(\xi_*) \in \mathcal{X}_{n+3}(S_h \times D_r^1)$ for $n \geq 6$. Moreover, due to Corollary 3.5.0.1 there exists an $r_3 > 0$ such that for $r > r_3$ there exists a unique normalized fundamental matrix $\mathbf{U} : S_h \times D_r^1 \to \mathbb{C}^{4\times 4}$ such that $\mathcal{L}(\mathbf{U}) = 0$ and $\mathbf{U} \asymp \hat{\mathbf{U}}$. Hence for $r > \max\left\{\frac{\pi\tan\theta_0}{1-\tan\theta_0}, r_3, r_2\right\}$ we can apply Theorem 2.4.2 and get a bounded operator $\mathcal{L}^{-1} : \mathfrak{X}_{n+3}(S_h \times D_r^1) \to \mathfrak{X}_n(S_h \times D_r^1)$ which is a right inverse of $\mathcal{L}$, i.e., $\mathcal{L}\mathcal{L}^{-1} = \mathrm{Id}$. Consequently, the function,

$$\xi_0 = \xi_* - \mathcal{L}^{-1}(\mathbf{Q}(\xi_*)), \tag{3.61}$$

belongs to the kernel of $\mathcal{L}$. Thus, due to Theorem 2.4.2 there exists a 2π-periodic analytic function $\mathbf{c}_0 : \mathbb{H}_{r-h} \to \mathbb{C}^4$, continuous on the closure of its domain, such that $\xi_0(\varphi,\tau) = \mathbf{U}(\varphi,\tau)\mathbf{c}_0(\tau-\varphi)$. The domain of $\mathbf{c}_0$ is a half plane,

$$\mathbb{H}_{r-h} = \{s \in \mathbb{C} \mid \mathrm{Im}(s) < -r + h\}.$$

Therefore equation (3.61) is equivalent to,

$$\xi_* = \mathcal{L}^{-1}(\mathbf{Q}(\xi_*)) + \mathbf{U}\mathbf{c}_0,$$

and the function ξ_* is a fixed point of the nonlinear operator,

$$\xi \mapsto \mathcal{L}^{-1}(\mathbf{Q}(\xi)) + \mathbf{U}\mathbf{c}_0, \tag{3.62}$$

which is defined in $\mathfrak{X}_n(S_h \times D_r^1)$. Now let $\rho = 2\,\|\mathbf{U}\mathbf{c}_0\|_n$. Similar estimates as in the proof of Theorem 3.4.1 show that the nonlinear operator (3.62) is contracting in

$\mathfrak{B}_\rho = \left\{\xi \in \mathfrak{X}_n(S_h \times D_r^1) \mid \|\xi\|_n \le \rho\right\}$ provided $r > r_4$ where,

$$r_4 > \frac{\left(16 \left\|\mathcal{L}^{-1}\right\|_{n,n+3} \|H\|_{C^3} \rho\right)^{\frac{1}{n-4}}}{\sin\theta_0}.$$

Therefore, by the contracting mapping theorem, the sequence $(\xi_k)_{k\ge 0}$ defined by,

$$\xi_{k+1} = \mathcal{L}^{-1}(\mathbf{Q}(\xi_k)) + \mathbf{U}\mathbf{c}_0, \qquad k \ge 0, \tag{3.63}$$

converges to ξ_*, i.e., $\|\xi_k - \xi_*\|_n \to 0$ as $k \to \infty$. Now define a new sequence of functions $\tilde{\xi}_k$ as follows,

$$\tilde{\xi}_k(\varphi,\tau) = e^{i(\tau-\varphi)}\mathbf{U}^{-1}(\varphi,\tau)\xi_k(\varphi,\tau), \quad \forall k \in \mathbb{N}_0. \tag{3.64}$$

In order to prove an exponential upper bound for ξ_* it is sufficient to prove that there exists an $C_* > 0$ such that,

$$C_k := \sup_{(\varphi,\tau)\in S_h\times D_r^1} \sum_{i=1}^{4} \left|\tilde{\xi}_{k,i}(\varphi,\tau)\right| < C_*, \quad \forall k \ge 0, \tag{3.65}$$

where $\tilde{\xi}_{k,i}$ are the components of the vector function $\tilde{\xi}_k$. Taking into account (3.64) and (3.65) it is not difficult to derive the following bound for $\|\xi_k\|_n$,

$$\|\xi_k\|_n \le 4K_{\mathbf{U}} \sup_{(\varphi,\tau)\in S_h\times D_r^1} \left|\tau^{n+4}e^{-i(\tau-\varphi)}\right| C_k.$$

Thus according to (3.60) and the previous estimate we obtain,

$$\|\mathbf{Q}(\xi_k)\|_{n+3} \le \frac{2^7 \|H\|_{C^3} K_{\mathbf{U}}^2 \sup_{(\varphi,\tau)\in S_h\times D_r^1} \left|\tau^{2n+8}e^{-2i(\tau-\varphi)}\right|}{(r\sin\theta_0)^n} C_k^2. \tag{3.66}$$

Now we construct another right inverse of $\mathcal{L}$ as follows. Using (3.64) and (3.65) and again similar estimates as in the proof of the Theorem 3.4.1 (in particular (3.52)) show that the components of $\mathbf{Q}(\xi_k)$ can be bounded by,

$$2\|H\|_{C^3} K_{\mathbf{U}} \left|e^{-2i(\tau-\varphi)}\tau^6\right| C_k^2,$$

valid in $S_h \times D_r^1$. Thus, if $\mu' > 0$ is any small positive real number we have,

$$\|\mathbf{Q}(\xi_k)\|_{\mu'} \le 8\|H\|_{C^3} K_{\mathbf{U}} \sup_{(\varphi,\tau)\in S_h\times D_r^1} \left|e^{-\mu' i(\tau-\varphi)}\tau^6\right| C_k^2. \tag{3.67}$$

Thus, for a given $1 > \mu_0 > \mu'$ we can apply Theorem 2.4.3 and obtain a bounded linear operator $\mathcal{L}_{\mu'}^{-1} : \mathfrak{Y}_{\mu'}(S_h \times D_r^1) \to \mathfrak{Y}_{\mu_0}(S_h \times D_r^1)$ which is a right inverse of $\mathcal{L}$, i.e., $\mathcal{L}\mathcal{L}_{\mu'}^{-1} = \mathrm{Id}$.

Note that $\mathcal{L}^{-1}(\mathbf{Q}(\xi_k)) - \mathcal{L}_{\mu'}^{-1}(\mathbf{Q}(\xi_k))$ belongs to the kernel of $\mathcal{L}$. It follows from Theorem 2.4.2 that there exists a 2π-periodic analytic function $\mathbf{c}_k : \mathbb{H}_{r-h} \to \mathbb{C}^4$, continuous on the closure of its domain, such that,

$$\mathbf{U}\mathbf{c}_k = \mathcal{L}^{-1}(\mathbf{Q}(\xi_k)) - \mathcal{L}_{\mu'}^{-1}(\mathbf{Q}(\xi_k)). \tag{3.68}$$

In order to prove the uniform bound (3.65) we rewrite the recursion formula in (3.63) as follows,

$$\xi_{k+1} = \mathcal{L}_{\mu'}^{-1}(\mathbf{Q}(\xi_k)) + \mathbf{U}\mathbf{c}_k + \mathbf{U}\mathbf{c}_0. \tag{3.69}$$

Now taking into account the relation (3.64) the previous equation is equivalent to,

$$\tilde{\xi}_{k+1} = e^{i(\tau-\varphi)}\mathbf{U}^{-1}\mathcal{L}_{\mu'}^{-1}(\mathbf{Q}(\xi_k)) + e^{i(\tau-\varphi)}\mathbf{c}_k + e^{i(\tau-\varphi)}\mathbf{c}_0. \tag{3.70}$$

The remaining steps of the proof are to estimate these functions in a proper way. In order to simplify the presentation of the subsequent estimates, it is convenient to introduce an adapted supremum norm as follows. Given a bounded analytic function $g = (g_1, \ldots, g_4) : S_h \times D_r^1 \times \to \mathbb{C}^4$ consider its norm $\|g\|_1$ defined by,

$$\|g\| = \sup_{(\varphi,\tau)\in S_h\times D_r^1} \sum_{i=1}^{4} |g_i(\varphi,\tau)| .$$

We also consider its usual induced norm on the space of 4 by 4 matrices valued functions $G = (G_{i,j}) : S_h \times D_r^1 \times \to \mathbb{C}^4$,

$$\|G\| = \max_{j=1,\ldots,4} \sup_{(\varphi,\tau)\in S_h\times D_r^1} \sum_{i=1}^{4} |G_{i,j}(\varphi,\tau)| .$$

Note that $\left\|\tilde{\xi}_k\right\| = C_k$ and for a given analytic function $\gamma : D_r^1 \to \mathbb{C}$ such that $\gamma(\tau) = O(\tau^{-3})$ we have,

$$\left\|\gamma\mathbf{U}^{-1}\right\| \leq 4K_{\mathbf{U}^{-1}} \sup_{\tau\in D_r^1} \left|\tau^3\gamma(\tau)\right| . \tag{3.71}$$

With this norm in mind it is not difficult to get the following inequalities,

$$\begin{aligned}\left\|\tau^n \mathcal{L}^{-1}(\mathbf{Q}(\xi_k))\right\| &\leq \left\|\mathcal{L}^{-1}(\mathbf{Q}(\xi_k))\right\|_n,\\ \left\|e^{(2-\mu_0)i(\tau-\varphi)}\mathcal{L}_{\mu'}^{-1}(\mathbf{Q}(\xi_k))\right\| &\leq \left\|\mathcal{L}_{\mu'}^{-1}(\mathbf{Q}(\xi_k))\right\|_{\mu_0}.\end{aligned} \tag{3.72}$$

Now let us estimate the terms in the right hand side of equation (3.70). Starting with the first term we get,

$$e^{i(\tau-\varphi)}\mathbf{U}^{-1}\mathcal{L}_{\mu'}^{-1}(\mathbf{Q}(\xi_k)) = e^{(\mu_0-1)i(\tau-\varphi)}\mathbf{U}^{-1}e^{(2-\mu_0)i(\tau-\varphi)}\mathcal{L}_{\mu'}^{-1}(\mathbf{Q}(\xi_k)).$$

Taking into account (3.71) and (3.72) we obtain the following estimate,

$$\begin{aligned}\left\|e^{i(\tau-\varphi)}\mathbf{U}^{-1}\mathcal{L}_{\mu'}^{-1}(\mathbf{Q}(\xi_k))\right\| &\leq \left\|e^{(\mu_0-1)i(\tau-\varphi)}\mathbf{U}^{-1}\right\| \left\|e^{(2-\mu_0)i(\tau-\varphi)}\mathcal{L}_{\mu'}^{-1}(\mathbf{Q}(\xi_k))\right\|,\\ &\leq 4K_{\mathbf{U}^{-1}} \sup_{(\varphi,\tau)\in S_h\times D_r^1}\left|\tau^3 e^{(\mu_0-1)i(\tau-\varphi)}\right| \left\|\mathcal{L}_{\mu'}^{-1}(\mathbf{Q}(\xi_k))\right\|_{\mu_0}.\end{aligned}$$

Thus, using (3.67) we get,

$$\left\|e^{i(\tau-\varphi)}\mathbf{U}^{-1}\mathcal{L}_{\mu'}^{-1}(\mathbf{Q}(\xi_k))\right\| \leq \bar{K}C_k^2, \tag{3.73}$$

where

$$\bar{K} = 2^5 K_{\mathbf{U}^{-1}}K_{\mathbf{U}}\left\|\mathcal{L}_{\mu'}^{-1}\right\|_{\mu_0,\mu'}\|H\|_{C^3}\sup_{(\varphi,\tau)\in S_h\times D_r^1}\left|\tau^9 e^{-(1-(\mu_0-\mu'))i(\tau-\varphi)}\right| < \infty, \tag{3.74}$$

since $n \geq 6$ and $0 < \mu' < \mu_0 < 1$. Now we deal with the second term of equation (3.70). Taking into account (3.68) we write,

$$\begin{aligned}\mathbf{c}_k &= \mathbf{U}^{-1}\mathcal{L}^{-1}(\mathbf{Q}(\xi_k)) - \mathbf{U}^{-1}\mathcal{L}_{\mu'}^{-1}(\mathbf{Q}(\xi_k)),\\ &= \tau^{-n}\mathbf{U}^{-1}\tau^n\mathcal{L}^{-1}(\mathbf{Q}(\xi_k)) - e^{-(2-\mu_0)i(\tau-\varphi)}\mathbf{U}^{-1}e^{(2-\mu_0)i(\tau-\varphi)}\mathcal{L}_{\mu'}^{-1}(\mathbf{Q}(\xi_k)).\end{aligned}$$

Thus, using the estimates (3.71) and (3.72) we can bound $\mathbf{c}_k$ as follows,

$$\begin{aligned}\|\mathbf{c}_k\| &\leq \left\|\tau^{-n}\mathbf{U}^{-1}\right\|\left\|\tau^n\mathcal{L}^{-1}(\mathbf{Q}(\xi_k))\right\| + \left\|e^{-(2-\mu_0)i(\tau-\varphi)}\mathbf{U}^{-1}\right\|\left\|e^{(2-\mu_0)i(\tau-\varphi)}\mathcal{L}_{\mu'}^{-1}(\mathbf{Q}(\xi_k))\right\|,\\ &\leq 4K_{\mathbf{U}^{-1}}\left(\sup_{\tau\in D_r^1}\left|\tau^{3-n}\right|\left\|\mathcal{L}^{-1}(\mathbf{Q}(\xi_k))\right\|_n\right.\\ &\qquad\left. + \sup_{(\varphi,\tau)\in S_h\times D_r^1}\left|\tau^3 e^{-(2-\mu_0)i(\tau-\varphi)}\right|\left\|\mathcal{L}_{\mu'}^{-1}(\mathbf{Q}(\xi_k))\right\|_{\mu_0}\right).\end{aligned} \tag{3.75}$$

Thus (3.66), (3.67) and (3.75) imply that,

$$\|\mathbf{c}_k\| \leq K_{\mathbf{c}_k} e^{-r+h} C_k^2, \tag{3.76}$$

where,

$$K_{\mathbf{c}_k} = 2^5 K_{\mathbf{U}^{-1}} K_{\mathbf{U}} \|H\|_{C^3} \left(\frac{2^4 \left\|\mathcal{L}^{-1}\right\|_{n,n+3} K_{\mathbf{U}} \sup_{(\varphi,\tau)\in S_h\times D_r^1} \left|\tau^{n+11} e^{-i(\tau-\varphi)}\right|}{r^n \sin^n \theta_0} \right.$$
$$\left. + \sup_{(\varphi,\tau)\in S_h\times D_r^1} \left|\tau^9 e^{-(1-(\mu_0-\mu'))i(\tau-\varphi)}\right| \left\|\mathcal{L}_{\mu'}^{-1}\right\|_{\mu_0,\mu'} \right) < \infty, \tag{3.77}$$

since $n \geq 6$ and $0 < \mu' < \mu_0 < 1$.

In order to complete the estimation of the terms of equation (3.70) we need the following simple result from complex analysis,

Claim 3.5.1.1. Let $\sigma > 0$ and $c : \mathbb{H}_\sigma \to \mathbb{C}$ an analytic function, 2π-periodic, continuous in the closure of $\mathbb{H}_\sigma$ and $\lim_{\mathrm{Im}(s)\to-\infty} c(s) = 0$. Then we can bound the function c as follows,

$$|c(s)| \leq \sup_{\mathrm{Im}(s)=-\sigma} |c(s)|\, e^{\mathrm{Im}(s)+\sigma}. \tag{3.78}$$

Proof. The proof is very simple as is just an application of the maximum modulus principle for analytic functions. □

Applying the previous result to each component of the 2π-periodic analytic vector function $\mathbf{c}_k = (c_{k,1}, \ldots, c_{k,4}) : \mathbb{H}_{r-h} \to \mathbb{C}^4$ we get,

$$|c_{k,i}(s)| \leq \sup_{\mathrm{Im}(s)=-r+h} |c_{k,i}(s)|\, e^{\mathrm{Im}(s)+r-h}, \quad i = 1, \ldots, 4.$$

Thus,

$$\sup_{(\varphi,\tau)\in S_h\times D_r^1} \left|e^{i(\tau-\varphi)} c_{k,i}(\tau-\varphi)\right| \leq \sup_{\mathrm{Im}(s)=-r+h} |c_{k,i}(s)|\, e^{r-h}, \quad i = 1, \ldots, 4,$$

and taking into account (3.76) we get,

$$\left\|e^{i(\tau-\varphi)}\mathbf{c}_k\right\| \leq K_{\mathbf{c}_k} C_k^2. \tag{3.79}$$

For the last term of equation (3.70) we know that $\mathbf{c}_0 = \mathbf{U}^{-1}\xi_0$ and taking into account (3.64) we conclude that $\left\| e^{i\tau}\mathbf{c}_0 \right\| = C_0$. Using the previous claim we can show that $C_0 < \infty$. Thus, it follows from equation (3.70) and the estimates (3.73) and (3.79) that,

$$C_{k+1} \leq \left(\bar{K} + K_{\mathbf{c}_k}\right) C_k^2 + C_0. \tag{3.80}$$

Note that both $\bar{K}$ and $K_{\mathbf{c}_k}$ which are given by expressions (3.74) and (3.77) respectively decay to zero as $r \to +\infty$. In fact for any $m \in \mathbb{N}$ it is easy to see that $\bar{K} = O(r^{-m})$ and $K_{\mathbf{c}_k} = O(r^{-m})$. Thus there exist $r_0 > 0$ sufficiently large such that for $r > r_0$ we have,

$$\left(\bar{K} + K_{\mathbf{c}_k}\right) C_0 \leq \frac{1}{4},$$

which together with (3.80) implies that $C_k \leq 2C_0$ for all $k \geq 0$. Consequently $\left\| e^{i(\tau-\varphi)}\mathbf{U}^{-1}\xi_* \right\| \leq 2C_0$. In order to finish the proof of the theorem note that the estimate (3.67) applied to ξ_* implies that $\mathbf{Q}(\xi_*) \in \mathfrak{Y}_{\mu'}(S_h \times D_r^1)$. Moreover, as $\xi_* - \mathcal{L}_{\mu'}^{-1}\left(\mathbf{Q}(\xi_*)\right) \in \mathrm{Ker}(\mathcal{L})$ then there exists a analytic 2π-periodic vector function $\mathbf{c}_* : \mathbb{H}_{r-h} \to \mathbb{C}^4$ such that $\xi_* = \mathbf{U}\mathbf{c}_* + \mathcal{L}_{\mu'}^{-1}\left(\mathbf{Q}(\xi_*)\right)$. Since $\lim_{\mathrm{Im}(s)\to-\infty} \mathbf{c}_*(s) = 0$, we can write its Fourier series as follows,

$$\mathbf{c}_*(s) = \sum_{m=1}^{\infty} \mathbf{c}_{*,m} e^{-ims},$$

where $\mathbf{c}_{*,m} \in \mathbb{C}^4$. Moreover, as

$$\mathcal{L}_{\mu'}^{-1}\left(\mathbf{Q}(\xi_*)\right) \in \mathfrak{Y}_{\mu_0}(S_h \times D_r^1),$$

we have that,

$$\xi_*(\varphi, \tau) = e^{-i(\tau-\varphi)}\mathbf{U}(\varphi,\tau)\Theta_* + O\left(e^{-(2-\mu_0)i(\tau-\varphi)}\right),$$

where $\Theta_* := \mathbf{c}_{*,1}$. This completes the proof of the Theorem. $\square$

Remark 3.5.1.1. One can repeat the arguments of the previous proof and obtain a similar estimate for the difference of the parametrisations defined on the upper connected component $S_h \times D_r^{1,+}$,

$$\mathbf{\Gamma}^+(\varphi,\tau) - \mathbf{\Gamma}^-(\varphi,\tau) = e^{i(\tau-\varphi)}\mathbf{U}(\varphi,\tau)\mathbf{\Theta}_*^+ + O(e^{(2-\mu_0)i(\tau-\varphi)}),$$

where $D_r^{1,+} = D_r^+ \cap D_r^- \cap \{\tau \in \mathbb{C} \mid \mathrm{Im}\tau > r\}$, $\mathbf{\Theta}_*^+ \in \mathbb{C}^4$ and $\mu_0 > 0$ is arbitrarily small.

Remark 3.5.1.2. Note that the previous Theorem provides an exponentially small upper bound for the difference $\mathbf{\Gamma}^+ - \mathbf{\Gamma}^-$. In fact, there exists an $M > 0$ such that,

$$\left\|\mathbf{\Gamma}^+(\varphi,\tau) - \mathbf{\Gamma}^-(\varphi,\tau)\right\| \leq M\,|\tau|^3\,e^{\mathrm{Im}(\tau)},$$

valid in $S_h \times D_r^1$.

3.5.1 Stokes Constant

In this subsection we use the asymptotic formula of Theorem 3.5.1 to construct an analytic invariant known as Stokes constant that measures the splitting distance of the complex invariant manifolds parametrised by $\mathbf{\Gamma}^\pm$. This constant is also related to the Stokes phenomenon where two difference analytic functions which possess a common asymptotic expansion in a common region differ by an exponentially small term. The Stokes constant is the normalized amplitude of this exponentially small term. In order to define this invariant, let us first prove two technical Lemmas which we will use later on. Let $\Delta(\varphi,\tau) = \mathbf{\Gamma}^+(\varphi,\tau) - \mathbf{\Gamma}^-(\varphi,\tau)$.

Lemma 3.5.1. *For every $v \in \mathbb{C}^4$ we have,*

$$\Omega(\mathbf{\Theta}_*^\pm, v) = \lim_{\mathrm{Im}(\tau)\to\pm\infty} \Omega(\Delta(\varphi,\tau), \mathbf{U}(\varphi,\tau)v)e^{\mp i(\tau-\varphi)},$$

where the convergence of the limit in the right hand side is uniform with respect to $\varphi \in S_h$.

Proof. According to Theorem 3.5.1 and Remark 3.5.1.1 we have the following asymptotic formula,

$$\Delta(\varphi,\tau) = e^{\pm i(\tau-\varphi)}\mathbf{U}(\varphi,\tau)\mathbf{\Theta}_*^\pm + O\left(e^{\pm(2-\mu_0)i(\tau-\varphi)}\right), \tag{3.81}$$

valid in $S_h \times D_r^{1,\pm}$ for $\mu_0 \in (0,1)$ very small, $r > 0$ sufficiently large. Now taking into account that $\mathbf{U}$ is a normalized fundamental matrix and formula (3.81) we get at once,

$$\begin{aligned}\Omega(\Delta(\varphi,\tau), \mathbf{U}(\varphi,\tau)v)e^{\mp i(\tau-\varphi)} &= \Omega(\mathbf{U}(\varphi,\tau)\mathbf{\Theta}_*^{\pm}, \mathbf{U}(\varphi,\tau)v) + O\left(e^{\pm(1-\mu_0)i(\tau-\varphi)}\right), \\ &= \Omega(\mathbf{\Theta}_*^{\pm}, v) + O\left(e^{\pm(1-\mu_0)i(\tau-\varphi)}\right).\end{aligned}$$

which proves the desired formula by taking the limit as $\mathrm{Im}(\tau) \to \pm\infty$. Moreover it is clear that the convergence is uniform with respect to $\varphi \in S_h$. ☐

Lemma 3.5.2. *The following limits exist, are independent of φ and the convergence is uniform in S_h,*

$$\Theta_0^{\pm} := \lim_{\mathrm{Im}(\tau)\to\pm\infty} \Omega(\Delta(\varphi,\tau), \partial_\varphi \mathbf{\Gamma}^-(\varphi,\tau))e^{\mp i(\tau-\varphi)} < \infty. \tag{3.82}$$

Moreover,

1. $\Theta_0^{\pm} = -\lim_{\mathrm{Im}(\tau)\to\pm\infty} \Omega(\Delta(\varphi,\tau), \partial_\tau \mathbf{\Gamma}^-(\varphi,\tau))e^{\mp i(\tau-\varphi)}$,
2. *If H is real analytic then,*

$$\Theta_0^+ = \begin{cases} -\overline{\Theta_0^-} & if\ \eta > 0, \\ \overline{\Theta_0^-} & if\ \eta < 0. \end{cases}$$

3. *For any other solutions $\tilde{\mathbf{\Gamma}}^{\pm} \in \mathfrak{X}_1(S_h \times D_{\tilde{r}}^{\pm})$ of equation* (3.4) *such that $\tilde{\mathbf{\Gamma}}^{\pm} \asymp \hat{\tilde{\mathbf{\Gamma}}}$ where $\hat{\tilde{\mathbf{\Gamma}}} \in \tau^{-1}\mathsf{T}_{\mathbb{C}}^4[[\tau^{-1}]]$ is a formal solution of equation* (3.4) *we have the following relation $\tilde{\Theta}_0^{\pm} = \Theta_0^{\pm}e^{\pm i(\tau_0-\varphi_0)}$ for some $(\varphi_0,\tau_0) \in \mathbb{C}^2$ where the definition of $\tilde{\Theta}_0^{\pm}$ is analogous to* (3.82) *for the parametrisations $\tilde{\mathbf{\Gamma}}^{\pm}$.*

Proof. That the limits (3.82) exist and are uniform with respect to φ follows from the previous Lemma with $v = (1,0,0,0)$. Now let us prove that

$$\Theta_0^- = -\lim_{\mathrm{Im}(\tau)\to-\infty} \Omega(\Delta(\varphi,\tau), \partial_\tau \mathbf{\Gamma}^-(\varphi,\tau))e^{i(\tau-\varphi)},$$

(the $+$ case being completely analogous). First note that (3.81) implies,

$$H(\mathbf{\Gamma}^+(\varphi,\tau)) = H(\mathbf{\Gamma}^-(\varphi,\tau)) + \nabla H(\mathbf{\Gamma}^-(\varphi,\tau))\Delta(\varphi,\tau) + O(e^{-2i(\tau-\varphi)}).$$

Now taking into account that $H(\mathbf{\Gamma}^{\pm}(\varphi,\tau)) = 0$ we get,

$$\lim_{\mathrm{Im}(\tau)\to-\infty} \nabla H(\mathbf{\Gamma}^-(\varphi,\tau))\Delta(\varphi,\tau)e^{i(\tau-\varphi)} = 0. \tag{3.83}$$

Moreover,

$$\nabla H(\mathbf{\Gamma}^-)\Delta = \Omega(X_H(\mathbf{\Gamma}^-),\Delta) = \Omega(\mathcal{D}\mathbf{\Gamma}^-,\Delta) = -\left(\Omega(\Delta,\partial_\varphi\mathbf{\Gamma}^-) + \Omega(\Delta,\partial_\tau\mathbf{\Gamma}^-)\right).$$

Thus, (3.83) yields,

$$\lim_{\mathrm{Im}(\tau)\to-\infty} \left(\Omega(\Delta(\varphi,\tau),\partial_\varphi\mathbf{\Gamma}^-(\varphi,\tau)) + \Omega(\Delta(\varphi,\tau),\partial_\tau\mathbf{\Gamma}^-(\varphi,\tau))\right)e^{i(\tau-\varphi)} = 0$$

which proves the desired equality.

Now suppose that H is real analytic and $\eta > 0$. Let us prove that $\overline{\Theta_0^-} = -\Theta_0^+$. Since Θ_0^- is defined by a limit as $\mathrm{Im}(\tau)\to-\infty$ we can take a sequence $\tau_n = -i\sigma_n$ where σ_n is any real sequence such that $\sigma_n\to+\infty$ as $n\to+\infty$. Then,

$$\Theta_0^- = \lim_{n\to+\infty}\Omega(\Delta(0,-i\sigma_n),\partial_\varphi\mathbf{\Gamma}^-(0,-i\sigma_n))e^{\sigma_n}.$$

Now it follows from Remark 3.2.2.2 that $\overline{\Delta(0,-i\sigma_n)} = \Delta(\pi,i\sigma_n)$ and $\overline{\partial_\varphi\mathbf{\Gamma}^-(0,-i\sigma_n)} = \partial_\varphi\mathbf{\Gamma}^-(\pi,i\sigma_n)$. Thus,

$$\begin{aligned}\overline{\Theta_0^-} &= \lim_{n\to+\infty}\Omega(\overline{\Delta(0,-i\sigma_n)},\overline{\partial_\varphi\mathbf{\Gamma}^-(0,-i\sigma_n)})e^{\sigma_n}\\ &= \lim_{n\to+\infty}\Omega(\Delta(\pi,i\sigma_n),\partial_\varphi\mathbf{\Gamma}^-(\pi,i\sigma_n))e^{i(-i\sigma_n-\pi)}e^{-i\pi}\\ &= -\Theta_0^+.\end{aligned}$$

Analogous considerations can be used to prove that $\overline{\Theta_0^-} = \Theta_0^+$ when $\eta < 0$.

Finally, let $\tilde{\mathbf{\Gamma}}^{\pm}\in\mathfrak{X}_1(S_h\times D^{\pm}_{\tilde{r}})$ be two solutions of equation (3.4) asymptotic to $\hat{\tilde{\mathbf{\Gamma}}}$. Then it follows from Theorem 3.2.2 that there exist $(\varphi_0,\tau_0)\in\mathbb{C}^2$ such that $\hat{\tilde{\mathbf{\Gamma}}}(\varphi,\tau) = \hat{\mathbf{\Gamma}}(\varphi+\varphi_0,\tau+\tau_0)$. Thus, uniqueness of solutions $\tilde{\mathbf{\Gamma}}^{\pm}\asymp\hat{\tilde{\mathbf{\Gamma}}}$ and $\mathbf{\Gamma}^{\pm}\asymp\hat{\mathbf{\Gamma}}$ allows us to conclude that $\tilde{\mathbf{\Gamma}}^{\pm}(\varphi,\tau) = \mathbf{\Gamma}^{\pm}(\varphi+\varphi_0,\tau+\tau_0)$. Therefore,

$$\begin{aligned}\tilde{\Theta}_0^{\pm} &= \lim_{\mathrm{Im}(\tau)\to\pm\infty}\Omega(\tilde{\mathbf{\Gamma}}^+(\varphi,\tau)-\tilde{\mathbf{\Gamma}}^-(\varphi,\tau),\partial_\varphi\tilde{\mathbf{\Gamma}}^-(\varphi,\tau))e^{\mp i(\tau-\varphi)}\\ &= \lim_{\mathrm{Im}(\tau)\to\pm\infty}\Omega(\Delta(\varphi+\varphi_0,\tau+\tau_0),\partial_\varphi\mathbf{\Gamma}^-(\varphi+\varphi_0,\tau+\tau_0))e^{\mp i(\tau+\tau_0-(\varphi+\varphi_0))}e^{\pm i(\tau_0-\varphi_0)}\\ &= \Theta_0^{\pm}e^{\pm i(\tau_0-\varphi_0)}.\end{aligned}$$

□

Theorem 3.5.2 (Stokes constant)**.** *Let $\mathfrak{H}_0$ be the space of analytic Hamiltonian functions $H : \mathcal{U} \to \mathbb{C}$ which have the same properties as described in the introduction of the present Chapter. For a given $H \in \mathfrak{H}_0$ the constants $\Theta_0^{\pm}$ define a functional $\mathcal{K}_0 : \mathfrak{H}_0 \to \mathbb{C}$ according to the formula,*

$$\mathcal{K}_0 = -\Theta_0^- \Theta_0^+ .$$

In other words, $\mathcal{K}_0$ is independent of the choice of the parametrisations $\mathbf{\Gamma}^{\pm}$. Moreover, $\mathcal{K}_0$ is independent of the coordinate system, i.e., if $\tilde{H} \in \mathfrak{H}_0$ is another Hamiltonian function which is conjugated to H, i.e., $\tilde{H} = H \circ \Psi$ for some analytic symplectic map Ψ which fixes the origin $\Psi(0) = 0$ then $\mathcal{K}_0(H) = \mathcal{K}_0(\tilde{H})$. The number $\sqrt{\mathcal{K}_0(H)}$ is known as the Stokes constant.

Proof. This Theorem follows directly from the previous Lemmas since all the freedom we have in the definition of the $\mathcal{K}_0$ comes from the freedom of the parametrisations $\mathbf{\Gamma}^{\pm}$. As the parametrisations are defined up to translation in (φ, τ) we get the desired conclusion which follows from the third item of the previous Lemma. The coordinate independence also follows from similar considerations. □

Remark 3.5.2.1. If H is real analytic then,

$$\mathcal{K}_0(H) = \begin{cases} \left|\Theta_0^-\right|^2 & \text{if } \eta > 0, \\ -\left|\Theta_0^-\right|^2 & \text{if } \eta < 0. \end{cases}$$

In the stable case, i.e. $\eta > 0$, the Stokes constant is equal to $\left|\Theta_0^-\right|$.

Remark 3.5.2.2. If the Stokes constant $\sqrt{\mathcal{K}_0(H)}$ does not vanish then the asymptotic formula (3.58) provides an exponentially small lower bound for the splitting distance $\|\mathbf{\Gamma}^+(\varphi, \tau) - \mathbf{\Gamma}^-(\varphi, \tau)\|$. Thus implying that H is non-integrable and that the normal form transformation Φ diverges.

Corollary 3.5.2.1. *If H is real analytic and X_H is reversible with respect to the involution* (3.16) *then there exist parametrisations $\mathbf{\Gamma}^{\pm} : S_h \times D_r^{\pm} \to \mathbb{C}^4$ which are symmetric* *in the sense that* $\overline{\mathbf{\Gamma}^{\pm}(\varphi,\tau)} = \mathcal{S}(\mathbf{\Gamma}^{\pm}(-\bar{\varphi},-\bar{\tau}))$ *such that the corresponding constant* Θ_0^- *is a purely imaginary number, i.e.,* $\mathrm{Re}(\Theta_0^-) = 0$*.*

Proof. It follows from Remark 3.2.2.2 and the reversibility of X_H that there exists a formal solution $\hat{\mathbf{\Gamma}} \in \tau^{-1}\mathsf{T}_{\mathbb{C}^4}[[\tau^{-1}]]$ of equation (3.4) such that,

$$\hat{\mathbf{\Gamma}}(\varphi,\tau) = \overline{\mathcal{S}(\hat{\mathbf{\Gamma}}(-\bar{\varphi},-\bar{\tau}))}. \tag{3.84}$$

This formal solution is unique up to translation $\varphi+\pi$, that is, if $\hat{\hat{\mathbf{\Gamma}}}$ is another formal solution of the same class satisfying (3.84) then there is a number $k \in \{0,1\}$ such that $\hat{\hat{\mathbf{\Gamma}}}(\varphi,\tau) = \hat{\mathbf{\Gamma}}(\varphi+k\pi,\tau)$. Now due to Theorem 3.4.1 and Theorem 3.4.2 there exist unique $\mathbf{\Gamma}^{\pm} : S_h \times D_r^{\pm} \to \mathbb{C}^4$ such that $\mathbf{\Gamma}^{\pm} \asymp \hat{\mathbf{\Gamma}}$. If we define $\tilde{\mathbf{\Gamma}}^{\pm}(\varphi,\tau) = \overline{\mathcal{S}(\mathbf{\Gamma}^{\pm}(-\bar{\varphi},-\bar{\tau}))}$ and taking into account that H is real analytic we conclude that the functions $\tilde{\mathbf{\Gamma}}^{\pm} : S_h \times D_r^{\pm} \to \mathbb{C}^4$ are solutions of equation (3.4) and due to (3.84) we also have that $\tilde{\mathbf{\Gamma}}^{\pm} \asymp \hat{\mathbf{\Gamma}}$. Thus, uniqueness of $\mathbf{\Gamma}^{\pm}$ implies that $\overline{\mathcal{S}(\mathbf{\Gamma}^{\pm}(-\bar{\varphi},-\bar{\tau}))} = \mathbf{\Gamma}^{\pm}(\varphi,\tau)$ yielding the first part of the corollary. As for the second part, taking into account the previous Theorem, we can write Θ_0^- as follows,

$$\Theta_0^- = \lim_{n\to+\infty} \Theta(0,-i\sigma_n)e^{\sigma_n},$$

where σ_n is any real sequence such that $\sigma_n \to +\infty$ as $n \to +\infty$. Thus,

$$\begin{aligned}
\overline{\Theta_0^-} &= \lim_{n\to+\infty} \Omega(\overline{\Delta(0,-i\sigma_n)}, \overline{\partial_\tau \mathbf{\Gamma}^-(0,-i\sigma_n)})e^{\sigma_n} \\
&= \lim_{n\to+\infty} \Omega(\mathcal{S}(\Delta(0,-i\sigma_n)), \mathcal{S}(\partial_\tau \mathbf{\Gamma}^-(0,-i\sigma_n)))e^{\sigma_n} \\
&= -\lim_{n\to+\infty} \Omega(\Delta(0,-i\sigma_n), \partial_\tau \mathbf{\Gamma}^-(0,-i\sigma_n))e^{\sigma_n} \\
&= -\Theta_0^-.
\end{aligned}$$

□

Remark 3.5.2.3. In fact the parametrisations $\mathbf{\Gamma}^{\pm}$ of the previous Corollary are uniquely defined by the reversibility up to a translation $\varphi+\pi$ in the first argument.

3.5.2 Analytic dependence of $\mathcal{K}_0$ on a parameter

Let $H : \mathcal{U} \times \mathbb{D}(\nu_0) \to \mathbb{C}^4$ be an analytic function where $\mathcal{U} \subset \mathbb{C}^4$ is an open connected neighbourhood of the origin and $\mathbb{D}(\nu_0)$ an open disc on the complex plane having radius $\nu_0 > 0$ and centered at 0. We also suppose that H is continuous on the closure of $\mathcal{U} \times \mathbb{D}(\nu_0)$. For $\mathbf{x} \in \mathcal{U}$ and $\nu \in \mathbb{D}(\nu_0)$ we shall write $H_\nu(\mathbf{x})$ instead of $H(\mathbf{x}, \nu)$ and say that H_ν is an one-parameter analytic family of Hamiltonian functions. Moreover, for each $\nu \in \mathbb{D}(\nu_0)$ we assume that each Hamiltonian function H_ν satisfies the assumptions of the previous Theorems and that the coefficient η which was defined in the introduction of the present Chapter and depends analytically on the parameter ν satisfies the non-degenerate condition,

$$\eta(\nu) \neq 0, \quad \text{for} \quad \nu \in \overline{\mathbb{D}(\nu_0)}. \tag{3.85}$$

Now by the theory of the previous sections (in particular Theorem 3.5.2) the function $\mathcal{K}_0 : \mathbb{D}(\nu_0) \to \mathbb{C}$ is well defined. Now we consider the following question: How regular is the function $\mathcal{K}_0$? The next Theorem provides the answer,

Theorem 3.5.3. *There exist $\nu_0 > 0$ and parametrisations $\mathbf{\Gamma}_\nu^\pm$ analytic with respect to $\nu \in \mathbb{D}(\nu_0)$ such that $\Theta_0^\pm : \mathbb{D}(\nu_0) \to \mathbb{C}$ are analytic functions.*

According to the definition of $\mathcal{K}_0$ (in Theorem 3.5.2) we conclude that $\mathcal{K}_0 : \mathbb{D}(\nu_0) \to \mathbb{C}$ is analytic since $\mathcal{K}_0$ is independent of the choice of the parametrisations.

Proof of Theorem 3.5.3. Tracing the proofs of Theorems 3.2.1, 3.2.2 and 3.2.3 it is not difficult to see that there exist formal series $\hat{\mathbf{\Gamma}}_\nu$ and $\hat{\mathbf{U}}_\nu$ such that the coefficients of the these formal series depend polynomially on a finite number of coefficients of H_ν which are assumed to be analytic with respect to ν. Thus the coefficients of both $\hat{\mathbf{\Gamma}}_\nu$ and $\hat{\mathbf{U}}_\nu$ are analytic with respect to ν. Note that the theory on the linear operators developed in Chapter 2 can be generalized to functions which are also analytic with respect to ν and following the proofs of Theorems 3.3.1 and 3.4.1 and the fact that the fundamental matrix $\mathbf{U}_0$ defined in (2.39) does not depend on ν we conclude that there exist a normalized fundamental matrix $\mathbf{U}_\nu$ and analytic parametrisations $\mathbf{\Gamma}_\nu^\pm$, all of which are analytic with respect to ν such that $\mathbf{U}_\nu \asymp \hat{\mathbf{U}}_\nu$ and $\mathbf{\Gamma}_\nu^\pm \asymp \hat{\mathbf{\Gamma}}_\nu$. Finally, let $\Delta_\nu = \mathbf{\Gamma}_\nu^+ - \mathbf{\Gamma}_\nu^-$, then according to the proof of Theorem 3.5.1 we conclude that,

$$\Delta_\nu = \mathbf{U}_\nu \mathbf{c}_\nu + \mathbf{R}_\nu, \tag{3.86}$$

where $\mathbf{c}_\nu$ is an analytic 2π-periodic vector function defined in a lower half complex plane, analytic with respect to ν, decaying to zero as $\operatorname{Im}\tau \to -\infty$ and $\mathbf{R}_\nu = O(e^{-(2-\mu_0)i(\tau-\varphi)})$ where the bound is uniform with respect to ν for some $0 < \mu_0 < 1$ very small. Now as in the proof of Theorem 3.5.1 we can represent $\mathbf{c}_\nu$ in Fourier series and conclude that,

$$\mathbf{c}_\nu(\tau - \varphi) = \Theta_\nu e^{-i(\tau-\varphi)} + O(e^{-2i(\tau-\varphi)}), \tag{3.87}$$

where the bound is uniform with respect to ν and,

$$\Theta_\nu = \frac{1}{2\pi}\int_{-i\sigma}^{2\pi - i\sigma} \mathbf{c}_\nu(s)e^{is}ds, \tag{3.88}$$

for some $\sigma > 0$. Clearly Θ_ν is analytic with respect to ν. Thus following the proof of Theorem 3.5.2 and taking into account (3.86), (3.87) and (3.88) we have that,

$$\Theta_0^-(\nu) := \lim_{\operatorname{Im}(\tau)\to-\infty} \Omega(\Delta_\nu(\varphi,\tau), \partial_\varphi \mathbf{\Gamma}_\nu^-(\varphi,\tau))e^{i(\tau-\varphi)} = -\Theta_{\nu,3}$$

where $\Theta_{\nu,3}$ is the third component of the vector Θ_ν. Thus Θ_0^- is an analytic function of ν. This concludes the proof as analogous considerations applied to Θ_0^+ yields analyticity in ν. □

3.5.3 The Stokes constant does not vanish identically

In this subsection we address the following question: Does the Stokes constant $\sqrt{\mathcal{K}_0}$ vanish identically? The answer is no. We shall construct an Hamiltonian satisfying the assumptions of Theorem 3.5.2 such that the corresponding Stokes constant does not vanish.

An important example

Let us define the following family H_ν of Hamiltonians,

$$H_\nu = -I_1 + I_2 + \eta I_3^2 + \nu q_2^5,$$

where $\eta > 0$, ν is in some disc of fixed radius around the origin and I_i, $i = 1, \ldots, 3$ are defined in (3.1). Notice that $H_0 = H^0$ as defined in (3.1) and moreover H^0 is integrable (where I_1 is first integral independent of H^0). We will often refer to subsection 2.4.2 of Chapter 2 for a detailed study of the parametrisations and corresponding variational equations of H^0.

Now according to Theorem 3.4.1 (resp. Theorem 3.4.2) there exist $r > 0$ and analytic parametrisations $\mathbf{\Gamma}_\nu^\pm : S_h \times D_r^\pm \to \mathbb{C}^4$ which are also analytic with respect to ν. As the parametrisations are analytic in ν we can write them as follows,

$$\mathbf{\Gamma}_\nu^\pm = \mathbf{\Gamma}_0 + \nu \xi_0^\pm + O(\nu^2), \tag{3.89}$$

where $\mathbf{\Gamma}_0$ is the parametrisation of H^0, i.e. $\mathcal{D}\mathbf{\Gamma}_0 = X_{H^0}(\mathbf{\Gamma}_0)$, which is defined in (2.36) and $\xi_0^\pm$ satisfy the following equation,

$$\mathcal{L}_0(\xi_0^\pm) = X_{q_2^5}(\mathbf{\Gamma}_0), \tag{3.90}$$

where $\mathcal{L}_0$ is the linear operator defined in (2.38). For our convenience, let us recall the form of $\mathbf{\Gamma}_0$,

$$\mathbf{\Gamma}_0(\varphi, \tau) = \left(\kappa\tau^{-2}\cos\varphi, \kappa\tau^{-2}\sin\varphi, \kappa\tau^{-1}\cos\varphi, \kappa\tau^{-1}\sin\varphi\right)^T.$$

The linear operator $\mathcal{L}_0$ has a normalized fundamental matrix $\mathbf{U}_0$, i.e. $\mathcal{L}_0(\mathbf{U}_0) = 0$, which can be found in (2.39). Thus, by Theorem 2.4.1 the linear operator $\mathcal{L}_0 : \mathfrak{X}_8(S_h \times D_r^\pm) \to \mathfrak{X}_8(S_h \times D_r^\pm)$ has trivial kernel and has an unique bounded right inverse $\mathcal{L}_0^{-1}$: $\mathfrak{X}_8(S_h \times D_r^\pm) \to \mathfrak{X}_7(S_h \times D_r^\pm)$ (see section 2.4.2 for the definition of the Banach spaces $\mathfrak{X}_p$). Notice that we have overloaded the notation of the linear operator $\mathcal{L}_0$ and its inverse since we write the same letter for the $-$ and $+$ case. Now a simple computation shows that,

$$X_{q_2^5}(\mathbf{\Gamma}_0) = \left(0,0,0,-\frac{5\kappa^4\sin^4\varphi}{\tau^8}\right)^T.$$

Thus, $X_{q_2^5}(\mathbf{\Gamma}_0) \in \mathfrak{X}_8(S_h \times D_r^{\pm})$ and we can invert equation (3.90) to get,

$$\xi_0^{\pm} = \mathcal{L}_0^{-1}(X_{q_2^5}(\mathbf{\Gamma}_0)).$$

In fact, following the proof of the Theorem 2.4.1 we can write explicit integrals for $\xi_0^{\pm}$ which read,

$$\xi_0^-(\varphi,\tau) = \mathbf{U}_0(\varphi,\tau)\int_{-\infty}^0 \mathbf{U}_0^{-1}(\varphi+s,\tau+s)X_{q_2^5}(\mathbf{\Gamma}_0(\varphi+s,\tau+s))ds,$$
$$\xi_0^+(\varphi,\tau) = -\mathbf{U}_0(\varphi,\tau)\int_0^{+\infty} \mathbf{U}_0^{-1}(\varphi+s,\tau+s)X_{q_2^5}(\mathbf{\Gamma}_0(\varphi+s,\tau+s))ds.$$

Our goal is to compute the Stokes constant $\sqrt{\mathcal{K}_0(\nu)}$. Recall that $\mathcal{K}_0(\nu)$ is analytic with respect to ν and by definition $\mathcal{K}_0(\nu) = -\Theta_0^-(\nu)\Theta_0^+(\nu)$ where $\Theta_0^{\pm}(\nu)$ are defined by the limits (3.82), depend on the parametrisations $\mathbf{\Gamma}_\nu^{\pm}$ and are also analytic with respect to ν. Moreover, it is not difficult to see that the family X_{H_ν} is reversible with respect to the involution $\mathcal{S}$ defined in (3.16). Thus, Remark 3.5.2.1 and Corollary 3.5.2.1 give that $\sqrt{\mathcal{K}_0(\nu)} = \left|\Theta_0^-(\nu)\right|$ where $\mathrm{Re}(\Theta_0^-(\nu)) = 0$. Moreover, since H_0 is integrable we know that $\mathcal{K}_0(0) = 0$. So in order to prove that $\sqrt{\mathcal{K}_0(\nu)}$ is non-zero for a certain ν it is sufficient to prove that the derivative of $\Theta_0^-(\nu)$ at $\nu = 0$ does not vanish. The following Lemma provides a formula for computing the first derivative,

Lemma 3.5.3. *Let $\Delta_0 = \xi_0^+ - \xi_0^-$. Then,*

$$\left.\frac{d\Theta_0^-}{d\nu}\right|_{\nu=0} = \lim_{\mathrm{Im}\,\tau\to-\infty} \Omega(\Delta_0(\varphi,\tau), \partial_\varphi\mathbf{\Gamma}_0(\varphi,\tau))e^{i(\tau-\varphi)}. \tag{3.91}$$

Let us postpone the proof of this Lemma to the end of the present subsection. In order to use the formula of the previous Lemma we have to compute the difference $\Delta_0 = \xi_0^+ - \xi_0^-$. It follows from (3.89) and the formulae for $\xi_0^{\pm}$ that,

$$\begin{gathered}\Delta_0 = \mathbf{U}_0\mathbf{c}_0 \quad \text{where} \quad \mathbf{c}_0(\varphi,\tau) = -\int_{-\infty}^{+\infty}\mathbf{F}_0(\varphi+s,\tau+s)ds, \quad \text{and,} \\ \mathbf{F}_0(\varphi,\tau) := \mathbf{U}_0^{-1}(\varphi,\tau)X_{q_2^5}(\mathbf{\Gamma}_0(\varphi,\tau))\end{gathered} \tag{3.92}$$

Moreover, from equations (3.90) we conclude that $\mathcal{L}_0(\Delta_0) = 0$ and Theorem 2.4.2 implies that $\mathbf{c}_0$ is in fact a 2π-periodic analytic vector function of a single variable, which is analytic on the union of two half planes $\mathbb{H}^-_{r-h} \cup \mathbb{H}^+_{r-h}$ where

$$\mathbb{H}^{\pm}_{r-h} = \{s \in \mathbb{C} \mid \mp \operatorname{Im} s < -r + h\}.$$

Taking into account the expressions for $\mathbf{U}_0$ and $\mathbf{\Gamma}_0$ a simple computations shows that,

$$\mathbf{F}_0(\varphi,\tau) = \left(-\frac{10\kappa^3 \cos\varphi \sin^4\varphi}{3\tau^7}, -\frac{10\kappa^5 \sin^5\varphi}{\tau^{11}}, -\frac{5\kappa^5 \cos\varphi \sin^4\varphi}{\tau^{10}}, \frac{3\kappa^3 \sin^5\varphi}{\tau^6}\right)^T.$$

Now since $\mathbf{U}_0$ is a normalized fundamental matrix it follows that,

$$\Omega(\Delta_0, \partial_\varphi \mathbf{\Gamma}_0) = \Omega(\mathbf{U}_0 \mathbf{c}_0, \partial_\varphi \mathbf{\Gamma}_0) = -c_{0,3}, \tag{3.93}$$

where $\mathbf{c}_0 = (c_{0,1}, \ldots, c_{0,4})^T$. Therefore, in order to compute $\left.\frac{d\Theta_0^-}{d\nu}\right|_{\nu=0}$ through formula (3.91) it is enough to compute the following integral,

$$c_{0,3}(\varphi,\tau) = \int_{-\infty}^{+\infty} \frac{5\kappa^5 \cos(\varphi+s)\sin^4(\varphi+s)}{(\tau+s)^{10}} ds,$$

where $(\varphi,\tau) \in S_h \times D^1_r$. Using the calculus of residues to compute the previous integral it is not difficult to get,

$$c_{0,3}(\varphi,\tau) = -\frac{5\kappa^5\pi}{2^3 9!} e^{-i(\tau-\varphi)} + \frac{3^{10} 5\kappa^5\pi}{2^4 9!} e^{-3i(\tau-\varphi)} - \frac{5^{10}\kappa^5\pi}{2^4 9!} e^{-5i(\tau-\varphi)}, \tag{3.94}$$

where $(\varphi,\tau) \in S_h \times D^1_r$. Note that $c_{0,3}$ only depends on $\tau - \varphi$ as predicted by the theory. Moreover it is analytic in $\mathbb{H}^-_{r-h}$ and 2π-periodic. Finally according to the formula (3.91), (3.93) and (3.94) we have that,

$$\left.\frac{d\Theta_0^-}{d\nu}\right|_{\nu=0} = -\lim_{\operatorname{Im}\tau \to -\infty} c_{0,3}(\varphi,\tau) e^{i(\tau-\varphi)} = \frac{5\kappa^5\pi}{2^3 9!}.$$

Recall that $\kappa^2 = -\frac{2}{\eta}$ and since $\eta > 0$ the previous expression imply that $\left.\frac{d\Theta_0^-}{d\nu}\right|_{\nu=0} \neq 0$. Consequently $\mathcal{K}_0(\nu)$ and the Stokes constant do not vanish identically.

Proof of Lemma 3.5.3. According to the definition of $\Theta_0^-(\nu)$ we have that,

$$\Theta_0^-(\nu) = \lim_{\operatorname{Im}\tau \to -\infty} \Omega(\Delta_\nu(\varphi,\tau), \partial_\varphi \mathbf{\Gamma}^-_\nu(\varphi,\tau)) e^{i(\tau-\varphi)}, \tag{3.95}$$

where $\Delta_\nu = \mathbf{\Gamma}_\nu^+ - \mathbf{\Gamma}_\nu^-$. Moreover, it follows from formulae (3.93) and (3.94) that,

$$F_0^* := \lim_{\operatorname{Im}\tau\to-\infty} \Omega(\Delta_0(\varphi,\tau), \partial_\varphi \mathbf{\Gamma}_0(\varphi,\tau))e^{i(\tau-\varphi)} < \infty. \tag{3.96}$$

Now we define the following auxiliary function,

$$R(\varphi,\tau,\nu) = \left(\Omega(\Delta_\nu(\varphi,\tau), \partial_\varphi \mathbf{\Gamma}_\nu^-(\varphi,\tau)) - \Omega(\Delta_0(\varphi,\tau), \partial_\varphi \mathbf{\Gamma}_0(\varphi,\tau))\nu\right) e^{i(\tau-\varphi)}.$$

Note that R is analytic in $S_h \times D_r^1 \times \mathbb{D}_{\nu'}$ for some $\nu' > 0$ and $\frac{dR}{d\nu}(\varphi,\tau,0) = 0$. Moreover, it follows from (3.95) and (3.96) that,

$$\lim_{\operatorname{Im}\tau\to-\infty} R(\varphi,\tau,\nu) = \Theta_0^-(\nu) - F_0^*\nu.$$

Now due to the uniform convergence of the limit we get at once,

$$0 = \frac{d}{d\nu}\lim_{\operatorname{Im}\tau\to-\infty} R(\varphi,\tau,\nu)\bigg|_{\nu=0} = \frac{d\Theta_0^-}{d\nu}(0) - F_0^*.$$

□

Generic Families

In the previous section we have constructed an Hamiltonian having non-zero Stokes constant. Now let $\mathfrak{H}_0$ denote the space of analytic Hamiltonian functions $H : \mathcal{U} \subset \mathbb{C}^4 \to \mathbb{C}$ that satisfy the properties described in the introduction of the present chapter. Then, we have the following result,

Corollary 3.5.3.1. *Given any analytic curve H_ν in $\mathfrak{H}_0$ where ν is defined in an open disc $\mathbb{D} \subset \mathbb{C}$, then for every $\epsilon > 0$ there is an ϵ-close analytic curve $F_\nu \in \mathfrak{H}_0$ to H_ν, i.e.*

$$\sup_{\mathbf{x}\in\mathcal{U},\nu\in\mathbb{D}} |H_\nu(\mathbf{x}) - F_\nu(\mathbf{x})| < \epsilon,$$

such that $\mathcal{K}_0(F_\nu)$ does not vanish on an open and dense subset of $\mathbb{D}$.

Proof. Given H_ν in $\mathfrak{H}_0$ and a point $\nu_0 \in \mathbb{D}$ there exists $H^* \in \mathfrak{H}_0$ such that $H^*(\mathbf{x}) - H_{\nu_0}(\mathbf{x}) = O(\|x\|^3)$ and $\mathcal{K}_0(H^*) \neq 0$. This simply follows from the discussion in the previous section and the fact that $\mathcal{K}_0$ is invariant under symplectic changes of coordinates. Thus we can define,

$$F_{\nu,\lambda} := H_\nu + \lambda(H^* - H_{\nu_0}) \in \mathfrak{H}_0.$$

Now it follows from $\mathcal{K}_0(F_{\nu_0,\lambda})$ being analytic with respect to λ and $F_{\nu_0,1} = H^*$ that for any $\epsilon > 0$ we can choose,

$$\delta < \left(\sup_{\mathbf{x} \in \mathcal{U}} |H^*(\mathbf{x}) - H_{\nu_0}(\mathbf{x})| \right)^{-1} \epsilon,$$

such that there is a $\lambda^* \in \mathbb{C}$ with $|\lambda^*| < \delta$ such that $\mathcal{K}_0(F_{\nu_0,\lambda^*}) \neq 0$. Then F_{ν,λ^*} is the desired family. $\square$

This result implies that for a given family $H_\nu \in \mathfrak{H}_0$ there exist another family $F_\nu \in \mathfrak{H}_0$ as close as we like to H_ν such that $\mathcal{K}_0(F_\nu)$ does not vanish on a open and dense set of the parameter ν. An important consequence is that F_ν is non-integrable for ν on a set which is open and has full Lebesgue measure.

Chapter 4

Splitting of Invariant Manifolds

In the present chapter we derive an asymptotic formula for the homoclinic invariant which measures the splitting of invariant manifolds near a Hamiltonian-Hopf bifurcation. The leading order of the asymptotic formula is given by a Stokes constant which was defined in chapter 3.

4.1 Introduction

Let $H_\epsilon : \mathcal{U} \subset \mathbb{R}^4 \to \mathbb{R}$ be an analytic family of two degrees of freedom Hamiltonians defined in a connected open neighbourhood $\mathcal{U}$ of the origin and analytic with respect to ϵ in $|\epsilon| < \epsilon_0$ for some $\epsilon_0 > 0$. Moreover, we suppose that the family of Hamiltonian vector fields X_{H_ϵ} has a common equilibrium point which we can assume to be at the origin ($X_{H_\epsilon}(0) = 0$ for every ϵ) that undergoes a Hamiltonian-Hopf bifurcation as described in section 2.2. Thus we can assume that H_ϵ has the following form,

$$H_\epsilon = -I_1 + I_2 - \epsilon I_3 + \eta I_3^2 + \text{high order terms},$$

where

$$I_1 = q_2 p_1 - q_1 p_2, \quad I_2 = \frac{q_1^2 + q_2^2}{2}, \quad I_3 = \frac{p_1^2 + p_2^2}{2}. \tag{4.1}$$

We also suppose that the normal form coefficient η is positive which corresponds to the stable case.

Recall that the matrix $DX_{H_\epsilon}(0)$ is assumed to have two pairs of complex conjugate eigenvalues $\pm\beta_\epsilon \pm i\alpha_\epsilon$ such that α_ϵ and β_ϵ are positive for $\epsilon > 0$. As $\epsilon \to 0^+$, β_ϵ converges to zero and α_ϵ converges to one. In fact we will show that $\alpha_\epsilon = O(1)$ and $\beta_\epsilon = O(\sqrt{\epsilon})$.

Thus, for $\epsilon > 0$ the equilibrium is hyperbolic and it has two dimensional stable (resp. unstable) manifold W^s_ϵ (resp. W^u_ϵ). Following the discussion in section 2.3 of chapter 2 we parametrize stable and unstable manifolds by solutions of the following nonlinear PDE,

$$\mathcal{D}_\epsilon \mathbf{\Gamma}^{s,u} = X_{H_\epsilon}(\mathbf{\Gamma}^{s,u}), \quad \text{where} \quad \mathcal{D}_\epsilon = \alpha_\epsilon \partial_\varphi + \beta_\epsilon \partial_z, \tag{4.2}$$

where we have omitted the dependence of $\mathbf{\Gamma}^{u,s}$ in ϵ to ease the notation. Now to solve equation (4.2) we require that $\mathbf{\Gamma}^{s,u}$ are 2π-periodic in φ and satisfy the following asymptotic conditions,

$$\lim_{z\to+\infty} \mathbf{\Gamma}^s(\varphi, z) = 0 \quad \text{and} \quad \lim_{z\to-\infty} \mathbf{\Gamma}^u(\varphi, z) = 0. \tag{4.3}$$

Even though these conditions do not define the parametrisations $\mathbf{\Gamma}^{s,u}$ uniquely, their freedom is restricted to a translation in their arguments by a constant, i.e., independent of (φ, z). Each of the derivatives $\partial_z \mathbf{\Gamma}^{s,u}$ and $\partial_\varphi \mathbf{\Gamma}^{s,u}$ defines a tangent vector field on $W^{s,u}_\epsilon$ and it can be checked that these vector fields are defined uniquely. Indeed, since $\mathbf{\Gamma}^{s,u}$ is defined uniquely up to a translation in (φ, z) plane, the tangent vector fields are independent from the freedom in the definition of $\mathbf{\Gamma}^{s,u}$. Moreover, the relation

$$\mathbf{\Gamma}^{s,u}(\varphi + \alpha_\epsilon t, z + \beta_\epsilon t) = \Phi^t_{H_\epsilon} \circ \mathbf{\Gamma}^{s,u}(\varphi, z), \tag{4.4}$$

where $\Phi^t_{H_\epsilon}$ denotes the Hamiltonian flow of H_ϵ, implies that $\partial_\varphi \mathbf{\Gamma}^{s,u}$ and $\partial_z \mathbf{\Gamma}^{s,u}$ are invariant under the restriction of the flow $\Phi^t_{H_\epsilon}\Big|_{W^{s,u}_\epsilon}$.

Given a homoclinic point $\mathbf{p}_\epsilon \in W^u_\epsilon \cap W^s_\epsilon$ we will show that it is possible to set $\mathbf{\Gamma}^{s,u}(0,0) = \mathbf{p}_\epsilon$ eliminating completely the freedom in the definition of the parametrisations. In a Hamiltonian system the symplectic form provides a natural tool for studying

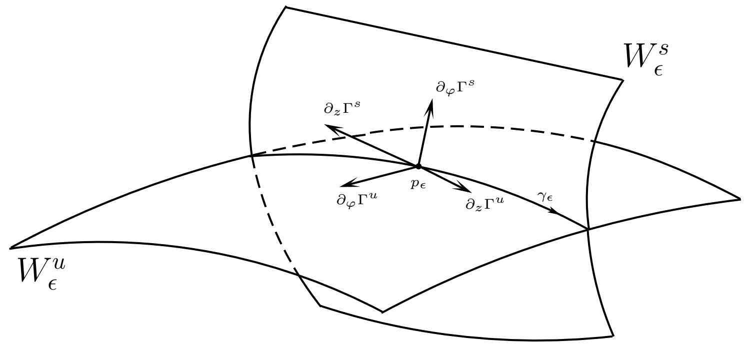

Figure 4.1: Illustration of stable and unstable manifolds, the symmetric homoclinic orbit γ_ϵ and the tangent vectors at the symmetric homoclinic point $\mathbf{p}_\epsilon$.

transversality of invariant manifolds. So we define the *homoclinic invariant* ω_ϵ of the homoclinic point $\mathbf{p}_\epsilon$ as follows,

$$\omega_\epsilon = \Omega(\partial_\varphi \mathbf{\Gamma}^s, \partial_\varphi \mathbf{\Gamma}^u)\Big|_{(\varphi,z)=(0,0)}. \tag{4.5}$$

It is relatively straightforward to check that ω_ϵ is an invariant: the definition leads to the same value for all points of the homoclinic trajectory $\gamma_\epsilon = \{\Phi^t_{H_\epsilon}(\mathbf{p}_\epsilon) : t \in \mathbb{R}\}$. We also note that the definition of ω_ϵ does not depend on the choice of coordinates. Moreover, since $\mathbf{\Gamma}^{s,u}$ belong to the energy level $\{H_\epsilon = 0\}$, which is three-dimensional, the inequality $\omega_\epsilon \neq 0$ implies the transversality of the homoclinic trajectory γ_ϵ.

Further, note that we have defined two vectors tangent to W^u_ϵ and another two vectors tangent to W^s_ϵ at $\mathbf{p}_\epsilon \in W^u_\epsilon \cap W^s_\epsilon$ and used a pair of them to define the homoclinic invariant (see Figure 4.1). Other pairs of tangent vectors give different definitions for the homoclinic invariant. However these are not independent as one can show that $W^{s,u}_\epsilon$ being Lagrangian manifolds imposes some relations between different definitions of ω_ϵ. In fact let us define,

$$\omega_{x,y} = \Omega(\partial_x \mathbf{\Gamma}^u(0,0), \partial_y \mathbf{\Gamma}^s(0,0)), \quad \text{where} \quad x, y \in \{\varphi, z\}\,.$$

Then the following relations are satisfied,

$$\alpha_\epsilon \omega_\epsilon + \beta_\epsilon \omega_{\varphi,z} = 0, \quad \alpha_\epsilon \omega_\epsilon + \beta_\epsilon \omega_{z,\varphi} = 0, \quad \alpha_\epsilon^2 \omega_\epsilon - \beta_\epsilon^2 \omega_{z,z} = 0.$$

The proof of the previous identities is very simple as it only uses the fact that stable and unstable manifolds are Lagrangian and the following formula,

$$\alpha_\epsilon \partial_\varphi \mathbf{\Gamma}^s(0,0) + \beta_\epsilon \partial_z \mathbf{\Gamma}^s(0,0) = \alpha_\epsilon \partial_\varphi \mathbf{\Gamma}^u(0,0) + \beta_\epsilon \partial_z \mathbf{\Gamma}^u(0,0).$$

Finally, note that the definition of the homoclinic invariant is a natural extension of the Lazutkin's invariant defined for homoclinic orbits of area-preserving maps [30] and it can be easily generalized to higher dimensional Hamiltonian systems.

In what follows we shall assume that the Hamiltonian vector field H_ϵ is time-reversible with respect to the linear involution,

$$\mathcal{S}(q_1, q_2, p_1, p_2) = (-q_1, q_2, p_1, -p_2). \tag{4.6}$$

That is $\mathcal{S} X_{H_\epsilon}(\mathbf{x}) = -X_{H_\epsilon}(\mathcal{S}\mathbf{x})$. Note that the normal form procedure preserves the reversibility given by $\mathcal{S}$. Let us denote the set of fixed points of the involution $\mathcal{S}$ by Fix($\mathcal{S}$). This set is known as the *symmetric plane*. It is clear that given an integral curve $\mathbf{x}(t)$ of X_{H_ϵ} then $\mathcal{S}(\mathbf{x}(-t))$ is also an integral curve of the same Hamiltonian vector field. In particular if $\mathbf{x}(0) \in$ Fix($\mathcal{S}$) then the curve $\mathbf{x}(t)$ is symmetric, i.e. $\mathbf{x}(t) = \mathcal{S}(\mathbf{x}(-t))$. If a symmetric curve $\mathbf{x}(t)$ belongs to the unstable manifold W_ϵ^u then $\mathbf{x}(t)$ is a *symmetric homoclinic orbit* and the point $\mathbf{x}(0)$ is called a *symmetric homoclinic point*.

The main result of this chapter in the following,

Theorem 4.1.1. *There exists a symmetric homoclinic point $p_\epsilon \in \mathrm{Fix}(\mathcal{S})$ belonging to a symmetric homoclinic orbit such that the corresponding homoclinic invariant has the following asymptotic formula,*

$$\omega_\epsilon = \pm 2 e^{-\frac{\pi \alpha_\epsilon}{2\beta_\epsilon}} (\omega_0 + O(\epsilon^{1-\mu})), \tag{4.7}$$

where $\omega_0 = \sqrt{\mathcal{K}_0}$ is the Stokes constant and $\mu > 0$ is arbitrarily small.

4.2 Formal Separatrix

In this section we construct an asymptotic series (formal separatrix) using the normal form Hamiltonian H_ϵ^{NF}. These series will provide approximations for the invariant manifolds $W_\epsilon^{s,u}$ and will be of fundamental importance in the analytic study of the invariant manifolds.

4.2.1 Base functions of the asymptotic series

Let us describe a useful class of functions that will be used throughout the present section. A function $f : \mathbb{C} \to \mathbb{C}$ belongs to this class if,

1. it is $2\pi i$-periodic.

2. it is analytic in $\mathbb{C}$ except for poles at $\frac{\pi}{2}i + k\pi i$, for $k \in \mathbb{Z}$.

3. $f \to 0$ as $\mathrm{Re}(z) \to \pm\infty$.

For instance, the function γ_0 defined by,

$$\gamma_0(z) = \sqrt{\frac{2}{\eta}} \frac{1}{\cosh(z)} \tag{4.8}$$

belongs to this class, as well its derivative $\dot{\gamma_0}$. It can be shown that any function f of this class can be written in the form,

$$f = p(\gamma_0) + \dot{\gamma_0} q(\gamma_0) \tag{4.9}$$

where p and q are polynomials in one variable and $p(0) = 0$. Indeed, suppose that $f(z)$ satisfy the properties above. Notice that the function $\tanh(z)$ is $i\pi$-periodic and analytic in $\mathbb{C}$ except for simple poles at $\frac{\pi}{2}i + k\pi i$ for $k \in \mathbb{Z}$. Writing the functions $f(z)$ and $\tanh(z)$ in Laurent series around the poles and comparing coefficients we can construct two polynomials $\hat{p}$ and $\hat{q}$ such that the function $f(z) - \big(\cosh^{-1}(z)\hat{p}(\tanh(z)) + \hat{q}(\tanh(z))\big)$ has no singularities and is bounded in $\mathbb{C}$. Hence must be equal to a constant, say $c \in \mathbb{C}$.

Taking into account the third property of f we deduce that $\hat{q}(x) = (1-x^2)r(x) + c$ where r is some polynomial. Thus,

$$f(z) = \cosh^{-1}(z)\hat{p}(\tanh(z)) + \cosh^{-2}(z)r(\tanh(z)). \tag{4.10}$$

Finally it is easy to check that $x = \gamma_0$ satisfy the differential equation $\ddot{x} = x - \eta x^3$ which can be written as an Hamiltonian system with Hamiltonian $\frac{y^2}{2} - \frac{x^2}{2} + \eta\frac{x^4}{4}$. From this observation we conclude that,

$$\dot{\gamma_0}^2 = \gamma_0^2 - \frac{\eta}{2}\gamma_0^4. \tag{4.11}$$

This relation can be used to simplify the expression (4.10) obtaining the desired representation (4.9). It will also be useful in the construction of the formal separatrix.

4.2.2 Formal Separatrix of the normal form

Recall from chapter 2 that by a formal near identity canonical change of coordinates Φ we can transform H_ϵ into its normal form,

$$H_\epsilon^{NF} = H_\epsilon \circ \Phi = H_\epsilon^0 + \sum_{3m+2j+2l\geq 5} a_{m,j,l} I_1^m I_3^j \epsilon^l, \qquad H_\epsilon^0 = -I_1 + I_2 - \epsilon I_3 + \eta I_3^2, \tag{4.12}$$

where I_i are given by (4.1). Now let $\hat{\mathcal{D}}_\delta$ denote the following formal differential operator,

$$\hat{\mathcal{D}}_\delta = \alpha\partial_\varphi + \beta\partial_z, \tag{4.13}$$

where $\alpha, \beta \in \mathbb{R}[[\delta]]$ such that,

$$\beta = \delta\sqrt{1 - \sum_{l\geq 2} a_{0,1,l}\delta^{2l-2}}, \qquad \alpha = 1 - \sum_{l\geq 1} a_{1,0,l}\delta^{2l}, \tag{4.14}$$

where $a_{i,j,l}$ are the normal form coefficients. The definition of the formal series β and α becomes clear in Lemma 4.4.1. Let h_δ^{NF} denote the normal form Hamiltonian H_ϵ^{NF} in the standard scaling (2.11). In this section we look for formal solutions of the nonlinear PDE,

$$\hat{\mathcal{D}}_\delta \hat{\mathbf{X}} = X_{h_\delta^{NF}}(\hat{\mathbf{X}}) \tag{4.15}$$

in the form of formal series in powers of δ,

$$\hat{\mathbf{X}}(\varphi, z) = \sum_{k\geq 0} X_k(\varphi, z)\delta^k. \tag{4.16}$$

If we substitute (4.16) into the equation (4.15) and collect terms of same order in δ then we get an infinite system of equations,

$$\begin{aligned}
&\partial_\varphi X_0 + X_{I_1}(X_0) = 0 \\
&\partial_\varphi X_1 + X_{I_1}(X_1) = -\partial_z X_0 + X_{I_2}(X_0) - X_{I_3}(X_0) + \eta X_{I_3^2}(X_0) \\
&\qquad\qquad\vdots \\
&\partial_\varphi X_k + X_{I_1}(X_k) = -\partial_z X_{k-1} + X_{I_2}(X_{k-1}) - X_{I_3}(X_{k-1}) + \eta \mathrm{d}X_{I_3^2}(X_0)X_{k-1} \\
&\qquad\qquad + G_k(X_0, \dots, X_{k-2}) \\
&\qquad\qquad\vdots
\end{aligned} \tag{4.17}$$

where G_k is a well defined polynomial function depending exclusively from a finite number of coefficients of the normal form h_δ^{NF}. Note that the normal form preserves the reversibility given by the linear involution $\mathcal{S}$ which we recall,

$$\mathcal{S}(q_1, q_2, p_1, p_2) = (-q_1, q_2, p_1, -p_2). \tag{4.18}$$

Also note that the normal form is rotationally symmetric, which follows from the fact that I_1 is an integral. Indeed the Hamiltonian vector field $X_{h_\delta^{NF}}$ commutes with the rotation R_φ defined by,

$$R_\varphi = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 & 0 \\ 0 & 0 & \cos(\varphi) & -\sin(\varphi) \\ 0 & 0 & \sin(\varphi) & \cos(\varphi) \end{pmatrix}. \tag{4.19}$$

Note that $-X_{I_1}$ is the infinitesimal generator of the group R_φ, i.e., $\partial_\varphi R_\varphi = -X_{I_1}(R_\varphi)$. The infinite system (4.17) can be solved recursively if we impose the following boundary conditions

$$X_k(\varphi + 2\pi, z) = X_k(\varphi, z), \quad \lim_{\mathrm{Re}(z)\to\pm\infty} X_k(\cdot, z) = 0, \quad \mathcal{S}(X_k(-\varphi, -z)) = X_k(\varphi, z). \tag{4.20}$$

From the first equation of (4.17) we deduce that $X_0 = R_\varphi \xi_0(z)$ where ξ_0 is an arbitrary function and R_φ is the rotation matrix (4.19). The second equation of (4.17) and the 2π-periodicity in φ imply that,

$$X_1 = R_\varphi \xi_1(z) \quad \text{and} \quad \partial_z \xi_0 = X_{I_2}(\xi_0) - X_{I_3}(\xi_0) + \eta X_{I_3^2}(\xi_0). \tag{4.21}$$

Taking into account that $\mathcal{S} R_{-\varphi} = R_\varphi \mathcal{S}$ we see that the last two conditions of (4.20) are equivalent to,

$$\lim_{\mathrm{Re}(z)\to\pm\infty} \xi_0(z) = 0 \quad \text{and} \quad \mathcal{S}(\xi_0(-z)) = \xi_0(z). \tag{4.22}$$

It is straightforward to check that $\xi_0 = (-\dot{\gamma}_0, 0, \gamma_0, 0)^T$ solves the second equation of (4.21) and satisfies conditions (4.22) where γ_0 is the base function defined in (4.8). Note that $X_0 = R_\varphi \xi_0$ is the parametrisation defined in (2.17). Moreover it is not difficult to see that $X_0(\varphi, z)$ and $X_0(\varphi + \pi, z)$ are the only reversible solutions that satisfy the boundary conditions (4.20). In the following Theorem we show that having fixed X_0 as above we can continue this process to solve the system (4.17) and obtain an unique solution that satisfy the boundary conditions (4.20).

Theorem 4.2.1 (Formal Separatrix of the normal form)**.** *Equation* (4.15) *has an unique non zero formal solution* $\hat{\mathbf{X}}$ *satisfying the conditions* (4.20) *and having the form,*

$$\hat{\mathbf{X}} = R_\varphi \left(\dot{\gamma}_0 \sum_{k\geq 0} \psi_k^1 \delta^{2k}, \sum_{k\geq 0} \phi_{k+1}^1 \delta^{2k+1}, \sum_{k\geq 0} \phi_k^2 \delta^{2k}, \dot{\gamma}_0 \sum_{k\geq 0} \psi_k^2 \delta^{2k+1} \right)^T, \tag{4.23}$$

where the coefficients ψ_k^i *are even polynomials in* γ_0 *of* $\deg(\psi_k^i) = 2k$ *and* ϕ_k^i *are odd polynomials in* γ_0 *of* $\deg(\phi_k^i) = 2k+1$*. Moreover* $\psi_0^1 = -1$ *and* $\phi_0^2 = \gamma_0$.

Proof. Let us suppose that $X_k(\varphi, z) = R_\varphi \xi_k(z)$ for all $k \geq 0$. We will justify this assumption at the end of the proof. Thus, if $\hat{\mathbf{X}}(\varphi, z) = R_\varphi \hat{\xi}(z)$ then equation (4.15) is equivalent to,

$$\begin{aligned} & \alpha \partial_\varphi R_\varphi \hat{\xi} + \beta \partial_z R_\varphi \hat{\xi} = X_{h_\delta^{NF}}(R_\varphi \hat{\xi}) \\ \Leftrightarrow \quad & -\alpha R_\varphi X_{I_1}(\hat{\xi}) + \beta R_\varphi \partial_z \hat{\xi} = R_\varphi X_{h_\delta^{NF}}(\hat{\xi}). \end{aligned}$$

Using the expression for the vector field $X_{h_\delta^{NF}}$ and the fact that $\alpha = 1 - \sum_{l\geq 1} a_{1,0,l}\delta^{2l}$ we get,

$$\beta\partial_z\hat{\xi} = \left(X_{I_2-I_3}(\hat{\xi}) + \eta X_{I_3^2}(\hat{\xi})\right)\delta + \sum_{\substack{3i+2j+2l=k\geq 5\\ i\neq 1 \text{ or } j\neq 0}} a_{i,j,l}X_{I_1^iI_3^j}\delta^{k-3}. \tag{4.24}$$

In the following we look for formal solutions of (4.24) of the form,

$$\hat{\xi}(z) = \sum_{k\geq 0}\xi_k(z)\delta^k. \tag{4.25}$$

In the variable $\hat{\xi}$ the boundary conditions (4.20) are equivalent to,

$$\lim_{\mathrm{Re}(z)\to\pm\infty}\xi_k(z) = 0 \quad \text{and} \quad \mathcal{S}(\xi_k(-z)) = \xi_k(z). \tag{4.26}$$

The last condition implies that the first and fourth components of ξ_k are odd functions and the second and third are even functions.

Substituting the series (4.25) into equation (4.24) and collecting terms of the same order in δ we obtain an infinite system of equations similar to (4.17) but without the rotation terms. Then at each order one has to compute solvability conditions which allow to solve the equations with respect to ξ_k. These solvability conditions are difficult to compute and there is a more convenient coordinate system such that the verification of these conditions and the construction of a formal solution becomes much simpler. In fact, taking advantage of the fact that h_δ^{NF} is formally integrable, where I_1 is a integral of motion, we consider the following change,

$$\begin{aligned} \xi^1 &= R\cos\theta - \tfrac{\Theta}{r}\sin\theta, \quad \xi^3 = r\cos\theta,\\ \xi^2 &= R\sin\theta + \tfrac{\Theta}{r}\cos\theta, \quad \xi^4 = r\sin\theta, \end{aligned} \tag{4.27}$$

where $\hat{\xi} = (\xi^1,\xi^2,\xi^3,\xi^4)$. Note that the integral I_1 is equal to Θ. In these new variables equation (4.24) takes the form,

$$\beta\partial_z\theta = -\frac{\Theta}{r^2}\delta - \sum_{\substack{3i+2j+2l=k\geq 5\\ i\neq 1 \text{ or } j\neq 0}} \frac{ia_{i,j,l}}{2^j}\Theta^{i-1}r^{2j}\delta^{k-3}, \quad \beta\partial_z r = -\delta R, \quad \beta\partial_z\Theta = 0, \tag{4.28}$$

$$\beta\partial_z R = \left(-\frac{\Theta^2}{r^3} - r + \eta r^3\right)\delta + \sum_{\substack{3i+2j+2l=k\geq 5\\ i\neq 1 \text{ or } j\neq 0}} \frac{2ja_{i,j,l}}{2^j}\Theta^i r^{2j-1}\delta^{k-3}. \tag{4.29}$$

Let us start with the third equation of (4.28). It follows that $\Theta(z) = \sum_{k\geq 0}\Theta_k\delta^k$ where $\Theta_k \in \mathbb{C}$. Taking into account the first condition of (4.26) we conclude that all Θ_k must vanish as $\mathrm{Re}(z) \to \pm\infty$. Hence $\Theta_k = 0$, $k \geq 0$.

We move on and consider now the second and fourth equations of (4.28). Taking into account that $\Theta = 0$, these two equations are equivalent to the following single equation,

$$\beta^2\partial_z^2 r = \left(r - \eta r^3\right)\delta^2 - \sum_{\substack{2j+2l=k\geq 5\\ j\geq 1}} \frac{2ja_{0,j,l}}{2^j} r^{2j-1}\delta^{k-2}. \tag{4.30}$$

In the following we construct a formal solution of (4.30) of the form,

$$r(z) = \sum_{k\geq 0} r_k(z)\delta^k. \tag{4.31}$$

Claim 4.2.1.1. Equation (4.30) has an unique non zero formal solution of the form (4.31) satisfying the boundary conditions

$$\lim_{\mathrm{Re}(z)\to\pm\infty} r_k(z) = 0, \quad r_k(z) = r_k(-z) \quad \text{and} \quad r_0(0) > 0. \tag{4.32}$$

Moreover, $r(z)$ only contains even powers of δ and its coefficients are odd polynomials in γ_0 with real coefficients,

$$r(z) = \sum_{k\geq 0} r_k(z)\delta^{2k}, \qquad \text{where} \quad r_k(z) = \sum_{l=0}^{k} r_{k,l}\gamma_0^{2l+1}, \quad r_{k,l} \in \mathbb{R}.$$

In particular $r_{0,0} = 1$.

Proof of Claim 4.2.1.1. Simplifying the summation indices in (4.30) we obtain an equivalent equation,

$$\beta^2 \partial_z^2 r = \left(r - \eta r^3\right)\delta^2 - \sum_{l\geq 2}\left(\sum_{j=1}^{l+1} \frac{2j a_{0,j,l+1-j}}{2^j} r^{2j-1}\right)\delta^{2l} \tag{4.33}$$

which we solve by substituting a formal power series of the form (4.31) into the equation and collect terms of the same order in δ. Let us recall the definition of β in (4.14),

$$\beta = \delta\sqrt{1 - \sum_{l\geq 1} a_{0,1,l+1}\delta^{2l}}. \tag{4.34}$$

Hence, $\beta^2 = \delta^2\left(1 - \sum_{l\geq 1} a_{0,1,l+1}\delta^{2l}\right)$. Now we are ready to start collecting coefficients. At the order δ^2 we obtain the following equation,

$$\partial_z^2 r_0 = r_0 - \eta r_0^3 \tag{4.35}$$

This equation has an unique solution satisfying the boundary conditions (4.32) which is,

$$r_0(z) = \gamma_0(z) = \sqrt{\frac{2}{\eta}}\frac{1}{\cosh(z)}$$

We move on to the next order in δ. Thus, collecting the coefficients of the same order in δ^3 we obtain the following equation,

$$\partial_z^2 r_1 = (1 - 3\eta r_0^2) r_1$$

This equation is linear with respect to r_1 and we rewrite it the following way,

$$L_0(r_1) = 0 \text{ where } L_0 = \left(\partial_z^2 - 1 + 3\eta\gamma_0^2\right).$$

It is not difficult to compute two independent solutions for the homogeneous equation $L_0 = 0$. In fact, one solution is $v_1 = \dot{\gamma_0}$. A second independent solution can be obtained using the well known theory of linear differential equations and it reads,

$$v_2 = \frac{3\eta}{2}\left(z\dot{\gamma_0} + \gamma_0\right) - \gamma_0^{-1}$$

Let $\mathcal{B}$ denote the linear space of polynomials in the variable γ_0 having real coefficients. It is not difficult to prove the following facts concerning the operator L_0,

1. $L_0 : \mathcal{B} \to \mathcal{B}$ is a linear map.

2. $L_0(\gamma_0^k) = (k-1)(k+1)\gamma_0^k - \frac{\eta}{2}(k-2)(k+3)\gamma_0^{k+2}$.

3. $\mathrm{Ker}\,(L_0) = \{0\}$.

4. If $\mathcal{B}^o$ is the subset of $\mathcal{B}$ consisting of odd polynomials, then $L_0(\mathcal{B}^o) \subseteq \mathcal{B}^o$.

5. If $g \in \mathcal{B}^o$ then equation $L_0(f) = g$ has a unique solution $f \in \mathcal{B}^o$ if and only if g does not contain the term γ_0. Moreover, if $\deg(g) = 2n+1$ then $\deg(f) = 2n-1$.

Thus, by item (3) we conclude that $r_1 = 0$. In order to proceed by induction we let $k \geq 2$ and collect all terms of the same order in δ^{k+2} in the equation (4.33). Thus,

$$\partial_z r_k - \sum_{l=1}^{\lfloor \frac{k}{2} \rfloor} a_{0,1,l+1} r_{k-2l} = (1 - \eta\gamma_0^2) r_k + G_k(r_0, \ldots, r_{k-1}),$$

where G_k is a polynomial with real coefficients. We rewrite the previous equation in the form,

$$L_0(r_k) = \sum_{l=1}^{\lfloor \frac{k}{2} \rfloor} a_{0,1,l+1} r_{k-2l} + G_k(r_0, \ldots, r_{k-1}). \tag{4.36}$$

where L_0 is the linear map defined above. For $k = 2$ the equation (4.36) reads,

$$L_0(r_2) = -a_{0,2,1}\gamma_0^3 - \frac{3}{4} a_{0,3,0}\gamma_0^5, \tag{4.37}$$

and due to item (5) there exists an unique $r_2 \in \mathcal{B}^o$ solving the previous equation such that $\deg(r_2) = 3$. Now we use induction on $k \geq 2$ and suppose that all coefficients r_m for $m \leq k$ have been uniquely determined by the equation (4.36) such that for m odd we have $r_m = 0$ and for even m we have $r_m \in \mathcal{B}^o$ and $\deg(r_m) = m + 1$. Let us consider the equation (4.36) for $k + 1$. There are two cases to distinguish. First, when $k + 1 = 2j + 1$ for some $j \in \mathbb{N}$ we have,

$$L_0(r_{k+1}) = 0,$$

due to the induction hypothesis and the fact that G_{k+1} only depends on r_i for odd i. According to item (3) the linear map L_0 has trivial kernel. Hence $r_{k+1} = 0$.

On the other hand, if $k+1=2j$ for some $j\in\mathbb{N}$ then,

$$L_0(r_{k+1})=\sum_{l=1}^{j}a_{0,1,l+1}r_{2(j-l)}+G_{k+1}(r_0,\ldots,r_k). \tag{4.38}$$

Now due to induction hypothesis it is not difficult to see that G_{k+1} is an odd polynomial in the variable γ_0, hence $G_{k+1}\in\mathcal{B}^o$. Moreover it can be checked that $\deg(G_{k+1})=k+4$ and

$$G_{k+1}(\gamma_0)=-\sum_{l=1}^{j}a_{0,1,l+1}\left[r_{2(j-l)}\right]_1\gamma_0+O(\gamma_0^3),$$

where $[.]_1$ denotes the coefficient of the term γ_0. Thus, we can rewrite equation (4.38) in the form,

$$L_0(r_{k+1})=g_{k+1}$$

where $g_{k+1}\in\mathcal{B}^o$ having $\deg(g_{k+1})=k+4$ and not containing the term γ_0. Thus, by item (5) of the properties of the linear map L_0 we conclude that there exists an unique $r_{k+1}\in\mathcal{B}^o$ such that $\deg(r_{k+1})=k+2$. Hence the claim is true. □

As a direct consequence of previous Claim and taking into account the second equation of (4.28) we conclude that,

$$R(z)=\sum_{k\geq 0}R_k(z)\delta^{2k},\quad \text{where}\quad R_k(z)=\dot{\gamma}_0\sum_{j=0}^{k}R_{k,j}\gamma_0^{2j},\quad R_{k,j}\in\mathbb{R}.$$

In particular $R_{0,0}=-r_{0,0}$. Note that the coefficients R_k satisfy,

$$\lim_{\mathrm{Re}(z)\to\pm\infty}R_k(z)=0\quad\text{and}\quad R_k(-z)=-R_k(z). \tag{4.39}$$

Finally, using the known formal solutions Θ and r we simplify the first equation of (4.28) and obtain,

$$\beta\partial_z\theta=-\sum_{\substack{j+l=i\geq 1\\ j\geq 1,l\geq 0}}\frac{a_{1,j,l}}{2^j}\left(\sum_{k\geq 0}r_k\delta^{2k}\right)^{2j}\delta^{2i}. \tag{4.40}$$

For this equation it is possible to compute a formal solution of the form,

$$\theta(z)=\sum_{k\geq 0}\theta_k(z)\delta^k. \tag{4.41}$$

Claim 4.2.1.2. Equation (4.40) has an unique non zero formal solution of the form (4.41) such that $\theta_k(-z) = -\theta_k(z)$. Moreover, $\theta(z)$ only contains odd powers of δ and its coefficients are of the form,

$$\theta(z) = \sum_{k\geq 0} \theta_k(z)\delta^{2k+1}, \qquad \text{where} \quad \theta_k(z) = \dot{\gamma}_0\gamma_0^{-1}\sum_{l=0}^{k}\theta_{k,l}\gamma_0^{2l}, \quad \theta_{k,l} \in \mathbb{R}.$$

In particular $\theta_{0,0} = \frac{a_{1,1,0}}{\eta} r_{0,0}^2$.

Proof of Claim 4.2.1.2. Due to Claim 4.2.1.1 we know that $r_k = \gamma_0 P_k$ where P_k is an even polynomial in the variable γ_0 such that $\deg(P_k) = 2k$. It is not difficult to see that,

$$\left(\sum_{k\geq 0} r_k\delta^{2k}\right)^{2j} = \gamma_0^{2j}\left(\sum_{k\geq 0} P_k\delta^{2k}\right)^{2j} = \gamma_0^{2j}\sum_{k\geq 0}\tilde{P}_k^{(j)}\delta^{2k},$$

for some even polynomials $\tilde{P}_k^{(j)}$ such that $\deg \tilde{P}_k^{(j)} = 2k$. Thus, the sum in the right hand side of equation (4.40) can be rewritten in the form,

$$\beta\partial_z\theta = -\sum_{k\geq 1}\left(\sum_{j=1}^{k}\tilde{b}_j\,\gamma_0^{2j}\right)\delta^{2k}, \tag{4.42}$$

where $\tilde{b}_j \in \mathbb{R}$. We know that $\beta^{-1} = \delta^{-1}\sum_{k\geq 0} h_k\delta^{2k}$ for some $h_k \in \mathbb{R}$. Hence, equation (4.42) is equivalent to,

$$\partial_z\theta = -\sum_{k\geq 0}\left(\sum_{j=1}^{k+1} b_j\,\gamma_0^{2j}\right)\delta^{2k+1},$$

where $b_j \in \mathbb{R}$. In particular we have, $b_1 = \frac{a_{1,1,0}}{2}$. The general formal solution of the previous equation reads,

$$\theta(z) = \theta_0 - \sum_{k\geq 0}\left(\sum_{j=1}^{k+1} b_j\int^z \gamma_0^{2j}\right)\delta^{2k+1},$$

for any $\theta_0 \in \mathbb{C}$. Since we are only interested in odd solutions, i.e. $\theta_k(-z) = -\theta_k(z)$, we can set $\theta_0 = 0$ and using the following formula,

$$\int_0^z \gamma_0^{2j} = -\dot{\gamma}_0\gamma_0^{-1}\sum_{i=0}^{j-1}\left(\frac{2}{\eta}\right)^{j-i}\frac{A_j A_{i+1}^{-1}}{2i+1}\gamma_0^{2i},$$

where

$$A_i = \prod_{l=1}^{i-1} \frac{2l}{2l+1} \quad \text{and} \quad A_1 = 1,$$

we get the desired form for the coefficients of $\theta(z)$. $\square$

At this point let us recall what we have proved. Equation (4.28) has a formal solution of the form,

$$\begin{aligned} \theta(z) &= \dot{\gamma_0}\gamma_0 \sum_{k\geq 0} T_k \delta^{2k+1}, \qquad & r(z) &= \gamma_0 \sum_{k\geq 0} Q_k \delta^{2k}, \\ \Theta(z) &= 0, & R(z) &= \dot{\gamma_0} \sum_{k\geq 0} P_k \delta^{2k}, \end{aligned} \tag{4.43}$$

such that T_k, Q_k and P_k are even polynomials of degree $2k$ in the variable γ_0. Moreover the solution is unique if $Q_0 > 0$. In particular this last condition implies that $Q_0 = 1$, hence $P_0 = -1$. Note that the formal solution $\theta(z)$ is independent from the condition $Q_0 > 0$. Indeed, equation (4.40) which defines $\theta(z)$ contains only even powers of the form r^{2j} and that is sufficient to show the independence.

In order to conclude the proof of Theorem 4.2.1, let us come back to the variable $\hat{\xi}$. First observe that,

$$\cos\theta(z) = \sum_{i\geq 0} (-1)^i (\dot{\gamma_0}\gamma_0^{-1}\delta)^{2i} \left(\sum_{k\geq 0} T_k \delta^{2k} \right)^{2i},$$

and taking into account the relation (4.11) and the fact that $\left(\sum_{k\geq 0} T_k\delta^{2k}\right)^{2i} = \sum_{k\geq 0} T_{i,k}\delta^{2k}$ where $T_{i,k}$ are even polynomials of degree $2k$, we can simplify the previous formula to get,

$$\cos\theta(z) = \sum_{i\geq 0}\sum_{k\geq 0} (-1)^i \left(1 - \frac{\eta}{2}\gamma_0^2\right)^i T_{i,k}\delta^{2(i+k)}.$$

Moreover, since $(-1)^i \left(1 - \frac{\eta}{2}\gamma_0^2\right)^i T_{i,k}$ is an even polynomial of degree $2(i+k)$ we can write the previous formula as follows,

$$\cos\theta(z) = \sum_{j\geq 0} W_j \delta^{2j}, \tag{4.44}$$

where W_j is an even polynomial in γ_0 of degree $2j$ and $W_0 = 1$. A similar formula holds for the sine function which reads,

$$\sin\theta(z) = \dot{\gamma}_0\gamma_0^{-2}\sum_{j\geq 0} Z_j\delta^{2j+1}, \tag{4.45}$$

where Z_j is an odd polynomial in γ_0 of degree $2j+1$. Now according to the change of variables (4.27) we have that,

$$\hat{\xi}(z) = \left(R(z)\cos\theta(z), R(z)\sin\theta(z), r(z)\cos\theta(z), r(z)\sin\theta(z)\right)^T,$$

is a formal solution of the equation (4.24). Using formulae (4.43), (4.44), (4.45) and (4.11) we can rewrite the components of $\hat{\xi}$ as follows,

$$R(z)\cos\theta(z) = \dot{\gamma}_0\sum_{k\geq 0}\psi_k^1\delta^{2k}, \qquad R(z)\sin\theta(z) = \sum_{k\geq 0}\phi_{k+1}^1\delta^{2k+1},$$
$$r(z)\cos\theta(z) = \sum_{k\geq 0}\phi_k^2\delta^{2k}, \qquad r(z)\sin\theta(z) = \dot{\gamma}_0\sum_{k\geq 0}\psi_k^2\delta^{2k+1},$$

where

$$\begin{aligned} \psi_k^1 &= \sum_{i+j=k} P_iW_j, & \phi_{k+1}^1 &= \left(1-\frac{\eta}{2}\gamma_0^2\right)\sum_{i+j=k} P_iZ_j, \\ \phi_k^2 &= \gamma_0\sum_{i+j=k} Q_iW_j, & \psi_k^2 &= \gamma_0^{-1}\sum_{i+j=k} Q_iZ_j. \end{aligned} \tag{4.46}$$

Note that $\xi_0 = (\dot{\gamma}_0\psi_1^1, 0, \phi_0^2, 0)^T$. Taking into account that $Q_0 = 1$, $P_0 = -1$ and $W_0 = 1$ we get that $\xi_0 = (-\dot{\gamma}_0, 0, \gamma_0, 0)$ as concluded in the introduction of the present subsection. Finally, at the beginning of this proof we assumed that $\hat{\mathbf{X}} = R_\varphi\hat{\xi}$. If $\hat{\mathbf{Y}}$ is any formal solution of (4.15) of the form (4.16) then its coefficients must satisfy the infinite system of equations (4.17). Since we require the functions involved to be 2π-periodic in φ then a simple induction argument shows that the coefficients of $\hat{\mathbf{Y}}$ must be of the form $R_\varphi\zeta_k(z)$. This concludes the proof of the Theorem. □

Remark 4.2.1.1. Inverting the standard scaling we obtain a formal separatrix $\hat{\mathbf{X}}_\delta$ which solves formally the equation,

$$\hat{\mathcal{D}}_\delta\hat{\mathbf{X}}_\delta = X_{H_\epsilon^{NF}}(\hat{\mathbf{X}}_\delta).$$

4.3 The Unstable Parametrisation

Let U be an open ball centered at $0 \in \mathbb{C}^4$ and $F = (F_1, F_2, F_3, F_4) : U \to \mathbb{C}^4$ an analytic vector field. We also assume that F is continuous on the closure of U. Suppose that F has an equilibrium point at the origin, i.e. $F(0) = 0$, and that the equilibrium point is hyperbolic with eigenvalues $\pm\beta \pm i\alpha$. Moreover suppose that the linear part of the vector field F is in the canonical form,

$$DF(0) = \begin{pmatrix} B^T & 0_{2\times 2} \\ 0_{2\times 2} & -B \end{pmatrix},$$

where B is a 2 by 2 Jordan block of the form,

$$B = \begin{pmatrix} \beta & \alpha \\ -\alpha & \beta \end{pmatrix}.$$

Since the equilibrium is hyperbolic, it follows form the stable (resp. unstable) manifold Theorem that there exists an analytic invariant stable (resp. unstable) immersed manifold $\mathcal{W}^s$ (resp. $\mathcal{W}^u$) such that orbits in this manifold converge to the equilibrium forward (resp. backward) in time at an exponential rate. In this section we parametrise the local unstable manifold $\mathcal{W}^u_{loc}$ by an analytic vector function Υ^u which satisfies the PDE,

$$\alpha\partial_\varphi \mathbf{x} + \beta\partial_z \mathbf{x} = F(\mathbf{x}). \tag{4.47}$$

An analogous result holds for the local stable manifold $\mathcal{W}^s_{loc}$ and in the following we will only present the details for the unstable case. We can rewrite equation (4.47) in the following equivalent form,

$$\alpha\partial_\varphi \mathbf{x} + \beta\partial_z \mathbf{x} = \Delta\mathbf{x} + R(\mathbf{x}), \tag{4.48}$$

where $\Delta = DF(0)$ and R is analytic in U, continuous on its closure and $R(\mathbf{x}) = O(\|\mathbf{x}\|^2)$.

Now let $\gamma \in \mathbb{R}$ and $h > 0$ (which we consider fixed throughout this section) and consider the following sets,

$$S_h = \{\varphi \in \mathbb{C} \mid \ |\mathrm{Im}(\varphi)| < h\}, \quad D^u_\gamma = \{z \in \mathbb{C} \mid \mathrm{Re}(z) < -\gamma\}.$$

Moreover, denote by $\mathfrak{X}$ the complex linear space of analytic maps $f : S_h \times D^u_\gamma \to \mathbb{C}^4$ which are 2π periodic in the variable φ, continuous on the closure of its domain and having finite norm,

$$\|f\|_{\mathfrak{X}} := \sup_{(\varphi,z)\in S_h\times D^u_\gamma} \left\|e^{-z} f(\varphi, z)\right\| < \infty$$

where $\|\cdot\|$ denotes the standard *infinity norm* defined in $\mathbb{C}^4$. The pair $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ is a complex Banach space. Let us prove two Lemmas which will be used to prove the main result of this subsection.

Lemma 4.3.1. *The linear PDE,*

$$(\alpha\partial_\varphi + \beta\partial_z)\,\xi = \Delta\xi$$

has a fundamental matrix solution Π *of the form,*

$$\Pi(\varphi, z) = \begin{pmatrix} e^z \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} & 0_{2\times 2} \\ 0_{2\times 2} & e^{-z} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \end{pmatrix}.$$

Moreover, it has the following properties:

1. $\Pi(0, 0) = \mathrm{Id}$,
2. $\Pi(\varphi_1 + \varphi_2, z_1 + z_2) = \Pi(\varphi_1, z_1)\Pi(\varphi_2, z_2)$,
3. $\Pi(\varphi, z)$ *is invertible for all* $(\varphi, z) \in \mathbb{C}^2$,
4. $\Pi^{-1}(\varphi, z) = \Pi(-\varphi, -z)$.

Proof. Verifying that $\Pi(\varphi, z)$ satisfies the linear equation is a straightforward computation. Moreover, it is not difficult to check the above properties. $\square$

Lemma 4.3.2. *Let* $\gamma \in \mathbb{R}$ *and* $\mathcal{L}_{\alpha,\beta} : \mathfrak{X} \to \mathfrak{X}$ *be the linear operator defined by,*

$$\mathcal{L}_{\alpha,\beta}(\xi) = (\alpha\partial_\varphi + \beta\partial_z)\,\xi - \Delta\xi.$$

The operator $\mathcal{L}_{\alpha,\beta}$ *has the following properties,* *1.* $\operatorname{Ker}(\mathcal{L}_{\alpha,\beta}) = \left\{ \Pi(\varphi, z) \begin{pmatrix} c \\ 0 \end{pmatrix} \mid c \in \mathbb{C}^2 \right\}$,

2. If $f \in \mathfrak{X}$ *and moreover* $f(\varphi, z) = O(e^{2z})$ *in* $S_h \times D^u_\gamma$ *then equation* $\mathcal{L}_{\alpha,\beta}(\xi) = f$ *has a general solution of the form,*

$$\xi(\varphi, z) = \Pi(\varphi, z) \begin{pmatrix} c \\ 0 \end{pmatrix} + \mathcal{L}^{-1}(f),$$

where $\mathcal{L}^{-1}$ *is defined by,*

$$\mathcal{L}^{-1}(f)(\varphi, z) = \int_{-\infty}^{0} \Pi(-s, -s) f(\varphi + s, z + s) ds. \tag{4.49}$$

Proof. Let us prove (1). Suppose that $\mathcal{L}_{\alpha,\beta}(\xi) = 0$ for $\xi \in \mathfrak{X}$. Let $\xi = \Pi \mathbf{c}$ where Π is the fundamental matrix of Lemma 4.3.1. Then according to the definition of $\mathcal{L}_{\alpha,\beta}$ and due to Lemma 4.3.1 we conclude that $(\alpha\partial_\varphi + \beta\partial_z)\mathbf{c} = 0$. Thus $\mathbf{c}(\varphi, z) = \mathbf{c}_0(z - \varphi)$ where $\mathbf{c}_0 : \mathbb{C} \to \mathbb{C}^4$ is an entire, 2π-periodic vector function. As $\xi(\varphi, z) = e^z \tilde{\xi}(\varphi, z)$ where $\tilde{\xi}$ is bounded in $S_h \times D^u_\gamma$ then $\mathbf{c}_0(\varphi - z) = e^z \Pi^{-1}(\varphi, z) \tilde{\xi}(\varphi, z) = e^z \Pi(-\varphi, -z) \tilde{\xi}(\varphi, z)$. Thus,

$$\mathbf{c}_0(\varphi - z) = \begin{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} & 0_{2\times 2} \\ 0_{2\times 2} & e^{2z} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \end{pmatrix} \tilde{\xi}(\varphi, z),$$

which implies that all components of $\mathbf{c}_0$ are bounded entire functions, hence must be equal to a constant, i.e. $\mathbf{c} \in \mathbb{C}^4$. Moreover, as the last two components of $\mathbf{c}_0$ decay to zero as $\operatorname{Re} z \to -\infty$ then these must be equal zero. Thus proving the desired result.

Finally, let us prove (2). Let $\mathcal{L}_{\alpha,\beta}(\xi) = f$ for $f \in \mathfrak{X}$ such that $f = O(e^{2z})$. Simple estimates show that,

$$\begin{aligned} \left\| \int_{-\infty}^{0} \Pi(-s, -s) f(\varphi + s, z + s) ds \right\| &\le \int_{-\infty}^{0} \left\| \Pi(-s, -s) f(\varphi + s, z + s) \right\| ds \\ &\le e^{2\operatorname{Re} z} \int_{-\infty}^{0} K e^{\operatorname{Re} s} ds, \end{aligned}$$

where $K > 0$ depends only on h and γ which define the set $S_h \times D^u_\gamma$. Therefore the integral (4.49) converges uniformly on $S_h \times D^u_\gamma$, thus defining an analytic function $\mathcal{L}^{-1}(f) \in \mathfrak{X}$ such that $\mathcal{L}_{\alpha,\beta}(\mathcal{L}^{-1}(f)) = f$. Since $\mathcal{L}^{-1}(f) - \xi \in \mathrm{Ker}(\mathcal{L}_{\alpha,\beta})$ then by item (1) there exists $c \in \mathbb{C}^2$ such that,

$$\mathcal{L}^{-1}(f) - \xi = \Pi \begin{pmatrix} c \\ 0 \end{pmatrix},$$

which concludes the proof of the Lemma. □

We are now ready to prove the following,

Theorem 4.3.1 (Unstable Parametrisation). *For every $c \in \mathbb{C}^2$ there exists $\gamma > 0$ such that equation* (4.47) *has an unique analytic solution $\mathbf{\Upsilon}^u : S_h \times D^u_\gamma \to \mathbb{C}^4$, which is 2π-periodic in φ, continuous on the closure of its domain and possessing the following asymptotics,*

$$\mathbf{\Upsilon}^u(\varphi, z) = \Pi(\varphi, z) \begin{pmatrix} c \\ 0 \end{pmatrix} + O(e^{2z}), \quad \text{in } S_h \times D^u_\gamma. \tag{4.50}$$

Proof. Let $c \in \mathbb{C}^2$ and $\gamma > 0$ (to be chosen later in the proof). We look for a solution ξ of equation (4.48) belonging to the Banach space $\mathfrak{X}$. To that end we rewrite equation (4.48) in the equivalent form,

$$\mathcal{L}_{\alpha,\beta}(\xi) = R(\xi), \tag{4.51}$$

where the linear operator $\mathcal{L}_{\alpha,\beta}$ acts in $\mathfrak{X}$ according Lemma 4.3.2. As $\xi \in \mathfrak{X}$ then standard Cauchy estimates applied to the map R which is defined in the open ball U yield that,

$$R(\xi) = O(e^{2z}), \text{ in } S_h \times D^u_\gamma.$$

for $\gamma > \gamma_1$ where $\gamma_1 > 0$ being sufficiently large. In the light of Lemma 4.3.2 we can invert $\mathcal{L}_{\alpha,\beta}$ in (4.51) and conclude that in order for ξ be a solution of (4.51) it must satisfy the integral equation,

$$\xi(\varphi, z) = \Pi(\varphi, z) \begin{pmatrix} c \\ 0 \end{pmatrix} + \int_{-\infty}^{0} \Pi(-s, -s) R(\xi(\varphi + s, z + s)) ds. \tag{4.52}$$

Let us denote the nonlinear operator in the right hand side of (4.52) by $\mathcal{G}(\xi)$. Note that a fix point of this operator yields a solution for (4.51), hence a solution for (4.67). We shall construct a fixed point of $\mathcal{G}$ using a contraction mapping argument. We first show that $\mathcal{G}$ leaves invariant a certain ball. Let $\mathfrak{B}_\rho$ denote a closed ball of radius $\rho > 0$,

$$\mathfrak{B}_\rho = \{\xi \in \mathfrak{X} \mid \|\xi\|_{\mathfrak{X}} \leq \rho\}.$$

Notice that,

$$\left\| \Pi \begin{pmatrix} c \\ 0 \end{pmatrix} \right\|_{\mathfrak{X}} \leq k_0 \|c\|,$$

where $k_0 > 0$ is some constant. If $\xi \in \mathfrak{B}_\rho$, then

$$\|\mathcal{G}(\xi)\|_{\mathfrak{X}} \leq k_0 \|c\| + \sup_{(\varphi,z)\in S_h\times D^u_\gamma} \left\| e^{-z} \int_{-\infty}^0 \Pi(-s,-s) R(\xi(\varphi+s, z+s)) ds \right\|.$$

Now since R is analytic in U and $R(\mathbf{x}) = O(\|\mathbf{x}\|^2)$ then simples estimates show that,

$$\|R(\xi(\varphi+s, z+s))\| \leq k_1 \|\xi(\varphi+s, z+s)\|^2 \leq k_1 e^{2\mathrm{Re}(z)} e^{2s} \|\xi\|^2_{\mathfrak{X}},$$

valid in $S_h \times D^u_\gamma$ where $\gamma > \gamma_2$ for $\gamma_2 > 0$ sufficiently large and $k_1 > 0$ is some constant. Thus,

$$\begin{aligned}
\|\mathcal{G}(\xi)\|_{\mathfrak{X}} &\leq k_0 \|c\| + \sup_{(\varphi,z)\in S_h\times D^u_\gamma} k_2 \int_{-\infty}^0 e^{-\mathrm{Re}(z)} e^{-s} \|R(\xi(\varphi+s, z+s))\| ds \\
&\leq k_0 \|c\| + k_1 k_2 \|\xi\|^2_{\mathfrak{X}} \sup_{z\in D^u_\gamma} e^{\mathrm{Re}(z)} \int_{-\infty}^0 e^s ds \qquad (4.53) \\
&\leq k_0 \|c\| + k_1 k_2 \|\xi\|^2_{\mathfrak{X}} e^{-\gamma},
\end{aligned}$$

where $k_2 > 0$ is some constant. Now let $\rho := 2k_0 \|c\|$, so if $\xi \in \mathfrak{B}_\rho$ then it follows from estimate (4.53) that,

$$\|\mathcal{G}(\xi)\|_{\mathfrak{X}} \leq \frac{\rho}{2} + k_1 k_2 \rho^2 e^{-\gamma},$$

and choosing $\gamma > \max\{\log(2k_1 k_2 \rho), \gamma_2, \gamma_1\}$ we conclude that $\|\mathcal{G}(\xi)\|_{\mathfrak{X}} \leq \rho$. Thus $\mathcal{G}(\mathfrak{B}_\rho) \subseteq \mathfrak{B}_\rho$. Now we show that $\mathcal{G}$ in contracting on the ball $\mathfrak{B}_\rho$. Given $\xi_1, \xi_2 \in \mathfrak{B}_\rho$ then

$$\mathcal{G}(\xi_1)(\varphi,z) - \mathcal{G}(\xi_2)(\varphi,z) = \int_{-\infty}^0 \Pi(-s,-s) \left(R(\xi_1(\varphi+s, z+s)) - R(\xi_2(\varphi+s, z+s)) \right) ds.$$

For $(\varphi, z) \in S_h \times D^u_\gamma$ the finite segment $\theta\xi_1(\varphi, z) + (1-\theta)\xi_2(\varphi, z))$ belongs to the open ball U and since $\mathfrak{B}_\rho \subseteq \mathfrak{X}$ is convex then Cauchy estimates yield,

$$
\begin{aligned}
\|R(\xi_1(\varphi, z)) - R(\xi_2(\varphi, z))\| &\leq e^{\mathrm{Re}(z)} \|\xi_1 - \xi_2\|_{\mathfrak{X}} \int_0^1 \|\mathrm{d}R(\theta\xi_1(\varphi, z) + (1-\theta)\xi_2(\varphi, z))\| \, d\theta \\
&\leq k_4 e^{2\mathrm{Re}(z)} \|\xi_1 - \xi_2\|_{\mathfrak{X}} ,
\end{aligned}
$$

where $k_4 > 0$ is some positive constant. Thus,

$$
\begin{aligned}
\|\mathcal{G}(\xi_1) - \mathcal{G}(\xi_2)\|_{\mathfrak{X}} &\leq \sup_{z \in D^u_\gamma} \int_{-\infty}^0 k_2 e^{-s} e^{-\mathrm{Re}(z)} \|R(\xi_1(\varphi + s, z + s)) - R(\xi_2(\varphi + s, z + s))\| \, ds \\
&\leq \sup_{z \in D^u_\gamma} \int_{-\infty}^0 k_4 k_2 e^{s} e^{\mathrm{Re}(z)} ds \, \|\xi_1 - \xi_2\|_{\mathfrak{X}} \\
&\leq \bar{k} e^{-\gamma} \|\xi_1 - \xi_2\|_{\mathfrak{X}} ,
\end{aligned}
$$

where $\bar{k} = k_2 k_4$. Choosing $\gamma > \max\left\{\log(2\bar{k}), \log(2k_1 k_2 \rho), \gamma_2, \gamma_1\right\}$ we get that,

$$
\|\mathcal{G}(\xi_1) - \mathcal{G}(\xi_2)\|_{\mathfrak{X}} \leq \frac{1}{2} \|\xi_1 - \xi_2\|_{\mathfrak{X}} ,
$$

for $\xi_1, \xi_2 \in \mathfrak{B}_\rho$. Thus, applying the contraction mapping theorem to the operator $\mathcal{G}$ we obtain the existence of an unique fixed point $\boldsymbol{\Upsilon}^u \in \mathfrak{B}_\rho$ of $\mathcal{G}$, i.e. $\boldsymbol{\Upsilon}^u = \mathcal{G}(\boldsymbol{\Upsilon}^u)$. Moreover, estimate (4.53) implies that,

$$
\boldsymbol{\Upsilon}^u(\varphi, z) = \Pi(\varphi, z) \begin{pmatrix} c \\ 0 \end{pmatrix} + O(e^{2z}), \text{ in } S_h \times D^u_\gamma.
$$

□

Remark 4.3.1.1. If $c = (0,0)^T$ then the unique analytic solution $\boldsymbol{\Upsilon}^u$ possessing the asymptotics (4.50) is the trivial solution, i.e., $\boldsymbol{\Upsilon}^u = 0$. Indeed, from the proof of the previous Theorem we know that $\boldsymbol{\Upsilon}^u = \mathcal{G}(\boldsymbol{\Upsilon}^u)$ and since $c = (0,0)^T$ then $\mathcal{G}(0) = 0$. Due to the uniqueness of the fixed point we conclude that $\boldsymbol{\Upsilon}^u = 0$.

Remark 4.3.1.2. If F is real analytic and $c \in \mathbb{R}^2$ then $\boldsymbol{\Upsilon}^u$ is real analytic in the half plane $\mathbb{R} \times (-\infty, -\gamma)$ for some $\gamma > 0$. Moreover for any $(\varphi_0, z_0) \in \mathbb{R} \times (-\infty, -\gamma)$ the orbit $\{\boldsymbol{\Upsilon}^u(\alpha t + \alpha_0, \beta t + z_0)\}_{t \in \mathbb{R}^-}$ belongs to the local unstable manifold $\mathcal{W}^u_{loc}$ of the equilibrium point. In fact $\mathbf{x}(t) = \boldsymbol{\Upsilon}^u(\alpha t + \alpha_0, \beta t + z_0)$ for $t \in \mathbb{R}^-$ is an integral curve of the vector field F and it spirals to the equilibrium as $t \to -\infty$ at an exponential rate $e^{\beta t}$. Thus we call $\boldsymbol{\Upsilon}^u$ an *unstable parametrisation*.

Remark 4.3.1.3. If we denote Φ^t the flow of the vector field F then the following relation holds,

$$\boldsymbol{\Upsilon}^u(\varphi + \alpha t, z + \beta t) = \Phi^t(\boldsymbol{\Upsilon}^u(\varphi, z)). \tag{4.54}$$

and we can use it to extend the domain of analyticity of the unstable parametrisation $\boldsymbol{\Upsilon}^u$ onto a larger domain in $\mathbb{C}^2$ until it leaves the domain U of the vector field F.

Remark 4.3.1.4. In Theorem 4.3.1 the freedom in the choice of the unstable parametrisation $\boldsymbol{\Upsilon}^u$ is given by the parameter c. In fact this is the only freedom we have. If $\tilde{\boldsymbol{\Upsilon}}^u$ is a different solution of equation (4.47) such that $\tilde{\boldsymbol{\Upsilon}}^u = O(e^z)$ then as in the proof of Theorem 4.3.1 there exists an unique $\tilde{c} \in \mathbb{C}^2$ such that,

$$\tilde{\boldsymbol{\Upsilon}}^u(\varphi, z) = \Pi(\varphi, z) \begin{pmatrix} \tilde{c} \\ 0 \end{pmatrix} + O(e^{2z}), \tag{4.55}$$

in $S_h \times D^u_{\tilde{\gamma}}$ for some $\tilde{\gamma} > 0$. Moreover, according to Lemma 4.3.1 we get that,

$$\boldsymbol{\Upsilon}^u(\varphi + \varphi_0, z + z_0) = \Pi(\varphi, z)\Pi(\varphi_0, z_0) \begin{pmatrix} c \\ 0 \end{pmatrix} + O(e^{2z}),$$

for $(\varphi + \varphi_0, z + z_0) \in S_h \times D^u_{\gamma}$. Comparing the previous equation with (4.55) we conclude that $\boldsymbol{\Upsilon}^u(\varphi + \varphi_0, z + z_0) = \tilde{\boldsymbol{\Upsilon}}^u(\varphi, z)$ if and only if,

$$\tilde{c} = e^{z_0} \begin{pmatrix} \cos\varphi_0 & -\sin\varphi_0 \\ \sin\varphi_0 & \cos\varphi_0 \end{pmatrix} c. \tag{4.56}$$

Equation (4.56) can be solved for (φ_0, z_0) and we conclude that the unstable parametrisation $\boldsymbol{\Upsilon}^u$ is uniquely defined up to a translation in (φ, z).

4.4 Approximation Theorems

In this section we provide explicit approximations for the unstable manifold W^u_ϵ of the equilibrium of H_ϵ in different regions. These approximations are constructed using the formal separatrix of the normal form.

4.4.1 Preliminaries

Given $n \in \mathbb{N}$ we normalize the Hamiltonian H_ϵ up to order $2n+4$ (see section 2.1.1 for more details about the normal form). After $2n$ steps of normalization we get,

$$H_{\epsilon,n} = H_\epsilon \circ \Phi_n = H_\epsilon^0 + \sum_{3m+2j+2l\geq 5}^{2n+4} a_{m,j,l} I_1^m I_3^j \epsilon^l + R_{\epsilon,n}, \tag{4.57}$$

where Φ_n is an analytic near identity canonical change of coordinates, $H_\epsilon^0 = -I_1 + I_2 - \epsilon I_3 + \eta I_3^2$, and I_i, $i = 1, 2, 3$ are given by (4.1). Moreover $R_{\epsilon,n}$ is a real analytic function defined in an open neighbourhood of the origin in $\mathbb{R}^4$, analytic with respect to ϵ and $R_{\epsilon,n} = O((|q|^{\frac{1}{2}} + |p| + |\epsilon|^{\frac{1}{2}})^{2n+5})$. In what follows it is convenient to complexify the Hamiltonian $H_{\epsilon,n}$, so we may assume that it is analytic in an open ball $B_{\sigma_n} \subseteq \mathbb{C}^4$ for some $\sigma_n > 0$ sufficiently small. The normal form coefficients $a_{m,j,l} \in \mathbb{R}$ are uniquely defined and the coefficient η in H_ϵ^0 is assumed to be positive which corresponds to the stable case.

Also, given two vector-functions $f, g : \Omega \subset \mathbb{C}^2 \to \mathbb{C}^4$ and $p \geq 0$ we write $g = O_p(f)$ if there exist $c_i > 0$, $i = 1, \ldots, 4$ such that,

$$|g_i(x)| \leq c_i \, |f_i(x)|^{p+1}, \; i = 1, 2 \quad \text{and} \quad |g_i(x)| \leq c_i \, |f_i(x)|^p, \; i = 3, 4, \tag{4.58}$$

valid in Ω where f_i and g_i denote the components of the corresponding functions.

Eigenvalues of $DX_{H_\epsilon}(0)$

The matrix $DX_{H_{\epsilon,n}}(0)$ has the same eigenvalues $\pm\beta_\epsilon \pm i\alpha_\epsilon$ as $DX_{H_\epsilon}(0)$ since these are preserved under the normal form procedure. Moreover, using the successive normalizations of H_ϵ we can prove the following,

Lemma 4.4.1. *For $\epsilon > 0$ the functions β_ϵ and α_ϵ can be expanded into convergent* *power series,*

$$\beta_\epsilon = \delta\sqrt{1-\sum_{l=2}^{\infty} a_{0,1,l}\delta^{2l-2}}, \qquad \alpha_\epsilon = 1-\sum_{l=1}^{\infty} a_{1,0,l}\delta^{2l},$$

where $\delta^2 = \epsilon$ and the coefficients in the series above are the normal form coefficients of H_ϵ.

Proof. Let $n \in \mathbb{N}$ and consider $H_{\epsilon,n}$ as defined (4.57). We scale variables according to the standard scaling (2.11) and change to complex variables given by the following relation,

$$z = q_1 + iq_2,\ w = p_1 + ip_2,\ \bar{z} = q_1 - iq_2,\ \bar{w} = p_1 - ip_2.$$

The map defined above does not preserve the canonical symplectic form. However, the following relation holds,

$$dq_1 \wedge dp_1 + dq_2 \wedge dp_2 = \frac{\delta^3}{2}\left(dz \wedge d\bar{w} + d\bar{z} \wedge dw\right),$$

and in the new variables we multiply the Hamiltonian by $2\delta^{-3}$ and use the canonical symplectic form to derive the Hamiltonian equations. The Hamiltonian $H_{\epsilon,n}$ in these new coordinates reads,

$$\begin{aligned}\tilde{h}_{\delta,n} = i\left(z\bar{w} - \bar{z}w\right) + \left(z\bar{z} - w\bar{w} + \frac{\eta}{2}(w\bar{w})^2\right)\delta \\ + \sum_{3m+2j+2l=k\geq 5}^{2n+4} \frac{a_{m,j,l}}{2^{j-1}}\left(\frac{z\bar{w}-\bar{z}w}{2i}\right)^m (w\bar{w})^j\delta^{k-3} + O(\delta^{2n+2}).\end{aligned} \tag{4.59}$$

Note that the eigenvalues of $DX_{H_\epsilon}(0)$ are the same as $DX_{\tilde{h}_{\delta,n}}(0)$. Now let $Z = (z, w, \bar{z}, \bar{w})$. We can write the Hamilton equations of (4.59) as follows,

$$\dot{Z} = A_\delta Z + O\left(\|Z\|^2\right),$$

where,

$$A_\delta = \begin{pmatrix} i\alpha_{\epsilon,n} & -\delta\mu_{\epsilon,n} & 0 & 0 \\ -\delta & i\alpha_{\epsilon,n} & 0 & 0 \\ 0 & 0 & -i\alpha_{\epsilon,n} & -\delta\mu_{\epsilon,n} \\ 0 & 0 & -\delta & -i\alpha_{\epsilon,n} \end{pmatrix} + O(\delta^{2n+2}),$$

and

$$\alpha_{\epsilon,n} = 1 - \sum_{l=1}^{n} a_{1,0,l}\delta^{2l}, \quad \mu_{\epsilon,n} = 1 - \sum_{l=1}^{n} a_{0,1,l+1}\delta^{2l}. \tag{4.60}$$

Since the spectrum of A_δ is invariant under complex conjugation and symmetric with respect to the imaginary axis, we can write its characteristic polynomial as follows,

$$\det(A_\delta - \lambda \mathrm{Id}_4) = \lambda^4 + b_2(\delta)\lambda^2 + b_0(\delta) \tag{4.61}$$

where $b_2(\delta)$ and $b_0(\delta)$ are analytic functions possibly having complex coefficients. A closer look to the determinant (4.61) gives,

$$b_2(\delta) = 2\alpha_{\epsilon,n}^2 - 2\delta^2\mu_{\epsilon,n} + O(\delta^{2n+2}), \quad b_0(\delta) = \left(\alpha_{\epsilon,n}^2 + \delta^2\mu_{\epsilon,n}\right)^2 + O(\delta^{2n+2}), \tag{4.62}$$

and using the quadratic formula it is not difficult to see that,

$$\sqrt{-\frac{b_2(\delta)}{2} + \sqrt{\left(\frac{b_2(\delta)}{2}\right)^2 - b_0(\delta)}},$$

is a root of the characteristic polynomial, hence an eigenvalue of A_δ. Moreover a simple computation shows that,

$$\left(\frac{b_2(\delta)}{2}\right)^2 - b_0(\delta) = -4\delta^2\alpha_{\epsilon,n}^2\mu_{\epsilon,n} + f_1(\delta),$$

where $f_1(\delta)$ is an analytic function such that $f_1(\delta) = O(\delta^{2n+2})$. Thus one can define an analytic function,

$$g(\delta) := 2i\alpha_{\epsilon,n}\delta\sqrt{\mu_{\epsilon,n} - \frac{f_1(\delta)}{4\alpha_{\epsilon,n}^2\delta^2}},$$

such that,

$$g^2(\delta) = \left(\frac{b_2(\delta)}{2}\right)^2 - b_0(\delta).$$

Now since $g(\delta) = 2i\alpha_{\epsilon,n}\delta\sqrt{\mu_{\epsilon,n}} + O(\delta^{2n+1})$ and bearing in mind (4.62) we have that,

$$-\frac{b_2(\delta)}{2} + g(\delta) = (i\alpha_{\epsilon,n} + \delta\sqrt{\mu_{\epsilon,n}})^2 + f_2(\delta)$$

where $f_2(\delta)$ is analytic and $f_2(\delta) = O(\delta^{2n+1})$. Putting all these observations together we conclude that,

$$\lambda_\epsilon := \left(i\alpha_{\epsilon,n} + \delta\sqrt{\mu_{\epsilon,n}}\right)\sqrt{1 + \frac{f_2(\delta)}{(i\alpha_{\epsilon,n} + \delta\sqrt{\mu_{\epsilon,n}})^2}},$$

is an eigenvalue of $DX_{H_\epsilon}(0)$ and is analytic with respect to δ. Moreover, it is not difficult to see that,

$$\lambda_\epsilon = i\alpha_{\epsilon,n} + \delta\sqrt{\mu_{\epsilon,n}} + O(\delta^{2n+1}).$$

Finally, taking into account the expressions (4.60), the fact that n is an arbitrary natural number and λ_ϵ is analytic we conclude that,

$$\lambda_\epsilon = i\left(1 - \sum_{l=1}^{\infty} a_{1,0,l}\delta^{2l}\right) + \delta\sqrt{1 - \sum_{l=2}^{\infty} a_{0,1,l}\delta^{2l-2}}.$$

□

Remark 4.4.0.5. Given $n \in \mathbb{N}$, the Hamiltonian $H_{\epsilon,n}$ after the standard scaling takes the form,

$$h_{\delta,n} = -I_1 + \left\{\frac{1}{2}I_2 - \frac{\epsilon}{2}I_3 + \frac{\eta}{4}I_3^2\right\}\delta + \sum_{3m+2j+2l=k\geq 5}^{2n+4} a_{m,j,l}I_1^m I_3^j \delta^{k-3} + O(\delta^{2n+2}).$$

Let us denote by h_δ^n the Hamiltonian $h_{\delta,n}$ truncated at order δ^{2n+2}. The eigenvalues of the matrix $DX_{h_{\delta,n}}(0)$ are $\pm\beta_\epsilon \pm i\alpha_\epsilon$ where α_ϵ and β_ϵ are analytic with respect to $\delta = \epsilon^2$ due to the previous Lemma. Then according to [7] (see Theorem 2 on pag. 233) it follows that the eigenvectors of $DX_{h_{\delta,n}}(0)$ will also depend analytically from δ. Consequently, there exists an analytic matrix T_δ such that,

$$\Delta_\delta = T_\delta^{-1} DX_{h_{\delta,n}}(0) T_\delta,$$

where

$$\Delta_\delta = \begin{pmatrix} B^T & 0 \\ 0 & -B \end{pmatrix} \text{ and } B = \begin{pmatrix} \beta_\epsilon & \alpha_\epsilon \\ -\alpha_\epsilon & \beta_\epsilon \end{pmatrix}. \tag{4.63}$$

Moreover, it is not difficult to see that the matrix $DX_{h_\delta^n}(0)$ has eigenvalues $\pm\beta_{\epsilon,n} \pm i\alpha_{\epsilon,n}$ where,

$$\alpha_{\epsilon,n} = 1 - \sum_{l=1}^{n} a_{1,0,l}\delta^{2l}, \quad \beta_{\epsilon,n} = \delta\sqrt{1 - \sum_{l=1}^{n} a_{0,1,l+1}\delta^{2l}}. \tag{4.64}$$

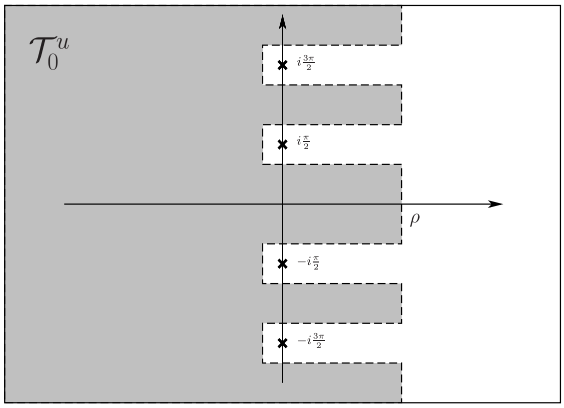

Figure 4.2: Domain $\mathcal{T}_0^u$.

Now according to previous Lemma we know that, $\beta_\epsilon = \beta_{\epsilon,n} + O(\delta^{2n+3})$ and $\alpha_\epsilon = \alpha_{\epsilon,n} + O(\delta^{2n+2})$. Thus, we can also transform the matrix $DX_{h_\delta^n}(0)$ to its canonical form,

$$\Delta_{\delta,n} = T_{\delta,n}^{-1} DX_{h_\delta^n}(0) T_{\delta,n}.$$

where the matrix $T_{\delta,n}$ is analytic with respect to δ and,

$$\Delta_{\delta,n} = \begin{pmatrix} B_n^T & 0 \\ 0 & -B_n \end{pmatrix} \text{ and } B_n = \begin{pmatrix} \beta_{\epsilon,n} & \alpha_{\epsilon,n} \\ -\alpha_{\epsilon,n} & \beta_{\epsilon,n} \end{pmatrix}.$$

Finally, analyticity in δ yields,

$$T_\delta = T_{\delta,n} + O(\delta^{2n+2}) \quad \text{and} \quad \Delta_\delta = \Delta_{\delta,n} + O(\delta^{2n+2}).$$

4.4.2 First approximation Theorem

In this subsection we prove that the unstable manifold W_ϵ^u can be parametrised by an analytic map $\mathbf{\Gamma}^u$ which is close to a partial sum of the formal separatrix $\hat{\mathbf{X}}_\delta$ and satisfies the following PDE,

$$\mathcal{D}_\epsilon \mathbf{\Gamma}^u = X_{H_{\epsilon,n}}(\mathbf{\Gamma}^u), \tag{4.65}$$

where recall from (4.2) that $\mathcal{D}_\epsilon = \alpha_\epsilon \partial_\varphi + \beta_\epsilon \partial_z$. More concretely, let $\rho, \sigma, h > 0$ and consider the following set,

$$\mathcal{T}_0^u(\rho, \sigma, h) = \{z \in \mathbb{C} \mid \mathrm{Re}(z) < \rho, X_0(\varphi, z - s) \in B_\sigma, \; \forall \, s \geq 0 \, \forall \varphi \in S_h\},$$

where $S_h = \{\varphi \in \mathbb{C} \mid |\mathrm{Im}\varphi| < h\}$, $B_\sigma \subset \mathbb{C}^4$ is an open ball centered at the origin having radius $\sigma > 0$ and X_0 is the leading order parametrisation given by (2.17). It is not difficult to see that has poles at $z = i\frac{\pi}{2} + i\pi k$, $k \in \mathbb{Z}$. Hence the domain $\mathcal{T}_0^u(\rho, \sigma, h)$ has the form similar to Figure 4.2.

In order to ease the notation we will occasionally drop the explicit dependence of the domain $\mathcal{T}_0^u(\rho, \sigma, h)$ on the parameters (ρ, σ, h). Now we are ready to prove the following,

Theorem 4.4.1. *Given $\rho, \sigma, h > 0$, for every $n \in \mathbb{N}$, there exists an analytic unstable parametrisation $\mathbf{\Gamma}^u : S_h \times \mathcal{T}_0^u(\rho, \sigma, h) \to \mathbb{C}^4$, 2π-periodic in φ, continuous on the closure of its domain and satisfying the PDE* (4.65) *such that*

$$\mathbf{\Gamma}^u = \mathbf{X}_\delta^n + O_{2n+2}(\delta),$$

valid in $S_h \times \mathcal{T}_0^u$ where $\mathbf{X}_\delta^n$ is a partial sum of the formal separatrix $\hat{\mathbf{X}}_\delta$ up to order δ^{2n+2} in the first two components and up to order δ^{2n+1} in the last two.

Proof. Since $DX_{H_{0,n}}(0)$ is not semisimple and we can not apply directly Theorem 4.3.1 to get an unstable parametrisation of $W^u_{\epsilon,loc}$. We overcome this difficulty by scaling variables according to the standard scaling (2.11). The Hamiltonian $H_{\epsilon,n}$ in the scaled variables reads,

$$h_{\delta,n} = h_\delta^n + O(\delta^{2n+2}), \tag{4.66}$$

where

$$h_\delta^n = -I_1 + \left(I_2 - I_3 + \eta I_3^2\right)\delta + \sum_{3m+2j+2l=k\geq 5}^{2n+4} a_{m,j,l} I_1^m I_3^j \delta^{k-3}.$$

Given $\sigma > 0$, for sufficiently small δ the domain of analyticity of the scaled Hamiltonian $h_{\delta,n}$ contains an δ-independent open ball $B_\sigma \subset \mathbb{C}^4$ centered at the origin and having radius σ.

Now following Remark 4.4.0.5 we can transform the linear part of the Hamiltonian vector field $X_{h_{\delta,n}}$ into its Jordan canonical form by a linear analytic change of variables,

$$\Delta_\delta = T_\delta^{-1} DX_{h_{\delta,n}}(0)T_\delta,$$

where Δ_δ is the matrix given in (4.63). Thus, we look for solutions of the following PDE,

$$\mathcal{D}_\epsilon \mathbf{x} = \Delta_\delta \mathbf{x} + F_\delta(\mathbf{x}), \tag{4.67}$$

where F_δ is analytic in B_σ, continuous on the closure of B_σ and $F_\delta(\mathbf{x}) = O(\|\mathbf{x}\|^2)$. We can now apply Theorem 4.3.1 and obtain for every $c \in \mathbb{C}^2$ an unique unstable parametrisation $\tilde{\mathbf{\Upsilon}}^u : S_h \times D_\gamma^u \to \mathbb{C}^4$, 2π-periodic in φ, continuous on the closure of its domain and satisfying the integral equation,

$$\tilde{\mathbf{\Upsilon}}^u = \Pi \begin{pmatrix} c \\ 0 \end{pmatrix} + \mathcal{L}^{-1}(F_\delta(\tilde{\mathbf{\Upsilon}}^u)), \tag{4.68}$$

where $\mathcal{L}^{-1}$ is given by Lemma 4.3.2 and $\gamma > 0$. Following Remark 4.3.1.3 we can extend the domain of analyticity of the unstable parametrisation $\tilde{\mathbf{\Upsilon}}^u$ onto a larger domain $\Omega \supseteq S_h \times D_\gamma^u$ of $\mathbb{C}^2$ until it leaves the open ball B_σ where the Hamiltonian $h_{\delta,n}$ is known to be analytic.

Let $\mathbf{\Upsilon}^u = T_\delta \circ \tilde{\mathbf{\Upsilon}}^u$. In the following we will construct an analytic map $\mathbf{X}_n$ (close to the formal series $\hat{\mathbf{X}}_\delta$ in the formal sense) that will approximate $\mathbf{\Upsilon}^u$ in a suitable subdomain of Ω. First note that the linearized system $DX_{h_\delta^n}(0)$ has eigenvalues $\pm\beta_{\delta,n} \pm i\alpha_{\delta,n}$ where $\alpha_{\delta,n}$ and $\beta_{\delta,n}$ are given by formulae (4.64). Also according to Remark 4.4.0.5 we have that

$$\beta_\epsilon + i\alpha_\epsilon = \beta_{\epsilon,n} + i\alpha_{\epsilon,n} + O(\delta^{2n+2}). \tag{4.69}$$

Now define $\mathcal{D}_{\epsilon,n} = \alpha_{\epsilon,n}\partial_\varphi + \beta_{\epsilon,n}\partial_z$. As in the proof of Theorem 4.2.1 we let $\mathbf{X}(\varphi, z) =$

$R_\varphi\xi(z)$ and note that,

$$\begin{aligned}\mathcal{D}_{\epsilon,n}\mathbf{X} - X_{h^n_\delta}(\mathbf{X}) &= \mathcal{D}_{\epsilon,n}R_\varphi\xi - X_{h^n_\delta}(R_\varphi\xi)\\ &= -\alpha_{\epsilon,n}R_\varphi X_{I_1}(\xi) + \beta_{\epsilon,n}R_\varphi\partial_z\xi - R_\varphi X_{h^n_\delta}(\xi)\\ &= R_\varphi\left(\beta_{\epsilon,n}\partial_z\xi - X_{\tilde{h}^n_\delta}(\xi)\right)\end{aligned} \tag{4.70}$$

where,

$$\tilde{h}^n_\delta = \left(I_2 - I_3 + \eta I_3^2\right)\delta + \sum_{\substack{3i+2j+2l=k\geq 5\\ i\neq 1 \text{ or } j\neq 0}}^{2n+4} a_{i,j,l}I_1^iI_3^j\delta^{k-3}.$$

Now changing to the polar coordinates (θ, r, Θ, R) as in the proof of Theorem 4.2.1 we define the following functions,

$$\begin{aligned}&\theta^{(n)}(z) := -\beta^{-1}_{\epsilon,n}\sum_{\substack{j+l=i\geq 1\\ j\geq 1, l\geq 0}}^{n}\frac{a_{1,j,l}}{2^j}\delta^{2i}\int_0^z\left(r^{(n)}(s)\right)^{2j}ds, \quad \Theta^{(n)} := 0,\\ &r^{(n)}(z) := \sum_{k=0}^{n} r_k(z)\delta^{2k}, \qquad R^{(n)}(z) := -\frac{\beta_{\epsilon,n}}{\delta}\partial_z r^{(n)}(z).\end{aligned} \tag{4.71}$$

where the coefficients r_k are defined in Claim 4.2.1.1 of Theorem 4.2.1 which are odd polynomials in the variable γ_0 (recall that $\gamma_0 = \sqrt{\frac{2}{\eta}\frac{1}{\cosh(z)}}$). Thus, it is clear that the functions $\theta^{(n)}$, $r^{(n)}$ and $R^{(n)}$ are analytic in $\mathbb{C}$ except for poles $z = i\frac{\pi}{2} + i\pi k$ for $k \in \mathbb{Z}$. Also from the proof of the same Theorem it follows that given $\rho, \sigma, h > 0$ we have that,

$$\beta^2_{\epsilon,n}\partial_z^2 r^{(n)} - \left(r^{(n)} - \eta(r^{(n)})^3\right)\delta^2 - \sum_{2j+2l=k\geq 5}^{2n+4}\frac{2ja_{0,j,l}}{2^j}(r^{(n)})^{2j-1}\delta^{k-2} = O(\delta^{2n+2}e^{3z}), \tag{4.72}$$

valid in the domain $\mathcal{T}^u_0(\rho, \sigma, h)$. Finally let us define the map $\mathbf{X}_n$ as follows,

$$\mathbf{X}_n := R_\varphi\xi_n, \tag{4.73}$$

where $\xi_n(z) = \left(R^{(n)}\cos\theta^{(n)}, R^{(n)}\sin\theta^{(n)}, r^{(n)}\cos\theta^{(n)}, r^{(n)}\sin\theta^{(n)}\right)^T$. Taking into account (4.70) and the estimate (4.72) it is not difficult to see that,

$$F_n := \mathcal{D}_{\epsilon,n}\mathbf{X}_n - X_{h^n_\delta}(\mathbf{X}_n) = O(\delta^{2n+2}e^{3z}), \quad \text{in} \quad S_h \times \mathcal{T}^u_0.$$

Moreover, it follows from the construction of the functions above and the last estimate that $\mathbf{X}_n$ coincides with the formal separatrix $\hat{\mathbf{X}}$ of Theorem 4.2.1 up to order δ^{2n}.

Now following Remark 4.4.0.5 there exists a linear analytic change of variables $T_{\delta,n}$ that transforms the linear part of the vector field $X_{h_\delta^n}$ into its Jordan canonical form,

$$\Delta_{\delta,n} = T_{\delta,n}^{-1} D X_{h_\delta^n}(0) T_{\delta,n}.$$

and moreover,

$$T_\delta = T_{\delta,n} + O(\delta^{2n+2}) \text{ and } \Delta_\delta = \Delta_{\delta,n} + O(\delta^{2n+2}). \tag{4.74}$$

Further, if $\tilde{\mathbf{X}}_n = T_{\delta,n}^{-1} \circ \mathbf{X}_n$ then it is not difficult to see that,

$$\mathcal{L}_{\alpha_{\epsilon,n},\beta_{\epsilon,n}}(\tilde{\mathbf{X}}_n) = F_\delta^n(\tilde{\mathbf{X}}_n) + \tilde{F}_n, \tag{4.75}$$

where,

$$F_\delta^n(\mathbf{x}) = T_{\delta,n}^{-1}\left(X_{h_\delta^n}(\mathbf{x}) - DX_{h_\delta^n}(0)\mathbf{x}\right) T_{\delta,n} \quad \text{and} \quad \tilde{F}_n = T_{\delta,n}^{-1} \circ F_n \circ T_{\delta,n},$$

and $\mathcal{L}_{\alpha_{\epsilon,n},\beta_{\epsilon,n}}$ is the linear operator defined in Lemma 4.3.2.

Now let $\Omega_0 = \Omega \cap (S_h \times \mathcal{T}_0^u(\rho,\sigma,h))$. Note that $F_\delta^n(\mathbf{x}) = O(\|\mathbf{x}\|^3)$ and standard Cauchy estimates yield $F_\delta^n(\tilde{\mathbf{X}}_n) = O(e^{3z})$ in Ω_0. Moreover, since $\tilde{F}_n = O(\delta^{2n+2}e^{3z})$ we can use Lemma 4.3.2 to rewrite equation (4.75) as follows,

$$\tilde{\mathbf{X}}_n = \Pi \begin{pmatrix} c_{\delta,n} \\ 0 \end{pmatrix} + \mathcal{L}^{-1}(F_\delta^n(\tilde{\mathbf{X}}_n)) + \mathcal{L}^{-1}(\tilde{F}_n), \tag{4.76}$$

where the constant $c_{\delta,n}$ is defined by the limit,

$$\begin{pmatrix} c_{\delta,n} \\ 0 \end{pmatrix} := \lim_{\operatorname{Re} z \to -\infty} \Pi(-\varphi,-z)\tilde{\mathbf{X}}_n(\varphi,z),$$

which converges uniformly with respect to $\varphi \in S_h$. Also note that it follows from the expressions (4.71) and (4.73) that $c_{\delta,n} \in \mathbb{R}^2[\delta]$.

Now we set $c := c_{\delta,n}$ in (4.68) and compute the difference (4.68)$-$(4.76),

$$\tilde{\mathbf{\Upsilon}}^u - \tilde{\mathbf{X}}_n = \mathcal{L}^{-1}(F_\delta(\tilde{\mathbf{\Upsilon}}^u) - F_\delta^n(\tilde{\mathbf{X}}_n)) - \mathcal{L}^{-1}(\tilde{F}_n), \tag{4.77}$$

where we have used the linearity of $\mathcal{L}^{-1}$. First we estimate the difference $F_\delta(\tilde{\mathbf{\Upsilon}}^u) - F_\delta^n(\tilde{\mathbf{X}}_n)$. Observe that,

$$F_\delta(\tilde{\mathbf{\Upsilon}}^u) - F_\delta^n(\tilde{\mathbf{X}}_n) = F_\delta(\tilde{\mathbf{\Upsilon}}^u) - F_\delta^n(\tilde{\mathbf{\Upsilon}}^u) + F_\delta^n(\tilde{\mathbf{\Upsilon}}^u) - F_\delta^n(\tilde{\mathbf{X}}_n). \tag{4.78}$$

Taking into account (4.66) and (4.74) we can deduce that,

$$F_\delta(\mathbf{x}) = F_\delta^n(\mathbf{x}) + O(\delta^{2n+2}\,\|\mathbf{x}\|^2),$$

and bearing in mind (4.68) we get the following upper bound for the first difference of the right hand side of (4.78),

$$F_\delta(\tilde{\mathbf{\Upsilon}}^u) - F_\delta^n(\tilde{\mathbf{\Upsilon}}^u) = O(\delta^{2n+2}e^{2z}), \quad \text{in} \quad \Omega_0.$$

Now we handle the second difference of (4.78). It follows from the Fundamental Theorem of Calculus that,

$$F_\delta^n(\tilde{\mathbf{\Upsilon}}^u) - F_\delta^n(\tilde{\mathbf{X}}_n) = \int_0^1 DF_\delta^n(s\tilde{\mathbf{\Upsilon}}^u + (1-s)\tilde{\mathbf{X}}_n)ds(\tilde{\mathbf{\Upsilon}}^u - \tilde{\mathbf{X}}_n).$$

Using Cauchy estimates for the function F_δ^n and the fact that both functions $\tilde{\mathbf{\Upsilon}}^u$ and $\tilde{\mathbf{X}}_n$ admit an upper bound of the type $O(e^z)$ in Ω_0 we can bound from above the integral in the previous formula by $O(e^{2z})$. Thus,

$$F_\delta^n(\tilde{\mathbf{\Upsilon}}^u) - F_\delta^n(\tilde{\mathbf{X}}_n) = O\left(e^{2z}(\tilde{\mathbf{\Upsilon}}^u - \tilde{\mathbf{X}}_n)\right) \quad \text{in} \quad \Omega_0.$$

Let $\mathbf{W} := \tilde{\mathbf{\Upsilon}}^u - \tilde{\mathbf{X}}_n$. Taking into account the upper bounds for the differences in (4.78) and the definition of $\mathcal{L}^{-1}$, it is not difficult to get the following estimates valid in Ω_0,

$$\begin{aligned}
\left\|e^{-2z}\mathcal{L}^{-1}(\tilde{F}_n)(\varphi,z)\right\| &\le k_0\delta^{2n+2},\\
\left\|e^{-2z}\mathcal{L}^{-1}\left(F_\delta(\tilde{\mathbf{\Upsilon}}^u) - F_\delta^n(\tilde{\mathbf{\Upsilon}}^u)\right)(\varphi,z)\right\| &\le k_1\delta^{2n+2},\\
\left\|e^{-2z}\mathcal{L}^{-1}\left(F_\delta^n(\tilde{\mathbf{\Upsilon}}^u) - F_\delta^n(\tilde{\mathbf{X}}_n)\right)(\varphi,z)\right\| &\le \int_{-\infty}^0 k_2 e^{3s}\left\|e^{-2(z+s)}\mathbf{W}(\varphi+s,z+s)\right\| ds,
\end{aligned} \tag{4.79}$$

where k_0, k_1 and k_2 are positive constants and $\|\cdot\|$ is the standard infinity norm in $\mathbb{C}^4$. Now for $t \leq \rho$ let us define,

$$w(t) := \sup_{(\varphi,z)\in\Omega_0,\ \mathrm{Re}\, z\leq t} \left\| e^{-2z}\mathbf{W}(\varphi,z)\right\|.$$

Taking into account the estimates (4.79) and equation (4.77) it is not difficult to derive the following inequality for $w(t)$,

$$w(t) \leq k_3\delta^{2n+2} + k_4\int_{-\infty}^{t} e^{3s}w(s)ds, \quad t\leq\rho.$$

An application of Gronwall Lemma yields,

$$w(\rho) \leq k_3 e^{\frac{k_4 e^{3\rho}}{3}}\delta^{2n+2}.$$

Thus, for $(\varphi,z)\in\overline{\Omega_0}$ we have that,

$$\tilde{\mathbf{\Upsilon}}^u(\varphi,z) = \tilde{\mathbf{X}}_n(\varphi,z) + O(e^{2z}\delta^{2n+2}). \tag{4.80}$$

Now we extend the domain of analyticity of $\tilde{\mathbf{\Upsilon}}^u$ to $S_h\times\mathcal{T}_0^u(\rho,\sigma,h)$ and conclude the same estimate (4.80) in that domain. The argument goes as follows. Recall that $\tilde{\mathbf{\Upsilon}}^u$ is analytic in a domain $\Omega\subset\mathbb{C}^2$ which contains the set $S_h\times D^u_\gamma$. Now suppose that $S_h\times\mathcal{T}_0^u(\rho,\sigma,h)$ is not a subset of Ω, that is, suppose there exist $(\varphi_0,z_0)\in S_h\times\mathcal{T}_0^u(\rho,\sigma,h)$ such that $(\varphi_0,z_0)\notin\Omega$. Define,

$$t^* := \inf\left\{t\in\mathbb{R}^- \mid (\varphi_0+t, z_0+t)\notin\Omega\right\}.$$

Note that the infimum exists since there is $t_0\in\mathbb{R}^-$ such that $(\varphi_0+t_0,z_0+t_0)\in S_h\times D^u_\gamma\subseteq\Omega$. Moreover, the set Ω is open in $\mathbb{C}^2$, thus its complement is closed. Hence, (φ_0+t^*,z_0+t^*) belongs to the complement of Ω and $(\varphi_0+t,z_0+t)\in\Omega_0$ for all $t<t^*$. Thus, we can use the estimate (4.80) to get,

$$\tilde{\mathbf{\Upsilon}}^u(\varphi_0+t^*,z_0+t^*) = \tilde{\mathbf{X}}_n(\varphi_0+t^*,z_0+t^*) + O(\delta^{2n+2}),$$

and bearing in mind the definition of $\mathcal{T}_0^u(\rho,\sigma,h)$ we conclude that for δ sufficiently small $\tilde{\mathbf{\Upsilon}}^u(\varphi_0+t^*,z_0+t^*)$ belongs to the open ball B_σ. Thus contradicting the fact that $(\varphi_0 + t^*, z_0 + t^*) \notin \Omega$. Thus, the unstable parametrisation $\tilde{\mathbf{\Upsilon}}^u$ is analytic in $S_h \times \mathcal{T}_0^u(\rho, \sigma, h)$ and the estimate (4.80) also holds in this set. Moreover,

$$\begin{aligned} \mathbf{\Upsilon}^u - \mathbf{X}_n &= T_\delta \circ \tilde{\mathbf{\Upsilon}}^u - T_{\delta,n} \circ \tilde{\mathbf{X}}_n \\ &= T_{\delta,n} \circ (\tilde{\mathbf{\Upsilon}}^u - \tilde{\mathbf{X}}_n) + O(\delta^{2n+2}) \\ &= O(e^{2z}\delta^{2n+2}) + O(\delta^{2n+2}) = O(\delta^{2n+2}), \end{aligned} \tag{4.81}$$

and if $\left[\hat{\mathbf{X}}\right]_{2n}$ denotes a partial sum of the formal separatrix up to order δ^{2n} then,

$$\mathbf{\Upsilon}^u - \left[\hat{\mathbf{X}}\right]_{2n} = \mathbf{\Upsilon}^u - \mathbf{X}_n + \mathbf{X}_n - \left[\hat{\mathbf{X}}\right]_{2n} = O(\delta^{2n+2}) + O(\delta^{2n+1}) = O(\delta^{2n+1}).$$

Finally, denoting by $\mathbf{\Gamma}^u$ the parametrisation $\mathbf{\Upsilon}^u$ in the unscaled variables (2.11) we get the desired result. □

4.4.3 Extension of the approximation towards the singularity $z = i\frac{\pi}{2}$

In the previous subsection we have constructed approximations for the unstable manifold W_ϵ^u near the equilibrium point. Then using a finite time stability argument we have extended the approximation until it leaves the domain of analyticity of the Hamiltonian vector field. Given $n \in \mathbb{N}$, the approximations $\mathbf{X}_\delta^n$ have singularities for complex $z \in \mathbb{C}$. In fact according to the definition of $\mathbf{X}_\delta^n$ in the statement of Theorem 4.4.1 we known that $\mathbf{X}_\delta^n = R_\varphi(\xi_1^n, \xi_2^n, \xi_3^n, \xi_4^n)$ where

$$\xi_1^n = \dot{\gamma}_0 \sum_{k=0}^{n} \psi_k^1 \delta^{2k+2}, \qquad \xi_2^2 = \sum_{k=0}^{n-1} \phi_{k+1}^1 \delta^{2k+3}, \tag{4.82}$$

$$\xi_3^n = \sum_{k=0}^{n} \phi_k^2 \delta^{2k+1}, \qquad \xi_4^n = \dot{\gamma}_0 \sum_{k=0}^{n-1} \psi_k^2 \delta^{2k+2}, \tag{4.83}$$

where ψ_k^i are even polynomials in γ_0 of $\deg(\psi_k^i) = 2k$ and ϕ_k^i are odd polynomials in γ_0 of $\deg(\phi_k^i) = 2k+1$. Recall that from the definition of γ_0 that it has simple poles for $z = i\frac{\pi}{2} + k\pi$ with $k \in \mathbb{Z}$. Thus the sum $\mathbf{X}_\delta^n$ grows in a neighbourhood of the singular point $z = i\frac{\pi}{2}$.

In this subsection we extend the approximation result obtained in the previous subsection for points δ-close to the singularity $z = i\frac{\pi}{2}$. For that end it is convenient to introduce a new variable τ which satisfies the relation,

$$z = \frac{\beta_\epsilon}{\alpha_\epsilon}\tau + i\frac{\pi}{2}. \tag{4.84}$$

According to Lemma 4.4.1 we known that $\alpha_\epsilon = O(1)$ and $\beta_\epsilon = O(\delta)$. Thus, change (4.84) fixes the singularity at $\tau = 0$ and for small δ augments a neighbourhood of the singularity by a factor of order δ^{-1}. In the new variable τ, the formal separatrix $\hat{\mathbf{X}}_\delta$ satisfies the following PDE,

$$\mathcal{D}\hat{\mathbf{X}}_\delta = \alpha_\epsilon^{-1} X_{H_\epsilon^{NF}}(\hat{\mathbf{X}}_\delta),$$

where $\mathcal{D}$ is the differential operator $\partial_\varphi + \partial_\tau$ used in chapter 3. This fact is very important and it will be used later on in the development of the theory. In order to extend the approximation given by $\mathbf{X}_\delta^n$ we first need to study its behaviour near the singular point.

Re-expansion of $\mathbf{X}_\delta^n$ around the singularity $i\frac{\pi}{2}$

In order to derive the Laurent series of $\mathbf{X}_\delta^n$ we first expand the base functions, γ_0 and $\dot{\gamma_0}$ around the singularity $i\frac{\pi}{2}$,

$$\gamma_0(\zeta + i\frac{\pi}{2}) = -i\sqrt{\frac{2}{\eta}}\frac{1}{\zeta}\left(1 + \sum_{k=1}^{\infty} a_k\zeta^{2k}\right), \quad \dot{\gamma_0}(\zeta + i\frac{\pi}{2}) = i\sqrt{\frac{2}{\eta}}\frac{1}{\zeta^2}\left(1 + \sum_{k=1}^{\infty} b_k\zeta^{2k}\right),$$

where $a_k, b_k \in \mathbb{C}$ and both functions are analytic in punctured disk $0 < |\zeta| < \pi$ (where the size of the disk is given by the distance to the closest singularity). In the following we will only deal with the function,

$$\xi_1^n = \dot{\gamma_0}\sum_{k=0}^{n} \psi_k^1 \delta^{2k+2}.$$

We compute its Laurent series in the new variable τ and the same procedure can be applied to the remaining components of $\mathbf{X}_\delta^n$. Let us present the details. Since ψ_k^1 is an even polynomial of degree $2k$ in the variable γ_0, we can write,

$$\psi_k^1 = \sum_{i=0}^{k} \psi_{k,i}^1 \gamma_0^{2i}, \quad \text{where} \quad \psi_{k,i}^1 \in \mathbb{R}.$$

According to the Laurent series of γ_0 we have,

$$\begin{aligned} \psi_k^1(\zeta + i\frac{\pi}{2}) &= \sum_{i=0}^{k} \psi_{k,i}^1 \left(-i\sqrt{\frac{2}{\eta}}\right)^{2i} \frac{1}{\zeta^{2i}} \left(1 + \sum_{j=0}^{\infty} a_j \zeta^{2j}\right)^{2i} \\ &= \frac{1}{\zeta^{2k}} \sum_{i=0}^{\infty} \hat{\psi}_{k,i}^1 \zeta^{2i}, \end{aligned} \tag{4.85}$$

for some $\hat{\psi}_{k,i}^1 \in \mathbb{C}$ and analytic in $0 < |\zeta| < \pi$. Now taking into account the Laurent series of $\dot{\gamma_0}$ we can expand the function ξ_1^n,

$$\begin{aligned} \xi_1^n(\zeta + i\frac{\pi}{2}) &= \sum_{k=0}^{n} \left(i\sqrt{\frac{2}{\eta}} \frac{1}{\zeta^2} (1 + \sum_{k=1}^{\infty} b_k \zeta^{2k}) \frac{1}{\zeta^{2k}} \sum_{i=0}^{\infty} \hat{\psi}_{k,i}^1 \zeta^{2i} \right) \delta^{2k+2} \\ &= \sum_{k=0}^{n} \frac{1}{\zeta^{2k+2}} \left(\sum_{i=0}^{\infty} \breve{\psi}_{k,i}^1 \zeta^{2i} \right) \delta^{2k+2}, \end{aligned}$$

for some $\breve{\psi}_{k,i}^1 \in \mathbb{C}$ and analytic in $0 < |\zeta| < \pi$. At this point we let $\zeta = \frac{\beta_\epsilon}{\alpha_\epsilon}\tau$ (according to formula (4.84)) and substitute into the previous series. First observe that due to Lemma 4.4.1 the quotient $\frac{\beta_\epsilon}{\alpha_\epsilon}$ is an odd function of δ and analytic in a sufficiently small open disk centered at $\delta = 0$. Moreover,

$$\left(\frac{\beta_\epsilon}{\alpha_\epsilon}\right)^{2i} = \delta^{2i} \sum_{j=0}^{\infty} h_{2i,j} \delta^{2j}.$$

It is convenient to write $\xi_1^n(\tau)$ for $\xi_1^n(\frac{\beta}{\alpha}\tau + i\frac{\pi}{2})$ in order to simplify the exposition. Thus,

$$\xi_1^n(\tau) = \sum_{k=0}^{n} \sum_{i=0}^{\infty} \breve{\psi}_{k,i}^1 \delta^{2i-(2k+2)} \sum_{j=0}^{\infty} h_{2(i-k-1),j} \delta^{2j} \tau^{2(i-k-1)} \delta^{2k+2}.$$

Note that the term δ^{2k+2} cancels. Now setting $i + j = m$ we can rearrange the sums in the previous formula as follows,

$$\xi_1^n(\tau) = \sum_{m=0}^{\infty} \left(\sum_{i+j=m} \sum_{k=0}^{n} h_{2(i-k-1),j} \breve{\psi}_{k,i}^1 \tau^{2(i-k-1)} \right) \delta^{2m}.$$

Finally we simplify the part inside the parenthesis in the previous formula. If $l = i-k-1$ then,

$$\begin{aligned}
\xi_1^n(\tau) &= \sum_{m=0}^{\infty} \delta^{2m} \sum_{l=-n-1}^{m-1} \tau^{2l} \sum_{\substack{i+j=m \\ i-k-1=l \\ k=0,\dots,n}} h_{2l,j} \breve{\psi}_{k,i}^1 \\
&= \sum_{m=0}^{\infty} \delta^{2m} \sum_{l=-n-1}^{m-1} \tilde{\psi}_{n,m,l}^1 \tau^{2l}
\end{aligned}$$

analytic for $|\delta|$ sufficiently small and $0 < |\tau| < \frac{\alpha_\epsilon}{\beta_\epsilon}\pi$.

Similar expansions can be obtained for the other components of $\mathbf{X}_\delta^n$ and we summarize the results in the form of a Lemma,

Lemma 4.4.2. *For any $n \in \mathbb{N}$ the functions ξ_i^n, $i = 1, \dots, 4$, have the following Laurent expansions around the singularity $i\frac{\pi}{2}$,*

$$\xi_1^n(\tau) = \sum_{m=0}^{\infty} \delta^{2m} \sum_{l=-n-1}^{m-1} \tilde{\psi}_{n,m,l}^1 \tau^{2l}, \qquad \xi_2^n(\tau) = \sum_{m=0}^{\infty} \delta^{2m} \sum_{l=-n-1}^{m-2} \tilde{\phi}_{n,m,l}^1 \tau^{2l+1},$$
$$\xi_3^n(\tau) = \sum_{m=0}^{\infty} \delta^{2m} \sum_{l=-n-1}^{m-1} \tilde{\phi}_{n,m,l}^2 \tau^{2l+1}, \quad \xi_4^n(\tau) = \sum_{m=0}^{\infty} \delta^{2m} \sum_{l=-n}^{m-1} \tilde{\psi}_{n,m,l}^2 \tau^{2l},$$

where τ is given by formula (4.84). *The coefficients $\tilde{\psi}_{n,m,l}^1, \tilde{\phi}_{n,m,l}^1, \tilde{\phi}_{n,m,l}^2, \tilde{\psi}_{n,m,l}^2$ belong to $\mathbb{C}$ and all series converge for $|\delta|$ sufficiently small and $0 < |\tau| < \frac{\alpha_\epsilon}{\beta_\epsilon}\pi$.*

Thus, $\mathbf{X}_\delta^n$ has poles at $z = i\frac{\pi}{2} + i\pi k$, $k \in \mathbb{Z}$ of order $2n+2$ in the first two components and of order $2n+1$ in the last two components.

Extension Theorem

Now given c_1, r_1 and ρ_1 positive real constants and $|\theta_1| < \frac{\pi}{4}$, consider the following set,

$$\begin{aligned}
D_1^u(\delta) = \{ \tau \in \mathbb{C} \mid\ & |\arg(\tau + r_1)| > \pi - \theta_1, \\
& -c_1\delta^{-1} < \mathrm{Re}(\tau) < \rho_1, |\mathrm{Im}(\tau)| < c_1\delta^{-1} \}.
\end{aligned}$$

Note that $D_1^u(\delta)$ is an open domain in $\mathbb{C}$ and is only defined for $\delta < \frac{c_1}{r}$. In the following we shall leave c_1, θ_1 and ρ_1 fixed. Moreover, in order not to overload the notation and if no confusion arises, we shall not write the explicit dependence of $D_1^u(\delta)$ on its parameters and taking into account the relation (4.84) we will shorten the notation by writing $\mathbf{X}_\delta^n(\varphi,\tau) = \mathbf{X}_\delta^n(\varphi,z)$ and $\mathbf{\Gamma}^u(\varphi,\tau) = \mathbf{\Gamma}^u(\varphi,z)$.

Theorem 4.4.2. *For any $n \in \mathbb{N}$ there exists an $r_1 > 0$ sufficiently large such that the unstable parametrisation $\mathbf{\Gamma}^u$ of Theorem 4.4.1 can be analytically extended onto $S_h \times D_1^u(\delta)$ such that,*

$$\mathbf{\Gamma}^u = \mathbf{X}_\delta^n + O_{2n+2}(\tau^{-1}), \quad \text{in} \quad S_h \times D_1^u(\delta).$$

Proof. In the new variable τ the unstable parametrisation $\mathbf{\Gamma}^u$ of Theorem 4.4.1 satisfies the following PDE,

$$\mathcal{D}\mathbf{\Gamma}^u = \alpha_\epsilon^{-1} X_{H_{\epsilon,n}}(\mathbf{\Gamma}^u), \tag{4.86}$$

where $\mathcal{D} = \partial_\varphi + \partial_\tau$. In order to extend the domain of analyticity of the unstable parametrisation and the estimate (4.94) onto the domain $D_1^u(\delta)$ we derive an new integral equation from which will follow a solution of the PDE (4.86) that will match the unstable parametrisation in a boundary domain. By the uniqueness of solutions of (4.86) it will provide the desired extension onto $D_1^u(\delta)$. Let us present the details. Let $\mathbf{Z} = \mathbf{\Gamma}^u - \mathbf{X}_\delta^n$. It follows from equation (4.86) that $\mathbf{Z}$ satisfies the PDE,

$$\mathcal{D}\mathbf{Z} = \alpha_\epsilon^{-1} X_{H_{\epsilon,n}}(\mathbf{Z} + \mathbf{X}_\delta^n) - \mathcal{D}\mathbf{X}_\delta^n. \tag{4.87}$$

Now we rewrite the previous equation as follows,

$$\mathcal{L}_0(\mathbf{Z}) = \mathbf{F}_0 + (\mathbf{F}_{1,0} + \mathbf{F}_{1,1})\mathbf{Z} + \mathcal{F}_2(\mathbf{Z}), \tag{4.88}$$

where $\mathcal{L}_0(\mathbf{Z}) = \mathcal{D}\mathbf{Z} - A_0\mathbf{Z}$ (the matrix A_0 is given by (2.37)) and,

$$\begin{aligned}
\mathbf{F}_0 &= \alpha_\epsilon^{-1} X_{H_{\epsilon,n}}(\mathbf{X}_\delta^n) - \mathcal{D}\mathbf{X}_\delta^n, \\
\mathbf{F}_{1,0} &= DX_{H_{0,n}}(\mathbf{X}_\delta^n) - A_0, \\
\mathbf{F}_{1,1} &= \alpha_\epsilon^{-1} DX_{H_{\epsilon,n}}(\mathbf{X}_\delta^n) - DX_{H_{0,n}}(\mathbf{X}_\delta^n), \\
\mathcal{F}_2(\mathbf{Z}) &= \alpha_\epsilon^{-1}\left(X_{H_{\epsilon,n}}(\mathbf{X}_\delta^n + \mathbf{Z}) - X_{H_{\epsilon,n}}(\mathbf{X}_\delta^n) - DX_{H_{\epsilon,n}}(\mathbf{X}_\delta^n)\mathbf{Z}\right).
\end{aligned} \tag{4.89}$$

Now for a given $\tau_0 \in \ell_{c_1}$ let us define,

$$\mathbf{Z}_{in}(\varphi, \tau) := \mathbf{U}_0(\varphi, \tau)\mathbf{U}_0^{-1}(\varphi - \tau + \tau_0, \tau_0)\mathbf{Z}(\varphi - \tau + \tau_0, \tau_0), \tag{4.90}$$

where ℓ_{c_1} is the left boundary of the set $D_1^u(\delta)$,

$$\ell_{c_1} = \left\{\tau \in \mathbb{C} \mid \quad \mathrm{Re}(\tau) = -c_1\delta^{-1}, \quad |\mathrm{Im}(\tau)| < c_1\delta^{-1}\right\}, \tag{4.91}$$

and $\mathbf{U}_0$ is the fundamental matrix of the linear operator $\mathcal{L}_0$ as defined in (2.38). Notice that $\mathbf{Z}_{in}(\varphi, \tau_0) = \mathbf{Z}(\varphi, \tau_0)$. Thus, equation (4.88) is equivalent to,

$$\mathbf{Z} = \mathbf{Z}_{in} + \mathcal{L}_0^{-1}(\mathbf{F}_0 + \mathbf{F}_{1,0}\mathbf{Z} + \mathbf{F}_{1,1}\mathbf{Z} + \mathcal{F}_2(\mathbf{Z})), \tag{4.92}$$

where $\mathcal{L}_0^{-1}$ is acting by the following formula,

$$\mathcal{L}_0^{-1}(\mathbf{Z})(\varphi, \tau) = \mathbf{U}_0(\varphi, \tau)\int_{\tau_0}^{\tau} \mathbf{U}_0^{-1}(\varphi - \tau + r, r)\mathbf{Z}(\varphi - \tau + r, r)dr, \tag{4.93}$$

and the path of the integral is a segment joining the points τ_0 and τ. In the following we will use equation (4.92) to extend the domain of analyticity of $\mathbf{Z}$. To that end we have to estimate the terms involved in equation that equation. Let us define a set Ω_{τ_0} as follows,

$$\Omega_{\tau_0} := \{(\varphi, \tau) \in S_h \times D_1^u(\delta) \mid \varphi - \tau + \tau_0 \in S_h,\ \lambda\tau + (1-\lambda)\tau_0 \in D_1^u(\delta),\ \forall\lambda \in [0,1]\}.$$

Note that Ω_{τ_0} is an open and connected subset of $S_h \times D_1^u(\delta)$. We need the following,

Claim 4.4.2.1. Formula (4.93) defines a bounded linear operator $\mathcal{L}_0^{-1} : \mathfrak{X}_{p+1}(\Omega_{\tau_0}) \to \mathfrak{X}_p(\Omega_{\tau_0})$ for $p \geq 3$.

Proof. That $\mathcal{L}_0^{-1}$ is linear it's clear from the definition. Now let $\xi \in \mathfrak{X}_{p+1}(\Omega_{\tau_0})$ then we can write $\xi = (\tau^{-p-2}\xi_1, \tau^{-p-2}\xi_2, \tau^{-p-1}\xi_3, \tau^{-p-1}\xi_4)$ where each ξ_i is bounded in Ω_{τ_0} for $i = 1, \ldots, 4$. Also from the definition of $\mathcal{L}_0^{-1}$ it is clear that $\mathcal{L}_0^{-1}(\xi)$ is an analytic function in Ω_{τ_0}, continuous on the closure of its domain and 2π-periodic in φ. Thus, it remains to show that $\left\|\mathcal{L}_0^{-1}(\xi)\right\|_p < \infty$.

Taking into account that $\mathbf{U}_0$ is a normalized fundamental matrix (in particular $\det \mathbf{U}_0 = 1$) then denoting by g_i each component of $\mathbf{U}_0^{-1}\xi$ then we get the following estimates,

$$|g_1(\varphi,\tau)| \leq \frac{M_{\mathbf{U}_0^{-1}} \|\xi\|_{p+1}}{|\tau|^p}, \quad |g_2(\varphi,\tau)| \leq \frac{M_{\mathbf{U}_0^{-1}} \|\xi\|_{p+1}}{|\tau|^{p+4}},$$
$$|g_3(\varphi,\tau)| \leq \frac{M_{\mathbf{U}_0^{-1}} \|\xi\|_{p+1}}{|\tau|^{p+3}}, \quad |g_4(\varphi,\tau)| \leq \frac{M_{\mathbf{U}_0^{-1}} \|\xi\|_{p+1}}{|\tau|^{p-1}},$$

valid in the set Ω_{τ_0} and $M_{\mathbf{U}_0^{-1}}$ some positive real constant. Note that $\|\xi\|_{p+1} < \infty$ by assumption. Now we estimate the integral in the formula of the definition of $\mathcal{L}_0^{-1}$. Let us handle the first component g_1. Thus, taking into account the estimate for g_1 we get,

$$\left|\int_{\tau_0}^{\tau} g_1(\varphi-\tau+r,r)dr\right| \leq \int_{\tau_0}^{\tau} \frac{M_{\mathbf{U}_0^{-1}} \|\xi\|_{p+1}}{|r|^p} |dr| \leq \int_{-\infty}^{0} \frac{M_{\mathbf{U}_0^{-1}} \|\xi\|_{p+1}}{|\tau+s|^p} ds.$$

Now using Lemma 2.4.1 we obtain the following estimate for the integral of g_1,

$$\left|\int_{\tau_0}^{\tau} g_1(\varphi-\tau+r,r)dr\right| \leq \frac{K_{p-1} M_{\mathbf{U}_0^{-1}} \|\xi\|_{p+1}}{|\tau|^{p-1}},$$

valid in Ω_{τ_0}. In the same way it is possible to obtain similar estimates for the other g_i's. Consequently,

$$\left\|\mathcal{L}_0^{-1}(\xi)\right\|_p \leq \bar{K} \|\xi\|_{p+1},$$

where $\bar{K} = (K_{p-1} + K_{p+3} + K_{p+2} + K_{p-2}) M_{\mathbf{U}_0^{-1}} M_{\mathbf{U}_0}$. □

Let us continue the proof of the Theorem. We start by estimating the function $\mathbf{Z}_{in}$ in Ω_{τ_0}. It follows from Theorem 4.4.1 that given $c_1 > 0$ sufficiently large the following estimate,

$$\mathbf{\Gamma}^u(\varphi,\tau) = \mathbf{X}_\delta^n(\varphi,\tau) + O_{2n+2}(\tau^{-1}), \tag{4.94}$$

holds on the segment ℓ_{c_1} which was defined in (4.91). Thus, according to the definition (4.90) we have that $\|\mathbf{Z}_{in}(\varphi,\tau)\| \leq C_{in}\delta^{2n-1}$ in Ω_{τ_0}. Thus $\mathbf{Z}_{in} \in \mathfrak{X}_{2n-2}(\Omega_{\tau_0})$.

Now, taking into account the definition of the formal separatrix $\hat{\mathbf{X}}$ it is not difficult to derive the following estimate,

$$\mathcal{D}_\epsilon \mathbf{X}^n - X_{h_{\delta,n}}(\mathbf{X}^n) = O(\delta^{2n+1}\cosh^{-2n-2}(z)),$$

where $\mathbf{X}^n$ is a truncation of $\hat{\mathbf{X}}$ at the order δ^{2n}. The previous estimate implies that $\|\mathbf{F}_0(\varphi,\tau)\| \leq C_0\,|\tau|^{-2n-2}$ in Ω_{τ_0}. Thus $\mathbf{F}_0 \in \mathfrak{X}_{2n-1}(\Omega_{\tau_0})$ and in light of the previous claim we conclude that $\mathcal{L}_0^{-1}(\mathbf{F}_0) \in \mathfrak{X}_{2n-2}(\Omega_{\tau_0})$.

Moreover, it is not difficult to derive the following upperbounds for the functions $\mathbf{F}_{1,0}$ and $\mathbf{F}_{1,1}$,

$$\|\mathbf{F}_{1,0}(\varphi,\tau)\| \leq C_{1,0}\,|\tau|^{-3} \quad \text{and} \quad \|\mathbf{F}_{1,1}(\varphi,\tau)\| \leq C_{1,1}\delta^2. \tag{4.95}$$

Indeed, the first estimate follows from the fact that,

$$\mathbf{X}^n_\delta = \mathbf{\Gamma}_0 + O_3(\tau^{-1}) \quad \text{and} \quad DX_{H_{0,n}}(\mathbf{\Gamma}_0) - A_0 = O(\tau^{-3}),$$

whereas the second estimate follows from,

$$\alpha_\epsilon^{-1} = 1 + O(\delta^2) \quad \text{and} \quad DX_{H_{\epsilon,n}}(\mathbf{X}^n_\delta) = DX_{H_{0,n}}(\mathbf{X}^n_\delta) + O(\delta^2).$$

Let $p \in \mathbb{N}$. The first estimate of (4.95) implies that $\delta^{-2}\mathbf{F}_{1,1}$ induces a bounded linear operator $\mathcal{F}_{1,1} : \mathfrak{X}_p(\Omega_{\tau_0}) \to \mathfrak{X}_p(\Omega_{\tau_0})$ acting by the formula $\mathcal{F}_{1,1}(\xi)(\varphi,\tau) = \delta^{-2}\mathbf{F}_{1,1}(\varphi,\tau)\xi(\varphi,\tau)$ with $\|\mathcal{F}_{1,1}\|_{p,p} \leq C_{1,1}$. Similarly the function $\mathbf{F}_{1,0}$ induces a bounded linear operator $\mathcal{F}_{1,0} : \mathfrak{X}_p(\Omega_{\tau_0}) \to \mathfrak{X}_{p+1}(\Omega_{\tau_0})$ acting according to the the formula $\mathcal{F}_{1,0}(\xi)(\varphi,\tau) = \mathbf{F}_{1,0}(\varphi,\tau)\xi(\varphi,\tau)$ with $\|\mathcal{F}_{1,1}\|_{p+1,p} \leq \frac{C_{1,0}}{r_1}$. Now we rewrite equation (4.92) as follows,

$$\left(\mathsf{Id} - \delta^2\mathcal{L}_0^{-1} \circ \mathcal{F}_{1,1}\right)\mathbf{Z} = \mathbf{Z}_{in} + \mathcal{L}_0^{-1}(\mathbf{F}_0) + \mathcal{L}_0^{-1} \circ \mathcal{F}_{1,0}(\mathbf{Z}) + \mathcal{F}_2(\mathbf{Z})).$$

Using the fact that $|\delta\tau|$ is bounded in $D_1^u(\delta)$ we conclude that $\delta\mathcal{L}_0^{-1} \circ \mathcal{F}_{1,1} : \mathfrak{X}_p(\Omega_{\tau_0}) \to \mathfrak{X}_p(\Omega_{\tau_0})$ is a bounded (independent of δ) linear operator. Thus, Neumann series can be used to prove that $\mathcal{L}_1 := \mathrm{Id} - \delta^2\mathcal{L}_0^{-1} \circ \mathcal{F}_{1,1}$ has a bounded inverse $\mathcal{L}_1^{-1} : \mathfrak{X}_p(\Omega_{\tau_0}) \to \mathfrak{X}_p(\Omega_{\tau_0})$ provided $\left\|\delta^2\mathcal{L}_0^{-1} \circ \mathcal{F}_{1,1}\right\|_{p,p} < 1$ which certainly holds for δ sufficiently small.

Furthermore, similar arguments as in the proof of Theorem 3.4.1 show that for δ sufficiently small,

$$X_{H_{\epsilon,n}}(\mathbf{X}^n_\delta + \mathbf{x}) - X_{H_{\epsilon,n}}(\mathbf{X}^n_\delta) - DX_{H_{\epsilon,n}}(\mathbf{X}^n_\delta)\mathbf{x} = O(\|\mathbf{x}\|^2),$$

for $\mathbf{x} \in B_{\sigma_n}$. Thus, according to the definition of $\mathcal{F}_2$ we have that $\mathcal{F}_2 : \mathfrak{X}_p(\Omega_{\tau_0}) \to \mathfrak{X}_{p+1}(\Omega_{\tau_0})$.

Thus for $n \geq 3$ we define a non-linear operator $\mathcal{H} : \mathfrak{X}_{2n-2}(\Omega_{\tau_0}) \to \mathfrak{X}_{2n-2}(\Omega_{\tau_0})$ acting according to the formula,

$$\mathcal{H}(\xi) = \mathcal{L}_1^{-1}(\mathbf{Z}_{in}) + \mathcal{L}_1^{-1} \circ \mathcal{L}_0^{-1}(\mathbf{F}_0) + \mathcal{L}_1^{-1} \circ \mathcal{L}_0^{-1} \circ \mathcal{F}_{1,0}(\xi) + \mathcal{L}_1^{-1} \circ \mathcal{L}_0^{-1} \circ \mathcal{F}_2(\xi), \quad (4.96)$$

and prove that it is contracting on a certain closed ball in $\mathfrak{X}_{2n-2}(\Omega_{\tau_0})$. Note that a fix point of $\mathcal{H}$ is a solution of equation (4.92).

First we show that there is $\mu > 0$ such that $\mathcal{H}(\mathfrak{B}_\mu) \subseteq \mathfrak{B}_\mu$ where,

$$\mathfrak{B}_\mu := \left\{\xi \in \mathfrak{X}_{2n-2}(\Omega_{\tau_0}) \mid \|\xi\|_{2n-2} \leq \mu\right\}.$$

In fact, similar estimates as in the proof of Theorem 3.4.1 show that for $r_1 > 0$ sufficiently large and δ small enough we have the following estimate,

$$\|\mathcal{F}_2(\xi)\|_{2n-1} \leq \frac{8 \|H_{\epsilon,n}\|_{C^3} \|\xi\|_{2n-2}^2}{(r_1 \sin\theta_1)^{2n-4}}, \quad (4.97)$$

for $\xi \in \mathfrak{X}_{2n-2}(\Omega_{\tau_0})$. Now let

$$\mu = 2 \left\|\mathcal{L}_1^{-1}\right\|_{2n-2,2n-2} \left(\|\mathbf{Z}_{in}\|_{2n-2} + \left\|\mathcal{L}_0^{-1}\right\|_{2n-2,2n-1} \|\mathbf{F}_0\|_{2n-1}\right).$$

Taking into account (4.96) and estimate (4.97), then for $\xi \in \mathfrak{B}_\mu$ we have that,

$$\|\mathcal{H}(\xi)\|_{2n-2} \leq \frac{\mu}{2} + \frac{C_{1,0} M \mu}{r_1} + \frac{8M \|H_{\epsilon,n}\|_{C^3} \mu^2}{r_1^{2n-4} \sin^{2n-4}\theta_1},$$

where

$$M = \left\|\mathcal{L}_1^{-1}\right\|_{2n-2,2n-2} \left\|\mathcal{L}_0^{-1}\right\|_{2n-2,2n-1}.$$

Thus for,

$$r_1 > 2C_{1,0}M + \frac{16M \|H_{\epsilon,n}\|_{C^3} \mu}{\sin^{2n-4}\theta_1}, \quad (4.98)$$

we get that $\|\mathcal{H}(\xi)\|_{2n-2} < \mu$ for $\xi \in \mathfrak{B}_\mu$. Thus $\mathcal{H}$ leaves invariant the closed ball $\mathfrak{B}_\mu$. Now let us prove that $\mathcal{H}$ is contracting in $\mathcal{B}_\rho$. Again, similar estimates as in the proof of Theorem 3.4.1 show that,

$$\|\mathcal{F}_2(\xi_2) - \mathcal{F}_2(\xi_1)\|_{2n-1} \leq \frac{8\mu \|H_{\epsilon,n}\|_{C^3}}{(r_1 \sin\theta_1)^{2n-4}} \|\xi_2 - \xi_1\|_{2n-2},$$

for $\xi_1, \xi_2 \in \mathfrak{B}_\mu$ and according to the definition of $\mathcal{H}$ and (4.98) we get at once,

$$\|\mathcal{H}(\xi_2) - \mathcal{H}(\xi_1)\|_{2n-2} < \frac{1}{2}\|\xi_2 - \xi_1\|_{2n-2},$$

which proves that $\mathcal{H}$ is contracting in $\mathfrak{B}_\mu$. Consequently there exists an unique fixed point $\xi_* \in \mathfrak{B}_\mu$ of $\mathcal{H}$ such that $\mathbf{X}_\delta^n + \xi_*$ solves equation (4.86). Thus, by the main local existence and uniqueness theorem for analytic PDE (see for instance [24]) we conclude that the function $\mathbf{X}_\delta^n + \xi_*$ extends the domain of analyticity of $\mathbf{\Gamma}^u$ onto the set Ω_{τ_0}. Moreover, since

$$S_h \times D_1^u(\delta) = \bigcup_{\tau_0 \in \ell_{c_1}} \Omega_{\tau_0},$$

we can repeat the same arguments for every $\tau_0 \in \ell_{c_1}$ and due to uniqueness of analytic continuation we get that,

$$\mathbf{\Gamma}^u = \mathbf{X}_\delta^n + O_{2n-2}(\tau^{-1}), \quad \text{in} \quad S_h \times D_1^u(\delta).$$

Finally increasing n we obtain,

$$\mathbf{\Gamma}^u - \mathbf{X}_\delta^n = \mathbf{\Gamma}^u - \mathbf{X}_\delta^{n+4} + \mathbf{X}_\delta^{n+4} - \mathbf{X}_\delta^n = O_{2n+2}(\tau^{-1}),$$

which proves the desired estimate on the set $S_h \times D_1^u(\delta)$. □

4.4.4 Complex Matching

In this subsection we construct different approximations for the parametrisations of the unstable manifold near the singularity. These approximations will be obtained by a method known as complex matching. Roughly speaking, they retain the essential behavior near the singularity, providing better estimates for the parametrisations in that region. Moreover, we will show that these approximations can distinguish the stable and unstable manifolds and can be used to capture the exponentially small splitting. In order to construct these approximations we first need to recall some the approximations provided by the formal separatrix $\hat{\mathbf{X}}_\delta$. According to Lemma 4.4.2 we can write the formal Laurent expansion of $\hat{\mathbf{X}}_\delta$ as follows,

$$\hat{\mathbf{X}}_\delta = \sum_{m=0}^{\infty} \hat{X}_m \delta^{2m}, \quad \text{where} \quad \hat{X}_m = R_\varphi(\tilde{\psi}^1_m, \tilde{\phi}^1_m, \tilde{\phi}^2_m, \tilde{\psi}^2_m), \tag{4.99}$$

such that,

$$\begin{aligned} \tilde{\psi}^1_m(\tau) &= \sum_{l\le m-1} \tilde{\psi}^1_{m,l}\tau^{2l}, \qquad \tilde{\phi}^1_m(\tau) = \sum_{l\le m-2} \tilde{\phi}^1_{m,l}\tau^{2l+1}, \\ \tilde{\phi}^2_m(\tau) &= \sum_{l\le m-1} \tilde{\phi}^2_{m,l}\tau^{2l+1}, \quad \tilde{\psi}^2_m(\tau) = \sum_{l\le m-1} \tilde{\psi}^2_{m,l}\tau^{2l}. \end{aligned} \tag{4.100}$$

Note that the formal series (4.99) satisfies equation $\mathcal{D}\hat{\mathbf{X}}_\delta = \alpha_\epsilon^{-1} X_{H_\epsilon^{NF}}(\hat{\mathbf{X}}_\delta)$. Now according to normal form theory there is a formal near identity canonical transformation Φ that puts $H_{\epsilon,n}$ into its formal normal form, i.e., $H_\epsilon^{NF} = H_{\epsilon,n} \circ \Phi$. The transformation Φ has the general form,

$$\begin{aligned} q &= Q + \sum_{2|i|+|j|+2l\ge 2n+3} \hat{\Phi}_{i,j,l} Q^i P^j \delta^{2l}, \\ p &= P + \sum_{2|i|+|j|+2l\ge 2n+4} \tilde{\Phi}_{i,j,l} Q^i P^j \delta^{2l}, \end{aligned} \tag{4.101}$$

written in multi-index notation where $\hat{\Phi}_{i,j,l}, \tilde{\Phi}_{i,j,l} \in \mathbb{R}^2$. The composition $\hat{\mathbf{\Gamma}} = \Phi \circ \hat{\mathbf{X}}_\delta$ is well defined (it converges in the formal sense) in the class of formal series since to compute a certain coefficient one only needs a finite number of previous coefficients. Moreover, taking into account (4.101), (4.99) and the formal series (4.100) we can write $\hat{\mathbf{\Gamma}}$ as follows,

$$\hat{\mathbf{\Gamma}} = \sum_{m\ge 0} \hat{\mathbf{\Gamma}}_m \delta^{2m}, \tag{4.102}$$

where $\hat{\mathbf{\Gamma}}_m \in \tau^{2m-1}\mathsf{T}^4_{\mathbb{C}}[[\tau^{-1}]]$ (see section 3.2 for a definition of these spaces) and most important,

$$\mathcal{D}\hat{\mathbf{\Gamma}} = \alpha_\epsilon^{-1} X_{H_{\epsilon,n}}(\hat{\mathbf{\Gamma}}). \tag{4.103}$$

Substituting the series (4.102) into the equation (4.103) and collecting terms of the same order in δ^{2m} we obtain an infinite system of equations relating the coefficients $\hat{\mathbf{\Gamma}}_m$. At the leading order δ^0 we get the following equation,

$$\mathcal{D}\hat{\mathbf{\Gamma}}_0 = X_{H_{0,n}}(\hat{\mathbf{\Gamma}}_0), \tag{4.104}$$

and for the remaining orders it is not difficult to derive,

$$\mathcal{D}\hat{\mathbf{\Gamma}}_m = DX_{H_{0,n}}(\hat{\mathbf{\Gamma}}_0)\hat{\mathbf{\Gamma}}_m + F_m(\hat{\mathbf{\Gamma}}_0, \ldots, \hat{\mathbf{\Gamma}}_{m-1}), \qquad m \geq 1, \tag{4.105}$$

where F_m is a well defined function that depends on a finite number of coefficients of $H_{\epsilon,n}$. The theory of chapter 3 can be used to obtain analytic solutions for the previous system of equations with prescribed asymptotics given by the formal series $\hat{\mathbf{\Gamma}}_m$. More concretely, we have the following,

Lemma 4.4.3. *There exists an $r > 0$ and an unique sequence of analytic functions $\{\mathbf{\Gamma}_m^-\}_{m \geq 0}$ solving the infinite system of equations (4.105) such that for every $m \geq 0$ and $N \geq 3$ we have that,*

$$\mathbf{\Gamma}_m^- - \left\langle \hat{\mathbf{\Gamma}}_m \right\rangle_N \in \mathfrak{X}_{N+1}(S_h \times D_r^-).$$

Proof. It follows from Theorem 3.4.1 that there exists an $r > 0$ sufficiently large and an unique analytic parametrisation $\mathbf{\Gamma}_0^- \in \mathfrak{X}_1(S_h \times D_r^{-1})$ such that $\mathcal{D}\mathbf{\Gamma}_0^- = X_{H_{0,n}}(\mathbf{\Gamma}_0^-)$ and $\mathbf{\Gamma}_0^- \asymp \hat{\mathbf{\Gamma}}_0$. Hence $\mathbf{\Gamma}_0^- - \left\langle \hat{\mathbf{\Gamma}}_0 \right\rangle_N \in \mathfrak{X}_{N+1}(S_h \times D_r^{-1})$ for all $N \geq 3$.

Now we can solve equations (4.105) using induction on $m \geq 1$. Let us start with $m = 1$. We are looking for a solution $\mathbf{\Gamma}_1^-$ of equation,

$$\mathcal{D}\mathbf{\Gamma}_1^- = DX_{H_{0,n}}(\mathbf{\Gamma}_0^-)\mathbf{\Gamma}_1^- + F_1(\mathbf{\Gamma}_0^-). \tag{4.106}$$

We seek such solution by setting $\mathbf{\Gamma}_1^- = \left\langle \hat{\mathbf{\Gamma}}_1 \right\rangle_N + \mathbf{Z}$ for some $N \geq 3$. Thus $\mathbf{Z}$ must satisfy,

$$\mathcal{L}(\mathbf{Z}) = \mathbf{R}_1, \quad \text{where} \quad \mathbf{R}_1 = \mathcal{D}\left\langle \hat{\mathbf{\Gamma}}_1 \right\rangle_N - DX_{H_{0,n}}(\mathbf{\Gamma}_0^-)\left\langle \hat{\mathbf{\Gamma}}_1 \right\rangle_N - F_1(\mathbf{\Gamma}_0^-), \tag{4.107}$$

and $\mathcal{L}(\mathbf{Z}) = \mathcal{D}\mathbf{Z} - DX_{H_{0,n}}(\mathbf{\Gamma}_0^-)\mathbf{Z}$. Since $\hat{\mathbf{\Gamma}}_1$ solves formally equation (4.106) we get that $\mathbf{R}_1 \in \mathfrak{X}_{N+1}(S_h \times D_r^-)$. Moreover, the results of chapter 3 (in particular Theorem 3.5.0.1) imply the existence of a normalized fundamental matrix $\mathbf{U}$ having the form (2.33) such that $\mathcal{L}(\mathbf{U}) = 0$. Thus according to Theorem 2.4.1 the linear operator $\mathcal{L} : \mathfrak{X}_N(S_h \times D_r^-) \to \mathfrak{X}_N(S_h \times D_r^-)$ has a bounded right inverse $\mathcal{L}^{-1} : \mathfrak{X}_{N+1}(S_h \times$

$D_r^-) \to \mathfrak{X}_N(S_h \times D_r^-)$ for $N \geq 3$. Since $R_1 \in \mathfrak{X}_{N+1}(S_h \times D_r^-)$, it follows that $\mathcal{L}^{-1}(R_1) \in \mathfrak{X}_N(S_h \times D_r^-)$, thus $\mathbf{\Gamma}_1^- := \left\langle \hat{\mathbf{\Gamma}}_1 \right\rangle_N + \mathcal{L}^{-1}(R_1)$ is the desired solution of equation (4.106). Moreover, its uniqueness follows from the fact that the kernel of $\mathcal{L}$ is trivial. As N is arbitrary we conclude that $\mathbf{\Gamma}_1^- \asymp \hat{\mathbf{\Gamma}}_1$.

Finally, in order to complete the induction it remains to show that we can repeat the same steps for $m \geq 2$. Since it does not present any difficulty we conclude the proof of the Lemma. □

Let $c_2 > 0$ be any fixed constant. Let $D_2^u(\delta)$ be a subset of $D_1^u(\delta)$ which is defined as follows,

$$D_2^u(\delta) = D_1^u(\delta) \cap \left\{ \tau \in \mathbb{C} \mid -c_2\delta^{-\frac{1}{2}} < \operatorname{Re}\tau < +\infty, \quad |\operatorname{Im}\tau| < c_2\delta^{-\frac{1}{2}} \right\},$$

and ℓ_{c_2} the left boundary of the set $D_2^u(\delta)$,

$$\ell_{c_2} = \left\{ \tau \in \mathbb{C} \mid \quad \operatorname{Re}(\tau) = -c_2\delta^{-\frac{1}{2}}, \quad |\operatorname{Im}(\tau)| < c_2\delta^{-\frac{1}{2}} \right\}. \tag{4.108}$$

Let us prove a preliminary result which will be used in the next theorem.

Lemma 4.4.4 (Complex Matching). *Given $n \in \mathbb{N}$, the following estimate holds,*

$$\mathbf{X}_\delta^n = \sum_{m=0}^{n} \mathbf{\Gamma}_m^- \delta^{2m} + O(\delta^{n+1}), \tag{4.109}$$

uniformly in the set $S_h \times \ell_{c_2}$.

Proof. It follows from the definition of $\mathbf{X}_\delta^n$ and the formal series $\hat{\mathbf{\Gamma}}_m$ that,

$$\mathbf{X}_\delta^n - \sum_{m=0}^{n} \left\langle \hat{\mathbf{\Gamma}}_m \right\rangle_{2n+2} \delta^{2m} = O(\delta^{n+1}), \quad \text{in} \quad S_h \times \ell_{c_2}.$$

Moreover, the previous Lemma implies that,

$$\sum_{m=0}^{n} \left(\left\langle \hat{\mathbf{\Gamma}}_m \right\rangle_{2n+2} - \mathbf{\Gamma}_m^- \right) \delta^{2m} = O(\delta^{n+1}), \quad \text{in} \quad S_h \times \ell_{c_2}$$

Putting together these estimates we get (4.109). □

We are now ready to prove our second approximation result.

Theorem 4.4.3. *Given $n \in \mathbb{N}$ there exists an $r_1 > 0$ such that the unstable parametrisation $\mathbf{\Gamma}^u$ of Theorem 4.4.2 can be approximated in $S_h \times D_2^u(\delta)$ as follows,*

$$\mathbf{\Gamma}^u = \sum_{m=0}^{n} \mathbf{\Gamma}_m^- \delta^{2m} + O((\tau\delta))^{2n+2},$$

where the functions $\mathbf{\Gamma}_m^-$ are given by Lemma 4.4.3.

Proof. According to Theorem 4.4.2, for every $n \geq 3$ there exists an unstable parametrisation $\mathbf{\Gamma}^u : S_h \times D_1^u(\delta) \to \mathbb{C}^4$ which is a solution of the following PDE,

$$\mathcal{D}\mathbf{\Gamma}^u = \alpha_\epsilon^{-1} X_{H_{\epsilon,n}}(\mathbf{\Gamma}^u), \tag{4.110}$$

such that,

$$\mathbf{\Gamma}^u = \mathbf{X}_\delta^n + O_{2n+2}(\tau^{-1}), \quad \text{in} \quad S_h \times D_1^u(\delta). \tag{4.111}$$

Now let $\mathbf{\Gamma}^n = \sum_{m=0}^{n} \mathbf{\Gamma}_m^- \delta^{2m}$ and define $\mathbf{Z} := \mathbf{\Gamma}^u - \mathbf{\Gamma}^n$. Since $\mathbf{\Gamma}^u$ satisfies equation (4.110) then it is not difficult to see that $\mathbf{Z}$ must satisfy the following equation,

$$\mathcal{L}(\mathbf{Z}) = \mathbf{F}_0 + \mathbf{F}_1\mathbf{Z} + \mathcal{F}_2(\mathbf{Z}), \tag{4.112}$$

where $\mathcal{L}(\mathbf{Z}) = \mathcal{D}\mathbf{Z} - DX_{H_{0,n}}(\mathbf{\Gamma}_0^-)\mathbf{Z}$ and moreover,

$$\begin{aligned}
\mathbf{F}_0 &= \alpha_\epsilon^{-1} X_{H_{\epsilon,n}}(\mathbf{\Gamma}^n) - \mathcal{D}\mathbf{\Gamma}^n, \\
\mathbf{F}_1 &= \alpha_\epsilon^{-1} DX_{H_{\epsilon,n}}(\mathbf{\Gamma}^n) - DX_{H_{0,n}}(\mathbf{\Gamma}_0^-), \\
\mathcal{F}_2(\mathbf{Z}) &= \alpha_\epsilon^{-1} \left(X_{H_{\epsilon,n}}(\mathbf{\Gamma}^n + \mathbf{Z}) - X_{H_{\epsilon,n}}(\mathbf{\Gamma}^n) - DX_{H_{\epsilon,n}}(\mathbf{\Gamma}^n)\mathbf{Z} \right).
\end{aligned}$$

Now for a given $\tau_0 \in \ell_{c_2}$ let us define,

$$\mathbf{Z}_{in}(\varphi, \tau) := \mathbf{U}(\varphi, \tau)\mathbf{U}^{-1}(\varphi - \tau + \tau_0, \tau_0)\mathbf{Z}(\varphi - \tau + \tau_0, \tau_0), \tag{4.113}$$

where ℓ_{c_2} is defined in (4.108) and $\mathbf{U}$ is a normalized fundamental matrix of $\mathcal{L}$, i.e., $\mathcal{L}(\mathbf{U}) = 0$, which exists due to Theorem 3.3.1. Notice that $\mathbf{Z}_{in}(\varphi, \tau_0) = \mathbf{Z}(\varphi, \tau_0)$. Thus, equation (4.112) is equivalent to,

$$\mathbf{Z} = \mathbf{Z}_{in} + \mathcal{L}^{-1}(\mathbf{F}_0) + \mathcal{L}^{-1}(\mathbf{F}_1\mathbf{Z}) + \mathcal{L}^{-1} \circ \mathcal{F}_2(\mathbf{Z}), \tag{4.114}$$

where $\mathcal{L}^{-1}$ is acting by the following formula,

$$\mathcal{L}^{-1}(\xi)(\varphi,\tau) = \mathbf{U}(\varphi,\tau)\int_{\tau_0}^{\tau} \mathbf{U}^{-1}(\varphi-\tau+r,r)\xi(\varphi-\tau+r,r)dr, \tag{4.115}$$

and the path of the integral is a segment joining the points τ_0 and τ.

It is possible to estimate the functions $\mathbf{F}_0$ and $\mathbf{F}_1$ as follows,

$$\|\mathbf{F}_0(\varphi,\tau)\| \leq C_0\delta^{2n+2}|\tau|^{2n-1} \quad \text{and} \quad \|\mathbf{F}_1(\varphi,\tau)\| \leq C_1\delta^2|\tau|, \tag{4.116}$$

valid in $S_h \times D_2^u(\delta)$ for some $C_0, C_1 > 0$. Indeed, both estimates follow from the fact that $\mathbf{\Gamma}_j^- = O_{2j-1}(\tau)$ for $j \geq 0$ and thus $\mathbf{\Gamma}^n = \mathbf{\Gamma}_0^- + O(\delta^2\tau)$ in $S_h \times D_2^u(\delta)$. Moreover, similar to Theorem 4.4.2 we define the set Ω_{τ_0} as follows,

$$\Omega_{\tau_0} := \{(\varphi,\tau) \in S_h \times D_2^u(\delta) \mid \varphi-\tau+\tau_0 \in S_h,\ \lambda\tau+(1-\lambda)\tau_0 \in D_2^u(\delta),\ \forall\lambda\in[0,1]\}.$$

Note that Ω_{τ_0} is an open and connected subset of $S_h \times D_2^u(\delta)$ and

$$S_h \times D_2^u(\delta) = \bigcup_{\tau_0 \in \ell_{c_2}} \Omega_{\tau_0}.$$

As in the proof of Theorem 4.4.2 we can show that formula (4.115) defines a bounded linear operator $\mathcal{L}^{-1} : \mathfrak{X}_{p+1}(\Omega_{\tau_0}) \to \mathfrak{X}_p(\Omega_{\tau_0})$ for $p \geq 3$.

Moreover, for $p \in \mathbb{N}$ it follows from the second estimate of (4.116) and the fact that $|\tau^2\delta|$ is bounded in $D_2^u(\delta)$ that we can defined a bounded linear operator $\mathcal{F}_1 : \mathfrak{X}_p(\Omega_{\tau_0}) \to \mathfrak{X}_{p+1}(\Omega_{\tau_0})$ defined by the formula $\mathcal{F}_1(\xi)(\varphi,\tau) = \mathbf{F}_1(\varphi,\tau)\xi(\varphi,\tau)$ for $\xi \in \mathfrak{X}_p(\Omega_{\tau_0})$. Moreover,

$$\|\mathcal{F}_1(\xi)\|_{p+1} \leq \frac{C_1}{r_1}\|\xi\|_p. \tag{4.117}$$

In order to estimate $\mathbf{Z}$ in the set $S_h \times D_2^u(\delta)$ we shall use a convergent iteration scheme for functions defined in Ω_{τ_0}. For $k \geq 0$, let $\mathbf{Z}_k : \Omega_{\tau_0} \to \mathbb{C}^4$ be the functions defined by the recursion formula,

$$\mathbf{Z}_{k+1} = \mathbf{Z}_{in} + \mathcal{L}^{-1}(\mathbf{F}_0) + \mathcal{L}^{-1}\circ\mathcal{F}_1(\mathbf{Z}_k) + \mathcal{L}^{-1}\circ\mathcal{F}_2(\mathbf{Z}_k), \quad \mathbf{Z}_0 = 0. \tag{4.118}$$

In the following we will show that $\mathbf{Z}_k \in \mathfrak{X}_4(\Omega_{\tau_0})$ for all $k \geq 0$ and that $\{\mathbf{Z}_k\}_{k\geq 0}$ is a Cauchy sequence. Let us start by estimating the functions $\mathbf{Z}_k$ for $k \geq 0$ in Ω_{τ_0}. For $k = 1$ we have that,

$$\mathbf{Z}_1 = \mathbf{Z}_{in} + \mathcal{L}^{-1}(\mathbf{F}_0). \tag{4.119}$$

It follows from Lemma 4.4.4 and Theorem 4.4.2 that,

$$\mathbf{\Gamma}^u = \mathbf{\Gamma}^n + O(\delta^{n+1}), \quad \text{in} \quad S_h \times \ell_{c_2}.$$

Taking into account the definition of $\mathbf{Z}_{in}$ we conclude that $\|\mathbf{Z}_{in}(\varphi,\tau)\| \leq C_{in}\delta^{n-\frac{1}{2}}$ in Ω_{τ_0} for some $C_{in} > 0$. Thus $\mathbf{Z}_{in} \in \mathfrak{X}_4(\Omega_{\tau_0})$ and,

$$\|\mathbf{Z}_{in}\|_4 \leq C_{in} \sup_{\tau\in D_2^u(\delta)} \left|\tau^2\delta\right|^{\frac{5}{2}} \delta^{n-3} = O(\delta^{n-3}). \tag{4.120}$$

Now, it follows from the first estimate in (4.116) that $\mathbf{F}_0 \in \mathfrak{X}_5(\Omega_{\tau_0})$ and,

$$\|\mathbf{F}_0\|_5 \leq C_0 \sup_{\tau\in D_2^u(\delta)} \left|\tau^2\delta\right|^{\frac{2n-1}{2}} \delta^{n+\frac{3}{2}} = O(\delta^{n+\frac{3}{2}}). \tag{4.121}$$

Thus, (4.119) and the estimates (4.120) and (4.121) imply that,

$$\|\mathbf{Z}_1\|_4 \leq \|\mathbf{Z}_{in}\|_4 + \left\|\mathcal{L}^{-1}\right\|_{4,5} \|\mathbf{F}_0\|_5 = O(\delta^{n-3}).$$

To prove an upper bound for $\mathbf{Z}_k$ with $k \geq 2$ we proceed by induction on $k \in \mathbb{N}$. Let us suppose that,

$$\|\mathbf{Z}_k\|_4 \leq 2\|\mathbf{Z}_1\|_4, \quad \text{for some } k \in \mathbb{N}.$$

Now we show that $\|\mathbf{Z}_{k+1}\|_4 \leq 2\|\mathbf{Z}_1\|_4$. Similar to the previous Theorem we can derive the following upper bound,

$$\|\mathcal{F}_2(\mathbf{Z}_k)\|_5 \leq \frac{8\|H_{\epsilon,n}\|_{C^3}\|\mathbf{Z}_k\|_4^2}{r_1^2\sin^2\theta_1}. \tag{4.122}$$

Thus (4.117), (4.118) and (4.122) imply that,

$$\begin{aligned}
\|\mathbf{Z}_{k+1}\|_4 &\leq \|\mathbf{Z}_1\|_4 + \left\|\mathcal{L}^{-1}\circ\mathcal{F}_1(\mathbf{Z}_k)\right\|_4 + \left\|\mathcal{L}^{-1}\circ\mathcal{F}_2(\mathbf{Z}_k)\right\|_4 \\
&\leq \|\mathbf{Z}_1\|_4 + \left\|\mathcal{L}^{-1}\right\|_{4,5}\left(\|\mathcal{F}_1(\mathbf{Z}_k)\|_5 + \|\mathcal{F}_2(\mathbf{Z}_k)\|_5\right) \\
&\leq \|\mathbf{Z}_1\|_4 + \left\|\mathcal{L}^{-1}\right\|_{4,5}\left(\frac{C_1}{r_1}\|\mathbf{Z}_k\|_4 + \frac{8\|H_{\epsilon,n}\|_{C^3}\|\mathbf{Z}_k\|_4^2}{r_1^2\sin^2\theta_1}\right)
\end{aligned}$$

Now using the induction hypothesis we conclude that,

$$\left\|\mathbf{Z}_{k+1}\right\|_4 \leq \left\|\mathbf{Z}_1\right\|_4 + \left\|\mathcal{L}^{-1}\right\|_{4,5} \left(\frac{2C_1}{r_1} \left\|\mathbf{Z}_1\right\|_4 + \frac{32 \left\|H_{\epsilon,n}\right\|_{C^3} \left\|\mathbf{Z}_1\right\|_4^2}{r_1^2 \sin^2 \theta_1} \right).$$

Choosing,

$$r_1 > \left\|\mathcal{L}^{-1}\right\|_{4,5} \left(2C_1 + \frac{32 \left\|H_{\epsilon,n}\right\|_{C^3} \left\|\mathbf{Z}_1\right\|_4}{\sin^2 \theta_1} \right), \tag{4.123}$$

we get $\left\|\mathbf{Z}_{k+1}\right\|_4 \leq 2 \left\|\mathbf{Z}_1\right\|_4$ as we wanted to prove. Now let us prove that the sequence $\{\mathbf{Z}_k\}_{k\geq 0}$ is Cauchy. First note that according to formula (4.118) we can write,

$$\left\|\mathbf{Z}_{k+1} - \mathbf{Z}_k\right\|_4 \leq \left\|\mathcal{L}^{-1}\right\|_{4,5} \left(\left\|\mathcal{F}_1(\mathbf{Z}_k - \mathbf{Z}_{k-1})\right\|_5 + \left\|\mathcal{F}_2(\mathbf{Z}_k) - \mathcal{F}_2(\mathbf{Z}_{k-1})\right\|_5 \right). \tag{4.124}$$

Similar considerations as in the proof of Theorem 4.4.2 show that,

$$\left\|\mathcal{F}_2(\mathbf{Z}_k) - \mathcal{F}_2(\mathbf{Z}_{k-1})\right\|_5 \leq \frac{16 \left\|\mathbf{Z}_1\right\|_4 \left\|H_{\epsilon,n}\right\|_{C^3}}{r_1^2 \sin^2 \theta_1} \left\|\mathbf{Z}_k - \mathbf{Z}_{k-1}\right\|_4 .$$

Thus, (4.117), (4.123), (4.124) and the previous estimate give,

$$\left\|\mathbf{Z}_{k+1} - \mathbf{Z}_k\right\|_4 \leq \frac{1}{2} \left\|\mathbf{Z}_k - \mathbf{Z}_{k-1}\right\|_4 ,$$

which implies that $\{\mathbf{Z}_k\}_{k\geq 0}$ is a Cauchy sequence in the Banach space $(\mathfrak{X}_4(\Omega_{\tau_0}), \left\|\cdot\right\|_4)$ and has limit $\mathbf{Z}$. Moreover, $\left\|\mathbf{Z}\right\|_4 \leq 2 \left\|\mathbf{Z}_1\right\|_4$ which taking into account (4.120) implies that $\mathbf{Z}(\varphi, \tau) = O(\delta^{n-3})$ in Ω_{τ_0}. Since $\tau_0 \in \ell_2$ is arbitrary we conclude that,

$$\mathbf{\Gamma}^u = \sum_{m=0}^{n} \mathbf{\Gamma}_m^- \delta^{2m} + O(\delta^{n-3}),$$

uniformly in the set $S_h \times D_2^u(\delta)$. Finally, substituting n by $2n+5$ in the previous estimate and taking into account that $\mathbf{\Gamma}_{n+1}^- \delta^{2n+2} = O((\tau\delta)^{2n+2})$ we conclude that,

$$\begin{aligned}
\mathbf{\Gamma}^u &= \sum_{m=0}^{2n+5} \mathbf{\Gamma}_m^- \delta^{2m} + O(\delta^{2n+2}) \\
&= \sum_{m=0}^{n} \mathbf{\Gamma}_m^- \delta^{2m} + \sum_{m=n+1}^{2n+5} \mathbf{\Gamma}_m^- \delta^{2m} + O(\delta^{2n+2}) \\
&= \sum_{m=0}^{n} \mathbf{\Gamma}_m^- \delta^{2m} + O((\tau\delta)^{2n+2}) + O(\delta^{2n+2}) \\
&= \sum_{m=0}^{n} \mathbf{\Gamma}_m^- \delta^{2m} + O((\tau\delta)^{2n+2}),
\end{aligned}$$

valid in $S_h \times D_2^u(\delta)$. This concludes the proof of the Theorem. □

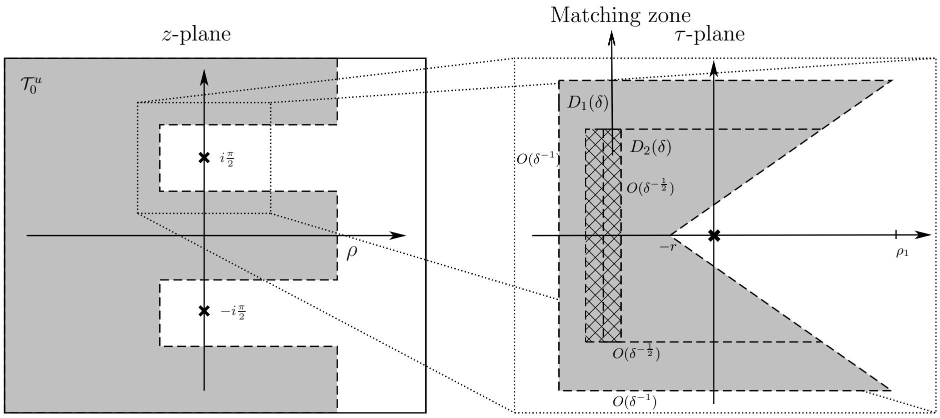

Figure 4.3: Overview of the regions of validity of the approximation results.

4.4.5 Summary of the approximation results

Let us collect the approximation results obtained until this point. By Theorem 4.4.1 the unstable manifold W_ϵ^u can be parametrised by an analytic function $\mathbf{\Gamma}^u : S_h \times \mathcal{T}_0^u(\rho, \sigma, h) \to \mathbb{C}^4$ which satisfies the equation $\mathcal{D}_\epsilon \mathbf{\Gamma}^u = X_{H_{\epsilon n}}(\mathbf{\Gamma}^u)$. The parametrisation $\mathbf{\Gamma}^u$ has real symmetry, i.e. it takes real values for $(\varphi, \tau) \in S_h \times \mathcal{T}_0^u \cap \mathbb{R}^2$. Thus when $\mathbf{\Gamma}^u$ is restricted to the reals it is real analytic (see Remark 4.3.1.2). Moreover, it is 2π-periodic in $\varphi \in S_h$. The set S_h is a strip in $\mathbb{C}$ of width h containing the real axis and the set $\mathcal{T}_0^u \subset \mathbb{C}$ has a shape similar to Figure 4.2. Furthermore we have proved in the same Theorem that, given $n \in \mathbb{N}$ the following estimate holds,

$$\mathbf{\Gamma}^u = \mathbf{X}_\delta^n + O_{2n+2}(\delta), \tag{4.125}$$

valid in $S_h \times \mathcal{T}_0^u$ where $\mathbf{X}_\delta^n$ is a partial sum of the formal separatrix $\hat{\mathbf{X}}_\delta$ up to order δ^{2n+2} in the first two components and up to order δ^{2n+1} in the last two. Recall (4.58) for a definition of the O_n notation.

In Theorem 4.4.2 we have extended the domain of analyticity of $\mathbf{\Gamma}^u$ and the estimate (4.125) until it reaches the boundary of an δ-neighbourhood of the singular point $z = i\frac{\pi}{2}$. It is convenient to present the estimate in terms of the τ variable which is related to the z variable according to formula,

$$z = \frac{\beta_\epsilon}{\alpha_\epsilon}\tau + i\frac{\pi}{2}. \tag{4.126}$$

Then Theorem 4.4.2 says that $\mathbf{\Gamma}^u$ can be analytically extended onto $S_h \times D_1^u(\delta)$ and estimated as follows,

$$\mathbf{\Gamma}^u = \mathbf{X}_\delta^n + O_{2n+2}(\tau^{-1}),$$

in the set $S_h \times D_1^u(\delta)$. As expected the approximation given by the formal separatrix deteriorates when z gets closer to $i\frac{\pi}{2}$.

In a region closer to the singular point $z = i\frac{\pi}{2}$ a more accurate approximation is given by Theorem 4.4.3. According to that Theorem the following estimate,

$$\mathbf{\Gamma}^u = \sum_{m=0}^{n} \mathbf{\Gamma}_m^- \delta^{2m} + O((\tau\delta))^{2n+2},$$

holds in $S_h \times D_2^u(\delta)$ where the functions $\mathbf{\Gamma}_m^-$ are given by Theorem 4.4.3.

We have obtained different approximations for the unstable parametrisation in different regions of $\mathbb{C}^2$ and in Figure 4.3 it is illustrated where these estimates are valid.

4.5 Stable Manifold

The theory presented in the previous sections concerns the unstable manifold W_ϵ^u of the equilibrium of the family H_ϵ, $\epsilon > 0$. We have constructed rather good approximations for this invariant manifold in different regions of the complexified phase space. Near the equilibrium, the approximations provided by the formal separatrix are quite accurate. In regions where the coefficients of the formal separatrix grow, i.e., near the singularities $i\frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$, we have constructed different approximations which account for the local behavior near the singularities and "glued" them together with the unstable parametrisation using a complex matching technique.

Analogous results can be obtained for the stable manifold W_ϵ^s. Let us define the following sets,

$$\mathcal{T}_0^s = \{z \mid -z \in \mathcal{T}_0^u\}, \quad D_1^s(\delta) = \{\tau \mid -\tau \in D_1^u(\delta)\}, \quad D_2^s(\delta) = \{\tau \mid -\tau \in D_2^u(\delta)\}.$$

Using the reversibility we can define the stable parametrisation as follows,

$$\mathbf{\Gamma}^s(\varphi, z) = \mathcal{S}(\mathbf{\Gamma}^u(-\varphi, -z)).$$

Recall that the formal separatrix $\hat{\mathbf{X}}_\delta$ is symmetric, i.e. $\mathcal{S}(\hat{\mathbf{X}}_\delta(-\varphi, -z)) = \hat{\mathbf{X}}_\delta(\varphi, z)$. Thus, similar to Theorem 4.4.1 we obtain,

Theorem 4.5.1. *For every $n \in \mathbb{N}$, the stable parametrisation $\mathbf{\Gamma}^s : S_h \times \mathcal{T}_0^s \to \mathbb{C}^4$ is analytic, 2π-periodic in φ, continuous on the closure of its domain, satisfy the PDE* (4.65) *and*

$$\mathbf{\Gamma}^s = \mathbf{X}_\delta^n + O_{2n+2}(\delta),$$

valid in $S_h \times \mathcal{T}_0^s$.

Continuing our analogy of results with the unstable case we have the following,

Theorem 4.5.2. *For any $n \in \mathbb{N}$ there exists an $r_1 > 0$ sufficiently large such that the stable parametrisation $\mathbf{\Gamma}^s$ can be analytically extended onto $S_h \times D_1^s(\delta)$ such that,*

$$\mathbf{\Gamma}^s = \mathbf{X}_\delta^n + O_{2n+2}(\tau^{-1}), \quad \text{in} \quad S_h \times D_1^s(\delta).$$

In a region closer to the singularity the stable parametrisation $\mathbf{\Gamma}^s$ can be approximated in $S_h \times D_2^s(\delta)$ as follows,

$$\mathbf{\Gamma}^s = \sum_{m=0}^{n} \mathbf{\Gamma}_m^+ \delta^{2m} + O((\tau\delta))^{2n+2},$$

where $\mathbf{\Gamma}_m^+(\varphi, \tau) = \mathcal{S}(\mathbf{\Gamma}_m^-(-\varphi, -\tau))$ solve the infinite system of equations (4.105) *and defined in $S_h \times D_r^+$ where $D_r^+ = \{\tau \in \mathbb{C} \mid -\tau \in D_r^-\}$.*

Now we consider the question of finding homoclinic orbits. A natural place to look for homoclinic points is the symmetric plane,

Lemma 4.5.1. *Given $n \in \mathbb{N}$, there exist functions $\varphi_0(\delta)$ and $z_0(\delta)$ analytic in $(-\delta_0, \delta_0)$ for some $\delta_0 > 0$ such that $\mathbf{\Gamma}^u(\varphi + \varphi_0(\delta), z + z_0(\delta)) \in$ Fix$(\mathcal{S})$ and moreover,*

$$\mathbf{\Gamma}^u(\varphi + \varphi_0(\delta), z + z_0(\delta)) = \mathbf{\Gamma}^u(\varphi, z) + O(\delta^{n+1}), \quad \text{in} \quad S_h \times \mathcal{T}_0^s. \qquad (4.127)$$

Proof. Tracing the proof of Theorem 4.4.1 it is possible to check that the unstable parametrisation $\mathbf{\Gamma}^u$ can be made real analytic with respect to δ in some open interval $(-\delta_0, \delta_0)$. Moreover, in the standard scaling the following estimate holds,

$$\mathbf{\Gamma}^u(\varphi, z) = \left[\hat{\mathbf{X}}\right]_n(\varphi, z) + O(\delta^{n+1}),$$

where $\left[\hat{\mathbf{X}}\right]_n$ denotes the sum of the formal separatrix of Theorem 4.2.1 up to order δ^n. Now consider the following function,

$$G(\varphi, z, \delta) = \mathcal{S}(\mathbf{\Gamma}^u(\varphi, z)) - \mathbf{\Gamma}^u(\varphi, z).$$

Due to the real analyticity of the unstable parametrisation, the function G is also real analytic. Moreover, as $\mathcal{S}(\left[\hat{\mathbf{X}}\right]_n(0,0)) = \left[\hat{\mathbf{X}}\right]_n(0,0)$ then $G(0,0,0) = 0$. Denote by G_i the components of the function G. Thus, by the Implicit Function Theorem it is sufficient to prove that,

$$d = \det \begin{pmatrix} \frac{\partial G_1}{\partial \varphi} & \frac{\partial G_1}{\partial \varphi} \\ \frac{\partial G_4}{\partial \varphi} & \frac{\partial G_4}{\partial \varphi} \end{pmatrix}\Bigg|_{\varphi=z=\delta=0} \neq 0.$$

Taking into account that $\left[\hat{\mathbf{X}}\right]_n = X_0 + O(\delta)$ and the definition of X_0 (see (2.17)) we conclude that $d = \frac{8}{\eta}$ and the result follows. Moreover, it is not difficult to see that $\varphi_0(\delta) = O(\delta^{n+1})$ and $z_0(\delta) = O(\delta^{n+1})$ and estimate (4.127) follows. $\square$

In the light of the previous Lemma and Remark 4.3.1.4 one can uniquely define a parametrisation of the unstable (resp. stable) manifold W^u_ϵ (resp. W^s_ϵ) by requiring,

$$\mathbf{\Gamma}^{u,s}(0,0) \in \mathsf{Fix}(\mathcal{S}).$$

Note that the approximations obtained in the previous sections are still valid due to estimate (4.127). Moreover, $\mathbf{\Gamma}^s(0,0) = \mathcal{S}(\mathbf{\Gamma}^u(0,0)) = \mathbf{\Gamma}^u(0,0)$ is a symmetric homoclinic point.

4.6 Measuring the splitting

In this section we proceed to measure the splitting of stable and unstable manifolds. Let us first derive some estimates for the difference $\mathbf{\Gamma}^s - \mathbf{\Gamma}^u$ that will be used throughout this section. Note that since $\mathcal{T}_0^s \cap \mathcal{T}_0^u \neq \emptyset$ then Theorems 4.4.1 and 4.5.1 imply that,

$$\mathbf{\Gamma}^s(\varphi, z) - \mathbf{\Gamma}^u(\varphi, z) = O_{2n+2}(\delta), \quad \forall n \in \mathbb{N}, \tag{4.128}$$

in $S_h \times \mathcal{T}_0^s \cap \mathcal{T}_0^u$. Now let us consider the following rectangles,

$$\begin{aligned} R_1(\delta) &= \left\{ z \in \mathbb{C} \mid |\mathrm{Re}\, z| < \rho_1 \delta, \quad 0 \leq \mathrm{Im}\, z < \frac{\pi}{2} - r_1 \delta \right\}, \\ R_2(\delta) &= R_1(\delta) \cap \left\{ z \in \mathbb{C} \mid \mathrm{Im}\, z > \frac{\pi}{2} - c_2 \delta^{1/2} \right\}, \end{aligned} \tag{4.129}$$

Note that $R_2(\delta) \subset R_1(\delta)$ for δ sufficiently small. According to the extension Theorems 4.4.2 and 4.5.2 we still have the following estimate,

$$\mathbf{\Gamma}^s(\varphi, z) - \mathbf{\Gamma}^u(\varphi, z) = O_{2n+2}(\tau^{-1}), \tag{4.130}$$

valid in $S_h \times R_1(\delta)$ where recall that $z = \frac{\beta_\epsilon}{\alpha_\epsilon}\tau + i\frac{\pi}{2}$. Note that the last estimate goes from $O(\delta^{2n+2})$ in the bottom part of $R_1(\delta)$ to $O(1)$ in the top part of $R_1(\delta)$. In a region closer to the singularity $i\frac{\pi}{2}$ we can get sharper upper bounds. It follows from Theorems 4.4.3 and 4.5.2 that,

$$\mathbf{\Gamma}^s(\varphi, z) - \mathbf{\Gamma}^u(\varphi, z) = \Delta_0(\varphi, \tau) + O((\delta\tau)^2),$$

in $S_h \times R_2(\delta)$ where $\Delta_0 = \mathbf{\Gamma}_0^+ - \mathbf{\Gamma}_0^-$. Now according to Theorem 3.5.1 we have that,

$$\Delta_0(\varphi, \tau) = O(\tau^3 e^{-i(\tau - \varphi)}),$$

in $S_h \times D_r^1$ where $D_r^1 = D_r^+ \cap D_r^- \cap \{\mathrm{Im}\tau < -r\}$ for $r > 0$ sufficiently large. Thus,

$$\mathbf{\Gamma}^s(\varphi, z) - \mathbf{\Gamma}^u(\varphi, z) = O(\tau^3 e^{-i(\tau - \varphi)}) + O((\delta\tau)^2), \tag{4.131}$$

in $S_h \times R_2(\delta)$. A sharper estimate of the difference in $S_h \times R_2(\delta)$ can be obtained as follows,

Lemma 4.6.1. *For any $0 < \mu < 1$ the following estimate holds,*

$$\mathbf{\Gamma}^s(\varphi, z) - \mathbf{\Gamma}^u(\varphi, z) = \Delta_0(\varphi, \tau) + O(e^{-\mu i(\tau-\varphi)}\delta^2) + O((\delta\tau)^4).$$

Proof. By Lemma 4.4.3 we known that the formal series $\mathbf{\Gamma}^{\pm} = \sum_{m\geq 0} \mathbf{\Gamma}^{\pm}_m \delta^{2m}$ satisfy the equation,

$$\mathcal{D}\mathbf{\Gamma}^{\pm} = \alpha_\epsilon^{-1} X_{H_{\epsilon,n}}(\mathbf{\Gamma}^{\pm}).$$

Recall that $\alpha_\epsilon = 1 - \sum_{l=1}^{\infty} a_{1,0,l}\delta^{2l}$. Now let us write $X_{H_{\epsilon,n}} = \sum_{m\geq 0} F_m \delta^{2m}$ where $F_0 = X_{H_{0,n}}$ and expand the previous equation in powers of δ. Collecting terms of the same order in δ^2 we get the following equation,

$$\mathcal{D}\mathbf{\Gamma}^{\pm}_1 = DF_0(\mathbf{\Gamma}^{\pm}_0)\mathbf{\Gamma}^{\pm}_1 + a_{1,0,1}F_0(\mathbf{\Gamma}^{\pm}_0) + F_1(\mathbf{\Gamma}^{\pm}_0).$$

Now we define $\Delta_1 = \mathbf{\Gamma}^{+}_1 - \mathbf{\Gamma}^{-}_1$ and rewrite the previous equation as follows,

$$\mathcal{L}(\Delta_1) = a_{1,0,1}(F_0(\mathbf{\Gamma}^{+}_0) - F_0(\mathbf{\Gamma}^{-}_0)) + F_1(\mathbf{\Gamma}^{+}_0) - F_1(\mathbf{\Gamma}^{-}_0) + \left(DF_0(\mathbf{\Gamma}^{+}_0) - DF_0(\mathbf{\Gamma}^{-}_0)\right)\mathbf{\Gamma}^{+}_1, \tag{4.132}$$

where $\mathcal{L}(\Delta_1) = \mathcal{D}\Delta_1 - DF_0(\mathbf{\Gamma}^{-}_0)\Delta_1$. Denote by R_1 the right hand side of equation (4.132). Taking into account that F_0 and F_1 are analytic and the estimates (4.131) and $\mathbf{\Gamma}^{\pm}_1 = O_{-1}(\tau^{-1})$ it is not difficult to conclude that $R_1 \in \mathfrak{Y}_{\mu'}(S_h \times D^1_r)$ for any $1 < \mu' < 2$. Note that by the result of chapter 3 the linear operator $\mathcal{L}$ has a fundamental matrix $\mathbf{U}$. According to Theorem 2.4.3 given $\mu'' > \mu'$ there exists a bounded linear operator $\mathcal{L}^{-1}_{\mu'} : \mathfrak{Y}_{\mu'}(S_h \times D^1_r) \to \mathfrak{Y}_{\mu''}(S_h \times D^1_r)$ such that $\mathcal{L}\mathcal{L}^{-1}_{\mu'} = \mathrm{Id}$. As $\Delta_1 = O(\tau^{-N})$ for all $N \in \mathbb{N}$, it follows from the fact that $\Delta_1 - \mathcal{L}^{-1}_{\mu'}(R_1) \in \ker(\mathcal{L})$ and Theorem 2.4.2 that there is an analytic 2π-periodic function $\mathbf{c}_1 : \{s \in \mathbb{C} : \mathrm{Im}(s) < -r + h\} \to \mathbb{C}^4$ such that $\Delta_1 - \mathcal{L}^{-1}_{\mu'}(R_1) = \mathbf{U}\mathbf{c}_1$. Since,

$$\lim_{\mathrm{Im}s\to-\infty} \mathbf{c}_1(s) = 0,$$

we can write $\mathbf{c}_1$ in Fourier series and conclude that $\mathbf{c}_1 = O(e^{-i(\tau-\varphi)})$. Thus $\Delta_1 = O(e^{-\mu i(\tau-\varphi)})$ where $\mu = 2 - \mu'$. Finally, as

$$\mathbf{\Gamma}^s(\varphi, z) - \mathbf{\Gamma}^u(\varphi, z) = \Delta_0(\varphi, \tau) + \Delta_1(\varphi, \tau)\delta^2 + O((\delta\tau)^4),$$

we get the desired result. $\square$

4.6.1 Derivation of an asymptotic formula

In this subsection we derive an asymptotic formula for the homoclinic invariant at the primary homoclinic point $\mathbf{\Gamma}^s(0,0) = \mathbf{\Gamma}^u(0,0)$. In order to derive the asymptotic formula we consider an auxiliary function defined by

$$\Theta(\varphi, z) = \Omega\left(\Delta(\varphi, z), \partial_\varphi \mathbf{\Gamma}^u(\varphi, z)\right) , \tag{4.133}$$

where $\Delta(\varphi, z) = \mathbf{\Gamma}^s(\varphi, z) - \mathbf{\Gamma}^u(\varphi, z)$ and Ω is the standard symplectic form. The homoclinic invariant of the primary homoclinic orbit is defined by (4.5) which takes the form

$$\omega_\epsilon = \Omega\big(\partial_\varphi \mathbf{\Gamma}^s, \partial_\varphi \mathbf{\Gamma}^u\big)\big|_{\varphi=z=0} . \tag{4.134}$$

Differentiating the definition of Θ at the origin and taking into account that $\Delta(0,0) = 0$ we get the relation:

$$\omega_\epsilon = \partial_\varphi \Theta(0,0).$$

Thus, we only need to estimate the function Θ and its derivative. Note that the function Θ satisfy the following PDE,

$$\mathcal{D}_\epsilon \Theta = \Omega(F(\Delta), \partial_\varphi \mathbf{\Gamma}^u), \tag{4.135}$$

where $F(\Delta) = X_{H_\epsilon}(\mathbf{\Gamma}^u + \Delta) - X_{H_\epsilon}(\mathbf{\Gamma}^u) - DX_{H_\epsilon}(\mathbf{\Gamma}^u)\Delta$. As $F(\Delta)$ is of second order in Δ, then Θ approximately satisfies the homogeneous equation $\mathcal{D}_\epsilon u = 0$. Thus, Θ is approximately equal to a $2\pi\beta_\epsilon$-periodic function depending on a single variable $\Theta(\varphi, z) \approx f(\alpha_\epsilon z - \beta_\epsilon \varphi)$. Periodicity allow us to write Θ in Fourier series,

$$\Theta(\varphi, z) \approx \sum_{k\in\mathbb{Z}} f_k e^{ik(\frac{\alpha_\epsilon}{\beta_\epsilon} z - \varphi)} ,$$

and we can estimate the function Θ by estimating the coefficients f_k using the standard integral formula. A rigorous argument that justifies the previous heuristic requires the method of flow box.

Flow box coordinates

The main idea of the method is to construct new coordinates valid in a suitable neighbourhood of a piece of the unstable manifold such that in the new coordinates the original flow is conjugated to a linear flow on a "cylinder". Let us be more precise. Given $r, c, \sigma > 0$, consider the following domain in the complex plane,

$$R(\delta) = \left\{ z \in \mathbb{C} \mid \ |\mathrm{Im}\, z| < \frac{\pi}{2} - r\delta, \quad |\mathrm{Re}\, z| < c\delta \right\}$$

and let $\mathcal{M}_\delta$ be the following domain in $\mathbb{C}^4$,

$$\mathcal{M}_\delta = S_h \times R(\delta) \times \left\{ (E_1, E_2) \in \mathbb{C}^2 \mid \ |E_1| + |E_2| < \delta^\sigma \right\}.$$

Then we have the following,

Theorem 4.6.1. *There exist $c, \sigma > 0$, $r > 0$ sufficiently large and $\delta_0 > 0$ such that if $\delta \in (0, \delta_0)$ then there exists a real analytic symplectic injective map $\Psi : \mathcal{M}_\delta \to \mathbb{C}^4$ such that:*

1. *Ψ is 2π-periodic in φ,*
2. *$\mathcal{D}_\epsilon \Psi = X_{H_\epsilon}(\Psi)$,*
3. *$\Psi(\varphi, z, 0, 0) = \mathbf{\Gamma}^u(\varphi, z)$,*
4. *$\left\| \Psi^{-1} \right\|_{C^2}$ is uniformly bounded (with respect to $\delta \in (0, \delta_0)$).*

The idea of constructing a flow box to study the splitting of invariant manifolds goes back to Lazutkin's original ideas when studying the splitting of separatrices of the standard map [47]. Here we will only give a sketch of its proof since it is a simple adaptation of the proof of Theorem 7.1 in [28]. There, it is constructed a symplectic diffeomorphism which conjugates the dynamics near a piece of the unstable separatrix of the standard map to a shift $(t, E) \mapsto (t + h, E)$ (see chapter 1 for an introduction to the splitting of separatrices of the standard map). One of the key ingredients in the proof of the theorem is to obtain a suitable description of solutions of the following variational equation,

$$\mathcal{D}_\epsilon u = DX_{H_\epsilon}(\mathbf{\Gamma}^u(\varphi, z))u. \tag{4.136}$$

Clearly the tangent vector fields $\partial_\varphi \mathbf{\Gamma}^u$ and $\partial_z \mathbf{\Gamma}^u$ satisfy the previous equation and since W_ϵ^u is Lagrangian it follows that,

$$\Omega(\partial_\varphi \mathbf{\Gamma}^u, \partial_z \mathbf{\Gamma}^u) = 0. \tag{4.137}$$

Now two other independent solutions u_1 and u_2 can be obtained using the method described in Appendix A. Together these four linear independent solutions form a symplectic fundamental solution $\Pi(\varphi, z)$ of equation (4.136). Moreover, u_1 and u_2 can be estimated in $S_h \times R(\delta)$ using the known estimates of $\mathbf{\Gamma}^u$ in that domain. Then we look for a solution of equation

$$\mathcal{D}_\epsilon \Psi = X_{H_\epsilon}(\Psi), \tag{4.138}$$

in the following form,

$$\Psi(\varphi, z, E_1, E_2) = \mathbf{\Gamma}^u(\varphi, z) + Z(\varphi, z, E_1, E_2),$$

subject to condition $Z(\varphi, z, 0, 0) = 0$. Thus, Z must satisfy the following integral equation,

$$\mathcal{L}(Z) = X_{H_\epsilon}(\mathbf{\Gamma}^u + Z) - X_{H_\epsilon}(\mathbf{\Gamma}^u) - DX_{H_\epsilon}(\mathbf{\Gamma}^u)Z, \tag{4.139}$$

where $\mathcal{L}(Z) = \mathcal{D}_\epsilon Z - DX_{H_\epsilon}(\mathbf{\Gamma}^u)Z$. This linear operator acts on the Banach space $\mathfrak{C}_\mu(S_h \times R(\delta))$ for $\mu > 0$ which consists of analytic functions $f : S_h \times R(\delta) \to \mathbb{C}^4$, 2π-periodic in φ, continuous on the closure of its domain and having finite norm,

$$\|f\|_{\mathfrak{C}_\mu} := \sup_{S_h \times R(\delta)} \|\cosh^\mu(z) f(\varphi, z)\| < \infty.$$

The linear operator $\mathcal{L}$ has kernel in $\mathfrak{C}_\mu(S_h \times R(\delta))$ which follows from the existence of a fundamental solution Π. Moreover, it is not difficult to construct a right inverse of $\mathcal{L}$ which we denote by $\mathcal{L}^{-1}$. Thus the problem of solving the integral equation (4.139) subject to condition $Z(\varphi, z, 0, 0) = 0$ is reduced to the problem of finding a fixed point,

$$Z = E_1 u_1 + E_2 u_2 + \mathcal{L}^{-1}(F(Z)),$$

where $F(Z)$ denotes the right hand side of equation (4.139). Now given $\mu > 0$ and $|E_1| + |E_2| < \delta^\sigma$ for some $\sigma(\mu) > 0$ and δ sufficiently small, it is possible to derive analogous estimates as in the proof of Theorem 7.1 in [28] to show that the non-linear operator in the right hand side of the previous equation is contracting in a suitable invariant closed ball (with radius possibly depending on μ) defined in $\mathfrak{C}_\mu(S_h \times R(\delta))$.

Then using a contraction mapping principle one can obtain the map Ψ. Note that Ψ as defined previously is not unique. In fact,

$$\Psi(\varphi + s_1(\alpha_\epsilon z - \beta_\epsilon \varphi), z + s_2(\alpha_\epsilon z - \beta_\epsilon \varphi), s_3(\alpha_\epsilon z - \beta_\epsilon \varphi), s_4(\alpha_\epsilon z - \beta_\epsilon \varphi)),$$

also satisfies equation (4.138) where s_i are $2\pi\beta_\epsilon$-periodic functions such that $s_i(0) = 0$. Since the map Ψ may not be symplectic, this freedom can be used to construct a new map $\tilde{\Psi}$ which has the desired properties stated in Theorem 4.6.1.

Now let us look at some consequences of the Theorem.

The splitting function

It follows from the second property of the Theorem, that in the new coordinates defined by the map Ψ the Hamiltonian flow of H_ϵ is conjugated to the linear motion given by,

$$\dot{\varphi} = \alpha_\epsilon, \quad \dot{z} = \beta_\epsilon, \quad \dot{E}_1 = 0, \quad \dot{E}_1 = 0.$$

Now let us define the splitting function as follows,

$$\Xi(\varphi, z) = E_1 \circ \mathbf{\Gamma}^s(\varphi, z), \tag{4.140}$$

where E_1 is the third component of the map Ψ^{-1}. Now we check the domain of validity of the function Ξ. According to Theorem 3.5.1 we have that,

$$\mathbf{\Gamma}_0^+(\varphi, \tau) - \mathbf{\Gamma}_0^-(\varphi, \tau) = O(\tau^3 e^{-i(\tau - \varphi)}),$$

in $S_h \times D_r^1$ for r sufficiently large. Consequently, Lemma 4.6.1, the previous estimate and estimates (4.128) and (4.130) imply that,

$$\mathbf{\Gamma}^s(\varphi, z) - \mathbf{\Gamma}^u(\varphi, z) = O(\delta^{\frac{2}{\mu}} \log^3 \delta^{-1}), \tag{4.141}$$

for $0 < \mu < 1$ arbitrarily close to 1, which is valid in the set $S_h \times \mathcal{D}(\delta)$ where,

$$\mathcal{D}(\delta) = R(\delta) \cap \left\{ |\mathrm{Im}\, z| < \frac{\pi}{2} - \frac{2}{\mu}\delta \log \delta^{-1} \right\}.$$

It is not difficult to see that the estimate (4.141) implies that,

$$\mathbf{\Gamma}^s(\varphi, z) - \mathbf{\Gamma}^u(\varphi, z) = O(\delta^2), \quad \text{in} \quad S_h \times \mathcal{D}(\delta). \tag{4.142}$$

Thus, provided $\sigma \leq 2$ in Theorem 4.6.1, the function $\Xi(\varphi, z)$ is well defined in the set $S_h \times \mathcal{D}(\delta)$.

Hereafter we shall assume that σ can be chosen such that the splitting function Ξ is well defined. This is not a serious assumption as is explained in the next subsection and it can be overcome by finer estimates for the difference (4.141).

Now let us study the splitting function and see that it provides a way to measure the splitting of the invariant manifolds. First of all note that,

$$\omega_\epsilon = -\partial_\varphi \Xi(0,0).$$

In fact, it follows directly from the third property of Ψ that

$$\Psi_*^{-1}\partial_\varphi \mathbf{\Gamma}^u(0,0) = \partial_\varphi \Psi^{-1}(\mathbf{\Gamma}^u(\varphi, z))\big|_{\varphi=z=0} = \partial_\varphi(\varphi, z, 0, 0)|_{\varphi=z=0} = (1,0,0,0),$$

and

$$\Psi_*^{-1}\partial_\varphi \mathbf{\Gamma}^s(0,0) = \partial_\varphi \Psi^{-1}(\mathbf{\Gamma}^s(\varphi, z))\big|_{\varphi=z=0}.$$

Finally, taking into account the definition of the homoclinic invariant and the fact that Ψ is a symplectic map we get,

$$\omega_\epsilon = \Omega(\Psi_*^{-1}\partial_\varphi \mathbf{\Gamma}^s(0,0), \Psi_*^{-1}\partial_\varphi \mathbf{\Gamma}^u(0,0)) = -\partial_\varphi \Xi(0,0).$$

This fact justifies why Ξ is known as the splitting function. Furthermore, since $\dot{E}_1 = 0$ it follows that,

$$\frac{d}{dt} E_1 \circ \mathbf{\Gamma}^s(\varphi + \alpha_\epsilon t, z + \beta_\epsilon t) = 0.$$

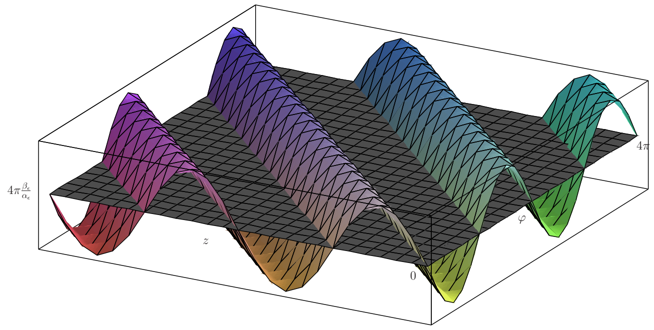

Figure 4.4: Illustration of the graph of the splitting function. The stable manifold "snakes" the unstable manifold which corresponds to the plane $E_1 = 0$.

Thus, $\mathcal{D}_\epsilon \Xi = 0$ and Ξ can be considered as a function of a single variable $\Xi(\varphi, z) = \Xi_0(\alpha_\epsilon z - \beta_\epsilon \varphi)$. Moreover, the 2π-periodicity in φ implies that Ξ_0 is in fact $2\pi\frac{\beta_\epsilon}{\alpha_\epsilon}$-periodic in z and its domain can be extended by periodicity to contain a strip $|\mathrm{Im}\, z| < \frac{\pi}{2} - \frac{2}{\mu}\delta \log \delta^{-1}$.

When Ξ is restricted to the reals, then a piece of the stable manifold is represented as the graph of Ξ while the unstable manifold in given by the plane $E_1 = 0$ as figure 4.4 illustrates.

Now we derive a formula that will be useful to estimate the function Ξ.

Lemma 4.6.2. *The following identity holds,*

$$\nabla E_1(\mathbf{\Gamma}^u(\varphi, z)) = (\partial_\varphi \mathbf{\Gamma}^u(\varphi, z))^T J\,.$$

Proof. According to the inverse function theorem we have that $(\mathrm{d}\Psi)^{-1} = \mathrm{d}\Psi^{-1}$. Moreover, given a symplectic matrix,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C and D are n-by-n matrices, then M^{-1} can be computed according to the following well known formula,

$$M^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}.$$

Thus, denoting by (q_1, q_2, p_1, p_2) the components of the map Ψ and taking into account the previous formula for the inverse of a symplectic matrix we have that,

$$\begin{pmatrix} \frac{\partial q_1}{\partial \varphi} & \frac{\partial q_1}{\partial z} & \frac{\partial q_1}{\partial E_1} & \frac{\partial q_1}{\partial E_2} \\ \frac{\partial q_2}{\partial \varphi} & \frac{\partial q_2}{\partial z} & \frac{\partial q_2}{\partial E_1} & \frac{\partial q_2}{\partial E_2} \\ \frac{\partial p_1}{\partial \varphi} & \frac{\partial p_1}{\partial z} & \frac{\partial p_1}{\partial E_1} & \frac{\partial p_1}{\partial E_2} \\ \frac{\partial p_2}{\partial \varphi} & \frac{\partial p_2}{\partial z} & \frac{\partial p_2}{\partial E_1} & \frac{\partial p_2}{\partial E_2} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\partial p_1}{\partial E_1} & \frac{\partial p_2}{\partial E_1} & -\frac{\partial q_1}{\partial E_1} & -\frac{\partial q_2}{\partial E_1} \\ \frac{\partial p_1}{\partial E_2} & \frac{\partial p_2}{\partial E_2} & -\frac{\partial q_1}{\partial E_2} & -\frac{\partial q_2}{\partial E_2} \\ -\frac{\partial p_1}{\partial \varphi} & -\frac{\partial p_2}{\partial \varphi} & \frac{\partial q_1}{\partial \varphi} & \frac{\partial q_2}{\partial \varphi} \\ -\frac{\partial p_1}{\partial z} & -\frac{\partial p_2}{\partial z} & \frac{\partial q_1}{\partial z} & \frac{\partial q_2}{\partial z} \end{pmatrix}.$$

Since $\Psi(\varphi, z, 0, 0) = \mathbf{\Gamma}^u(\varphi, z)$ and equating the third row of the previous matrices we get the desired identity. □

Now the last property of Theorem 4.6.1 implies that we can use Taylor series around $\mathbf{\Gamma}^u(\varphi, z)$ to expand the splitting function as follows,

$$\Xi(\varphi, z) = \nabla E_1(\mathbf{\Gamma}^u(\varphi, z)) \cdot (\mathbf{\Gamma}^s(\varphi, z) - \mathbf{\Gamma}^u(\varphi, z)) + O(\|\mathbf{\Gamma}^s(\varphi, z) - \mathbf{\Gamma}^u(\varphi, z)\|^2).$$

Thus, taking into account the identity of the previous Lemma we conclude that,

$$\Xi(\varphi, z) = -\Theta(\varphi, z) + O(\|\mathbf{\Gamma}^s(\varphi, z) - \mathbf{\Gamma}^u(\varphi, z)\|^2), \tag{4.143}$$

where recall that Θ is the auxiliary function defined in (4.133). Now let us estimate the function Θ on the line,

$$\ell(\delta) = \partial\mathcal{D}(\delta) \cap \left\{ \operatorname{Im} z = \frac{\pi}{2} - \frac{2}{\mu}\delta \log \delta^{-1} \right\}.$$

Recall that $0 < \mu < 1$ is arbitrarily close to 1. In the following we shall use both variables z and τ which are related through the formula $z = \frac{\beta_\epsilon}{\alpha_\epsilon}\tau + i\frac{\pi}{2}$. According to Theorem 4.4.3, for $(\varphi, z) \in S_h \times \ell(\delta)$ we have that,

$$\begin{aligned} \mathbf{\Gamma}^u(\varphi, z) &= \mathbf{\Gamma}_0^-(\varphi, \tau) + O(\delta^2 \log^2 \delta^{-1}), \quad \text{in} \quad S_h \times \ell(\delta), \\ \partial_\varphi \mathbf{\Gamma}^u(\varphi, z) &= \partial_\varphi \mathbf{\Gamma}_0^-(\varphi, \tau) + O(\delta^2 \log^2 \delta^{-1}), \quad \text{in} \quad S_h \times \ell(\delta), \end{aligned} \tag{4.144}$$

where the last estimate for the derivative follows from standard Cauchy estimates. Moreover, Lemma 4.6.1 implies that,

$$\mathbf{\Gamma}^s(\varphi, z) - \mathbf{\Gamma}^u(\varphi, z) = \mathbf{\Gamma}_0^+(\varphi, \tau) - \mathbf{\Gamma}_0^-(\varphi, \tau) + O(\delta^{4-\mu_1}), \quad \text{in} \quad S_h \times \ell(\delta), \tag{4.145}$$

where $\mu_1 > 0$ is an arbitrarily small. Thus, taking into account the definition of $\Theta(\varphi, z)$, the previous estimates (4.144) and (4.145) we get,

$$\Theta(\varphi, z) = \Omega(\mathbf{\Gamma}_0^+(\varphi, \tau) - \mathbf{\Gamma}_0^-(\varphi, \tau), \partial_\varphi \mathbf{\Gamma}_0^-(\varphi, \tau)) + O(\delta^{4-\mu_2}), \tag{4.146}$$

valid in $S_h \times \ell(\delta)$ where $\mu_2 > 0$ is arbitrarily small. Now according to Theorem 3.5.1 we have that,

$$\Omega(\mathbf{\Gamma}_0^+(\varphi, \tau) - \mathbf{\Gamma}_0^-(\varphi, \tau), \partial_\varphi \mathbf{\Gamma}_0^-(\varphi, \tau)) = \Theta_0^- e^{-i(\tau-\varphi)} + O\left(e^{-(2-\mu_0)i(\tau-\varphi)}\right), \tag{4.147}$$

for $\mu_0 > 0$ arbitrarily small, valid in $S_h \times D_r^1$. Also note that from Corollary 3.5.2.1 we have $\operatorname{Re} \Theta_0^- = 0$ (a consequence of reversibility). Putting estimates (4.146) and (4.147) together and changing to variable z we get,

$$\Theta(\varphi, z) = e^{-\frac{\pi\alpha_\epsilon}{2\beta_\epsilon}} \Theta_0^- e^{-i(\frac{\alpha_\epsilon}{\beta_\epsilon} z - \varphi)} + O(\delta^{4-\mu_3}),$$

on the line $S_h \times \ell(\delta)$, where $\mu_3 > 0$ is arbitrarily small. Thus, taking into account the previous estimate, (4.142) and (4.143) we have the following estimate for the splitting function,

$$\Xi(\varphi, z) = -e^{-\frac{\pi\alpha_\epsilon}{2\beta_\epsilon}} \Theta_0^- e^{-i(\frac{\alpha_\epsilon}{\beta_\epsilon} z - \varphi)} + O(\delta^{4-\mu_4}),$$

valid in $S_h \times \ell(\delta)$, where $\mu_4 > 0$ is arbitrarily small. Since $e^{i\frac{\alpha_\epsilon}{\beta_\epsilon} z} = O(e^{-\frac{\pi\alpha_\epsilon}{2\beta_\epsilon}})$ on the line $\ell(\delta)$ and moreover $\Theta_0^- = \pm i\sqrt{\mathcal{K}_0}$ (see Remark 3.5.2.1) where $\mathcal{K}_0$ is the Stokes constant of H_0, then the following estimate is still valid,

$$\begin{aligned} \Xi(\varphi, z) &= \mp i e^{-\frac{\pi\alpha_\epsilon}{2\beta_\epsilon}} \sqrt{\mathcal{K}_0} \left(e^{-i(\frac{\alpha_\epsilon}{\beta_\epsilon} z - \varphi)} - e^{i(\frac{\alpha_\epsilon}{\beta_\epsilon} z - \varphi)}\right) + O(\delta^{4-\mu_4}) \\ &= \pm 2 e^{-\frac{\pi\alpha_\epsilon}{2\beta_\epsilon}} \sqrt{\mathcal{K}_0} \sin\left(\frac{\alpha_\epsilon}{\beta_\epsilon} z - \varphi\right) + O(\delta^{4-\mu_4}), \end{aligned} \tag{4.148}$$

in $S_h \times \ell(\delta)$. Taking into account that $\Xi(\varphi, z)$ is real analytic then the same estimate holds in the set,

$$S_h \times \left(\partial \mathcal{D}(\delta) \cap \left\{ \operatorname{Im} z = -\frac{\pi}{2} + \frac{2}{\mu} \delta \log \delta^{-1} \right\} \right).$$

Now, since $\Xi(\varphi, z)$ is $2\pi \frac{\beta_\epsilon}{\alpha_\epsilon}$-periodic in z then using a maximum modulus principle we conclude that,

$$\Xi(0, z) = \pm 2 e^{-\frac{\pi \alpha_\epsilon}{2\beta_\epsilon}} \sqrt{\mathcal{K}_0} \sin\left(\frac{\alpha_\epsilon}{\beta_\epsilon} z \right) + O(\delta^{4-\mu_4}),$$

valid in the strip $|\operatorname{Im} z| \leq \frac{\pi}{2} - \frac{2}{\mu} \delta \log \delta^{-1}$.

Using this bound for the splitting function we are now ready to get a lower bound for the homoclinic invariant. The argument is based on estimating the Fourier coefficients of Ξ in a suitable way. It goes as follows: consider the following function,

$$g(\varphi, z) = \Xi(\varphi, z) \mp 2 e^{-\frac{\pi \alpha_\epsilon}{2\beta_\epsilon}} \sqrt{\mathcal{K}_0} \sin\left(\frac{\alpha_\epsilon}{\beta_\epsilon} z - \varphi \right).$$

It has the same properties as Ξ and moreover $g(0,0) = 0$. Now we expand the function g into Fourier series, i.e.,

$$g(\varphi, z) = \sum_{k \in \mathbb{Z}} g_k e^{ik(\frac{\alpha_\epsilon}{\beta_\epsilon} z - \varphi)},$$

where coefficients of the series can be expressed in terms of Fourier integrals:

$$g_k = \frac{\alpha_\epsilon}{2\pi\beta_\epsilon} \int_0^{\frac{2\pi\beta_\epsilon}{\alpha_\epsilon}} e^{-ik\frac{\alpha_\epsilon}{\beta_\epsilon} z} g(0, z) dz .$$

Following the common procedure of Fourier Analysis, we shift the contour of integration to $\operatorname{Im} z = i\rho$ where $\rho = \frac{\pi}{2} - \frac{2}{\mu} \delta \log \delta^{-1}$ to get,

$$\begin{aligned} g_k &= \frac{\alpha_\epsilon}{2\pi\beta_\epsilon} \int_{i\rho}^{i\rho + \frac{2\pi\beta_\epsilon}{\alpha_\epsilon}} e^{-ik\frac{\alpha_\epsilon}{\beta_\epsilon} z} g(0, z) dz \\ &= \frac{\alpha_\epsilon}{2\pi\beta_\epsilon} e^{\frac{k\alpha_\epsilon \rho}{\beta_\epsilon}} \int_0^{\frac{2\pi\beta_\epsilon}{\alpha_\epsilon}} e^{-ik\frac{\alpha_\epsilon}{\beta_\epsilon} z} g(0, i\rho + z) dz . \end{aligned}$$

Thus, for $k \leq -1$ we can estimate g_k as follows,

$$|g_k| \leq e^{\frac{k\alpha_\epsilon \rho}{\beta_\epsilon}} \sup_{|\mathrm{Im}\, z| \leq \rho} |g(0,z)|\,, \quad k \leq -1.$$

Analogously, by shifting the contour of integration to $\mathrm{Im}\, z = -i\rho$ we get for $k \geq 1$ the following estimate,

$$|g_k| \leq e^{-\frac{k\alpha_\epsilon \rho}{\beta_\epsilon}} \sup_{|\mathrm{Im}\, z| \leq \rho} |g(0,z)|\,, \quad k \geq 1.$$

Taking into account these estimates for the Fourier coefficients we obtain for $(\varphi, z) \in \mathbb{R}^2$ that,

$$\begin{aligned}
|g(\varphi,z) - g_0| &\leq \sum_{k \in \mathbb{Z} - \{0\}} |g_k| \\
&\leq 2 \sup_{|\mathrm{Im}\, z| \leq \rho} |g(0,z)| \sum_{k \geq 1} e^{-\frac{k\alpha_\epsilon \rho}{\beta_\epsilon}} \\
&\leq 2 \sup_{|\mathrm{Im}\, z| \leq \rho} |g(0,z)| \, \frac{e^{-\frac{\alpha_\epsilon \rho}{\beta_\epsilon}}}{1 - e^{-\frac{\alpha_\epsilon \rho}{\beta_\epsilon}}}.
\end{aligned}$$

Finally, taking into account that,

$$\sup_{|\mathrm{Im}\, z| \leq \rho} |g(0,z)| = O(\delta^{4-\mu_4}) \quad \text{and} \quad e^{-\frac{\alpha_\epsilon \rho}{\beta_\epsilon}} = O(e^{-\frac{\pi \alpha_\epsilon}{2\beta_\epsilon}} \delta^{-2-\mu_5}),$$

where $\mu_5 > 0$ is arbitrarily small we conclude that,

$$|g(\varphi,z) - g_0| = O(e^{-\frac{\pi \alpha_\epsilon}{2\beta_\epsilon}} \delta^{2-\mu_4-\mu_5}), \quad \text{for} \quad (\varphi,z) \in \mathbb{R}^2.$$

Thus,

$$|g(\varphi,z)| = |g(\varphi,z) - g(0,0)| \leq |g(\varphi,z) - g_0| + |g(0,0) - g_0| = O(e^{-\frac{\pi \alpha_\epsilon}{2\beta_\epsilon}} \delta^{2-\mu_4-\mu_5}),$$

which implies that,

$$\Xi(\varphi,z) = \pm 2 e^{-\frac{\pi \alpha_\epsilon}{2\beta_\epsilon}} \sqrt{\mathcal{K}_0} \sin\left(\frac{\alpha_\epsilon}{\beta_\epsilon} z - \varphi\right) + O(e^{-\frac{\pi \alpha_\epsilon}{2\beta_\epsilon}} \delta^{2-\mu_6}),$$

for $\mu_6 > 0$ arbitrarily small. At last, taking into account that $\omega_\epsilon = -\partial_\varphi \Xi(0,0)$ we obtain the desired asymptotic formula for the homoclinic invariant. This completes the proof of Theorem 4.1.1.

4.6.2 Finer estimates for $\mathbf{\Gamma}^s - \mathbf{\Gamma}^u$ and an asymptotic expansion for the homoclinic invariant

In order to define the splitting function in the previous subsection, we have assumed that the domain of definition of the symplectic map Ψ was large enough to contain a piece of the stable parametrisation $\mathbf{\Gamma}^s$. More precisely we have assumed that $|E_1|+|E_2| < \delta^\sigma$ where $\sigma \leq 2$. As mentioned previously, this is not a serious restriction and it can be overcome by finer estimates for the inner differences $\mathbf{\Gamma}_k^+ - \mathbf{\Gamma}_k^-$. In fact, using the methods developed in Chapter 3 it is possible to prove that,

$$\mathbf{\Gamma}_k^+(\varphi,\tau) - \mathbf{\Gamma}_k^-(\varphi,\tau) = O(\tau^{N_k} e^{-i(\tau-\varphi)}), \quad \text{in} \quad S_h \times D_r^1, \tag{4.149}$$

where $D_r^1 = D_r^+ \cap D_r^- \cap \{\mathrm{Im}\tau < -r\}$ for $r > 0$ sufficiently large and $N_k \in \mathbb{N}$. Now, taking into account Theorems 4.4.3 and 4.5.2 we have that,

$$\mathbf{\Gamma}^s(\varphi,z) - \mathbf{\Gamma}^u(\varphi,z) = \sum_{k=0}^{n} (\mathbf{\Gamma}_k^+(\varphi,\tau) - \mathbf{\Gamma}_k^-(\varphi,\tau))\delta^{2k} + O((\delta\tau)^{2n+2}) \tag{4.150}$$

valid in $S_h \times R_2(\delta)$ where recall that $z = \frac{\beta_\epsilon}{\alpha_\epsilon}\tau + i\frac{\pi}{2}$ and $R_2(\delta)$ is defined in (4.129). Thus, estimates (4.149) and (4.150) imply that,

$$\mathbf{\Gamma}^s(\varphi,z) - \mathbf{\Gamma}^u(\varphi,z) = O(\delta^\sigma \log^3 \delta^{-1}),$$

valid in the set $S_h \times \mathcal{D}_\sigma(\delta)$ where,

$$\mathcal{D}_\sigma(\delta) = \left\{ |\mathrm{Im}\, z| < \frac{\pi}{2} - \sigma\delta \log \delta^{-1}, \quad |\mathrm{Re}\, z| < c\delta \right\},$$

for some $c > 0$. Thus, given any $\sigma > 0$ the splitting function (4.140) is well defined. Finally, similar considerations as in the previous subsection and taking into account the finer estimates for the differences (4.149) we can derive an asymptotic expansion for the homoclinic invariant which we conjecture as follows,

$$\omega_\epsilon \asymp \pm 2 e^{-\frac{\pi\alpha_\epsilon}{2\beta_\epsilon}} \sum_{k\geq 0} \omega_k \delta^{2k}, \quad \omega_k \in \mathbb{R}, \tag{4.151}$$

where $\omega_0 = \sqrt{\mathcal{K}_0}$.

In the next chapter we perform numerical experiments that support the validity of the asymptotic formula and asymptotic expansion of the homoclinic invariant.

4.7 Conclusion

The goal of this chapter was to prove Theorem 4.1.1. Its proof depends on several different results obtained in the previous and present chapter. In this section we shall briefly describe the main steps of the proof. The strategy is as follows:

1. **Parametrization of the invariant manifolds.** We show in Theorem 4.3.1 that it is possible to parametrize stable and unstable manifolds by solutions of equation (4.2). This parametrization is initially defined in a complex neighbourhood of the equilibrium point.

2. **Approximation near the equilibrium.** We prove that any truncation of the *formal separatrix* (see Theorem 4.2.1) of the normal form H_ϵ^{NF} provides a good approximation of the stable and unstable manifolds in a neighbourhood of the equilibrium point. This is the content of Theorem 4.4.1.

3. **Analytic continuation of the parametrizations towards the singular points.** The approximations provided by the formal separatrix have singularities at $z = i\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$. We show in Theorem 4.4.2 that it is possible to extend the approximation and the domain of analyticity of the parametrizations up to a δ-neighbourhood of the singular points.

4. **Complex matching near the singularity.** The approximations provided by the truncations of the formal separatrix grow near the singularity $z = i\frac{\pi}{2}$. Instead of improving the existent approximations we construct different approximations using the method of complex matching (see Theorem 4.4.3). Roughly speaking, the new approximations retain the essential behavior near the singularity, providing better estimates for the parametrisations in that region. The leading order of the approximation is given by the parametrizations $\mathbf{\Gamma}_0^{\pm}$ which are studied in chapter 3. These new approximations distinguishes between stable and unstable manifolds.

5. **Flow box coordinates and the splitting function.** This is the last step of

the proof and is developed in section 4.6.1 of the present chapter. Using a flow box (see Theorem 4.6.1) and the upper bounds provided by the approximations of the complex matching method we are able to get an asymptotic formula for the splitting. The main point here is periodicity of a certain splitting function (4.140), that allow us to use standard arguments in Fourier analysis to capture the exponential smallness of the splitting.

Chapter 5

Numerical Investigation of Homoclinic Phenomenon

In this chapter we study the asymptotic formula of the homoclinic invariant from a numerical point of view. Our example is the Swift-Hohenberg equation. We perform several numerical experiments that support the validity of the asymptotic formula and obtain the same Stokes constant using two completely different methods. All computations were performed using Maple Software with high-precision arithmetic.

5.1 The generalized Swift-Hohenberg equation

The generalized Swift-Hohenberg equation (GSHE),

$$u_t = \epsilon u + \kappa u^2 - u^3 - (1+\Delta)^2 u \tag{5.1}$$

is widely used to model nonlinear phenomena in various areas of modern Physics including hydrodynamics, pattern formation and nonlinear optics (e.g. [12, 40]). This equation (with $\kappa = 0$) was originally introduced by Swift and Hohenberg [72] in a study of thermal fluctuations in a convective instability.

In the following we consider u to be one dimensional and study stationary solutions of (5.1) which satisfy the ordinary differential equation

$$\epsilon u + \kappa u^2 - u^3 - (1 + \partial_x^2)^2 u = 0 \,. \tag{5.2}$$

Obviously this equation has a reversible symmetry (if $u(x)$ satisfy the equation then $u(-x)$ also does). It is well known that for small negative ϵ this equation has two symmetric homoclinic solutions [34] similar to the ones shown on Figure 5.1. In this chapter we study from a numerical point of view the transversality of the homoclinic solutions, which implies, by the results of the previous chapters, the existence of multi-pulse homoclinic solutions and a small scale chaos. Recently, similar computations for the Swift-Hohenberg equation have been performed by S. J. Chapman and G. Kozyreff in [18] where they study localised patterns emerging from a subcritical modulation instability using the multiple-scales analysis beyond all orders. Our methods extend those of [18] as they can be applied to any Hamiltonian system near a Hamiltonian-Hopf bifurcation. Moreover, our dynamical system approach provides more insight about the divergence of the asymptotic expansions derived in [18] and gives a rigorous framework to study transversal homoclinic orbits for the Swift-Hohenberg equation.

In order to describe the homoclinic phenomena it is convenient to rewrite the equation (5.2) in the form of an equivalent Hamiltonian system [8, 48]:

$$\begin{aligned} \dot{q_1} &= q_2 & \dot{p_1} &= p_2 - \epsilon q_1 - \kappa q_1^2 + q_1^3 \\ \dot{q_2} &= p_2 - q_1 & \dot{p_2} &= -p_1 \,, \end{aligned} \tag{5.3}$$

where the variables are defined by the following equalities

$$u = q_1, \quad u' = q_2, \quad -(u' + u''') = p_1 \quad \text{and} \quad u + u'' = p_2 \tag{5.4}$$

and the Hamiltonian function has the form

$$H_\epsilon = p_1 q_2 - p_2 q_1 + \frac{p_2^2}{2} + \epsilon \frac{q_1^2}{2} + \kappa \frac{q_1^3}{3} - \frac{q_1^4}{4}. \tag{5.5}$$

The system (5.3) is reversible with respect to the involution,

$$S : (q_1, q_2, p_1, p_2) \to (q_1, -q_2, -p_1, p_2).$$

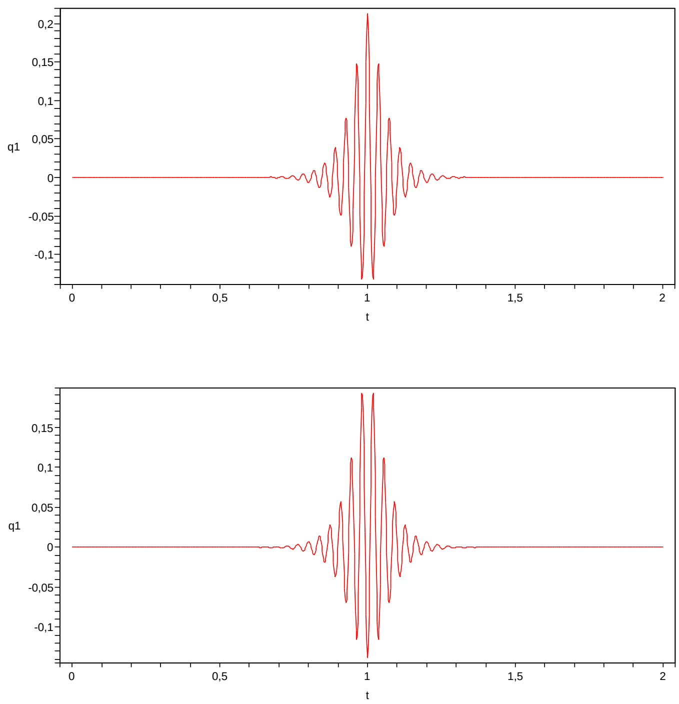

Figure 5.1: Two primary symmetric homoclinic solutions of the scalar stationary GSHE ($\epsilon = -0.05$).

The origin is an equilibrium of the system and the eigenvalues of the linearized vector field are

$$\left\{ \pm\sqrt{-1+\sqrt{\epsilon}}, \pm\sqrt{-1-\sqrt{\epsilon}} \right\} .$$

If $\epsilon < 0$, the eigenvalues form a quadruple $\pm\beta_\epsilon \pm i\alpha_\epsilon$ where

$$\begin{aligned} \beta_\epsilon &= \frac{\sqrt{2\sqrt{1-\epsilon}-2}}{2} = \sqrt{-\frac{\epsilon}{4}}\,(1+O(\epsilon))\,, \\ \alpha_\epsilon &= \frac{\sqrt{2\sqrt{1-\epsilon}+2}}{2} = 1+O(\epsilon)\,. \end{aligned}$$

At $\epsilon = 0$ the eigenvalues collide forming two purely imaginary eigenvalues $\pm i$ of multiplicity two. Moreover, the corresponding linearization of the vector field is not semisimple. Thus, the equilibrium point of system (5.3) undergoes a Hamiltonian-Hopf bifurcation described in section 2.2. In general position there are two possible scenarios of the bifurcation depending on the sign of a certain coefficient of a normal form (see section 2.2). In the Swift-Hohenberg equation both scenarios are possible and depend on the value of the parameter κ. In the following we shall consider the case when the equilibrium is stable at the moment of the bifurcation which corresponds to $|\kappa| > \sqrt{\frac{27}{38}}$ as shown in [8]. For the degenerate case $\kappa = \sqrt{\frac{27}{38}}$, interesting phenomena known as *snaking* takes place [77].

When $\epsilon < 0$ is small, the equilibrium is a saddle-focus and we can parametrise the invariant manifolds $W_\epsilon^{u,s}$ by solutions of the PDE (4.2) (see also discussion in section 2.3 of chapter 2). In the case of the Swift-Hohenberg equation the system of PDE (4.2) can be conveniently replaced by a single scalar PDE of higher order,

$$(1 + \mathcal{D}_\epsilon^2)^2 u = \epsilon u + \kappa u^2 - u^3 \,, \tag{5.6}$$

where we recall that $\mathcal{D}_\epsilon$ denotes the following differential operator,

$$\mathcal{D}_\epsilon = \alpha_\epsilon \partial_\varphi + \beta_\epsilon \partial_z.$$

Let us use $u^\pm$ to denote the first component of $\mathbf{\Gamma}^u$ and $\mathbf{\Gamma}^s$ respectively, then $u^\pm$ satisfies the equation (5.6). Its other components can be restored using (5.4). The Swift-Hohenberg equation is reversible and we assume that

$$u^+(\varphi, z) = u^-(-\varphi, -z),$$

and $\mathbf{\Gamma}^s(0,0) = \mathbf{\Gamma}^u(0,0) \in \mathrm{Fix}(S)$ is the primary symmetric homoclinic point. We recall the definition of homoclinic invariant,

$$\omega_\epsilon = \Omega(\partial_\varphi \mathbf{\Gamma}^s(0,0), \partial_\varphi \mathbf{\Gamma}^u(0,0)). \tag{5.7}$$

In the case of the Swift-Hohenberg equation the formula above can be rewritten in terms of u^-:

$$\omega_\epsilon = 2\partial_\varphi \left((u^-)^2 + u^- \mathcal{D}_\epsilon^2 u^- \right) ,$$

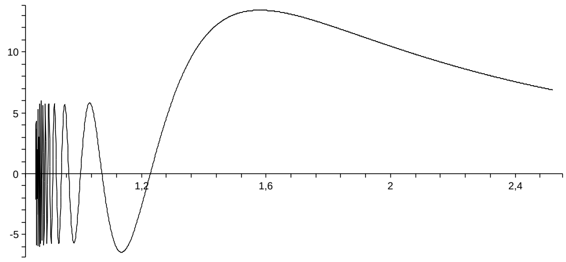

Figure 5.2: Graph of the function $\mathrm{Im}(\Theta_0^-(\kappa))$ for $\kappa > \sqrt{\frac{27}{38}}$.

where the derivatives are evaluated at $(\varphi, z) = (0, 0)$. The theory of the previous chapters implies that,

$$\omega_\epsilon = \pm 2 e^{-\frac{\pi \alpha_\epsilon}{2\beta_\epsilon}} \left(\omega_0 + O(\epsilon^{1-\mu})\right) , \tag{5.8}$$

where $\omega_0 = \left|\Theta_0^-(\kappa)\right|$ (see section 3.5 of chapter 3 for a definition of Θ_0^-) and $\mu > 0$ is arbitrarily small. This formula implies the transversality of the homoclinic orbit for all small values of ϵ provided the splitting coefficient ω_0 does not vanish. This constant is known as the Stokes constant and due to the reversibility is a purely imaginary number (see Corollary 3.5.2.1). Figure 5.2 gives an idea about its behaviour as a function of the parameter κ.

5.1.1 Normal form of the Swift-Hohenberg equation

Let us compute the normal form for the Swift-Hohenberg equation. As a first step the quadratic part of the Hamiltonian (5.5) is normalised with the help of a linear symplectic transformation (similar to [11]):

$$T = \begin{pmatrix} 0 & -1/4\sqrt{2} & -1/2\sqrt{2} & 0 \\ 1/4\sqrt{2} & 0 & 0 & 1/2\sqrt{2} \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 0 \end{pmatrix},$$

which transforms (5.5) into

$$\begin{aligned} H_\epsilon = &-(q_2p_1 - q_1p_2) + \frac{1}{2}(q_1^2+q_2^2) + \frac{1}{4}p_1^2\epsilon - \frac{\sqrt{2}}{12}\kappa p_1^3 + \frac{1}{4}q_2p_1\epsilon - \frac{\sqrt{2}}{8}\kappa q_2 p_1^2 + \\ &\frac{1}{16}q_2^2\epsilon - \frac{\sqrt{2}}{16}\kappa q_2^2 p_1 - \frac{\sqrt{2}}{96}\kappa q_2^3 - \frac{1}{16}p_1^4 - \frac{1}{8}q_2p_1^3 - \frac{3}{32}q_2^2p_1^2 - \frac{1}{32}q_2^3p_1 - \frac{1}{256}q_2^4, \end{aligned} \tag{5.9}$$

where we keep the same notation for the variables. Note that the involution S in the new coordinates takes the form

$$\mathcal{S} : (q_1, q_2, p_1, p_2) \to (-q_1, q_2, p_1, -p_2). \tag{5.10}$$

Now, with the quadratic part in normal form, we can apply the standard normal form procedure to normalize the Hamiltonian (5.9) up to any order: There is a near identity canonical change of variables Φ_n which normalizes all terms of order less than equal to n and transforms the Hamiltonian to the following form:

$$H_\epsilon = H_\epsilon^n + \text{higher order terms} \tag{5.11}$$

where

$$H_\epsilon^n = -I_1 + I_2 + \sum_{\substack{3i+2j+2l\geq 4 \\ i+j\geq 1}}^{n} a_{i,j,l} I_1^i I_3^j \epsilon^l$$

with

$$I_1 = q_2p_1 - q_1p_2, \qquad I_2 = \frac{q_1^2+q_2^2}{2}, \qquad I_3 = \frac{p_1^2+p_2^2}{2}.$$

This normalization preserves the reversibility with respect to the involution (5.10). In the case of the GSHE the normal form up to the order five has the form (see Appendix B for more details about the change of variables)

$$H_\epsilon^5 = -I_1 + \left(I_2 + \frac{1}{4}\epsilon I_3 + \eta I_3^2\right) + \left(\frac{1}{8}\epsilon I_1 + \mu\, I_1 I_3\right) .$$

The leading part of the normal form includes two parameters which can be explicitly expressed in terms of the original parameter κ:

$$\eta = 4\left(\frac{19}{576}\kappa^2 - \frac{3}{128}\right) \quad \text{and} \quad \mu = 2\left(\frac{65}{864}\,\kappa^2 - \frac{3}{64}\right) .$$

The geometry of the invariant manifolds depends on the sign of η. In the case of GSHE, if

$$|\kappa| > \sqrt{\frac{27}{38}},$$

then $\eta > 0$ and the truncated normal form has a continuum of homoclinic orbits among which exactly two are reversible, i.e., symmetric with respect to the involution (5.10).

In order to describe the geometry of the invariant manifolds near the bifurcation it is convenient to introduce the new parameter $\epsilon = -4\delta^2$ and perform the standard scaling (2.11) which we recall for convenience:

$$q_1 = \delta^2 Q_1, \quad q_2 = \delta^2 Q_2, \quad p_1 = \delta P_1, \quad p_2 = \delta P_2 .$$

This change of variables is not symplectic, nevertheless it preserves the form of the Hamiltonian equations since the symplectic form gains a constant factor δ^3, so we have to multiply the Hamiltonian by δ^{-3} in order to return back to the standard symplectic form. The Hamiltonian H_ϵ^n is transformed into,

$$h_\delta^n = -I_1 + \left(I_2 - I_3 + \eta I_3^2\right)\delta + \left(-\frac{1}{2}I_1 + \mu\, I_1 I_3\right)\delta^2 + O(\delta^3).$$

This Hamiltonian system has an equilibrium at the origin characterized by a quadruple of complex eigenvalues $\pm i\alpha_{n,\epsilon} \pm \beta_{n,\epsilon}$, where $\alpha_{n,\epsilon} = 1 + \frac{1}{2}\delta^2 + O(\delta^4)$ and $\beta_{n,\epsilon} = \delta - \frac{1}{2}\delta^3 + O(\delta^5)$.

The equilibrium has a real two dimensional stable and two dimensional unstable manifolds. We parametrise these manifolds by solutions of the PDE:

$$(\alpha_{n,\epsilon}\partial_\varphi + \beta_{n,\epsilon}\partial_z)\,\mathbf{X}_n = X_{h_\delta^n}(\mathbf{X}_n). \tag{5.12}$$

The function $\mathbf{X}_n(\varphi, z)$ is real-analytic, converges to zero as $z \to \pm\infty$ and is 2π-periodic in φ. Taking into account the rotational symmetry of the normal form Hamiltonian, we can look for the solution of this equation in the form:

$$\begin{aligned}\mathbf{X}_n(\varphi, z) = \big(&R_n(z)\cos(\theta_n(\varphi, z)), R_n(z)\sin(\theta_n(\varphi, z)),\\ &r_n(z)\cos(\theta_n(\varphi, z)), r_n(z)\sin(\theta_n(\varphi, z))\big),\end{aligned}$$

where $R_n(z)$, $r_n(z)$ and $\theta_n(\varphi, z)$ are real analytic functions. In particular, for $n = 5$ we get the following system of equations:

$$\begin{aligned}\beta_{5,\epsilon} R_5' = -\delta r_5 \left(1 - \eta r_5^2\right), \qquad \beta_{5,\epsilon} r_5' = -\delta R_5\,,\\ \left(\alpha_{5,\epsilon}\partial_\varphi + \beta_{5,\epsilon}\partial_z\right)\theta_5 = 1 + \frac{\delta^2}{2}(1 - \mu r_5^2)\,.\end{aligned}$$

From these equations we conclude, if

$$\beta_{5,\epsilon} = \delta \qquad \alpha_{5,\epsilon} = 1 + \frac{\delta^2}{2}\,,$$

then

$$\begin{aligned}r_5 = \sqrt{\frac{2}{\eta}}\frac{1}{\cosh z}\,, \qquad R_5 = \sqrt{\frac{2}{\eta}}\frac{\sinh z}{\cosh^2 z}\\ \theta_5 = \varphi - \frac{\delta^2\mu}{2}\int^z r_5^2 dz = \varphi - \frac{\delta^2\mu}{\eta}\frac{\sinh z}{\cosh z}\,.\end{aligned}$$

We see that $(r(z), R(z))$ runs over a homoclinic loop when z varies from $-\infty$ to $+\infty$.

In general the parameterization $\mathbf{X}_n$ is the unique solution of (5.12) such that $R_n(0) = 0$ and $\theta_n(\varphi, 0) = \varphi$. Thus, $\mathbf{X}_n(\varphi, z)$ belongs to the symmetry plane associated with the involution (5.10) if and only if $z = 0$ and $\varphi = 0$ or $\varphi = \pi$. Therefore, there are exactly 2 symmetric homoclinic points. Let us call these homoclinic orbits the primary reversible homoclinic orbit.

In chapter 4 (see Theorem 4.4.1) it was shown that the functions $\mathbf{X}_n$ approximate reasonably well the parametrisations $u^\pm$ in a neighbourhood of the equilibrium. Transforming $\mathbf{X}_5(\varphi, z)$ back to the original coordinates we obtain the following approximation:

$$u_\epsilon^-(\varphi,z) = -\frac{1}{\sqrt{\eta}}\frac{\cos(\varphi)}{\cosh(z)}\delta \tag{5.13}$$
$$+\left(\frac{9\kappa+\kappa\cos(2\varphi)}{18\eta}\frac{1}{\cosh^2(z)} - \frac{1}{\sqrt{\eta}}\left(\frac{\mu}{\eta}+\frac{1}{2}\right)\frac{\sin(\varphi)\sinh(z)}{\cosh^2(z)}\right)\delta^2 + O(\delta^3),$$

where $\epsilon = -4\delta^2$. Since the function in the right-hand-side of the equation is even, it also approximates the stable manifold represented by $u_\epsilon^+(\varphi,z) = u^-(-\varphi,-z)$.

5.1.2 Stokes constant

Let us study invariant manifolds of (5.2) for $\epsilon = 0$. Following (2.19) it is convenient to parametrise these invariant manifolds by solutions of the following PDE:

$$(1+(\partial_\varphi+\partial_\tau)^2)^2 u = \kappa u^2 - u^3\,. \tag{5.14}$$

The results of chapter 3 imply that this equation has an unique analytic solution $u = u_0^-$ with the following asymptotic behaviour:

$$u_0^-(\varphi,\tau) = \frac{P_1(\varphi)}{\tau} + \frac{P_2(\varphi)}{\tau^2} + O(\tau^{-3})$$

in the set

$$\tau \in D_r^- = \{\tau \,:\, |\arg(\tau+r)| > \theta_0\}\ ,$$

where θ_0 is a small fixed constant and r is sufficiently large and

$$P_1 = \frac{i\cos(\varphi)}{\sqrt{\eta}}\,, \qquad P_2 = \frac{i}{\sqrt{\eta}}\left(\frac{\mu}{\eta}+\frac{1}{2}\right)\sin(\phi) - \frac{\kappa\,\cos(2\,\phi)}{18\eta} - \frac{\kappa}{2\eta}\,. \tag{5.15}$$

The function u_0^- is 2π-periodic in φ.

The equation (5.14) has a second solution $u = u_0^+$ with

$$u_0^+(\tau,\varphi) = \overline{u_0^-(-\overline{\tau},-\overline{\varphi})}\,.$$

It has the same asymptotic behaviour as u_0^- but is defined in a different sector, more precisely, it is defined for τ such that $-\tau \in D_r^-$. The solutions $u_0^\pm$ have a common asymptotics on the intersection of their domains but they do not typically coincide (see Theorem 3.5.1). The difference of these two solutions can be described in the following way. We can restore 4-dimensional vectors $\mathbf{\Gamma}_0^{\pm}$ using equations (5.4) with $'$ replaced by $\partial_\varphi + \partial_\tau$. In particular, the first component of $\mathbf{\Gamma}_0^{\pm}$ coincides with $u_0^{\pm}$. The functions $\mathbf{\Gamma}_0^{\pm}$ are parametrisations of the stable and unstable manifolds and satisfy the following non-linear PDE,

$$D\Gamma = X_{H_0}(\Gamma), \quad \text{where} \quad D = \partial_\varphi + \partial_\tau. \tag{5.16}$$

Let

$$\Delta_0(\varphi,\tau) = \mathbf{\Gamma}_0^{+}(\varphi,\tau) - \mathbf{\Gamma}_0^{-}(\varphi,\tau),$$

and

$$\theta_0(\varphi,\tau) = \Omega\big(\Delta_0(\varphi,\tau), \partial_\varphi \mathbf{\Gamma}_0^{+}(\varphi,\tau)\big)\,,$$

where Ω is the standard symplectic form. Then according to Lemma 3.5.2 and Theorem 3.5.1 there is a purely imaginary number $\Theta_0^-(\kappa)$ such that

$$\theta_0(\varphi,\tau) = \Theta_0^-(\kappa)e^{-i(\tau-\varphi)} + O(e^{-(2-\mu_0)i(\tau-\varphi)}), \tag{5.17}$$

as $\operatorname{Im}\tau \to -\infty$ and for very small $\mu_0 > 0$. The constant $\Theta_0^-(\kappa)$ is known as the *Stokes (or splitting) constant*. The Stokes constant of the Swift-Hohenberg equation can be defined by the following limit:

$$\Theta_0^-(\kappa) := \lim_{\operatorname{Im}(\tau)\to-\infty} \theta_0(\varphi,\tau)e^{i(\tau-\varphi)}\,. \tag{5.18}$$

We note that the value of the Stokes constant cannot be obtained from our arguments. Fortunately the numerical evaluation of this constant is reasonably easy. Figure 5.2 shows the values of $\operatorname{Im}\Theta_0^-(\kappa)$ plotted against κ for $\kappa > \kappa_0 = \sqrt{\frac{27}{38}}$. The picture suggests that the Stokes constant vanishes infinitely many times and that its zeros accumulate to κ_0.

5.2 Numerical methods

In this section we present numerical methods that support the validity of the asymptotic formula (5.8). The procedure is based on comparison of two different methods for evaluation of the Stokes constants. The first method relies on the definition (5.18) and involves the GSHE with $\epsilon = 0$ only. The second method evaluates the homoclinic invariant for $\varepsilon \neq 0$ and relies on the validity of the asymptotic expansion (4.151) to extrapolate the values of the (normalised) homoclinic invariant towards $\varepsilon = 0$ in order to get ω_0.

5.2.1 Computation of the Stokes constant

Let us describe the first method for computing the Stokes constant. We set $\tau = -i\sigma$, $\varphi = 0$ and rewrite equation (5.17) in the form:

$$\Theta_0^- = \theta_0(0, -i\sigma)e^{\sigma} + O(e^{-(1-\epsilon_0)\sigma}). \tag{5.19}$$

A method for the computation of the Stokes constant

Let us proceed as follows:

1. The first step is to construct a good approximation of stable and unstable manifolds. This approximation is given by a finite sum $\mathbf{\Gamma}_N$ of the unique formal separatrix $\hat{\mathbf{\Gamma}}_0$ of (5.16),

   $$\mathbf{\Gamma}_N(\varphi, \tau) = \sum_{k=1}^{N} \Gamma_k(\varphi)\tau^{-k}\,,$$

   where

   $$\Gamma_k(\varphi) = \sum_{j=-k}^{k} \Gamma_{k,j} e^{ji\varphi} \text{ with } \Gamma_{k,j} \in \mathbb{C}^4,$$

   that approximates the parametrisations $\mathbf{\Gamma}_0^{\pm}$ in the following sense

   $$\mathbf{\Gamma}_0^{\pm}(\varphi, z) - \mathbf{\Gamma}_N(\varphi, \tau) = O(\tau^{-N-1})\,. \tag{5.20}$$

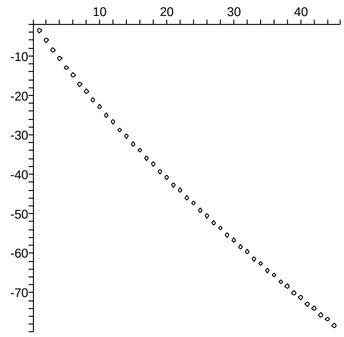

Figure 5.3: Graph of $\log_{10}\left(\frac{\max_j |\Gamma_{k,j}|}{(350\pi)^k}\right)$.

The natural number N can be chosen using the *astronomers recipe*. It simply chooses N such that for fixed τ and φ it minimizes $\left|\Gamma_{N+1}(\varphi)\tau^{-N-1}\right|$, that is, the least term of the series $\hat{\mathbf{\Gamma}}_0(\varphi,\tau)$ (see Figure 5.3).

2. A point on the unstable manifold (resp. stable manifold) can be represented in the coordinates (φ,τ). In order to obtain a point close to the unstable manifold we fix a positive real number $\sigma \in \mathbb{R}^+$ and a sufficiently large $d \in \mathbb{R}^+$ and define $z_0^- = \mathbf{\Gamma}_N(-d, -i\sigma - d)$ and a tangent vector $v_0^- = \partial_\varphi \mathbf{\Gamma}_N(-d, -i\sigma - d)$. Analogously, for the stable manifold we define $z_0^+ = \mathbf{\Gamma}_N(d, -i\sigma + d)$ and $v_0^+ = \partial_\varphi \mathbf{\Gamma}_N(d, -i\sigma + d)$.

3. The next step is to measure the difference of stable and unstable manifolds at the point $(\varphi,\tau) = (0, -i\sigma)$. Taking into account the periodicity in φ we set d equal to a multiple to 2π and integrate numerically the ODE,

$$\begin{aligned} z' &= X_{H_0}(z), \\ v' &= DX_{H_0}(z)v, \end{aligned} \tag{5.21}$$

forward in time with $t \in [0,d]$ and initial conditions $z^-(0) = z_0^-, v^-(0) = v_0^-$ and then backward in time with $t \in [-d,0]$ and initial conditions $z^+(0) = z_0^+, v^+(0) = v_0^+$.

4. Finally we evaluate,

$$\hat{\Theta}(\sigma) = \Omega(z^+(-d) - z^-(d), v^-(d))e^{\sigma}. \tag{5.22}$$

Remark 5.2.0.1. The stable and unstable manifolds have the same asymptotic expansion and the difference $z^+(-d) - z^-(d)$ is known to be exponentially small (see Theorem 3.5.1), i.e. comparable with e^{σ}. Thus the system (5.21) has to be integrated with great accuracy. In the case of GSHE an excellent integrator can be constructed using a high order Taylor series method.

Numerical results

In all current computations we have used a Taylor series method, which is incorporated in the Maple Software, to integrate the equations of motion (5.21). The method uses an adaptive step procedure controlled by a local error tolerance which was set to 10^{-D}, where D is the number of significant digits used in the computations. The order of the method has been automatically defined using the formula $\max(22, \lfloor 1.5D \rfloor)$. Having fixed $\kappa = 2$ we have computed the first 45 coefficients of the formal separatrix $\hat{\mathbf{\Gamma}}_0$ with 60 digits precision. Taking into account (5.20) we see that the error committed by the approximation $\mathbf{\Gamma}_N$ is approximately of the order of the first missing term (see Figure 5.3). Using double precision (16 digits) we have integrated numerically the equations (5.21) to obtain $\hat{\Theta}(\sigma)$ for values of σ uniformly distributed in the interval $[20, 28.89]$. The initial conditions were computed using $d = 350\pi$ and the first 9 terms of the formal series $\hat{\mathbf{\Gamma}}_0$. The results are depicted in Figure 5.4. The expected errors are bounded by the red curves. This implies in particular that the method is numerically stable, that is, the propagation errors due to integration do not increase drastically. There are several sources of errors that affect the accuracy of the computation of the Stokes constant, namely:

- Approximation of stable and unstable manifolds given by the function $\mathbf{\Gamma}_N$;
- Errors due to the numerical integration;

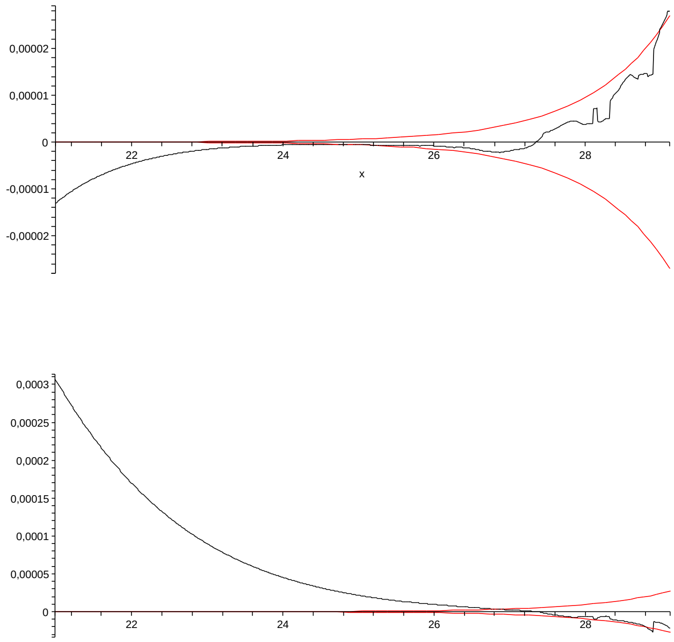

Figure 5.4: The top figure represents the graph of the function $\mathrm{Im}(\hat{\Theta}(\sigma))e^{\sigma} - 10.472161956944$ and the bottom figure represents the graph of the function $\mathrm{Re}(\hat{\Theta}(\sigma))e^{\sigma}$. When σ is around 25 the rounding errors become visible and the convergence stops. The red curves represent the magnitude of the rounding errors.

- Rounding errors.

The first and the second source of errors can be made small compared to the rounding errors, which can be roughly estimated by,

$$\frac{C}{|\eta|\,\sigma^2}10^{-D}e^{\sigma}, \tag{5.23}$$

where D is the number of digits used in the computations and C is some real positive constant which reflects the propagation of rounding errors. Using this estimate we have provided bounds for the rounding errors which can be observed in Figure 5.4. The constant C can be estimated by fitting (5.23) to the points $\left|\hat{\Theta}(\sigma)\right|$ for $\sigma \geq 25$. Using the method of least squares we have concluded that $C \approx 16.7$.

With double arithmetic precision the method previously described allows the computation of 7 to 8 correct digits of the Stokes constant. In fact the rounding errors in computing $\hat{\Theta}(\sigma)$ from formula (5.22) grow accordingly to (5.23) whereas the neglected terms of the formula (5.19) decrease like $C_1 e^{-\sigma}$, where C_1 is some positive constant. Hence the optimum is attained when both contributions are of the same order. The constant C_1 can be estimated by fitting the function $C_0 + C_1 e^{-\sigma}$ to the points $\left|\hat{\Theta}(\sigma)\right|$ for $\sigma \leq 24$. Using the method of least squares we have obtained that $C_1 \approx 17305.75$. Using this information we can determine the value σ^* where both contributions are essentially of the same order. This means that σ^* must satisfy the equation,

$$(e^{-\sigma})^2 = \frac{C}{|\eta|\,\sigma^2\,C_1}10^{-D},$$

which implies that,

$$\left|\Theta_0 - \hat{\Theta}(\sigma^*)\right| \approx \frac{816}{\sigma^*}10^{-\frac{D}{2}}.$$

In this way it is possible to obtain 8 correct digits for the Stokes constant using only double precision (16-digits precision). In Table 5.1 we have listed the values of $\hat{\Theta}(\sigma^*)$ evaluated at the optimum σ^* for higher computer precisions. The digits in bold correspond to correct digits of the Stokes constant. We also note that the numerics suggest that Θ_0^- is pure imaginary which agrees with Corollary 3.5.2.1.

D	σ^*	$\mathrm{Re}(\hat{\Theta}(\sigma^*))$	$\mathrm{Im}(\hat{\Theta}(\sigma^*))$
16	24.68	2.7e-05	**10.4721614**3901571
20	29.46	7.8e-07	**10.47216195**3423286113
24	34.21	1.6e-08	**10.4721619569**069446924024
28	38.95	3.1e-10	**10.472161956944**13924682820786
32	43.67	5.3e-12	**10.47216195694439**6725504278408504
36	48.37	8.5e-14	**10.4721619569443983**419527788851129556
40	53.07	1.2e-15	**10.472161956944398358**12989263311456886391
44	57.76	1.8e-17	**10.4721619569443983582**84180684468467819622191
48	62.45	2.6e-19	**10.4721619569443983582855**084356725900717201861670
52	67.12	3.5e-21	**10.472161956944398358285521**30242825730920048239485015
56	71.80	4.7e-23	**10.47216195694439835828552143**0879142372532568396894067732
60	76.46	6.2e-25	**10.4721619569443983582855214320**209319731283197852962601326570
64	81.13	8.0e-27	**10.472161956944398358285521432031**66495538939445255794702026972749
68	85.79	1.0e-28	**10.47216195694439835828552143203190**0047829633854060398152634432422925

Table 5.1: Stokes constant evaluated at the optimum σ^* for different computer precisions. In the computations we have used $d = 350\pi$ and $N = 40$

$d \backslash N$	10	20	30
100π	10.47216215179386	10.47216215183208	10.47216215181955
150π	10.47216131335742	10.47216131335746	10.47216131335772
200π	10.47216144775669	10.47216144775671	10.47216144775682
250π	10.47216149546998	10.47216149546998	10.47216149547027
300π	10.47216132022817	10.47216132022820	10.47216132022773
350π	10.47216138600882	10.47216138600883	10.47216138600868

Table 5.2: Comparison of the value of $\mathrm{Im}(\hat{\Theta}(25))$ for different values of parameters N and d.

Finally, let us mention that in the process of computing the Stokes constant we have made several choices for the parameters. Namely, the number of terms N used to compute $\mathbf{\Gamma}_N$ and the parameter d which were used in computing the initial conditions of step (ii) of the numerical scheme. In fact the results are independent of these particular choices and Table 5.2 demonstrates the robustness of the numerical method.

5.2.2 High precision computations of the homoclinic invariant

In this section we present a numerical method for the computation of the homoclinic invariant as defined in (5.7) for the Swift-Hohenberg equation with $\kappa = 2$ and $\epsilon < 0$. This section follows the ideas of [32] originally developed for the study of exponentially small phenomena for area-preserving maps.

In order to compute the homoclinic invariant (5.7) we need to compute two tangent vectors at the symmetric homoclinic point $\mathbf{\Gamma}^s(0,0)$. Using the fact that the system is reversible we can obtain the stable tangent vector $\partial_\varphi \mathbf{\Gamma}^s$ by applying the reverser to the unstable tangent vector $\partial_\varphi \mathbf{\Gamma}^u$. The unstable tangent vector $\partial_\varphi \mathbf{\Gamma}^u$ lives in the tangent plane of the unstable manifold at the symmetric homoclinic orbit. Thus an easy way to compute this tangent vector is to approximate the primary homoclinic orbit near the equilibrium point by the following expansion,

$$\mathbf{\Gamma}_N^u(\varphi, z) = \sum_{k=1}^{N} e^{kz} \left(\mathbf{c}_k(\epsilon) + \sum_{j\geq 1}^{k} \mathbf{a}_{k,j}(\epsilon)\cos(j\varphi) + \mathbf{b}_{k,j}(\epsilon)\sin(j\varphi) \right), \tag{5.24}$$

and then use the variational equations,

$$\begin{aligned} \mathbf{x}' &= X_{H_\epsilon}(\mathbf{x}), \\ \mathbf{v}' &= DX_{H_\epsilon}(\mathbf{x})\mathbf{v}, \end{aligned} \tag{5.25}$$

to transport the tangent vector $\partial_\varphi \mathbf{\Gamma}_N^u$ along the primary homoclinic orbit until it hits the symmetric plane $\mathrm{Fix}(S)$ defined by $\{q_2 = 0, p_1 = 0\}$. Let us present the details of the method.

A method for the computation of the homoclinic invariant

1. The first step is to determine the coefficients of (5.24). To that end we take a new expansion,

$$u_N(\varphi, z) = \sum_{k=1}^{N} e^{kz} \left(c_k(\epsilon) + \sum_{j\geq 1}^{k} a_{k,j}(\epsilon)\cos(j\varphi) + b_{k,j}(\epsilon)\sin(j\varphi) \right),$$

   and substitute into the equation,

$$\left((\alpha_\epsilon \partial_\varphi + \beta_\epsilon \partial_z)^2 + 1\right)^2 u = \epsilon u + 2u^2 - u^3, \tag{5.26}$$

   and collect the terms of the same order in e^{kz}. In this way it is possible to determine coefficients c_k, $a_{k,j}$ and $b_{k,j}$. It is not difficult to see that the coefficients $a_{1,1}$ and $b_{1,1}$ satisfy no relations and that all other coefficients depend from these two. So we define,

$$a_{1,1} = r_0\cos(\psi_0) \quad \text{and} \quad b_{1,1} = r_0\sin(\psi_0).$$

Now recall that the first component of $\mathbf{\Gamma}^u$ solves equation (5.26) and due to the asymptotic behavior (5.13) we conclude that for $z << 0$ and $\delta << 1$ it is approximately,

$$e^z \left(-\frac{2\delta}{\sqrt{\eta}} \cos(\varphi) + \frac{\delta^2}{\sqrt{\eta}} \left(1 + \frac{2\mu}{\eta} \right) \sin(\varphi) \right) + \mathcal{O}(e^{2z}), \tag{5.27}$$

where $\epsilon = -4\delta^2$. Next we "*match*" the leading order of $u_N(\phi, s)$ with the expression (5.27) and conclude that ψ_0 and r_0 must satisfy,

$$\begin{aligned} \psi_0 &= \arctan\left(-\left(1 + \frac{2\mu}{\eta}\right) \frac{\delta}{2} \right), \\ r_0 &= \frac{2\delta}{\sqrt{\eta}} \sqrt{1 + \left(1 + \frac{2\mu}{\eta}\right)^2 \frac{\delta^2}{4}}. \end{aligned} \tag{5.28}$$

Taking into account (5.4) we reconstruct $\mathbf{\Gamma}_N^u$ from u_N and due to the "*matching*" (5.28) we have,

$$\mathbf{\Gamma}^u(t,t) \approx \mathbf{\Gamma}_N^u(t,t), \text{ as } t \to -\infty, \ \delta \to 0.$$

That is, for small values of δ, the expansion $\mathbf{\Gamma}_N^u$ provides a good approximation of the primary homoclinic orbit near the equilibrium point.

2. The second step is to improve the accuracy of the approximation of the symmetric homoclinic point, provided by $\mathbf{\Gamma}_N^u$. Given small δ and sufficiently large $T_0 > 0$ we want to determine (T, ψ) such that,

$$\mathbf{x}' = X_{H_\epsilon}(\mathbf{x}), \qquad \mathbf{x}(0;\psi) = \mathbf{\Gamma}_N^u(-\alpha_\epsilon T_0, -\beta_\epsilon T_0; \psi),$$

subject to,

$$\mathbf{x}(T;\psi) \in \mathrm{Fix}(S). \tag{5.29}$$

This problem can be solved using Newton method. Starting from (T_0, ψ_0) we obtain a sequence of points (T_i, ψ_i),

$$\begin{pmatrix} T_{i+1} \\ \psi_{i+1} \end{pmatrix} = \begin{pmatrix} T_i \\ \psi_i \end{pmatrix} - \begin{pmatrix} \frac{\partial q_2}{\partial T}(T_i;\psi_i) & \frac{\partial q_2}{\partial \psi}(T_i;\psi_i) \\ \frac{\partial p_1}{\partial T}(T_i;\psi_i) & \frac{\partial p_1}{\partial \psi}(T_i;\psi_i) \end{pmatrix}^{-1} \begin{pmatrix} q_2(T_i;\psi_i) \\ p_1(T_i;\psi_i) \end{pmatrix}, \tag{5.30}$$

that converges to a limit (T_*, ψ_*) such that $\mathbf{x}(T_*; \psi_*) \in \mathrm{Fix}(S)$, provided (T_0, ψ_0) is sufficiently close to (T_*, ψ_*) (see [16]). The derivatives in (5.30) can be computed using the variational equations along the orbit $\mathbf{x}(t; \psi)$. Later we will see that formulae (5.28) provide sufficiently accurate initial "guesses" yielding the convergence of the Newton method.

3. Having obtained in the previous step an accurate approximation of the symmetric homoclinic point, the last step is to integrate numerically the system,

$$\begin{aligned} \mathbf{x}' &= X_{H_\epsilon}(\mathbf{x}), & \mathbf{x}(0;\psi) &= \mathbf{\Gamma}_N^u(-\alpha_\epsilon T_0, -\beta_\epsilon T_0; \psi_*), \\ \mathbf{v}' &= DX_{H_\epsilon}(\mathbf{x})\mathbf{v}, & \mathbf{v}(0;\psi) &= \alpha_\epsilon \partial_\varphi \mathbf{\Gamma}_N^u(-\alpha_\epsilon T_0, -\beta_\epsilon T_0; \psi_*), \end{aligned}$$

   and evaluate the homoclinic invariant,

$$\hat{\omega} = \Omega(\mathbf{v}(T_*, \psi_*), S(\mathbf{v}(T_*, \psi_*))).$$

Numerical results

We have considered a finite set $\mathcal{I}$ consisting of points in the interval $\epsilon \in [-\frac{1}{10}, -\frac{1}{1000}]$ and computed the homoclinic invariant for those points using the method previously described. For all points in $\mathcal{I}$ the magnitude of the homoclinic invariant ranges from 10^{-5} to 10^{-45}. Thus, in all numerical integrations we have used a high order Taylor method which allows to perform the numerical integration with very high precision. We have computed the coefficients of the expansion (5.24) up to $N = 5$ and for each $\epsilon \in \mathcal{I}$ we have chosen T_0 sufficiently large so that $\mathbf{\Gamma}_N^u(-\alpha_\epsilon T_0, -\beta_\epsilon T_0)$ approximates the unstable manifold within the required precision. The initial point (T_0, ψ_0) used in Newton method proved to be very close to (T_*, ψ_*) and its relative error can be observed in Figure 5.5. After computing the homoclinic invariant we have normalized it using the formula,

$$\bar{\omega}(\epsilon) = \frac{\omega_\epsilon}{2} e^{\frac{\pi \alpha_\epsilon}{2\beta_\epsilon}}.$$

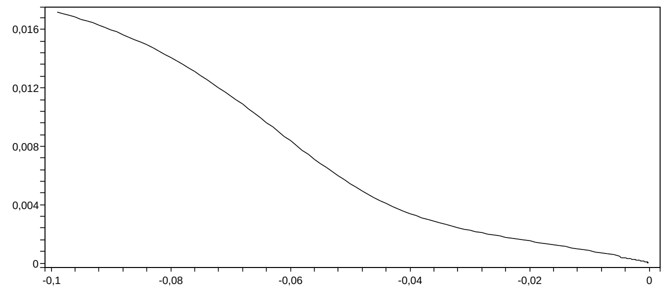

Figure 5.5: Relative error of (T_0, ψ_0) depending on $\epsilon \in \mathcal{I}$.

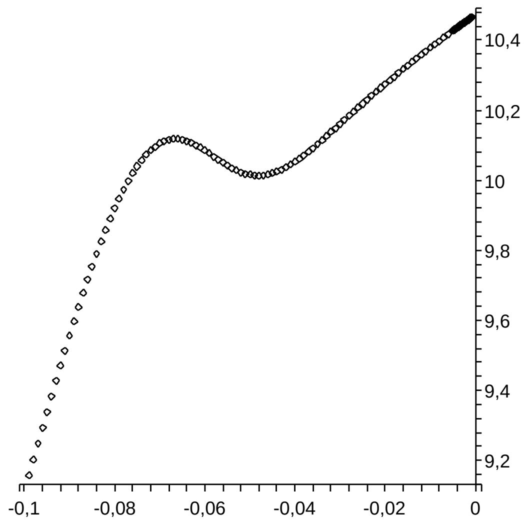

Figure 5.6: Graph of the function $\bar{\omega}(\epsilon)$.

	$\bar{\omega}_0$	$\bar{\omega}_1$	$\bar{\omega}_2$
5	10.47216195694	8.979943127	- 42.60110
6	10.472161956944	8.979943127	- 42.601100
7	10.4721619569443	8.9799431275	- 42.6011004
8	10.47216195694439	8.97994312752	- 42.60110043
9	10.472161956944398	8.9799431275209	- 42.601100432
10	10.4721619569443983	8.9799431275210	- 42.601100432
11	10.4721619569443983	8.9799431275210	- 42.601100432
12	10.4721619569443983	8.9799431275210	- 42.6011004327
	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
5	152.88	- 774.4	3.8×10^3
6	152.888	- 774.2	3.8×10^3
7	152.887	- 774.40	3.80×10^3
8	152.88795	- 774.39	3.814×10^3
9	152.88795	- 774.394	3.813×10^3
10	152.887958	- 774.3944	3.8138×10^3
11	152.887958	- 774.3944	3.813×10^3
12	152.887958	- 774.3944	3.813×10^3

Table 5.3: Coefficients of the estimated polynomials for different subsets of $\mathcal{P}$ and different degrees.

The behaviour of the function $\bar{\omega}(\epsilon)$ can be observed in Figure 5.6. It possible to see that it is approaching the value of the Stokes constant computed in the previous section. Moreover, it is approaching this value in an approximately linear fashion, supporting the validity of the asymptotic formula (5.8). Taking into account the conjecture (4.151) for ω_ϵ we investigate the validity of the following asymptotic expansion for $\bar{\omega}(\epsilon)$,

$$\bar{\omega}(\epsilon) \asymp \sum_{k\geq 0} \bar{\omega}_k \epsilon^k. \tag{5.31}$$

To that end, we have taken 14 points evenly spaced in the interval $[-2.7 \times 10^{-3}, -1.4 \times 10^{-3}]$ and computed the corresponding normalized homoclinic invariant with more than 40 correct digits. Let us denote this set of homoclinic invariants by $\mathcal{P}$. Then, in order to get the first few coefficients of the asymptotic expansion (5.31) we have fitted a partial sum of the asymptotic expansion to the points of $\mathcal{P}$. Here we have used as many points as the number of unknown coefficients. Moreover, following [32] we have performed the following tests to evaluate the validity of the asymptotic expansion:

1. Interpolating different partial sums to different subsets of $\mathcal{P}$ should give essentially

the same results for the coefficients.

2. The constant term of the interpolating polynomial should coincide with the value of the Stokes constant computed in the previous section.

3. The interpolating polynomial should reasonably approximate $\bar{\omega}(\epsilon)$ outside the interval $[-2.7 \times 10^{-3}, -1.4 \times 10^{-3}]$, in the sense that it agrees with the main property of an asymptotic expansion:
$$\left|\bar{\omega}(\epsilon) - \sum_{k\geq 0}^{n-1} \bar{\omega}_k \epsilon^k\right| \leq C\epsilon^n, \ \forall \epsilon \in [\epsilon_0, 0)\,,$$
for some $C > 0$ and $\epsilon_0 < 0$.

For the first test we have considered all possible subsets of $\mathcal{P}$ having only 6 consecutive elements and interpolated these points by polynomials of degree 5. Then for each coefficient, we have extracted the part of the number which is equal to all polynomials. We have repeated this process for polynomials of degree 6 up to degree 12. The results are summarized in Table 5.3, where it is possible to see that there is a good agreement between the coefficients of the different interpolating polynomials of different subsets of $\mathcal{P}$. We can also infer from Table 5.3 that the results are numerically stable. Thus, we have the following estimates for the first 6 coefficients of (5.31):

$\bar{\omega}_0 = 10.4721619569443983\ldots$ $\bar{\omega}_1 = 8.9799431275210\ldots$ $\bar{\omega}_2 = -42.601100432\ldots$

$\bar{\omega}_3 = 152.887958\ldots$ $\bar{\omega}_4 = -774.3944\ldots$ $\bar{\omega}_5 = 3.813\ldots \times 10^3$

Furthermore, it is clear that the coefficient $\bar{\omega}_0$ coincides (up to 18 digits) with the value of the Stokes constant which we recall,

$$\left|\Theta_0^-\right| = 10.47216195694439835828552143203190\ldots$$

Moreover, in Figure 5.7 we see that the relative error of the asymptotic expansion does not exceed 0.06 in the hole interval $\left[-\frac{1}{10}, 0\right]$. Thus, our numerical results provide a satisfactory numerical evidence that supports the correctness of the asymptotic expansion (4.151) for the homoclinic invariant.

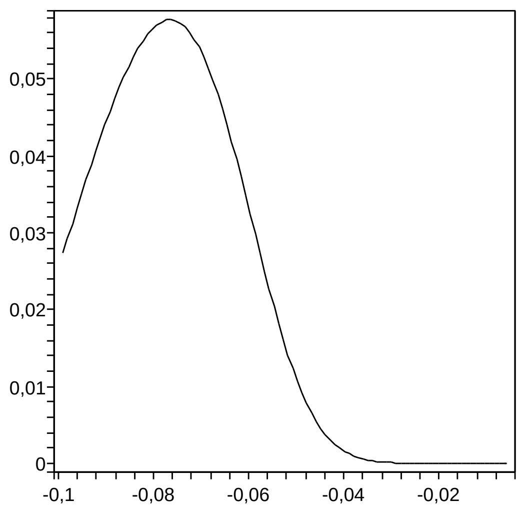

Figure 5.7: Relative error of the asymptotic expansion of $\bar{\omega}(\epsilon)$.

Appendix A

Solutions of linear Hamiltonian systems

Let us consider the following system of linear differential equations,

$$\dot{x} = A(t)x, \tag{A.1}$$

such that $A(t)$ is an $2n$-by-$2n$ Hamiltonian matrix, i.e. $A(t) = JS(t)$ where $S(t)$ is a non-degenerate symmetric matrix and J is the canonical skew-symmetric matrix. We also assume that $S(t)$ is at least C^1. In the following let us omit the dependence of time for simplicity.

Solutions of (A.1) form an $2n$-dimensional linear space and it is well known that there is a fundamental matrix solution $\Pi(t)$ which is symplectic for all t [58]. Let us suppose that we know n linear independent solutions of (A.1), say v_i, $i = 1, \ldots, n$, such that,

$$v_i^T J v_j = 0, \quad \forall\, i, j = 1, \ldots, n. \tag{A.2}$$

Now consider the problem of finding n solutions u_i $i = 1, \ldots, n$ that combined with the v_i's span the linear space of solutions of equation (A.1) and satisfy,

$$\Pi^T J\, \Pi = J, \tag{A.3}$$

where $\Pi = [v_1, \ldots, v_n, u_1, \ldots, u_n]$. This last condition is equivalent to saying that Π is a symplectic matrix. Let us restate the problem in block form. We start by rewriting the matrices A and Π as follows,

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \quad \text{and} \quad \Pi = \begin{pmatrix} V_q & U_q \\ V_p & U_p \end{pmatrix}, \tag{A.4}$$

where $A_{i,j}$, V_q, V_p, U_q and U_p are n-by-n matrices. Suppose that V_q and V_p are known which are formed by the v_i's in the obvious way. Thus, finding solutions u_i $i = 1, \ldots, n$ of (A.1) is equivalent to finding matrices U_q and U_p such that,

$$\begin{aligned} \dot{U}_q &= A_{1,1}U_q + A_{1,2}U_p, \\ \dot{U}_p &= A_{2,1}U_q + A_{2,2}U_p, \end{aligned} \tag{A.5}$$

subject to the condition,

$$\begin{pmatrix} V_q & U_q \\ V_p & U_p \end{pmatrix}^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} V_q & U_q \\ V_p & U_p \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{A.6}$$

Since A is non-singular then either $A_{1,2}$ or $A_{2,2}$ is non-singular. By the same reasoning, since $\{v_1, \ldots, v_n\}$ are linear independent, then either V_q or V_p is non-singular. Without lost of generality let us assume that both $A_{1,2}$ and V_q are non-singular matrices. Then the following formulae,

$$U_q = V_qC, \qquad U_p = V_pC + (V_q^{-1})^T, \qquad \dot{C} = V_q^{-1}A_{1,2}(V_q^{-1})^T, \tag{A.7}$$

define matrices U_q and U_p that solve the desired problem. Let us derive the previous formulae. Condition (A.6) is equivalent to,

$$V_q^TV_p = V_p^TV_q, \quad U_q^TU_p = U_p^TU_q \quad \text{and} \quad V_q^TU_p - V_p^TU_q = I. \tag{A.8}$$

Since V_q is invertible, we deduce from the last equality of (A.8) that,

$$U_p = (V_q^{-1})^T + (V_q^{-1})^TV_p^TU_q. \tag{A.9}$$

Substituting the previous expression for U_p into the first equation of (A.5) we get,

$$\begin{aligned}\dot{U}_q &= A_{1,1}U_q + A_{1,2}\left((V_q^{-1})^T + (V_q^{-1})^T V_p^T U_q\right) \\ &= \left(A_{1,1} + A_{1,2}(V_q^{-1})^T V_p^T\right) U_q + A_{1,2}(V_q^{-1})^T.\end{aligned} \tag{A.10}$$

Now the homogeneous equation,

$$\dot{u} = \left(A_{1,1} + A_{1,2}(V_q^{-1})^T V_p^T\right) u,$$

has a fundamental solution V_q. Indeed, since $V_q^T V_p = V_p^T V_q$ and $\dot{V}_q = A_{1,1}V_q + A_{1,2}V_p$ by hypothesis, then

$$\dot{V}_q - \left(A_{1,1} + A_{1,2}(V_q^{-1})^T V_p^T\right) V_q = \dot{V}_q - A_{1,1}V_q - A_{1,2}(V_q^{-1})^T V_q^T V_p = 0.$$

Thus, by the method of variation of constants $U_q = V_q C$ solves equation (A.10) where C satisfies,

$$\dot{C} = V_q^{-1} A_{1,2}(V_q^{-1})^T.$$

Finally, according to equation (A.9) and $V_q^T V_p = V_p^T V_q$ we get,

$$U_p = V_p C + (V_q^{-1})^T. \tag{A.11}$$

Now using the fact that $A_{1,2}$ is symmetric it is not difficult to conclude that $U_q^T U_p = U_p^T U_q$. Consequently, formulae (A.7), V_q and V_p define a symplectic fundamental matrix solution Π of equation (A.1).

Appendix B

Transformation of GSHE to the normal form

In order to normalize H_ϵ up to order 5, we have used the method of Lie series to determine Hamiltonians F_i, $i = 0, \ldots, 4$ which generate a near identity canonical map $\Psi_5 = \Phi^1_{F_0} \circ \Phi^1_{F_1} \circ \Phi^1_{F_2} \circ \Phi^1_{F_3} \circ \Phi^1_{F_4}$ where

$$
\begin{aligned}
F_0 &= \epsilon \left(-\frac{5}{32}\, q_1\, p_1 + \frac{3}{32}\, q_2\, p_2 + \frac{1}{8}\, p_1\, p_2 \right) \\
F_1 &= \frac{7}{216}\, \kappa\, \sqrt{2} {q_1}^2 p_2 + \frac{95}{216}\, \kappa\, \sqrt{2} q_1\, q_2\, p_1 + \frac{17}{72}\, \kappa\, \sqrt{2} q_1\, {p_1}^2 + \frac{5}{36}\, \kappa\, \sqrt{2} q_1\, {p_2}^2 + \\
&\quad \frac{175}{432}\, \kappa\, \sqrt{2} {q_2}^2 p_2 + \frac{1}{36}\, \kappa\, \sqrt{2} q_2\, p_1\, p_2 - \frac{1}{12}\, \kappa\, \sqrt{2} {p_1}^2 p_2 - \frac{1}{18}\, \kappa\, \sqrt{2} {p_2}^3 \\
F_2 &= \left(-\frac{517}{20736}\, \kappa^2 + \frac{29}{512} \right) q_1\, {p_1}^3 + \left(-\frac{217}{20736}\, \kappa^2 + \frac{17}{512} \right) q_1\, p_1\, {p_2}^2 + \\
&\quad \left(\frac{2327}{20736}\, \kappa^2 - \frac{31}{512} \right) q_2\, {p_1}^2 p_2 + \left(-\frac{19}{512} + \frac{2027}{20736}\, \kappa^2 \right) q_2\, {p_2}^3 + \\
&\quad \left(-\frac{5}{128} + \frac{7}{192}\, \kappa^2 \right) {p_1}^3 p_2 + \left(\frac{19}{576}\, \kappa^2 - \frac{3}{128} \right) p_1\, {p_2}^3 \\
F_3 &= \epsilon \left(-\frac{143}{1152}\, \kappa\, \sqrt{2} {p_1}^2 p_2 - \frac{167}{1728}\, \kappa\, \sqrt{2} {p_2}^3 \right) \\
F_4 &= -\frac{2}{1215}\, \sqrt{2} \kappa\, \left(37\, \kappa^2 - 27 \right) {p_2}^5 - \frac{1}{648}\, \sqrt{2} \kappa\, \left(-45 + 52\, \kappa^2 \right) {p_1}^4 p_2 - \\
&\quad \frac{1}{243}\, \sqrt{2} \kappa\, \left(-27 + 34\, \kappa^2 \right) {p_1}^2 {p_2}^3
\end{aligned}
\tag{B.1}
$$

Using an algebraic manipulator it is not difficult to see that Ψ_5 transforms H_ϵ into the desired form.

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