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Philippe A. Naert*
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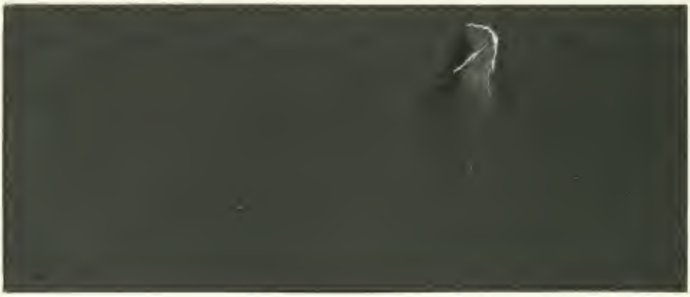
MASSACHUSETTS
INSTITUTE OF TECHNOLOGY
50 MEMORIAL DRIVE
CAMBRIDGE, MASSACHUSETTS 02139



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* Assistant Professor of Management Science, Alfred P. Sloan School of Management, M.I.T.

** Special Doctoral Student, Alfred P. Sloan School of Management, M.I.T.

INTRODUCTION

A logically consistent market share model should predict market shares that are between zero and one, and sum to one.

Few authors have worried about this type of problem in empirical studies, mainly because of the usual interest in a particular brand. It is then implicitly assumed that if predicted market share for that brand is MS_t, the other firms combined will get $(1 - MS_t)$. This may seem pretty obvious, yet it is not. If we were to estimate the market share response functions for the other brands as well, we would often find that the sum of the market shares is not one. The problem becomes particularly apparent to the reader when the response functions for all brands are estimated simultaneously. We will therefore make constant reference to an article by Neil E. Beckwith in a recent issue of JMR [2], in which he reported an application of the joint generalized least squares method (joint GLS) [15] to the estimation of linear market share response functions of various competing brands.

We will first examine the restrictions on the explanatory variables and the parameters which are implied by the market share additivity constraint. The restrictions on the disturbance terms are discussed next. In the final section we conclude that in order for market share functions to be logically consistent, their functional form almost invariably should be intrinsically nonlinear.[1]

The appendix contains some additional comments specific to Beckwith's article.

IMPLICATIONS OF THE ADDITIVITY CONSTRAINT

Zellner's method will usually lead to considerable gains in efficiency of the estimators provided that disturbances of the different equations are contemporaneously highly correlated. ([15], pp. 353-4). This condition is satisfied in the case of market share response functions for various brands competing in an oligopolistic market since, for example, a positive disturbance for one brand in a particular time period implies that at least one other brand will have a negative disturbance in that same period. Thus it would appear that Zellner's method may be profitably applied for joint estimation of a set of market share response functions.

However, Zellner's method does not guarantee that the sum of the market shares predicted from the estimated functions will add up to one. Yet such a constraint is necessary if a logically consistent model is desired. This seems particularly important when the estimated parameters are used to determine the optimal advertising expenditures for the individual brands, as in Beckwith's article.

First we will examine the implications of imposing a sum constraint on the dependent variable in a set of linear functions. We will then apply the results to the specific model proposed by Beckwith. Consider the following linear function (e.g. market share function for brand i):[2]

(1) $$y_{it} = \beta_{i1} X_{i1t} + \beta_{i2} X_{i2t} + \dots + \beta_{ip} X_{ipt}, \text{ for } i=1,\dots,n$$

Let $$X'_{jt} = (X_{1jt}, X_{2jt},\dots,X_{ijt},\dots, X_{njt}), \text{ for } j=1,\dots, p.$$

$$y_t' = (y_{1t}, y_{2t}, \ldots, y_{nt})$$

$$\beta_j' = (\beta_{1j}, \beta_{2j}, \ldots, \beta_{nj}) \text{ , for } j = 1, \ldots, p$$

Let j = 1 correspond to the constant term in the regression, that is,

$$X_{i1t} = 1 \quad \text{for } i = 1, \ldots, n$$
$$t = 1, \ldots, T$$

Finally let u be a nx1 column vector of ones,

$$u' = (1, 1, \ldots, 1)$$

Theorem:[3] The necessary and sufficient conditions for a linear model,

$$y_{it} = \sum_{j=1}^{p} \beta_{ij} X_{ijt},$$ to predict sum-constrained dependent variables, i.e.: $u'y_t = r$, are the following:

1) the explanatory variables should be sum-constrained

   i.e.: $u' X_{jt} = c_j$, for all j;

2) the coefficients, excluding the constant term, should be equal across equations, i.e.: $\beta_{ij} = \beta^j$, for all i and all $j \neq 1$;

3) $u' \beta_1 + \sum_{j=2}^{p} \beta^j c_j = r.$[4]

Sufficient Condition: If $u'X_{jt} = c_j$ for all j, and $\beta_{ij} = \beta^j$ for all i and for j = 2, . . . , p, and if $u'\beta_1 + \sum_{j=2}^{p} \beta^j c_j = r$, it follows that $u'y_t = r$

Proof: First we take the sum of the dependent variables

(2) $$u'y_t = \sum_{j=1}^{p} X'_{jt}\,\beta_j$$

X_{1t} is a vector of ones, and thus $X'_{1t}\,\beta_1 = u'\,\beta_1$. For all j not equal to one β_j is a vector of constants β^j, i.e. $\beta'_j = (\beta^j, \beta^j, \ldots, \beta^j)$, and therefore we can write

$$X'_{jt}\,\beta_j = \beta^j\,u'\,X_{jt}.$$

With $u'X_{jt} = c_j$, we obtain after substitution in (2),

(3) $$u'y_t = u'\beta_1 + \sum_{j=2}^{p} \beta^j c_j.$$

With the right hand side of (3) equal to r, the proof of sufficiency is complete.

Necessary Condition: Suppose now that the dependent variables are sum constrained, i.e. $u'y_t = r$. To be shown is that $u'y_t = r$ implies $u'X_{jt} = c_j$ for all j, $\beta_{ij} = \beta^j$ for all i and for j=2, . . . , p, and $u'\beta_1 + \sum_{j=2}^{p} \beta^j c_j = r$.

Proof: 1) $u'X_{jt} = c_j$ for all j.

Consider two sets of admissible values for the independent variables $\overset{\circ}{X}_{jt}$, and X^*_{jt}, that is

(4) $$r = \sum_{j=1}^{p} \overset{\circ}{X}{}'_{jt}\,\beta_j\ , \text{ and}$$

(5) $$r = \sum_{j=1}^{p} X^{*\prime}_{jt}\,\beta_j.$$

Subtracting (5) from (4), we obtain

$$(6) \qquad 0 = \sum_{j=1}^{p} (X^{\circ\prime}_{jt} - X^{*\prime}_{jt}) \beta_j .$$

Without loss of generality, we can consider the following

$$X^{\circ}_{jt} \neq X^{*}_{jt} \quad \text{for } j = k \neq 1$$

$$X^{\circ}_{jt} = X^{*}_{jt} \quad \text{for } j \neq k.$$

Equation (6) then simplifies to

$$(7) \qquad 0 = \sum_{i=1}^{n} \beta_{ik} (X^{\circ}_{ikt} - X^{*}_{ikt}).$$

Without loss of generality we can assume $\beta_{ik} \neq 0$ only for $i = s$.[5] Suppose now that X_{jt} is not sum constrained. The following could then be admissible X_{kt} vectors

$$X^{\circ}_{ikt} = X^{*}_{ikt} \quad \text{for } i \neq s$$

$$X^{\circ}_{ikt} \neq X^{*}_{ikt} \quad \text{for } i = s.$$

Equation (7) now becomes

$$0 = \beta_{sk} (X^{\circ}_{skt} - X^{*}_{skt}).$$

Since $\beta_{sk} \neq 0$, and $X^{\circ}_{skt} \neq X^{*}_{skt}$, this is a contradiction and hence

$$u'X_{jt} = c_j \quad \text{for } j = 2, \ldots, p.$$

Note that for j=1, $u'X_{jt}$ is by definition constant and equal to n.

2) $\underline{\beta_{ij} = \beta^j \text{ for } j=2, \ldots, p}$

Consider the following vectors for the explanatory variables

$$\overset{\circ}{X}_{jt} \neq \overset{*}{X}_{jt} \quad \text{for } j = k \neq 1$$

$$\overset{\circ}{X}_{jt} = \overset{*}{X}_{jt} \quad \text{for } j \neq k$$

Assume that the $\overset{\circ}{X}_{jt}$ vectors satisfy the sum constraint conditions derived in part (1). Let $\overset{*}{X}_{jt}$ be defined as follows

$$\overset{*}{X}_{ikt} = \overset{\circ}{X}_{ikt} \quad \text{for } i \neq w,v$$

$$(8) \qquad \overset{*}{X}_{wkt} = \overset{\circ}{X}_{wkt} + \Delta, \text{ with } \Delta \neq 0$$

$$\overset{*}{X}_{vkt} = \overset{\circ}{X}_{vkt} - \Delta.$$

Thus, the $\overset{*}{X}_{jt}$ vectors also satisfy the sum constraint conditions. Now assume that

$$\beta_{wk} \neq \beta_{vk}.$$

Substituting (8) into (7) we find

$$0 = \Delta\,(\beta_{vk} - \beta_{wk})$$

With $\Delta \neq 0$, assuming $\beta_{vk} \neq \beta_{wk}$ results in a contradiction. Thus

$$\beta_{ij} = \beta^j \text{ for } j=2, \ldots, p$$

3) $$\underline{u'\beta_1 + \sum_{j=2}^{p} \beta^j c_j = r}$$

Substituting the results of parts (1) and (2) into equation (2), we obtain

$$u'y_t = u'\beta_1 + \sum_{j=2}^{p} \beta^j c_j .$$

Given $u'y_t = r$, the proof is complete.

Let us now apply the theorem to the specific model proposed by Beckwith. Market share, $MS_{i,t}$ is a function of lagged market share, $MS_{i,t-1}$ and advertising share $AS_{i,t}$:

(9) $$MS_{i,t} = \lambda_i \, MS_{i,\,t-1} + \gamma_i \, AS_{i,t} + \varepsilon_{i,t} .$$

The constraint on the dependent variable is

$$\sum_{i=1}^{n} MS_{i,t} = 1.$$

By definition, the explanatory variables are also sum constrained,

$$\sum_{i=1}^{n} MS_{i,t-1} = 1 \qquad \text{and} \qquad \sum_{i=1}^{n} AS_{i,t} = 1.$$

Following the theorem, however, we should also have

$$\lambda_i = \lambda \text{ for all } i,$$

(10) $$\gamma_i = \gamma \text{ for all } i,$$

$$\lambda + \gamma = 1.$$

The conditions on λ and γ are not satisfied in Beckwith's article. For example, considering the IZEF estimates in his Table 2, the range of values for λ_i is from 0.9814 to 1.0068, for γ_i from -0.0030 to 0.0133, and for $\lambda_i + \gamma_i$ from 0.9891 to 1.0136. Clearly, the various λ_i estimates are

$\Phi = \Omega \otimes I$ note however that Ω is singular since if the market shares sum to one, $\sum_{i=1}^{n} \varepsilon_{i,t} = 0$ (see [9], p. 1203); as a result one of the brands observations ought to be deleted from the set, so that the method should be applied to n-1 brands.

RESTRICTIONS ON THE DISTURBANCE TERMS

Market share is a quantity obviously confined to the interval from zero to one. This natural restriction limits the set of distributions capable of describing the behavior of the disturbances (see [9], p. 1205) since $0 \leq MS_{i,t} \leq 1$ implies:

$$-1 \leq -(\lambda \cdot MS_{i,t-1} + \gamma \cdot AS_{i,t}) \leq \varepsilon_{i,t} \leq 1-(\lambda \cdot MS_{i,t-1} + \gamma \cdot AS_{i,t}) \leq 1$$

which indicates that the normality assumption is irrelevant.

As pointed out by Theil ([13], pp. 629 et seq.), this aspect is awkward in regression-type situations and therefore a monotonic transformation is usually applied to this particular kind of dependent variable so that the new dependent variable constructed is then defined over the $[-\infty, +\infty]$ range. In fact, many transformations have this property. The logit is one of them and in our case the first step amounts to defining a new variable

$$m_{i,t} = \frac{MS_{i,t}}{1-MS_{i,t}},$$

a quantity with range $[0, \infty]$ since when $MS_{i,t} = 0$, $m_{i,t} = 0$ and when $MS_{i,t} = 1$, $m_{i,t} = \frac{1}{0} = \infty$.

The log of $m_{i,t}$ is then defined over the $[-\infty, +\infty]$ - interval since

(log 0) = $-\infty$ and (log $+\infty$) = $+\infty$. Adjusting the original model accordingly we obtain the equation to estimate. In Beckwith's case we come up with:

$$\log\left[\frac{MS_{i,t}}{1-MS_{i,t}}\right] = \log\left[\frac{\lambda \cdot MS_{i,t-1} + \gamma \cdot AS_{i,t}}{1-(\lambda \cdot MS_{i,t-1} + \gamma \cdot AS_{i,t})}\right]$$

or

$$\log\left[\frac{MS_{i,t}}{1-MS_{i,t}}\right] = \log(\lambda \cdot MS_{i,t} + \gamma \cdot AS_{i,t}) - \log(1-\lambda MS_{i,t-1} - \gamma \cdot AS_{i,t})$$

which has now an intrinsically nonlinear form.

Another example of the same transformation is presented in the next section.

INTRINSICALLY NONLINEAR FORMS: A NECESSITY ?

Now one might argue that the requirements $\lambda_i = \lambda$ and $\gamma_i = \gamma$ for all i, are not very appealing. Indeed they do not allow for differentiation between brands, in the way market shares respond to advertising decisions. We come then to the conclusion that if we want a logically consistent model which allows for differences in the response parameters between the various brands, the model structure should simply be nonlinear.

We may wonder at this point whether such a structure may easily be defined. Usually when market share functions are nonlinear, they are of such a form as to become linear upon transformation. For example, multiplicative functions are widely used in empirical work. The multiplicative equivalent of the expected value of Beckwith's market share function is,

(11) $$MS_{i,t} = MS_{i,t-1}^{\lambda_i} \; AS_{i,t}^{\gamma_i}.$$

For a multiplicative response function such as (11) it is not possible to determine meaningful restrictions on the explanatory variables and on the parameters, which will guarantee that market shares add up to one.[6] More complex functions are generally needed.

For example, with advertising and lagged market share as the only explanatory variables, let

$$(12) \qquad MS_{i,t} = \lambda\, MS_{i,t-1} + (1-\lambda) \frac{A_{i,t}^{\gamma_i}}{\sum_{k=1}^{n} A_{k,t}^{\gamma_k}},$$

where $A_{i,t}$ is brand i's advertising expenditures. It is readily seen that the sum of the market shares as defined in (12) is one, and this without specific constraints on the γ_i parameters. Unfortunately such formulations have some problems of their own. Determining the values of the parameters is less straightforward than in the linear case, although many nonlinear programming procedures are now available which make nonlinear estimation much less of a problem. However, the statistical properties of nonlinear estimators are much weaker. These are probably the main reasons why in empirical work one has usually avoided intrinsically nonlinear functional forms. Only a few applications of nonlinear estimation techniques were reported in the marketing literature, among them: Glen Urban's product line study [14] and Kuehn, Mcguire and Weiss' estimation of Kuehn's model [6].

Some readers may now ask themselves whether it is possible to design any logically consistent market share model which may be linearized. The answer is Yes but . . . Rather "heroic" assumptions have to be made and non trivial transformations have to be devised.

Let us illustrate this point further. Suppose the consumption pattern on a two-brand market may be described by a Markov-type matrix:

$$\tau \begin{bmatrix} \lambda_i & \sigma_i \\ \sigma_c & \lambda_c \end{bmatrix}^{\tau+1}$$

where λ_i is the proportion of consumers loyal to brand i,

σ_c is the proportion of consumers switching from brand c to brand i;

λ_c and σ_i are defined similarly but refer to brand c.

We can define the impact of the competitive forces on the consumption habits by making λ_i, σ_i, λ_c and σ_c explicit functions of the two brands marketing-mix. Considering only the effect of advertising (A), we could postulate:

$$\lambda_{i,t} = 1-e^{-\alpha_i A^r_{i,t}} \qquad \sigma_{c,t} = e^{-\alpha_c A^r_{c,t}}$$

(13)

$$\sigma_{i,t} = e^{-\alpha_i A^r_{i,t}} \qquad \lambda_{c,t} = 1-e^{-\alpha_c A^r_{c,t}}$$

where: $A^r_{i,t} = A_{i,t}/A_{c,t}$ and $A^r_{c,t} = A_{c,t}/A_{i,t}$

from which we deduce the nonlinear market share equation:

$$(14) \quad E\left[MS_{i,t} \mid A_{i,t}, A_{c,t}\right] = \left[1-(e^{-\alpha_i A^r_{i,t}} + e^{-\alpha_c A^r_{c,t}})\right] \cdot MS_{i,t-1} + e^{-\alpha_c A_{c,t}}.$$

and at equilibrium, assuming no change in the brands advertising budgets until equilibrium is reached, then:

$$(15)\quad E\left[MS_{i,e} \mid A_i,\ A_c\right] = \frac{\sigma_c}{1-\lambda_i + \sigma_c} = \frac{e^{-\alpha_c A_c^r}}{e^{-\alpha_i A_i^r} + e^{-\alpha_c A_c^r}}$$

If we are willing to assume that consumption patterns are sufficiently stable, which is very often the case when the primary demand reaches the maturity stage and when product innovations do not occur, and provided that the unit time period is sufficiently long, then the equilibrium may be achieved within period t and we may retain equation (15)[7]. Applying the logit transformation to it we obtain:

$$\log\left[\frac{MS_{i,t}}{1-MS_{i,t}}\right] = \log\left[\frac{e^{-\alpha_c A_{c,t}^r}}{e^{-\alpha_i A_{i,t}^r}}\right] = \log\left(e^{-\alpha_c A_{c,t}^r}\right) - \log\left(e^{-\alpha_i A_{i,t}^r}\right)$$

which means that we are now able to apply the OLS technique to

$$(16)\quad \log\left[\frac{MS_{i,t}}{1-MS_{i,t}}\right] = \alpha_i A_{i,t}^r - \alpha_c A_{c,t}^r + \varepsilon_{i,t}$$

The error term $\varepsilon_{i,t}$ is thus defined over the range $[-\infty, +\infty]$ and if we consider $MS_{i,t}$ as a binomial frequency[8] then $\varepsilon_{i,t}$ is asymptotically normally distributed ([13], p. 636).

It should be clear that the linearization of equation (15) is critically conditional upon the exponential specification chosen for λ_i, σ_i, λ_c and σ_c. It should also be obvious that a lot of very restrictive assumptions have to be made in order to select equation (15) instead of the intrinsically nonlinear form (14).

CONCLUSION

In this paper we derived restrictions on explanatory variables, parameters, and disturbances implied by an additivity constraint on the dependent variable, for example, market share. These restrictions are such that linear market share structures do not allow for differences in the response parameters for various brands. We argued that to be logically consistent market share models should generally be intrinsically nonlinear.

This does not mean, however, that in future empirical work, intrinsically linear market share functions should no longer be used. We should recognize the fact that linear models are easier and less expensive to estimate, the statistical results are believed to be more straightforward to interpret, and when used as aids in decision making, optimal allocation rules are simpler to derive and to apply. Predicted market share values from an intrinsically linear model may often provide sufficiently close approximations. Moreover, when estimated parameters are used as prior estimates, which are subsequently adjusted by managerial judgment, such an approximation is perhaps even more acceptable. For an interesting discussion on these issues we refer to Lambin [7, pp. 120-21].

APPENDIX

MISCELLANEOUS REMARKS ON BECKWITH'S APPROACH

Suppose the linear market share model were acceptable, i.e. the constraints defined above are satisfied, other specification tests are still to be performed. Although Beckwith does not completely neglect this aspect, the tests he carried through ([2], p. 171) are not very relevant.

Including additional lagged dependent variables ($MS_{i,t-2}$ and $MS_{i,t-3}$) should not throw much light on the model specification. On the contrary, examining alternative lag distributions, e.g. the Pascal distribution ([13], pp. 264-267), or checking whether the geometric decline starts at the first lag or at the k-th one are questions to be dealt with seriously. Thus, for example, when the time period is short-one month, in Beckwith's case - it may be unrealistic to assume that the successive advertising coefficients decline immediately.[9]

Moreover Beckwith does not pay any attention to the problem of autocorrelation which may prevail among the disturbances of each brand equation and may indicate misspecification or measurement errors in the variables (see [8], p. 504). In a working paper upon which Beckwith's article is based, however, the residuals of the brand market-share model are tabulated and the Durbin-Watson statistics d computed (see [1], p. 23). The latter are of course biased due to the presence of the lagged dependent variable on the right-hand side of the equations.[10] But provided that these statistics were adjusted for gaps in the data set[11], we can use them to derive the new DURBIN statistic h, which in the case of independence of the disturbances is asymptotically distributed as a standardized normal variate (see [3], p. 419).

$$h = a \cdot \left(\frac{n}{1 - n.\hat{V}(\lambda)} \right)^{1/2}$$

in which a is the autocorrelation coefficient of residuals,

n is the number of observations,

and $\hat{V}(\lambda)$ is the estimate of the variance of the lagged dependent variable coefficient given by least-squares analysis.

The h statistics computed on Beckwith's IZEF brand market-share model[12] are reported in Table 1. Brand A and B's h statistics are greater than the critical 10% value and may indicate autocorrelation.

TABLE 1

DURBIN STATISTICS

Brand	a	h
A	-0.235	-1.3904
B	0.260	1.5380
C	0.135	0.7989
D	-0.085	-0.5028
E	-0.170	-1.0059

*Critical values at the 10% and 5% levels, under H_o: a=o, are 1.282 and 1.645, respectively.

One should be careful, however, in using the h statistics for testing autocorrelation of disturbances in a Koyck model. In fact, the h statistics tests the non-autocorrelation against the alternative hypothesis of a first-order Markov dependence scheme. If we now assume that disturbances ($e_{i,t}$) in the original model:

$$MS_{i,t} = \gamma \cdot \sum_{\tau=o}^{\infty} \lambda^{\tau} AS_{i,t-\tau} + e_{i,t}$$

are independent,when we apply Koyck's transformation the resulting disturbances ($\varepsilon_{i,t}$) get correlated and their autocorrelation scheme does not follow a first-order Markov process, since $\varepsilon_{i,t} = e_{i,t} - \lambda \cdot e_{i,t-1}$, in which case the h statistics is irrelevant.

Griliches ([4], pp. 41-42) liscusses an instrumental variable approach which does not depend on a particular structure of the disturbances and should yield a consistent estimator of λ. Applied to Beckwith's model this two-stage approach amounts to first estimate the form:

$$(17) \quad MS_{i,t} = b_1 \cdot AS_{i,t} + b_2 \cdot AS_{i,t-1} + b_3 \cdot AS_{i,t-2} + \dots + v_{i,t}.$$

including additional lagged $AS_{i,t-\tau}$ terms as long as they contribute to the explanation of the market share, and then re-estimate the usual Koyck-transformed equation in which $\widehat{MS}_{i,t-1}$ predicted by (17) has been substituted for $MS_{i,t-1}$, that is

$$(18) \quad MS_{i,t} = \lambda \cdot \widehat{MS}_{i,t-1} + \gamma \cdot AS_{i,t} + \varepsilon_{i,t} + \lambda \cdot v_{i,t-1}$$

The estimated λ and γ are then consistent provided that $AS_{i,t}$ is a true exogenous variable (at least with respect to $MS_{i,t}$).

The above discussion indicates that Beckwith has paid little attention to the problems of model specification, disturbance autocorrelation and estimates consistency.[13]

On the other hand, he focused considerably on the testing for normality of the errors distribution, while as he stated it himself this assumption has only a minor importance: ". . . Most of the properties of this estimator (IZEF) are not dependent upon assumed normality" (see [2], p. 169). Furthermore as we noted it earlier, testing the normality assumptions is irrelevant in this case.

More surprising is his testing of residuals zero mean (see [2], p. 171), since it is not true, in general, that:

either the OLS residuals have zero sum when the equation contains no constant term (see [13], p. 40);

or the GLS and joint GLS residuals have zero sum even when the equation contains a constant term, due to the use of the covariance matrix of disturbances in the estimation method (see [13], p. 239)

The market-share model specification and estimation is not the only matter of concern for us in Beckwith's paper. The normative inferences he deduced are also subject to caution.

The author omitted to mention that the optimal advertising level he proposed (see [2], p. 173) was derived under the following restrictive assumptions:

1) Cournot behavior of the decision-makers, i.e. each firm determines its optimal level regardless of the possible reactions of its competitors:

$$\frac{dA^o_{i,t}}{dA^*_{i,t}} = 0$$

where $A^o_{i,t}$ is the amount spent on advertising by brand i's competitors.

2) Stable primary demand, i.e. M = a constant, while in fact Beckwith himself wrote: "Consumption was seasonal, generally increasing over the period studied, and believed to be constant per capita." (see[2], p. 169) -so that in deriving the optimal level for advertising expenditures the fact that M changes with the increase in the number of potential consumers should be taken into account. So for example, if the potential market grows at a rate r, a month, the industry demand can be represented by:

$$M_\alpha = M_t (1 + r)^{\alpha - t}$$

and the derived optimal advertising is:

$$A^*_{i,t} = \text{MAX}\left[o, \left[\frac{M_t \gamma_i m_i (1 + d_i)}{(1 + d_i) - (1 + r)\lambda_i} A^o_{i,t} \right]^{1/2} - A^o_{i,t} \right]$$

where d_i is the monthly cost of capital and m_i the incremental unit profit margin.

Needless to say that the constraint $(1 + r) \lambda_i < (1+d_i)$ is to be met if the above formula is to make sense. Given Beckwith's estimates and data this amounts to:

$$r < \frac{1 + d_i}{\lambda_i} - 1$$

using the largest estimate of λ_i, i.e.: 0.9979 and given $d_i = 0.008$, r should be less than 0.010121.

Footnotes

[1]By intrinsically nonlinear forms we mean equations which cannot be linearized.

[2]In fact, we should add a disturbance term $\varepsilon_{i,t}$. Since we will be dealing with expectations $\varepsilon_{i,t}$ will be deleted.

[3]A similar theorem was derived by Schmalensee ([12], pp. 47-49). In his demonstration, he makes use of the properties of the row kernel of the matrix. Our proof is more concise and generalizes Schmalensee's theorem.

[4]Note that r could be time-dependent, that is, the constraint on y_t is $u'y_t = r_t$. For example, if the dependent variable were brand sales, $q_{i,t}$, their sum should add up to industry sales, Q_t, which may be time dependent. In the formulation of the theorem the c_j are then replaced by $c_{j,t}$. The proof is not affected except that r becomes r_t and c_j becomes $c_{j,t}$.

[5]If β_{ik} were zero for all i and k, none of the X's would be explanatory variables. If at least one of the X's is a true explanatory variable, s and k can always be chosen such that $\beta_{sk} \neq o$.

[6]For example in (11), by definition $\sum_{i=1}^{n} MS_{i,t-1} = 1$, and $\sum_{i=1}^{n} AS_{i,t} = 1$. With $\lambda_i = o$, and $\gamma_i = 1$ for all i, the sum of the expected market shares is one. However, these restrictions are, at least a priori, not very meaningful.

[7]Provided of course that no change in the advertising strategy of both firms occurs during period t. Similar assumptions were implicitly made by Hartung and Fisher [5]. We are also presently investigating similar forms as well as nonlinear structures applied to the appraisal of distribution network performance [11] .

[8]An acceptable assumption since we consider the $MS_{i,t}$ as conditional equilibrium market shares.

[9]For a study of some of these problems, see [10].

[10]The residuals are less correlated than the disturbances and therefore "it is not possible to make a correct assessment of the correlation of the errors using the autocorrelation coefficients of the residuals" (see [8], p. 558).

[11]Since Beckwith eliminated 13 observations corresponding to "unusual transients" (see [2], p 170), 13 gaps should be considered in the computation of the Durbin-Watson statistic d,

$$d = \sum_{g=1}^{13} \left[\frac{\sum_{\tau = t_g}^{T_g - 1} (e_{\tau+1} - e_\tau)^2}{\sum_{\tau = t_g}^{T_g} e_\tau^2} \right]$$

where $[t_g, T_g]$ delimits the observations between gaps; the autocorrelation coefficient is then deduced via:

$$a \cong 1 - \frac{d}{2} .$$

[12]Although the residuals of the equations estimated separately by OLS are not available we can use the IZEF residuals since we are only interested in their degree of autocorrelation (while the joint GLS method only adjusts for contemporaneous covariances across equations).

[13]"Correlation of the errors may greatly bias estimation of the parameters of an autoregressive model, when this estimation is carried out by the method of least-squares" ([8], p. 558).

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