Particle Accelerators
1973, Vol. 5, pp. 227–235

A FLUX THEOREM FOR THE DESIGN OF MAGNET COIL ENDS†

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Analytical expressions are developed for the Fourier expansion coefficients integrated along the axis of the field of iron-free multipole magnets, or magnets having an infinitely permeable concentric circular shield. The method makes use of the mutual inductance of the magnet and an idealized, infinitely-long cos $n\theta$ current sheet concentric with and interior to the magnet. The expressions obtained are used to verify a previously-known ideal end structure and also to compute the harmonics in a real magnet end on which measurements have been made.

1. INTRODUCTION

The purpose of this paper is to develop analytical means to calculate the end effects of essentially two-dimensional accelerator and beam handling magnets with three-dimensional end coil distributions. In fact, means are developed to deal with this problem in an empirical way as well. The result is really an elaboration of a reciprocal theorem proved by Robinson[1] for use in the design of magnetic pickup electrodes.

The problem has received attention at many accelerator centers, notably by Lambertson,[1] Meuser,[2] and Green[3] at the Lawrence Berkeley Laboratory, and Coupland[4] at the Rutherford Laboratory, but the present treatment is developed in a different though presumably equivalent way.

We are interested in calculating the quantities

$$q_n = \frac{1}{n!}\int_{-\infty}^{\infty} \frac{\partial^{n-1}}{\partial x^{n-1}} B_y(s)\bigg|_{x=0} \mathrm{d}s, \qquad (1)$$

where the q_n are $1/n$ of the Taylor expansion coefficients of the integral value of vertical field component B_y along the s-axis, x is distance on the median plane from the magnet axis, and s is the distance along the magnet axis. MKS units are used throughout. (We are not interested here in 'rotated' multipoles arising from midplane antisymmetric coils, although the development includes them up to a point.) Our concern with the q_n alone is equivalent to the assumption that the effect of the field on a traversing particle is equivalent to that of a field represented by a single axial component of vector potential acting over a finite distance. This is strictly true only if the particle does not change radial or azimuthal position while traversing the field, i.e., the particle momentum is infinitely great

An integral expression for the q_n in terms of magnet coil parameters is obtained as follows: (1) We define a hypothetical coil of infinite length and infinitesimal thickness which produces a perfect $2n$-pole field. The field of this 'linear n-pole' is obtained in terms of its parameters. (2) We show that the flux produced by the accelerator magnet which links the linear n-pole is expressible in terms of the q_n defined above. (3) We calculate the flux produced by the linear n-pole which links the accelerator magnet coil. Since the mutual inductance theorem states that these two flux linkages are equal when the coil currents are equal, the q_n are thus calculated from magnet coil parameters. To the extent that a coil representing the ideal linear n-pole can be manufactured, the result (2) expresses the q_n in terms of coil parameters and measured linear n-pole flux change.

The integral expression obtained is true for any three-dimensional coil configuration, with or without an infinite-μ circular iron shield, but in the present paper it is applied only to infinitesimally thin coils lying on a circular cylinder, with or without a concentric iron shield.

Although the formalism developed strictly applies only to complete coils, it will be seen that by the artifice of putting in ideal 'bedstead ends' in transition planes, the ends may be calculated independently of the main or s-independent sections of the windings.

†Work done under the auspices of the U.S. Atomic Energy Commission.

2. LINEAR n-POLES

We choose a coil radius a, less than the radius of the accelerator magnet coil, and an angular current density $\mathrm{d}i_s/\mathrm{d}\theta = IN_0 \cos n\theta$. There are thus (N_0/a) $\cos n\theta$ turns per unit length of circumference, each carrying current I. The scalar potential is (subscripted 'a' is a coefficient, not the radius)

$$\begin{aligned}\psi_n &= a_n r^n \sin n\theta \qquad r \leq a \\ &= b_n r^{-n} \sin n\theta \qquad r \geq a.\end{aligned}$$

Using $B = -\nabla\psi$ and applying boundary conditions at $r = a$ yields

$$a_n = \mu_0 IN_0/2na^n, \quad b_n = -\mu_0 N_0 Ia^n/2n.$$

If a circular cylindrical iron shell of radius R and infinite permeability concentric with the linear n-pole surrounds it, the potential in the region $a \leq r \leq R$ has two terms, one in r^{-n} as before, the other in r^n. To obtain the coefficients, it is convenient to use the easily verified fact that a current filament at radius a is imaged at R^2/a (if the current return is on the axis). The coefficient of the new term is then just the a_n with a replaced by R^2/a, giving

$$\psi_n = \frac{\mu_0 N_0 Ia^n}{2n} \sin n\theta \left(\frac{r^n}{R^{2n}} - \frac{1}{r^n}\right) \tag{2}$$

which satisfies the boundary condition $\psi_n = 0$ at $r = R$. Then

$$B_r = -\frac{\mu_0 N_0 Ia^n}{2} \sin n\theta \left(\frac{r^{n-1}}{R^{2n}} + \frac{1}{r^{n+1}}\right) \tag{3a}$$

$$B_\theta = -\frac{\mu_0 N_0 Ia^n}{2} \cos n\theta \left(\frac{r^{n-1}}{R^{2n}} - \frac{1}{r^{n+1}}\right). \tag{3b}$$

The equivalent boundary condition, $B_\theta = 0$ at $r = R$ is also satisfied. The interior potential, though not needed in the present development, from the continuity in B_r at $r = a$, simply increases in the ratio $1 + (a/R)^{2n}$, i.e., a factor of 2 if $R = a$.

3. THE INTERIOR FIELD OF THE ACCELERATOR MAGNET, AND FLUX LINKING THE LINEAR n-POLE

The interior field of the magnet is obtainable from a scalar potential $\varphi(r, \theta, s)$ via $B = -\nabla\varphi$, and φ satisfies Laplace's equation. Owing to the finite extent of the coil, B is zero as s approaches $\pm\infty$. The potential φ is defined to be zero as $s \to \pm\infty$. Let

$$E(r, \theta) = \int_{-\infty}^{\infty} \varphi \, \mathrm{d}s.$$

Then

$$\frac{\partial E}{\partial r} = \int_{-\infty}^{\infty} \frac{\partial \varphi}{\partial r} \mathrm{d}s = -\int_{-\infty}^{\infty} B_r \, \mathrm{d}s,$$

and

$$\frac{1}{r}\frac{\partial E}{\partial \theta} = -\int_{-\infty}^{\infty} B_\theta \, \mathrm{d}s.$$

The validity of differentiation under the integral is assumed, though proof is obvious if the variables are separable. Then E satisfies the two-dimensional Laplacian,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E}{\partial \theta^2} = 0$$

as can be shown by direct substitution:

$$\begin{aligned}&\frac{1}{r}\frac{\partial}{\partial r}\left(r\int_{-\infty}^{\infty}\frac{\partial\varphi}{\partial r}\mathrm{d}s\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\int_{-\infty}^{\infty}\varphi\,\mathrm{d}s \\ &= \int_{-\infty}^{\infty}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial s^2}\right]\mathrm{d}s \\ &\quad - \int_{-\infty}^{\infty}\frac{\partial^2\varphi}{\partial s^2}\mathrm{d}s.\end{aligned}$$

This is zero since the integrand of the first integral on the right is everywhere zero and the second integral is $B_s|_{-\infty}^{\infty} = 0$. The solution to $\nabla^2 E = 0$ near the origin in two dimensions is

$$E = \sum q_n r^n \sin n\theta + p_n r^n \cos n\theta.$$

The field is also obtainable via $B = \nabla \times A$ from a vector potential A which can be assumed zero at $\pm\infty$. The flux enclosed by a closed path c is given by

$$F_c = \int_S\int B \cdot \mathrm{d}S = \int_S\int (\nabla \times A) \cdot \mathrm{d}S = \oint_c A \cdot \mathrm{d}c,$$

where S is the surface which has c as its periphery. The sense of $\mathrm{d}S$ is positive when S lies on the left as c is traversed. For c choose two lines at (r_1, θ_1) and (r_2, θ_2) parallel to the s-axis, with crossover at sufficiently large $\pm s$ that A is zero. The flux between them is

$$F_{12} = \int_{-\infty}^{\infty} A_s(r_2, \theta_2)\, ds - \int_{-\infty}^{\infty} A_s(r_1, \theta_1)\, ds,$$

where A_s is the axial component of A. Either of these integrals along a line at (r, θ) is a two-dimensional potential since it is independent of path. We define such an integral as $F(r, \theta)$ and further observe that

$$\int B_r\, ds = \int (\nabla \times A)_r\, ds = \int \left(\frac{1}{r}\frac{\partial A_s}{\partial \theta} - \frac{\partial A_\theta}{\partial s}\right) ds$$
$$= \frac{1}{r}\frac{\partial}{\partial \theta}\int A_s\, ds - A_\theta \Big|_{-\infty}^{\infty} = \frac{1}{r}\frac{\partial F}{\partial \theta}$$

and

$$\int B_\theta\, ds = \int \left(\frac{\partial A_r}{\partial s} - \frac{\partial A_s}{\partial r}\right) ds$$
$$= A_r \Big|_{-\infty}^{\infty} - \frac{\partial}{\partial r}\int A_s\, ds = -\frac{\partial F}{\partial r}.$$

The same assumption as to differentiability under the integral is made.

But then

$$\frac{1}{r}\frac{\partial F}{\partial \theta} = -\frac{\partial E}{\partial r} \tag{4a}$$

and

$$\frac{\partial F}{\partial r} = \frac{1}{r}\frac{\partial E}{\partial \theta}, \tag{4b}$$

which are the Cauchy–Riemann equations in polar coordinates. Accordingly, E and F are conjugate functions and $\Phi = F + iE$ is an analytic function of $z = x + iy$. The solution for F is

$$\Sigma(q_n r^n \cos n\theta - p_n r^n \sin n\theta), \tag{5a}$$

and

$$\Phi = \Sigma(q_n + ip_n) z^n. \tag{5b}$$

From $A = (\mu_0/4\pi)\int J\, dv/r$ (Green's solution to Poisson's equation), A_s has the same symmetry properties as J_s, so if J_s has midplane symmetry, so does A_s and the p_n are all zero.

With the mathematics reduced to two-dimensional[5] form, a great deal of prior work is immediately applicable.[6–9] Although the reduction to two dimensions was done for the case of fields interior to a coil array, the treatment also applies to exterior fields in iron and current-free space. The same solutions are obtained, but with n negative. Following Beth,[8] we denote by G the complex integrated field:

$$G = G_x + iG_y,$$

where

$$G_x = \int_{-\infty}^{\infty} B_x\, ds \quad \text{and} \quad G_y = \int_{-\infty}^{\infty} B_y\, ds.$$

Then

$$G = \frac{\partial \Phi}{\partial z} = \sum n(q_n + ip_n) z^{n-1}$$

and

$$\frac{\partial^{n-1} G}{\partial z^{n-1}}\bigg|_{z=0} = n!\,(q_n + ip_n)$$

i.e., since

$$\frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{d}{dz} = \frac{d}{dz}$$

this is equivalent to

$$\left(\frac{\partial^{n-1}}{\partial x^{n-1}}\int_{-\infty}^{\infty} B_y\right)_{r=0} ds = n!\, q_n$$

and

$$\left(\frac{\partial^{n-1}}{\partial x^{n-1}}\int_{-\infty}^{\infty} B_x\right)_{r=0} ds = n!\, p_n$$

which justifies the statement made in the introduction.

The flux linked by a linear n-pole of radius a interior to the accelerator magnet is easily calculated using the flux potential F obtained above. The fact that a wire and its return constitute a closed loop enclosing flux is not needed—the specification of angular turns density by $N_0 \cos n\theta$ includes the winding direction sense; a $(-)$ sign indicates the wire can be interpreted as a return. The flux linkage is then

$$F_n = \int_{-\pi}^{\pi} F N_0 \cos n\theta\, d\theta$$
$$= \int_{-\pi}^{\pi} N_0 \sum_m (q_m a^m \cos m\theta - p_m a^m \sin m\theta) \cos n\theta\, d\theta$$
$$= \pi N_0\, q_n\, a^n. \tag{6}$$

To the extent that one can construct a perfect n-pole, this expression is useful for measurements of harmonic content by the mutual inductance method.

4. FLUX FROM THE INTERIOR LINEAR n-POLE LINKED BY THE ACCELERATOR MAGNET

The major problem in characterizing the magnet ends is that of an analytic representation for the ends. This is somewhat simplified by the present procedure since the field produced by a linear n-pole has no s-component, making the flux linkage calculation easier. However, windings with both radial and azimuthal variation can introduce intractable problems of integration, making analytic results difficult to obtain. The general flux linkage expressions presented will be applied analytically only to the simple and physically realizable case of 'smooth' ends, defined as those which are developed in the same circular cylinder of radius b as the main parallel winding. An important exception to this is the ideal bedstead end in which all the crossovers lie in a plane perpendicular to the s-axis and which has no flux linkage with a linear n-pole. Such ends are difficult to realize with real conductors since a mathematical right-angle bend can only be approached in practice. The importance of this end lies in its conceptual use as a termination to either the parallel windings or a more physically realistic end, thus allowing one to compute ends separately from the main winding.

For each element of current we assign a parameter λ which, perhaps with other constants, completely characterizes the shape of the coil end. Then we specify the end coil shape by a single parameter family of space curves defined as the intersection of two surfaces $G_1(\lambda, r, \theta, s = 0)$ and $G_2(\lambda, r, \theta, s) = 0$. A simple example of this is illustrated in Figure 1, in which $G_2 = 0$ is a circular cylinder and $G_1 = 0$ consists of a family of pairs of planes, one inclined to the s-axis, the other parallel to it.

The flux linked by an element ('end turn') specified by λ, due to the linear n-pole is

$$f_{n,\lambda} = \iint_{S_\lambda} B \cdot \mathrm{d}S.$$

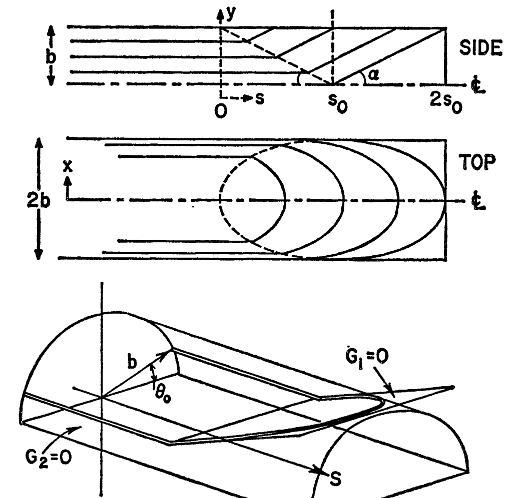

FIGURE 1 The upper half winding of a dipole magnet end, lying in a circular cylinder, which develops no higher harmonics.

The total flux linkage is

$$F_n' = \int_\lambda N(\lambda) f_{n,\lambda} \,\mathrm{d}\lambda.$$

Two ways of developing $f_{n,\lambda}$ are possible: the first, as a line integral around S, appears simpler if there is a radial variation. The second, as an area integral, was the first tried in the present study, and appears to give somewhat easier integrals in the important case of $r = b$. The first development proceeds as follows: As in Section 3,

$$f_{n,\lambda} = \iint_{S_\lambda} B \cdot \mathrm{d}S = \int_{c_\lambda} A_s \,\mathrm{d}s,$$

where c_λ is the space curve defined by $G_1 = 0$ and $G_2 = 0$. The Cauchy–Riemann equations can be used to transform the scalar potential of the linear n-pole into the vector potential, giving

$$A_s = \frac{\mu_0 N_0 I a^n \cos n\theta}{2n}\left(\frac{r^n}{R^{2n}} + \frac{1}{r^n}\right).$$

Then

$$F_n' = \int_\lambda N(\lambda)\,\mathrm{d}\lambda \int_{c_\lambda} \frac{\mu_0 N_0 I a^n \cos n\theta}{2n}\left(\frac{r^n}{R^{2n}} + \frac{1}{r^n}\right)\mathrm{d}s.$$

By the mutual inductance theorem, $F_n' = F_n$ if the currents are equal, and $F_n = \pi N_0 q_n a^n$. Hence

$$q_n = \frac{\mu_0 I}{2\pi n}\int_\lambda N(\lambda)\,\mathrm{d}\lambda \int_{c\lambda} \cos n\theta\left(\frac{r^n}{R^{2n}}+\frac{1}{r^n}\right)\mathrm{d}s; \quad (7)$$

with G_1 and G_2 specified the q_n can now be computed. Notice that the properties of the linear n-pole, namely N_0 and a, do not appear.

The second development for $f_{n,\lambda}$ is as follows:

$$\iint_{S_\lambda} B\cdot \mathrm{d}S = \iint_{S_\lambda} B_r\, r\,\mathrm{d}\theta\,\mathrm{d}s + \iint_{S_\lambda} B_\theta\,\mathrm{d}r\,\mathrm{d}s$$
$$= \int_{S_\lambda} B_r\, rs\,\mathrm{d}\theta + \int_{S_\lambda} B_\theta\, s\,\mathrm{d}r.$$

Using the expressions for B_r and B_θ obtained in Section 2,

$$q_n = \frac{\mu_0 I}{2\pi}\int N(\lambda)\mathrm{d}\lambda\left[-\int \sin n\theta\left(\frac{r^n}{R^{2n}}+\frac{1}{r^n}\right)s\,\mathrm{d}\theta + \int \cos n\theta\left(\frac{r^{n-1}}{R^{2n}}-\frac{1}{r^{n+1}}\right)s\,\mathrm{d}r\right], \quad (8)$$

again with $G_1=0$ and $G_2=0$. If $G_2(r,\theta,s,\lambda)=0$ becomes $r-b=0$, the second integral is zero (i.e., no azimuthal flux is intercepted by the winding). In this case, an especially convenient choice for λ is the value θ_0 at which a current element lies in the 'main parallel winding' before making the transition to 'end'. The distribution in θ_0 is then dictated by the function of the magnet in question, i.e., $\cos n\theta_0$ if an ideal n-pole, or perhaps a step-function approximation to $\cos n\theta_0$. With this choice, and $G_1(\theta,s,\theta_0)\cdot = 0$

$$q_n = -\frac{\mu_0 I}{2\pi}\left(\frac{b^n}{R^{2n}}+\frac{1}{b^n}\right)\int_{\theta_0} N(\theta_0)\,\mathrm{d}\theta_0 \int_{S_{\theta_0}} \sin n\theta\, s\,\mathrm{d}\theta. \quad (9)$$

The specification of $G_1(\theta,s,\theta_0)=0$ is a compromise between mechanical engineering and field shaping. Some reasonable engineering demands are that there be no discontinuities in $\mathrm{d}s/\mathrm{d}\theta$ at $\theta=\theta_0$ (no sharp bends) and

$$\left.\frac{\mathrm{d}s}{\mathrm{d}N}\right|_{\theta=\pi/2n} \geqq \left.\frac{\mathrm{d}p}{\mathrm{d}N}\right|_{s=0}, \quad (10)$$

where $p=r\theta$ is distance in the azimuthal direction, i.e., that the coils at the end be packed no tighter than on the sides.

5. INTEGRAL LIMITS AND THE EVALUATION OF THE q_n WITH SOME SYMMETRIC ANALYTIC FORMS FOR G_1 AND G_2

We consider a magnet winding on a circular cylinder of radius b required to produce a field with $2m$-pole symmetry, that is, the winding sense is the same as $\cos m\theta$: + for 'go' and − for 'return', i.e., the sense reverses at alternate poles. The limits in the integral in Eq. (9) then become

$$q_n = -Q_n \sum_{i=1}^{2m}(-1)^i \int_{\pi(i-1)/m}^{\pi(2i-1)/2m} N(\theta_0)\,\mathrm{d}\theta_0 \times \int_{\pi(i-1)/m+\theta_0}^{\pi i/m-\theta_0} s(\theta,\theta_0)\sin n\theta\,\mathrm{d}\theta,$$

where

$$Q_n = \frac{\mu_0 I}{2\pi}\left(\frac{b^n}{R^{2n}}+\frac{1}{b^n}\right).$$

This can be simplified by arguments based on symmetry. First, $s(\theta,\theta_0)$ is symmetric in θ around $(2j-1)\pi/2m$ by construction. Second, if $n\geq m$ and $n/m=$ odd integer, then $\sin n\theta$ is symmetric around $(2j-1)\pi/2m$; if $n/m=$ even integer, it is antisymmetric. This can be shown by expanding $\sin n[(2j-1)\pi/2m+\varphi]$ and $\sin n[(2j-1)\pi/2m-\varphi]$ with $n/m=k$. With these results

$$q_n = -4mQ_n\int_0^{\pi/2m} N(\theta_0)\,\mathrm{d}\theta_0 \times \int_{\theta_0}^{\pi/2m} s(\theta,\theta_0)\sin n\theta\,\mathrm{d}\theta, \quad k \text{ odd} \quad (11)$$
$$= 0, \quad k \text{ even}.$$

Similar symmetry arguments applied to Eq. (7) yield for the line integral formulation, with r not constant, but with $G_2=0$ having 2-m pole symmetry.

$$q_n = -\frac{2m\mu_0 I}{n\pi}\int_0^{\pi/2m} N(\theta_0)\,\mathrm{d}\theta_0 \times \int_{c_{\theta_0}} \cos n\theta\left(\frac{r^n}{R^{2n}}+\frac{1}{r^n}\right)\mathrm{d}s, \quad (12)$$

where c_{θ_0} is that part of the boundary lying between

$\theta = \theta_0$ and $\theta = \pi/2m$. With s = constant in Eq. (11) and $N(\theta_0) = N_0 \cos m\theta_0$ the result is immediate; this is just the usual bedstead end. Further consideration leads one to the conclusion that any end configurations which are tractable analytically will be made up of linear combinations of simple functions. That is, we may evaluate a number of simple functions, then take combinations to create a physically realizable end in which we may also balance out higher multipole contributions. Some choices for $s(\theta, \theta_0)$ and the resulting coefficients are given in Table I, for dipoles ($m = 1$). In Table I u is equal to $(-1)^{(n+1)/2}$ and has the value plus or minus 1.

TABLE I
Harmonic content of ends specified by some simple functions

$s(\theta, \theta_0)$	$-q_1/N_0 Q_1$	$-q_n/N_0 Q_n$
K_1	$K_1 \pi$	0
$K_2 \theta_0$	$K_2(\pi^2 - 1)/4$	$-\frac{4K_2}{n(n^2-1)^2}[n^2 + 1 + 2nu]$
$K_3 \sin \theta_0$	$4K_3/3$	$-4K_3/[n(n^2-4)]$
$\pm K_4 \cos \theta_0$	$-8K_4 u/3$	$8K_4 u[n^2(n^2-4)]$
$K_5 \sin \theta$	$4K_5/3$	$-8K_5/[n(n^2-4)]$
$\pm K_6 \cos \theta$	$4K_6/3$	$4K_6 u/[n^2-4]$

We notice immediately that the combination $s = s_0(1 + \sin\theta - 2\sin\theta_0)$ gives $q_n = 0$ for all n greater than 1. This is a configuration described to one of the authors by G. Lambertson some years back and apparently it or something similar has been patented in Britain by J. H. Coupland. A sketch of this end (which we will term L-C for Lambertson-Coupland) is shown in Figure 1.

As a short digression, we note that the method followed by those authors shows that the contributions to q_n come only from current elements in the z-direction. If iron is present then one must show (but has not) that the contributions to q_n from iron magnetization correspond to those directly from the coil. That is, in the language of images, that the image currents of current elements in the r or θ directions do not have z components. In the present work that difficulty is overcome by the two-dimensional character of the field of the linear n-pole. The essential agreement of the two methods may in fact be taken as a proof that the magnetization may be represented by three-dimensional images in the case of a cylindrical iron surface.

Resuming the discussion of Figure 1, the turns, following the sharp bend at the dashed line, follow a space curve which is a section of an ellipse. The angle α which the dashed plane of bends makes with the median plane is equal to that which the plane containing the element $\theta_0 = 0$ makes with the median plane. In practice, depending on conductor thickness and the angle α, the required sharpness of bend might be hard to realize. A more difficult problem arises with windings made up of ribbon conductors: a ribbon cannot follow an arbitrary space curve. Although the center of gravity of an individual ribbon might be made to follow a curve of the L-C sort, it is possible that the twist which must be put in the ribbon would lead to spacing errors at $\theta = \pi/2$. If we denote azimuthal distance along the periphery by p, then the spacing at $\theta = \pi/2$ is given by $\mathrm{d}s/\mathrm{d}N = 1/(\mathrm{d}N/\mathrm{d}p \cdot \mathrm{d}p/\mathrm{d}s)$, where $\mathrm{d}N/\mathrm{d}p = N_0 \cos\theta_0/b$ and $\mathrm{d}p/\mathrm{d}s = -b/(2s_0\cos\theta_0)$. Thus $\mathrm{d}s/\mathrm{d}N = -2s_0/N_0$, i.e., the spacing along the top is uniform. Present construction practice at Brookhaven of the main parallel winding requires the ribbons to lie essentially in planes containing the s-axis, thus the required sharp bend is easily realizable with ribbon conductors although the spatial curve of the L-C end is not.

6. CORRECTION OF EXISTING ENDS

Especially in the case of ribbon-wound magnets, the end shape, as already stated, is a compromise with mechanical requirements. Hence one may ask, given an existing satisfactory mechanical design, can it be modified to give satisfactory magnetic performance? Green[3] described a procedure used on quadrupoles which is essentially the following. From Eq. (7) for the q_n given in Section 4, we note that provided r and θ are held constant, the harmonic content of a given element is proportional to its length. This linearity implies that, given an arbitrary harmonic content, one can always find an $s(\theta_0)$ (an axial spacing distribution) which will exactly null it. (If the number of elements is finite, the number of harmonics which can be nulled is also finite.) However, the solution so obtained may not satisfy the axial spacing requirements, Eq. (10) of Section IV. But given a sufficient number of elements, it may be hoped that at least a few of the lowest order components can be nulled; those which are not nulled are at least minimized, as the following example shows.

A proposed accelerator dipole magnet[10] contains ribbon conductors grouped into six equally-sized blocks of different current densities to approximate a cos θ distribution. Models containing four blocks have been made, and measurements made of the harmonic content of the ends.[11] The models have an overall length of about 39 cm, main winding length of 26 cm with a mean winding radius of 3.3 cm. Figure 2 is a photograph of such a four-block, ribbon-wound-end from the side, and Figure 3 shows the same end from the top. Figure 4 is line drawings of a vertical section (plane containing the s-axis and a point at $\theta = \pi/2$) of the end and a section perpendicular to the axis through the main winding. A space curve approximating the 'center of gravity' of the various blocks (but having no radial variation) is given by

$$s(\theta, \theta_0) = K_3 R(1 - K_4 \sin \theta_0) \times [1 - (K_1 - \sin \theta)^2/(K_1 - \sin \theta_0)^2]^{1/2},$$

where R is the radius of the center of gravity of the winding and K_1, K_3, and K_4 are constants chosen to fit the slope dy/ds at the end, the axial extent, and axial spacing. The curves are a family of intersections of elliptical cylinders with axis parallel to the x-axis at $s = 0$ with the circular cylinder on which the windings lie. This very crude approximation to a quite complex reality gives calculated harmonics which agree well with measured harmonic content, as shown in Table II in terms of measuring coil voltage ratios. The harmonic coil[12] integrates over the entire end, being a single-turn, finite-length approximation to the 'linear n-pole' described earlier.

TABLE II
Calculated and measured harmonic coil voltages

n	3	5	7	9
$\varepsilon_n/\varepsilon_1$ measured	0.0386	0.0096	0.0038	0.0008
$\varepsilon_n/\varepsilon_1$ calculated	0.0333	0.0094	0.0025	0.00079

The ε_1 of Table II is the signal voltage of the coil when centered in the main parallel winding, not the dipole contribution of one end, which, though not measured, is calculated to be 18.6 per cent of the main winding. The remarkably good agreement of measurement with calculation indicates that the harmonic content is not very sensitive to curve shape.

The integrations of Eq. (11) required to compute the model harmonics were performed numerically. With four blocks of current, one can change three axial spacings, giving three parameters to adjust to null the 6, 10 and 14-pole harmonics. A computer program was written which computes the harmonic content of arbitrary lengths of filaments centered on

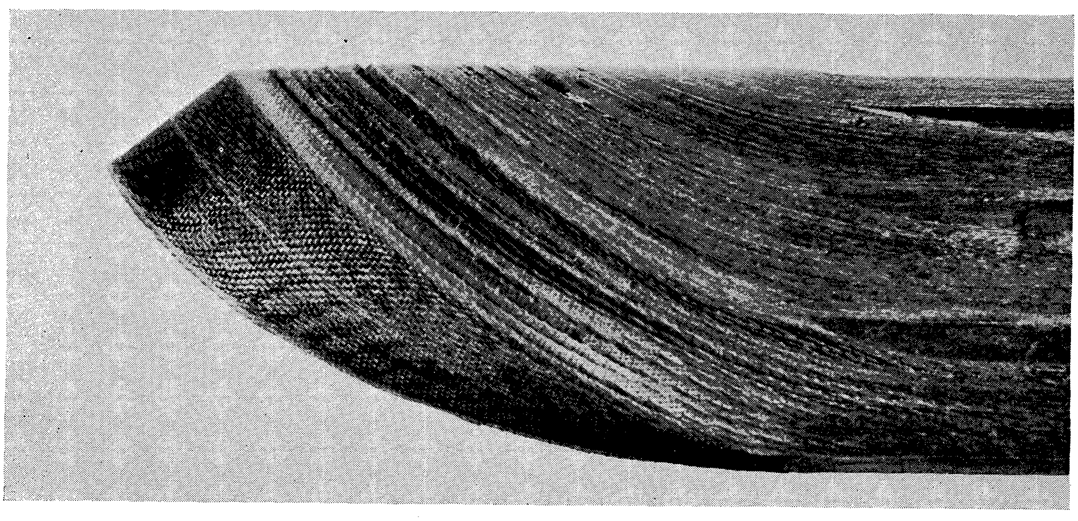

FIGURE 2 Side view of the end of a half of a four-block, ribbon-wound magnet. The tapered ends of Fibreglas-epoxy wedges towards the right mark the transition from straight-section to end.

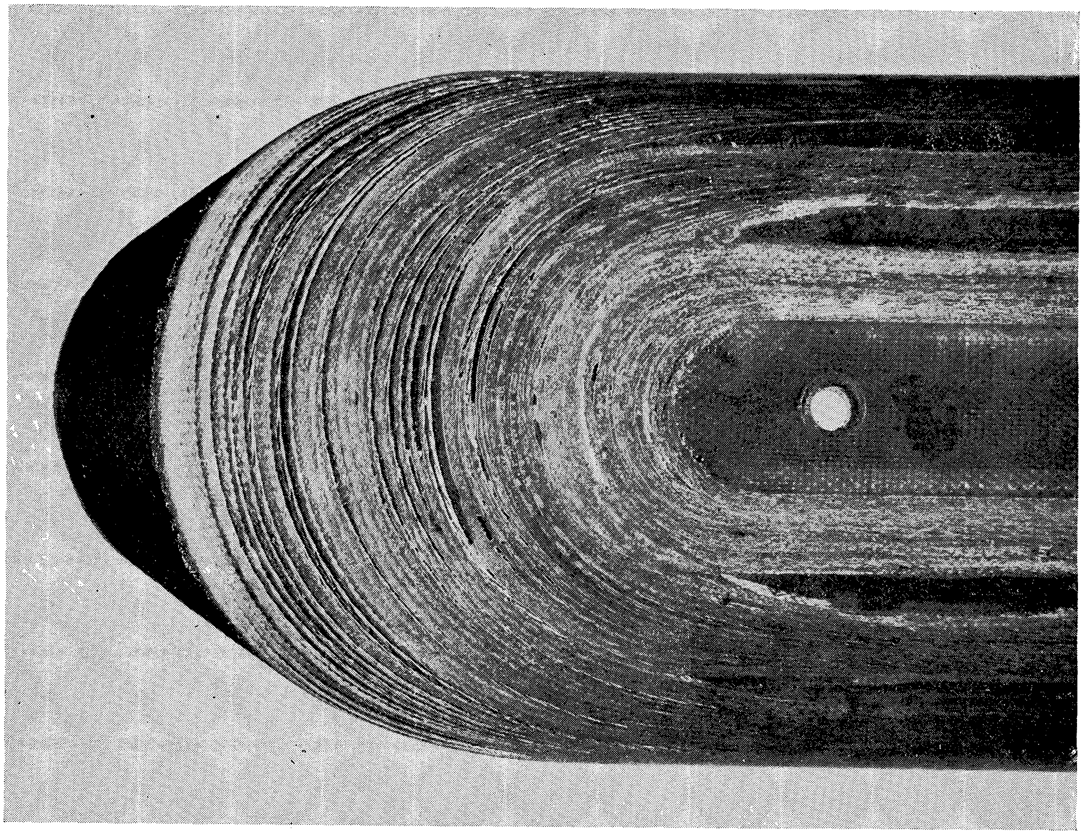

FIGURE 3 Top view of the end shown in Figure 2. Some of the turns in the blocks nearest the pole (the centerline of this view) carry no current.

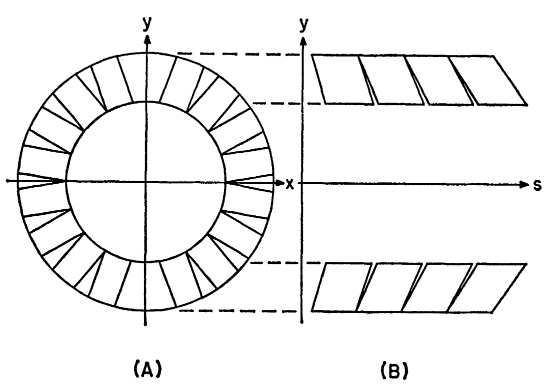

FIGURE 4 View (A) is a cross section of the two-dimensional part of the winding, and view (B) shows a cross section of the blocks at the end. To simplify computations, the current is assumed to lie in a filament located at the center of area of each block.

the blocks in the straight section, using a summation over the four values of θ_0 instead of the integration shown in Eq. (9). The three functions are set equal to the measured or calculated harmonics, and the three equations solved for the unknown spacings using a numerical matrix inversion process. When applied to the measured and calculated harmonics of the model, the spacings of Table III resulted.

TABLE III

Computed axial spacings (in centimeters) for correction of a four-block model

Spacing number (from center out)	1	2	3
Spacing from measured harmonics	1.2	−0.2	3.1
Spacing from calculated harmonics	0.3	0.1	2.6

One spacing computed from the measurements of Table III is negative, though small. The appropriate procedure would be to set this spacing to zero, then compute the two remaining spacings required to null the 6 and 10-pole harmonics. Further measurements could then be made, and the correction process repeated if necessary. The number of iterations of the correction process required will depend on the accuracy of the computed correction matrix.

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Received 3 April 1973