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ANOMALOUS PARTICLE CREATION AND ITS APPLICATIONS TO QCD

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Abstract

We give a simple introduction to spectral asymmetry and index theory techniques and their use in the determination of anomalous quantum numbers, with special emphasis on the applications to strong interaction physics. We introduce the spectral asymmetry method, we explain its relation to index theory, and we demonstrate its use by computing the baryon number of a Skyrmion. We then discuss the axial anomaly from the same viewpoint, paying particular attention to various issues of regularization, especially in the infrared. Finally, we apply this formalism to the computation of anomalous quantum numbers induced by topologically nontrivial field configurations in the QCD vacuum, such as instantons, and we discuss the possible phenomenological relevance of such effects on quantum numbers of the nucleon, such as its axial charge.

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1. Nonperturbative Methods in QCD

For all its phenomenological success, QCD is still only under theoretical control in the very peculiar kinematic domain where it is amenable to perturbative treatment. In particular, whereas in perturbative QCD a systematic set of computational techniques is available, in the nonperturbative domain only specific problems can be handled using an array of heterogeneous methods. In these lectures we shall discuss a particular set of nonperturbative computational tools, which have been now known for some time (usually in the context of mathematical and condensed matter physics) having in mind their application to the physics of strong interactions. These techniques provide a powerful method to compute quantum numbers induced by the coupling to a classical (external) background; they are particularly useful to understand the origin of quantum numbers of semiclassical field configurations, such as the topologically nontrivial configurations which are believed to control the physics of the QCD ground state (instantons), and those which are used to describe baryons in topological soliton models such as the Skyrme model. A notable reason why such techniques are interesting is that they provide a nonperturbative handle on the physics of the axial anomaly: this allows to understand several subtleties related to the renormalization of the anomaly, such as its infrared sensitivity, and to exploit the peculiar features of anomalous effects, which link the high-energy properties of QCD with its infrared (nonperturbative) dynamics.

In the next section we shall give a general introduction to the spectral asymmetry method for the computation of induced quantum numbers, relate it to index theorems, and apply it to the determination of quantum numbers of a skyrmion. In Sect. 3 we shall work out in some detail a completely solvable 1+1 dimensional example: this will allow us to derive the axial anomaly equation and the related charge-creation process; we will take advantage of the spectral asymmetry approach to tackle some subtle issues of infrared regularization of the anomaly. Finally, in Sect. 4 we shall apply the formalism to anomalous particle creation in QCD, and in particular its contribution to the quantum numbers of the nucleon.

2. Vacuum–induced quantum numbers and the spectral asymmetry

When a quantized field is coupled to an external classical background its elementary excitations may acquire quantum numbers which are induced by the coupling. This effect may have surprising consequences: for example it may turn out that in a purely bosonic theory there exist states that carry half-integer spin, or that in a theory where all fields carry integer fermion number some states have half-integer fermion number [1]. Because the ground state of QCD is believed to be dominated by field fluctuations which may be treated in the semiclassical approximation [2] (instantons [3]), this effect is of potential phenomenological relevance for the computation of quantum numbers of strongly-interacting physical states. Furthermore, it is closely related to the physics of the axial anomaly [4], as we shall discuss in the next section.

2.1. The spectral asymmetry method

Consider a system of Dirac fermions, described by the Lagrangian

$$\mathcal{L} = \bar{\psi}\gamma^0 \left(i\frac{\partial}{\partial t} - H \right) \psi \equiv \bar{\psi} \not{D} \psi, \tag{2.1}$$

where the Hamiltonian H contains the coupling to an external field.[1] In general, in the presence of a coupling, the fermion number of physical states will receive a contribution induced by the background, and it will differ from that which is naively given by its fermion content: thus, for instance, the fermion number of the vacuum will not vanish.

This can be shown directly by computing [5] the canonical, normal ordered fermionic charge

$$Q = \int d\vec{x} \; :\psi^\dagger(\vec{x})\psi(\vec{x}): \; = \int d\vec{x} \, \frac{1}{2} \left[\psi^\dagger(\vec{x}), \psi(\vec{x}) \right]. \tag{2.2}$$

In the last step the normal ordering has been performed explicitly: indeed, taking the commutator is the only choice of operator ordering such that Q is charge-conjugation odd. Let us now expand the field over a Fock state basis of eigenstates of the Hamiltonian:

$$\psi(x) = \lim_{s\to 0} \sum\!\!\!\!\!\!\!\int_k \left(b_k u_k(\vec{x}) + d_k^\dagger v_k(x) \right) |\lambda_k|^{-s/2}, \tag{2.3}$$

where u_k are eigenstates of H with positive (negative) eigenvalues λ_k, we have symbolically indicated with $\sum\!\!\!\!\!\!\int_k$ the summation over a continuous and/or discrete spectrum labelled by k, and we have introduced an explicit ζ-function regulator $|\lambda_k|^{-s/2}$ (which is eventually removed) in order to regularize the spectral sum. The coefficients of the expansion are the usual creation and annihilation operators; notice that these create eigenstates of the full Hamiltonian H (which contains a coupling to external fields) and not free states.

Substituting the expansion Eq. (2.3) in the expression Eq. (2.2) of the fermion charge gives

$$Q = Q^c - \frac{1}{2} \lim_{s\to 0} \eta_H(s); \tag{2.4}$$

$$Q^c = \sum\!\!\!\!\!\!\!\int_k \left(b_k^\dagger b_k - d_k^\dagger d_k \right), \tag{2.5}$$

$$\eta(s) = \sum\!\!\!\!\!\!\!\int_k \frac{\mathrm{sign}\lambda_k}{|\lambda_k|^s} \equiv \mathrm{Tr} \frac{\mathrm{sign}H}{|H|^s}. \tag{2.6}$$

[1] Although the considerations which follow would apply equally well to bosons, we consider fermions both because they are phenomenologically of more direct interest, and because they are somewhat simpler to handle.

The first term on the r.h.s. of Eq. (2.4) is the “naive” one, which we would have found in the absence of a background; it is equal to the canonical charge built out of creation and annihilation operators. This term is supplemented with an extra contribution, proportional to the quantity $\eta_H(s)$ defined in Eq. (2.6), which is constructed from the eigenvalues of the Hamiltonian and can thus be thought of as a functional of the background field. This quantity is called the spectral asymmetry of the operator H, since, roughly speaking, it measures the difference between the overall number of positive and negative eigenvalues of H, or, a bit more precisely, it is a ζ-function regularization of it. In the last step in Eq. (2.6) the spectral asymmetry has been symbolically expressed as a functional trace over the spectrum of H.

The vacuum expectation value of Q can now be expressed in terms of the spectral asymmetry: if the Dirac vacuum is defined as the state which is annihilated by the annihilation operators b_k and d_k then

$$\langle Q \rangle = -\frac{1}{2} \lim_{s\to 0} \eta_H(s). \tag{2.7}$$

Notice that it is the requirement that Q has the correct symmetry properties (specifically, that it is charge-conjugation odd) which fixes uniquely its vacuum expectation value.

The (generally nonzero) vacuum charge Eq. (2.7) can be understood physically [6] as a charge carried by the Dirac sea: if the charge of the Dirac sea is defined to be zero when no background is present, then the same definition implies that in general the sea has charge given by Eq. (2.7). A rough and ready way to see how this works is the following: call n^+ (n^-) the total number of positive (negative) energy states; of course, in general such numbers will be infinite, but assume that there exists a suitable regularized definition. Assuming further for simplicity that there are no zero modes (i.e., vanishing energy states) the total number of eigenstates is $n = n^+ + n^-$. The total number of eigenstates does not depend on the background, thus if $n_0^\pm$ are the numbers of states when *no* background is present, then

$$n = n^+ + n^- = n_0^+ + n_0^-; \tag{2.8}$$

moreover, in the absence of a background the spectrum of H is symmetric about zero, so that $n_0^+ = n_0^-$, hence $n = 2n_0^+ = 2n_0^-$. Now, the vacuum is defined by filling negative energy states, hence the vacuum charge equas the number of negative energy states — the charge carried by the Dirac sea. Since, however, the charge must vanish for a free theory, we choose our operator ordering so that the vacuum charge is equal to zero in the absence of background. This is accomplished by subtracting this quantity from the charge, thereby regulating its divergence. Putting everything together we get thus

$$\begin{aligned}\langle Q \rangle &= n^- - n_0^- = n^- - \frac{1}{2}\left(n_0^+ + n_0^-\right) = n^- - \frac{1}{2}\left(n^+ + n^-\right) = \\ &= -\frac{1}{2}\left(n^+ - n^-\right).\end{aligned} \tag{2.9}$$

The last step is recognized to be a crude form of the spectral asymmetry, properly defined in Eq. (2.6). More precisely, Eq. (2.9) is correct when the number of states is is finite; in such case it coincides with the spectral asymmetry Eq. (2.7), which however generalizes it to the case of infinite n.

The generation of a vacuum charge induced by a background can also be understood without invoking a canonical approach, directly from the path integral [7]. Indeed, the vacuum charge can be obtained by functional differentiation of the generating functional with respect to a source μ:

$$\begin{aligned}\langle Q\rangle &= \frac{\delta}{\delta\mu}\ln\int D\psi D\bar{\psi}\, e^{i\int dx\,\mathcal{L}+\mu(x)\bar{\psi}\gamma^0\psi}\Big|_{\mu=0}\\ &= \frac{\delta}{\delta\mu}\mathrm{Tr}\ln[i\not{D}+\gamma^0\mu]\big|_{\mu=0}\\ &= \mathrm{Tr}\frac{1}{i\frac{\partial}{\partial t}-H}.\end{aligned} \tag{2.10}$$

Wick-rotating to Euclidean space, and rewriting the functional trace as an integration over plane-wave eigenfunctions of the time-derivative operator, and a trace over the eigenstates of H, leads to

$$\langle Q\rangle = -i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\mathrm{Tr}\left(\frac{1}{\omega+iH}-\frac{1}{\omega+iH_0}\right), \tag{2.11}$$

where in the last step we have regulated by explicitly subtracting out the charge in the absence of background: H_0 denotes the Hamiltonian when no background is present. Computing the integrals by Cauchy's theorem leads to

$$\langle Q\rangle = \mathrm{Tr}\left[\Theta(H)-\Theta(-H_0)\right]. \tag{2.12}$$

If we use again the fact that the total number of states is left unchanged by the presence of the background it follows that $\mathrm{Tr}[\Theta(H)+\Theta(-H)] = \mathrm{Tr}[\Theta(H_0)+\Theta(-H_0)]$, and Eq. (2.12) is recognized to provide once again an (unregulated) expression of the spectral asymmetry — if we had been more careful in introducing a regularization from the beginning, then the result Eq. (2.6) would have been recovered.

As the Lagrangian derivation suggests, this method is actually quite general, and may be used to compute any vacuum-induced quantum number, and not only the vacuum induced charge, by evaluating a suitably weighted spectral asymmetry. Thus, for instance, the vacuum-induced angular momentum will be computed by weighting the spectral asymmetry with the angular momentum operator J:

$$\langle J\rangle = \lim_{s\to 0}\mathrm{Tr}\frac{J\,\mathrm{sign}H}{|H|^s}, \tag{2.13}$$

and likewise for the expectation value of other operators.

2.2. An example: the quantum numbers of the Skyrmion

As a first example of application of the spectral asymmetry method, we sketch its application to the computation of the baryon number and axial charge of a Skyrme soliton. The quantum numbers carried by a soliton can be determined by coupling the fields which carry the soliton excitation to a test fermion, and then determining the variation induced by the background on the quantum numbers carried by the fermionic vacuum, i.e., the state which contains no explicit excitations of the test fermion.

The fermion-Skyrmion coupling is dictated by the quantum numbers carried by the Skyrmion, and has the sigma model form [8]

$$\mathcal{L}_s = \bar{\psi}\left(i\,\partial\!\!\!/ - \mu e^{\frac{2i}{f_\pi} i\gamma_5 \pi^a \lambda_a}\right)\psi. \tag{2.14}$$

When expanded in powers of the field $U(x)$ this Lagrangian generates a quark mass term with mass μ, and couplings to an increasing number of pions, with decay constant fixed by f_π. The Skyrmion field is

$$U = e^{\frac{2i}{f_\pi}\pi^a\lambda_a}, \tag{2.15}$$

and a soliton is obtained when the pion field $\pi^a(x)$, viewed as a mapping from space to the group SU(2) (on which the isospin generators λ^a act) is topologically nontrivial. This means the following: if the physical space is viewed as a sphere S_3 (by considering spatial infinity as one point, which is possible provided the fields fall off rapidly enough at infinity), then, recalling that the group manifold of SU(2) (i.e. the space spanned by U) is also isomorphic to a sphere S_3, the field $U(x)$ provides a map $S_3 \to S_3$. Now, such maps fall in equivalence classes; each class contains maps which can be deformed smoothly into each other, but not into the maps in other classes. These classes, called homotopy classes, form a group, $\pi_3[\mathrm{SU}(2)] = \mathbb{Z}$. The field is nontrivial if it belongs to a class which is not the same as that of the trivial map $U(x) = 1\!\!1$; this happens when as x winds on the space S^3, $U(x)$ also winds on the group manifold.

We can rewrite the Lagrangian of Eq. (2.14) in the general form of Eq. (2.1) by defining the Hamiltonian

$$H_S = -i\gamma^0\gamma^i\partial_i + \mu\gamma^0\left[UP_R + U^\dagger P_L\right]; \quad P_R = \frac{1\!\!1 + i\gamma_5}{2}\; P_L = \frac{1\!\!1 - i\gamma_5}{2}, \tag{2.16}$$

where P_R, P_L are chiral projectors[2]. The charge carried by the Skyrmion is then just given by Eq. (2.7), with the spectral asymmetry which pertains the Hamiltonian Eq.(2.16). This in turn can be easily computed from its representation Eq. (2.11), by expanding the

[2] We define $\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma_5^\dagger$.

integrand in powers of the Skyrme field and its derivatives [9]. The leading order term is found to be

$$\begin{aligned}\langle Q\rangle &= -\int \frac{d\omega}{2\pi}\mathrm{Tr}\left(\frac{H}{\omega^2+H^2}-\frac{H_0}{\omega^2+H^2}\right)\\ &= \frac{1}{24\pi^2}\int d^3x\,\epsilon^{ijk}\mathrm{Tr}\left(U^{-1}\partial_i U U^{-1}\partial_j U U^{-1}\partial_k U\right).\end{aligned} \tag{2.17}$$

The expression Eq. (2.17) of the charge of the Skyrmion is recognized as an explicit form of the winding number of $U(x)$. This is a number which classifies the homotopy classes of maps, by taking one and the same value for all maps in the same class, and can be chosen to take (all) integer values by a suitable choice of normalization. The present calculation shows that this number is actually equal to the fermion number (baryon number) of the soliton field. Therefore, the 1-skyrmion (1-antiskyrmion) configuration is defined to be that which has $\langle Q\rangle = 1$ ($\langle Q\rangle = -1$).

Along the same lines we may compute the axial charge of the Skyrmion, namely

$$\langle Q_5\rangle \equiv \langle \int d^3x\,\psi^\dagger i\gamma_5\psi\rangle = \lim_{s\to 0}\mathrm{Tr}\frac{i\gamma_5\,\mathrm{sign}\,H_S}{|H_S|^s}. \tag{2.18}$$

The result follows immediately from the observation that the spectral asymmetry satisfies

$$\lim_{s\to 0}\eta_H(s) = \lim_{s\to 0}\eta_{HP_R}(s) + \lim_{s\to 0}\eta_{HP_L}(s), \tag{2.19}$$

i.e., the spectral asymmetry of the Hamiltonian decomposes into the sum of the spectral asymmetries of its chiral projections (defined as in Eq. (2.16)). This in turn is a consequence of the fact that

$$\int\frac{d\omega}{2\pi}\mathrm{Tr}\frac{1}{\omega+iH(P_R+P_L)} = \int\frac{d\omega}{2\pi}\mathrm{Tr}\frac{H}{\omega^2+H^2}(P_R+P_L). \tag{2.20}$$

But since for the Skyrme Hamiltonian Eq. (2.16) $\lim_{s\to 0}\eta_{H_S P_R}(s) = \lim_{s\to 0}\eta_{H_S P_L}(s)$, then

$$\begin{aligned}\langle Q\rangle &= \eta[HP_R]+\eta[HP_L] = 2\eta[HP_R]\\ \langle Q_5\rangle &= \eta[HP_R]-\eta[HP_L] = 0,\end{aligned} \tag{2.21}$$

where we denote for short with $\eta[H]$ the $s\to 0$ limit of the spectral asymmetry of the operator H. Hence, the axial charge of the Skyrmion vanishes identically.

The computation of the axial charge induced by background gauge fields is especially instructive, since the relevant spectral asymmetry is then determined by an index theorem. In view of the connection between induced quantum numbers and anomalies, this explains the relation between the anomaly and the index theorem, as we shall see in detail in the next section. Let us thus consider fermions minimally coupled to a background gauge potential A_μ, which will be in general a matrix acting on the gauge group, i.e. a linear combination $A_\mu = A^a_\mu \lambda^a$, where λ^a are generators of the gauge group, with Lagrangian and Hamiltonian given respectively by[3]

$$\begin{aligned} \mathcal{L} &= \bar{\psi} i \not{D} \psi; \quad \not{D} = \gamma^\mu \left(\partial_\mu + A_\mu\right) \\ H &= -i\gamma^0\gamma^i \left(\partial_i + A_i\right), \end{aligned} \tag{2.22}$$

in the $A_0 = 0$ gauge, which we shall use throughout.

In a representation of the Dirac matrices where $i\gamma_5 = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix}$ the Dirac Hamiltonian has the block-diagonal form

$$H = \begin{pmatrix} i \not{D}_3 & 0 \\ 0 & -i \not{D}_3 \end{pmatrix}; \quad \not{D}_3 = \sigma^i \left(\partial_i + A_i\right), \tag{2.23}$$

where σ^i are two by two matrices which satisfy the Clifford algebra $\{\sigma^i, \sigma^j\} = \delta^{ij}$, such as the usual Pauli matrices (and all unitary transformation thereof), and $i \not{D}_3$ may be viewed as a Dirac operator in three dimensions (by considering the three space dimensions as a Euclidean three-dimensional spacetime). It then follows immediately that the axial charge, defined as in Eq. (2.18) [but with the Dirac Hamiltonian Eq. (2.23)] is

$$\langle Q_5 \rangle = -\frac{1}{2}\eta[i\gamma_5 H] = -\eta[i \not{D}_3], \tag{2.24}$$

while of course $\eta[H] = 0$. Otherwise stated, because the Hamiltonian is block diagonal with respect to eigenstates of $i\gamma_5$, and its projections over the two eigenstates of $i\gamma_5$ (namely $\pm i \not{D}_3$) are the opposite of each other, the spectral asymmetries of these two projections are also the opposite of each other; hence

$$\begin{aligned} \langle Q \rangle &= 0 \\ \langle Q_5 \rangle &= -\eta[i \not{D}_3]. \end{aligned} \tag{2.25}$$

[3] We assume the normalization of λ^i to be chosen in such a way that they are antihermitian; the potentials A^a_μ are then real vector fields

The spectral asymmetry of the three-dimensional Dirac operator $i\,\not{D}_3$ can now be expressed in close form as a functional of the gauge potentials A, using the powerful mathematical methods of index theory. The relevant construction [10] starts by relating $i\,\not{D}_3$ to a new Dirac operator $i\,\not{D}_4$, which acts on a 4-dimensional Euclidean spacetime. This is defined as follows: first, given the gauge potential $A_i(\vec{x})$ on which $i\,\not{D}_3$ depends, construct a one-parameter family of gauge potentials $A^i(\tau,\vec{x})$, where $0 \le \tau \le 1$, such that $A^i(\tau=0,\vec{x}) = 0$ and $A^i(\tau=1,\vec{x}) = A^i$. This is therefore an interpolation between the vacuum and the given gauge potential. Then, construct the Dirac operator in 4 dimensions

$$i\,\not{D}_4 = \sigma_1 \otimes \mathbb{1}\, i\frac{\partial}{\partial\tau} + \sigma_2 \otimes i\,\not{D}_3, \tag{2.26}$$

where $\sigma \otimes \tau$ is the 4×4 matrix obtained replacing each of the elements τ_{ij} of the 2×2 matrix τ with the 2×2 matrix obtained multiplying by τ_{ij} the elements of σ. It is easy to check that Eq. (2.26) defines a *bona fide* 4-dimensional Euclidean Dirac operator, in the sense that it can be written as $\not{D}_4 = \gamma^\mu(\partial_\mu + A_\mu)$, where $\{\gamma^\mu,\gamma^\nu\} = \delta^{\mu\nu}$. The four-dimensional space on which this operator acts can be viewed as a cylinder, which has the three-dimensional space we started from as a "basis" and the segment spanned by τ as "height"; the three-dimensional space we started from is the boundary of this cylinder.

Then, the Atiyah-Patodi-Singer index theorem states that

$$\begin{aligned} -\frac{1}{2}\eta[i\,\not{D}_3] &= -\Omega_0[A] + \mathrm{ind}\, i\,\not{D}_4 \\ \Omega_0[A] &= -\frac{1}{8\pi^2}\mathrm{tr}\int d^3x\,\epsilon^{ijk}\left(A_i\partial_j A_k + \frac{2}{3}A_iA_jA_k\right), \end{aligned} \tag{2.27}$$

where the trace is performed over the gauge group matrices λ^a contained in the potentials A^μ. Here $\mathrm{ind}\, i\,\not{D}_4$ is the index of the operator $i\,\not{D}_4$, and it is determined by the chirality of its zero modes. Indeed, zero modes (eigenvectors with vanishing eigenvalue) of $i\,\not{D}_4$ are right-handed or left-handed (i.e., they are eigenstates of $i\gamma_5$ with eigenvalue equal to ± 1), because $i\gamma_5$ anticommutes with $i\,\not{D}_4$; the index is defined as the difference between the number of right-handed minus the number of left-handed zero modes. Thus, Eq. (2.27) determines the spectral asymmetry as a functional of A, given by $\Omega_0[A]$ (called the Chern-Simons functional of A), up to an integer, equal to the index of $i\,\not{D}_4$.

The reason why Eq. (2.27) is called an index theorem is that it can be viewed, in reverse, as a determination of the index of the operator $i\,\not{D}_4$. Indeed, note that

$$\begin{aligned} \Omega_0 &= Q(x) \\ Q(x) &= -\frac{1}{16\pi^2}\mathrm{tr}\int_{M_4} d^4x\,\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}, \end{aligned} \tag{2.28}$$

where $F^{\mu\nu}$ is the field strength computed from the potential A^μ, and the integration is extended to the four-dimensional (Euclidean) space M_4 on which $i\,\not{D}_4$ acts. Observing that $i\, \not{D}_3$ is just the restriction of this operator to the boundary of M_4, Eq. (2.27) is seen to determine the index of $i\, \not{D}_4$ in terms of a functional of the gauge potentials, $Q(x)$, and a boundary correction, the spectral asymmetry $\eta[i\, \not{D}_3]$.[4]

The index theorem Eq. (2.27) can then be generalized to any manifold with or without boundary by writing it in the form

$$\text{ind}\; i\, \not{D}_4 = Q(x) - \frac{1}{2}\sum_i \eta[i\, \not{D}_3^i], \tag{2.29}$$

where the sum runs over all the disconnected components which the boundary will in general have. The simplest example is the case of boundaryless manifolds, in which the spectral asymmetry term is missing; this is then the original Atyiah-Singer index theorem [11]. The meaning of the index of $i\, \not{D}_4$ can be clarified by considering a second example, namely that when the operator $i\, \not{D}_3\, [\tau]$ is a function of a parameter τ which interpolates between two potentials $A(\tau_1) = A^{(1)}$, and $A(\tau_2) = A^{(2)}$ [of which Eq. (2.27) is the special case in which $A^{(1)} = 0$ and $A^{(1)} = A$]. Then, the boundary consists of two disconnected components, i.e. the endpoints $\tau = \tau_1, \tau_2$, and Eq. (2.29) gives

$$\text{ind}\; i\, \not{D}_4 = -\left(\Omega_0[A_2] - \Omega_0[A_1]\right) - \frac{1}{2}\Big(\eta[i\, \not{D}_3^i\, [A_2]] - \eta[i\, \not{D}_3^i\, [A_1]]\Big), \tag{2.30}$$

i.e. the index equals the variation of the combination on the r.h.s. of Eq. (2.27) as τ varies. Furthermore, it can be shown[5] [4] that $i\, \not{D}_4\, [\tau]$ has a zero mode if and only if, diagonalizing $i\, \not{D}_3$ at fixed τ, there is an eigenvalue of $i\, \not{D}_3$ which changes sign as τ varies from τ_1 to τ_2; if the zero mode is right-handed (left-handed) the eigenvalue changes from positive to negative (negative to positive), provided a state is defined to be right-handed (left-handed) if it is an eigenstate of $i\gamma_5$ with eigenvalue $+1$ (-1). More in general, if we define a signature s_i of the i-th level crossing as $s_i = +1$ $(s_i = -1)$ if one level crosses from positive to negative (negative to positive) then

$$\begin{aligned} \text{ind}\, i\, \not{D}_4 &= s \\ s &\equiv \sum_i s_i, \end{aligned} \tag{2.31}$$

where the sum runs over all level crossings. The quantity s also known as the spectral flow of the family of operators $i\, \not{D}_3\, [\tau]$.

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[4] Eq. (2.27) is correct only provided the fields fall-off at spatial infinity fast enough that the three-dimensional space can be viewed as a sphere (i.e. at least as $1/x$).

[5] We will demonstrate this explicitly in Sect. 3.2 in a two-dimensional setting.

Finally, it is interesting to notice that the gauge invariance of the spectral asymmetry (which is manifest, since is computed from the gauge invariant spectrum of the Hamiltonian) is realized in a nontrivial way in Eq. (2.27). Indeed, the Chern-Simons functional Ω_0 is not, in general, gauge invariant, but rather it is only gauge invariant modulo integers: the same kind of topological structure that allows the existence of Skyrmion field Eq. (2.15) also implies that in SU(2) or SU(3) gauge theories (in fact, for any SU(N) with $N > 1$) there exist gauge transformations which cannot be continuously deformed into the identity. Upon the action of such transformations, Ω_0 varies by an integer, so that it can be actually used to classify these transformations into equivalence classes, by putting in the same class all transformations such that Ω_0 changes by the same amount. But then this means that $\mathrm{ind}\, i\, \not{D}_4$ in Eq. (2.27) must also be gauge-noninvariant, in such a way as to exactly compensate the gauge noninvariance of Ω_0. Restricting instead to gauge transformations which can be continuously connected to the identity (i.e., those which can be obtained by exponentiating a linear combination of generators of the group), then it is easy to verify [4] that Ω_0 is gauge invariant (and the index is as well).[6]

These somewhat formal results are actually very powerful: they embody all the physics of the axial anomaly, and they provide an easy computational approach to it. This is what we will discuss in the next section, by working out an explicit two-dimensional example.

3. Induced quantum numbers and anomalies

In the previous section we discussed the quantum numbers induced by the presence of a background field on a test particle (specifically a fermion). The presence of such quantum numbers followed from symmetry requirements: for example a vacuum charge Eq. (2.7) is induced by demanding that the second-quantized charge operator Eq. (2.2) be odd under charge conjugation, and that the vacuum carry no charge. Likewise, imposing symmetry requirements may force the generation of a time-dependent quantum number, which in turn may signal the breaking of a symmetry. This phenomenon, where it is not possible to maintain simultaneously a pair of classical symmetries in a quantized theory, and enforcing one requires the other to be broken, is known as anomaly. We will study it, as particular case of particle creation, in the simple setting of a completely solvable two-dimensional model.

[6] An explicit construction of one-dimensional homotopically nontrivial gauge transformations will be discussed in Sect. 2.2 below.

We consider massless fermions coupled to an electromagnetic field in 1+1 dimensions. This is a special case of the Lagrangian and Hamiltonian Eq. (2.22), where the index i takes one single value (corresponding to the single space coordinate), and the gauge potentials A_μ are just functions, and not matrices. It is convenient to choose the representation of the Dirac matrices where

$$\gamma^0 = \sigma^1,\ \ \gamma^1 = i\sigma^2; \quad i\gamma_5 = -\sigma_3. \tag{3.1}$$

It then follows that the Hamiltonian can be rewritten as (denoting with A the nonvanishing component of the gauge potential, in the gauge $A_0 = 0$)

$$\begin{aligned} H &= -\sigma_3 \left(\frac{1}{i}\frac{d}{dx} - A(x,t) \right) \\ &= -\begin{pmatrix} D_1 & 0 \\ 0 & -D_1 \end{pmatrix}; \quad D_1 = \left(\frac{1}{i}\frac{d}{dx} - A(x,t) \right), \end{aligned} \tag{3.2}$$

which has the same block-diagonal structure encountered in the four-dimensional case Eq. (2.23); hence, the vacuum axial charge is again given in terms of the spectral asymmetry by Eq. (2.25), with $i\not{D}_3$ replaced by D_1 Eq. (3.2). Also, it is possible to derive a two-dimensional index theorem of the form Eq. (2.27):

$$-\frac{1}{2}\eta[H] = -\Omega_0 + \mathrm{ind}\, i\not{D}_2, \tag{3.3}$$

where now Ω_0 is the one-dimensional Chern-Simons term

$$\Omega_0 = \frac{1}{2\pi}\int_0^L dx\, A(x;t), \tag{3.4}$$

and the index is that of the two-dimensional Euclidean Dirac operator

$$i\not{D}_2 = \sigma^1 i\frac{\partial}{\partial\tau} + \sigma^2 D_1. \tag{3.5}$$

The theory with Hamiltonian Eq. (3.2) has classically both a vector and an axial symmetry, leading respectively to the conservation of the vector and axial currents

$$j^\mu = \bar\psi\gamma^\mu\psi; \quad j_5^\mu = \bar\psi\gamma^\mu i\gamma_5\psi \tag{3.6}$$

and the associated charges Q and Q_5. When the theory is quantized, however, the symmetry is spoilt (axial anomaly): an axial charge is induced in the vacuum, according to Eq.s (2.25), (2.27) [with Chern-Simons term (3.4)], but the induced charge is time-dependent, so that $\dot{Q}_5 \neq 0$.

In our model this can be seen explicitly since the spectrum of the Hamiltonian Eq. (3.2) is trivial to determine. It is convenient to put the system in a (one-dimensional) box of length L, in order to deal with a discrete spectrum. Antiperiodic boundary conditions are imposed (it can be shown [12] that this is required by Fermi statistics); the eigenvectors and eigenvalues of D_1 are then just given by

$$\psi_k = \frac{1}{\sqrt{L}} \exp i \left(x E_k + \int_0^x dx' \, A(x';t) \right) \tag{3.7}$$

$$E_k = \frac{(2k+1)\pi}{L} - \frac{2\pi}{L} \Omega_0(t) \tag{3.8}$$

while the eigenvectors of H (3.2) are

$$\psi_k^+ = \begin{pmatrix} 0 \\ \psi_k \end{pmatrix} \quad \psi_k^- = \begin{pmatrix} \psi_k \\ 0 \end{pmatrix}, \tag{3.9}$$

with eigenvalues $\pm E_k$, respectively. Notice that the eigenvectors ψ^- (ψ^+) are right-handed (left-handed), respectively, i.e. they are eigenstates of $i\gamma_5$ with eigenvalue $+1$ (-1).

The origin of the non-conservation of Q_5 can be understood along the lines of the simple argument of Sect.2.2, in terms of the charge carried by the Dirac sea. Indeed, consider the special case of a constant (in space) potential A. Then, the spectrum of Eq. (3.8) reduces to $E = \pm(k - A)$, where k denotes the set of momentum eigenvalues of the free problem. Now, choose A so that a constant electric field $\mathcal{E}$ is generated: $A = \mathcal{E}t$. If at $t = 0$ the Dirac sea is defined in the usual way, i.e. by filling all states with $E < 0$ (i.e. $k > 0$ for the $+$ eigenvalues and $k < 0$ for the $-$ eigenvalues), then at $t = \Delta$ all states with $k < \mathcal{E}\Delta$ will have the "wrong" filling prescription. That is, all $+$ states with $k < \mathcal{E}\Delta$ will have $E < 0$ but will be empty (because they had $E > 0$ at $t = 0$ when the filling prescription was defined), and analogously all $-$ states with $k < \mathcal{E}\Delta$ will have $E > 0$ but will be filled. Otherwise stated, there is an induced charge due to the presence of the background; this charge depends on time and therefore spoils charge conservation. Because the right-handed and left-handed spectra are opposite to each other, the rate of creation of the respective charges is the same, but with opposite sign. As a consequence, the charge $Q = Q_R + Q_L$ is still conserved, but the axial charge $Q_5 = Q_R - Q_L$ is not.

This qualitative argument is made quantitative [13] by using the expression of the induced charge in terms of the spectral asymmetry of Eq. (2.25). Indeed, the asymmetry of the spectrum Eq. (3.8) of the operator D_1 Eq. (3.2) can be easily computed by relating it to the generalized Riemann ζ function $\zeta(s,z)$:

$$\begin{aligned} \eta[H] &= \lim_{s\to 0} - \sum_{k=-\infty}^{\infty} \left(\frac{2\pi}{L}\right)^{-s} \left| k + \Omega_0(t) + \frac{1}{2} \right|^{-s} \operatorname{sign}\left(k + \Omega_0(t) + \frac{1}{2} \right) \\ &= \lim_{s\to 0} - \sum_{k=-\infty}^{\infty} \left(\frac{2\pi}{L}\right)^{-s} \left[\zeta\left(s, \left\{\Omega_0 + \frac{1}{2}\right\}\right) - \zeta\left(s, 1 - \left\{\frac{1}{2} - \Omega_0\right\}\right)\right], \end{aligned} \tag{3.10}$$

where $\zeta(s,z)=\sum_{k=0}^{\infty}(k+z)^{-s}$, and $\{x\}$ is the fractional part of x, i.e

$$\{x\}\equiv x-[x] \tag{3.11}$$

if $[x]$ denotes the largest integer smaller than or equal to x (integer part of x). Noting that $\zeta(0,z)=\frac{1}{2}-z$ the spectral asymmetry can be determined explicitly:

$$\eta(s=0)=2\left(\frac{1}{2\pi}\int_0^L dx' A(x')-\left[\Omega_0+\frac{1}{2}\right]\right). \tag{3.12}$$

If we relate the vacuum charge to the spectral asymmetry using Eq. (2.25), then Eq. (3.10) gives the vacuum charge in terms of A, and thus in particular the variation of the charge as A is varied as a function of time. This, however, determines the variation of the charge carried by the system only provided the system does not change its state as $A(t)$ is varied: thus, for instance, if at $t=0$ the system is in its vacuum state use of Eq. (3.12) in Eq. (2.25) gives the charge at time t only if the system is then still in the vacuum state. This, in general is not the case, because the system may actually jump to excited states as A is varied. To understand this, and thus determine the rate of charge creation in general, it is convenient to compare the result of the explicit computation Eq. (3.10)–(3.12) to the index theorem discussed in Sect. (2.3).

3.2. The index theorem and the anomalous charge

The explicit evaluation of the spectral asymmetry given in Eq. (3.12) provides a realization of the 1+1 dimensional version of the index theorem Eq. (3.3). This is obvious if we recall that the index is equal to the spectrakl flow s Eq. (2.31) of the operator D_1 as the field A is smoothly varied from the vacuum $A=0$ to the final configuration $A(t)$: since the eigenvalues have the simple form Eq. (3.8), if $t_i\le t\le t_f$ this is just equal to

$$s=\left[\Omega_0(t)+\frac{1}{2}\right]=\operatorname{ind} i\,\not{D}_2\,. \tag{3.13}$$

In the present 1+1 dimensional case it is also easy to see explicitly that each crossing corresponds to a zero mode of the operator $\not{D}_2$ Eq. (3.5). Indeed, a zero mode ψ_0 of $i\,\not{D}_2$ is a (normalizable) solution of $i\,\not{D}_2\,\psi_0=0$, i.e. of

$$i\frac{\partial}{\partial\tau}\psi_0(\tau,x)=-\sigma^1\sigma^2 D_1\psi_0(\tau,x). \tag{3.14}$$

This equation can be solved by rescaling the range of τ from $0\le\tau\le1$ to $-\infty\le\tau\le\infty$, so that the interpolation from $A[\tau=-\infty]=0$ to $A[\tau=\infty]=A(t)$ is infinitely slow, and Eq. (3.14) may be solved in the adiabatic approximation, which is exact in this limit. All solutions to Eq. (3.14) are then given in terms of the eigenvalues $E_k(\tau)$ and eigenvectors $\psi_k^{\pm}(\tau,x)$ [of the form (3.7)-(3.9)] of $D_1(\tau)$ by

$$\psi_0^{(k)} = e^{\pm\int_0^\tau E_k(\tau')d\tau'}\psi_k^{\pm}. \tag{3.15}$$

But this solution is normalizable over the given range of τ if and only if $\lambda(-\infty) > 0$ ($\lambda(-\infty) < 0$) and $\lambda(\infty) < 0$ ($\lambda(\infty) > 0$) for the right-handed (left-handed) solutions ψ_k^+ (ψ_k^-). Since the chirality of $\psi_0^{(k)}$ is the same as that of $\psi_k^{\pm}$ it follows that indeed level crossings of eigenvalues of D_1 are in one-to-one correspondence with zero modes of $i\,\not{D}_2$, and the signature of the crossing is equal to the handedness of the zero mode so that the spectral flow is given by the index of $i\,\not{D}_2$ according to Eq. (2.31).

The interpretation of the two terms in the expression Eq. (3.12) of the spectral asymmetry is now immediate: each time an eigenvalue changes sign from negative (positive) to positive (negative) the spectral asymmetry varies discontinuously, increasing (decreasing) by two units, in agreement with its definition as a regularization of the difference in number of positive and negative eigenvalues. This discontinuous variation is given by the second term on the r.h.s. of Eq. (3.12), i.e. by the index contribution to the spectral asymmetry. When there are no level crossings, the asymmetry is a smooth function of the spectrum, which in turn depends smoothly on the background field, and thus is a smooth function of t if the background is. The corresponding continuous dependence of the spectral asymmetry on the background is given by the first term on the r.h.s. of Eq. (3.12), i.e. by the Chern-Simons term Ω_0 in the index formula. Separating the spectral asymmetry in its continuous and discontinuous parts we thus get

$$\begin{aligned}
\eta &= \eta_c + \eta_d \\
&\quad \eta_c = 2\Omega_0[A] \\
&\quad \eta_d = -2\left[\Omega_0 + \frac{1}{2}\right] = -2\,\mathrm{ind}\, i\,\not{D}_2\ .
\end{aligned} \tag{3.16}$$

We can now address the question we started from, namely, what is the charge created as $A(t)$ varies. Clearly, if one eigenvalue changes sign, a system which was prepared in the vacuum jumps to an excited state: if a negative eigenvalue becomes positive the system is left in a state where a positive eigenvalue is filled, and conversely. It follows that a system prepared in the vacuum is left in an excited state with charge $Q = -s$, where s is the spectral flow Eq. (2.31). In the present case, crossings always occur in pairs of opposite chirality, hence, after n crossings, the charge Q is unchanged, but the axial charges is given by $Q_5 = -2s$. On top of this, the vacuum charge itself varies proportionally to the spectral asymmetry according to Eq. (2.25). The total charge created as the background varies between, say, $A(t_1)$ and $A(t_2)$ is thus given by the sum of two contributions, the variation of the vacuum-induced charge, and the charge due to transition to excited states;

both contributions come in pairs of opposite handedness which cancel in the vector charge but add in the axial charge. Whereas the former contribution is given by Eq. (2.25), the latter is equal to the number of level crossings, hence

$$\begin{aligned}\Delta Q_5(t_1,t_2) =&2\left\{-\frac{1}{2}\left(\eta[D_1(t_2)]-\eta[D_1(t_1)]\right)-\left(\left[\Omega_0(t_2)+\frac{1}{2}\right]-\left[\Omega_0(t_1)+\frac{1}{2}\right]\right)\right\}\\ =&2\left[\eta_c(t_2)-\eta_c(t_1)\right]\\ =&\frac{1}{\pi}\int_0^L dx\,\left[A(x,t_2)-A(x,t_1)\right]=-2\Omega_0.\end{aligned}\tag{3.17}$$

In other words, the total charge created along the flow equals the *continuous* part of the spectral asymmetry. This can be equivalently expressed as

$$\eta_c=\tilde{\eta}\equiv-\lim_{s\to 0}\left(\frac{2\pi}{L}\right)^{-s}\sum_{k=-\infty}^{\infty}\left|k+\Omega_0+\frac{1}{2}\right|^{-s}\text{sign}\left(k+\frac{1}{2}\right)\equiv\text{Tr}\frac{\text{sign}\,(H_0)}{|H(t)|^s},\tag{3.18}$$

where H_0 is the Hamiltonian in the absence of background; i.e., as a modified spectral asymmetry defined using the vacuum filling prescription.

The total axial charge created Eq. (3.17) appears thus to be a smooth function of time. It is straightforward to calculate its derivative, which equals

$$\frac{d}{dt}\Delta Q_5=-2\dot{\Omega}_0=-\frac{1}{2\pi}\int d^2x\,\epsilon^{\mu\nu}F_{\mu\nu};\tag{3.19}$$

as a consequence, the axial current Eq. (3.6) is no longer conserved in the quantized theory, rather,

$$\partial_\mu j_5^\mu=-\frac{1}{\pi}{}^*F\equiv-\frac{1}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}.\tag{3.20}$$

This, as advertized, is the two-dimensional axial anomaly equation [4].

It is interesting to check explicitly the gauge invariance of ΔQ_5 Eq. (3.17). Upon gauge transformation, the gauge potentials transform according to[7]

$$A^g(x,t)=A(x,t)+\partial_x g(x,t)\tag{3.21}$$

so that

$$\Omega_0^g=\Omega_0+\frac{1}{2\pi}\left[g(L,t)-g(0,t)\right].\tag{3.22}$$

Because upon gauge transformation $\psi^g=e^{-ig(x,t)}$, if one requires that the boundary conditions on fermions be preserved by the transformation (for example the antiperiodic

[7] We consider transformation which preserve the condition $A_0=0$. The more general case does not bring in any new features.

boundary conditions which follow from Fermi statistics) it follows that g must satisfy the condition

$$g(L,t) - g(0,t) = 2\pi n; \quad n \in \mathbb{Z} \ , \tag{3.23}$$

which in turn implies

$$\Omega_0^g(t) = \Omega_0(t) + n; \quad n \in \mathbb{Z} \ . \tag{3.24}$$

Otherwise stated, the functions $g(x,t)$ for each t fall in equivalence classes labelled by the integer n, associated to the boundary condition they satisfy. This is again a manifestation of a nontrivial topology (homotopy), just like that discussed in Sect. 2.2 for the Skyrmion (in that case of the flavor group, in the present case of the color group, however). Indeed, the gauge group is isomorphic to a one-sphere S_1, i.e. a circle, and thus is infinitely connected: maps $g: S_1 \to S_1$ from a circle onto it fall into equivalence classes (homotopy classes), which form a group, the fundamental group $\pi_1[U(1)]$, isomorphic to the integers $\mathbb{Z}$. Because space is compact ($0 \leq x \leq L$, with fixed boundary conditions) it may be viewed as a circle, and then the functions $g(x,t)$, viewed as functions of x for each fixed t provide such maps. Furthermore, because n in Eq. (3.23) is constrained to be an integer, functions $g(x,t)$ associated to distinct values of n cannot be smoothly deformed into each other. It follows that n must be independent of t, and thus the integer n actually classifies equivalence classes of functions $g(x,t)$.

Now, upon homotopically trivial gauge transformations (i.e. those which preserve the boundary condition trivially, with $n=0$) the two terms on the r.h.s. of the expression Eq. (3.12) of the spectral asymmetry are separately gauge invariant, i.e., the index and the Chern-Simons term on the r.h.s. of the index formula Eq. (3.3) are separately invariant. However, if $n \neq 0$ then, because of Eq. (3.24), the Chern-Simons is not gauge invariant, but rather, it varies by an integer. However, the index, which is expressed by to Eq. (3.13) in terms of the Chern-Simons term itself, varies by the same integer, so that the spectral asymmetry (and thus the vacuum-induced charge) remains gauge invariant. What about the total charge created at time t, $\Delta Q_5(t)$ Eq. (3.17) ? The continuous part of the spectral asymmetry, being proportional to the Chern-Simons term, is not gauge invariant, however, because n is t-independent the created charge still is:

$$\Delta Q_5^g(t_1,t_2) = 2\left[(\eta_c(t_2)+n) - (\eta_c(t_1)+n)\right] = \Delta Q_5(t_1,t_2). \tag{3.25}$$

It is apparent that many of the simple results derived in this section rely crucially on having considered massless fermions on a compact space. If either or both of these simplifying assumptions are relaxed several subtleties arise, essentially because one has then to deal with more complicated spectra, and suitably generalize the definition of induced charge and spectral asymmetry. These complications are necessarily present in a realistic treatment, and are related to the way the axial anomaly manifests infrared and ultraviolet divergences of the theory. Tackling them will thus give us a handle on problems which are present in realistic four-dimensional theories.

The derivation presented in the previous section shows that ultimately the origin of the anomaly can be traced to the definition of the charge operator through normal ordering according to Eq. (2.2): a charge is generally induced by the presence of background fields which distort the positive-negative energy symmetry of the vacuum spectrum, but the corresponding contributions to the fermion charge actually cancel because of the U(1) symmetry of the Lagrangian; however, this cancellation cannot be separately enforced for right-handed and left-handed states, so that the axial charge is not conserved.[8]

If the theory had a finite number of states charge nonconservation would only be manifest when an energy eigenstate changes sign; due however to the presence of an infinite numer of states the need to regularize the ultraviolet divergence of the sum over states Eq. (2.6) induces a nonvanishing spectral asymmetry and thus anomalous charge creation even when no level crossings occur. If we consider a system defined on open, rather than compact space, the spectrum is continuous (or at least in general it will have a continuum component), and this divergence will be accordingly harder to handle. In particular, it would seem superficially that the spectral asymmetry cannot be computed if the Hamiltonian Eq. (3.2) is taken to act on open space, $-\infty \leq x \leq \infty$: if we let $L \to \infty$ in Eq. (3.7) the eigenvectors are generic plane waves, with eigenvalues $E_k = k$, $-\infty \leq k \leq \infty$, so that the spectral asymmetry reduces to $\int_{-\infty}^{\infty} dk \, \frac{\mathrm{sign}\, k}{k^s}$. This is ill-defined, because the continuum spectrum extends all the way to vanishing energy: it will diverge in the infrared if $s \geq 1$, as required in order to regulate the functional trace in the ultraviolet.

We may, however, define the open space problem as the $L \to \infty$ limit of the compactified one. In such case, the continuous part of the spectral asymmetry (and hence the anomalous charge) is given by

$$\eta_c = \lim_{L\to\infty} \lim_{s\to 0} \left(\frac{2\pi}{L}\right)^{-s} \sum_{k=-\infty}^{\infty} \left| k + \Omega_0 + \frac{1}{2} \right|^{-s} \mathrm{sign} \left(k + \frac{1}{2} \right). \tag{3.26}$$

In order for this definition to make sense we must check that the two limits commute. If the $s \to 0$ limit is taken first, then, using Eq. (3.16) we get

$$\eta_c = \lim_{L\to\infty} \frac{1}{\pi} \int_0^L dx \, A(x,t) = \frac{1}{\pi} \int_0^\infty dx \, A(x,t). \tag{3.27}$$

[8] It is of course possible to modify the definition of normal-ordering in Eq. (2.2) in such a way that axial charge be conserved at the expense of violating vector charge conservation: it is enough to define the normal ordering with opposite signs for the left-handed and right-handed parts of the spectrum. This is equivalent to adopting an opposite filling prescription (positive energy states filled and negative energy states empty) for, say, left-handed modes. Then, the roles of vector and axial charges are interchanged. However, since with this choice the vector current is not conserved, gauge invariance is lost and the theory cannot be meaningfully quantized.

If we take the $L \to \infty$ limit first, we get instead

$$\eta_c = \lim_{s\to 0}\lim_{L\to\infty} -\left(\frac{2\pi}{L}\right)\left[\zeta\left(s, \Omega_0+\frac{1}{2}\right)-\zeta\left(s,\frac{1}{2}-\Omega_0\right)\right]. \tag{3.28}$$

Now Ω_0 grows with L (the average value of A being fixed), hence we may use the asymptotic expansion $\zeta(s,x) \underset{x\to\infty}{\sim} \frac{1}{\Gamma(s)}x^{1-s}\Gamma(s-1)+O\left(x^{-s}\right)$ to get

$$\eta_c = \lim_{s\to 0}\lim_{L\to\infty} -\left(\frac{2\pi}{L}\right)^{-s}\frac{1}{s-1}2(\Omega_0)^{1-s} = 2\Omega_0(t), \tag{3.29}$$

in agreement with Eq. (3.27).

Hence, it is possible to define the spectral asymmetry (and thus the anomalous charge) for a continuous spectrum, but then infrared and ultraviolet divergences of the spectral asymmetry Eq. (3.26) must be regulated independently: the latter by the usual ζ-function regulator, keeping s large, and the former by putting the system in a box of size L (so that the spectrum is discretized) and recovering the continuum problem in the limit [13]. This regulates the infrared divergence because the lowest-energy modes are then lifted to have energy of order $\sim \frac{1}{L}$. In fact, in the large L limit the summation in Eq. (3.26) is well approximated by an integral, so that the naive expression of the spectral asymmetry as an integral over all k is recovered; however, the lowest-energy level has energy of order $\frac{2\pi}{L}$, so that the spectral integration does not reach zero, which is enough to regulate the infrared divergence:

$$\eta_c = \lim_{s\to 0}\lim_{L\to\infty}\frac{L}{2\pi}\left[\int_{-\infty}^{\Omega_0-\epsilon(L)}+\int_{\Omega_0+\epsilon(L)}^{\infty}\right]dk\,\frac{\operatorname{sign}k}{\left|k-\frac{2\pi}{L}\left(\Omega_0-\frac{1}{2}\right)\right|^{s}}, \tag{3.30}$$

where $\epsilon(L)=\frac{2\pi}{L}\{\Omega_0-\frac{1}{2}\}$. Of course computing the integral explicitly leads back to the result Eq. (3.29). It is interesting to notice that in the $L\to\infty$ limit charge creation is due to the fact that level crossings occur at all times; the infrared divergence which we have been discussing is then related to the discontinuous behavior of the spectral asymmetry when level crossings occur. The spectral asymmetry proper, i.e. η, obtained from η_c Eq. (3.30) by the replacement $\operatorname{sign}k \to \operatorname{sign}\left(k-\frac{2\pi}{L}\left(\Omega_0-\frac{1}{2}\right)\right)$, vanishes as $\frac{1}{L}$ as $L\to\infty$.

A common alternative way of regulating infrared divergences consists of adding a mass term to the Hamiltonian which will lift the lowest energy (and then taking $m\to 0$). This case is also interesting to discuss *per se* since it leads us to evaluate the anomalous charge for massive fermions. We thus replace the block-diagonal Hamiltonian Eq. (3.2) with

$$H^m = \begin{pmatrix} -D_1 & m \\ m & D_1 \end{pmatrix} = H + m\gamma_0, \tag{3.31}$$

whose eigenvalues and eigenvectors are

$$\widetilde{\psi}_k^{m\pm} = a_k^\pm \psi_k^+ + b_k^\pm \psi_k^- \tag{3.32}$$

$$E^m{}_k^\pm = \pm\sqrt{E_k^2 + m^2}, \tag{3.33}$$

where $a_k^\pm$ and $b_k^\pm$ are functions of m and E_k which are easily determined by diagonalization of the two by two matrix $\langle\psi_k^\pm|H^m|\psi_k^\pm\rangle$. In the spectrum Eq. (3.33) level crossings never occur: this suggests then that in the continuum limit the anomalous charge and the axial anomaly would vanish for massive fermions. Indeed, if we now compute the axial charge according to Eq. (2.24) we get

$$\begin{aligned}\eta[i\gamma_5 H^m] &= \lim_{s\to 0} \mathrm{Tr} i\gamma_5 \frac{\mathrm{sign}\, H^m}{|H^m|^s} = \lim_{s\to 0} \mathrm{Tr} i\gamma_5 \frac{H^m}{((H^m)^2)^{s+\frac{1}{2}}} = \lim_{s\to 0} \mathrm{Tr} i\gamma_5 \frac{H + m\gamma_0}{(H^2+m^2)^{s+\frac{1}{2}}} \\ &= \lim_{s\to 0} \mathrm{Tr} i\gamma_5 \frac{H}{(H^2+m^2)^{s+\frac{1}{2}}} = \lim_{s\to 0} 2 \sum\!\!\!\!\!\!\int_k \frac{E_k}{\sqrt{E_k^2+m^2}} \frac{1}{(E_k^2+m^2)^s},\end{aligned} \tag{3.34}$$

i.e. the result in the massive case is obtained from the massless expression (the spectral asymmetry Eq. (2.25)) with the two replacements $\mathrm{sign}\, E_k \to \frac{E_k}{\sqrt{E_k^2+m^2}}$ and $E_k^{-s} \to \left(E_k^2+m^2\right)^{-s}$. In the continuum limit this reduces to

$$\eta[H^m] = \lim_{s\to 0} \lim_{L\to\infty} \left(\frac{L}{2\pi}\right)^{-2s} \int dk\, \frac{k+\frac{1}{2}-\Omega_0}{\left[\left(k+\frac{1}{2}-\Omega_0\right)^2+m^2\right]^{s+\frac{1}{2}}}. \tag{3.35}$$

Now if we let $m \to 0$ in Eq. (3.35) we recover the results discussed previously; however, if we keep $m > 0$, then even in the limit there is no infrared singularity (which remains regulated by m), and we may let $L \to \infty$; however then we may shift the integration variable and obtain

$$\eta[H^m] = \lim_{s\to 0} \lim_{L\to\infty} \left(\frac{L}{2\pi}\right)^{-2s} \int dk\, \frac{k}{(k^2+m^2)^{s+\frac{1}{2}}} = 0. \tag{3.36}$$

Hence indeed the spectral asymmetry in the continuum limit vanishes for massive fermions; furthermore, this implies that the chiral ($m \to 0$) limit does not commute with the continuum limit. Finally, since in the massive case no level crossings may occur in the spectrum Eq. (3.33) as Ω_0 is varied, the spectral asymmetry is a continuous function of the background, i.e. $\eta = \eta_c$. This would seem to imply that charge creation and the anomaly disappear for any $m > 0$ and are only present in the strict chiral limit, with the somewhat paradoxical implication that the chiral limit of the axial charge and its time derivative is not smooth.

This however is not the case: the anomaly persists even with nonvanishing mass, and the chiral limit is smooth. The resolution of the paradox is based on two observations: first, in the massive case the axial charge is not conserved even at the classical level, and furthermore, in the massive case the time dependence of the background induces transitions of the system form the vacuum to excited states even in the absence of level crossings.

The first point is trivial: if we add a mass term to the Hamiltonian according to Eq. (3.31) the chiral symmetry of the Lagrangian is broken explicitly; as a consequence the classical axial current satisfies

$$\partial_\mu j_5^{\mu\,\mathrm{cl}} = 2im\bar\psi\gamma_5\psi. \tag{3.37}$$

It follows that once the theory is quantized the time dependence of the vacuum charge will be given by

$$\langle \dot Q_5\rangle = \langle im\bar\psi\gamma_5\psi\rangle + \langle \dot Q_5^{\mathrm{an}}\rangle, \tag{3.38}$$

where the first contribution is due to charge nonconservation at the classical level according to Eq. (3.37), and Q_5^{an} denotes a possible anomalous contribution induced at the quantum level, which if $m=0$ is given by the anomaly Eq. (3.19)-(3.20), and in the massive case is still to be determined. Now, the spectral asymmetry $\eta[H^m]$ Eq. (3.34) gives the total vacuum charge: hence the vanishing of it in general does not imply that the anomalous charge vanishes, i.e. that both contributions on the r.h.s. of Eq. (3.38) separately vanish, but rather that, in general, they cancel.

Now, the first term on the r.h.s. of Eq. (3.38) can be computed through essentially the same steps as performed in Eq. (2.10) to determine the vacuum charge, i.e. differentiating the path integral with respect to a source for $\bar\psi\gamma_5\psi$, and then Wick-rotating to Euclidean space:

$$\langle im\bar\psi\gamma_5\psi\rangle = \mathrm{Tr} im\gamma_5\frac{1}{i\,\not{D}+im} = im \sum\!\!\!\!\!\!\int_k \frac{\bar\psi\gamma_5\psi}{\lambda_k+im}, \tag{3.39}$$

where $i\,\not{D}$ is the (massless) two-dimensional Dirac operator of the theory, with eigenvalues and eigenvectors $i\,\not{D}\,\psi_k = \lambda_k\psi_k$, and the mass term is imaginary after Wick rotation. Now, because $\{i\gamma_5,\gamma^\mu\}=0$ it follows that $\{i\gamma_5, i\,\not{D}\}=0$, and hence acting with $i\gamma_5$ on an eigenvector yields a new eigenvector with opposite eigenvalue: $i\gamma_5\psi_k=\psi_{-k}$, $\lambda_{-k}=-\lambda_k$; unless $\lambda_k=0$ in which case $i\gamma_5\psi_k^\pm = \pm\psi_k^\pm$. But eigenvectors associated to different eigenvalues are orthogonal, hence only the contribution from zero modes to the trace survives; the latter equals the number of zero modes weighted by their chirality, i.e., the index of $i\,\not{D}$:

$$\langle im\bar\psi\gamma_5\psi\rangle = \mathrm{ind}\, i\,\not{D}\,. \tag{3.40}$$

In the continuum limit the spectral asymmetry $\eta[D_1]$ of the *massless* Dirac operator vanishes, as it is clear letting $L\to 0$ in Eq. (3.12); the index theorem Eq. (2.27), (3.3) then implies $\mathrm{ind}\, i\, \not{D} = \Omega_0$. It follows that the vacuum charge Eq. (3.38) will indeed vanish in the continuum limit, in agreement with Eq. (3.36), provided the anomalous charge is equal to $\langle \dot{Q}_5^{\mathrm{an}} \rangle = -2\Omega_0$. But this is exactly the same as the anomalous charge in the massless case, Eq. (3.17). We thus conclude that in fact the anomalous charge and its time dependence as given by the anomaly equation are unchanged by the addition of a mass term; however, in the presence of mass the anomaly equation generalizes to

$$\partial_\mu j_5^\mu = 2im\bar{\psi}\gamma_5\psi - \frac{1}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}, \tag{3.41}$$

i.e., it acquires an extra contribution which, when averaged over the vacuum in the continuum limit, exactly removes the anomalous contribution so that the total vacuum charge vanishes.

If we identify the vacuum charge with the charge induced by the background on the system, this would still imply a non-smooth behavior of the charge in the chiral limit. But this is not the case since, as we already mentioned, when the mass is unequal to zero the background induces transition of the system from the vacuum to excited states even in the absence of level crossings. This can be seen immediately by inspection of the expression Eq. (3.32) of the eigenvectors of the massive Hamiltonian: the eigenvectors are linear combinations of pairs of eigenstates of the massless Hamiltonian $\psi_k^\pm$ with coefficients which depend on E_k and thus on the background. If the latter is time dependent, the coefficients also are, thus the, say, positive energy eigenstate $\psi_k^+(t)$ will be in general a linear combination of the two states $\psi_k^\pm(t')$ at a different time $t' \neq t$. Therefore, a system such that all states E^{m-} are filled at time t_0 will have nonvanishing overlap with the state at time t where also states E^{m+} are filled, and will thus in general undergo transitions to this state with a nonzero rate.

This rate can be computed exactly [14], and will depend in general on the rate of time variation of the potential A (i.e., on the strength of the electric field $\partial_0 A$). If the transition is very slow on the scale set by the mass m then we may neglect it: this is the adiabatic approximation, which consists of assuming that the system remains in the state in which it has been prepared. In this limit the rate of charge creation is given by Eq. (3.38), i.e. it vanishes. The opposite limit obtains when m is small on the scale set by the transition rate, i.e. compared to the size of the electric field (in natural units). We know already the result in such case, because in this limit we may neglect the mass term, so that the problem reduces to that discussed in sect. 2.2: there is only one contribution to charge creation, which is determined by the continuous part of the spectral asymmetry Eq. (3.18).

We may, however, check this explicitly by computing the contribution to the charge due to transitions to excited states in this case. This turns out to be quite easy, because in the limit of very rapid transitions we may use the sudden approximation, which is the opposite extreme as the adiabatic approximation. Whereas in the adiabatic limit one assumes that the system remains in the state in which it was prepared in the sudden limit the system is assumed to undergo with unit probability transitions to all states with which the initial state has nonvanishing overlap, so that the probability of each of these transitions is given by the overlap itself. In our case, this means that if the system is prepared at $t = t_0$ by filling all negative-energy states $\widetilde{\psi}_k^{m-}$ [as given by Eq. (3.32)with $a_k^- = a_k^-(t_0)$] then at time t it will undergo transitions to states in which the positive energy states $\widetilde{\psi}_k^{m+}$ are filled with a rate given by $a_k^+(t)a_k^-(t_0) + b_k^+(t)b_k^-(t_0)$ (and likewise for negative energy states). It follows that the charge of the system at time t — the charge of the Dirac sea — is actually not the number of negative energy states: rather, it is given by summing over all states (positive and negative energy), with a weight given by the transition probability from filled states (i.e. those which had negative energy at time t_0). That is, by replacing in Eq. (2.9) $n^- \to Q^-$, where Q^- is the charge of the filled states:

$$\begin{aligned} Q^-(t_0,t) = & \sum\!\!\!\!\!\!\!\int_k |\langle \widetilde{\psi}_k^m(t)|\widetilde{\psi}_k^{m-}(t_0)\rangle|^2 = \mathrm{Tr}\left(\sum\!\!\!\!\!\!\!\int_k |\widetilde{\psi}_k^{m-}(t_0)\rangle\langle\widetilde{\psi}_k^{m-}(t_0)| =\right) \\ & = \mathrm{Tr}\left[\Theta(-H(t_0))\right], \end{aligned} \tag{3.42}$$

where the sum extends to all states Eq. (3.33). This is recognized to be the same as the expression [Eq. s (2.12), (2.9)] of the charge in terms of a spectral asymmetry Eq. (2.6), but with the replacement $H(t) \to H(t_0)$. A rerun of the steps which lead from the charge of the Dirac sea to the spectral asymmetry thus shows that $Q^-(t_0,t)$ is simply found by evaluating evaluating the spectral asymmetry with the sign function computed with respect to the initial Hamiltonian $H(t_0)$.

Hence in the sudden approximation the expression for the axial charge created in the massive case becomes

$$\Delta^m Q_5(t_1,t_2) = -\left(\eta_{t_0}[D_1(t_2)] - \eta_{t_0}[D_1(t_1)]\right) \tag{3.43}$$

where η_{t_0} denotes the spectral asymmetry defined from the filling prescription at time t_0 as discussed above:

$$\eta_{t_0}[D_1(t)] = \lim_{s\to 0} \mathrm{Tr}\frac{\mathrm{sign}\, H^m(t_0)}{|H^m(t)|^s} = \lim_{s\to 0} \sum\!\!\!\!\!\!\!\int_k \frac{E^k(t_0)}{\sqrt{E(t_0)_k^2 + m^2}} \frac{1}{(E_k^2(t) + M^2)^s}. \tag{3.44}$$

We may evaluate this explicitly, by assuming e.g. that at t_0 no background is present, $A(t_0) = 0$. We then have

$$\begin{aligned} \Delta^m Q_5(t_0,t) &= -\lim_{s\to 0}\lim_{L\to\infty}\left(\frac{L}{2\pi}\right)^{-s}\int_{-\infty}^{\infty} dk\, \frac{k}{\sqrt{k^2+m^2}} \frac{1}{\left[(k + +\frac{1}{2} - \Omega_0)^2 + m^2\right]^s} \\ &= -\lim_{s\to 0}\lim_{L\to\infty}\left(\frac{L}{2\pi}\right)^{-s}\int_0^{\infty}\frac{dt}{\Gamma(s)}t^{s-1}\int_{-\infty}^{\infty} dk\, \frac{k}{\sqrt{k^2+m^2}} e^{-t\left[(k+\frac{1}{2}-\Omega_0)^2+m^2\right]} \\ &= -\lim_{t\to 0}\lim_{L\to\infty}\int_{-\infty}^{\infty} dk\, \frac{k}{\sqrt{k^2+m^2}} e^{-t\left[(k+\frac{1}{2}-\Omega_0)^2+m^2\right]}. \end{aligned} \tag{3.45}$$

In the last step we have used the fact that since $\Gamma(s)$ has a simple pole with residue equal to one at $s = 0$, in the limit $s \to 0$ only the residue of the pole of the t integral survives; this in turn comes from the lower end of the integration range $\int_0^\epsilon dt\, t^{s-1} e^{-t\lambda} = \frac{1}{s} \lim_{t\to 0} e^{-t\lambda}[1 + O(s)]$. The last integral is easily evaluated in the limit and gives

$$\Delta^m Q_5(t_0, t) = \lim_{L\to\infty} 2\left(\Omega_0 + \frac{1}{2}\right) = 2\Omega_0. \tag{3.46}$$

This is what we set out to prove: the result of a calculation in the massive case in the sudden approximation is the same as the massless result Eq. (3.17).

In summary, we have shown that the rate of charge creation in the massive case depends on the relative size of the mass and the electric field (the time derivative of the gauge potential). If the mass is very large, then charge creation is suppressed, if it is very small it is given by Eq. (3.17). The result in the limit of small mass can be found using the sudden approximation, and reproduces that found in the massless case, thereby showing that the massless limit is smooth. The result in the case of large mass can be found in the adiabatic approximation, and shows that there is no charge creation. In fact, in the massless case the adiabatic approximation is exact (since there are no scales); the charge created is then given by the modified spectral asymmetry according to Eq. (3.17). If we use the mass as an infrared regulator we are interested in taking $m \to 0$ eventually. In order to obtain a correct result, we must then either use the sudden approximation, or introduce an extra infrared regulator (such as box-quantization), and then let $m \to 0$ before we use the adiabatic approximation.

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