Howe's Rank and Dual Pair Correspondence in Semistable Range

by

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ABSRACT

Let G be a classical group of type I. For an irreducible unitary representation, Howe defined the notion of rank in analytic terms. On the algebraic side, there is the theory of primitive ideals and associated variety. In the first part of this thesis, we relate Howe's rank with the associated variety.

In the second part, We study the Bargmann-Segal model of the oscillator representation. Based on this model, we construct an analytic compactification of the symplectic group. We also construct an analytic compactification of the orthogonal group. All the compactifications are compact symmetric spaces.

In the third part, we define semistable range in the dual pair correspondence, and give an explicit construction of the dual pair correspondence in the semistable range. Finally, we prove the nonvanishing theorems of the dual pair correspondence in the semistable range for $(O_{p,q}, Sp_{2n}(\mathbb{R}))$. Our proof is based on some density theorems on some compact symmetric spaces.

Thesis Supervisor: David Vogan
Title: Professor of Mathematics

The LORD is my shepherd; I shall not want.
He maketh me to lie down in green pastures:
he leadeth me beside the still waters.
He restoreth my soul:
he leadeth me in the paths of righteousness for his name's sake.
Yea, though I walk through the valley of the shadow of death, I will fear no evil:
for thou art with me; thy rod and thy staff they comfort me.
Thou preparest a table before me in the presence of mine enemies:
thou anointest my head with oil; my cup runneth over.
Surely goodness and mercy shall follow me all the days of my life:
and I will dwell in the house of the LORD for ever.

—Psalm 23

For with much wisdom comes much sorrow;
the more knowledge, the more grief.

—Ecclesiastes 1:18

But Jesus called the children to him and said,
"Let the little children come to me,
for the kingdom of God belongs to such as these.
I tell you the truth,
anyone who will not receive the kingdom of God like a little child will never enter it."

—Luke 18:16-17

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Above all, living in a confused age, I found the divine words the light for my feet and ever shining on my way....

Contents

1 Introduction

Let G be a connected semisimple noncompact group. Let $\hat{G}$ be the unitary dual of G, and $\hat{G}_{ad}$ be the admissible dual of G. Langlands gave a nice classification of $\hat{G}_{ad}$ for linear semisimple groups. Currently, there are two important unsolved problems in the representation theory of G. One is to classify all the irreducible unitary representations. Another is to construct these representations. There are two major techniques to construct an admissible representation, namely, parabolic induction and cohomological induction [Vogan0]. In a lot of cases, unitarity can be determined once the construction is under way. However, there is one class of mysterious representations, the so called unipotent representations that can not be constructed in either approach. Originally, this thesis was aimed at a better understanding of the unipotent representations. One available tool to study the unipotent representation is the dual pair correspondence of Howe. In this thesis, we study Howe's rank and dual pair correspondence in hoping that these studies could lead to a better understanding of unitary representations in general and unipotent representations in particular.

1.1 Dual Pair Correspondence and Lower Rank Representations

The oscillator Representation (also called symplectic spinor, metaplectic representation) is probably the most intensively studied unipotent representation. It was studied by Bargmann, Segal, Shale and Weil in the sixties. The oscillator representation is a unitary representation of the *metaplectic group*, the double covering of the symplectic group Sp. Two major models of the oscillator representation were built along the way, namely the Schrödinger model and Bargmann-Fock-Segal model. We denote such a representation by ω. We always have $\omega(\epsilon) = -1$, where $\{1, \epsilon\}$ is the preimage of the identity under the metaplectic covering. In this thesis, if G is a subgroup of Sp, we will use $\widetilde{G}$ to denote the preimage of G under the metaplectic covering.

Following the work of Siegel, Weil reformulated the theory of theta-series in order to study automorphic forms. Roughly speaking, a pair of reductive subgroups (G_1, G_2) of $Sp(V, \Omega)$ is said to be a *reductive dual pair* if each of G_1 and G_2 is the centralizer of the other. Let $(\omega, \mathcal{P})$ be the Harish-Chandra module of the oscillator representation of $\widetilde{Sp(V, \Omega)}$. Let $\mathcal{R}(\tilde{G}, \omega)$ be the space of irreducible admissible representations of $\widetilde{G}$ which occur as quotient of ω (in a proper category). Howe proved that, for classical real groups, $\mathcal{R}(\widetilde{G_1 G_2}, \omega)$ yields an one-to-one correspondence between $\mathcal{R}(\widetilde{G_1}, \omega)$ and $\mathcal{R}(\widetilde{G_2}, \omega)$. This is often called Howe's correspondence or the dual pair correspondence [Howe1]. We denote it by ω. Howe proved that for $\pi \in \mathcal{R}(\tilde{G}_1, \omega)$, $\omega(\pi)$ can be regarded as a unique quotient of a natural module $\omega_0(\pi)$.

The first success of using the dual pair correspondence to construct unipotent representations came about in J-S Li's thesis. Li constructed a class of interesting singular unitary representations often called *lower rank representations* in the sense of Howe. Roughly speaking, we say G_1 is in the stable range of G_2 if the rank of $\mathfrak{g}_{1\mathbb{C}}$ is less or equal to the real rank of G_2. Let $\widetilde{G}$ be the preimage of G under the metaplectic covering of Sp, and $\hat{\tilde{G}}(\epsilon)$ be those unitary representations satisfying $\pi(\epsilon) = -1$. Li proved that for type I dual pairs the dual pair correspondence yields a one to one correspondence between the unitary dual $\widehat{\widetilde{G_1}}(\epsilon)$ and the lower rank unitary representations of $\widetilde{G_2}(\epsilon)$ up to a central character [Li1] [Li2]. By utilizing the nice geometry of the stable range dual pairs, Li succeeded in constructing Howe's quotient using Mackey's theory and proved the unitarity using the mixed model of the oscillator representation. Of course, for nonstable range dual pairs, Mackey's theory and the mixed model would not work.

1.2 Invariants Associated with a Representation

Generally speaking, classification problems are approached by constructing invariants. In representation theory, some of the natural objects to study are the invariants associated with an equivalence class of irreducible representations. Of course, one hopes that these studies could shed some lights on the classification and construction of unitary representations. The first and foremost important invariant is the infinitesimal character, studied by Harish-Chandra and others. Then along this line, Langlands studied the growth condition on the matrix coefficients of an irreducible representation and gave a classification of all the irreducible admissible representations. However, the problem of constructing irreducible representations is still not completely understood.

To unveil the algebraic structure of an irreducible representation, Vogan studied the Gelfand-Kirillov dimension for Harish-Chandra modules. Along this line, one can build a few geometric invariants, for example, associated variety, asymptotic cycle and wave front set. Of course all these invariants are tied up with the orbit method developed by Kirillov, Kostant and Vogan which we will not discuss here. It suffices to say that associated variety is the right object to study in order to understand the unipotent representations.

Roughly speaking, for every irreducible admissible representation π, there is a Harish-Chandra module V_π associated with it. This module is irreducible as a $U(\mathfrak{g})$ module. Associated with V_π is the annihilator $Ann(V_\pi)$. Since $U(\mathfrak{g})$ has a natural filtration, $Ann(V_\pi)$ inherits a filtration from $U(\mathfrak{g})$. The associated variety $\mathcal{V}(Ann(V_\pi))$ can be defined as the associated variety of $gr(Ann(V_\pi))$ in $\mathfrak{g}_{\mathbb{C}}^*$. For a reductive Lie algebra $\mathfrak{g}_{\mathbb{C}}$, It is well-known that for π irreducible, $\mathcal{V}(Ann(V_\pi))$ is a closure of a single nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}^*$. Since $\mathfrak{g}_{\mathbb{C}}^*$ can be identified with $\mathfrak{g}_{\mathbb{C}}$ through an invariant bilinear form on $\mathfrak{g}_{\mathbb{C}}$, sometimes we will regard $\mathcal{V}(Ann(V_\pi))$ as a subvariety of $\mathfrak{g}_{\mathbb{C}}$. We use $\mathcal{R}(\mathcal{O})$ to denote the set of irreducible representation π such that $\mathcal{V}(Ann(V_\pi))$ is equal to $\mathcal{O}$.

Thus it is now an interesting problem to see what are the possible associated varieties for lower rank representations constructed by J-S Li. It is even more interesting to see if dual pair correspondence can produce all the unipotent representations.

1.3 Associated Variety and Howe's Rank

The notion of rank of a unitary representation was introduced by Howe for $G = Sp_{2n}(\mathbb{R})$. Let N be the (Abelian) nilradical of the maximal parabolic subgroup P corresponding to the roots $\{e_1 - e_2, \ldots, e_{n-1} - e_n\}$. In this case, $\hat{N}$ can be regarded as the space of symmetric bilinear forms. For a unitary representation π of G, we consider its restriction on N. According to Stone's theorem, $\pi|_N$ is uniquely determined by a spectral measure $\mu_N(\pi)$. Howe defined the notion of N-rank of π to be the highest rank of the support of $\mu_N(\pi)$ regarded as symmetric bilinear forms. Later, Jian-Shu Li extended the ZN_k-rank to all the type I classical groups (see Definition 2.1). For type II classical groups, namely, $GL(n, D)$ ($D = \mathbb{R}, \mathbb{C}, \mathbb{H}$), the unitary dual is more or less well-understood (see [Vogan]). In this thesis, We will only consider type I classical groups. We prove the following theorem relating ZN_k-rank with associated variety.

Theorem 1.1 *Let (π, H) be an irreducible unitary representation of a type I classical group G. Then*

1. *for $G = Sp_{2n}(\mathbb{R}), U(p,q)$, ZN_k-rank of (π, H) equals* $\min(k, rank(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))))$;
2. *for $G = O_{p,q}$, ZN_k-rank of (π, H) equals* $\min(k, rank(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))))$ *if k is even,* $\min(k-1, rank(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))))$ *if k is odd;*
3. *for $G = O^*(2n), Sp(p,q)$, ZN_k-rank of (π, H) equals* $\min(k, \frac{1}{2} rank(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))))$;
4. *for $G = Sp(n, \mathbb{C})$, ZN_k-rank of (π, H) equals* $\min(k, \frac{1}{2} rank(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))))$;
5. *for $G = O(n, \mathbb{C})$, ZN_k-rank of (π, H) equals* $\min(k, \frac{1}{2} rank(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))))$ *when k is even, and* $\min(k-1, \frac{1}{2} rank(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))))$ *when k is odd.*

Since $\mathfrak{g}$ is a classical Lie algebra, its complexification $\mathfrak{g}_{\mathbb{C}}$ can be regarded as a matrix Lie algebra. $rank(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi)))$ here is defined to be the maximal rank of the elements in $\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))$.

Now for $G = Sp_{2n}(\mathbb{R})$, we know that the set of complex nilpotent orbits is in one to one correspondence with the partitions of $2n$

$$\lambda_1 \geq \lambda_2 \ldots \geq \lambda_k > 0, \qquad \sum_1^k \lambda_i = 2n$$

such that odd parts occur with even multiplicities. We call such a partition a symplectic partition, we denote the nilpotent orbit corresponding to such partition by $\mathcal{O}_\lambda$. Very briefly, the lower rank representations are the unipotent representations attached to those nilpotent orbits $\mathcal{O}_\lambda$ with $k > n$.

Also one can easily observe that if we write a sympletic partition as (the rows of) a Young diagram. By deleting the first column, we obtain a (smaller) orthogonal partition. Here an orthogonal partition is a partition where even parts occur with odd multiplicities. Conversely, by deleting the first column of an orthogonal partition, we obtain a (smaller) symplectic partition. One remarkable phenomenon proved by Przebinda is that dual pair correspondence in the stable range actually descends to a correspondence between the nilpotent orbits of sympletic groups and nilpotent orbits of orthogonal groups (see [Przebinda]). Explicitly, for the dual pair $(O_{p,q}, Sp_{2n}(\mathbb{R}))$ with $n > p+q$, the orbit correspondence takes an arbitrary orthogonal partition of $p+q$ to a symplectic partition of $2n$ by adding a first column of size $2n-p-q > n$. Therefore even if we assume a complete description of $\widehat{O_{p,q}}$, we can only hope to construct those irreducible unitary representations attached to the nilpotent orbits $\mathcal{O}_\lambda$ with $k > n$. Thus it is necessary for us to study the dual pair correspondence in non-stable range. If the dual pair correspondence was well-understood in the non-stable range, one may hope to build a zig-zag construction model for each unipotent representation via dual pair correspondence.

1.4 Dual Pair Correspondence in the Semistable Range

For a nonstable range dual pair, not much is known about Howe's correspondence. For instance, we do not know much about $\mathcal{R}(\widetilde{G}, \omega)$, we do not know how to construct $\omega(\pi)$ from π, and we do not know whether unitarity is preserved. The main theme in my thesis is to search for those irreducible representations occuring in $\mathcal{R}(\tilde{G}, \omega)$ and to finish the construction of $\omega(\pi)$ for π in a certain range.

Let $(G, G') \subseteq Sp(V, \Omega)$ be a reductive dual pair. Let $\pi \in \mathcal{R}(\widetilde{G}, \omega)$. Following J-S Li, we define formally the averaging operator

$$\mathcal{L}_{\widetilde{G}} : \mathcal{P}^c \otimes V_\pi \rightarrow Hom_{\mathfrak{g}, \widetilde{K}}(\mathcal{P}, V_\pi)$$

by

$$\mathcal{L}_{\widetilde{G}}(\psi \otimes v)(\phi) = \int_{\widetilde{G}} (\phi, \omega(g)\psi)\pi(g) v dg \qquad (\psi \in \mathcal{P}^c, \phi \in \mathcal{P}, v \in V_\pi)$$

We observe that the image of the averaging operator is in fact a $(\mathfrak{g}', \widetilde{K}')$-module. The first problem settled in this thesis is when $\mathcal{L}_{\widetilde{G}}(\mathcal{P}^c \otimes V_\pi)$ is well-defined. Roughly speaking, we say that π is in the semistable range of $(\tilde{G}, \widetilde{G'})$ if $\mathcal{L}_{\widetilde{G}}$ is well-defined and $\pi(\epsilon) = -1$. In this thesis, we give a precise description of the semistable range for $(Sp_{2n}, O_{p,q})$ in terms of the growth condition on the matrix coefficients. Then the Langlands parameters in the semistable range can be read off from the growth condition. We will prove that

Theorem 1.2 (Construction) *Suppose that π is in the semistable range of $\tilde{G}$. If $\mathcal{L}_{\tilde{G}}$ does not vanish, then $\pi \in \mathcal{R}(\widetilde{G}, \omega)$. The converse is also true. Furthermore, the image of $\mathcal{L}_{\tilde{G}}$ is irreducible and isomorphic to the dual representation of $\omega(\pi)$ (in the category of Harish-Chandra modules).*

From this theorem, our construction of $\omega(\pi)$ will be complete if we can prove nonvanishing of $\mathcal{L}_{\tilde{G}}$ for a particular π. In this thesis, we will prove

Theorem 1.3 *Suppose $p + q \leq 2n + 1$. Let π be an irreducible admissible representation of $\widetilde{O_{p,q}}$ such that each of its leading exponents v satisfies that $Re(v) + 2\rho - (n, \ldots, n)$ is a strictly negative combination of simple roots. Here 2ρ is the sum of restricted roots (with multiplicity). Suppose $\pi(\epsilon) = -1$. Then π is in the semistable range. In addition, either $\pi \in \mathcal{R}(\widetilde{O_{p,q}}, \omega)$ or $\pi \otimes \chi \in \mathcal{R}(\widetilde{O_{p,q}}, \omega)$, and the dual Harish-Chandra module of $\omega(\pi)$ or $\omega(\pi \otimes \chi)$ can be constructed using the averaging operator. Here χ is a one-dimensional character of $\widetilde{O_{p,q}}$.*

The case of averaging over $\widetilde{Sp_{2n}(\mathbb{R})}$ to obtain representations of $\widetilde{O_{p,q}}$ is a little subtle. Roughly speaking, one can no longer expect that the dual pair correspondence in the semistable range be an injection for an individual dual pair $(Sp_{2n}(\mathbb{R}), O_{p,q})$. One easy counterexample is when $O_{p,q}$ is compact. However, we can consider the disjoint union of $\mathcal{R}(\widetilde{O_{p,q}}, \omega_{p,q})$ where $p+q=m$ is fixed. In this thesis, we will prove

Theorem 1.4 *Suppose $p+q \geq 2n$. Let π be an irreducible admissible representation of $\widetilde{Sp_{2n}}$ such that $\pi(\epsilon) = -1$. Suppose that each of its leading exponents v satisfies that $Re(v) + 2\rho - (\frac{p+q}{2}, \ldots, \frac{p+q}{2})$ is a strictly negative combination of simple roots. Then π is in the semistable range. Let $\omega_{p,q}$ be the underlying oscillator representation for $(Sp_{2n}, O_{p,q})$. Then there exists $p'+q' = p+q$, such that $\pi \in \mathcal{R}(\widetilde{Sp_{2n}}, \omega_{p',q'})$. Hence, the dual Harish-Chandra module of $\omega_{p',q'}(\pi)$ can be constructed using the averaging operator.*

Certainly, these two theorems can help us to get our hands on the dual pair correspondence in the semistable range. Also semistable range seems to be the right range to study for unipotent representation. However, our investigation is far from complete.

1.5 Compactification of $SO_{p,q}$ and $Sp_{2n}(\mathbb{R})$ and Some Density Theorems

Let X be an analytic manifold. We say that $(i, \overline{X})$ is an analytic compactification of X if there exist a compact ananlytic manifold $\overline{X}$ and an analytic embedding

$$i : X \to \overline{X}$$

such that $i(X)$ is dense in $\overline{X}$. In this thesis (Chapter 6), we prove that

Theorem 1.5 (Compactification of $Sp_{2n}(\mathbb{R})$) *There exists an analytic embedding:*

$$\mathcal{H} : Sp_{2n}(\mathbb{R}) \to U(2n)/O_{2n}(\mathbb{R})$$

The image is open dense in $U(2n)/O_{2n}(\mathbb{R})$. If f is a K-finite matrix coefficient of an irreducible unitary representation of $Sp_{2n}(\mathbb{R})$, then f can be extended into a continuous function on $U(2n)/O_{2n}(\mathbb{R})$.

Theorem 1.6 (Compactification of $O_{p,q}$) *There exists an analytic compactification $(\mathcal{H}_1, O_{p+q})$ of $O_{p,q}$.*

This theorem is proved as Theorem 10.3. We shall remark here that the compactification we defined here is different from the construction of T.Oshima [Oshima].

One of the main ideas in proving the nonvanishing theorems is to relate Howe's dual pair correspondence to the harmonic analysis of compact symmetric spaces. Roughly speaking, the integration kernel of G in the Bargmann-Segal model yields a compactification from the group G to a *compact symmetric space*. Thus many questions about dual pair correspondence can be converted into questions about this compactification and questions about functions on the compact symmetric space.

For example, for every $X, Y \in Mat(p+q, n, \mathbb{C})$, we define a function on the compact group SO_{p+q} by

$$F_{X,Y}(g) = Tr(X^t g Y), \qquad (g \in SO_{p+q})$$

Let R_n be the linear span of the functions

$$\{F^i_{X,Y} \mid X, Y \in Mat(p+q, n, \mathbb{C}), i \in \mathbb{N}\}$$

The nonvanishing of $L_{\widetilde{G}}$ for $G = SO_{p,q}$ is closely related to the density of R_n in L^∞-functions of SO_{p+q}. In this thesis, we prove that

Theorem 1.7 (Density theorem for SO_{p+q}) *Let $\mathcal{O}_{SO_{p+q}}$ be the space of regular functions on SO_{p+q}. If $n \geq \frac{p+q-1}{2}$, then $R_n = \mathcal{O}_{SO_{p+q}}$.*

Notice that $U(2n)/O_{2n}$ can be identified with

$$\mathcal{S}_{2n} = \{UU^t \mid U \in U(2n)\}$$

For $X \in Mat(2n, p, \mathbb{C})$, we define an algebraic function on $\mathcal{S}$ by

$$F_X(s) = Tr(X^t s X) \qquad (s \in \mathcal{S}_{2n})$$

Let R_p be the linear span of

$$\{F^i_X \mid X \in Mat(2n, p), i \in \mathbb{N}\}$$

and $\overline{R_p}$ be its conjugation. Let $R_p \otimes \overline{R_q}$ be the space of functions on $\mathcal{S}_{2n}$ spanned by the product of functions in R_p and $\overline{R_q}$. The nonvanishing of $\mathcal{L}_{\widetilde{G}}$ for $G = Sp_{2n}(\mathbb{R})$ and $G' = O_{p,q}$ is closely related to the density of $R_p \otimes \overline{R_q}$ in the space of L^∞-functions on $\mathcal{S}_{2n}$. In this thesis, we prove

Theorem 1.8 (Density Theorem for $\mathcal{S}_n$) *If $l \geq n$, then $\oplus_{i=0}^{l} R_i \otimes \overline{R_{l-i}}$ is equal to $\mathcal{O}_{\mathcal{S}_n}$.*

In fact we develop a model for the ring of regular functions on $\mathcal{S}_n$.

Theorem 1.9 *The ring of regular functions on S_n is spanned by the functions of the following form:*

$$Tr(X_1^t s X_1)^{i_1} \ldots Tr(X_k^t s X_k)^{i_k} \overline{Tr(X_{k+1}^t s X_{k+1})}^{i_{k+1}} \ldots \overline{Tr(X_n^t s X_n)}^{i_n} \qquad (i_j \in \mathbb{Z}, s \in S, X_j \in \mathbb{C}^n)$$

To summarize, the dual pair correspondence is defined algebraically. We study the dual pair correspondence using the analytic tool $\mathcal{L}_{\widetilde{G}}$. Then we use compactification to convert the questions about $\mathcal{L}_{\widetilde{G}}$ to some purely algebraic questions about the density of some function space.

The following is what is covered in this thesis. In Chapter 2, we present the structure theory of parabolic subgroups for a type I classical group. In Chapter 3, we investigate the relationship between the associated variety of M and the H-associated variety of M where M is a $U(\mathfrak{g})$ module. In Chapter 4, we study the Lie algebra action under the framework of direct integral for Abelian Lie groups. We show that for a unitary representation of a connected Abelian Lie group G, the associated variety is the algebraic closure of the support of its spectral measure. In Chapter 5, we compute the ZN_k-rank using associated varieties. In Chapter 6, we review the Bargmann-Segal model and construct the analytic compactification of $Sp_{2n}(\mathbb{R})$. In Chapter 7, we review the dual pair correspondence of Howe. In Chapter 8, we study the growth condition of the matrix coefficients of the oscillator representation. We also investigate the growth condition for the convergence of $\mathcal{L}_{\widetilde{G}}$. In Chapter 9, we study the algebraic properties of the averaging operator $\mathcal{L}_{\widetilde{G}}$ and prove the construction theorem. In Chapter 10, we study the compactification of $O_{p,q}$ and prove the density theorem for SO_{p+q}, thus the nonvanishing of $\mathcal{L}_{\widetilde{SO_{p,q}}}$. In Chapter 11, we we prove the density theorem for $U(n)/O_n$ and investigate the nonvanishing of $\mathcal{L}_{\widetilde{Sp_{2n}(\mathbb{R})}}$.

2 Structure Theory on Classical Groups of Type I

In this section, we summarize some results about the structure of parabolic subgroups of a classical group of type I.

2.1 Type I classical Groups

Definition 2.1 *A type I classical group $G(V)$ consists of the following data.*

- *A division algebra D of a field $\mathbb{F}$ with involution $\sharp$, and $a^\sharp b^\sharp = (ba)^\sharp$;*
- *A (right) vector space V over D, with a nondegenerate (D-valued) sesquilinear form $(,)_\epsilon$, $\epsilon = \pm 1$, i.e.,*

$$(u, v) = \epsilon(v, u)^\sharp \qquad (u, v \in V)$$
$$(u\lambda, v) = (u, v)\lambda \qquad (u, v \in V, \lambda \in D);$$

- *G is the isometry group of $(,)$, i.e.,*

$$g.(u\lambda) = (g.u)\lambda \qquad (\lambda \in D, u \in V, g \in G)$$
$$(gu, gv) = (u, v) \qquad (u, v \in V).$$

Here we allow $\sharp$ to be trivial. We call the identity component of G *connected* classical group of type I. For $\mathbb{F} = \mathbb{C}$, $\sharp$ trivial, we obtain all the complex simple groups of type I, namely, $Sp_{2n}(\mathbb{C})$, and $O(n, \mathbb{C})$. If $D = \mathbb{H}$, $\mathbb{F} = \mathbb{R}$, $\sharp$ the usual involution, we obtain $Sp(p, q)$ and $O^*(2n)$ depending on the sesquilinear form. For $\mathbb{F} = \mathbb{R}$, $D = \mathbb{C}$ and $\sharp$ the usual conjugation, we obtain $U(p, q)$ depending on the signature of the Hermitian form. For $\mathbb{F} = \mathbb{R}$, $D = \mathbb{R}$ with trivial involution, we obtain $Sp_{2n}(\mathbb{R})$ and $O_{p,q}(\mathbb{R})$. If $(V, (,))$ is implicitly understood, we write G or $G(n)$ if $V \cong D^n$. Let V_0 be a linear subspace of V, we write $V_0^\perp$ for the orthogonal complement of V_0 in V. If $(,)$ is nondegenerate on V_0, we let $G(V_0)$ denote the subgroup of G consisting of elements which acts by identity on $V_0^\perp$. from our scope.

2.2 Flags and Parabolic Subgroups

Definition 2.2 *A flag $\mathcal{F}$ of $V = D^n$ is a sequence of strictly increasing (D-)linear subspaces of V*

$$0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_k \subsetneq V$$

such that

$$V_i^\perp = V_{k+1-i}.$$

Suppose $\dim(V_i) = d_i$. *$\mathcal{F}$ is said to be a flag of type*

$$\mathcal{I} = (0 < d_1 < d_2 < \ldots < d_k < n) \qquad (d_i \in \mathbb{N}).$$

We denote the space of flags of type $\mathcal{I}$ by $\mathcal{B}_{\mathcal{I}}$. We fix once for all a maximal set of linearly independent vectors

$$\{e_1, e_2, \dots, e_r, e_1^*, e_2^*, \dots, e_r^*\} \qquad (e_i, e_i^* \in V)$$

such that

$$(e_i, e_j) = 0 = (e_i^*, e_j^*), \qquad (e_i, e_j^*) = \delta_{ij}$$

where r is the real rank of G. For each integer $1 \le i \le r$, we let X_i be the linear span of $\{e_1, \dots, e_i\}$, and X_i^* be the linear span of $\{e_1^*, \dots, e_i^*\}$. We set $W_i = X_i \oplus X_i^*$. We define a map $\tau \in G$ as follows

$$\tau(e_i) = e_i^*, \qquad \tau(e_i^*) = \epsilon e_i \qquad (i \in [1, r]),$$

$$\tau|_{W_r^\perp} = id.$$

Let $\mathcal{I}_0 = \{0 < 1 < 2 < \dots < r \le n - r < n - r + 1 < \dots < n - 1 < n\}$. We fix a flag

$$\mathcal{F}_0 = \{0 \subsetneqq X_1 \subsetneqq \dots \subsetneqq X_r \subseteq X_r^\perp \subsetneqq \dots \subsetneqq X_1^\perp \subsetneqq V\}$$

For an arbitrary $\lambda = (\lambda_1, \dots \lambda_r) \in (\mathbb{R}^+)^r$, we define a linear isomorphism $A(\lambda) \in GL_D(V)$ as follows,

$$A(\lambda) e_i = \lambda_i e_i; \qquad A(\lambda) e_i^* = \lambda_i^{-1} e_i^* \qquad (i \in [1, r])$$

$$A(\lambda) u = u \qquad (u \in W_r^\perp).$$

It is easy to check that $A(\lambda) \in G(V)$. Let A be the group consisting of all $A(\lambda)$. Then A is a maximal connected split Abelian subgroup of $G(V)$.

For $h = (h_1, \dots, h_r) \in \mathbb{R}^r$, we may also define $a(h) \in End_D(V)$ such that

$$a(h) e_i = h_i e_i, \qquad a(h) e_i^* = -h_i e_i^* \qquad (i \in [1, r])$$

$$a(h) u = u \qquad (u \in W_r^\perp).$$

It is easy to see that the Lie algebra $\mathfrak{a}$ of A consists of all $a(h)$. Let $\Delta(\mathfrak{g}, \mathfrak{a})$ be the restricted root system. For $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$, let $\mathfrak{g}_\alpha$ be the root space. Then we have

$$\tau(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha} \qquad (\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})).$$

Theorem 2.1 *The isotropy group $P_0 = G_{\mathcal{F}_0}$ is a minimal parabolic subgroup of G. Its Levi factor is*

$$\begin{aligned} MA =& P_0 \cap \tau(P_0) \\ =& \{g \in G(V) \mid g.X_i = X_i,\ g.X_i^* = X_i^*,\ g.W_r^\perp = W_r^\perp\} \\ =& \{g \in G(V) \mid g.(e_i D) = e_i D,\ g.(e_i^* D) = e_i^* D,\ g.W_r^\perp = W_r^\perp\} \end{aligned} \tag{2.1}$$

Similarly, we can define a flag $\mathcal{F}_{\mathcal{I}}$ of type

$$\mathcal{I} = \{0 < i_1 < i_2 \ldots < i_k < n\}$$

by

$$V_j = X_{i_j} \qquad (j \leq \frac{k+1}{2})$$

$$V_j = X_{i_{k+1-j}}^{\perp} \qquad (j \geq \frac{k+1}{2}).$$

Of course, we assume that for every $j \in [1, k]$, $i_j + i_{k+1-j} = n$.

Theorem 2.2 *$P_{\mathcal{I}} = G_{\mathcal{F}_{\mathcal{I}}}$ are all the parabolic subgroups containing P_0. If $G \neq O_{1,1}, O(2, \mathbb{C})$ (in these two cases, no proper parabolic subgroup exists), the maximal parabolic subgroups correspond to $\mathcal{I} = \{0 < k \leq n-k < n\}$.*

Proof: We will only sketch a proof here. Obviously $P_{\mathcal{I}} \supseteq P_0$. Now we observe that for $G \neq O_{1,1}, O(2, \mathbb{C})$, $P_{\mathcal{I}}$ and $P_{\mathcal{I}'}$ are different if $\mathcal{I} \neq \mathcal{I}'$. The cardinality of all the $\mathcal{I}$'s is 2^r. But the cardinality of parabolic groups containing P_0 is also 2^r. Thus $P_{\mathcal{I}}$ exhaust all the parabolic subgroups containing P_0.

Observe that $P_{\mathcal{I}} \supseteq P_{\mathcal{I}'}$ if and only if $\mathcal{I}'$ is a refinement of $\mathcal{I}$. Therefore the maximal parabolic subgroups correspond to $\mathcal{I} = \{0 < k \leq n-k < n\}$. Q.E.D.

2.3 Maximal Parabolic Subgroups and Grading

We denote the maximal parabolic subgroup $P_{\{0<k\leq n-k<n\}}$ by P_k.

Theorem 2.3 *The Levi factor $M_{\mathcal{I}} A_{\mathcal{I}}$ can be given by*

$$P_{\mathcal{I}} \cap \tau(P_{\mathcal{I}}) = \{g \in G(V) \mid g.X_{i_j} = X_{i_j}; g.X_{i_j}^* = X_{i_j}^*\}$$

For P_k maximal parabolic, let $M_k A_k N_k$ be the Langlands decomposition. Then A_k is 1-dimensional.

$$A_k = \{a_t, t \in \mathbb{R}^+ \mid a(t)|_{X_K} = t; a(t)|_{X_K^*} = t^{-1}; a(t)|_{W_k^{\perp}} = 1\}$$

$$M_k A_k = \{g \in G(V) \mid g.X_k = X_k; gX_k^* = X_k^*\} \cong GL_D(k) \times G(W_k^{\perp})$$

Now we fix an $h_k \in \mathfrak{a}_k$, such that h_k is identity on X_k, and -1 on X_k^*, and zero on $W_k^{\perp}$. Then V can be decomposed into eigenspaces of h_k

$$V_{-1} = X_k^* \qquad V_1 = X_k \qquad V_0 = W_k^{\perp}$$

Thus $\mathfrak{g}$ can be decomposed into eigenspaces of h_k as follows.

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

where

$$\mathfrak{g}_0 = \{x \in \mathfrak{g} \mid x.X_k \subseteq X_k; x.X_k^* \subseteq X_k^*; x.W_k^\perp \subseteq W_k^\perp\}$$
$$\mathfrak{g}_1 = \{x \in \mathfrak{g} \mid x.X_k = 0; x.W_k^\perp \subseteq X_k; x.X_k^* \subseteq W_k^\perp\}$$
$$\mathfrak{g}_2 = \{x \in \mathfrak{g} \mid x.X_k = 0; x.W_k^\perp = 0; x.X_k^* \subseteq X_k\}$$
$$\mathfrak{g}_{-i} = \tau(\mathfrak{g}_i) \qquad (i = 1, 2)$$

Moreover

$$\mathfrak{g}_0 = \mathfrak{m}_k \oplus \mathfrak{a}_k \qquad \mathfrak{g}_1 \oplus \mathfrak{g}_2 = \mathfrak{n}_k$$

Notation: Since our argument is valid for every k, $\mathfrak{g}_i$ will denote the i-eigenspace of $ad(h_k)$ for a fixed (**implicit**) k.
Notice that

$$x \in \mathfrak{g}_2 \Longleftrightarrow x|_{X_k \oplus W_k^\perp} = 0;\ (x.u, v) + (u, x.v) = 0\ (\forall\ u, v \in X_k^*)$$

If we define a sesquilinear form on X_k^* to be

$$B_x(u, v) = (x.u, v) \qquad (u, v \in X_k^*)$$

then

$$B_x(u, v) = -\epsilon B_x(v, u)^\sharp$$

Therefore $\mathfrak{g}_2$ can be identified with a space of sesquilinear forms $(,)_{-\epsilon}$ on X_k^*. Similarly, $\mathfrak{g}_2^*$ can be identified with a space of sesquilinear forms $(,)_{-\epsilon}$ on X_k.

Now for every $x \in \mathfrak{g}_1$, we define $C_x \in Hom_D(W_k^\perp, X_k)$ to be the restriction of x on $W_k^\perp$. Since

$$(x.u, v) + (u, x.v) = 0 \qquad (u \in W_k^\perp, v \in X_k^*)$$

Then for each $v \in X_k^*$, $x.v$ can be uniquely determined by

$$(u, x.v) = -(x.u, v) = -(C_x u, v) \qquad (u \in W_k^\perp) \tag{2.2}$$

because that $(,)$ restricted to $W_k^\perp$ is nondegenerate. Conversely, for each $C_x \in Hom_D(W_k^\perp, X_k)$, we may define an $x \in Hom_D(V, V)$ such that

$$x.X_k = 0, \qquad x|_{W_k^\perp} = C_x$$

and Equation 2.2 holds. To summarize, we have shown that

$$C : \mathfrak{g}_1 \cong Hom_D(W_k^\perp, X_k)$$

is an isomorphism.

Similarly, for each $x \in \mathfrak{g}_1$, we may define $D_x \in Hom_D(X_k^*, W_k^\perp)$ to be the restriction of x on X_k^*. We also have

$$D : \mathfrak{g}_1 \cong Hom_D(X_k^*, W_k^\perp)$$

Theorem 2.4 *Let C be the restriction of $\mathfrak{g}_1$ on $W_k^{\perp}$. Let D be the restriction of $\mathfrak{g}_1$ on X_k^*. Then $C : \mathfrak{g}_1 \to Hom_D(W_k^{\perp}, X_k)$ and $D : \mathfrak{g}_1 \to Hom_D(X_k^*, W_k^{\perp})$ are bijections. Moreover*

$$[x, y] = C_x D_y - C_y D_x \qquad (x, y \in \mathfrak{g}_1)$$

Theorem 2.5 *Let G_0 be the Levi factor of the maximal parabolic subgroup P_k as defined in Theorem 2.3. Then $\mathfrak{g}_1$ is an irreducible G_0-module. Suppose $\mathfrak{g}_2 \neq \{0\}$. Then $\mathfrak{g}_2$ is the center of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$.*

Proof: Recall that $\mathfrak{g}_0 \cong End_D(X_k^*) \oplus \mathfrak{g}(W_k^{\perp})$. The adjoint action of G_0 on $\mathfrak{g}_1$ can be identified with the action of $GL_D(X_k^*) \times G(W_k^{\perp})$ on $Hom_D(X_k^*, W_k^{\perp})$. But X_k^* as an $GL_D(X_k^*)$-module is irreducible, and $W_k^{\perp}$ as a $G(W_k^{\perp})$-module is also irreducible. Thus $Hom_D(X_k^*, W_k^{\perp})$ is an irreducible $GL_D(X_k^*) \times G(W_k^{\perp})$-module. In other words, $\mathfrak{g}_1$ is an irreducible G_0-module.

Since $\mathfrak{g}$ is a Lie algebra, we have

$$[\mathfrak{g}_1, \mathfrak{g}_2] = \mathfrak{g}_3 = \{0\}, \qquad [\mathfrak{g}_2, \mathfrak{g}_2] = \mathfrak{g}_4 = \{0\}$$

Thus $\mathfrak{g}_2$ is in the center of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. Observe that $Z(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \cap \mathfrak{g}_1$ is a $\mathfrak{G}_0$-module. Thus either $Z(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \cap \mathfrak{g}_1 = \mathfrak{g}_1$, i.e., $\mathfrak{g}_1$ Abelian, or $Z(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \cap \mathfrak{g}_1 = \{0\}$. Suppose that $\mathfrak{g}_1$ is Abelian. Let α be the simple restricted root such that $\mathfrak{g}_{-\alpha}$ is not contained in $\mathfrak{g}_0$. Then $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_1 \oplus \mathfrak{g}_2$. From root decomposition, either $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_1$ or $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_2$. If $\mathfrak{g}_\alpha$ lies in $\mathfrak{g}_1$, and $\mathfrak{g}_1$ Abelian, then $\mathfrak{g}_2 = \{0\}$. This is a contradiction. Otherwise $\mathfrak{g}_\alpha$ lies in $\mathfrak{g}_2$. This implies that $\mathfrak{g}_1 = \{0\}$. In both cases, we have

$$Z(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \cap \mathfrak{g}_1 = \{0\}$$

Thus

$$Z(\mathfrak{g}_1 \oplus \mathfrak{g}_2) = \mathfrak{g}_2$$

Q.E.D.

Theorem 2.6 *P_k acts on $\mathfrak{g}_2^*$ with finitely many orbits. The orbits are uniquely determined by the rank and the signature of the corresponding sesquilinear form.*

Proof: It is well-known that a Hermitian or (skew-Hermitian) sesquilinear form on D^k can be determined by its signature and rank up to the action of $GL_D(k)$. But $GL_D(k) \subseteq M_k A_k$, and $G(W_k^{\perp})$ acts trivially on X_k, X_k^*, thus trivially on $\mathfrak{g}_2^*$. In addition, from weight decomposition, $\mathfrak{n}_k$ has to act trivially on $\mathfrak{g}_2^*$. Thus N_k acts trivially on $\mathfrak{g}_2^*$. Therefore P_k acts through $GL_D(k)$ on $\mathfrak{g}_2$, and the orbits are determined by their rank and their signature of the corresponding sesquilinear forms. Q.E.D.

We define the rank of any subset S of $\mathfrak{g}_2^*$ to be the maximal rank of the elements of S regarded as sesquilinear form.

3 Associated Variety under Restriction

A filtered (noncommutative) algebra $\mathcal{D}$ over $\mathbb{C}$ is an algebra endowed with a filtration $\{\mathcal{D}_i\}_{i\in\mathbb{Z}}$ such that

$$\mathcal{D}_i \cdot \mathcal{D}_j \subseteq \mathcal{D}_{i+j} \qquad (i, j \in \mathbb{Z})$$

Let $gr(\mathcal{D}) = \oplus \mathcal{D}_{i+1}/\mathcal{D}_i$ be the associated graded algebra. Let $\sigma_i : \mathcal{D}_i \rightarrow \mathcal{D}_i/\mathcal{D}_{i-1}$ be the natural projection. Throughout this paper, our filtered algebra will be assumed to have the following property:

1. $\mathcal{D}_0 = \mathbb{C}1$, where 1 is the identity element;
2. $\mathcal{D}_{-1} = \{0\}$;
3. $gr(\mathcal{D})$ is a commutative affine algebra.

Notice that $gr(\mathcal{D})$ being commutative is equivalent to

$$[\mathcal{D}_i, \mathcal{D}_j] \subseteq \mathcal{D}_{i+j-1}$$

3.1 Associated Variety and Restrictions

Definition 3.1 *Let $spec(\mathcal{D})$ be the maximal spectrum of $gr(\mathcal{D})$. Suppose that $\mathcal{I}$ is a (left) ideal of $\mathcal{D}$. Then $\mathcal{I}$ inherits a filtration from $\mathcal{D}$, i.e.,*

$$\mathcal{I}_i = \mathcal{D}_i \cap \mathcal{I} \qquad (i \in \mathbb{N})$$

Let $gr(\mathcal{I})$ be the graded algebra of $\mathcal{I}$. Then $gr(\mathcal{I})$ is an ideal of $gr(\mathcal{D})$. Let $\mathcal{V}(\mathcal{I})$ be the set of maximal ideals in $gr(\mathcal{D})$ containing $gr(\mathcal{I})$. $\mathcal{V}(\mathcal{I})$ is called the associated variety of $\mathcal{I}$.

Now suppose that $\mathcal{C}$ is a subalgebra of $\mathcal{D}$ with identity. $\mathcal{C}$ inherits a filtration from $\mathcal{D}$. Thus we have an injection:

$$j : gr(\mathcal{C}) \rightarrow gr(\mathcal{D})$$

Suppose that $gr(\mathcal{C})$ is an affine, (automatically) commutative algebra. Then the associated map on the spaces of spectrum is

$$j^* : spec(\mathcal{D}) \rightarrow spec(\mathcal{C})$$

Theorem 3.1 *Let M be a $\mathcal{D}$-module, N a linear subspace of M. Let $\mathcal{C}$ be a subalgebra of $\mathcal{D}$. Let $Ann_{\mathcal{D}}(N)$ be the annihilator of N. Then $Ann_{\mathcal{D}}(N)$ is a left ideal of $\mathcal{D}$ and*

$$j^*\mathcal{V}(Ann_{\mathcal{D}}(N)) \subseteq \mathcal{V}(Ann_{\mathcal{C}}(N))$$

Proof: Let $I = gr(Ann_{\mathcal{C}}(N))$, and $J = gr(Ann_{\mathcal{D}}(N))$. Suppose $f \in I$ is homogeneous of degree k. Then there exists $U \in Ann_{\mathcal{C}}(N) \subseteq Ann_{\mathcal{D}}(N)$, such that

$$\sigma_k(U) = f$$

This implies that

$$j(f) \in J$$

Therefore $j(I) \subseteq J$. Let L be the ideal generated by $j(I)$ in $gr(\mathcal{D})$. It follows immediately that

$$\mathcal{V}(L) \supseteq \mathcal{V}(J)$$

By inspection of the definitions, this amounts to

$$(j^*)^{-1}(\mathcal{V}(I)) \supseteq \mathcal{V}(J)$$

This is equivalent to

$$\mathcal{V}(I) \supseteq j^*(\mathcal{V}(J))$$

Q.E.D.

3.2 Associated Variety of $U(\mathfrak{g})$-modules

Now let $\mathcal{D} = U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ with complex coefficients. Since $U(\mathfrak{g})$ has a natural filtration

$$\mathbb{C} \cdot 1 \subseteq U_1(\mathfrak{g}) \subseteq U_2(\mathfrak{g}) \subseteq \ldots \subseteq U_i(\mathfrak{g}) \subseteq \ldots$$

the associated graded algebra $gr(U(\mathfrak{g}))$ can be identified with the symmetric algebra $S(\mathfrak{g})$. Thus

$$spec(U(\mathfrak{g})) = \mathfrak{g}^*_{\mathbb{C}}$$

Here $\mathfrak{g}^*_{\mathbb{C}}$ is the complex dual of $\mathfrak{g}$. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Then j^* is simply the projection of $\mathfrak{g}^*_{\mathbb{C}}$ onto $\mathfrak{h}^*_{\mathbb{C}}$ (through restriction). Under this setting, we have

Theorem 3.2 *Let M be a $\mathfrak{g}$-module. Let N be a linear subspace of M. Then*

$$j^*(\mathcal{V}(Ann_{U(\mathfrak{g})}(N))) \subseteq \mathcal{V}(Ann_{U(\mathfrak{h})}(N))$$

Now we are interested in the following equation.

$$cl(j^*(\mathcal{V}(Ann_{U(\mathfrak{g})}(N)))) = \mathcal{V}(Ann_{U(\mathfrak{h})}(N))$$

At this stage, we only have a very limited understanding about the behavior of j^* for associated varieties. Nevertheless, we have the following theorem.

Theorem 3.3 *Suppose a is a semisimple element in an arbitrary Lie algebra $\mathfrak{g}$ with only real eigenvalues, i.e., $ad(a)$ possesses an eigenspace decomposition:*

$$\mathfrak{g} = \mathfrak{g}_{-r_1} \oplus \ldots \oplus \mathfrak{g}_r$$

Let r be the maximal eigenvalue. Suppose $r > 0$. Let $\mathfrak{h} = \mathfrak{g}_r$. Then $\mathfrak{h}$ is Abelian. Let M be a $\mathfrak{g}$-module, and N a subspace of M such that $a.N \subseteq N$. Then

$$\mathcal{V}(Ann_{U(\mathfrak{h})}(N)) = cl(j^*(\mathcal{V}(Ann_{U(\mathfrak{g})}(N))))$$

where $cl(j^*(\mathcal{V}(Ann_{U(\mathfrak{g})}(N))))$ is the algebraic closure of $j^*(\mathcal{V}(Ann_{U(\mathfrak{g})}(N)))$.

Proof: First of all, under the eigendecomposition of $ad(a)$, we have

$$[\mathfrak{g}_r, \mathfrak{g}_r] = \mathfrak{g}_{2r} = \{0\}$$

Therefore $\mathfrak{h} = \mathfrak{g}_r$ is Abelian. Now it suffices to show that

$$\mathcal{V}(Ann_{U(\mathfrak{h})}(N)) \subseteq cl(j^*(\mathcal{V}(Ann_{U(\mathfrak{g})}(N))))$$

Suppose that $f \in S^i(\mathfrak{h})$ vanishes on $cl(j^*(\mathcal{V}(Ann_{U(\mathfrak{g})}(N))))$. In other words, $j(f) = f$ vanishes on $V(Ann_{U(\mathfrak{g})}(N))$. Thus there exists $n \in \mathbb{N}$, such that $f^n \in gr(Ann_{U(\mathfrak{g})}(N))$. Therefore,

$$\exists\ P \in U_{ni}(\mathfrak{g}) \cap Ann_{U(\mathfrak{g})}(N), P = P_0 + P_1$$

where

$$P_0 \in U_{ni}(\mathfrak{h}), P_1 \in U_{ni-1}(\mathfrak{g}), \sigma_{ni}(P_0) = f^n$$

Since $ad(a)$ is semisimple, $U_{ni}(\mathfrak{g})$ is completely reducible as $ad(a)-$module. Also notice that N is an a-module. Thus $Ann_{U(\mathfrak{g})}(N)$ is also an $ad(a)-$module. Now $U_{ni}(\mathfrak{g}) \cap Ann_{U(\mathfrak{g})}(N)$ possesses an eigen (weight) decomposition with respect to $ad(a)$

$$U_{ni}(\mathfrak{g}) \cap Ann_{U(\mathfrak{g})}(N) = \oplus_{k \in \mathbb{R}} (U_{ni}(\mathfrak{g}) \cap Ann_{U(\mathfrak{g})}(N))_k$$

This implies that every eigencomponent of P with respect to $ad(a)$ is again in $Ann_{U(\mathfrak{g})}(N)$. Since $\mathfrak{h}$ is of the highest weight in $\mathfrak{g}$, by comparing the weight of P_0 with the weights of $U_{ni-1}(\mathfrak{g})$, we can see that the highest weight component of P is P_0. Thus

$$P_0 \in Ann_{U(\mathfrak{h})}(N)$$

In addition,

$$\sigma_{ni}(P_0) = f^n \in gr(Ann_{U(\mathfrak{h})}(N))$$

This implies that f vanishes at $\mathcal{V}(Ann_{U(\mathfrak{h})}(N))$. Q.E.D.

Now under the setting from section (1), we have the following theorem.

Theorem 3.4 *Let $\mathfrak{g}_{\mathbb{C}}$ be the classical Lie algebra of type I. Let M be a $\mathfrak{g}$-module. Let j^* be the canonical projection from $\mathfrak{g}_{\mathbb{C}}^*$ onto $\mathfrak{g}_{2\mathbb{C}}{}^*$. Then*

$$\mathcal{V}(Ann_{U(\mathfrak{g}_2)}(M)) = cl(j^*(Ann_{U(\mathfrak{g})}(M)))$$

We will end this section with the following definition.

Definition 3.2 *Let N be a connected closed subgroup of G. Let π be a unitary representation of G. We call $\mathcal{V}(Ann_{U(\mathfrak{n})}(\pi))$ the N-associated variety.*

Let $N_G(N)$ be the normalizer of N in G. One can easily see that the N-associated variety is $N_G(N)$-stable.

4 Associated Variety and Support: Abelian Case

Let G be a locally compact Abelian group. Let $\hat{G}$ be the set of unitary characters of G endowed with the Gelfand topology. Then $\hat{G}$ is a locally compact Abelian group under pointwise multiplication. Let H be a unitary representation of G. Then the Lie algebra $\mathfrak{g}$ acts on the smooth vectors in $\mathcal{H}^{\infty}$. On the one hand, we have the theory of abelian harmonic analysis available. We can study the support of H as a closed subset of $\hat{G}$. On the other hand, we also have the theory of commutative algebras available. We can study the associated variety of H^{∞} as a Zariski-closed subvariety of $\mathfrak{g}_{\mathbb{C}}^{*}$. In this chapter, we will see how these two invaraints are related.

4.1 Stone's Theorems and Spectral Integrals

Theorem 4.1 (Stone) *If H is a Hilbert space and μ a regular projection-valued Borel measure on $\hat{G}$, then the equation*

$$T_g = \int_{\hat{G}} \xi(g) d\mu(\xi) \qquad (g \in G) \tag{4.3}$$

defines a unitary representation T of G on H. Conversely, every unitary representation of G determines a unique regular projection-valued Borel measure μ on H such that Equation 4.3 holds.

We define the support of a unitary representation H of G to be the (closed) support of the projection-valued measure μ. In other words, $supp_G(\pi)$ is the complement of the biggest open subset U of $\hat{G}$ such that $\mu(U) = 0$. Equivalently, $supp_G(\pi)$ is the smallest closed subset K of $\hat{G}$ such that $\mu(K) = id$. Of course if we remove the closedness of $supp_G(\pi)$, $supp_G(\pi)$ is only unique up to a set of measure zero.

Theorem 4.2 *Make the same assumption as in Stone's theorem. For any $v \in H$, there exists a vector valued regular Borel measure μ_v such that*

$$T_g(v) = \int_{\hat{G}} \xi(g) d\mu_v(\xi) \qquad (g \in G)$$

For every $u, v \in H$, there exists a complex regular Borel measure $\mu_{u,v}$ such that

$$(T_g(u), v) = \int_{\hat{G}} \xi(g) d\mu_{u,v}(\xi) \qquad (g \in G)$$

Proof: For arbitrary Borel measurable set $K \subseteq \hat{G}$, we define

$$\mu_v(K) = \mu(K).v$$

$$\mu_{u,v}(K) = (\mu(K)u, v)$$

It is easy to check that both measures inherit regularity. Q.E.D.

Suppose G is a connected Abelian Lie group and $\mathfrak{g}$ is the (real) Lie algebra of G. Let $\mathfrak{g}^*$ be the real dual of $\mathfrak{g}$. Each $\xi \in \hat{G}$ corresponds to a smooth function $\xi(g)$ on G. We can define

$$\xi(x) = \frac{d}{dt}\xi(\exp(tx)) \qquad (x \in \mathfrak{g})$$

This defines a map from $\hat{G}$ to $\mathfrak{g}^*_{\mathbb{C}}$. Since

$$\xi(\exp(tx))\overline{\xi(\exp(tx))} = 1$$

Thus

$$\xi(x) + \overline{\xi(x)} = 0$$

This implies that $\xi(x) \in i\mathbb{R}$. We denote the pure imaginary dual by $i\mathfrak{g}^*$. Then we have defined a map from $\hat{G}$ to $i\mathfrak{g}^*$. Now, we want to study the Lie algebra action π of $\mathfrak{g}$. We recall the following definition of spectral integral.

Definition 4.1 *Let (μ, X) be a projection-valued spectral measure on a Hilbert space. Let $f : X \to \mathbb{C}$ be a μ-measurable function. Then we may find a sequence $\{A_n\}$ of pairwise disjoint measurable sets such that*

- $\cup_1^\infty A_n = X$;
- *f is $\mu-$essentially bounded on each A_n*

Let $H_n = range(P_n)$, $T_n = \int_{A_n} f d\mu$. Then there exists a unique normal operator $T = \Sigma T_n$ on $\oplus H_n$. T is often written as $\int f d\mu$, called the spectral integral of f.

Now we begin with the following theorems in page 118 of [Fell&Doran].

Theorem 4.3 *Suppose $f : \hat{G} \to \mathbb{C}$ is a $\mu-$measurable function. Let*

$$T_f = \int_{\hat{G}} f d\mu$$

Then $v \in Dom(T_f)$ if and only if

$$\int |f(\xi)|^2 d\mu_{v,v}(\xi) < \infty$$

In this case,

$$\|T_f v\|^2 = \int |f(\xi)|^2 d\mu_{v,v}(\xi)$$

$$(T_f v, u) = \int f(\xi) d\mu_{v,u}(\xi) \qquad (u \in H)$$

Theorem 4.4 *Let f_1, f_2 be μ-measurable functions on $\hat{G}$. Then*

$$(\int f_1 d\mu)(\int f_2 d\mu) \subset \int f_1 f_2 d\mu$$

$$(\int f_1 d\mu)^* = \int \overline{f_1} d\mu$$

4.2 Abelian Lie Algebra Action

In general, derivative can be defined in a Banach space. Suppose $c(t)$ is a continuous curve in a Hilbert space H. If there exists a vector v in H such that

$$\forall\ u \in H, \qquad (v, u) = \frac{d}{dt}|_{t=0}(c(t), u)$$

We say $c(t)$ is differentiable at 0 and v is said to be the derivative of $c(t)$ at 0. Now we can prove the following theorem.

Theorem 4.5 *Let (π, H) be a unitary representation of a connected Abelian Lie group G. Let μ be the projection-valued regular Borel measure from Stone's theorem. We denote the Lie algebra $\mathfrak{g}$ actions by π. Then*

$$\int_{\hat{G}} \xi(X) d\mu(\xi) \subset \pi(X) \qquad (X \in \mathfrak{g})$$

Proof: Let $T_X = \int_{\hat{G}} \xi(X) d\mu(\xi)$. Suppose $u \in Dom(T_X)$. First we want to show that $\forall\ v \in H$,

$$(T_X u, v) = \frac{d}{dt}(\pi(\exp(tX))u, v)$$

We would like to interchange the integration and differentiation, obtaining

$$\begin{aligned} \frac{d}{dt}(\pi(\exp(tX))u, v) =& \frac{d}{dt} \int \xi(\exp(tX)) d\mu_{u,v}(\xi) \\ =& \int \frac{d}{dt}\xi(\exp(tX)) d\mu_{u,v}(\xi) \\ =& \int \xi(X) d\mu_{u,v}(\xi) \end{aligned} \tag{4.4}$$

To show that the integration is interchangeable with the differentiation, first we observe that

$$|\frac{d}{dt}\xi(\exp(tX))| = |\frac{d}{dt}\exp(t\xi(X))| \leq |\xi(X)| \qquad (\xi \in \hat{G}).$$

For a complex measure μ, we define $|\mu|(U)$ to be the supremum of $\{\Sigma_{j=1}^{m} |\mu(E_j)|\}$, where $\{E_j\}_1^m$ is any measurable partition of U. Since

$$|(\mu(U)u, v)|^2 = |(\mu(U)u, \mu(U)v)|^2 \leq \|\mu(U)u\|^2 \|\mu(U)v\|^2$$

we have

$$|\mu_{u,v}|(U)^2 \leq |\mu_{u,u}|(U)|\mu_{v,v}|(U) = \mu_{u,u}(U)\mu_{v,v}(U)$$

Therefore

$$\begin{aligned}(\int |\xi(X)| d|\mu_{u,v}|(\xi))^2 \leq & (\int |\xi(X)|^2 d\mu_{u,u}(\xi))(\int d\mu_{v,v}(\xi)) \\ = & (\int |\xi(X)|^2 d\mu_{u,u}(\xi))\|v\|^2\end{aligned} \tag{4.5}$$

From Theorem 4.3, $u \in Dom(T_X)$ implies that

$$\int |\xi(X)|^2 d\mu_{u,u}(\xi) < \infty$$

Hence $\xi(X)$ as a function on $\hat{G}$ is absolutely integrable with respect to $\mu_{u,v}$. But $\frac{d}{dt}\xi(\exp(tX))$ is dominated by $|\xi(X)|$. Thus integration and differentiation are interchangeable. We obtain

$$\frac{d}{dt}(\pi(\exp(tX))u, v) = (T_x u, v)$$

Of course, here we have not proved that $u \in Dom(\pi(X))$. In fact we have

$$|\frac{d}{dt}(\pi(\exp(tX))u, v)|^2 \leq (\int |\xi(X)| d|\mu_{u,v}|(\xi))^2 \leq (\int |\xi(X)|^2 d\mu_{u,u}(\xi))\|v\|^2$$

From here we see that $\pi(X)u$ can be defined abstractly as a linear functional on H such that

$$(\pi(X)u, v) = \frac{d}{dt}(\pi(\exp(tX))u, v) \qquad (\forall\ v \in H)$$

Now $\pi(X)u \in H$ is well-defined. Therefore $u \in Dom(\pi(X))$. Q.E.D.

Now for $X_1, X_2, \dots, X_n \in \mathfrak{g}$, we define

$$T_{X_1 X_2 \dots X_n} = \int_{\hat{G}} \xi(X_1)\xi(X_2)\dots\xi(X_n) d\mu(\xi)$$

We can extend this definition by linearity to all $D \in U(\mathfrak{g})$. One can easily obtain the following theorem about the universal enveloping algebra $U(\mathfrak{g})$.

Theorem 4.6 *Let (π, H) be a unitary representation of a connected Abelian Lie group G, and μ its projection-valued regular Borel measure. Suppose $X_1, X_2, \dots, X_n \in \mathfrak{g}$. Then*

$$T_{X_1} T_{X_2} \dots T_{X_n} \subset \pi(X_1 X_2 \dots X_n)$$

$$T_{X_1 X_2 \dots X_n} \supset T_{X_1} T_{X_2} \dots T_{X_n}$$

Since $U(\mathfrak{g})$ is commutative, we may identify it with $S(\mathfrak{g})$. Thus for every $\xi \in \mathfrak{g}^*$, $D \in U(\mathfrak{g})$, $\xi(D)$ is well-defined. We will also denote $\xi(D)$ by $D(\xi)$, just to indicate the fact that D can be regarded as a function on $\mathfrak{g}^*$.

4.3 Spaces of Smooth Vectors

Theorem 4.7 *If $u \in Dom(T_D)$ for every $D \in U(\mathfrak{g})$, then u is smooth. Furthermore,*

$$\pi(D)u = T_D u$$

Proof: Suppose $u \in Dom(T_D)$ for every $D \in U(\mathfrak{g})$. Then we know that

$$|\frac{d}{dt_i}\xi(\exp(t_1X_1 + \ldots t_iX_i))\xi(X_{i+1})\ldots\xi(X_n)| \leq |\Pi_i^n \xi(X_j)|$$

By a similar argument of interchanging integration with differentiation from Theorem 4.5, we have

$$\begin{aligned}(\pi(X_1X_2\ldots X_n)u, v) =& \frac{d}{dt_1}\frac{d}{dt_2}\cdots\frac{d}{dt_n}\int_{\hat{G}}\xi(\exp(\Sigma_1^n t_iX_i))d\mu_{u,v}(\xi) \\ =& \int \xi(X_1X_2\ldots X_n)d\mu_{u,v}(\xi)\end{aligned} \tag{4.6}$$

Thus u is smooth and $\pi(D)u = T_D u$. Q.E.D.

Before we continue on, we want to examine the definition of the annihilator of a unitary representation for an arbitrary Lie group G.

Theorem 4.8 *Let (π, H) be a unitary representation of a Lie group G. Let M be any dense subset of the space of smooth vectors H^∞. Then*

$$Ann_{U(\mathfrak{g})}(H^\infty) = Ann_{U(\mathfrak{g})}(M)$$

Proof: If $D \in U(\mathfrak{g})$, and $\pi(D)H^\infty = 0$, then $\pi(D)M = 0$. Thus

$$Ann_{U(\mathfrak{g})}(M) \supseteq Ann_{U(\mathfrak{g})}(H^\infty)$$

If $D \in Ann_{U(\mathfrak{g})}(M)$, then

$$\forall\ u \in M, v \in H^\infty, (\pi(D)u, v) = 0$$

Since $\mathfrak{g}$ act as skew-adjoint operators, i.e.,

$$\forall\ X \in \mathfrak{g}, \pi(X)^* = \pi(-X)$$

we have

$$(\pi(D)u, v) = (u, \pi(D)^*v) = 0 \qquad (u \in M, v \in H^\infty)$$

Since M is dense in H^∞, M is dense in H. Hence $\pi(D)^*v = 0$. We have

$$(\pi(D)u, v) = (u, \pi(D)^*v) = 0 \qquad (u \in H^\infty, v \in H^\infty)$$

Thus for every $u \in H^\infty$, $\pi(D)u = 0$ Therefore

$$D \in Ann_{U(\mathfrak{g})}(H^\infty)$$

This implies that

$$Ann_{U(\mathfrak{g})}(M) \subseteq Ann_{U(\mathfrak{g})}(H^\infty)$$

Q.E.D.

Thus we may define $Ann_{U(\mathfrak{g})}(\pi)$ to be the annihilator of any smooth dense subset M of H. In particular, in our context, for G an Abelian Lie group, we choose

$$M = \{\int_{\hat{G}} f(\xi) d\mu_u(\xi) \mid f \in B_c(\hat{G}), u \in H\}$$

where $B_c(\hat{G})$ is the space of bounded measurable functions with compact support. M here has some property similar to Gårding space.

4.4 Associated Variety and Support

Theorem 4.9 *Let (π, H) be a unitary representation of a connected Abelian Lie group G, μ its projection-valued regular Borel measure. Then M is dense in H, and $M \subseteq H^\infty$. Suppose $D \in U(\mathfrak{g}) = S(\mathfrak{g})$ such that*

$$D(\xi) = 0 \qquad (\xi \in supp_G(\pi))$$

Then $D \in Ann_{U(\mathfrak{g})}(\pi)$.

Proof: We will show that $M \subseteq Dom(T_D)$ for every $D \in U(\mathfrak{g})$. $\forall\ f \in B_c(\hat{G}), u \in H, D \in S(\mathfrak{g})$, let $v = (\int f(\xi) d\mu(\xi))u$. Then for every $U \subset \hat{G}$ measurable, we have

$$\mu_{v,v}(U) = (\int_U d\mu(\xi)v, v) = \int_U |f(\xi)|^2 d\mu_{u,u}(\xi)$$

This implies that

$$d\mu_{v,v}(\xi) = |f(\xi)|^2 d\mu_{u,u}(\xi)$$

We have

$$\int |D(\xi)|^2 d\mu_{v,v}(\xi) = \int |D(\xi)f(\xi)|^2 d\mu_{u,u}(\xi) \tag{4.7}$$

coverges since f is compactly supported. Thus

$$(\int f(\xi) d\mu_(\xi))u \in Dom(T_D) \subseteq Dom(\pi(D)) \qquad (\forall\ D \in U(\mathfrak{g}))$$

Therefore $\int f(\xi) d\mu_u(\xi) \in H^\infty$. We have

$$M \subseteq H^\infty$$

Notice that $1_{\hat{G}}$ can be approximated by bounded functions $\{f_i\}_1^\infty$ with compact support. Since μ is regular, $u \in H$ can be approximated by $\int f_i(\xi)d\mu_u(\xi)$. Therefore M is dense in H. Now suppose

$$D(\xi) = 0 \qquad (\forall\ \xi \in supp_G(\pi))$$

Then we have

$$\pi(D)(\int f(\xi)d\mu(\xi))u = (\int D(\xi)d\mu(\xi))(\int f(\xi)d\mu(\xi))u = (\int D(\xi)f(\xi)d\mu(\xi))u = 0$$

Hence $D \in Ann_{U(\mathfrak{g})}(M) = Ann_{U(\mathfrak{g})}(\pi)$. Q.E.D.

Theorem 4.10 *Let (π, H) be a unitary representation of a connected Abelian Lie group G, μ its projection-valued regular Borel measure. If $D \in Ann_{U(\mathfrak{g})}(\pi)$, then*

$$D(supp_G(\pi)) = 0$$

Proof:

1. First, we want to show that

$$D(supp_G(\pi)) = 0 \qquad (a.e.\mu)$$

Suppose not. Then there exist a complex number $a \neq 0$, a compact $K \subset supp_G(\pi)$, $\mu(K) \neq 0$, such that

$$|D(\xi) - a| < \frac{1}{2}|a| \qquad (\xi \in K)$$

It follows that

$$\begin{aligned} \|\int_K D(\xi)d\mu(\xi) - a\mu(K)\| &= \|\int_K (D(\xi) - a)d\mu(\xi)\| \\ &\le \|\int_K |D(\xi) - a|d\mu(\xi)\| \\ &\le \|\int_K \frac{1}{2}|a|d\mu(\xi)\| \\ &\le \frac{1}{2}|a|\|\mu(K)\| \end{aligned} \tag{4.8}$$

Thus $\int_K D(\xi)d\mu(\xi) \neq 0$. On the other hand, for every $v \in H$, $(\int_K d\mu(\xi))v \in \cap_{D\in U(\mathfrak{g})}Dom(T_D)$, we have

$$0 = \pi(D)(\int_K d\mu(\xi))v = T_D(\int_K d\mu(\xi))v = (\int_K D(\xi)d\mu(\xi))v$$

This is a contradiction.

2. Therefore, we have $\mu(zero(D) \cap supp_G(\pi)) = id$. Notice that for a connected Abelian Lie group G, the Gelfand topology is just the induced Euclidean topology. Thus $zero(D) = \{\xi \in \hat{G} \mid D(\xi) = 0\}$ is closed. Therefore $zero(D) \cap supp_G(\pi)$ is closed. According to the minimality of $supp_G(\pi)$, we have

$$zero(D) \cap supp_G(\pi) = supp_G(\pi)$$

Thus $zero(D) \supseteq supp_G(\pi)$. Hence

$$D(supp_G(\pi)) = 0$$

Q.E.D.

What we have shown is that for $D \in U(\mathfrak{g})$,

$$D(supp_G(\pi)) = 0 \Longleftrightarrow D \in Ann_{U(\mathfrak{g})}(\pi)$$

But

$$D \in Ann_{U(\mathfrak{g})}(\pi) \Longleftrightarrow D(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))) = 0$$

Thus we have

Theorem 4.11 *Suppose that (π, H) is a unitary representation of a connected Abelian Lie group G. If we identify $\hat{G}$ with a subset of $i\mathfrak{g}^*$, then*

$$\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi)) = cl(supp_G(\pi))$$

5 N-spectrum and N-associated variety

In this chapter, we introduce the notions of N-spectrum and rank. We will prove Theorem 1.1.

5.1 Spectrum, Associated Variety and Rank

Let G be a locally compact Lie group, H be a closed subgroup. Let $\hat{G}$ be the unitary dual of G. Suppose that G and H are type I groups. Take a unitary representation (π, H) of G and consider its restriction to H. According to the direct integral theory [Wallach0] Ch 14.9 and 14.10, $\pi|_H$ is equivalent to a projection-valued Borel measure $\mu_H(\pi)$ on $\hat{H}$. R. Howe called such a measure the $H-$spectrum of π. Under the Fell topology, the (closed) support of $\mu_H(\pi)$ is called the geometric H-spectrum [Howe0]. Let $N_G(H)$ be the normalizer of H in G. Then $supp(\mu_H(\pi))$ is $N_G(H)$-stable.

To study H-spectrum, we have to have a well-understood unitary dual $\hat{H}$. For H nilpotent or solvable of type I, $\hat{H}$ is well-understood to some extent. For H connected Abelian, $\hat{H}$ can be identified with a subset of $i\mathfrak{h}^*$. In this chapter, we will identify it with a subset of $\mathfrak{h}^*$.

In spite of the fact that the unitary dual $\hat{G}$ is difficult to understand, the associated variety of a representation is well-understood. In particular, we have

Theorem 5.1 (Borho-Brylinski-Joseph) *Suppose $\mathfrak{g}$ is a reductive Lie algebra, M a simple $\mathfrak{g}$-module. Then $\mathcal{V}(Ann_{U(\mathfrak{g})}(M))$ is the closure of a single coadjoint orbit.*

Now concerning a linear reductive Lie group G with finitely many components, we can employ Mackey machine to show that for any irreducible unitary representation (π, H) of G, π splits into finitely many irreducible representations when restricted to the identity component G_0, namely,

$$\pi = \pi_1 \oplus \pi_2 \oplus \ldots \oplus \pi_s$$

Furthermore, G/G_0 permutes these irreducible factors. A more careful examination shows that the Harish-Chandra modules of π_i's are related by the algebra isomorphisms of $U(\mathfrak{g})$ defined by the adjoint action of G/G_0. Thus $\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi_i))$ are related by automorphisms of $\mathfrak{g}$ defined by G/G_0. In fact, $\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))$ is exactly the union of G/G_0-orbit on any chosen $\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi_i))$. More precisely, we have

$$\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi)) = \sqcup_{xG_0 \in G/G_0} Ad(x)(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi_1)))$$

Thus, for the rest of this paper, even though some of the classical Lie groups G are not connected, we may prove our results for the identity component G_0 first. Then all the results can be generalized to G.

Recall that the H-associated variety is $N_G(H)$-stable. Thus one may investigate the relationship between the H-associated variety and the H-spectrum. According to Theorem 4.11, we have

Theorem 5.2 *Let $(\pi, \mathcal{H})$ be a unitary representation of a type I classical group G. Then the ZN_k-associated variety of π is the algebraic closure of the geometric ZN_k-spectrum of π.*

Since $\mathfrak{g}$ is a reductive linear Lie algebra, $\mathfrak{g}^*$ can be identified with $\mathfrak{g}$. If we regard $\mathfrak{g}$ as a subset of $Hom_D(V, V)$, then j^* can be regarded as the (eigen)-projection of $\mathfrak{g}$ onto $\mathfrak{g}_{-2} \cong \tau(zn_k)$. For any subset S of $\mathfrak{g}^*$, we define $rank(S)$ to be the $\max\{rank_D(X) \mid X \in S\}$.

Recall that the parabolic subgroup P_k acts on $\mathfrak{z}\mathfrak{n}_k^*$ with finitely many orbits and that $\mathfrak{z}\mathfrak{n}_k^*$ can be identified with a subspace of sesquilinear forms. Howe and Li defined the ZN_k-rank to be the rank of $supp(\mu_{ZN_k}(\pi))$ regarded as sesqulinear forms. Notice that for each $x \in Hom_D(X_k, X_k^*)$ the rank of the linear transform x is the same as the rank of the bilinear form B_x defined in chapter 1. Therefore the ZN_k-rank coincides with $rank(supp(\mu_{ZN_k}(\pi)))$. In the rest of this paper, we will compute the ZN_k-rank using associated variety.

5.2 Complexification and $\mathbb{C}$-Rank

Now for a type I classical group $G(V)$, for every $x \in \mathfrak{g}$, we may define a sesquilinear form B_x such that

$$B_x(u, v) = (x.u, v) \qquad (u, v \in V)$$

Then

$$B_x(u, v) = -\epsilon B_x(v, u)^\sharp$$

Thus $\mathfrak{g}$ can be identified with a space of sesquilinear forms. Compatibly, we have the following list regarding $\mathfrak{g}_{-2}$ and its complexification:

1. $G = U(p, q)$, $\mathfrak{z}\mathfrak{n}_k^*$ is the space of $k \times k$ skew-Hermitian matrices, its complexification is the space of $k \times k$ complex matrices;

2. $G = O_{p,q}$, $\mathfrak{z}\mathfrak{n}_k^*$ is the space of $k \times k$ real skew-symmetric matrices, its complexification is the space of $k \times k$ complex skew-symmetric matrices;

3. $G = Sp_{2n}(\mathbb{R})$, $\mathfrak{z}\mathfrak{n}_k^*$ is the space of $k \times k$ real symmetric matrices, its complexification is the space of $k \times k$ complex symmetric matrices;

4. $G = O^*(2n)$, $\mathfrak{z}\mathfrak{n}_k^*$ is the space of sesquilinear forms on $\mathbb{H}^k$, such that

   $$(u, v) = (v, u)^\sharp \qquad (u, v \in \mathbb{H}^k)$$

   Let $(u, v) = A(u, v) + jB(u, v)$ with A and B complex-valued. Then

   $$A(v, u) + jB(v, u) = (A(u, v) + jB(u, v))^\sharp = \overline{A(u, v)} - jB(u, v)$$

   Therefore

   $$A(u, v) = \overline{A}(v, u) \qquad B(u, v) = -B(v, u)$$

   Now $B(u, v)$ is a (right) $\mathbb{C}$-bilinear form. If we fix a basis $\{(e_i, je_i)\}_1^k$ for $\mathbb{H}^k$, $\mathfrak{z}\mathfrak{n}_k^*$ can be identified with

   $$\{\begin{pmatrix} U & V \\ -\bar{V} & \bar{U} \end{pmatrix} \mid U^t = -U, \bar{V} = V^t\}$$

   Thus the complexification of $\mathfrak{z}\mathfrak{n}_k^*$ can be identified with the space of $2k \times 2k$ complex skew-symmetric matrices.

5. $G = Sp(p,q)$, $\mathfrak{z}\mathfrak{n}_k^*$ can be identified with a space of $2k \times 2k$ symmetric matrices, its complexification is the space of $2k \times 2k$ complex symmetric matrices.

6. $G = O(n, \mathbb{C})$, $\mathfrak{z}\mathfrak{n}_k^*$ is the space of $k \times k$ complex skew-symmetric matrices. It can be identified with

$$\{\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mid A^t = -A, B^t = -B, A, B \in End_{\mathbb{R}}(\mathbb{R}^k)\}$$

Therefore $\mathfrak{z}\mathfrak{n}_{k\mathbb{C}}^*$ can be identified with

$$\{\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mid A^t = -A, B^t = -B, A, B \in End_{\mathbb{C}}(\mathbb{C}^k)\}$$

7. $G = Sp(n, \mathbb{C})$, $\mathfrak{z}\mathfrak{n}_k^*$ can be identified with

$$\{\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mid A^t = A, B^t = B, A, B \in End_{\mathbb{R}}(\mathbb{R}^k)\}$$

and $\mathfrak{z}\mathfrak{n}_{k\mathbb{C}}^*$ can be identified with

$$\{\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mid A^t = A, B^t = B, A, B \in End_{\mathbb{C}}(\mathbb{C}^k)\}$$

For any $S \subseteq \mathfrak{z}\mathfrak{n}_{k\mathbb{C}}^*$, we write $rank_{\mathbb{C}}(S)$ for the maximal rank of the elements in S under this setting. We call it the $\mathbb{C}$-rank of S. Thus, we have

$$rank(supp(\mu_{ZN_k}(\pi))) = rank_{\mathbb{C}}(supp(\mu_{ZN_k}(\pi))) \qquad (G = U(p,q), O_{p,q}, Sp_{2n}(\mathbb{R})) \tag{5.9}$$

$$2rank(supp(\mu_{ZN_k}(\pi))) = rank_{\mathbb{C}}(supp(\mu_{ZN_k}(\pi))) \qquad (G = Sp(n,\mathbb{C}), O(n,\mathbb{C}), Sp(p,q), O^*(2n)) \tag{5.10}$$

In this setting, taking the algebraic closure of a subset of sesquilinear form would not change $\mathbb{C}$-rank of such a subset.

But, from Theorem 4.11, $\mathcal{V}(Ann_{U(\mathfrak{z}\mathfrak{n}_{\mathfrak{k}})}(\pi))$ is the algebraic closure of $supp(\mu_{ZN_k}(\pi))$. Therefore

$$rank_{\mathbb{C}}(\mathcal{V}(Ann_{U(\mathfrak{z}\mathfrak{n}_k)}(\pi))) = rank_{\mathbb{C}}(supp(\mu_{ZN_k}(\pi)))$$

Again, from Theorem 3.4 $\mathcal{V}(Ann_{U(\mathfrak{z}\mathfrak{n}_{\mathfrak{k}})}(\pi))$ is the algebraic closure of $j^*(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi)))$, where $j^* : \mathfrak{g}_{\mathbb{C}} \to (\mathfrak{z}\mathfrak{n}_k)_{\mathbb{C}}^*$ is the canonical projection. Thus

$$rank_{\mathbb{C}}(supp(\mu_{ZN_k}(\pi))) = rank_{\mathbb{C}}(j^*(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi)))) \tag{5.11}$$

5.3 Rank and Associated Variety: Real Groups

Now we restrict our attention to those non-complex groups, $O_{p,q}$, $U(p,q)$, $Sp_{2n}(\mathbb{R})$, $O^*(2n)$, $Sp(p,q)$. We will deal with complex groups at the end. According to [C-M] Ch 5.1, each nilpotent orbit in a (complex) simple Lie algebra $\mathfrak{g}(m) \subseteq End_{\mathbb{C}}(\mathbb{C}^m)$ is parametrized by a certain partition

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0)$$

of m. We denote the adjoint orbit corresponding to λ by $\mathcal{O}_\lambda$. Then

$$rank_{\mathbb{C}}(\mathcal{O}_\lambda) = m - l$$

Theorem 5.3 *Let $S \subseteq \mathfrak{g}(m)$. Then*

$$rank_{\mathbb{C}}(j^*(S)) \leq \min(r_k, rank_{\mathbb{C}}(S))$$

where $r_k = rank_{\mathbb{C}}(\mathfrak{z}\mathfrak{n}_k^*)$. In particular,

$$rank_{\mathbb{C}}(j^*(\mathcal{O}_\lambda)) \leq \min(r_k, rank_{\mathbb{C}}(\mathcal{O}_\lambda))$$

Proof: It suffices to show that

$$rank_{\mathbb{C}}(j^*(S)) \leq rank_{\mathbb{C}}(S)$$

Recall that $V = X_k^* \oplus X_k \oplus W_k^\perp$. Let $P : V \rightarrow X_k^*$ be the canonical projection. Notice that

$$\begin{aligned} rank_{\mathbb{C}}(j^*(S)) &= \max\{rank_{\mathbb{C}}(j^*(x).X_k) \mid x \in S\} \\ &= \max\{rank_{\mathbb{C}}(P(x.X_k)) \mid x \in S\} \\ &\leq \max\{rank_{\mathbb{C}}(x.X_k^*) \mid x \in S\} \\ &\leq \max\{rank_{\mathbb{C}}(x.V) \mid x \in S\} \\ &= rank_{\mathbb{C}}(S) \end{aligned} \tag{5.12}$$

Q.E.D.

Now we have to treat Type A,C and Type B, D Lie algebras differently. We will follow the convention in [C-M] Ch 6.2 regarding the order of nilpotent orbits.

Theorem 5.4 (Type A,C $\mathfrak{g}_{\mathbb{C}}$) $rank_{\mathbb{C}}(j^*(\mathcal{O}_\lambda)) = \min(k, rank_{\mathbb{C}}(\mathcal{O}_\lambda))$

Proof: If $rank_{\mathbb{C}}(\mathcal{O}_\lambda) \geq k$, then $\lambda \geq (1^{m-2k}, 2^k)$. Thus

$$cl(\mathcal{O}_\lambda) \supseteq cl(\mathcal{O}_{(1^{m-2k}, 2^k)})$$

Since $\mathfrak{g}_{-2}$ is nilpotent and satisfies

$$rank(X) \leq k; \; X^2 = 0 \qquad (\forall \;\; X \in \mathfrak{g}_{-2})$$

we have

$$\mathfrak{g}_{-2} \subseteq cl(\mathcal{O}_{(1^{m-2k},2^k)})$$

Therefore

$$cl(j^*(\mathcal{O}_\lambda)) \supseteq j^*(cl(\mathcal{O}_\lambda)) \supseteq j^*(cl(\mathcal{O}_{(1^{m-2k},2^k)})) \supseteq j^*(\mathfrak{g}_{-2}) \supseteq \mathfrak{g}_{-2}$$

Hence $rank_{\mathbb{C}}(j^*(\mathcal{O}_\lambda)) = k$. If $rank_{\mathbb{C}}(\mathcal{O}_\lambda) = s < k$, then $\lambda \geq (1^{m-2s}, 2^s)$. Therefore

$$cl(\mathcal{O}_\lambda) \supseteq cl(\mathcal{O}_{(1^{m-2s},2^s)})$$

Thus

$$cl(j^*(\mathcal{O}_\lambda)) \supseteq j^*(cl((\mathcal{O}_\lambda))) \supseteq j^*(cl(\mathcal{O}_{(1^{m-2s},2^s)}))$$

But $rank_{\mathbb{C}}(cl(\mathcal{O}_{(1^{m-2s},2^s)}) \cap \mathfrak{g}_{-2}) = s$, because the elements in $\mathfrak{g}_{-2}$ of rank s are all contained in $\mathcal{O}_{(1^{m-2s},2^s)}$. Therefore

$$rank_{\mathbb{C}}(j^*(\mathcal{O}_\lambda)) \geq rank_{\mathbb{C}}(j^*(cl(\mathcal{O}_{(1^{m-2s},2^s)})) \cap \mathfrak{g}_{-2}) = rank_{\mathbb{C}}(cl(\mathcal{O}_{(1^{m-2s},2^s)})) \cap \mathfrak{g}_{-2}) = s$$

Combined with Theorem 5.3, we have

$$rank_{\mathbb{C}}(j^*(\mathcal{O}_\lambda)) = \min(k, rank_{\mathbb{C}}(\mathcal{O}_\lambda))$$

Q.E.D.

Theorem 5.5 (Type B,D $\mathfrak{g}_{\mathbb{C}}$) *$rank_{\mathbb{C}}(j^*(\mathcal{O}_\lambda))$ is always even and it is equal to* $\min(r_k, rank_{\mathbb{C}}(\mathcal{O}_\lambda))$.

Proof: For $O_{p,q}$, the $\mathbb{C}$-rank of a real skew-symmetric form is always even. For $O^*(2n)$, the $\mathbb{C}$-rank of an $\mathbb{H}$-sesquilinear form is also even. Thus $rank_{\mathbb{C}}(j^*(\mathcal{O}_\lambda))$ is always even. Recall that the partitions corresponding to Type B, D nilpotent orbits satisfy that even parts occur with even multiplicity. In other words, if we delete the first column in the Young diagram, then odd parts occur with even multiplicity. Therefore, $rank_{\mathbb{C}}(\mathcal{O}_\lambda)$ has to be even as well. The rest of the proof is the same as the proof for type A, C groups. Q.E.D.

5.4 Rank and Associated Variety: Complex Groups

Now we want to deal with complex groups $O(n,\mathbb{C})$ and $Sp(n,\mathbb{C})$. In these cases, $\mathfrak{g}_{\mathbb{C}}$ is not simple. However, once we regard $\mathfrak{g}$ as a real matrix Lie algebra, $\mathfrak{g}_{\mathbb{C}}$ is still a matrix algebra. Thus the $\mathbb{C}$-rank of $\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))$ is still valid. Let $WF(\pi)$ be the wave front set of π as defined in [Howe4]. Recall that

$$cl(WF(\pi)) = \mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))$$

We will use these ideas to prove the following theorem.

Theorem 5.6

$$rank_{\mathbb{C}}(j^*(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi)))) = \min(r_k, rank_{\mathbb{C}}(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))))$$

where $r_k = rank_{\mathbb{C}}(\mathfrak{z}\mathfrak{n}_k^*)$.

Proof: First of all, $WF(\pi)$ is already a finite union of complex nilpotent orbits in $\mathfrak{g}$. From the real cases in the last section, we see that in the complex Lie algebra $\mathfrak{g}$

$$rank_{\mathbb{C}}(j^*(WF(\pi))) = \min(rank_{\mathbb{C}}(\mathfrak{z}\mathfrak{n}_k), rank_{\mathbb{C}}(WF(\pi)))$$

Now in the complexification $\mathfrak{g}_{\mathbb{C}}$, every quantity in this equation is doubled. Therefore this equation still holds in $\mathfrak{g}_{\mathbb{C}}$. Notice that

$$cl(j^*(WF(\pi))) = cl(j^*(cl(WF(\pi)))) = cl(j^*(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))))$$

We obtain

$$rank_{\mathbb{C}}(j^*(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi)))) = \min(rank_{\mathbb{C}}(\mathfrak{z}\mathfrak{n}_k), rank_{\mathbb{C}}(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))))$$

Q.E.D.

From Equation 5.11, we see that

$$rank_{\mathbb{C}}(supp(\mu_{ZN_k}(\pi))) = rank_{\mathbb{C}}(j^*(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))))$$

We come to our conclusion.

- For $G = Sp_{2n}, U(p,q)$, according to Equation 5.9 and Theorem 5.4, Howe's ZN_k-rank of (π, H) equals $\min(k, rank(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))))$.
- For $G = O_{p,q}$, according to Equation 5.9 and Theorem 5.5, Howe's ZN_k-rank of (π, H) equals $\min(k, rank(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))))$ for k even, and $\min(k-1, rank(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))))$ for k odd.
- For $G = Sp(p,q)$, according to Equation 5.10 and Theorem 5.4, Howe's ZN_k-rank of (π, H) equals $\min(k, \frac{1}{2}rank_{\mathbb{C}}(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))))$.
- For $G = O^*(n, \mathbb{C})$, according to Equation 5.10 and Theorem 5.5, Howe's ZN_k-rank of (π, H) equals $\min(k, \frac{1}{2}rank_{\mathbb{C}}(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))))$.
- For $Sp(n, \mathbb{C})$, according to Equation 5.10 and Theorem 5.4, Howe's ZN_k-rank of (π, H) equals $\min(k, \frac{1}{2}rank_{\mathbb{C}}(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))))$.
- For $G = O(n, \mathbb{C})$, according to Equation 5.10 and Theorem 5.5, Howe's ZN_k-rank of (π, H) equals $\min(k, \frac{1}{2}rank_{\mathbb{C}}(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))))$ when k is even,
and $\min(k-1, \frac{1}{2}rank_{\mathbb{C}}(\mathcal{V}(Ann_{U(\mathfrak{g})}(\pi))))$ when k is odd.

Thus Theorem 1.1 is proved.

6 Compactification of the Symplectic Group

First of all, let X be an analytic manifold. We say $(i, \overline{X})$ is an analytic compactification of X, if $\overline{X}$ is a compact analytic manifold and

$$i : X \to \overline{X}$$

is an embedding, such that $i(X)$ is open dense in $\overline{X}$. Let G be the standard symplectic group. Then G has a KAK decomposition, where K is $U(n) = Sp_{2n}(\mathbb{R}) \cap SO_{2n}(\mathbb{R})$ and $A \cong \mathbb{R}^n$. Let K^o be the opposite group. Then G has a $K^o \times K$ action. For the symmetric space $Y = U(2n)/O_{2n}(\mathbb{R})$, one can also define a $K \times K$ action on Y, where $K \times K$ is embedded diagonally into $U(2n)$. We define a group isomorphism $\tau : K^o \times K \to K \times K$ by

$$\tau(k_1, k_2) = (k_1^{-1}, \overline{k_2}) \qquad (k_1, k_2 \in K)$$

Thus $K^o \times K$ can be identified with $K \times K$ through τ. In this chapter, we prove the following theorem.

Theorem 6.1 *There exists an $U(n) \times U(n)$-equivariant analytic embedding:*

$$\mathcal{H} : Sp_{2n}(\mathbb{R}) \to U(2n)/O_{2n}(\mathbb{R})$$

The image is open dense in $U(2n)/O_{2n}(\mathbb{R})$. If f is a K-finite matrix coefficient of an irreducible unitary representation of $Sp_{2n}(\mathbb{R})$, then f can be extended into a continuous function on $U(2n)/O_{2n}(\mathbb{R})$.

Bargmann-Segal model is the "minimal" unitary representation of the double covering of $Sp_{2n}(\mathbb{R})$. The underlying Hilbert space is the space of L^2-analytic functions with respect to the Gaussian measure. Then the group action of $\widetilde{Sp_{2n}(\mathbb{R})}$ can be expressed as integration operators. We observed some nice structure in the integration kernel which leads to the compactification $\mathcal{H}$.

In fact, $U(2n)/O_{2n}$ can be realized as a space of matrices. Let $\mathcal{S}_{2n}$ be the space of symmetric unitary matrices of the following form

$$\{X^t X \mid X \in U(2n)\}$$

If $2n$ is fixed, we will write $\mathcal{S}$. Now $g \in U(2n)$ acts on $\mathcal{S}$ by

$$\tau(g) : s \to g s g^t \qquad (s \in \mathcal{S})$$

We compute the isotropic subgroup at the identity,

$$U(2n)_I = \{U^t U = I \mid U \in U(2n)\} = O_{2n}$$

Therefore $\mathcal{S}$ can be identified with $U(2n)/O_{2n}$. Hence the compactification of $Sp_{2n}(\mathbb{R})$ can be represented by $\mathcal{S}$.

In this Chapter, we provide the exact formula for the compactification of $Sp_2(\mathbb{R}) \cong SL(2, \mathbb{R})$.

Theorem 6.2

$$\mathcal{H}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} \frac{(a-d)+(b+c)i}{(a+d)-(c-b)i} & \frac{-2i}{(a+d)-(c-b)i} \\ \frac{-2i}{(a+d)-(c-b)i} & \frac{(a-d)-(b+c)i}{(a+d)-(c-b)i} \end{pmatrix} \in \mathcal{S}_2$$

6.1 Bargmann-Segal Model

Let V be an $n-$dimensional complex Hilbert space with the standard inner product $(*, *)$. Let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal basis of V. We write

$$(u, v) = Re(u, v) + iIm(u, v) \qquad (u, v \in V)$$

Then $\Omega(u, v) = Im(u, v)$ is a real symplectic form on V. Notice that

$$i\Omega(ie_j, e_k) = iIm(ie_j, e_k) = i\delta_j^k$$

Thus we may fix a $\mathbb{R}-$basis

$$\{ie_1, ie_2, \dots, ie_n, e_1, \dots, e_n\} = \{\xi_1, \xi_2, \dots, \xi_n, e_1, \dots, e_n\}$$

If we regard V as a real vector space under such a basis, then Ω is the standard symplectic form, and $Re(,)$ is the standard (real) inner product. From now on, whenever we regard V as a complex space, we will add a subscript $\mathbb{C}$. For a linear endomorphism g of V, without the subscript $\mathbb{C}$, g will be a real linear transform. However $g_{\mathbb{C}}$ will be a complex linear transform of V.

Let $O_{2n}(\mathbb{R})$ be the subgroup of $GL(V)$ fixing $Re(,)$, and $Sp_{2n}(\mathbb{R})$ be the subgroup of $GL(V)$ fixing $\Omega(,)$. Let $U(n)$ be the subgroup of $GL(V)$ fixing the (complex) inner product $(,)$. Then

$$U(n) = O_{2n}(\mathbb{R}) \cap Sp_{2n}(\mathbb{R})$$

In terms of real basis, the complex multiplication by imaginary i can be identified with left multiplication by

$$J = \begin{pmatrix} O & I \\ -I & 0 \end{pmatrix}$$

For arbitrary $g \in Sp_{2n}(\mathbb{R})$, g can be decomposed into

$$g = C_g + A_g$$

where C_g commutes with J, A_g anticommutes with J. Thus $C_g \in End_{\mathbb{C}}(V)$, and A_g is complex-conjugate linear. Explicitly,

$$C_g = \frac{1}{2}(g - JgJ) \qquad A_g = \frac{1}{2}(g + JgJ)$$

It is known that $C_g \in GL_{\mathbb{C}}(V)$ [R-R]. Let $\Pi_{\mathbb{C}}(V)$ be the set of $T \in GL_{\mathbb{C}}(V)$ for which $Re(Tv, v)$ is strictly positive for all nonzero $v \in V$. According to [Blattner], $\Pi_{\mathbb{C}}(V)$ is a contractible open domain of the identity in $GL_{\mathbb{C}}(V)$. Consequently, there is a unique continuous function

$$\det^{\frac{1}{2}} : \Pi_{\mathbb{C}}(V) \to \mathbb{C}$$

such that

$$\det^{\frac{1}{2}}(id) = 1 \qquad (\det^{\frac{1}{2}}(T))^2 = \det T \quad (T \in \Pi_{\mathbb{C}}(V))$$

Notice here $\det T$ is the determinant of T as a complex matrix, since $T \in \Pi_{\mathbb{C}}(V)$. Now we define

$$Z_g = C_g^{-1} A_g \qquad (g \in Sp(V, \Omega))$$

It can be shown that [R-R]

Theorem 6.3 *$I - Z_{g_1} Z_{g_2} \in \Pi_{\mathbb{C}}(V)$ for $g_1, g_2 \in Sp(V, \Omega)$.*

Let $Mp(V, \Omega)$ be the double cover of $Sp_{2n}(\mathbb{R})$, called the metaplectic group. Sometimes, we denote it by $\widetilde{Sp_{2n}}(\mathbb{R})$. There is in fact a nice way to represent this group [R-R].

Theorem 6.4

$$Mp(V, \Omega) = \{(\lambda, g) \mid g \in Sp(V, \Omega), \lambda \in \mathbb{C}, \lambda^2 \det_{\mathbb{C}}(C_g) = 1\}$$

In addition, the multiplicative structure is given by

$$(\lambda_1, g_1)(\lambda_2, g_2) = (\lambda_1 \lambda_2 (\det_{\mathbb{C}}^{\frac{1}{2}}(I - Z_{g_1} Z_{g_2^{-1}}))^{-1}, g_1 g_2)$$

Now we will construct the Bargmann-Segal model. Let dx be the Euclidean measure on V. Let

$$d\mu(x) = \exp(-\frac{1}{2}(x, x)) dx$$

be the Gaussian measure. Let $\mathcal{P}_n$ or simply $\mathcal{P}$ be the polynomial ring on $V_{\mathbb{C}}$. We define an inner product on $\mathcal{P}$ by

$$(f, g) = \int_V f(x) \overline{g(x)} d\mu(x) \qquad (f, g \in \mathcal{P})$$

Let $\|f\|^2 = (f, f)$. Let $\mathcal{F}$ be the completion of $\mathcal{P}$ under $\| * \|$. Then $\mathcal{F}$ is exactly the space of square Gaussian integrable analytic functions. In particular, $\| * \|$-covergence implies pointwise convergence.

Theorem 6.5 (Bargmann-Segal model) *Let $(\lambda, g) \in Mp(V, \Omega)$. For every $f \in \mathcal{F}$, we define*

$$\omega(\lambda, g) f(z) = \int_V \lambda \exp \frac{1}{4}(2(C_g^{-1} z, w) - (z, Z_{g^{-1}} z) - (Z_g w, w)) f(w) d\mu(w)$$

Then ω is a faithful unitary representation of $Mp(V, \Omega)$. Let

$$\mathcal{H}(g, z, w) = 2(C_g^{-1} z, w) - (z, Z_{g^{-1}} z) - (Z_g w, w)$$

If $g \neq g'$, then as functions of complex variables z and w

$$\mathcal{H}(g, z, w) \neq \mathcal{H}(g', z, w)$$

Proof: A proof of the first part of the theorem can be found in [R-R]. Suppose $g \neq g'$, but

$$\mathcal{H}(g, z, w) = \mathcal{H}(g', z, w)$$

Then $C_g = C_{g'}$. Let $\lambda \in \mathbb{C}$, such that

$$\lambda^2 \det_{\mathbb{C}}(C_g) = 1$$

Then $(\lambda, g), (\lambda, g') \in Mp(V, \Omega)$, and

$$\omega(\lambda, g) = \omega(\lambda, g')$$

This implies that $g = g'$, a contradiction. Q.E.D.

6.2 Some structure theory

Since $K = U(n)$ is a maximal compact subgroup of $Sp_{2n}(\mathbb{R})$, we can choose

$$A = \{diag(\lambda_1, \dots, \lambda_n, \lambda_1^{-1}, \dots, \lambda_n^{-1}) \mid \lambda_i \in \mathbb{R}^+\}$$

to be the maximal split Abelian subgroup. Then $Sp_{2n}(\mathbb{R})$ possesses a KAK decomposition.

Theorem 6.6 *For $g \in Sp_{2n}(\mathbb{R})$, let $g = k_1 a k_2$ be a KAK decomposition. Let $a = \exp(H)$, $H \in \mathfrak{a}$. Then we have*

$$C_g = k_1 \cosh(H) k_2; \qquad A_g = k_1 \sinh(H) k_2$$

$$Z_g = k_2^{-1} \tanh(H) k_2; \qquad Z_{g^{-1}} = -k_1 \tanh(H) k_1^{-1}$$

where

$$\cosh(H) = \frac{\exp(H) + \exp(-H)}{2}; \qquad \sinh(H) = \frac{\exp(H) - \exp(-H)}{2}$$

$$\tanh(H) = (\cosh(H))^{-1} \sinh(H)$$

Proof: In $Sp_{2n}(\mathbb{R})$, the action of K commutes with J. Thus

$$\begin{aligned} C_g =& \frac{1}{2}(g - JgJ) \\ =& \frac{1}{2}(k_1 a k_2 - J k_1 a k_2 J) \\ =& \frac{1}{2}(k_1 a k_2 - k_1 J a J k_2) \\ =& k_1 C_a k_2 \end{aligned} \tag{6.13}$$

Similarly, we have

$$A_g = k_1 A_a k_2$$

Thus

$$Z_g = C_g^{-1} A_g = (k_1 C_a k_2)^{-1}(k_1 A_a k_2) = k_2^{-1}(C_a^{-1} A_a) k_2$$

Since $g^{-1} = k_2^{-1} a^{-1} k_1^{-1}$, we have

$$Z_{g^{-1}} = k_1 (C_{a^{-1}})^{-1} A_{a^{-1}} k_1^{-1}$$

Now a simple computation shows that

$$JaJ = -a^{-1} \qquad (a \in A)$$

Thus

$$C_a = \frac{\exp(H) + \exp(-H)}{2} = \cosh(H)$$
$$A_a = \frac{\exp(H) - \exp(-H)}{2} = \sinh(H)$$
$$Z_a = C_a^{-1} A_a = \tanh(H)$$
$$Z_{a^{-1}} = \tanh(-H) = -\tanh(H)$$

Therefore

$$C_g = k_1 \cosh(H) k_2; \qquad A_g = k_1 \sinh(H) k_2$$
$$Z_g = k_2^{-1} \tanh(H) k_2; \qquad Z_{g^{-1}} = -k_1 \tanh(H) k_1^{-1}$$

Q.E.D.

We define

$$\operatorname{sech}(H) = (\cosh(H))^{-1}, \qquad \coth(H) = (\tanh(H))^{-1}$$

Combined with Theorem 6.5, we have

Theorem 6.7 *Let $(\lambda, g) \in Mp(V, \Omega)$ and $g = k_1 \exp(H) k_2$ in $Sp_{2n}(\mathbb{R})$. Then*

$$\mathcal{H}(g, z, w) = 2(\operatorname{sech}(H) k_1^{-1} z, k_2 w) + (k_1^{-1} z, \tanh(H) k_1^{-1} z) - (\tanh(H) k_2 w, k_2 w)$$

In particular, the right hand side does not depend on the KAK decomposition.

Recall that C_a is always complex linear, and A_a complex-conjugate linear. Suppose

$$H = diag(H_1, \dots, H_n, -H_1, \dots, -H_n) \qquad (H_i \in \mathbb{R})$$

We write

$$H_{\mathbb{C}} = diag(H_1, H_2, \dots, H_n)$$

Then

$$(\operatorname{sech}(H) z, w) = (\operatorname{sech}(H_{\mathbb{C}}) z, w)$$

Now we want to compute $(\tanh(H)z, w)$. Let $z = iy + x$ with $x, y \in \mathbb{R}^n$. Then

$$\tanh(H)z = \tanh(H)(x + iy) = i\tanh(H_{\mathbb{C}})y - \tanh(H_{\mathbb{C}})x = -\tanh(H_{\mathbb{C}})\overline{z}$$

Therefore

$$\begin{aligned}(\tanh(H)z, w) &= -(\tanh(H_{\mathbb{C}})\overline{z}, w)\\ (z, \tanh(H)w) &= -(z, \tanh(H_{\mathbb{C}})\overline{w})\end{aligned}$$

Now

$$\begin{aligned}\mathcal{H}(g, z, w) =&2(\mathrm{sech}(H_{\mathbb{C}})k_1^{-1}z, k_2w) - (k_1^{-1}z, \tanh(H_{\mathbb{C}})\overline{k_1^{-1}z}) + (\tanh(H_{\mathbb{C}})\overline{k_2w}, k_2w)\\ =&2\overline{w^tk_2^t}\mathrm{sech}(H_{\mathbb{C}})k_1^{-1}z - z^t\overline{k_1}\tanh(H_{\mathbb{C}})k_1^{-1}z + \overline{w^tk_2^t}\tanh(H_{\mathbb{C}})\overline{k_2w}\\ =&(iz^t, \overline{w^t})\begin{pmatrix}\overline{k_1} & 0\\ 0 & \overline{k_2^t}\end{pmatrix}\begin{pmatrix}\tanh(H_{\mathbb{C}}) & -i\mathrm{sech}(H_{\mathbb{C}})\\ -i\mathrm{sech}(H_{\mathbb{C}}) & \tanh(H_{\mathbb{C}})\end{pmatrix}\begin{pmatrix}k_1^{-1} & 0\\ 0 & \overline{k_2}\end{pmatrix}\begin{pmatrix}iz\\ \overline{w}\end{pmatrix}\end{aligned} \tag{6.14}$$

Definition 6.1 *We define*

$$\mathcal{H}(k_1\exp(H)k_2) = \begin{pmatrix}\overline{k_1} & 0\\ 0 & \overline{k_2^t}\end{pmatrix}\begin{pmatrix}\tanh(H_{\mathbb{C}}) & -i\mathrm{sech}(H_{\mathbb{C}})\\ -i\mathrm{sech}(H_{\mathbb{C}}) & \tanh(H_{\mathbb{C}})\end{pmatrix}\begin{pmatrix}k_1^{-1} & 0\\ 0 & \overline{k_2}\end{pmatrix}$$

Notice that k_1, k_2 are unitary. One critical observation is that the images of $\mathcal{H}$ are symmetric unitary matrices. Therefore this definition of $\mathcal{H}$ is the uniquely determined by the following equation

$$\mathcal{H}(g, z, w) = (iz^t, \overline{w^t})\mathcal{H}(g)\begin{pmatrix}iz\\ \overline{w}\end{pmatrix} \tag{6.15}$$

Theorem 6.8 *The map $\mathcal{H}$ is a continuous injection from $Sp_{2n}(\mathbb{R})$ into $U(2n)$.*

Proof: First of all, if $\mathcal{H}(g) = \mathcal{H}(g')$, then

$$\mathcal{H}(g, z, w) = \mathcal{H}(g', z, w) \qquad (\forall\ \ z, w \in V)$$

According to Theorem 6.5, we have $g = g'$. Therefore $\mathcal{H}$ is an injection. Since that the maps $g \to C_g^{-1}$, $g \to Z_g$, and $g \to Z_{g^{-1}}$ are all continuous, for every $z, w \in V$, $g \to \mathcal{H}(g, z, w)$ is continuous. From Equation 6.15 and by linearity, every entry of the matrix $\mathcal{H}(g)$ is a continuous function of $Sp_{2n}(\mathbb{R})$. Therefore, $\mathcal{H}(g)$ is continuous as well. Q.E.D.

6.3 Analytic Properties of $\mathcal{H}$

We define $\mathbb{T}_n$ in $U(2n)$ to be the space of matrices of the following form

$$T(\theta) = \begin{pmatrix}diag(\cos\theta_1, \ldots, \cos\theta_n) & diag(-i\sin\theta_1, \ldots, -i\sin\theta_n)\\ diag(-i\sin\theta_1, \ldots, -i\sin\theta_n) & diag(\cos\theta_1, \ldots, \cos\theta_n)\end{pmatrix} \qquad (\theta = (\theta_1, \ldots, \theta_n))$$

We want to analyze the map $\kappa : A \to \mathbb{T}^n$, defined to be the restriction of $\mathcal{H}$ on A. Without loss of generality, let $n = 1$. Then

$$\kappa(\exp H) = \begin{pmatrix} \tanh H & -i\text{sech}H \\ -i\text{sech}H & \tanh H \end{pmatrix} \in \mathbb{T}$$

κ can be regarded as a homeomorphism from $\mathbb{R}$ to $(0, \pi)$. Therefore, θ can be regarded as a continuous function of H, and H can also be regarded as a continuous function of θ. Notice that from $\tanh H = \cos\theta$, we obtain

$$(\text{sech}(H))^2 dH = -\sin\theta d\theta$$

Therefore

$$\frac{d\theta}{dH} = -\text{sech}(H) \neq 0 \qquad \frac{dH}{d\theta} = -\csc\theta \neq 0$$

Since all these functions are (real) analytic, κ is an analytic embedding from A to $\mathbb{T}^n$. From here one may guess that $\mathcal{H}$ is in fact an embedding. However, nothing can be proved since $K^o \otimes K$ acts on $Sp_{2n}(\mathbb{R})$ with singularities.

Let $\mathcal{S} = \{U^t U \mid U \in U(2n)\}$ be a subset of $U(2n)$. $\kappa(A)$ is contained in $\mathcal{S}$. Thus $\mathcal{H}(Sp_{2n}(\mathbb{R}))$ is in fact contained in $\mathcal{S}$. We obtain

Lemma 6.1

- *Let $U(2n)$ act on $\mathcal{S}$ by*

  $$g \to U^t g U \qquad (g \in \mathcal{S}, U \in U(2n))$$

  Then $\mathcal{S} \cong U(2n)/O_{2n}(\mathbb{R})$;

- *$\mathcal{H}$ is a continuous map into $\mathcal{S}$;*

- *Let $U(n)^o$ be the opposite group of $U(n)$. Let $U(n)^o \times U(n)$ act on $Sp_{2n}(\mathbb{R})$ by left and right multiplications respectively. Let $\tau : U(n)^o \times U(n) \to U(n) \times U(n)$ be a group isomorphism defined as follows:*

  $$\tau(k_1, k_2) = (k_1^{-1}, \overline{k_2})$$

  If we identify these two groups through τ, then $\mathcal{H}$ is equivariant with respect to these two group actions.

Now we want to compute $d\mathcal{H} : TSp_{2n}(\mathbb{R}) \to T\mathcal{S}$. Let $g(t)$ be a germ of a smooth curve near $g \in Sp_{2n}(\mathbb{R})$. Let dg be the tangent vector represented by this germ $g(t)$. Here $Sp_{2n}(\mathbb{R})$ is contained in the space of $2n \times 2n$ matrices. We may engage all our discussion in the space of $2n \times 2n$ matrices. Thus the tangent vector dg in $Sp_{2n}(\mathbb{R})$ can be identified with a $2n \times 2n$ matrix. This is the perspective we take in interpreting all the equations here. From $gg^{-1} = 1$, we obtain

$$(dg)g^{-1} + g(dg^{-1}) = 0$$

Therefore

$$dg^{-1} = -g^{-1}(dg)g^{-1}$$

We have the following lemma.

Lemma 6.2 *1.* $dg^{-1} = -g^{-1}dgg^{-1}$*;*

2. $dZ_g = -C_g^{-1}dC_gZ_g + C_g^{-1}dA_g$*; where*

$$dC_g = \frac{1}{2}(dg - J(dg)J) \qquad dA_g = \frac{1}{2}(dg + J(dg)J)$$

3. $dZ_{g^{-1}} = -C_{g^{-1}}^{-1}dC_{g^{-1}}Z_{g^{-1}} + C_{g^{-1}}^{-1}dA_{g^{-1}}$*;*

4. $dC_g^{-1} = -C_g^{-1}dC_gC_g^{-1}$*;*

5. $dC_{g^{-1}} = -\frac{1}{2}(g^{-1}(dg)g^{-1} - Jg^{-1}(dg)g^{-1}J)$*;*

6. $dA_{g^{-1}} = -\frac{1}{2}(g^{-1}(dg)g^{-1} + Jg^{-1}(dg)g^{-1}J)$

Now we can compute $d\mathcal{H}$. Let $\mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g}$ be the Cartan decomposition with $K = U(n)$. In fact, it can be shown that

Lemma 6.3 $\mathfrak{k}$ *is complex linear, and* $\mathfrak{p}$ *is complex-conjugate linear as morphisms of* V*.*

For an arbitrary $g \in Sp_{2n}(\mathbb{R})$, let $g = k \exp p$ with $p \in \mathfrak{p}$. Because of the action of $U(n)^o \times U(n)$, without loss of generality, we may assume that $g = \exp H$, $H \in \mathfrak{a}$. Now we have the following theorem.

Theorem 6.9 $(d\mathcal{H})_g : T_g(Sp_{2n}(\mathbb{R})) \to T_{\mathcal{H}(g)}(\mathcal{S})$ *is bijective.*

Proof:

1. First notice that

$$\dim(\mathcal{S}) = \dim(U(2n)) - \dim(O_{2n}(\mathbb{R})) = \frac{2n(2n+1)}{2} = \dim(Sp_{2n}(\mathbb{R}))$$

   It suffices to show that the kernel of $(d\mathcal{H})_g$ is trivial.

2. Let dg be the equivalence class of the germ $g \exp tk$ with $k \in \mathfrak{k}$. Then we may write $dg = gk$.

$$\begin{aligned} dC_g =& \frac{1}{2}(dg - J(dg)J) \\ =& \frac{1}{2}(gk - JgkJ) \\ =& \frac{1}{2}(g - JgJ)k \\ =& C_gk \end{aligned} \tag{6.16}$$

   Similarly,

$$dC_{g^{-1}} = -kC_{g^{-1}}$$

$$dA_g = A_g k \qquad dA_{g^{-1}} = -kA_{g^{-1}}$$

Thus

$$dC_g^{-1} = -C_g^{-1}(dC_g)C_g^{-1} = -kC_g^{-1}$$

$$dZ_g = -C_g^{-1}(dC_g)Z_g + C_g^{-1}(dA_g) = -kZ_g + Z_g k$$

$$dZ_{g^{-1}} = -C_{g^{-1}}^{-1}(dC_{g^{-1}})C_{g^{-1}}^{-1}A_{g^{-1}} + C_{g^{-1}}^{-1}dA_{g^{-1}} = C_{g^{-1}}^{-1}kA_{g^{-1}} - C_{g^{-1}}^{-1}kA_{g^{-1}} = 0$$

Therefore

$$d(z, Z_{g^{-1}}z) = 0 \tag{6.17}$$

$$d\mathcal{H}(g,z,w) = 2(-kC_g^{-1}z, w) + ((kZ_g - Z_g k)w, w) \tag{6.18}$$

Since $C_g \in GL_{\mathbb{C}}(V)$, we can see that

$$d\mathcal{H}(g,z,w) = 0 \ (\forall\ z,w \in V) \Longrightarrow (-kC_g^{-1}z, w) = 0 \ (\forall\ z,w \in V) \Longleftrightarrow k = 0 \tag{6.19}$$

3. On the other hand, let dg be the equivalence class of the germ $g \exp tp$ with $p \in \mathfrak{p}$. Then we may write $dg = gp$. We have $pJ = -Jp$. Thus

$$dC_g = \frac{1}{2}(dg - JdgJ) = \frac{1}{2}(gp - JgpJ) = \frac{1}{2}(gp + JgJp) = A_g p$$

$$dA_g = C_g p \qquad dA_{g^{-1}} = -pC_{g^{-1}} \qquad dC_{g^{-1}} = -pA_{g^{-1}}$$

Then

$$dC_g^{-1} = -C_g^{-1}(dC_g)C_g^{-1} = -Z_g p C_g^{-1}$$

$$dZ_g = -C_g^{-1}(dC_g)Z_g + C_g^{-1}(dA_g) = -Z_g p Z_g + p$$

$$\begin{aligned} dZ_{g^{-1}} &= -C_{g^{-1}}^{-1}(dC_{g^{-1}})Z_{g^{-1}} + C_{g^{-1}}^{-1}dA_{g^{-1}} \\ &= C_{g^{-1}}^{-1}pA_{g^{-1}}Z_{g^{-1}} - C_{g^{-1}}^{-1}pC_{g^{-1}} \\ &= C_{g^{-1}}^{-1}p(A_{g^{-1}}Z_{g^{-1}} - C_{g^{-1}}) \end{aligned} \tag{6.20}$$

Suppose $g = \exp H$, $H \in \mathfrak{a}$. Then

$$\begin{aligned} A_{g^{-1}}Z_{g^{-1}} - C_{g^{-1}} &= \sinh(H)\tanh(H) - \cosh(H) \\ &= (\cosh(H))^{-1}(\sinh(H)^2 - \cosh(H)^2) \\ &= -(\cosh(H))^{-1} \end{aligned} \tag{6.21}$$

because $\mathfrak{a}$ is commutative. Therefore

$$d(z, Z_{g^{-1}}z) = (z, -\mathrm{sech}(H)(p)\mathrm{sech}(H)z) = -(z, \mathrm{sech}(H_{\mathbb{C}})(p)\mathrm{sech}(H_{\mathbb{C}})z)$$

Suppose under the real basis $\{ie_j, e_j\}_1^n$,

$$p = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}, \qquad (A^t = A, B^t = B) \tag{6.22}$$

Therefore

$$p(yi + x) = i(Ay + Bx) + (By - Ax) = (Bi - A)(x - iy) = (Bi - A)(\overline{x + iy})$$

We see that

$$\begin{aligned} d(z, Z_{g^{-1}}z) &= -\ (z, \mathrm{sech}(H_{\mathbb{C}})p\,\mathrm{sech}(H_{\mathbb{C}})z) \\ &= -\ (z, \mathrm{sech}(H_{\mathbb{C}})(Bi - A)\mathrm{sech}(H_{\mathbb{C}})\overline{z}) \\ &= -\ z^t\mathrm{sech}(H_{\mathbb{C}})(-Bi - A)\mathrm{sech}(H_{\mathbb{C}})z \\ &= z^t\mathrm{sech}(H_{\mathbb{C}})(A + Bi)\mathrm{sech}(H_{\mathbb{C}})z \end{aligned} \tag{6.23}$$

Since $A + Bi$ is a symmetric matrix and $\mathrm{sech}(H_{\mathbb{C}})$ is invertible, we have

$$d(z, Z_{g^{-1}}z) = 0 \ \ (\forall\ z \in V) \Longleftrightarrow A + Bi = 0 \Longleftrightarrow p = 0 \tag{6.24}$$

4. For an arbitrary $X = k + p \in \mathfrak{g}$, $g = \exp H$, Let $g(t) = g\exp(tX)$ be a fixed germ. Let p be defined as in Equation 6.22. Suppose that $d\mathcal{H}(g, z, w) = 0$. Then from Equation 6.17 and Equation 6.23, we see that

$$d(z, Z_{g^{-1}}z) = z^t\mathrm{sech}(H_{\mathbb{C}})(A + Bi)\mathrm{sech}(H_{\mathbb{C}})z$$

Thus

$$d(z, Z_{g^{-1}}z) = 0(\forall\ z \in V) \Longrightarrow p = 0$$

Now $X = k$. From Equation 6.19, we see that $k = 0$. Therefore, $X = 0$. Thus we have proved that

$$d\mathcal{H}(g, z, w) = 0\ (\forall\ z, w \in V) \Longrightarrow X = 0$$

This implies that

$$d\mathcal{H}(g) = 0 \Longrightarrow X = 0$$

5. Since $Sp_{2n}(\mathbb{R})$ is a Lie group, the tangent space $T_g(Sp_{2n}(\mathbb{R}))$ can be identified with those germs

$$g\exp(tX) \qquad (X \in \mathfrak{g})$$

Thus

$$d\mathcal{H}|_g : T_g(Sp_{2n}(\mathbb{R})) \to T_{\mathcal{H}(g)}(\mathcal{S})$$

is injective. Because of the left and right K-action, this is true for all $g \in Sp_{2n}(\mathbb{R})$.

Q.E.D.

This shows that $\mathcal{H}$ is an immersion, locally homeomorphism. It is also one-to-one. Thus $\mathcal{H}$ is a homeomorphism from $Sp_{2n}(\mathbb{R})$ onto an open submanifold of $\mathcal{S}$.

Theorem 6.10 *$\mathcal{H} : Sp_{2n}(\mathbb{R}) \to \mathcal{S}$ is analytic.*

Proof: In this proof V will be regarded as a real vector space. Then $\mathcal{S}$ is an analytic submanifold of $B(V \oplus V, \mathbb{C})$, the space of symmetric complex-valued bilinear forms on $V \oplus V$. It suffices to show that

$$\mathcal{H} : Sp_{2n}(\mathbb{R}) \to B(V \oplus V, \mathbb{C})$$

is analytic. Recall that under the real basis $\{ie_j, (j = 1, \ldots, n), e_j, (j = 1, \ldots, n)\}$, multiplication by i can be regarded as left multiplication by J, and taking conjugation can be regarded as left multiplication by

$$B = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

Therefore

$$\begin{aligned} 2(C_g^{-1} z, w) &= \bar{w}^t C_g^{-1} z = 2\bar{w}^t C_g^{-1}(-J) iz \\ (z, Z_{g^{-1}} z) &= \overline{z^t \overline{Z_{g^{-1}}}^t} z = -(iz^t) B \overline{Z_{g^{-1}}}^t (iz) \\ (Z_g w, w) &= \overline{w^t} Z_g w = \overline{w^t} Z_g B \overline{w} \end{aligned}$$

Since the maps $g \to g^{-1}$, $g \to C_g^{-1}$, $g \to Z_g$ are all real analytic, we conclude that

$$\mathcal{H} : Sp_{2n}(\mathbb{R}) \to B(V \oplus V, \mathbb{C})$$

is analytic. Q.E.D.

Now we have shown that $d\mathcal{H}_g$ is bijective and $\mathcal{H} : Sp_{2n}(\mathbb{R}) \to \mathcal{S}$ is analytic and one-to-one. From the classical theorem on inverse functions (see page 21 [Varadarajan]), we have

Theorem 6.11 *$\mathcal{H} : Sp_{2n}(\mathbb{R}) \to \mathcal{S}$ is an analytic embedding.*

Now, we obtain

Theorem 6.12 *Let G be an arbitrary group with a faithful representation into $Sp_{2n}(\mathbb{R})$. Suppose the closure of $\mathcal{H}(G)$, denoted by $\overline{G}$, is a compact smooth submanifold of $\mathcal{S}$. Then $(\mathcal{H}|_G, \overline{G})$ is an analytic compactification of G.*

6.4 Generalized Cartan decomposition and Some Remarks

For a subgroup H of G, let $N_G(H), Z_G(H)$ be the normalizer and centralizer of H in G. For a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, let $N_G(\mathfrak{h})$ and $Z_G(\mathfrak{h})$ be the normalizer and centralizer of $\mathfrak{h}$ in G. Suppose G is a compact connected Lie group. Let σ, τ be a pair of commuting involutions of G. Let K and H be the fixed point sets of σ and τ respectively. Let $\mathfrak{p}$ be the -1 eigenspace of σ, and $\mathfrak{q}$ the -1 eigenspace of τ. Let $\mathfrak{t}_{\mathfrak{pq}}$ be the maximal Abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. Let $T_{\mathfrak{pq}}$ be the analytic group of $\mathfrak{t}_{\mathfrak{pq}}$. We define the Weyl group to be

$$W_{\mathfrak{pq}} = N_K(\mathfrak{t}_{\mathfrak{pq}})/Z_K(\mathfrak{t}_{\mathfrak{pq}}) \cong N_H(\mathfrak{t}_{\mathfrak{pq}})/Z_H(\mathfrak{t}_{\mathfrak{pq}})$$

Theorem 6.13 (Generalized Cartan Decomposition) *G possesses a $KT_{\mathfrak{pq}}H$ decomposition. In other words,*

$$m : K \times T_{\mathfrak{pq}} \to G/H$$

is surjective. In addition, for $g = kth$, t is unique up to $W_{\mathfrak{pq}}$ and a multiplication of $T_{\mathfrak{pq}} \cap Z_K(\mathfrak{t}_{\mathfrak{pq}})Z_H(\mathfrak{t}_{\mathfrak{pq}})$.

This theorem is essentially due to Hoogenboom (see page 194 in [H-S]). Now for $G = U(2n)$, let

$$\sigma(x) = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} x \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \qquad (x \in U(2n))$$

$$\tau(x) = \overline{x} \qquad (x \in U(n))$$

It is obvious that

$$\tau\sigma = \sigma\tau$$

and

$$K = U(n) \times U(n) \qquad H = O_{2n}(\mathbb{R})$$

$$\mathfrak{p} = \{\begin{pmatrix} 0 & A \\ -\overline{A}^t & 0 \end{pmatrix} \mid A \in gl(n, \mathbb{C})\}$$

$$\mathfrak{q} = \{iB \mid B^t = B, B \in gl(2n, \mathbb{R})\}$$

Thus

$$\mathfrak{p} \cap \mathfrak{q} = \{\begin{pmatrix} 0 & iA \\ iA^t & 0 \end{pmatrix} \mid A \in gl(n, \mathbb{R})\}$$

We may choose $T_{\mathfrak{pq}} = \mathbb{T}^n \subseteq U(2n)$. Then

$$\mathfrak{t}_{\mathfrak{pq}} = \{\mathfrak{t}_\theta = \begin{pmatrix} 0 & -diag(i\theta_1, \ldots, i\theta_n) \\ -diag(i\theta_1, \ldots, i\theta_n) & 0 \end{pmatrix} \mid \theta_i \in [0, 2\pi]\}$$

Then $W_{\mathfrak{pq}}$ is simply the Weyl group of type B_n Lie algebra. More precisely, $W_{\mathfrak{pq}}$ acts on $\mathfrak{t}_\theta$ by permuting θ_i's and changing the sign of θ_i's. We identify $\mathbb{T}^n$ with $(\mathbb{T})^n$. According to the generalized Cartan decomposition, we obtain,

Theorem 6.14 *$U(2n)$ possesses an $K\mathbb{T}^nH$ decomposition, where $K = U(n) \times U(n)$ embedded diagonally, and $H = O_{2n}(\mathbb{R})$. In addition, for $g = kth$, $t = \exp \mathfrak{t}_\theta$ is unique up to a reordering of θ and sign changes of θ_i's. If we define $\psi : K \times \mathbb{T}^n \to \mathcal{S}$ by*

$$\psi(k, t) = ktk^t \in \mathcal{S} \cong U(2n)/H \qquad (k \in K, t \in \mathbb{T}^n)$$

then ψ is surjective.

In particular, due to the action of Weyl group, we may assume that $\sin\theta_i \geq 0$, i.e.,

$$\theta_i \in [0, \pi]$$

We observe that

$$Im(\mathcal{H}) = \psi(K \times (\mathcal{H}(A)) = \psi(K \times \{\mathrm{t}_\theta \mid \theta_i \in (0, \pi)\})$$

is dense in $\mathcal{S}$. Combined with Theorem 6.11 we have shown

Theorem 6.15 (Compactification of $Sp_{2n}(\mathbb{R})$) *$\mathcal{H} : Sp_{2n}(\mathbb{R}) \to \mathcal{S}$ is an analytic compactification.*

Let $d_{Sp_{2n}}g$ be a Haar measure of $Sp_{2n}(\mathbb{R})$. Let $d_{U(2n)}s$ be a $U(2n)$-invariant measure of $\mathcal{S}$. Then both measures can be regarded as volumes forms. In addition, these volume forms are nondegenerate over every point because of the group action. Thus we may pull back $d_{U(2n)}s$ to a volume form $\mathcal{H}^*(d_{U(2n)}s)$ on $Sp_{2n}(\mathbb{R})$. We define $\frac{d\mathcal{H}(g)}{dg}$ to be the unique function satisfying

$$\mathcal{H}^*(d_{U(2n)}\mathcal{H}(g)) = \frac{d\mathcal{H}(g)}{dg} d_{Sp_{2n}}g$$

Since $\mathcal{H}$ is an analytic embedding, $\frac{d\mathcal{H}(g)}{dg}$ is an analytic function, and it is always positive. Conversely, since $\mathcal{S} - \mathcal{H}(Sp_{2n}(\mathbb{R}))$ is of measure zero with respect to $d_{U(2n)}s$, we can write

$$d_{Sp_{2n}}g = (\frac{d\mathcal{H}(g)}{dg})^{-1}\mathcal{H}^* d_{U(2n)}s$$

Theorem 6.16 *For every $g \in Sp_{2n}(\mathbb{R})$,*

$$\frac{d\mathcal{H}(g)}{dg} > 0$$

Finally, we will derive some applications of this compactifiaction. For any function, $f \in C(Sp_{2n}(\mathbb{R})$, let $\bar{f}$ be the push-forward, defined to be

$$\bar{f}(s) = f(\mathcal{H}^{-1}(s)), \qquad (s \in Im(\mathcal{H}))$$

and zero otherwise.

Theorem 6.17 *Let (π, H) be a nontrivial irreducible unitary representation of $Sp_{2n}(\mathbb{R})$. Let $f(g) = (\pi(g)u, v)$ be a matrix coefficient, with u, v K−finite. Then $\bar{f}$ is continuous, and real analytic over $Im(\mathcal{H})$.*

Proof: Since u, v are K-finite, $f(g)$ is real analytic (See [Knapp1] page 210). It suffices to show that if

$$\lim_{i\to\infty} s^i = s, \qquad (s \in \mathcal{S} - Im(\mathcal{H}), s^i = \mathcal{H}(g^i))$$

then

$$\lim_{i\to\infty} f(g^i) = 0$$

Let $K = U(n)$ be the maximal compact subgroup in $Sp_{2n}(\mathbb{R})$. Let $g = k \exp p$ be the Cartan decomposition. We define

$$\|g\| = Trace(pp)$$

Let $g = k_1 \exp H k_2 = k_1 k_2 \exp(Ad(k_1^{-1})H)$. Then

$$\|g\| = Trace(H^2) = 2\sum_{i=1}^{n} H_j^2$$

We say $g^i \to \infty$ if $\|g^i\| \to \infty$. Notice that we have the following commutative diagram

$$\begin{array}{ccc} K^o \times K \times A & \xrightarrow{KAK} & Sp_{2n}(\mathbb{R}) \\ \downarrow{\scriptstyle \tau\otimes\kappa} & & \downarrow{\scriptstyle \mathcal{H}} \\ K \times K \times \mathbb{T}^n & \xrightarrow{\psi} & \mathcal{S} \end{array} \tag{6.25}$$

Let $s^i = \psi((k_1^i, k_2^i), t^i)$, $s = \psi((k_1, k_2), t)$. Then $t^i = \exp(\mathfrak{t}_{\theta^i})$ and $t = \exp(\mathfrak{t}_\theta)$ can be chosen such that

$$\lim_{i\to\infty} t_i = t_0$$

Since $s \notin Im(\mathcal{H})$, $t = \exp \mathfrak{t}_\theta \notin Im(\mathcal{H})$. This simply means that for some $j \in [1, n]$, $\sin(\theta_j) = 0$. This implies that

$$\lim_{i\to\infty} \sin(\theta_j^i) = 0$$

Recall that $\sin(theta_j) = \cosh(H_j)$. Thus

$$\lim_{i\to\infty} H_j(\theta^i) = \infty$$

Therefore

$$\lim_{i\to\infty} \|g^i\| = \lim_{i\to\infty} \sum_{k=1}^{n} H_k(\theta^i)^2 \geq \lim_{i\to\infty} H_j(\theta^i)^2 = \infty$$

Thus $g_i \to \infty$. From Theorem 5.4 [Borel-Wallach], we know that f vanishes at ∞. Therefore

$$\lim_{i\to\infty} f(g^i) = 0$$

Thus $\bar{f}$ is continuous. Q.E.D.

We will compute exactly the compactification of $Sp_2(\mathbb{R}) = SL(2, \mathbb{R})$.

Theorem 6.18

$$\mathcal{H}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} \frac{(a-d)+(b+c)i}{(a+d)-(c-b)i} & \frac{-2i}{(a+d)-(c-b)i} \\ \frac{-2i}{(a+d)-(c-b)i} & \frac{(a-d)-(b+c)i}{(a+d)-(c-b)i} \end{pmatrix} \in \mathcal{S}_2$$

Proof: Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Then

$$C_g = \frac{1}{2}\begin{pmatrix} a+d & b-c \\ c-b & a+d \end{pmatrix} \qquad A_g = \frac{1}{2}\begin{pmatrix} a-d & b+c \\ b+c & d-a \end{pmatrix}$$

We obtain

$$\det C_g = \frac{a^2+b^2+c^2+d^2+2ad-2bc}{4} = \frac{a^2+b^2+c^2+d^2+2}{4}$$

Let $\xi = a^2+b^2+c^2+d^2+2$. We have

$$C_g^{-1} = \frac{2}{\xi}\begin{pmatrix} a+d & c-b \\ b-c & a+d \end{pmatrix} \qquad Z_g = C_g^{-1}A_g = \frac{1}{\xi}\begin{pmatrix} a^2-d^2+c^2-b^2 & 2(ab+cd) \\ 2(ab+cd) & b^2-c^2+d^2-a^2 \end{pmatrix}$$

Recall that the real basis of $V = \mathbb{C}$ is $i, 1$. We obtain

$$2(C_g^{-1}z, w) = \frac{4}{\xi}((a+d)+(c-b)i)z\overline{w} = \frac{-4i}{(a+d)+(b-c)i}(iz)\overline{w}$$

$$(Z_g w, w) = \frac{1}{\xi}((b^2-c^2+d^2-a^2)+2(ab+cd)i)\overline{ww} = \frac{(b+ai)^2+(d+ci)^2}{\xi}\overline{ww} = \frac{(d-a)+(c+b)i}{(d+a)-(c-b)i}\overline{ww}$$

By interchanging $a \leftrightarrow d, b \leftrightarrow -b, c \leftrightarrow -c$, we obtain

$$(Z_{g^{-1}}z, z) = \frac{(a-d)-(c+b)i}{(a+d)-(b-c)i}\overline{zz}$$

Thus

$$(z, Z_{g^{-1}}z) = \overline{(Z_{g^{-1}}z, z)} = \frac{(a-d)+(b+c)i}{(a+d)+(b-c)i}zz$$

Therefore, from

$$\mathcal{H}(g, z, w) = (iz, \overline{w})\mathcal{H}(g)\begin{pmatrix} iz, \overline{w} \end{pmatrix}$$

We obtain that

$$\mathcal{H}(g) = \begin{pmatrix} \frac{(a-d)+(b+c)i}{(a+d)+(b-c)i} & \frac{-2i}{(a+d)+(b-c)i} \\ \frac{-2i}{(a+d)+(b-c)i} & \frac{(d-a)+(c+b)i}{(d+a)+(b-c)i} \end{pmatrix}$$

It is easy to check that $\mathcal{H}(g) \in \mathcal{S}$. Q.E.D.

7 Dual Pair Correspondence

In this chapter,we will review the fundemental theory on the dual pair correspondence of Howe [Howe2]. We will use $(\pi, \mathcal{H}_\pi)$ to denote the Hilbert representation and (π, V_π) to denote its Harish-Chandra module. We will refer to π for both Hilbert representation and Harish-Chandra module. In all cases $\mathfrak{g} = L(G)$ is the Lie algebra of G, and $\mathfrak{g}_\mathbb{C}$ is the complexification of $\mathfrak{g}$.

Let

$$C(X) = \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} X \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \qquad (X \in Sp_{2n})$$

be a transform from $Sp_{2n}(\mathbb{R})$ to $Sp_{2n}(\mathbb{C})$. We write

$$Sp = C(Sp_{2n}(\mathbb{R})) = Sp_{2n}(\mathbb{C}) \cap U(n,n)$$

throughout this chapter. In the later chapters, Sp will be the standard symplectic group.

7.1 Dual Pairs

We follow the definition in [Howe1]. Let (G, G') be a pair of subgroups of the symplectic group $Sp(W)$. We say (G, G') is a reductive dual pair if

1. G and G' act absolutely reductively on W;

2. G is the centralizer of G', G' is the centralizer of G in $Sp(W)$

If (G, G') is a reductive dual pair in $Sp(V, \Omega)$, and if $V = V_1 \oplus V_2$ is a direct sum decomposition such that $\Omega(V_1, V_2) = 0$ and V_1, V_2 are $G \cdot G'$ invariant, then we say that (G, G') is reducible. Here $\Omega|_{V_i}$ is automatically nondegenerate. If we restrict (G, G') to V_i, we obtain a dual pair (G_i, G'_i) in $Sp(V_i, \Omega|_{V_i})$. Then G can be identified with $G_1 \times G_2$ and G' can be identified with $G'_1 \times G'_2$. We say that (G, G') is the direct sum of (G_1, G'_1) and (G_2, G'_2). Essentially, every reductive dual pair can be decomposed as the direct sum of irreducible ones. All the irreducible dual pairs are classified in [Howe1].

Theorem 7.1 *There are two types of irreducible reductive dual pairs. (G, G') is a dual pair of type I means that there exist*

1. *a division algebra D over F,with involution $\natural$;*

2. *D-modules V and V', with non-degenerate $\natural$-sesqulinear form $(,)$ and $(,)'$; one $\natural$-hermitian, and the other $\natural$-skew-Hermitian; G and G' are the isometry groups respectively;*

3. *$W = V \otimes_D V'$, $< a \otimes b, c \otimes d >= Tr_{D/F}((a,c)_1 (b,d)_2^\natural)$, then $<,>$ is a symplectic form on W, and $Sp = Sp(W)$ is the isometry group of $<,>$*

(G, G') is a dual pair of type II means that there exist two GG'-stable Langragian subspaces X and Y such that $W = X \oplus Y$ and there exist

1. *a division algebra D over F;*

2. *left D-module* V_1 *and right D-module* V_2*;*

3. $X = V_1 \otimes_D V_2$*;*

4. (G, G') *is identified with* $GL_D(V_1), GL_D(V_2)$ *in* $Sp(W)$

For real classical groups, all the dual pairs were made explicit in [Howe2].

Theorem 7.2 (Howe) *There are seven families of dual pairs.*

1. $D = \mathbb{R}$, $\sharp = id$, $(O(p,q), Sp_{2n}(\mathbb{R})) \subseteq Sp_{2(p+q)n}$

2. $D = \mathbb{C}$, $\sharp = id$, $(O(p,\mathbb{C}), Sp_{2n}(\mathbb{C})) \subseteq Sp_{4pn}$

3. $D = \mathbb{C}$, $\sharp =$ *conjugation*, $(U(p,q), U(r,s)) \subseteq Sp_{2(p+q)(r+s)}$

4. $D = \mathbb{H}, \sharp =$ *conjugation*, $(Sp(p,q), O^*(2n)) \subseteq Sp_{4n(p+q)}$

5. $D = \mathbb{R}$, $(GL(m,\mathbb{R}), GL(n,\mathbb{R})) \subseteq Sp_{2nm}$

6. $D = \mathbb{C}$, $(GL(m,\mathbb{C}), GL(n,\mathbb{C})) \subseteq Sp_{4mn}$

7. $D = \mathbb{H}$, $(GL(m,\mathbb{H}), GL(n,\mathbb{H})) \subseteq Sp_{8mn}$

The first four are type I classical groups, and the last three are type II classical groups.

Let $\widetilde{Sp}$ be the metaplectic cover of Sp. For any subgroup G of Sp, we use $\tilde{G}$ to denote the preimage of G under the metaplectic cover, regardless of the fundamental group of G. Let $\{1, \epsilon\}$ be the preimage of the identity in Sp. Let $(\omega, \mathcal{H}_o)$ be the oscillator representation. Then we will always have

$$\omega(\epsilon) = -1$$

Thus we may use $\hat{\tilde{G}}(\epsilon)$ ($\hat{\tilde{G}}_{ad}(\epsilon)$)to denote the unitary (admissible) dual of $\tilde{G}$ such that $\pi(\epsilon) = -1$ holds. We shall keep in mind that this equation is always true in this thesis.

Definition 7.1 (stable range) *A type I dual pair* (G, G') *is in the stable range if the real rank* r *of* G *is greater or equal to* $dim_D(V')$.

7.2 Structure Theory on Dual Pairs of Sp

1. We fix a Cartan decomposition $sp = u \oplus S$.

$$U = \left\{ \begin{pmatrix} X & 0 \\ 0 & (X^{-1})^t \end{pmatrix} \mid X \in U(n) \right\}$$

$$u = \left\{ \begin{pmatrix} A + iB & 0 \\ 0 & A - iB \end{pmatrix} \mid A^t = -A, B^t = B \in gl_n(\mathbb{R}) \right\}$$

$$u_{\mathbb{C}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} \mid A \in gl_n(\mathbb{C}) \right\}$$

$$S = \left\{ \begin{pmatrix} 0 & A+iB \\ A-iB & 0 \end{pmatrix} \mid A^t = A, B^t = B \in gl_n(\mathbb{R}) \right\}$$

$$S_{\mathbb{C}}^{+} = \left\{ \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \mid A^t = A, complex \right\}$$

$$S_{\mathbb{C}}^{-} = \left\{ \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \mid A^t = A, complex \right\}$$

We shall notice here that

$$S = \{\overline{X} = X^t \mid X \in S_{\mathbb{C}}\}$$

Later we will use this structure to identify $\mathfrak{p}$.

2. Let $e_{i,j}$ be the $2n \times 2n$ matrix with 1 on its (i,j)-th entry and 0 otherwise. Let

$$X_{i,-j} = e_{i,j} - e_{n+j,n+i}, \qquad X_{i,j} = e_{i,n+j} + e_{j,n+i}, \qquad X_{-i,-j} = e_{n+i,j} + e_{n+j,i}$$

3. Let $\mathcal{P}$ be the polynomial algebra of n variables $(z_1, z_2, \dots, z_n)$. Let $\mathcal{P}^{\leq m}$ be the polynomials of degree less or equal to m. There exists an action ω of $U(sp_{2n})$ on $\mathcal{P}$, often called Fock model.

$$\omega(X_{i,-j}) = z_i \partial z_j + \frac{1}{2}\delta_{i,j}$$
$$\omega(X_{i,j}) = -z_i z_j$$
$$\omega(X_{-i,-j}) = \partial z_i \partial z_j$$

This representation splits into two irreducible subrepresentation $\mathcal{P}_+$, $\mathcal{P}_-$, the spaces of even polynomials and odd polynomials.

4. Let (G, G') be an irreducible dual pair. There exists a Cartan involution θ of Sp, such that $\theta|_G$, $\theta|_{G'}$ are Cartan involutions of G and G'. This allows us to write

$$K = U \cap G, \qquad \mathfrak{p} = S \cap \mathfrak{g}, \qquad \mathfrak{k} = u \cap \mathfrak{g}$$
$$K' = U \cap G', \qquad \mathfrak{p}' = S \cap \mathfrak{g}', \qquad \mathfrak{k}' = u \cap \mathfrak{g}'$$

5. **Fact**: There exist reductive dual pairs (K, M'), (K', M) in Sp, furthermore

$$K \subseteq G \subseteq M, \qquad K' \subseteq G' \subseteq M'$$

For an exhaustive proof of this fact, see [Howe2],Ch 5. Let

$$m_{\mathbb{C}}^{\pm} = m_{\mathbb{C}} \cap S_{\mathbb{C}}^{\pm}, \qquad m_{\mathbb{C}}^{0} = m_{\mathbb{C}} \cap u_{\mathbb{C}}$$
$$m_{\mathbb{C}}^{\prime,\pm} = m_{\mathbb{C}}' \cap S_{\mathbb{C}}^{\pm}, \qquad m_{\mathbb{C}}^{\prime,0} = m_{\mathbb{C}}' \cap u_{\mathbb{C}}$$

We have

$$m_{\mathbb{C}}^{\prime,\pm} = (S_{\mathbb{C}}^{\pm})^{K}, \qquad m_{\mathbb{C}}^{\prime,0} = (u_{\mathbb{C}})^{K}$$
$$m_{\mathbb{C}}^{\pm} = (S_{\mathbb{C}}^{\pm})^{K'}, \qquad m_{\mathbb{C}}^{0} = (u_{\mathbb{C}})^{K'}$$

6. **Fact**: Let $M^0 = U^{K'}$, and $M'^{,0} = U^K$. Then $(M^0, M'^{,0})$ is a dual pair in Sp, and

$$K \subseteq M^0, \qquad K' \subseteq M'^{,0}$$

Furthermore,

$$m_{\mathbb{C}}^+ \oplus m_{\mathbb{C}}^- = \mathfrak{p}_{\mathbb{C}} \oplus m_{\mathbb{C}}^+ = \mathfrak{p}_{\mathcal{C}} \oplus m_{\mathbb{C}}^-$$
$$m_{\mathbb{C}}'^{,+} \oplus m_{\mathbb{C}}'^{,-} = \mathfrak{p}_{\mathbb{C}}' \oplus m_{\mathbb{C}}'^{,+} = \mathfrak{p}_{\mathcal{C}}' \oplus m_{\mathbb{C}}'^{,-}$$

This fact was also proved in [Howe2] Ch 5.

7.3 Structure of $\mathcal{P}$

For a compact group $K \subseteq Sp$, let $\mathcal{R}(\widetilde{K}, \mathcal{P})$ be the set of $\widetilde{K}$ types in $\mathcal{P}$. For a pair of commuting compact groups $(L, L') \subseteq Sp$, $\sigma \in \mathcal{R}(\widetilde{L}, \mathcal{P})$, $\sigma' \in \mathcal{R}(\widetilde{L'}, \mathcal{P})$, we use $\sigma \otimes \sigma'$ to denote the tensor product of σ and σ' regarded as a representation of $\widetilde{LL'}$. Notice here

$$1 \to \{(1,1), (\epsilon, \epsilon)\} \to \widetilde{L} \times \widetilde{L}' \to \widetilde{LL'} \to 1$$

is an exact sequence. And

$$\sigma(\epsilon) \otimes \sigma'(\epsilon) = (-1)(-1) = 1$$

Therefore $\sigma \otimes \sigma'$ is indeed a representation of $\widetilde{LL'}$. We use $\mathcal{P}_\sigma$ to denote the σ-isotypic subspace of $\mathcal{P}$. For arbitrary L-module V, we denote the σ-isotypic subspace by V_σ. The main reference here is [Howe2].

1. For any $\tau \in \mathcal{R}(\widetilde{M^0}, \mathcal{P})$, there exists a unique $\tau' \in \mathcal{R}(\widetilde{M'^{,0}}, \mathcal{P})$ such that

   $$\mathcal{P}_\tau \cong \tau \otimes \tau'$$

   We often denote τ' by $\omega(\tau)$. The map

   $$\omega : \mathcal{R}(\widetilde{M^0}, \mathcal{P}) \longrightarrow \mathcal{R}(\widetilde{M'^{,0}}, \mathcal{P})$$

   is a one to one correspondence. The inverse map is given by interchanging M^0 with $M'^{,0}$. In particular,

   $$\mathcal{P} = \oplus_{\tau \in \mathcal{R}(\tilde{M}^0, \mathcal{P})} \mathcal{P}_\tau = \oplus_{\tau \in \mathcal{R}(\tilde{M}^0, \mathcal{P})} \tau \otimes \omega(\tau) = \oplus_{\tau \in \mathcal{R}(\tilde{M}^0, \mathcal{P})} \mathcal{P}_{\tau, \omega(\tau)}$$

2. For $\sigma \in \mathcal{R}(\widetilde{K}, \mathcal{P})$, we have

   $$\mathcal{P}_\sigma = \sigma \otimes \omega(\sigma) \tag{7.26}$$

   where $\omega(\sigma)$ is an irreducible unitary Harish-Chandra module of $\tilde{M}'$. In other words, we have

   $$\mathcal{P} = \oplus_{\sigma \in \mathcal{R}(\tilde{K}, \mathcal{P})} \sigma \otimes \omega(\sigma)$$

3. Recall that $(S^{-})^K = m'^{,-}$. Let

$$\mathcal{H}(K) = \{p \in \mathcal{P} \mid \omega(x)p = 0 \qquad \forall x \ \in m'^{,-}\}$$

Also notice that the K-invariant algebraic differential operators on $\mathcal{P}$ with constant coefficient are generated by $m'^{,-}$, and under the oscillator representation $m'^{,-}$ is all the 2nd order K-invariant differential operators with constant coefficient(see [howe3],Theorem 2). Thus we may call $\mathcal{H}(K)$ the K-harmonic functions.

Theorem 7.3 (Howe) *For $\sigma \in \hat{\widetilde{K}}$, $\mathcal{P}_\sigma = U(\mathfrak{g}')\mathcal{H}(K)_\sigma$. $\mathcal{H}(K)_\sigma$ consists of homogenous polynomials of lowest degree (denoted by $\deg(\sigma)$) in $\mathcal{P}_\sigma$. In addition,*

$$\exists\ \tau' \in \mathcal{R}(\widetilde{M^{0,\prime}}, \mathcal{P}), \qquad \mathcal{H}(K)_\sigma \cong \sigma \otimes \tau' \tag{7.27}$$

For a proof, see [Howe2] Prop 3.1.

4. **Theorem 7.4 (Howe)** *There exists a unique $\sigma' \in \mathcal{R}(\widetilde{K}', \mathcal{P})$, such that*

$$\mathcal{H}(K)_\sigma \cap \mathcal{H}(K') \cong \sigma \otimes \sigma'$$

We denote such a space by $\mathcal{H}_{\sigma,\sigma'}$. We have

$$(\sigma \in \mathcal{R}(\widetilde{K}, \mathcal{P}), \sigma' \in \mathcal{R}(\widetilde{K}', \mathcal{P}), \tau \in \mathcal{R}(\widetilde{M^0}, \mathcal{P}), \tau' \in \mathcal{R}(\widetilde{M'^{,0}}, \mathcal{P}))$$

mutually determine one another under the relationship we have defined in Equations 7.26 and 7.27.

See [Howe2] Ch 3.

5. **Theorem 7.5 (Howe)**

$$\mathcal{P} = U(\mathfrak{g})U(\mathfrak{g}')(\mathcal{H}(K) \cap \mathcal{H}(K'))$$

Let $\mathcal{N}$ be an arbitrary $(\mathfrak{g}, \mathfrak{g}', \widetilde{KK'})$ submodule of $\mathcal{P}$. Suppose $\mathcal{P}^{\leq d-1} \subseteq \mathcal{N}$, but $\mathcal{P}^{\leq d} \nsubseteq \mathcal{N}$. Then

$$d = \min\{\deg(\sigma) \mid \sigma \in \mathcal{R}(\widetilde{K}, \mathcal{P}/\mathcal{N})\} = \min\{\deg(\sigma') \mid \sigma' \in \mathcal{R}(\widetilde{K}', \mathcal{P}/\mathcal{N})\}$$

If σ is of minimal degree, then

$$\mathcal{H}_{\sigma,\sigma'} \cong \sigma \otimes \sigma' \in \mathcal{R}(\widetilde{KK'}, \mathcal{P}/\mathcal{N})$$

and σ' is of minimal degree as well. In addition, no other irreducible representation of $\widetilde{K}'$ occurring in $\mathcal{H}(K)_\sigma$ occurs in $\mathcal{P}/\mathcal{N}$.

See [Howe2] Ch 4 for proof.

7.4 Howe's Correspondence

Definition 7.2 *Let $\mathcal{R}(\mathfrak{g}, \widetilde{K}, \mathcal{P})$ be the set of isomorphism classes of irreducible $(\mathfrak{g}, \widetilde{K})$-modules which can be represented as $\mathcal{P}/\mathcal{N}$, for some $(\mathfrak{g}, \widetilde{K})$ submodule $\mathcal{N}$. For $\pi \in \mathcal{R}(\mathfrak{g}, \widetilde{K}, \mathcal{P})$, let $\mathcal{N}_\pi$ be the intersection of all such $\mathcal{N}$ satisfying $\mathcal{P}/\mathcal{N} \cong V_\pi$. Then $\mathcal{P}/\mathcal{N}_\pi$ is a $(\mathfrak{g} \times \mathfrak{g}', \widetilde{KK'})$-module. We write*

$$\mathcal{P}/\mathcal{N}_\pi \cong \pi \otimes \omega_0(\pi)$$

where $\omega_0(\pi)$ is a $(\mathfrak{g}', \widetilde{K}')$-module.

Notice that this definition is consistent with our earlier definition of ω for the dual pairs $(M^0, M'^{,0})$ and (K, M') where $\omega = \omega_0$.

Theorem 7.6 (Howe) *The $(\mathfrak{g}', \widetilde{K}')$-module $\omega_0(\pi)$ is finitely generated, admissible, quasisimple. It has a composition series of finite length. $\omega_0(\pi)$ has a unique irreducible quotient $\omega(\pi)$. The correspondence*

$$\omega : \pi \to \pi'$$

defines a bijection from $\mathcal{R}(\mathfrak{g}, \widetilde{K}, \mathcal{P})$ to $\mathcal{R}(\mathfrak{g}', \widetilde{K}', \mathcal{P})$.

A proof of this theorem can be found in [Howe2]. ω is often called Howe's correspondence, or dual pair correspondence. Sometimes we will write $\mathcal{R}(\mathfrak{g}, \widetilde{K}, \mathcal{P})$ as $\mathcal{R}(\widetilde{G}, \omega)$.

Theorem 7.7 *Let $\sigma \in \mathcal{R}(\widetilde{K}, \mathcal{P}/\mathcal{N}_\pi)$ be of minimal degree in $\mathcal{P}$. Then $\mathcal{H}_{\sigma,\sigma'}$ generate $\mathcal{P}/\mathcal{N}_\pi$ as $(\mathfrak{g}, \mathfrak{g}')$ module, and $\sigma' \in \omega(\pi)$.*

Proof: The first statement was proved in [Howe2]. Let $\mathcal{N} \supseteq \mathcal{N}_\pi$ be the unique $(\mathfrak{g}, \mathfrak{g}')$ module such that

$$\mathcal{P}/\mathcal{N} = \pi \otimes \omega(\pi)$$

To show that $\sigma' \in \omega(\pi)$, it suffices to show that

$$\mathcal{H}_{\sigma,\sigma'} \cap \mathcal{N} = \{0\}$$

Otherwise, since $\mathcal{H}_{\sigma,\sigma,}$ is an irreducible $\widetilde{K} \times \widetilde{K}'$ module,

$$\mathcal{H}_{\sigma,\sigma'} \subseteq \mathcal{N}$$

This implies that $\mathcal{P} \subseteq \mathcal{N}$, which contradicts our assumption. Q.E.D.

8 Matrix Coefficient:Convergence

Let (G, G') be a reductive dual pair in Sp. Let ω be the oscillator representation of $\widetilde{Sp}$. Let $(\pi, \mathcal{H})$ be an irreducible representation of $\widetilde{G}$. Let $\mathcal{P}$ be the Harish-Chandra module of ω and V be the Harish-Chandra module of π. We define a bilinear form $(,)_\pi$ on $\mathcal{P} \otimes V \otimes \mathcal{P}^c \otimes V^c$ as follows.

$$(\phi \otimes u, \psi \otimes v)_\pi = \int_{\widetilde{G}} (\omega(g)\phi, \psi)(\pi(g)u, v) dg \qquad (\phi, \psi \in \mathcal{P}, u, v \in V)$$

We will examine the convergence of this integral. Roughly speaking, π is said to be in the semistable range of (G, G') if this integral converges for every (u, v, ϕ, ψ). We will only restrict our attention to real reductive groups with compact center.

8.1 Structure Theory

Let G be a real reductive group with compact center. Let KAN be its Iwasawa decomposition and $\mathfrak{a}$ the Lie algebra of A.

1. $G = K \exp \mathfrak{p}$—the Cartan decomposition, θ—the Cartan involution;
2. $(,)$—invariant real bilinear form on $\mathfrak{g}$, positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$;
3. Σ_r^+—Positive restricted roots with respect to N;
4. Δ—Simple roots;
5. $\mathfrak{a}^+ = \{\alpha(a) > 0 \mid \alpha \in \Delta\}$;
6. $\mathfrak{a}_{\mathbb{C}}^+ = \{Re\,(\alpha(a)) > 0 \mid \alpha \in \Delta\}$;
7. $A^+ = \exp(\mathfrak{a}^+)$;
8. $\mathfrak{g} = \oplus \mathfrak{g}_\lambda$; $\mathfrak{g}_\lambda$ root spaces;
9. $\rho = \frac{1}{2} \sum_{\lambda \in \Sigma_r^+} \dim(\mathfrak{g}_\lambda)\lambda$;
10. $\alpha \geq \beta$ if $\alpha - \beta = \Sigma_{\alpha_i \in \Delta} n_i \alpha_i$ for n_i non-negative integers or one of n_i is not an integer;
11. $\alpha \succeq \beta$ if $\alpha = \beta + \Sigma_{\alpha_i \in \Delta} c_i \alpha_i$ for c_i nonnegative; $\alpha \succ \beta$ if at least one $c_i > 0$;
12. $M = Z_K(\mathfrak{a})$, the centralizer of $\mathfrak{a}$ in K;
13. $\mathfrak{b}$ is the maximal torus in $\mathfrak{m}$;
14. $W(\mathfrak{a}, \mathfrak{g})$—the Weyl group;

Theorem 8.1 (KAK Decomposition) *Every $g \in G$ has a decomposition of the form $k_1 a k_2$ with $k_1, k_2 \in K, a \in A$. The element a is often written as $a(g)$ or $\exp(H(g))$, and it is unique up to conjugation by Weyl group $W(\mathfrak{a}, \mathfrak{g})$. When $a \in A^+$, g is called regular. Under the KAK decomposition, the Haar measure*

$$dg = \prod_{\lambda \in \Sigma_r^+} (\exp(\lambda(H)) - \exp(-\lambda(H)))^{\dim \mathfrak{g}_\lambda} dk_1 dH dk_2$$

In short we will write

$$dg = \Delta(H) dk_1 dH dk_2$$

Furthermore,

$$\Delta(H) \neq 0 \qquad (H \in \mathfrak{a}^+)$$

The Haar measure of the singular(non-regular) elements is zero.

The KAK decomposition can be computed as follows. According to Cartan decompostion, for any $g \in G$, we may decompose it uniquely as $k(g)\exp(p(g))$. And $p(g) \in \mathfrak{p}$ can be written as $Ad(k)H(g)$, where $H(g) \in \mathfrak{a}$. Thus, we obtain the KAK decomposition of g:

$$g = k(g) k \exp(H(g)) k^{-1}$$

One advantage of this definition is that we can define a norm on $g \in G$, namely

$$\|g\| = (p(g), p(g)) = (H(g), H(g))$$

One immediate result is that

Theorem 8.2 *The set $\{g \in G, \|g\| \leq r\}$ is compact.*

Now we may speak of $g_n \to \infty$ if $\|g_n\| \to \infty$. For $H \in \overline{\mathfrak{a}^+}$, it can be proved that

$$\Delta(H) \leq c \exp(2\rho(H))$$

If g is not regular, i.e., $a(g) \notin A^+$, we say g is on the wall.

Theorem 8.3 *Under NAK decomposition, the Haar measure is given by*

$$dg = \exp(2\rho(\log(a)) dn da dk \qquad (g = nak)$$

where da, dn, dk are Haar measures of A,N,K respectively. The measures da and dn can be identified by the exponential mappings with Lebesgue measures on $\mathfrak{a}$ and $\mathfrak{n}$.

8.2 Matrix Coefficient

We will always assume that the Hilbert norm on $\mathcal{H}_\pi$ is K-invariant.

Definition 8.1 *Let $(\pi, \mathcal{H}_\pi)$ be a representation of a real reductive group G. For $(u,v) \in \mathcal{H}_\pi$, we define*

$$\pi_{u,v}(g) = (\pi(g)u, v)$$

$\pi_{u,v}$ is called a matrix coefficient. We will assume u, v lie in the Harish-Chandra module V_π from now on.

Now let (τ_1, U),(τ_2, V) be the finite dimensional K-modules generated by u,v respectively. We may define $\pi_{U,V}(g) \in Hom_{\mathbb{C}}(U, V)$, a matrix-valued matrix coefficient by

$$\pi_{U,V}(g)(u) = P_V \pi(g) u \qquad (u \in U)$$

where P_V is the projector (of Hilbert spaces) onto $V \subseteq V_\pi$. Since P_V commutes with $\pi(K)$, we have

$$\begin{aligned} \pi_{U,V}(k_1 a k_2) =& P_V \pi(k_1)\pi(a)\pi(k_2) \\ =& \pi(k_1) P_V \pi(a) \pi(k_2) \\ =& \tau_1(k_1) P_V \pi(a) \tau_2(k_2) \end{aligned} \tag{8.28}$$

We will write $\tau = ((\tau_1, U), (\tau_2, V))$ for the pair of representations of K, and $\pi_\tau = \pi_{U,V} : G \to Hom_{\mathbb{C}}(U, V)$. Let $\mathcal{H}_\pi$ be an irreducible admissible representation with infinitesimal character λ. Let χ_λ be the corresponding character of $U(\mathfrak{g})^{\mathfrak{g}}$. Then π_τ satisfies a class of differential equations defined by

$$\pi(z)\pi_\tau(g) = \chi(\lambda)(z)\pi_\tau(g), \qquad z \in U(\mathfrak{g})^{\mathfrak{g}}, g \in G$$

Suppose $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$. For $H \in \mathfrak{a}$, we define

$$\alpha(H) = (\alpha_1(H), \alpha_2(H), \dots, \alpha_r(H))$$

We have the following theorem:

Theorem 8.4 ([Knapp1] Theorem 8.32) *For a pair (τ_1, τ_2), matrix coefficient π_τ has the following asymptotic expansion*

$$\pi_\tau(\exp(H)) = \sum F_{v-\rho}(\exp(H)) \qquad (H \in \mathfrak{a}^+)$$

$$F_{v-\rho}(\exp(H)) = \sum_{q \in \mathbb{N}^r} c_{v,q} \alpha(H)^q \exp((v-\rho)H), \qquad (c_{v,q} \in Hom\,(U, V))$$

Here the summation is over finite number subset of $\mathbb{N}^r$. This expansion converges on $\mathfrak{a}^+$, converges absolutely on any compact subset of $\mathfrak{a}^+$.

If $F_{v-\rho}(\exp(H)) \neq 0$ for some $H \in \mathfrak{a}^+$, then $v - \rho$ is said to be an exponent of π_τ. A leading exponent $v - \rho$ is an exponent of π_τ, such that $v \geq v'$ for any exponent $v' - \rho$ of π_τ. The corresponding term $F_{v-\rho}$ in the asymptotic expansion will be called a leading term of π_τ.

Theorem 8.5 ([Knapp1] Theorem 8.33) *Suppose that π is an irreducible admissible representation of G with infinitesimal character λ, τ a pair of finite dimensional subrepresentation of V_π. If $v - \rho$ is a leading exponent of π_τ, then*

$$v = w.\lambda|_{\mathfrak{a}}$$

for some $w \in W(\mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{b}_{\mathbb{C}}; \mathfrak{g}_{\mathbb{C}})$

We say that $v - \rho$ is a leading exponent of π if $v \geq v'$ for every exponent $v' - \rho$ of any π_τ.

8.3 Asymptotic Behaviors

Theorem 8.6 ([Knapp1] Theorem 8.47) *Suppose π is an admissible representation of G. Then the following conditions on a weight $v_0 \in \mathfrak{a}^*$ are equivalent:*

1. *We have Re $(v) \preceq v_0$ for every leading exponent $v - \rho$ of π;*
2. *There is an integer $q \geq 0$ such that for every $u, v \in V_\pi$, there exists a constant C such that*
$$\pi_{u,v}(\exp(H)) \leq C \exp((v_0 - \rho)H)(1 + \|H\|)^q, \qquad (\forall\ H \in \overline{\mathfrak{a}^+})$$
3. *There is an integer $q \geq 0$ such that for every $u, v \in V_\pi$, there exists a constant C such that*
$$\pi_{u,v}(g) \leq C \exp((v_0 - \rho)(H(g)))(1 + \|H(g)\|)^q, \qquad (\forall\ g \in G)$$

From now on, C or c will be used as symbolic constant. Let L and R represent left and right regular representation of G. According to [Knapp1] 8.47,

Theorem 8.7 *Suppose π is an admissible representation of G and v_0 satisfies one of the conditions in Theorem 8.6. Let $x \in U(\mathfrak{g})$. Then the estimates of Theorem 8.6 hold for every $L(x)\pi_{u,v}$ and $R(x)\pi_{u,v}$.*

Now let $f(g)$ be a (smooth) function on G. We would like to give a necessary condition on f such that

$$\forall\ \ x \in \mathfrak{g}, \qquad \int_G L(x)f(g)dg = 0$$

Theorem 8.8 *For $x \in [\mathfrak{g}, \mathfrak{g}]$, suppose that f and $L(x)f$ satisfy the property that*

$$\|f(g)\| \leq c\exp(v_0(H(g)))(1 + \|H\|)^q$$

$$\|L(x)f(g)\| \leq c(x)\exp(v_0(H(g)))(1 + \|H\|)^q \qquad \forall\ x \in [\mathfrak{g}, \mathfrak{g}]$$

$c(x), c, q \geq 0$ *are constants and*

$$v_o + 2\rho = \sum_{\alpha_i \in \Delta} c_i \alpha_i, \qquad (c_i < 0)$$

Then

$$\int_G L(x) f(g) dg = 0$$

Proof:

1. First we use KAK decomposition to show that $L(x)f(g)$ is integrable with respect to Haar measure of G. From our assumption, we have

$$\begin{aligned} \int_G L(x) f(g) dg &\leq c \int_{KA^+K} \exp(v_o(H))(1 + \|g\|)^q \Delta(H) dK_1 dH dk_2 \\ &\leq c \int_{\mathfrak{a}^+} \exp((v_0 + 2\rho)H)(1 + \|H\|)^q dH \\ &\leq c \int_{\alpha_1(H)<0,\dots,\alpha_r(H)<0} \exp(\sum_{\alpha_i \in \Delta} c_i \alpha_i(H))(1 + \|H\|)^q dH \end{aligned} \tag{8.29}$$

converges absolutely

2. We claim that there exists a $c_0 < 0$ such that

$$\forall\ H \in \mathfrak{a}^+ \qquad c_0 \|H\|^{\frac{1}{2}} \geq \sum_{\alpha_i \in \Delta} c_i \alpha_i(H); \tag{8.30}$$

Notice that

$$\sum_{\alpha_i \in \Delta} c_i \alpha_i(H) = \text{nonzero constant}$$

$$\alpha_i(H) \geq 0 \qquad (\alpha_i \in \Delta)$$

define a convex compact polytope, and this polytope does not contain 0. Thus we can choose

$$c_0 = \max \{ \frac{\sum_{\alpha_i \in \Delta} c_i \alpha_i(H)}{\|H\|^{\frac{1}{2}}} \}$$

in this polytope. Since the numerator and denominator are homogenous of degree 1, this c_0 satisfies equation 8.30.

3. Now let $x_1 \in \mathfrak{n}$. Since the curve $\exp(t_1 x_1)$ is always closed, $\exp(t_1 x_1) g$ is closed for every $g \in G$. In addition, $\exp(t_1 x_1) g$ is homeomorphic to $\mathbb{R}^+$. On the other hand, for $r > 0$, the set

$$\begin{aligned} C_r = \{ g \in G : \|f(g)\| \geq r \} &\subseteq \{ g \in G : c \exp((v_0 + 2\rho) H(g))(1 + \|g\|)^q \geq r \} \\ &\subseteq \{ g \in G : c \exp(c_0 \|g\|)(1 + \|g\|)^q \geq r \} \\ &\subseteq \{ g \in G : \|g\| \leq c_r \} \end{aligned} \tag{8.31}$$

is compact.

Therefore $C_r \cap \{\exp(t_1x_1)g\} = \{\|\exp(t_1x_1)g\| \geq r\}$ is compact. Hence

$$f(exp(\sum_i t_1x_1)ak)|_{-\infty}^{+\infty} = 0$$

4. We will use NAK decomposition to compute $\int_G L(x_1)f(g)dg$. Notice that there exists a parametrization

$$n = \exp(t_1x_1)\exp(t_2x_2)\cdots\exp(t_mx_m)$$

With

$$\mathfrak{n} = \oplus\mathbb{R}x_i, \qquad (t_1, t_2, \dots, t_m) \in \mathbb{R}^m$$

such that

$$dn = dx_1dx_2\dots dx_m$$

Thus

$$\begin{aligned}
&\int_G L(x_1)f(nak)\ exp(2\rho(loga))dndadk \\
=&\int \frac{d}{dt_1}f(\exp(t_1x_1)\dots\exp(t_mx_m)aK)\exp(2\rho(loga))dt_1\dots dt_mdadk \\
=&\int dk\int da\int dt_2dt_3\dots dt_m\int\frac{d}{dt_1}f(\exp(t_1x_1)\dots\exp(t_mx_m)ak)dt_1(\text{Fubini's theorem}) \\
=&\int dk\int da\int f(\exp(\sum_i t_ix_i)ak)|_{-\infty}^{+\infty}dt_2\dots dt_m \\
=&0
\end{aligned} \tag{8.32}$$

5. We can change our choice on $\mathfrak{n}$ by a conjugation of $k \in K$. Thus for every $k \in K$, $x_1 \in \mathfrak{n}$,

$$\int_G L(Ad(k)x_1)f(g)dg = 0$$

Since $[\mathfrak{g}, \mathfrak{g}]$ is the linear span of $\{Ad(k)x_1\|x_1 \in \mathfrak{n}, k \in K\}$, we have for every $x \in [\mathfrak{g}, \mathfrak{g}]$,

$$\int_G L(x)f(g)dg = 0$$

Q.E.D.

This is the hard part. In fact, we have shown that

Theorem 8.9 *Suppose that that $L(x)f$ is integrable with respect to dg for every $x \in [\mathfrak{g}, \mathfrak{g}]$, and f is bounded by a positive function $I(g)$, such that, for arbitrary $r > 0$, $\{g \in G : I(g) \geq r\}$ is compact. Then*

$$\forall\ \ x \in [\mathfrak{g}, \mathfrak{g}] \qquad \int_G L(x)f(g) = 0$$

On the other hand, recall that for each K-finite smooth function f on K, according to Peter-Weyl theorem, f can be written as

$$f = \sum_{\sigma \in \hat{K}} f_\sigma$$

where $f_\sigma \in C(K)_\sigma$. Here K acts on $C(K)$ by left translation, $C(K)_\sigma$ is the $\sigma-$isotypic subspace of $C(K)$. If σ is not trivial on K_0, the identity component of K, we have

$$\int_K L(x) f_\sigma(k) dk = 0 \qquad x \in \mathfrak{k}$$

If σ is trivial on K_0, we will still have

$$\int_K L(x) f_\sigma(k) dk = 0 \qquad x \in \mathfrak{k}$$

Thus we have the following theorem

Theorem 8.10 *Suppose $x \in \mathfrak{k}$, $f \in C(G)$ is K-finite, and $L(x)f$ is integrable with respect to the Haar measure. Then*

$$\int L(x) f(g) dg = 0$$

Proof: Notice that every integral of $L(x)f$ over Kg for a fixed $g \in G$ vanishes. According to Fubini's theorem,

$$\begin{aligned} \int L(x) f(g) dg &= \int_{K\backslash G} \int_K L(x) f(kg) dk d[g] \\ &= 0 \end{aligned} \tag{8.33}$$

Here $d[g]$ is the right G-invariant measure on $K\backslash G$. Q.E.D.

We have proved that

Theorem 8.11 *Let G be a real reductive group with compact center. For every $x \in \mathfrak{g}$, suppose that f and $L(x)f$ satisfy*

$$\|f(g)\| \leq c \exp(v_0(H(g)))(1 + \|H\|)^q$$

$$\|L(x) f(g)\| \leq c(x) \exp(v_0(H(g)))(1 + \|H\|)^q$$

for some constants $c(x), c, q \geq 0$ and

$$v_o + 2\rho = \sum_{\alpha_i \in \Delta} c_i \alpha_i \qquad (c_i < 0)$$

Then,

$$\int_G L(x) f(g) dg = 0$$

8.4 Matrix coefficient of the Oscillator Representation

Definition 8.2 (Schrödinger Model) *There exists a unitary representation ω of $\widetilde{Sp_{2n}}$ on $L^2(\mathbb{R}^n)$ such that*

$$\omega(X_{k,-j}) = x_k\partial x_j + \frac{1}{2}\delta_k^j;$$

$$\omega(X_{k,j}) = ix_kx_j \ (k \neq j) \qquad \omega(X_{k,k}) = \frac{i}{2}x_k^2$$

$$\omega(X_{-k,-j}) = i\partial x_k\partial x_j \ (k \neq j) \qquad \omega(X_{-k,-k}) = \frac{i}{2}\partial^2 x_k$$

Let $GL(n)$ be the subgroup of $Sp_{2n}(\mathbb{R})$ of the following form

$$\begin{pmatrix} A & 0 \\ 0 & (A^{-1})^t \end{pmatrix}$$

Then

$$\widetilde{GL(n)} = \{(g,\xi) \mid g \in GL(n), \xi^2 = \det g\}$$
$$\omega(g,\xi)(f)(x) = \xi f(g^t x)$$

See [Wallach] Ch 5.3 and 5.5 for more details. The isometry between the Schrödinger Model and the Segal-Bargmann model was discovered by Bargmann in [Bargmann].

Let $K = Sp_{2n}(\mathbb{R}) \cap O_{2n}(\mathbb{R})$ be the maximal compact subgroup. Then $\widetilde{K}$ the maximal subgroup of $\widetilde{Sp}$ is also connected. Thus the $\widetilde{K}$-finite vectors are those $\mathfrak{k}$-finite vectors.

Theorem 8.12 *Let $\mu(x) = \exp(-\frac{1}{2}\|x\|^2)$. Then the Harish Chandra module in the Schrödinger model is given by $\mu(x)\mathcal{P}$, where $\mathcal{P}$ is the polynomial algebra of n real variables.*

Proof: We want to show that $\{X_{i,-j} - X_{j,-i}\}_1^n, \{X_{i,j} - X_{-i,-j}\}_1^n$ act on $\mathcal{P}^{\leq m}$.
For $p(x) \in \mathcal{P}$, and $i \neq j$, we have

$$\begin{aligned}(x_ix_j - \partial x_i\partial x_j)(\mu(x)p(x)) =& x_ix_j\mu(x)p(x) - x_ix_j\mu(x)p(x) + x_i\mu(x)\partial_jp(x)) \\ &+ x_j\mu(x)\partial_ip(x) - \mu(x)\partial_i\partial_jp(x) \\ =&(x_i\partial_jp(x) + x_j\partial_ip(x) - \partial_i\partial_jp(x))\mu(x)\end{aligned} \tag{8.34}$$

Thus $\mathcal{P}^{\leq m}$ is preserved by the action of $\{x_ix_j - \partial x_i\partial x_j\}_{i\neq j}$.
For $i = j$, we have,

$$\begin{aligned}(x_i^2 - \partial^2 x_i)(\mu(x)p(x)) =& x_i^2\mu(x)p(x) + \mu(x)p(x) - x_i^2\mu(x)p(x) + x_i\mu(x)\partial_ip(x) \\ &- \mu(x)\partial_i^2p(x) + x_i\mu(x)\partial_ip(x) \\ =&(2x_i\partial_ip(x) + p(x) - \partial_i^2p(x))\mu(x)\end{aligned} \tag{8.35}$$

Again, $\mathcal{P}^{\leq m}$ is preserved. Therefore, $\mathcal{P}^{\leq m}$ is preserved by $\{X_{i,j} - X_{-i,-j}\}$. Notice that $X_{i,-j} - X_{j,-i}$ acts homogeneously on $\mathcal{P}$. Thus $\mathcal{P}^{\leq m}$ is preserved by $\mathfrak{k}$. It is well-known that the Harish-Chandra module of the oscillator representation can be decomposed as the direct sum of irreducible highest $\widetilde{K}$-module with highest weights $(n+\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, where n is a nonnegative integer. From the equations we acquire, it is easy to see that the highest weight of $\mathcal{P}^{\leq m}/\mathcal{P}^{<m}$ is exactly $(m+\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. Therefore $\mu(x)\mathcal{P}$ is the Harish-Chandra module of the Schrödinger model. Q.E.D.

Even though the Schrödinger model has certain properties similar to the Fock model, their Harish-Chandra modules are different. For example, it is difficult to write down exactly the $\widetilde{K}$-types in Schrödinger model, since the $\widetilde{K}$ action does not preserve homogeneity. Let g be an element of G. For simplicity, we use $\widetilde{g}$ to denote any preimage of g under the metaplectic covering when it causes no confusion. Now we can compute the matrix coefficients of the Schrödinger model. We follow the multi-index convention.

Theorem 8.13 *Let $a = diag(a_1, a_2, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}) \in A^+ \subseteq Sp_{2n}$, i.e.,*

$$a_1 > a_2 > \ldots > a_n > 1$$

and let

$$(a, \xi) \in \widetilde{Sp_{2n}} \qquad (\xi^2 = \prod_1^n a_i)$$

Then $\widetilde{A} \cong (A, \pm 1)$. We write $sgn(a, \xi) = sgn(\xi)$. Then

$$\begin{aligned} \omega_{\alpha,\beta}(\widetilde{a}) &= (\omega(\widetilde{a})x^{\alpha}\mu(x), x^{\beta}\mu(x)) \\ &= sgn(\widetilde{a})c_{\alpha,\beta}\prod_{i=1}^{n} a_i^{-\beta_i-1/2}(1+a_i^{-2})^{-(\alpha_i+\beta_i+1)/2} \\ &= sgn(\widetilde{a})c_{\alpha,\beta}\prod_{i=1}^{n} a_i^{\alpha_i+1/2}(1+a_i^{2})^{-(\alpha_i+\beta_i+1)/2} \end{aligned} \tag{8.36}$$

These formulae yield the asymptotic expansions for $\{(a_1, \ldots, a_n) \mid a_i > 1\}$ and $\{(a_1, \ldots, a_n) \mid a_i < 1\}$ respectively. Both domains contain $n!$ Weyl chambers of A. Moreover, the leading exponent of the oscillator representation is

$$(-\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2})$$

and the infinitesimal character is given by

$$(n-\frac{1}{2}, n-\frac{3}{2}, \ldots, \frac{1}{2})$$

Proof: Let $C_\alpha = \int x^\alpha \mu(x)dx$. Then

$$\begin{aligned}\omega_{\alpha,\beta}(\widetilde{a}) =& sgn(\xi)(\prod_{i=1}^{n} a_i)^{\frac{1}{2}}(a^\alpha x^\alpha \exp(-\frac{1}{2}\sum a_i^2 x_i^2), x^\beta \mu(x)) \\ =& sgn(\xi)(\prod_{i=1}^{n} a_i)^{\frac{1}{2}} a^\alpha \int x^{\alpha+\beta} \exp(-\frac{1}{2}\sum (a_i^2+1)x_i^2)dx \\ =& sgn(\xi)(\prod_{i=1}^{n}(a_i))^{\frac{1}{2}} a^\alpha \prod_{i=1}^{n}(a_i^2+1)^{-\frac{\alpha_i+\beta_i+1}{2}} \int x^{\alpha+\beta}\mu(x)dx \\ =& sgn(\xi)c_{\alpha+\beta}\prod_{i=1}^{n}(a_i)^{\alpha_i+\frac{1}{2}}(a_i^2+1)^{-\frac{\alpha_i+\beta_i+1}{2}} \\ =& sgn(\xi)c_{\alpha+\beta}\prod_{i=1}^{n}(a_i)^{-\beta_i-\frac{1}{2}}(a_i^{-2}+1)^{-\frac{\alpha_i+\beta_i+1}{2}}\end{aligned} \tag{8.37}$$

Thus the unique leading exponent of the oscillator representation is

$$v = (-\frac{1}{2}, -\frac{1}{2}, \dots, -\frac{1}{2})$$

Therefore the infinitesimal character is given by

$$v + \rho = (n - \frac{1}{2}, n - \frac{3}{2}, \dots, \frac{1}{2})$$

Q.E.D.

Finally, we will give a theorem on the growth condition of the matrix coefficients of the Schrödinger model.

Theorem 8.14 *Let $g = k_1 \exp(H(g))k_2$ with $H(g) \in \overline{\mathfrak{a}^+}$ be the KAK decomposition of $\widetilde{Sp_{2n}}(\mathbb{R})$. Then for every u, v in the Harish-Chandra module of the Schödinger model, there exists a constant c such that*

$$|\omega_{u,v}(g)| \leq c \exp(-\frac{1}{2}\sum_{i=1}^{n} H_i(g)) \qquad (H(g) = diag(H_1(g), \dots, H_n(g), -H_1(g), \dots - H_n(g)))$$

The same estimates hold for every $L(x)\omega_{u,v}(g)$ and $R(x)\omega_{u,v}(g)$, where $x \in \mathfrak{g}$ and L and R represent the left and right regular actions.

Proof: We only sketch a proof here. For an arbitrary $w \in \mu(x)\mathcal{P}$, since $\{x^\alpha \mu(x)\}_{\alpha \in \mathbb{N}^n}$ is a basis for $\mu(x)\mathcal{P}$, we write

$$w = \sum w_\alpha x^\alpha \mu(x)$$

Notice that

$$\omega_{u,v}(g) = (\omega(g)u, v) = (\omega(\exp(H(g)))\omega(k_2)u, \omega(k_1^{-1})v)$$

Suppose the finite dimensional K-subspace spanned by u and v is contained in $\mu(x)\mathcal{P}^{\leq m}$. According to the previous theorem, for every pair $(\omega(k_2)u, \omega(k_1^{-1})v)$, there exists a constant $C(k_2, k_1)$ such that

$$|(\omega(\exp(H(g)))\omega(k_2)u, \omega(k_1^{-1})v)| \leq C(k_2, k_1)\exp(-\frac{1}{2}\sum_{i=1}^{n} H_i(g))$$

Here we may choose

$$C(k_2, k_1) = \sum_{\{\alpha,\beta|\sum \alpha_i \leq m, \sum \beta_j \leq m\}} |(\omega(k_2)u)_\alpha||(\omega(k_1^{-1})v)_\beta|C_{\alpha,\beta}$$

Now since $(k_2, k_1) \in K \times K$ is compact, and $C(k_2, k_1)$ is a continuous function, $c = \max\{C(k_2, k_1) \mid k_2 \in K, k_1 \in K\}$ exists. Then we have

$$|(\omega(\exp(H(g)))\omega(k_2)u, \omega(k_1^{-1})v)| \leq c\exp(-\frac{1}{2}\sum_{i=1}^{n} H_i(g))$$

The first statement is proved. Since ω is unitary,

$$L(x)(\omega(g)u, v) = (\omega(x)\omega(g)u, v) = -(\omega(g)u, \omega(x)v) = -\omega_{u,\omega(x)v}(g)$$

$$R(x)(\omega(g)u, v) = (\omega(g)\omega(x)u, v) = \omega_{\omega(x)u,v}(g)$$

Then the first statement implies the second statement. Q.E.D.

For oscillator representation, the growth condition estimate in this theorem is in fact stronger that the growth condition in Theorem 8.6.

8.5 Convergence

We will study the restriction of the matrix coefficients of the oscillator representation to the dual pair (G, G'). For a representation π of $\widetilde{G}$, for $\phi, \psi \in \mathcal{P}$, $u, v \in V_\pi$, we may formally define a bilinear form on $(\mathcal{P} \otimes V_\pi, \mathcal{P}^c \otimes V_\pi^c)$ as follows

$$(\psi \otimes v, \phi \otimes u)_\pi = \int_{\widetilde{G}} (\omega(g)\psi, \phi)(\pi(g)v, u)dg \tag{8.38}$$

Now we want to study the convergence of this integral.

In general, let V_ω and V_π be two Harish-Chandra modules of an arbitrary reductive group G. Let $\{x^\alpha\}$ be a fixed orthonormal basis of V_ω and $\{v_i\}$ be a fixed orthonormal basis of V_π.

Observe that

$$
\begin{aligned}
&(\psi \otimes v, \phi \otimes u) \\
&= \int_G (\omega(g)\psi, \phi)(\pi(g)v, u)dg \\
&= \int_{K \times \mathfrak{a}^+ \times K} (\omega(k_1)\omega(\exp(H))\omega(k_2)\psi, \phi)(\pi(k_1)\pi(\exp(H))\pi(k_2)v, u)\Delta(H)dk_1 dH dk_2 \\
&= \int_{\mathfrak{a}^+} \Delta(H)dH \int_{K \times K} (\omega(\exp(H))\omega(k_2)\psi, \omega(k_1^{-1})\phi)(\pi(\exp(H))\pi(k_2)v, \pi(k_1^{-1})u)dk_1 dk_2 \\
&= \int_{\mathfrak{a}^+} \Delta(H)dH \sum_{\alpha,\beta,i,j} (\omega(\exp(H))x^\alpha, x^\beta)(\pi(\exp(H))v_i, v_j)H(\phi, u; \beta, j)H(\psi, v; \alpha, i)
\end{aligned}
\tag{8.39}
$$

where

$$
\sum_{\beta,j} H(\phi, u; \beta, j)x^\beta \otimes v_j = \int_K \omega(k)\phi \otimes \pi(k)u dk
$$

$$
\sum_{\alpha,i} H(\psi, v; \alpha, i)x^\alpha \otimes v_i = \int_K \omega(k)\psi \otimes \pi(k)v dk
$$

Once we choose an orthonormal basis of the K-types of V_π, there are only finite number of (α, β, i, j)'s such that $H(\psi, v; \alpha, i)$ and $H(\phi, u; \beta, j)$ are non-zero. Thus the integral (8.38) converges for all u, ϕ, v, ψ if and only if

$$
I(\alpha, \beta, i, j) = \int_{\mathfrak{a}^+} (\omega(\exp(H))x^\alpha, x^\beta)(\pi(\exp(H))v_i, v_j)\Delta(H)dH
$$

converges for all α, β, i and j.

Let $G = O_{p,q}$ be the orthogonal group fixing the symmetric form defined by

$$
I_{p,q} = \begin{pmatrix} 0_p & 0 & I_p \\ 0 & I_{q-p} & 0 \\ I_p & 0 & 0_p \end{pmatrix}
$$

and $G' = Sp_{2n}$ be the standard symplectic group. Now as a dual pair in $Sp_{2(p+q)n}$, we may choose $\mathbb{R}^{2n(p+q)}$ to be $M(p+q, 2n)$, such that G acts by left multiplication and G' acts by (inverse) right multiplication. We denote both actions on $M(p+q, n)$ by m. We may realize the Schrödinger model on

$$
L^2(M(p+q, n)) = L^2(x_{ij}, i = 1, \ldots, p+q, j = 1, \ldots, n)
$$

Now let $a = diag(a_1, a_2, \ldots a_p, 1, \ldots, 1, a_1^{-1}, \ldots, a_p^{-1})$. We define

$$
A^+ = \{a \mid a_1 > a_2 > \ldots > a_p > 1\} \subseteq G
$$

Let $b = diag(b_1^{-1}, b_2^{-1}, \ldots, b_n^{-1}, b_1, \ldots, b_n)$. We define

$$
A'^+ = \{b \mid b_1 > b_2 > \ldots > b_n > 1\} \subseteq G'
$$

Thus we have

$$m(a)e_{i,j} = \begin{cases} a_i e_{i,j} & i = 1, \dots, p \\ e_{i,j} & i = p+1, \dots, q \\ a_i^{-1} e_{i,j} & i = q+1, \dots, p+q \end{cases}$$

$$m(b)e_{i,j} = b_j e_{i,j} \qquad (i = 1, \dots, p+q)(j = 1, \dots, n)$$

These formulae indicate that the embedding of A and A' into $GL(M(p+q,n))$ are simply the left multiplication and the (inverse) right multiplication.

$$m(ab)e_{i,j} = \begin{cases} a_i b_j e_{i,j} & i = 1, \dots, p \\ b_j e_{i,j} & i = p+1, \dots, q \\ a_i^{-1} b_j e_{i,j} & i = q+1, \dots, p+q \end{cases}$$

We can easily deduce the following theorem from Theorem 8.13.

Theorem 8.15

$$\begin{aligned}\omega_{\alpha,\beta}(\widetilde{ab}) = sgn(\widetilde{ab})c_{\alpha,\beta} \prod_{i=1,j=1}^{p,n} & a_i^{\alpha_{i,j}-\alpha_{q+i,j}} b_j^{\alpha_{i,j}+\alpha_{q+i,j}+1} (a_i^2 b_j^2 + 1)^{-\frac{\alpha_{i,j}+\beta_{i,j}+1}{2}} (a_i^{-2} b_j^2 + 1)^{-\frac{\alpha_{q+i,j}+\beta_{q+i,j}+1}{2}} \\ \prod_{i=p+1,j=1}^{q,n} & b_j^{\alpha_{i,j}+\frac{1}{2}} (b_j^2 + 1)^{-\frac{\alpha_{i,j}+\beta_{i,j}+1}{2}} \\ \omega_{\alpha,\beta}(\widetilde{a}) = sgn(\widetilde{a})c_{\alpha,\beta} \prod_{i=1,j=1}^{p,n} & a_i^{-(\alpha_{q+i,j}+\beta_{i,j}+1)} (a_i^{-2} + 1)^{-\frac{\alpha_{i,j}+\alpha_{q+i,j}+\beta_{i,j}+\beta_{q+i,j}+2}{2}} \\ \omega_{\alpha,\beta}(\widetilde{b}) = sgn(\widetilde{b})c_{\alpha,\beta} \prod_{i=1,j=1}^{p,n} & b_j^{-(\beta_{q+i,j}+\beta_{i,j}+1)} (b_j^{-2} + 1)^{-\frac{\alpha_{i,j}+\alpha_{q+i,j}+\beta_{i,j}+\beta_{q+i,j}+2}{2}} \\ \prod_{i=p+1,j=1}^{q,n} & b_j^{-\beta_{i,j}-\frac{1}{2}} (b_j^{-2} + 1)^{-\frac{\alpha_{i,j}+\beta_{i,j}+1}{2}} \end{aligned} \tag{8.40}$$

Combined with Theorem 8.14, we obtain

Theorem 8.16 *Let $g = k_1 \exp(H(g))k_2$ with $H(g) \in \overline{\mathfrak{a}^+}$ be the KAK decomposition of $\widetilde{O_{p,q}}$. Let $g' = k_1' \exp(H(g'))k_2'$ with $H(g') \in \overline{\mathfrak{a}'^+}$ be the $K'A'K'$ decomposition of $\widetilde{Sp_{2n}}(\mathbb{R})$. Then there exists a constant c such that*

$$|\omega_{u,v}(\widetilde{g})| \leq c \exp(-n \sum_{i=1}^{p} H_i(g)) \qquad (g \in \widetilde{G}, H_i(g) = \ln a_i(g))$$

$$|\omega_{u,v}(\widetilde{g}')| \leq c \exp(-\frac{p+q}{2} \sum_{j=1}^{n} H_j(g)) \qquad (g \in \widetilde{G}', H_j(g) = \ln b_j(g))$$

The same estimates hold for $L(x)\omega_{u,v}(g)$ and $R(x)\omega_{u,v}(g')$ for every $x \in \mathfrak{sp}_{2n(p+q)}$.

By abuse of notation, we write $n = (n, n, \dots, n)$. Now we can prove the following theorem concerning $I(\alpha, \beta, i, j)$.

Theorem 8.17 *Suppose that for every leading exponent v of an irreducible admissible representation π of $\widetilde{O_{p,q}}$, $2\rho + Re\, v - n$ is a strictly negative combination of the simple roots, i.e.,*

$$2\rho + Re\, v - n \preceq \sum_{\alpha_i \in \Delta} c_i \alpha_i = \lambda \qquad (c_i < 0)$$

Then $I(\alpha, \beta, i, j)$ converges for every α,β,i,j. Thus, $(,)_\pi$ is well-defined.

Proof: Recall that for arbitrary matrix coefficient, we have

$$|(\pi(\exp(H))v_i, v_j)| \leq c \exp((\lambda - 2\rho + n)(H))(1 + \|H\|)^q \qquad (H \in \mathfrak{a}^+)$$

Thus

$$\begin{aligned} I(\alpha, \beta, i, j) \leq & \int_{\mathfrak{a}^+} c_{\alpha,\beta} \exp((\lambda - 2\rho + n)(H))(1 + \|H\|)^q \\ & \exp(-\sum_{i=1,j=1}^{p,n} (\alpha_{q+i,j} + \beta_{i,j} + 1)H_i) \prod_{i=1,j=1}^{p,n} (\exp(-2H_i) + 1)^{\frac{\alpha_{i,j} + \alpha_{q+i,j} + \beta_{i,j} + \beta_{q+i,j} + 2}{2}} \Delta(H) dH \\ \leq & \int_{\mathfrak{a}^+} c \exp((\lambda - 2\rho + n)(H))(1 + \|H\|)^q \exp(-\sum_{i=1}^{p} nH_i) \exp(2\rho(H))(1 + \|H\|)^q dH \\ \leq & \int_{\mathfrak{a}^+} c \exp(\lambda(H))(1 + \|H\|)^q dH \\ = & \int_{\alpha_1(H)>0, \dots, \alpha_p(H)>0} c \exp(\sum_1^p c_i \alpha_i(H))(1 + \|H\|)^q d\alpha_1 H \dots d\alpha_p(H) \\ < & \infty \end{aligned} \tag{8.41}$$

Thus $I(\alpha, \beta, i, j)$ is always well-defined. And $(,)_\pi$ is well-defined as well .
Q.E.D.

Definition 8.3 *We say that π is in the semistable range of the dual pair $(O_{p,q}, Sp_{2n})$, if for every leading exponent v of an irreducible admissible representation π of $\widetilde{O_{p,q}}$, there exist $c_i < 0 (i = 1, 2, \dots, p)$ such that*

$$2\rho + Re(v) - n \preceq \sum_{\alpha_i \in \Delta} c_i \alpha_i$$

We denote the set of representations in the semistable range by $\mathcal{R}_{ss}(O_{p,q}, Sp_{2n})$.

At this moment, we do not know whether $\mathcal{R}_{ss}(O_{p,q}, Sp_{2n})$ is contained in $\mathcal{R}(\widetilde{O_{p,q}}, \omega)$. We will show later that under a natural condition

$$\mathcal{R}_{ss}(O_{p,q}, Sp_{2n}) \subseteq \mathcal{R}(\widetilde{O_{p,q}}, \omega)$$

(up to a central character). The question of whether an irreducible representation π belongs to $\mathcal{R}(\widetilde{O}_{p,q}, \omega)$ can be read off from the Langlands parameters of π.

Theorem 8.18 *For every leading exponent v of an irreducible admissible representation π of Sp_{2n}, if $-\frac{p+q}{2} + Re\,(v) + 2\rho$ is a strictly negative combination of the simple roots, i.e.,*

$$-\frac{p+q}{2} + Re\,(v) + 2\rho \preceq \sum_{\alpha_i \in \Delta} c_i \alpha_i = \lambda \qquad (c_i < 0)$$

then $I(\alpha, \beta, i, j)$ converges for every α, β, i, j. Thus, $(,)_\pi$ is well-defined.

Proof: Notice that from Theorem 8.6 we have

$$|(\pi(\exp(H))v_i, v_i)| \leq c\exp(\lambda - 2\rho + \frac{p+q}{2})(H)(1 + \|H\|)^q \qquad (H \in \mathfrak{a}^+)$$

From Theorem 8.15

$$|\omega_{\alpha,\beta}(\exp(H))| \leq c\exp(-\frac{p+q}{2}(H))$$

Therefore,

$$\begin{aligned} I(\alpha, \beta, i, j) &\leq \int_{\mathfrak{a}^+} c\exp((\lambda - 2\rho + \frac{p+q}{2})(H))(1 + \|H\|)^q \exp(-\frac{p+q}{2}H)\Delta(H) \\ &\leq \int_{\mathfrak{a}^+} c\exp(\lambda(H))(1 + \|H\|)^q dH \\ &= \int_{\alpha_1(H)>0,\dots,\alpha_n(H)>0} c\exp(\sum_1^n c_i\alpha_i(H))d\alpha_1(H)\dots d\alpha_n(H) \\ &< \infty \end{aligned} \tag{8.42}$$

Thus, $I(\alpha, \beta, i, j)$ is well-defined. And $(,)_\pi$ is also well-defined. Q.E.D.

Definition 8.4 *Let π be an irreducible admissible representation of $\widetilde{Sp_{2n}(\mathbb{R})}$. If there exists $c_i < 0$, such that*

$$-\frac{p+q}{2} + Re\,(v) + 2\rho \preceq \sum_{\alpha_i \in \Delta} c_i \alpha_i$$

for every leading exponent v of π, then we say that π is in the semi-stable range of $(Sp_{2n}, O_{p,q})$. We denote the space of representations in the semistable range by $\mathcal{R}_{ss}(Sp_{2n}, O_{p,q})$.

Notice that from this definition semistable range only depends on $p+q$, not on the pair (p, q). Thus if π is in the semistable range of $O_{p,q}$, then it is also in the semistable range of $O_{p',q'}$ for $p'+q' = p+q$. For this reason, sometimes we will denote the semistable range by $\mathcal{R}_{ss}(Sp_{2n}, p+q)$.

9 Semi-stable range

In this chapter, we will define the averaging operator

$$\mathcal{L}_{\tilde{G}} : \mathcal{P}^c \otimes V_\pi \to Hom_{\mathbb{C}}(\mathcal{P}, V_\pi)$$

for a fixed π in the semi-stable range. And we will further show that if $\mathcal{L}_{\tilde{G}} \neq 0$, the image of $\mathcal{L}_{\tilde{G}}$ is always irreducible as a $(\mathfrak{g}', \tilde{K}')$ module. Throughout this chapter, for an arbitrary $\mathfrak{g}$-module W with a K action, W_K will be the subspace of K-finite vectors. Here all the homomorphisms are complex linear.

9.1 Averaging Operator

For a Harish-Chandra module (π, V), let (π^c, V^c) be the same real module as V, with $\mathbb{C}$ acting from the right conjugate linearly. Let (π^h, V^h) be the Hermitian dual, and (π^*, V^*) the complex dual. We have

$$V_K^h = (V_K^*)^c = (V^c)_K^*$$
$$(V_K^h)_K^h = V = (V_K^*)_K^*$$

From our definition of semistable range,

$$\pi \in \mathcal{R}_{ss}(G, G') \Leftrightarrow \pi^c \in \mathcal{R}_{ss}(G, G')$$

$$(\pi^*, (V_\pi^*)_K) \in \mathcal{R}_{ss}(G, G') \Leftrightarrow \pi \in \mathcal{R}_{ss}(G, G')$$

In general, (π^h, V^h) will stand for the Hermitian dual space with the Hermitian dual action. (π^*, V^*) will stand for the dual linear space of V with the dual action. Occasionally, we will just use π^h and π^* to denote the dual spaces of (π, V) is the category of Harish-Chandra modules.

Definition 9.1 *We define* $i : \mathcal{P}^c \otimes V_\pi \to Hom(\mathcal{P}, V_\pi)$ *by*

$$i(\psi \otimes v)(\phi) = (\phi, \psi)v \qquad (\psi, \phi \in \mathcal{P}, v \in V_\pi)$$

$\mathfrak{g}$ *acts on the left by* $\omega^c(g) \otimes \pi(g)$*, acts on the right by* $(\omega)^*(g) \otimes \pi(g)$*;* $\mathfrak{g}'$ *acts on the left by* $\omega^c(g')$*, acts on the right by* $(\omega)^*(g')$*.*

Since $\mathcal{P}^c \hookrightarrow (\mathcal{P})^*$ is an embedding of $(\mathfrak{g}, \mathfrak{g}')$ modules, i is a map of $(\mathfrak{g}, \mathfrak{g}')$ modules. Frequently, we will neglect the map i when we identify "vector valued" functions in $\mathcal{P}^c \otimes V$, with certain "vector valued" distributions in $Hom(\mathcal{P}, V)$.

Definition 9.2 *Formally (to be made precise later), we define the averaging operator*

$$\mathcal{L}_G : \mathcal{P}^c \otimes V_\pi \longrightarrow Hom(\mathcal{P}, V_\pi)$$

as follows

$$\mathcal{L}_G(\psi \otimes v)(\phi) = \int_G (\phi, \omega(g)\psi)\pi(g)v dg \qquad (\psi, \phi \in \mathcal{P}, v \in V_\pi)$$

Theorem 9.1 *Suppose (π, V_π) is an irreducible admissible Harish-Chandra module of G with a $K-$invariant inner product $(,)$ and $(\omega, \mathcal{P})$ a unitary Harish-Chandra module of G. Suppose that $(,)_\pi : (\mathcal{P} \otimes V_\pi) \otimes (\mathcal{P} \otimes V_\pi)^c \to \mathbb{C}$ is well-defined. In other words, the integral (8.38) converges for all ϕ, ψ, u, v. Let σ be a K-type in π, P_σ be the projector from V_π onto its σ-isotypic subspace. Then*

$$\mathcal{L}_G(\psi \otimes v)\phi = \sum_{\sigma \in \hat{K}} \int_G (\phi, \omega(g)\psi) P_\sigma(\pi(g)v)$$

is well-defined. Moreover, $\mathcal{L}_G$ does not depend on the choice of the inner product $(,)$ on V_π.

Proof:

1. First we define $\mathcal{L}_G^0 : \mathcal{P}^c \otimes V_\pi \to Hom(\mathcal{P}, V_\pi^h)$ by

$$\mathcal{L}_G^0(\psi \otimes v)\phi(u) = \int_G (\phi, \omega(g)\psi)(\pi(g)v, u)dg = \int_G \overline{(\omega(g)\psi, \phi)}(\pi(g)v, u)dg, \qquad (\psi, \phi \in \mathcal{P}, v, u \in V_\pi)$$

   Since $\mathcal{L}_G^0(\psi \otimes v)\phi$ is a conjugate linear functional on V_π, it lies in V_π^h.

2. Next, $\mathcal{L}_G^0$ is K-equivariant.

$$\begin{aligned} \mathcal{L}_G^0(\psi \otimes v)(\omega(k)\phi)(u) &= \int_G (\omega(k)\phi, \omega(g)\psi)(\pi(g)v, u)dg \\ &= \int_G (\phi, \omega(k^{-1}g)\psi)(\pi(g)v, u)dg \\ &= \int_G (\phi, \omega(g)\psi)(\pi(kg)v, u)dg \\ &= \int_G (\phi, \omega(g)\psi)(\pi(g)v, \pi(k^{-1})u)dg \\ &= (\pi^h(k)\mathcal{L}_G^0(\psi \otimes v)(\phi))u \end{aligned} \tag{9.43}$$

   Thus $\mathcal{L}_G^0 : \mathcal{P}^c \otimes V_\pi \to Hom_K(\mathcal{P}, V_\pi^h)$. But $\mathcal{P}$ are K-finite vectors, therefore $\mathcal{L}_G^0(\psi \otimes v)\phi$ lies in $(V_\pi^h)_K$.

3. If we identify $(V_\pi^h)_K$ with V_π under the inner product $(,)$, we can define $\mathcal{L}_G$ as the composition of $\mathcal{L}_G^0$ with this identification. More precisely, for a fixed $\sigma \in \hat{K}$, and $\phi \in \mathcal{P}_\sigma$, we define

$$\mathcal{L}_G(\psi \otimes v)\phi = \int_G (\phi, \omega(g)\psi) P_\sigma(\pi(g)v)dg$$

   It is easy see that, for all $u \in V_\pi$, we have

$$\mathcal{L}_G^0(\psi \otimes v)\phi(u) = (\mathcal{L}_G(\psi \otimes v)\phi, u) \qquad (\phi \in \mathcal{P}_\sigma)$$

4. In general, we define

$$\mathcal{L}_G(\psi \otimes v)\phi = \sum_{\sigma \in \hat{K}} \int_G (\phi, \omega(g)\psi) P_\sigma(\pi(g)v) \tag{9.44}$$

   Since for any $\phi \in \mathcal{P}$, ϕ is K-finite. Therefore ϕ is contained in a finite direct sum of $\mathcal{P}_\sigma$'s. Thus only finitely many terms occur in the direct sum (9.44). Hence $\mathcal{L}_G$ is a well-defined map from $\mathcal{P}^c \otimes V_\pi$ to $Hom(\mathcal{P}, V_\pi)$.

5. Since the projector P_σ does not depend on the choice of the K-invariant inner product $(,)$ on V_π, $\mathcal{L}_G : \mathcal{P}^c \otimes V_\pi \to Hom(\mathcal{P}, V_\pi)$ does not depend on the choice of $(,)$ on V_π.

Q.E.D.

Notice that we do not assume (π, V_π) is unitary here. However we assume $\pi|_K$ is unitary. In fact, if we define the matrix coefficients of π to be of the form $g \to \delta(\pi(g)f)$ with $\delta \in (V_\pi^*)_K, f \in V_\pi$, then we do not need the inner product structure on V_π. Nevertheless, we stick with our original definition of matrix coefficient, since it is more commonly used in the literature. Suppose that $\pi \in \mathcal{R}_{ss}(G, G')$. Then $(,)_\pi$ is well-defined. Immediately, we obtain

Theorem 9.2 *If π is in the semistable range of (G, G'), then*

$$\mathcal{L}_{\tilde{G}}(\psi \otimes v)\phi = \sum_{\sigma \in \hat{\tilde{K}}} \int_{\tilde{G}} (\phi, \omega(g)\psi) P_\sigma \pi(g) v dg$$

is well-defined.

I should remark here that the concept of averaging operator is not new in the compact Lie group theory. In fact, various forms of averaging operators are used in studying the geometry and topology of homogenous spaces. However, for noncompact Lie groups, the concept of averaging operator is less commonly used due to the difficulty in determining the convergence. In the next section, we will examine the properties of the averaging operator we have defined.

9.2 Properties of the Averaging Operator

We will assume that π is in the semistable range from now on unless stated otherwise.

Theorem 9.3 *Let P_0 be the projector onto the trivial $\widetilde{K}$-type of $\mathcal{P}^c \otimes V_\pi$. Then*

$$\mathcal{L}_{\tilde{G}} = \mathcal{L}_{\tilde{G}} \circ P_0$$

Proof: We have

$$\begin{aligned}\mathcal{L}_{\widetilde{G}}(\omega(k)\psi \otimes \pi(k)v)(\phi) &= \sum_{\sigma\in\hat{\widetilde{K}}} \int_{\widetilde{G}} (\phi, \omega(g)\omega(k)\psi) P_\sigma(\pi(g)\pi(k)v)dg \\ &= \sum_{\sigma\in\hat{\widetilde{K}}} \int_{\widetilde{G}} (\phi, \omega(g)\psi) P_\sigma(\pi(g)v)dg \\ &= \mathcal{L}_{\widetilde{G}}(\psi\otimes v)(\phi)\end{aligned} \tag{9.45}$$

Thus

$$\mathcal{L}_{\widetilde{G}}(\omega(k)\psi \otimes \pi(k)v) = \mathcal{L}_{\widetilde{G}}(\psi \otimes v)$$

Therefore $\mathcal{L}_{\widetilde{G}}$ is only nonzero for the trivial $\widetilde{K}$-types of $\mathcal{P}^c \otimes V_\pi$. We have

$$\mathcal{L}_{\widetilde{G}} = \mathcal{L}_{\widetilde{G}} \circ P_0$$

Q.E.D.

Theorem 9.4 *$\mathcal{L}_{\widetilde{G}}$ is a map of $(\mathfrak{g}', \widetilde{K}')$ modules.*

Proof: For $x' \in \mathfrak{g}'$, we have

$$\begin{aligned}\mathcal{L}_{\widetilde{G}}(\omega(x')\psi \otimes v)(\phi) &= \sum_{\sigma\in\hat{\widetilde{K}}} \int_{\widetilde{G}} (\phi, \omega(g)\omega(x')\psi) P_\sigma(\pi(g)v)dg \\ &= \sum_{\sigma\in\hat{\widetilde{K}}} \int_{\widetilde{G}} (\phi, \omega(x')\omega(g)\psi) P_\sigma(\pi(g)v)dg \\ &= \sum_{\sigma\in\hat{\widetilde{K}}} \int_{\widetilde{G}} (-\omega(x')\phi, \omega(g)\psi) P_\sigma(\pi(g)v)dg \\ &= (\omega^*(x')\mathcal{L}_{\widetilde{G}}(\psi\otimes v))(\phi)\end{aligned} \tag{9.46}$$

For $k \in \widetilde{K}'$, a similar statement can be proved. Therefore, $\mathcal{L}_{\widetilde{G}}$ is a $(\mathfrak{g}', \widetilde{K}')$ map. Q.E.D.

Theorem 9.5 *If π is in the semistable range of (G, G'), then*

$$\mathcal{L}_{\widetilde{G}} : \mathcal{P}^c \otimes V_\pi \to (Hom_{\,\mathfrak{g},\widetilde{K}}(\mathcal{P}, V_\pi))_{\widetilde{K}'}$$

In particular, if $\mathcal{L}_{\widetilde{G}}(\mathcal{P}^c \otimes V_\pi) \neq 0$, then $\pi \in \mathcal{R}(\mathfrak{g}, \widetilde{K}, \mathcal{P})$.

Proof: From Theorem 9.1, $\mathcal{L}_{\widetilde{G}}$ is $\widetilde{K}$ equivariant. From Theorem 9.3 wee see that under $\mathcal{L}_{\widetilde{G}}$ only the trivial isotypic space is mapped nonzero. Thus

$$\mathcal{L}_{\widetilde{G}}(\mathcal{P}^c \otimes V_\pi) \subseteq Hom_{\,\widetilde{K}}(\mathcal{P}, V_\pi)$$

Now it suffices to show that $\forall\, x \in \mathfrak{g}$,

$$\mathcal{L}_{\widetilde{G}}(\psi \otimes v)(\omega(x)\phi) = \pi(x)(\mathcal{L}_{\widetilde{G}}(\psi \otimes v)\phi)$$

This is equivalent to $\forall\, u \in V_\pi$,

$$\begin{aligned} \int_{\widetilde{G}} (\omega(x)\phi, \omega(g)\psi)(\pi(g)v, u)dg &= \int_{\widetilde{G}} (\phi, \omega(g)\psi)(\pi(x)\pi(g)v, u)dg \\ \Leftrightarrow \int_{\widetilde{G}} \frac{d}{dt}(\phi, \omega(\exp(tx))\omega(g)\psi)(\pi(\exp(tx))\pi(g)v, u)dg &= 0 \\ \Leftrightarrow \int_{\widetilde{G}} L(x)((\phi, \omega(g)\psi)(\pi(g)v, u))dg &= 0 \end{aligned} \tag{9.47}$$

Here L is the left regular action of $\mathfrak{g}$ on the smooth functions on $\widetilde{G}$. We write $F(g) = \omega_{\psi,\phi}(g)\pi_{v,u}(g)$. We will prove

$$\int_{\widetilde{G}} L(x)F(g)dg = 0$$

for the dual pair $(O_{p,q}, Sp_{2n}(\mathbb{R}))$. For all the other pairs, we can proceed similarly. From the semistability, there exist $v_o \in \mathfrak{a}^*$ and $c_i < 0$ such that

$$v_0 + 2\rho - n = \sum_{\alpha_i \in \Delta} c_i \alpha_i$$

From Theorem 8.6, we have

$$|\pi_{v,u}(g)| \leq c\exp(v_o(H(g)))(1 + \|H(g)\|)^q$$

According to Thorem 8.7, we have

$$|L(x)\pi_{v,u}(g)| \leq c\exp(v_o(H(g)))(1 + \|H(g)\|)^q$$

From Theorem 8.16, we have

$$|\omega_{\psi,\phi}(g)| \leq c\exp(-n(H(g)))$$

$$|L(x)\omega_{\psi,\phi}(g)| \leq c\exp(-n(H(g)))$$

Thus,

$$|F(g)| \leq c\exp((v_0 - n)(H(g)))(1 + \|H(g)\|)^q$$

$$|L(x)F(g)| \leq c\exp((v_0 - n)(H(g)))(1 + \|H(g)\|)^q$$

According to theorem 8.11,

$$\int L(x)f(g)dg = 0$$

Now we have shown that

$$\mathcal{L}_{\widetilde{G}}(\mathcal{P}^c \otimes V_\pi) \subseteq Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, V_\pi)$$

Since $\mathcal{L}_{\widetilde{G}}$ is a $(\mathfrak{g}', \widetilde{K}')$-map, and $\mathcal{P}^c$ is $\widetilde{K}'$-finite,

$$\mathcal{L}_{\widetilde{G}} : \mathcal{P}^c \otimes V_\pi \to (Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, V_\pi))_{\widetilde{K}'}$$

Finally, if $\mathcal{L}_{\widetilde{G}}(\mathcal{P}^c \otimes V_\pi) \neq 0$, then $(Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, V_\pi))_{\widetilde{K}'} \neq 0$. Thus $\pi \in \mathcal{R}(\mathfrak{g}, \widetilde{K}, \mathcal{P})$. Q.E.D.

Later we will give more precise information about the nonvanishing of $\mathcal{L}_{\widetilde{G}}$ under some quite general assumption. Just as a byproduct, we have

Theorem 9.6 *For every $x \in \mathfrak{g}$, $\psi \in \mathcal{P}^c$,$v \in V$, we have*

$$\mathcal{L}_{\widetilde{G}}(\omega(x)\psi \otimes v + \psi \otimes \pi(x)v) = 0$$

In other words, $\omega(x)\psi \otimes v + \psi \otimes \pi(x)v \in ker(\mathcal{L}_{\widetilde{G}})$.

Proof: Use $R(x)$ instead of $L(x)$ in the proof of the last theorem. Q.E.D.

9.3 Howe's Quotient and its Dual

In representation theory, sometimes it is easier to deal with submodules than quotients. For Howe's quotient, we have

Theorem 9.7 *$\pi \in \mathcal{R}(\mathfrak{g}, \widetilde{K}, \mathcal{P})$ if and only if $(\pi^*, (V_\pi^*)_{\widetilde{K}})$ can be embedded as a $(\mathfrak{g}, \widetilde{K})$-submodule of $\mathcal{P}^*$.*

Proof: Suppose there exists a $(\mathfrak{g}, \widetilde{K})$ module map:

$$i : \mathcal{P} \to V_\pi$$

such that $\mathcal{P}/Ker(i) \cong \pi$. Thus $\pi^* \cong (\mathcal{P}/ker(i))^*$. But $(\mathcal{P}/ker(i))^* \subseteq \mathcal{P}^*$. Therefore, $(\pi^*, (V_\pi^*)_{\widetilde{K}}$ can be embedded as $(\mathfrak{g}, \widetilde{K})$-submodule of $\mathcal{P}^*$. On the other hand, suppose there exists a $(\mathfrak{g}, \widetilde{K})$-submodule

$$j : (V_\pi^*)_{\widetilde{K}} \hookrightarrow \mathcal{P}^*$$

Let $\mathcal{N} = \cap_{v \in (V_\pi^*)_{\widetilde{K}}} ker(j(v))$. Then $j((V_\pi^*)_{\widetilde{K}})$ lies in $(\mathcal{P}/\mathcal{N})^*$, and

$$i : \mathcal{P}/\mathcal{N} \to (V_\pi^*)_{\widetilde{K}}^*$$

is an embedding of $(\mathfrak{g}, \widetilde{K})$-modules. Since $\mathcal{P}/\mathcal{N}$ is $\widetilde{K}$-finite, thus the image of i sits in $((V_\pi^*)_{\widetilde{K}}^*)_{\widetilde{K}} \cong V_\pi$. But V_π is already irreducible, so

$$\mathcal{P}/\mathcal{N} \cong V_\pi$$

Q.E.D. Let

$$\{0\} = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V$$

be a composition series of Harish-Chandra modules, such that V_i/V_{i-1} is irreducible. Then

$$V_K^* = (V/V_0)_K^* \supseteq (V/V_1)_K^* \supseteq \cdots \supseteq (V/V_{n-1})_K^* \supseteq (V/V_n)_K^* = \{0\}$$

is a composition series, with each subquotient irreducible.

Theorem 9.8 *If (π, V_π) is the unique irreducible quotient of a Harish-Chandra module (π_0, V_{π_0}) of finite length, then $(\pi^*, (V_\pi^*)_K)$ is the unique irreducible submodule of $(\pi_0^*, (V_{\pi_0}^*)_K)$.*

Theorem 9.9

$$V_{\omega_0(\pi)}^* \cong Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, V_\pi)$$

$$(V_{\omega_0(\pi)}^*)_{\widetilde{K}'} \cong \oplus_{\sigma' \in \mathcal{R}(\widetilde{K}', \mathcal{P})} Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}_{\sigma'}, V_\pi)$$

Proof: First let $\Phi \in Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, V_\pi)$. Then $\ker\Phi \supseteq \mathcal{N}_\pi$ as defined in Definition 7.2. Thus Φ can be regarded as an element in

$$Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}/\mathcal{N}_\pi, V_\pi) \cong Hom_{\mathfrak{g},\widetilde{K}}(V_\pi \otimes V_{\omega_0(\pi)}, V_\pi) \cong (V_{\omega_0(\pi)})^*$$

On the other hand, for every $\Phi \in Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}/\mathcal{N}_\pi, V_\pi)$, there exists a unique (still denoted by) $\Phi \in Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, V_\pi)$ such that $\Phi(\mathcal{N}_\pi) = 0$. Therefore,

$$V_{\omega_0(\pi)}^* \cong Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, V_\pi)$$

For the second statement, we can simply take the $\widetilde{K}'$-finite subspace. We have

$$(V_{\omega_0(\pi)}^*)_{\widetilde{K}} \cong \oplus_{\sigma' \in \mathcal{R}(\widetilde{K}', \mathcal{P})} Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}_{\sigma'}, V_\pi)$$

Q.E.D.

Notice that

$$Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}_{\sigma'}, V_\pi) \cong (Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, V_\pi))_{\sigma'^*}$$

For every $\Phi \in Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}_{\sigma'}, V_\pi)$, because of the direct sum decomposition, we may extend it to an element in $Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, V_\pi)$. In our future discussion, we will identify $Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}_{\sigma'}, V_\pi)$ with its extension in $Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, V_\pi)$.

Lemma 9.1 (Reciprocity) *Let $(\sigma, V_\sigma) \in \mathcal{R}(\widetilde{K}, V_\pi)$ be of minimal degree in $\mathcal{P}$. Let $(\sigma', V_{\sigma'})$ be as in Theorem 7.4. Then*

$$\dim(Hom_{\widetilde{K}}(V_\sigma, V_\pi)) = \dim(Hom_{\widetilde{K}'}(V_{\sigma'}, V_{\omega(\pi)})) = \dim(Hom_{\widetilde{K}'}(V_{\sigma'}, V_{\omega_0(\pi)}))$$

Proof: Notice that $\mathcal{P}_{\sigma'} = U(\mathfrak{g})\mathcal{H}(K')_{\sigma'}$. Thus the restriction from $\mathcal{P}_{\sigma'}$ to $\mathcal{H}(K')_{\sigma'}$

$$Res : Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}_{\sigma'}, V_\pi) \rightarrow Hom_{\widetilde{K}}(\mathcal{H}(K')_{\sigma'}, V_\pi)$$

is injective. From section 7.3,

$$\exists\, \tau \in \mathcal{R}(\widetilde{M}^0, \mathcal{P}) \qquad \mathcal{H}(K')_{\sigma'} \cong V_{\sigma'} \otimes V_\tau$$

and if $\alpha \in \mathcal{R}(\widetilde{K}, V_\tau)$ and $\alpha \neq \sigma$ in $\hat{\hat{\widetilde{K}}}$, then according to Theorem 7.5

$$\alpha \notin \mathcal{R}(\widetilde{K}, \pi)$$

Thus, we can further have an injection (still denoted by Res)

$$Res : Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}_{\sigma'}, V_\pi) \to Hom_{\widetilde{K}}(\mathcal{H}_{\sigma,\sigma'}, V_\pi) \cong V_{\sigma'}^* \otimes Hom_{\widetilde{K}}(V_\sigma, V_\pi)$$

Therefore we have

$$\begin{aligned}
\dim(Hom_{\widetilde{K}'}(V_{\sigma'}, V_{\omega_0(\pi)})) &= \dim(Hom_{\widetilde{K}'}(V_{\sigma'}^*, (V_{\omega_0(\pi)}^*)_{\widetilde{K}'})) \\
&= \dim(Hom_{\widetilde{K}'}(V_{\sigma'}^*, (Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, V_\pi))_{\widetilde{K}'})) \\
&= \dim(Hom_{\widetilde{K}'}(V_{\sigma'}^*, Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}_{\sigma'}, V_\pi))) \\
&\leq \dim(Hom_{\widetilde{K}'}(V_{\sigma'}^*, V_{\sigma'}^* \otimes Hom_{\widetilde{K}}(V_\sigma, V_\pi))) \\
&= \dim(Hom_{\widetilde{K}}(V_\sigma, V_\pi))
\end{aligned} \tag{9.48}$$

For more or less the same reasons, we have

$$\begin{aligned}
\dim(Hom_{\widetilde{K}}(V_\sigma, V_{\omega_0(\omega(\pi))})) &\leq \dim(Hom_{\widetilde{K}'}(V_{\sigma'}, V_{\omega(\pi)})) \\
&\leq \dim(Hom_{\widetilde{K}'}(V_{\sigma'}, V_{\omega_0(\pi)})) \\
&\leq \dim(Hom_{\widetilde{K}}(V_\sigma, V_\pi)) = \dim(Hom_{\widetilde{K}}(V_\sigma, V_{\omega(\omega(\pi))})) \\
&\leq \dim(Hom_{\widetilde{K}}(V_\sigma, V_{\omega_0(\omega(\pi))}))
\end{aligned} \tag{9.49}$$

Thus,

$$\dim(Hom_{\widetilde{K}}(V_\sigma, V_\pi)) = \dim(Hom_{\widetilde{K}'}(V_{\sigma'}, V_{\omega(\pi)})) = \dim(Hom_{\widetilde{K}'}(V_{\sigma'}, V_{\omega_0(\pi)}))$$

Q.E.D.

Corollary 9.1 *The map*

$$Res : Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}_{\sigma'}, V_\pi) \to Hom_{\widetilde{K}}(\mathcal{H}_{\sigma,\sigma'}, V_\pi)$$

is a bijection. In other words, every map Ψ in $Hom_{\widetilde{K}}(\mathcal{H}_{\sigma,\sigma'}, V_\pi)$ can be extended to a map $\widetilde{\Psi}$ in $Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}_{\sigma'}, V_\pi)$ such that

$$\widetilde{\Psi}|_{\mathcal{H}_{\sigma,\sigma'}} = \Psi$$

Since $(\omega_0(\pi), V_{\omega_0(\pi)})$ is a finite generated quasisimple $(\mathfrak{g}', \widetilde{K}')$-module with a unique irreducible quotient $V_{\omega(\pi)}$, $(\omega_0(\pi)^*, (Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, V_\pi))_{\widetilde{K}'})$ is also a finitely generated quasisimple $(\mathfrak{g}', \widetilde{K}')$-module with a unique irreducible submodule equivalent to $(V_{\omega(\pi)}^*)_{\widetilde{K}'}$. For notational purpose, we will denote such a submodule by $(V_{\omega(\pi)}^*)_{\widetilde{K}'}$.

Corollary 9.2 *$(Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, V_\pi))_{\widetilde{K}'}$ is a finitely generated quasi-simple $(\mathfrak{g}', \widetilde{K}')$ module. There exists a unique irreducible submodule $\omega(\pi)^*_{\widetilde{K}'}$. Let $\sigma \in V_\pi$, such that $deg(\sigma)$ is minimal among all the $\widetilde{K}$-types of π. Let σ' be the unique $\widetilde{K}'$-type such that*

$$\mathcal{H}_{\sigma,\sigma'} = \mathcal{H}(K') \cap \mathcal{H}(K)_\sigma$$

Then $(V^*_{\omega(\pi)})_{\widetilde{K}'}$ is generated by $Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}_{\sigma'}, V_\pi)$ as a $U(\mathfrak{g}')$ module.

Proof: From the last Lemma, we have

$$Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}_{\sigma'}, V_\pi) \cong (V^*_{\omega_0(\pi)})_{\sigma'^*} \cong (V^*_{\omega(\pi)})_{\sigma'^*}$$

Thus $Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}_{\sigma'}, V_\pi)$ is contained in the irreducible submodule $(V^*_{\omega(\pi)})_{\widetilde{K}'}$. But $(V^*_{\omega(\pi)})_{\widetilde{K}'}$ is already irreducible. Therefore, it is generated by $Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}_{\sigma'}, V_\pi)$. Q.E.D.

9.4 Irreducibility

Now we come back to the averaging operator $\mathcal{L}$. To show that the image of our averaging operator is irreducible in $(Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, V_\pi))_{\widetilde{K}'}$, it suffices to show that the image is exactly $(V^*_{\omega(\pi)})_{\widetilde{K}'}$, or alternatively, $\mathcal{L}_{\widetilde{G}}(\mathcal{P}^c \otimes V_\pi)$ is generated by $Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}_{\sigma'}, V_\pi)$. We shall focus on the $\widetilde{K}'$-type σ' of minimal degree.

Lemma 9.2 *Suppose π is in the semi-stable range of (G, G') and $\mathcal{L}_{\widetilde{G}} \neq 0$. Let $\sigma \in \mathcal{R}(\widetilde{K}, \pi)$ with minimal degree in $\mathcal{P}$. Then*

$$\mathcal{L}_{\widetilde{G}}((P_{\sigma'})^c \otimes V_\pi) \neq 0$$

Proof: Since $\mathcal{L}_{\widetilde{G}}$ is a $\widetilde{K}'$-equivariant map, by taking the $(\sigma')^c$-isotypic submodule, we have

$$\mathcal{L}_{\widetilde{G}} : (\mathcal{P}_{\sigma'})^c \otimes V_\pi \to Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}_{\sigma'}, V_\pi)$$

If $\mathcal{L}_{\widetilde{G}}((\mathcal{P}_{\sigma'})^c \otimes V_\pi) = 0$, in other words,

$$Im(\mathcal{L}_{\widetilde{G}}) \cap Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}_{\sigma'}, V_\pi) = \{0\}$$

then

$$Im(\mathcal{L}_{\widetilde{G}}) \cap (V^*_{\omega(\pi)})_{\widetilde{K}'} = \{0\}$$

But $(V^*_{\omega(\pi)})_{\widetilde{K}'}$ is the unique irreducible submodule of $(Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, V_\pi))_{\widetilde{K}'}$. Thus

$$Im(\mathcal{L}_{\widetilde{G}}) = 0$$

This contradicts that $\mathcal{L}_{\widetilde{G}} \neq 0$. Q.E.D.

We shall make one comment here. In our proof, we identified $(\mathcal{P}_{\sigma'})^c$ with $(\mathcal{P}^c)_{\sigma'^c}$ just for simplicity. In general, (π^c, V^c) only differs from (π, V) in the complex structure, they are identical as $(\mathfrak{g}, K)$-modules.

Lemma 9.3 *Suppose π is in the semistable range of (G, G') and $\mathcal{L}_{\widetilde{G}}(\mathcal{P} \otimes V_\pi) \neq 0$. Then the Hermitian dual $(\pi^h, V^h_{\widetilde{K}}) \in \mathcal{R}(\mathfrak{g}, \widetilde{K}, \mathcal{P})$ and the dual $(\pi^*, V^*_{\widetilde{K}}) \in \mathcal{R}(\mathfrak{g}, \widetilde{K}, \mathcal{P}^c)$.*

Proof: We define $\mathcal{L}^1_{\widetilde{G}} : \mathcal{P}^c \rightarrow V^*_\pi \otimes Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, \pi)$ by

$$\mathcal{L}^1_{\widetilde{G}}(\psi)(v)(\phi) = \int_{\widetilde{G}} (\phi, \omega(g)\psi)\pi(g) v dg \qquad (\psi \in \mathcal{P}^c, \phi \in \mathcal{P}, v \in V_\pi)$$

Notice that $L^1_{\widetilde{G}}$ is $(\mathfrak{g}, \widetilde{K})$-equivariant with respect to $(\mathcal{P}^c, V^*_\pi)$, and $(\mathfrak{g}', \widetilde{K}')$-equivariant with respect to $(\mathcal{P}^c, Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, \pi))$. Thus

$$\mathcal{L}^1_{\widetilde{G}} : \mathcal{P}^c \rightarrow (V^*_\pi)_{\widetilde{K}} \otimes (Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, \pi))_{\widetilde{K}'}$$

$\mathcal{L}_{\widetilde{G}} \neq 0$ implies that $\mathcal{L}^1_{\widetilde{G}} \neq 0$. Therefore

$$(\pi^*, (V^*_\pi)_{\widetilde{K}}) \in \mathcal{R}(\mathfrak{g}, \widetilde{K}, \mathcal{P}^c)$$

Thus

$$(\pi^h, (V^h_\pi)_{\widetilde{K}}) \in \mathcal{R}(\mathfrak{g}, \widetilde{K}, \mathcal{P})$$

This is equivalent to

$$(\pi^*, V^*_{\widetilde{K}}) \in \mathcal{R}(\mathfrak{g}, \widetilde{K}, \mathcal{P}^c)$$

Q.E.D.

Theorem 9.10 (Irreducibility) *Suppose π is in the semi-stable range of (G, G') and $\mathcal{L}_{\widetilde{G}} \neq 0$. Then*

$$Im(\mathcal{L}_{\widetilde{G}}) \cong (V^*_{\omega(\pi)})_{\widetilde{K}'}$$

is irreducible.

Proof: From the lemma, we have $(V^*_\pi)_{\widetilde{K}} \in \mathcal{R}(\mathfrak{g}, \widetilde{K}, \mathcal{P}^c)$. Let $\mathcal{P}^c/\mathcal{N}$ be the maximal quotient defined in Definition 7.2. Since $\mathcal{L}^1_{\widetilde{G}}$ is a $(\mathfrak{g}, \widetilde{K})$ map from $\mathcal{P}^c$ to

$$(V^*_\pi)_{\widetilde{K}} \otimes (Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, \pi))_{\widetilde{K}'}$$

$\mathcal{L}^1_{\widetilde{G}}$ descends to a map from $\mathcal{P}^c/\mathcal{N}$ to

$$(V^*_\pi)_{\widetilde{K}} \otimes (Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, \pi))_{\widetilde{K}'}$$

Let $\sigma \in \pi$ be a $\widetilde{K}$-type of minimal degree in $\mathcal{P}$. Then $\sigma^* \cong \sigma^c \in \mathcal{R}(\widetilde{K}, \pi^*)$ is of minimal degree in $\mathcal{P}^c$. Thus $\mathcal{P}^c/\mathcal{N}$ is generated as a $(\mathfrak{g}' \times \mathfrak{g}, \widetilde{KK'})$-module by $(\mathcal{H}_{\sigma,\sigma'})^c$ according to Theorem 7.7. Therefore $\mathcal{L}^1_{\widetilde{G}}(\mathcal{P}^c)$ is generated as a $(\mathfrak{g} \times \mathfrak{g}', \widetilde{KK'})$-module by $\mathcal{L}^1_{\widetilde{G}}(\mathcal{H}^c_{\sigma,\sigma'})$, as a $(\mathfrak{g}', \widetilde{K}')$-module by $\mathcal{L}^1_{\widetilde{G}}((\mathcal{P}_{\sigma'})^c)$. Notice that

$$\mathcal{L}_{\widetilde{G}}((P_{\sigma'})^c \otimes V_\pi) = \mathcal{L}^1_{\widetilde{G}}((\mathcal{P}_{\sigma'})^c)(V_\pi) \subseteq (Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, \pi))_{(\sigma')^c} = Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}_{\sigma'}, V_\pi) \qquad (9.50)$$

But $(V^*_{\omega(\pi)})_{\widetilde{K}'}$ is generated by $Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}_{\sigma'}, V_\pi)$. $\mathcal{L}_{\widetilde{G}}(\mathcal{P}^c \otimes V_\pi)$ is contained in $(V^*_{\omega(\pi)})_{\widetilde{K}'}$. But $(V^*_{\omega(\pi)})_{\widetilde{K}'}$ is already irreducible, and $Im(\mathcal{L}_{\widetilde{G}}) \neq 0$. Therefore

$$Im(\mathcal{L}_{\widetilde{G}}) \cong (V^*_{\omega(\pi)})_{\widetilde{K}'}$$

Q.E.D.

9.5 Invariant Hermitian Structure

Let π be a unitary representation of $\widetilde{G}$. This is equivalent to say that there exists an inner product on V_π such that

$$\forall\, k \in \widetilde{K},\ u, v \in V_\pi \qquad (ku, kv) = (u, v)$$

$$\forall\, x \in \mathfrak{g},\ u, v \in V_\pi \qquad (xu, v) + (u, xv) = 0$$

Let $\mathcal{R} = ker(\mathcal{L}_{\widetilde{G}})$ be the radical. Then

$$\mathcal{L}_{\widetilde{G}} : (\mathcal{P}^c \otimes V_\pi)/\mathcal{R} \to Hom_{\mathfrak{g},\widetilde{K}}(\mathcal{P}, V_\pi)$$

is injective. Since $Im(\mathcal{L}_{\widetilde{G}})$ is irreducible and isomorphic to $(V^*_{\omega(\pi)})_{\widetilde{K}'}$, $(\mathcal{P}^c \otimes V_\pi)/\mathcal{R}$ is irreducible and isomorphic to $(V^*_{\omega(\pi)})_{\widetilde{K}}$.

Definition 9.3 *We define an Hermitian form on $(\mathcal{P}^c \otimes V_\pi)/\mathcal{R}$ by*

$$([\phi \otimes u], [\psi \otimes v]) = \int_{\widetilde{G}} (\psi, \omega(g)\phi)(\pi(g)u, v)dg$$

We can easily show that

1. $([\phi \otimes u], [\psi \otimes v]) = (L_{\widetilde{G}}(\phi \otimes u)\psi, v)$
2. $(,)$ on $(\mathcal{P}^c \otimes V_\pi)/\mathcal{R}$ is Hermitian.
$$([\phi \otimes u], [\psi \otimes v]) = \overline{([\psi \otimes v], [\phi \otimes u])}$$
3. $(,)$ on $(\mathcal{P}^c \otimes V_\pi)/\mathcal{R}$ is $\mathfrak{g}'$ invariant.
$$\forall x\ \in \mathfrak{g}',\ (\omega(x)[\phi \otimes u], [\psi \otimes v]) + ([\phi \otimes u], \omega(x)[\psi \otimes v]) = 0$$

Theorem 9.11 *Suppose $\mathcal{L}_{\widetilde{G}}(\mathcal{P}^c \otimes V) \neq 0$. If π is unitary and in the semistable range of (G, G'), then the Hermitian form $(,)$ on $(\mathcal{P}^c \otimes V_\pi)/\mathcal{R} \cong (V^*_{\omega(\pi)})_{\widetilde{K}'}$ is $\mathfrak{g}'$ invariant.*

10 Non-Vanishing Theorem for $(O_{p,q}, Sp_{2n}(\mathbb{R}))$

The dual pair correspondence for real reductive pair $(O_{p,q}, Sp_{2n}(\mathbb{R}))$ is a one-to-one correspondence between $\mathcal{R}(\widetilde{O_{p,q}}, \omega)$ and $\mathcal{R}(\widetilde{Sp_{2n}(\mathbb{R})}, \omega)$. Philosophically speaking, this provides a tool to study representations of a "bigger" group through representations of a "smaller" group. In this chapter, we will assume $p+q \leq 2n+1$. Thus we regard $\widetilde{O_{p,q}}$ as the smaller group and $\widetilde{Sp_{2n}(\mathbb{R})}$ as the bigger group. However, one essential question that needs to be answered here is what representations are contained in $\mathcal{R}(\widetilde{O_{p,q}}, \omega)$. In this chapter, we will show that roughly all the representations (up to a central character) in the semistable range $\mathcal{R}_{ss}(O_{p,q}, Sp_{2n}(\mathbb{R}))$ are contained in $\mathcal{R}(\widetilde{O_{p,q}}, \omega)$. We will first study the Bargmann-Segal model (see Ch 1,2 [R-R]) in the frame of the dual pair $(O_{p,q}, Sp_{2n}(\mathbb{R}))$. Then we will proceed to show that for π within the semistable range and $p+q \leq 2n+1$, either $\mathcal{L}_{\widetilde{O_{p,q}}}(\mathcal{P}^c \otimes V_\pi)$ or $\mathcal{L}_{\widetilde{O_{p,q}}}(\mathcal{P}^c \otimes V_{\pi\otimes\chi})$ is not vanishing. Here χ is a central character of $\widetilde{O_{p,q}}$. Thus according to Theorem 9.10, $\mathcal{L}_{\widetilde{G}}(\mathcal{P}^c \otimes V_\pi)$ yields an irreducible representation of G', namely $(\omega(\pi)^*, (V^*_{\omega(\pi)})_{\widetilde{K'}})$. Throughout this chapter, we will fix p, q, n.

10.1 Bargmann-Segal Model and $(O_{p,q}, Sp_{2n})$ pairs

Let $S_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$, and $W = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$ Let V_1 be a real vector space of $p+q$ dimension. Let $(,)_1$ be the nondegenerate form

$$(x, y) = x^t S_{p,q} y \qquad (x, y \in V_1)$$

Let (V_2, ω) be a symplectic space such that

$$\omega(x, y) = x^t W y \qquad (x, y \in V_2)$$

1. Let $V = Hom_{\mathbb{R}}(V_1, V_2) = Mat(p+q, 2n, \mathbb{R})$. Let

   $$\Omega(X, Y) = Tr(S_{p,q} Y W X^t) = Tr(X^t S_{p,q} Y W) \qquad (X, Y \in Mat(p+q, 2n, \mathbb{R}))$$

   Then

   $$\Omega(Y, X) = -\Omega(X, Y)$$

   and Ω is nondegenerate. Thus V is a symplectic space. Let $Sp(V, \Omega)$ be the symplectic group fixing Ω. We define the left multiplication

   $$L : O_{p,q} \to Sp(V, \Omega)$$

   and the right multiplication

   $$R : Sp_{2n}(\mathbb{R}) \to Sp(V, \Omega)$$

   It is easy to check that $L(O_{p,q})$ and $R(Sp_{2n}(\mathbb{R}))$ fix Ω. And $L(O_{p,q})$ commutes with $R(Sp_{2n}(\mathbb{R}))$. Therefore $(O_{p,q}, Sp_{2n}(\mathbb{R}))$ is a dual pair in $Sp(V, \Omega)$.

2. Suppose $p \leq q$. Let

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in \begin{pmatrix} M(p,n) & M(p,n) \\ M(q,n) & M(q,n) \end{pmatrix}$$

Then

$$\Omega(X,Y) = Tr(X_{12}Y_{11}^t - X_{11}Y_{12}^t - X_{22}Y_{21}^t + X_{21}Y_{22}^t)$$

3. Let $V_{\mathbb{C}} = Mat(p,n,\mathbb{C}) \oplus Mat(q,n,\mathbb{C})$. Here $V_{\mathbb{C}}$ is not the complexification of V. It is V itself regarded as a complex linear space. The exact identification will be discussed later. For $v \in V_{\mathbb{C}}$, we may either write $v = (v_1, v_2)$ or $v = v_1 + v_2$ where $v_1 \in Mat(p,n,\mathbb{C})$ and $V_2 \in Mat(q,n,\mathbb{C})$. Let $v_i = Re(v_i) + iIm(v_i)$. We define

$$(u,v) = Tr(u_1\overline{v_1^t}) + Tr(u_2\overline{v_2^t})$$

It can be computed that

$$Re(u,v) = Tr(Re(u_1)Re(v_1^t) + Im(u_1)Im(v_1^t) + Re(u_2)Re(v_2^t) + Im(u_2)Im(v_2^t))$$

$$Im(u,v) = Tr(Im(u_1)Re(v_1^t) - Re(u_1)Im(v_1^t) + Im(u_2)Re(v_2^t) - Re(u_2)Im(v_2^t))$$

Now in order that $\Omega(u,v) = Im(u,v)$, we let

$$Im(u_1) = X_{12} \quad Re(v_1) = Y_{11} \quad Im(v_1) = Y_{12} \quad Re(u_1) = X_{11}$$

$$Im(u_2) = X_{21} \quad Re(v_2) = Y_{22} \quad Im(v_2) = Y_{21} \quad Re(u_2) = X_{22}$$

Thus we may identify V with $V_{\mathbb{C}}$ as follows:

$$C : (u_1, u_2) \to \begin{pmatrix} Re(u_1) & Im(u_1) \\ Im(u_2) & Re(u_2) \end{pmatrix}$$

$$C^{-1} : \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \to (X_{11} + iX_{12}, X_{22} + iY_{21})$$

4. Now the complex multiplication as linear transform in V can be written as

$$\begin{pmatrix} Re(u_1) & Im(u_1) \\ Im(u_2) & Re(u_2) \end{pmatrix} \to \begin{pmatrix} -Im(u_1) & Re(u_1) \\ Re(u_2) & -Im(u_2) \end{pmatrix}$$
$$= S_{p,q} \begin{pmatrix} Re(u_1) & Im(u_1) \\ Im(u_2) & Re(u_2) \end{pmatrix} W$$

Let $A_n = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$. Then the complex conjugation can be written as

$$\begin{pmatrix} Re(u_1) & Im(u_1) \\ Im(u_2) & Re(u_2) \end{pmatrix} \to \begin{pmatrix} Re(u_1) & -Im(u_1) \\ -Im(u_2) & Re(u_2) \end{pmatrix}$$
$$= S_{p,q} \begin{pmatrix} Re(u_1) & Im(u_1) \\ Im(u_2) & Re(u_2) \end{pmatrix} A_n$$

5. Now we fix a maximal split Abelian subalgebra $\mathfrak{a}_1$ of $\mathfrak{o}_{p,q}$

$$\mathfrak{a}_1 = \left\{ H(\lambda) = \begin{pmatrix} 0_p & \lambda & 0_{p,q-p} \\ \lambda & 0_p & 0_{p,q-p} \\ 0_{q-p,p} & 0_{q-p,p} & 0_{q-p} \end{pmatrix} \mid \lambda = diag(\lambda_1, \ldots, \lambda_p) \right\}$$

Then the positive Weyl chamber $\mathfrak{a}_1^+$ is given by those λ such that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_p > 0$$

For $O_{p,p}$, we do need the disconnectedness of $O_{p,q}$ in order to produce such a Weyl Chamber. The maximal split Abelian subgroup is of the following form

$$L(\exp H(\lambda)) = L \begin{pmatrix} \cosh\lambda & \sinh\lambda & 0 \\ \sinh\lambda & \cosh\lambda & 0 \\ 0 & 0 & I_{q-p} \end{pmatrix}$$

Here L indicates how $O_{p,q}$ is embedded into $Sp(V,\Omega)$. From now on we regard all the $L(g)\,(g \in O_{p,q})$ as elements in $Sp(V,\Omega)$ abstractly, and g as a standard matrix form representing $L(g)$. In all cases, our discussion will be in $Sp(V,\Omega)$, our matrix or group manipulation may be based on $O_{p,q}$.

6. We also fix a maximal split Abelian subalgebra $\mathfrak{a}_2$ of $\mathfrak{sp}_{2n}(\mathbb{R})$

$$\mathfrak{a}_2 = \left\{ \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix} \mid \mu = diag(\mu_1, \ldots, \mu_n) \right\}$$

Then the positive Weyl chamber $\mathfrak{a}_2^+$ is given by those μ such that

$$\mu_1 > \mu_2 > \cdots \mu_n > 0$$

The maximal split Abelian subgroup is of the following form

$$R(\exp(\mu)) = \begin{pmatrix} \exp(\mu) & 0 \\ 0 & \exp(-\mu) \end{pmatrix}$$

Now let $(\omega, \mathcal{H})$ be the Bargmann-Segal model for $Sp(V,\Omega)$ and $\mathcal{P}$ be the Harish-Chandra module of $\mathcal{H}$.

10.2 Bargmann-Segal kernel for $O_{p,q}$

We continue on with the structure theory. Let $g = k_1 \exp H(\lambda) k_2$ be the KAK decomposition of $O_{p,q}$, where $K = O_p \times O_q$. Let $k_i = (U_i, V_i) \in O_p \times O_q$.

1. Recall that J is the complex multiplication of i on $V_{\mathbb{C}}$. We compute

$$J(L(g))J(x) = S_{p,q}(L(g)S_{p,q}xW)W = L(-S_{p,q}gS_{p,q})x \qquad (x \in V)$$

2. For $L(\lambda) \in \mathfrak{a}$, we have

$$J(L(\exp H(\lambda)))J = L\left(\begin{pmatrix} -\cosh\lambda & \sinh\lambda & 0 \\ \sinh\lambda & -\cosh\lambda & 0 \\ 0 & 0 & -I_{q-p} \end{pmatrix}\right)$$

3.

$$C_{L(\exp H(\lambda))} = L\left(\begin{pmatrix} \cosh\lambda & 0 & 0 \\ 0 & \cosh\lambda & 0 \\ 0 & 0 & I_{q-p} \end{pmatrix}\right)$$

We denote it by $L(\cosh\lambda)$.

4.

$$A_{L(\exp H(\lambda))} = L\left(\begin{pmatrix} 0 & \sinh\lambda & 0 \\ \sinh\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

We denote it by $L(\sinh\lambda)$.

5.

$$Z_{L(\exp H(\lambda))} = L\left(\begin{pmatrix} 0 & \tanh(\lambda) & 0 \\ \tanh(\lambda) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

We denote it by $L(\tanh\lambda)$.

6. $Z_{L(\exp(-\lambda))} = -L(\tanh\lambda)$.

Let $z = (z_1, z_2), w = (w_1, w_2) \in V_{\mathbb{C}}$. Recall from Theorem 6.7 that

$$\mathcal{H}(g, z, w)) = 2(\mathrm{sech}(H)k_1^{-1}z, k_2w) + (k_1^{-1}z, \tanh(H)k_1^{-1}z) - (\tanh(H)k_2w, k_2w)$$

In our setting, we have

$$\begin{aligned}(L(\cosh\lambda)^{-1}k_1^{-1}z, k_2w) =& (\mathrm{sech}(\lambda)U_1^{-1}z_1, U_2w_1) + (\begin{pmatrix} \mathrm{sech}\lambda & 0 \\ 0 & I_{q-p} \end{pmatrix} V_1^{-1}z_2, V_2w_2) \\ =& Tr(\overline{w_1}^t U_2^t \mathrm{sech}\lambda U_1^t z_1) + Tr(\overline{w_2^t} V_2^t \begin{pmatrix} \mathrm{sech}\lambda & 0 \\ 0 & I_{q-p} \end{pmatrix} V_1^t z_2)\end{aligned} \tag{10.51}$$

To compute $L(\tanh(\lambda))$, first we consider $u = (u_1, u_2)$.

$$\begin{aligned}L(\tanh(\lambda))\begin{pmatrix} Re(u_1) & Im(u_1) \\ Im(u_2) & Re(u_2) \end{pmatrix} &= \begin{pmatrix} 0 & \tanh(\lambda) & 0 \\ \tanh(\lambda) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} Re(u_1) & Im(u_1) \\ Im(u_2) & Re(u_2) \end{pmatrix} \\ &= \begin{pmatrix} (\tanh\lambda, 0)Im(u_2) & (\tanh\lambda, 0)Re(u_2) \\ \begin{pmatrix} \tanh\lambda \\ 0 \end{pmatrix} Re(u_1) & \begin{pmatrix} \tanh\lambda \\ 0 \end{pmatrix} Im(u_1) \end{pmatrix}\end{aligned} \tag{10.52}$$

Thus in terms of $u \in V_{\mathbb{C}}$, we have

$$L(\tanh\lambda)(u) = (i(\tanh\lambda, 0)\overline{u_2}, i\begin{pmatrix} \tanh\lambda \\ 0 \end{pmatrix}\overline{u_1})$$

Now we obtain

$$\begin{aligned}(k_1^{-1}z, L(\tanh\lambda)k_1^{-1}z) =&(U_1^{-1}z_1, i(\tanh\lambda, 0)V_1^{-1}\overline{z_2}) + (V_1^{-1}z_2, i\begin{pmatrix} \tanh\lambda \\ 0 \end{pmatrix}U_1^{-1}\overline{z_1}) \\ =&-2iTr(z_2^tV_1\begin{pmatrix} \tanh\lambda \\ 0 \end{pmatrix}U_1^tz_1)\end{aligned} \tag{10.53}$$

Similarly, we obtain

$$\begin{aligned}(L(\tanh\lambda)k_2w, k_2w) = \overline{(k_2w, L(\tanh\lambda)k_2w)} =&2iTr(\overline{w_2^t}V_2^t\begin{pmatrix} \tanh\lambda \\ 0 \end{pmatrix}U_2\overline{w_1}) \\ =&2iTr(\overline{w_1^t}U_2^t(\tanh\lambda, 0)V_2\overline{w_2})\end{aligned} \tag{10.54}$$

To sum up, we have

$$\begin{aligned}\mathcal{H}(g, z, w) =&2Tr\left(\left(\overline{w_1^t}U_2^t, -iz_2^tV_1\right)\begin{pmatrix} \mathrm{sech}\lambda & -\tanh\lambda & 0 \\ \tanh\lambda & \mathrm{sech}\lambda & 0 \\ 0 & 0 & I_{q-p} \end{pmatrix}\begin{pmatrix} U_1^tz_1 \\ iV_2\overline{w_2} \end{pmatrix}\right) \\ =&2Tr\left(\left(z_1^tU_1, i\overline{w_2^t}V_2^t\right)\begin{pmatrix} \mathrm{sech}\lambda & \tanh\lambda & 0 \\ -\tanh\lambda & \mathrm{sech}\lambda & 0 \\ 0 & 0 & I_{q-p} \end{pmatrix}\begin{pmatrix} U_2\overline{w_1} \\ -iV_1^tz_2 \end{pmatrix}\right) \\ =&2Tr\left(\left(z_1^t, i\overline{w_2^t}\right)\begin{pmatrix} U_1 & 0 \\ 0 & V_2^t \end{pmatrix}\begin{pmatrix} \mathrm{sech}\lambda & \tanh\lambda & 0 \\ -\tanh\lambda & \mathrm{sech}\lambda & 0 \\ 0 & 0 & I_{q-p} \end{pmatrix}\begin{pmatrix} U_2 & 0 \\ 0 & V_1^t \end{pmatrix}\begin{pmatrix} \overline{w_1} \\ -iz_2 \end{pmatrix}\right)\end{aligned} \tag{10.55}$$

We observe that

$$\begin{pmatrix} \mathrm{sech}\lambda & \tanh\lambda & 0 \\ -\tanh\lambda & \mathrm{sech}\lambda & 0 \\ 0 & 0 & I_{q-p} \end{pmatrix} \in O_{p+q}$$

Definition 10.1 *We define $\mathcal{H}_1 : O_{p,q} \to O_{p+q}$ by*

$$\mathcal{H}_1(g) = \begin{pmatrix} U_1 & 0 \\ 0 & V_2^t \end{pmatrix}\begin{pmatrix} \mathrm{sech}\lambda & \tanh\lambda & 0 \\ -\tanh\lambda & \mathrm{sech}\lambda & 0 \\ 0 & 0 & I_{q-p} \end{pmatrix}\begin{pmatrix} U_2 & 0 \\ 0 & V_1^t \end{pmatrix}$$

where $g = k_1 \exp H(\lambda)k_2$, and $k_i = (U_i, V_i) \in O_p \times O_q$.

Thus for $g \in O_{p,q}$, we have

$$\mathcal{H}(g,z,w) = 2Tr\left(\left(z_1^t, i\overline{w_2^t}\right)\mathcal{H}_1(g)\left(\begin{array}{c}\overline{w_1}\\ -iz_2\end{array}\right)\right) \tag{10.56}$$

Therefore, the group action of $(\xi, g) \in \widetilde{O_{p,q}}$ on $\mathcal{H}$ is given by

$$\omega(\xi,g)f(z) = \int_V \xi \exp\left(\frac{1}{2}\left(z_1^t, i\overline{w_2^t}\right)\mathcal{H}_1(g)\left(\begin{array}{c}\overline{w_1}\\ -iz_2\end{array}\right)\right) f(w)d\mu(w)$$

Since w_1, w_2, z_1, z_2 can all be chosen arbitrarily, $\mathcal{H}(g,z,w)$ determines $\mathcal{H}_1(g)$ uniquely, and vice versa. From Theorem 6.7 and Theorem 6.5, we see that $\mathcal{H}_1$ is well-defined and injective.

Let O_p^o, O_q^o be the opposite group of O_p, O_q. We define a group involution

$$\tau : O_p \times O_q \times O_p^o \times O_q^o \rightarrow O_p \times O_q \times O_p^o \times O_q^o$$

by $\tau(U_1, V_1, U_2, V_2) = (U_1, V_2^t, U_2, V_1^t)$. Then we may identify $O_p \times O_q \times O_p^o \times O_q^o$ with $O_p \times O_q \times O_p^o \times O_q^o$ through τ. In that sense $\mathcal{H}_1$ is a $O_p \times O_q \times O_p^o \times O_q^o$-equivariant map.

10.3 The Compactification $\mathcal{H}_1$

We shall prove here that $\mathcal{H}_1$ is an analytic compactification of $O_{p,q}$. Let $\mathbb{T}_p$ be a compact torus consisting of elements of the following form

$$T(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & I_{q-p}\end{pmatrix} \qquad (\theta \in (-\pi, \pi]^p)$$

For each θ_i, we may define an element

$$\begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix} \in \mathbb{T}_1$$

Then $\mathbb{T}_p$ can be identified with direct product of p copies of $\mathbb{T}_1$. We set

$$\begin{pmatrix} \text{sech}\lambda & \tanh\lambda & 0 \\ -\tanh\lambda & \text{sech}\lambda & 0 \\ 0 & 0 & I_{q-p}\end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & I_{q-p}\end{pmatrix}$$

Since $\cos\theta_i(\lambda_i) = \text{sech}\lambda_i > 0$, $\theta_i(\lambda_i)$ can be regarded as a smooth homeomorphism from $\mathbb{R}^1$ to $(-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore θ can be regarded as a smooth homeomorphism from $\mathbb{R}^p$ to $(-\frac{\pi}{2}, \frac{\pi}{2})^p$. We observe that $\mathcal{H}_1$ is a "map" from the "reductive symmetric pair" $(O_{p,q}, O_p \times O_q)$ (of noncompact type) to the "reductive symmetric pair" $(O_{p+q}, O_p \times O_q)$ (of compact type). We would like to see if $\mathcal{H}_1$ is an analytic compactification of $O_{p,q}$. We recall some definitions and basic facts about symmetric spaces from [Helgason] Ch 4.3.

Definition 10.2 (symmetric pair) *Let G be a connected reductive Lie group, H a closed subgroup. Let σ be an involution of G such that*

$$(G^\sigma)_0 \subseteq H \subseteq G^\sigma$$

where G^σ is the fixed point set of σ, $(G^\sigma)_0$ the identity component of G^σ. The pair (G, H) is called a reductive symmetric pair. If $Ad_G(H)$ is compact, (G, H) is said to be a Riemannian reductive symmetric pair.

We will only be interested in Riemannian symmetric pairs and Riemannian symmetric spaces. According to [Helgason] Ch 4.3, a Riemannian reductive symmetric pair yields a Riemannian globally symmetric space G/K, and every Riemannian reductive globally symmetric space can be obtained from a Riemannian reductive symmetric pair.

Definition 10.3 (Weyl group) *Let (G, K) be a Riemannian reductive symmetric pair. Let $<,>$ be an invariant real symmetric bilinear form on $\mathfrak{g}$ such that $(,)_{\mathfrak{k}}$ is negative definite. Let $\mathfrak{p} = \mathfrak{k}^\perp$. Let $\mathfrak{h}_{\mathfrak{p}}$ be a maximal Abelian subspace of $\mathfrak{p}$. Let $H = \exp \mathfrak{h}_{\mathfrak{p}}$ be the corresponding Abelian subgroup. Let M, M' be the centralizer and normalizer of $\mathfrak{h}_{\mathfrak{p}}$ in K respectively. In other words,*

$$M = \{k \in K \mid Ad(k)h = h \ \forall h \in \mathfrak{h}_{\mathfrak{p}}\}$$

$$M' = \{k \in K \mid Ad(k)\mathfrak{h}_{\mathfrak{p}} \subseteq \mathfrak{h}_{\mathfrak{p}}\}$$

$W(G, K) = M'/M$ is called the Weyl group of (G, K).

Definition 10.4 (Regular and Singular Points) *Let (G, K) be a Riemannian reductive symmetric pair, and $X = G/K$. Let $\Sigma(\mathfrak{g}, \mathfrak{h}_{\mathfrak{p}})$ be the root system, Σ^+ positive roots of Σ. We define*

$$\Phi : K/M \times H \to G/K$$

as follows.

$$\forall\ [k] \in K/M, h \in H, \qquad \Phi([k], h) = [Ad(k)h]_K$$

Then for a generic point $x \in G/K$, $\Phi^{-1}(x)$ is finite. We call such a point regular. We use X_r to denote the set of regular points. If $\Phi^{-1}(x)$ is not finite, we say x is singular.

Theorem 10.1 (Symmetric decomposition) *Every Riemannian reductive symmetric pair (G, K) induces a decomposition of G into KHK. For an arbitrary $x \in G$, $H(x)$ is unique up to a conjugation of $W(G, K)$ and a multiplication of $K \cap H$.*

Most of the proof can be found in [Helgason2] Ch 1.5, section 2 and [Helgason] Ch 7.3. and [Helgason3] Ch 7.8. Notice for G noncompact, this decomposition is nothing more than KAK decomposition, and the results are well-known. In all cases, we will use $d_G g$ to denote the Haar measure of G, and $d_G x$ to denote a fixed G−invariant measure of X.

Now back to our pair $(O_{p+q}, O_p \times O_q)$, let $S(O_p \times O_q)$ be the normal subgroup of $O_p \times O_q$ with determinant 1. Then we obtain an injective map

$$SO_{p+q}/S(O_p \times O_q) \to O_{p+q}/O_p \times O_q$$

It is not difficult to see that this map is in fact surjective. Thus we may identify $O_{p+q}/O_p \times O_q$ with $SO_{p+q}/S(O_p \times O_q)$. The symmetric decomposition holds for $(O_{p+q}, O_p \times O_q)$-pair. We fix $H = \mathbb{T}_p$ and $K = O_p \times O_q$. Observe that

- $H \cap K \cong (\mathbb{Z}/2\mathbb{Z})^p$. More explicitly, let $A = diag(\pm 1, \pm 1, \dots, \pm 1) \in O_p$. Then $diag(A, A, I_{q-p}) \in H \cap K$.
- $W(O_{p+q}, O_p \times O_q)$ acts on $\mathbb{T}_p \cong \mathbb{T}_1 \times \mathbb{T}_1 \times \dots \times \mathbb{T}_1$ by permutations and transposes on each factor $\mathbb{T}_1$.

Theorem 10.2 ($K\mathbb{T}_pK$ decomposition) *Every $g \in O_{p+q}$ can be decomposed into $k_1 T(\theta(g)) k_2$, such that*

$$\pi/2 \geq \theta_1(g) \geq \theta_2(g) \geq \dots \theta_p(g) \geq 0$$

Proof: First of all g can be decomposed into $k_1 T(\theta) k_2$. Applying a multiplication of $H \cap K$ on $T(\theta)$, we may assume that $\cos(\theta_i) \geq 0$, i.e.,

$$\pi/2 \geq \theta_i \geq -\pi/2 \qquad (i = 1 \dots, p)$$

Again, applying a conjugation by $W(O_{p+q}, O_p \times O_q)$, we may assume that

$$\pi/2 \geq \theta_1 \geq \theta_2 \geq \dots \geq \theta_p \geq 0$$

Q.E.D.

Now the image of $\mathcal{H}_1$ consists of

$$\{k_1 T(\theta) k_2 \mid k_1, k_2 \in O_p \times O_q, \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})^p\}$$

Since the set $\{T(\theta) \mid \theta_i \in [0, \frac{\pi}{2})\}$ is already dense in

$$\{T(\theta) \mid \pi/2 \geq \theta_1(g) \geq \theta_2(g) \geq \dots \theta_p(g) \geq 0\}$$

according to the $K\mathbb{T}_pK$ decomposition, $\mathcal{H}_1(O_{p,q})$ is dense in O_{p+q}. We may further prove that

Theorem 10.3 *$\mathcal{H}_1$ is an analytic compactification from $O_{p,q}$ to O_{p+q}.*

Proof: Let $(\mathcal{H}, \mathcal{S})$ be the compactification of $Sp(V, \Omega)$ as defined by Equation 6.15. Then we have for $g \in O_{p,q}$

$$(iz^t, \overline{w^t}) \mathcal{H}(g) \begin{pmatrix} iz \\ \overline{w} \end{pmatrix} = 2Tr(z_1^t, i\overline{w_2^t}) \mathcal{H}_1(g) \begin{pmatrix} \overline{w_1} \\ -iz_2 \end{pmatrix}$$

Therefore there exists a smooth embedding

$$i : O_{p+q} \to \mathcal{S}$$

such that the following diagram commutes

$$\begin{array}{ccc} O_{p,q} & \xrightarrow{L} & Sp(V,\Omega) \\ \downarrow{\scriptstyle \mathcal{H}_1} & & \downarrow{\scriptstyle \mathcal{H}} \\ O_{p+q} & \xrightarrow{i} & \mathcal{S} \end{array} \tag{10.57}$$

Now, $i(O_{p+q})$ is automatically a closed analytic subvariety of $\mathcal{S}$. Thus the closure of $\mathcal{H}(L(O_{p,q}))$ is exactly $i(O_{p+q})$. Therefore according to Theorem 6.12, $\mathcal{H}_1$ is an analytic compactification of $O_{p,q}$. Q.E.D.

Now we summarize some properties of $\mathcal{H}_1$ which we are going to use in the following corollary.

Corollary 10.1 *The closure of $\mathcal{H}_1(SO_{p,q})$ is SO_{p+q}. Therefore SO_{p+q} is an analytic compactification of $SO_{p,q}$.*

Corollary 10.2 *Let $d_{O_{p,q}}g$ and $d_{O_{p+q}}g$ be the Haar measure of $O_{p,q}$ and O_{p+q}. Then under the compactification $\mathcal{H}_1$, for every $g \in O_{p,q}$,*

$$\frac{d\mathcal{H}_1(g)}{dg} \neq 0$$

where

$$d_{O_{p+q}}\mathcal{H}_1(g) = \frac{d\mathcal{H}_1(g)}{dg} d_{O_{p,q}}g$$

Same holds for $SO_{p,q}$.

Notice here, if we regard the Haar measures of $O_{p,q}$ and O_{p+q} as invariant volume forms, then $\frac{d\mathcal{H}_1(g)}{dg}$ is simply the ratio (a function) between the pull back of the invariant volume form of O_{p+q} and the invariant volume form of $O_{p,q}$. It is not equal to zero at any point since $d\mathcal{H}_1$ is nondegenerate. In the future, we will denote such a function by $\det \mathcal{H}_1$. Of course, for the inverse, we will simply have

$$d_{O_{p,q}}g = (\det \mathcal{H}_1^{-1}) d_{O_{p+q}}(\mathcal{H}_1(g))$$

Here

$$(\det \mathcal{H}_1^{-1})(\mathcal{H}_1(g)) = (\det \mathcal{H}_1(g))^{-1} \qquad (g \in O_{p,q})$$

10.4 Nonvanishing theorem

Recall that the dual pair correspondence is a one-to-one correspondence between $\mathcal{R}(\widetilde{O_{p,q}}, \omega)$ and $\mathcal{R}(\widetilde{Sp_{2n}(\mathbb{R})}, \omega)$. We have shown that for π in the semistable range $\mathcal{R}_{ss}(O_{p,q}, Sp_{2n}(\mathbb{R}))$, if $\mathcal{L}_{\widetilde{O_{p,q}}}(\mathcal{P}^c \otimes V_\pi)$ is not vanishing, then $\pi \in \mathcal{R}(\widetilde{O_{p,q}}, \omega)$. Of course, one can easily see that

$$\mathcal{L}_{\widetilde{O_{p,q}}}(\mathcal{P}^c \otimes V_\pi) = 0 \Leftrightarrow \forall\ \psi, \phi \in \mathcal{P}, u, v \in V_\pi, \int_{\widetilde{O_{p,q}}} (\phi, \omega(g)\psi)(\pi(g)v, u) dg = 0$$

Through the preparation in the last few sections, we are ready to study this bilinear form $(,)_\pi$ in details.

Let us look at the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & SO_{p,q} & \longrightarrow & O_{p,q} & \xrightarrow{\det} & \{\pm 1\} & \longrightarrow & 1 \\ \| & & \pi \uparrow & & \pi \uparrow & & & & \\ 1 & \longrightarrow & \widetilde{SO_{p,q}} & \longrightarrow & \widetilde{O_{p,q}} & \xrightarrow{\chi} & \{\pm 1\} & \longrightarrow & 1 \end{array} \tag{10.58}$$

Since $SO_{p,q}$ is a normal subgroup of $O_{p,q}$, $\widetilde{SO_{p,q}}$ is a normal subgroup of $\widetilde{O_{p,q}}$. This uniquely defines a character

$$\chi : \widetilde{O_{p,q}} \to \pm 1$$

We begin with the following lemma.

Lemma 10.1 *The following are equivalent.*
1) $\mathcal{L}_{\widetilde{O_{p,q}}}(\mathcal{P}^c \otimes V_\pi) \neq 0$ *or* $\mathcal{L}_{\widetilde{O_{p,q}}}(\mathcal{P}^c \otimes V_{\pi \otimes \chi}) \neq 0$
2) $\mathcal{L}_{\widetilde{SO_{p,q}}}(\mathcal{P}^c \otimes V_\pi) \neq 0$

Proof: Since χ is a central character, we may regard the Harish-Chandra module $V_{\pi\otimes\chi}$ as V_π with the same underlying space but different actions.

1. We will prove that $\mathcal{L}_{\widetilde{SO_{p,q}}}(\mathcal{P}^c \otimes V_\pi) = 0$ if and only if $\mathcal{L}_{\widetilde{O_{p,q}}}(\mathcal{P}^c \otimes V_\pi) = 0$ and $\mathcal{L}_{\widetilde{O_{p,q}}}(\mathcal{P}^c \otimes V_{\pi\otimes\chi}) = 0$. Let g_0 be an arbitrary element in $\widetilde{O_{p,q}}$ but not in $\widetilde{SO_{p,q}}$, $\psi, \phi \in \mathcal{P}$ and $v \in V_\pi$. Then we have

$$\begin{aligned} \mathcal{L}_{\widetilde{O_{p,q}}}(\psi \otimes v)(\phi) &= \int_{\widetilde{O_{p,q}}} (\phi, \omega(g)\psi)(\pi(g)v) dg \\ &= \int_{\widetilde{SO_{p,q}}} (\phi, \omega(g)\psi)(\pi(g)v) + (\phi, \omega(g)\omega(g_0)\psi)(\pi(g)\pi(g_0)v) dg \\ &= \mathcal{L}_{\widetilde{SO_{p,q}}}(\psi \otimes v)(\phi) + \mathcal{L}_{\widetilde{SO_{p,q}}}(\omega(g_0)\psi \otimes \pi(g_0)v)(\phi) \end{aligned} \tag{10.59}$$

2. Similarly, for $\pi \otimes \chi$, $v \in V_{\pi \times \chi}$, we have

$$\begin{aligned}
\mathcal{L}_{\widetilde{O_{p,q}}}(\psi \otimes v)(\phi) &= \int_{\widetilde{O_{p,q}}} (\phi, \omega(g)\psi)(\pi \otimes \chi)v \\
&= \int_{\widetilde{SO_{p,q}}} (\phi, \omega(g)\psi)(\pi \otimes \chi)(g)v + (\phi, \omega(g)\omega(g_0)\psi)((\pi \otimes \chi)(g)(\pi \otimes \chi)(g_0)v)dg \\
&= \int_{\widetilde{SO_{p,q}}} (\phi, \omega(g)\psi)\pi(g)v - (\phi, \omega(g)\omega(g_0)\psi)(\pi(g)\pi(g_0)v)dg \\
&= \mathcal{L}_{\widetilde{SO_{p,q}}}(\psi \otimes v)(\phi) - \mathcal{L}_{\widetilde{SO_{p,q}}}(\omega(g_0)\psi \otimes \pi(g_0)v)(\phi)
\end{aligned} \tag{10.60}$$

3. Suppose that $\mathcal{L}_{\widetilde{SO_{p,q}}}(\mathcal{P}^c \otimes V_\pi) = 0$. Then from the computation above, we have

$$\mathcal{L}_{\widetilde{O_{p,q}}}(\mathcal{P}^c \otimes V_\pi) = 0$$

$$\mathcal{L}_{\widetilde{O_{p,q}}}(\mathcal{P}^c \otimes V_{\pi \otimes \chi}) = 0$$

4. Suppose that

$$\mathcal{L}_{\widetilde{O_{p,q}}}(\mathcal{P}^c \otimes V_\pi) = 0$$

$$\mathcal{L}_{\widetilde{O_{p,q}}}(\mathcal{P}^c \otimes V_{\pi \otimes \chi}) = 0$$

Then we will have

$$2\int_{\widetilde{SO_{p,q}}} (\phi, \omega(g)\psi)(\pi(g)v)dg = \int_{\widetilde{O_{p,q}}} (\phi, \omega(g)\psi)\pi(g)vdg + \int_{\widetilde{O_{p,q}}} (\phi, \omega(g)\psi)(\pi \otimes \chi)vdg = 0$$

Thus

$$\mathcal{L}_{\widetilde{SO_{p,q}}}(\mathcal{P}^c \otimes V_\pi) = 0$$

Q.E.D.

Of course, there may be much easier but more abstract way to prove this lemma by using the direct sum decomposition with respect to $\widetilde{SO_{p,q}}$. We chose this proof just to illuminate what is behind the abstract approach. Now this lemma allows us to reduce the study of $\mathcal{L}_{\widetilde{O_{p,q}}}$ to the study $\mathcal{L}_{\widetilde{SO_{p,q}}}$. For a measure space (X, μ), let $L^1(X, d\mu)$ be the space of integrable function on X.

Theorem 10.4 (Nonvanishing Theorem) *Suppose π is in the semistable range $\mathcal{R}_{ss}(O_{p,q}, Sp_{2n}(\mathbb{R}))$. If $n \geq \frac{p+q-1}{2}$, then*

$$\mathcal{L}_{\widetilde{SO_{p,q}}}(\mathcal{P}^c \otimes V_\pi) \neq 0$$

Thus either $\pi \in \mathcal{R}(\widetilde{O_{p,q}}, \omega)$ or $\pi \otimes \chi \in \mathcal{R}(\widetilde{O_{p,q}}, \omega)$. In addition the dual representation under the dual pair correspondence can be constructed through $\mathcal{L}_{\widetilde{O_{p,q}}}$.

Proof: We will prove this theorem by contradiction. We write $G = SO_{p,q}$.

1. Let $\widetilde{g}$ be an element of $\widetilde{G}$. We can write $\widetilde{g} = (\xi, g)$ with $\det_{\mathbb{C}}(C_{L(g)}) = \xi^{-2}$ and $g \in SO_{p,q}$. Suppose $\mathcal{L}_{\widetilde{G}}(\mathcal{P}^c \otimes V_\pi) = 0$. In other words, we have

$$\forall \ \ \alpha, \beta \in \mathbb{N}^{n(p+q)}, u, v \in V_\pi$$

$$\int_{\widetilde{G}} (z^\beta, \omega(\widetilde{g}) z^\alpha)(\pi(\widetilde{g})u, v) d\widetilde{g} = 0$$

Taking conjugation, we have

$$\begin{aligned} 0 &= \int_{\widetilde{G}} (\omega(\widetilde{g}) z^\alpha, z^\beta)(v, \pi(\widetilde{g})u) d\widetilde{g} \\ &= \int_{\widetilde{G}} \int_{z,w \in \mathbb{C}^{n(p+q)}} K(z, w, \widetilde{g}) w^\alpha \overline{z}^\beta d\mu(z) d\mu(w) d\widetilde{g} \end{aligned} \tag{10.61}$$

where

$$\begin{aligned} K(z, w, \widetilde{g}) &= \xi(v, \pi(\widetilde{g})u) \exp\left(\frac{1}{2} Tr(z_1^t, i\overline{w_2}^t) \mathcal{H}_1(\widetilde{g}) \begin{pmatrix} \overline{w_1} \\ -iz_2 \end{pmatrix}\right) \\ &= F(\widetilde{g}) W(z, w, g) \\ F(\widetilde{g}) &= \xi(v, \pi(\widetilde{g})u) \\ W(z, w, g) &= \exp\left(\frac{1}{2} Tr(z_1^t, i\overline{w_2}^t) \mathcal{H}_1(g) \begin{pmatrix} \overline{w_1} \\ -iz_2 \end{pmatrix}\right) \end{aligned} \tag{10.62}$$

Notice that $\epsilon(\xi, g) = (-\xi, g)$. Since we always assume that $\pi(\epsilon) = -1$,

$$F(\widetilde{g}) = \xi(v, \pi(\widetilde{g})u) = (-\xi)(v, \pi(\epsilon\widetilde{g})u) = F(\epsilon\widetilde{g})$$

Thus $F(\widetilde{g})$ and $K(z, w, \widetilde{g})$ can be regarded as functions on $SO_{p,q}$, we will write as $F(g)$ and $K(z, w, g)$.

2. Claim: The integration $d\mu(z)d\mu(w)$ and dg in Equation 10.61 are interchangeable.

From Fubini's theorem, it suffices to show that $K(z, w, g) w^\alpha \overline{z}^\beta$ is integrable with respect to $d\mu(z)d\mu(w)dg$. Since $\mathcal{H}_1(g) \in SO_{p+q}$, we have

$$\begin{aligned} & \| Tr(z_1^t, i\overline{w_2}^t) \mathcal{H}(g) \begin{pmatrix} \overline{w_1} \\ -iz_2 \end{pmatrix} \| \\ \leq & \|(z_1, iw_2)\| \|(\overline{w_1}, -iz_2)\| \\ = & \sqrt{\|z_1\|^2 + \|w_2\|^2} \sqrt{\|w_1\|^2 + \|z_2\|^2} \\ \leq & \frac{1}{2} (\|z_1\|^2 + \|w_2\|^2 + \|z_2\|^2 + \|w_1\|^2) \end{aligned} \tag{10.63}$$

Therefore,

$$\|W(z,w,g)\| \leq \exp\frac{1}{4}(\|z\|^2 + \|w\|^2)$$

Hence $W(z,w,g)w^\alpha\overline{z}^\beta$ is integrable with respect to $d\mu(z)d\mu(w)$. On the other hand, suppose that $g = k_1 \exp H(\lambda) k_2$ with $H(\lambda) \in \mathfrak{a}^+$. Recall that

$$C_{L(\exp H(\lambda))} = L\left(\begin{pmatrix} \cosh\lambda & 0 & 0 \\ 0 & \cosh\lambda & 0 \\ 0 & 0 & I_{q-p} \end{pmatrix}\right)$$

Therefore

$$\det_{\mathbb{C}}(C_{L(\exp(H(\lambda)))}) = \prod_{i=1}^{p} \cosh(\lambda_i)^n \prod_{i=1}^{p} \cosh(\lambda_i) = \prod_{i=1}^{p} \cosh(\lambda_i)^{2n}$$

Hence

$$\|\xi\| = \prod_{i=1}^{p} \cosh(\lambda_i)^n \leq C \prod_{i=1}^{p} \exp(-n\lambda_i) = C\exp(-nH(\lambda))$$

Because of the semistable condition, we have

$$|F(g)| = |\xi(v, \pi(\widetilde{g})u)| \in L^1(SO_{p,q}, d_{SO_{p,q}}g)$$

Thus

$$|W(z,w,g)F(g)w^\alpha\overline{z}^\beta| = |K(z,w,g)w^\alpha\overline{z}^\beta| \in L^1(d\mu(z)d\mu(w)dg)$$

We have

$$0 = \int_{\mathbb{C}^{n(p+q)}\times\mathbb{C}^{n(p+q)}} \left(\int_G K(z,w,g)w^\alpha\overline{z}^\beta dg\right)d\mu(z)d\mu(w)$$

3. Claim: $\int_G K(z,w,g)dg$ is holomorphic with respect to z and antiholomorphic with respect to w.

It suffices to show that $\int_G K(z,w,g)dg$ is infinitely differentiable with respect to z and $\overline{w}$. Notice that $K(z,w,g)$ is infinitely differentiable with respect to z and $\overline{w}$. We shall examine the following equation

$$\frac{d}{dz_i}\int_G K(z,w,g)dg = \int_G \frac{d}{dz_i}K(z,w,g)dg$$

Here z_i is a single complex variable in z.
For (z,w) in a compact set, $\exists\ C$ such that

$$\begin{aligned} \left|\frac{d}{dz_i}K(z,w,g)\right| &= \left|F(g)\frac{d}{dz_i}W(z,w,g)\right| \\ &\leq |F(g)|\|(z,w)\|\exp\frac{1}{4}\|(z,w)\|^2 \\ &\leq C|F(g)| \\ \|(z,w)\| &= \sqrt{\|z\|^2 + \|w\|^2} \end{aligned} \tag{10.64}$$

But

$$|F(g)| \in L^1(SO_{p,q}, d_{SO_{p,q}}g)$$

By the dominated convergence theorem, the integration and differentiation are interchangeable. And similarly, we can show that $\int_G K(z,w,g)dg$ is infinitely differentiable with respect to z and $\overline{w}$.

4. Claim: For each $z, w \in V$, $\int_G K(z,w,g)dg = 0$.

   Notice that $\int_G K(z,w,g)dg$ possesses a power series expansion, namely

$$\int_G K(z,w,g)dg = \sum k(\lambda,\mu) z^\lambda \overline{w}^\mu$$

   We obtain $\forall \;\; \alpha, \beta \in \mathbb{N}^{(p+q)n}$

$$0 = \int_{\mathbb{C}^{n(p+q)} \times \mathbb{C}^{n(p+q)}} (\int_G K(z,w,g)dg) w^\alpha \overline{z}^\beta d\mu(z) d\mu(w) = \sum_{\lambda,\mu} \int_G k(\lambda,\mu) w^\alpha \overline{z}^\beta \overline{w}^\mu z^\lambda d\mu(z) d\mu(w)$$

   The interchangeability can be guaranteed by an integrable function which dominates the power series

$$\sum_{\lambda,\mu} k(\lambda,\mu) w^\alpha \overline{z}^\beta \overline{w}^\mu z^\lambda$$

   We choose

$$\exp \frac{1}{4}(\|z\|^2 + \|w\|^2)(\int_G |F(g)|dg)|w^\alpha||\overline{z}^\beta| \in L^1(\mathbb{C}^{n(p+q)} \times \mathbb{C}^{n(p+q)}, d\mu(z)d\mu(w))$$

   Then, according to orthogonality of the basis $\{w^\alpha, z^\beta\}$ (see [Bargmann]), we have

$$\forall \; \alpha, \beta, \;\; k(\alpha,\beta) = 0$$

   This implies that $\int_G K(z,w,g)dg = 0$.

5. Now, we fix (z,w). We have

$$\begin{aligned} 0 &= \int_G K(z,w,g)dg \\ &= \int_G F(g) \exp\left(\frac{1}{2} Tr(z_1^t, i\overline{w_2}^t) \mathcal{H}_1(g) \begin{pmatrix} \overline{w_1} \\ -iz_2 \end{pmatrix}\right) dg \\ &= \int_{SO_{p+q}} F(\mathcal{H}_1^{-1}(g)) \exp\left(\frac{1}{2} Tr(z_1^t, i\overline{w_2}^t) g \begin{pmatrix} \overline{w_1} \\ -iz_2 \end{pmatrix}\right) \frac{d\mathcal{H}_1^{-1}(g)}{dg} dg \\ &= \int_{SO_{p+q}} \tilde{F}(g) \exp\left(\frac{1}{2} Tr(z_1^t, i\overline{w_2}^t) g \begin{pmatrix} \overline{w_1} \\ -iz_2 \end{pmatrix}\right) dg \end{aligned} \tag{10.65}$$

where $\tilde{F}(g) = F(\mathcal{H}_1^{-1}(g))\frac{d\mathcal{H}_1^{-1}(g)}{dg}$ is a function defined on $\mathcal{H}_1(G)$. $\frac{d\mathcal{H}_1^{-1}(g)}{dg} = \det \mathcal{H}^{-1}$ is continuous and positive on $\mathcal{H}_1(G)$. Thus $\widetilde{F}(g)$ is continuous on $\mathcal{H}(G)$. Moreover

$$F(g) = \widetilde{F}(\mathcal{H}_1(g))\frac{d\mathcal{H}(g)}{dg} \qquad (g \in G)$$

We will use this fact in the proof. Let $z = 0, w = 0$. Then we have

$$\tilde{F}(g) \in L^1(G, dg)$$

Finally, one may "polarize" Equation 10.65 by changing (z_1, w_2) to $(sz_1, s_w 2)$ where $s \in \mathbb{R}^+$. We obtain

$$0 = \int_{SO_{p+q}} \tilde{F}(g) \exp\left(s\frac{1}{2}Tr(z_1^t, i\overline{w_2}^t)g\begin{pmatrix} \overline{w_1} \\ -iz_2 \end{pmatrix}\right) dg \qquad (s \in \mathbb{R}^+) \tag{10.66}$$

6. Claim: $\forall m,\ \int_{SO_{p+q}} \tilde{F}(g)(Tr(z_1^t, i\overline{w_2}^t)g\begin{pmatrix} \overline{w_1} \\ -iz_2 \end{pmatrix})^n dg = 0$
It suffices to prove that

Lemma 10.2 *Suppose X is compact and μ is a Borel measure on X. If $\phi \in L^1(X, d\mu)$, $\psi \in C(X)$, such that*

$$\forall\ s \geq 0; \qquad \int_X \phi(x) \exp s\psi(x) d\mu = 0$$

then

$$\forall\ m \qquad \int_X \phi(x)\psi(x)^m d\mu(x) = 0$$

Proof: Since we have

$$\exp s\psi(x) = \sum \frac{s^m \psi(x)^m}{m!}$$

and this power series is bounded absolutely by $\exp(s \max(\|\psi(x)\|))$, we can interchange the power series expansion with integration

$$0 = \int_X \phi(x) \sum \frac{s^m}{m!}\psi(x)^m dx = \sum \frac{s^m}{m!} \int_X \phi(x)\psi(x)^m d\mu(x)$$

This implies that for all m,

$$\int_X \phi(x)\psi(x)^m d\mu(x) = 0$$

Q.E.D.

By applying this lemma to Equation 10.66, we obtain

$$\forall\ m \in \mathbb{N} \qquad \int_{SO_{p+q}} \tilde{F}(g)(Tr(z_1^t, i\overline{w_2}^t)g\begin{pmatrix} \overline{w_1} \\ -iz_2 \end{pmatrix})^m dg = 0$$

7. Now we quote the density theorem which we will prove in the next section.

Theorem 10.5 (Density Theorem) *Suppose $2n+1 \geq p+q$. Then the linear span of*

$$\{\left(Tr(z_1^t, i\overline{w_2}^t)g\begin{pmatrix} \overline{w_1} \\ -iz_2 \end{pmatrix}\right)^m \mid z, w \in V, m \in \mathbb{N}\}$$

is equal to $\mathcal{O}_{SO_{p+q}}$.

From this density theorem, we see that the linear span of

$$\{\left(Tr(z_1^t, i\overline{w_2}^t)g\begin{pmatrix} \overline{w_1} \\ -iz_2 \end{pmatrix}\right)^m\}_{z,w,m}$$

is dense in $C(SO_{p+q})$ under the uniform norm. Thus we have

$$\forall\, \psi \in \mathcal{C}(SO_{p+q}), \quad \int \tilde{F}(g)\psi(g)dg = 0$$

This implies

$$\tilde{F}(g) = 0 \ a.e.$$

Otherwise, suppose the measure of $\{g \in SO_{p+q} \mid \tilde{F}(g) \neq 0\}$ is not zero. Then we may construct a C^∞ compactly supported function ϕ, such that

$$\int_{SO_{p+q}} \phi(g)\tilde{F}(g)dg \neq 0$$

Therefore $\widetilde{F}(G)$ is zero almost everywhere. But $\tilde{F}$ is continuous on $\mathcal{H}_1(G)$. Hence $\tilde{F} = 0$ on $\mathcal{H}_1(G)$. Then for every $g \in G$

$$F(g) = \tilde{F}(\mathcal{H}_1(g))\frac{d\mathcal{H}_1(g)}{dg} = 0$$

Recall that for $\widetilde{g} = (\xi, g)$, $F(g) = \xi(v, \pi(\widetilde{g})u)$ and $\xi \neq 0$. Therefore

$$\forall\, \widetilde{g} \in \widetilde{G}, \qquad (v, \pi(\widetilde{g})u) = 0$$

Since u, v are arbitrary, this contradicts the fact that π is a representation of $\widetilde{G}$. Thus

$$\mathcal{L}_{\widetilde{SO_{p,q}}}(\mathcal{P}^c \otimes \pi) \neq 0$$

8. By the last lemma, we have either

$$\mathcal{L}_{\widetilde{O_{p,q}}}(\mathcal{P}^c \otimes \pi) \neq 0 \qquad or \qquad \mathcal{L}_{\widetilde{O_{p,q}}}(\mathcal{P}^c \otimes \pi \otimes \chi) \neq 0$$

Thus either $\pi \in \mathcal{R}(\widetilde{O_{p,q}}, \omega)$ or $\pi \otimes \chi \in \mathcal{R}(\widetilde{O_{p,q}}, \omega)$. Furthermore, if $\pi \in \mathcal{R}(\widetilde{O_{p,q}}, \omega)$, the corresponding $\omega(\pi)$ can be constructed by taking dual of $\mathcal{L}_{\widetilde{O_{p,q}}}(\mathcal{P}^c \otimes V_\pi)$ in the category of Harish-Chandra modules. The same is true if $\pi \otimes \chi \in \mathcal{R}(\widetilde{O_{p,q}}, \omega)$.

Q.E.D.

In fact, we have proved that if $F(\widetilde{g})$ is a continuous function of $\widetilde{G}$, $F(\epsilon\widetilde{g}) = -F(\widetilde{g})$, and

$$\int_{\widetilde{G}} (\omega(\widetilde{g})z^{\alpha}, z^{\beta})F(\widetilde{g})d\widetilde{g} = 0 \qquad (\forall\ \alpha, \beta \in \mathbb{N}^{(p+q)n})$$

then $F(\widetilde{g}) \equiv 0$.

10.5 Density Theorem

Now we would like to prove the density theorem. Let $\mathcal{O}_{SO_{p+q}}$ be the space of regular functions on SO_{p+q}. Let SO_{p+q} act on $\mathcal{O}_{SO_{p+q}}$ by left translation. Then $\mathcal{O}_{SO_{p+q}}$ is the space of SO_{p+q}-finite functions on SO_{p+q}. For every $X, Y \in Mat(p+q, n)$, we define a function

$$F_{X,Y}(g) = Tr(X^t gY), \qquad (g \in SO_{p+q})$$

in $C(SO_{p+q})$. Of course $F_{X,Y}$ can also be regarded as a function on $Mat(p+q, p+q)$. Here the base field can be either $\mathbb{R}$ or $\mathbb{C}$. Let R_n be the linear span of

$$\{F^i_{X,Y} \mid X, Y \in Mat(p+q, n), i \in \mathbb{N}\}$$

Since we can define a filtration,

$$Mat(p+q, n) \hookrightarrow Mat(p+q, n+1)$$

by mapping $Mat(p+q, n)$ into the first n columns of $Mat(p+q, n+1)$, and seting the last column to be zero. This induces a natural filtration

$$R_1 \subseteq \ldots \subseteq R_n \subseteq \ldots \subseteq \mathcal{O}_{SO_{p+q}}$$

On the other hand, if $n = p+q$, then $\{F_{X,Y} \mid X, Y \in Mat(p+q, p+q)\}$ as functions on $Mat(p+q, p+q)$ exhaust all the linear functions on $Mat(p+q, p+q)$. Therefore

$$\{F^i_{X,Y} \mid X, Y \in Mat(p+q, p+q), i \in \mathbb{N}\}$$

spans the space of regular functions on $Mat(p+q, p+q)$. Thus

$$R_{p+q} = \mathcal{O}_{SO_{p+q}}$$

Therefore there exists an "n" such that

$$R_n = \mathcal{O}_{SO_{p+q}}; \qquad R_{n-1} \neq \mathcal{O}_{SO_{p+q}}$$

We will restate the density theorem.

Theorem 10.6 (Density Theorem) *If* $2n+1 \geq p+q$, *then* $R_n = \mathcal{O}_{SO_{p+q}}$.

We will use mainly the highest weight theory to prove this theorem. Let us first recall a lemma [Kumar], also known as the Parthasarathy-Ranga Rao-Varadarajan Conjecture.

Lemma 10.3 (Kumar) *Let V_λ and V_μ be the irreducible representations of the complex semisimple Lie algebra $\mathfrak{g}$ with highest weight λ and μ respectively. Let $W(\mathfrak{g})$ be the Weyl group of $\mathfrak{g}$. Let w_1 be an arbitrary element in W, and $w_2 \in W$ such that $\eta = w_2(\lambda + w_1(\mu))$ is dominant. Then $V_\lambda \otimes V_\mu$ contains an irreducible subrepresentation V_η of highest weight η, i.e.,*

$$V_\eta \subseteq V_\lambda \otimes V_\mu$$

There is actually a stronger conjecture of Kostant, also proved by Kumar, about the multiplicity of V_η. Since our argument will be based on Kumar's lemma and highest weight computation, we will not give a special name for this chosen V_η in $V_\lambda \otimes V_\mu$. Since the highest weights are different in type B and type D groups, we will treat them differently. We will always use $S(V)$ to denote the symmetric algebra of V and $S^i(V)$ to denote the $i-$th symmetric power of V.

Theorem 10.7 **(SO_{2m+1})** *Let $\mathbb{C}^{2m+1}$ be the standard representation of SO_{2m+1}. We choose the standard Cartan subgroup and dominant chamber $\{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0\}$. Then every irreducible representation V_λ of SO_{2m+1} with the integral highest weight $\lambda = (\lambda_1 \geq \lambda_2 \ldots \lambda_m \geq 0)$ can be realized as a subrepresentation of $S^{\|\lambda\|}(\oplus^m(\mathbb{C}^{2m+1}))$. Here*

$$\|\lambda\| = \sum_1^m \lambda_i$$

Proof:

1. Let v_1 be the highest weight vector for $\mathbb{C}^{2m+1}$. Then $v_1^{\lambda_1}$ is a highest weight vector of $S^{\lambda_1}(\mathbb{C}^{2m+1})$. Thus there must exist an irreducible submodule $V_{(\lambda_1,0,\ldots,0)}$ of $S^{\lambda_1}(\mathbb{C}^{2m+1})$ with highest weight $(\lambda_1, 0, \ldots, 0)$.

2. Recall that
$$S^n(\oplus^m(\mathbb{C}^{2m+1})) = \oplus_{n_1+n_2\cdots+n_m=n} \otimes_1^m S^{n_i}(\mathbb{C}^{2m+1})$$
Here $n_i \in \mathbb{N}$ could be zero. We will proceed inductively on i to show that there exists
$$V_{(\lambda_1\geq\lambda_2\cdots\geq\lambda_i,0,\cdots 0)} \subseteq \otimes_1^i S^{\lambda_k}(\mathbb{C}^{2m+1})$$

3. Suppose that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_i \geq \lambda_{i+1} \geq 0$ and
$$V_{(\lambda_1\geq\lambda_2\cdots\geq\lambda_i,0,\ldots 0)} \subseteq \otimes_1^i S^{\lambda_k}(\mathbb{C}^{2m+1})$$
And also we know that
$$V_{(\lambda_{i+1},0,\ldots,0)} \subseteq S^{\lambda_{i+1}}(\mathbb{C}^{2m+1})$$

According to Kumar's lemma, if we choose w_1 to be the permutation $(1 \quad i+1)$ in the Weyl group, then there exists

$$V_{(\lambda_1 \geq \ldots \lambda_i \geq \lambda_{i+1} \geq 0)} \subseteq \otimes_1^{i+1} S^{\lambda_k}(\mathbb{C}^{2m+1})$$

4. Thus by induction, for every $\lambda_1 \geq \ldots \geq \lambda_m \geq 0$, there always exists

$$V_{(\lambda_1 \geq \ldots \geq \lambda_m \geq 0)} \subseteq \otimes_1^m S^{\lambda_k}(\mathbb{C}^{2m+1}) \subseteq S^{\|\lambda\|}(\oplus^m(\mathbb{C}^{2m+1}))$$

Q.E.D.

For SO_{2m}, the integral highest weights are given by

$$\lambda = (\lambda_1 \geq \lambda_2 \ldots \geq \lambda_{m-1} \geq |\lambda_m|) \qquad (\lambda_i \in \mathbb{N})$$

Theorem 10.8 **(SO_{2m})** *Every irreducible representation of SO_{2m} with the highest weight $\lambda = (\lambda_1 \geq \lambda_2 \ldots \geq \lambda_{m-1} \geq |\lambda_m|)$ occurs as a subrepresentation of $S^{\|\lambda\|}(\oplus^m(\mathbb{C}^{2m}))$.*

Proof:

1. Actually the same argument from the proof for SO_{2m+1} proves that for every $\lambda = (\lambda_1 \geq \ldots \geq \lambda_m \geq 0)$, there exists a subrepresentation

$$V_\lambda \subseteq S^{\|\lambda\|}(\oplus^m(\mathbb{C}^{2m}))$$

2. Let O'_{2m} be the other component of O_{2m}. Notice that $S^{\|\lambda\|}(\oplus^m(\mathbb{C}^{2m}))$ is automatically a representation of O_{2m}. We may look at $O'_{2m}V_\lambda \subseteq S^{\|\lambda\|}(\oplus^m(\mathbb{C}^{2m}))$ which is also a representation of SO_{2m}. In fact,

$$O'_{2m}V_\lambda \cong V_{(\lambda_1, \lambda_2, \ldots, \lambda_{m-1}, -\lambda_m)}$$

Thus $V_{(\lambda_1, \lambda_2, \ldots, \ldots, \lambda_{m-1}, -\lambda_m)}$ also occurs in $S^{\|\lambda\|}(\oplus^m(\mathbb{C}^{2m}))$.

3. To sum up, for every $\lambda = (\lambda_1 \geq \lambda_2 \ldots \geq \lambda_{m-1} \geq |\lambda_m|)$, V_λ occurs in $S(\oplus(\mathbb{C}^{2m}))$.

Q.E.D.

One major fact that motivates this proof is that, unlike the other exterior products of $\mathbb{C}^{2m}$ which are irreducible, the m-th exterior product of $\mathbb{C}^{2m}$ splits. In fact,

$$\Lambda^m(\mathbb{C}^{2m}) \cong V_{(1,1,\ldots,1,1)} \oplus V_{(1,1,\ldots,1,-1)}$$

Now we can sum up the results for type B and D groups in the following theorem.

Theorem 10.9 (Type B, D) *Suppose $n \geq \frac{p+q-1}{2}$. Every irreducible representation of SO_{p+q} occurs as a subrepresentation of the $i-th$ symmetric power $S^i(M(p+q, n))$ for some i.*

We will spend the rest of this section to prove the density theorem. $Mat(p+q,n)$ will be denoted by M. For a fixed i, we write R_n^i for the linear span of $\{F_{X,Y}^i \mid X, Y \in M\}$. If we define $F^1 : M \times M \to C(SO_{p+q})$ by

$$F^1(X,Y) = F_{X,Y}$$

then R_n^1 is simply the space of matrix coefficients of the standard representation of SO_{p+q}. Now we can further define $F^i : M \times M \to C(SO_{p+q})$ to be

$$F^i(X,Y) = (F_{X,Y})^i$$

For $i \geq 2$, F^i is no longer linear. However, we have the following commutative diagram.

$$\begin{array}{ccc} M \times M & \xrightarrow{F^i} & C(SO_{p+q}) \\ \Big\downarrow \Delta^i \times \Delta^i & & \Big\| \\ (\otimes^i M) \otimes (\otimes^i M) & \xrightarrow{\mathcal{F}^i} & C(SO_{p+q}) \end{array} \tag{10.67}$$

where Δ^i is the diagonal map, and $\mathcal{F}^i$ is the linear extension of F^i. $\mathcal{F}^i$ can be written explicitly as follows

$$\mathcal{F}^i((X_1 \otimes X_2 \ldots \otimes X_i) \otimes (Y_1 \otimes Y_2 \ldots \otimes Y_i)) = F_{X_1,Y_1} F_{X_2,Y_2} \ldots F_{X_i,Y_i} \in \mathcal{O}_{SO_{p+q}} \qquad (X_j, Y_j \in M)$$

It is easy to see that the linear span of $\Delta^i(M)$ in $\otimes^i(M)$ is $S^i(M)$. Therefore

$$R_n^i = \mathcal{F}^i(S^i(M) \otimes S^i(M))$$

It is easy to see that $\mathcal{F}^i$ is a linear SO_{p+q}-equivariant map. Now we need a lemma.

Lemma 10.4 *R_n^i is equal to the linear span of matrix coefficients of $S^i(M)$.*

Proof: We define an inner product on $\otimes^i M$

$$(Y_1 \otimes Y_2 \otimes \ldots \otimes Y_i, X_1 \otimes X_2 \otimes \ldots X_i) = \prod_{j=1}^{i} Tr \overline{X_j^t} Y_j \qquad (X_j, Y_j \in M)$$

Then this inner product is invariant under SO_{p+q}. For simplicity, we use $g.v$ to denote the action of SO_{p+q} on M, and $\otimes^i g$ to denote the action on the tensor product $\otimes^i M$. Notice that

$$\begin{aligned} &\mathcal{F}^i((X_1 \otimes X_2 \ldots \otimes X_i) \otimes (Y_1 \otimes Y_2 \ldots \otimes Y_i))(g) \\ =&(g.Y_1 \otimes g.Y_2 \otimes \ldots \otimes g.Y_i, \overline{X_1} \otimes \overline{X_2} \otimes \ldots \otimes \overline{X_i}) \end{aligned} \tag{10.68}$$

Therefore the function $F_{X,Y}^i$ is precisely given by the matrix coefficient

$$((\otimes^i g).(Y \otimes Y \otimes \ldots \otimes Y), (\overline{X} \otimes \overline{X} \otimes \ldots \overline{X}))$$

Since $S^i(M)$ is the linear span of

$$Y \otimes Y \otimes \ldots \otimes Y \qquad (Y \in M)$$

R_n^i is contained in the linear span of matrix coefficients of $S^i(M)$. By the same argument, one can check easily that the converse is also true. Q.E.D.

One direct implication is that R_n is spanned by the matrix coefficients of $S(M)$. From our Theorem 10.9, every irreducible representation of SO_{p+q} can be embedded as a subrepresentation of $S(M) = S(\oplus^n \mathbb{C}^{p+q})$ if $n \geq \frac{p+q-1}{2}$. Thus the matrix coefficients of every irreducible representation occur in R_n for $n \geq \frac{p+q-1}{2}$. Thus R_n contains the matrix coefficients of every irreducible representation of SO_{p+q}. According to the Peter-Weyl theorem, $\mathcal{O}_{SO_{p+q}}$ is spanned by the matrix coefficients of all the irreducible representation. Therefore $R_n = \mathcal{O}_{SO_{p+q}}$ and R_n is dense in $C(SO_{p+q})$. This finishes the proof of the density theorem. Q.E.D.

Finally, we want to formulate a conjecture along this line.

Conjecture 1 *If $p+q \geq 2n+2$, then $R_n \neq \mathcal{O}_{SO_{p+q}}$.*

10.6 Some Conjectures

We say that π is an irreducible admissible representation of $\widetilde{O_{p,q}}$ with bounded matrix coefficients if for every $u, v \in V_\pi$, there exists a constant $C_{u,v}$ such that

$$|(\pi(g)u, v)| \leq C_{u,v}, \qquad (u, v \in V_\pi)$$

Theorem 10.10 *If $(O_{p,q}, Sp_{2n})$ is in the stable range, i.e.,*

$$n \geq p+q$$

and π is an irreducible admissible representation of $\widetilde{O_{p,q}}$ with bounded matrix coefficients, then

$$\pi \in \mathcal{R}_{ss}(O_{p,q}, Sp_{2n})$$

Proof: Recall that the half of the sum of the positive roots of $O_{p,q}$

$$\rho_{p,q} = \sum_{i=1}^{p} (\frac{q-p}{2} + i - 1)e_i \qquad (e_i(H) = H_i)$$

and the simple roots are given by

$$(e_p - e_{p-1}, e_{p-1} - e_{p-2}, \ldots, e_2 - e_1, e_1), \qquad (p \neq q)$$

$$(e_p - e_{p-1}, e_{p-1} - e_{p-2}, \ldots, e_2 - e_1, e_1 + e_2), \qquad (p = q)$$

Then

$$2\rho_{p,q} - n = ((p+q-2) - n)e_p + ((p+q-4) - n)e_{p-1} + \ldots + (q-p-n)e_1$$

Since $n+2-p-q \geq 2$, $(n+2-p-q)e_p$ is already a strictly positive combination of simple roots, and it is easy to see that $n - 2\rho_{p,q}$ can be written as a strictly positive combination of simple roots. Thus $2\rho_{p,q} - n$ can be written as a strictly negative combination of simple roots. From the boundedness of matrix coefficients of π, each of its leading exponents v has to satisfy:

$$Re\, v \preceq 0$$

Therefore $Re\, v + 2\rho_{p,q} - n$ is still a strictly negative combination of simple roots. Thus

$$\pi \in \mathcal{R}_{ss}(O_{p,q}, Sp_{2n})$$

Q.E.D.

Thus we proved that

Theorem 10.11 *Suppose $n \geq p+q$ and π is an irreducible Harish-Chandra module of $\widetilde{O_{p,q}}$ with bounded matrix coefficients. Then either $\pi \in \mathcal{R}(\widetilde{O_{p,q}}, \omega)$ or $\pi \otimes \chi \in \mathcal{R}(\widetilde{O_{p,q}}, \omega)$.*

Once we confine our attention to unitary representations, there is a stronger result of Li [Li1] [Li2], which stated that the dual representations under the dual pair correspondence for different (unordered) pairs $p+q \leq n$ and $p'+q' \leq n$ are different. Therefore these dual representations (also called lower rank representations) can be classified by a pair (p,q) such that $p+q \leq n$ and an irreducible unitary representation of $O_{p,q}$.

Motivated by Li's result, we formulate the following conjecture:

Conjecture 2 *Suppose $p+q = m \leq 2n+1$, and $p \leq q$. Let $\omega_{p,q} : \mathcal{R}(\widetilde{O_{p,q}}, \omega) \rightarrow \mathcal{R}(\widetilde{Sp_{2n}}, \omega)$ be the dual pair correspondence. Then $\omega_{p,m-p}(\mathcal{R}_{ss}(\widetilde{O_{p,m-p}}, \omega))$ does not intersect with other $\omega_{p',m-p'}(\mathcal{R}_{ss}(\widetilde{O_{p',m-p'}}, \omega))$, i.e.,*

$$\omega_{p,m-p}(\mathcal{R}_{ss}(\widetilde{O_{p,m-p}}, \omega)) \cap \omega_{p',m-p'}(\mathcal{R}_{ss}(\widetilde{O_{p',m-p'}}, \omega)) = \emptyset \qquad (p+p' \neq m, p \neq p')$$

A stronger conjecture can be formulated as follows.

Conjecture 3 *If the unordered pair $(p,q) \neq (p',q')$ and $p+q \leq p'+q' \leq 2n+1$, then*

$$\omega_{p,q}(\mathcal{R}_{ss}(\widetilde{O_{p,q}}, \omega)) \cap \omega_{p',q'}(\mathcal{R}_{ss}(\widetilde{O_{p',q'}}, \omega)) = \emptyset$$

Of course I am less positive about it.

11 Nonvanishing theorem for $(Sp_{2n}, O_{p,q})$

Let $\hat{G}$ be the unitary dual of G, and $\hat{G}_{ad}$ the admissible dual of G. The proof of nonvanishing theorem of the dual pair correspondence in the semistable range of $(O_{p,q}, Sp_{2n})$ relies on matrix coefficient estimation and manipulation of a "compact" kernel $K(z, w, g)$. From now on we will assume $p + q \geq 2n$. We will regard $O_{p,q}$ as the "bigger" group and $Sp_{2n}(\mathbb{R})$ as the "smaller" group. One might conjecture that a similar result as Theorem 10.4 holds for $(Sp_{2n}, O_{p,q})$.

Conjecture: Let π be in the semistable range of $(Sp_{2n}, O_{p,q})$. If $p + q \geq 2n$, then

$$\mathcal{L}_{\widetilde{Sp_{2n}}}(\mathcal{P}^c \otimes \pi) \neq 0$$

In particular,

$$\pi \in \mathcal{R}(\widetilde{Sp_{2n}}, \omega)$$

$$\mathcal{R}_{ss}(Sp_{2n}, O_{p,q}) \subseteq \mathcal{R}(\widetilde{Sp_{2n}}, \omega)$$

However, this conjecture turns out not to be true. For example, we can take $O_{p,q}$ to be the compact O_q. Then the irreducible representations of $\widetilde{O_q}$ are parametrized by integrable highest weights. We let

$$p = 0, \qquad q = 4n + 2$$

Then

$$\rho_{Sp_{2n}} = ne_1 + (n-1)e_2 + \ldots e_1 \qquad (e_i(H) = H_i)$$

$$-\frac{p+q}{2} + 2\rho_{Sp_{2n}} = -(e_n + 3e_{n-1} + \ldots + (2n-1)e_1)$$

Thus $-\frac{p+q}{2} + 2\rho_{Sp_{2n}}$ is a strictly negative combination of simple roots. Suppose π is an irreducible unitary representation of $Sp_{2n}(\mathbb{R})$. Then each of its leading exponent v satisfies

$$Re\, v \preceq 0$$

Thus $Re\, v - \frac{p+q}{2} + 2\rho_{Sp_{2n}}$ is a strictly negative combination of simple roots. This proves that

$$\widehat{\widetilde{Sp_{2n}}}(\epsilon) \subseteq \mathcal{R}_{ss}(Sp_{2n}, O_{4n+2})$$

However, there exists no one-to-one map between $\widehat{\widetilde{Sp_{2n}}}(\epsilon)$ and $\widehat{\widetilde{O_{4n+2}}}$. This shows that there exists no injective map

$$\omega : \mathcal{R}_{ss}(Sp_{2n}, O_{4n+2}) \to \widehat{\widetilde{O_{4n+2}}}$$

On the other hand, dual pair correspondence is a one to one correspondence

$$\omega : \mathcal{R}(\widetilde{Sp_{2n}}, \omega) \to \widehat{\widetilde{O_{4n+2}}}$$

Therefore

$$\mathcal{R}_{ss}(Sp_{2n}, O_{4n+2}) \not\subseteq \mathcal{R}(\widetilde{Sp_{2n}}, \omega)$$

Recall that the definition of semistable range of $(Sp_{2n}, O_{p,q})$ only depends on the pair $(n, p+q)$. This suggests that instead of individual $O_{p,q}$, we may consider the disjoint union of representations of the real forms of $O(p+q, \mathbb{C})$, i.e.,

$$\mathcal{R}_{ss}(Sp_{2n}, m) \hookrightarrow \sqcup_{p+q=m} \widehat{\widetilde{O_{p,q}}}_{ad}$$

We will first compute the integration kernel $K(z, w, g)$ and derive some properties of $K(z, w, g)$. Then we will establish the relation between nonvanishing theorems and density theorems. Finally, we will prove some density theorems. The ideas used in this chapter are similar to the ideas used in last chapter.

11.1 Setting

To begin with, we want to focus on the $Sp_{2n}(\mathbb{R})$ action on $V = Mat(p+q, n, \mathbb{R})$. However, in the setting from the last chapter, $Sp_{2n}(\mathbb{R})$ acts from the right. Because of the right action, we need to recompute everything we have done in the last chapter. Thus we choose to look at the left action τ of $Sp_{2n}(\mathbb{R})$ on $Mat(2n, p+q, \mathbb{R})$. Let $V = Mat(2n, p+q, \mathbb{R})$ and

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in \begin{pmatrix} Mat(n,p,\mathbb{R}) & Mat(n,q,\mathbb{R}) \\ Mat(n,p,\mathbb{R}) & Mat(n,q,\mathbb{R}) \end{pmatrix}$$

We define

1. The symplectic form

$$\begin{aligned} \Omega(X,Y) &= -Tr(X^t W Y S_{p,q}) \\ &= -Tr\begin{pmatrix} X_{11}^t & X_{21}^t \\ X_{12}^t & X_{22}^t \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}\begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix} \\ &= -Tr(X_{11}^t Y_{21} - X_{21}^t Y_{11} + X_{22}^t Y_{12} - X_{12}^t Y_{22}) \end{aligned} \tag{11.69}$$

2. Complex multiplication of i

$$JX = -WXS_{p,q} = \begin{pmatrix} -X_{21} & X_{22} \\ X_{11} & -X_{12} \end{pmatrix}$$

3. Identification of V with $Mat(n,p,\mathbb{C}) \oplus Mat(n,q,\mathbb{C})$ by

$$X = (X_1, X_2) \leftrightarrow X = \begin{pmatrix} Re(X_1) & Im(X_2) \\ Im(X_1) & Re(X_2) \end{pmatrix}$$

$$X_1 = X_{11} + iX_{21}; \qquad X_2 = X_{22} + iX_{12}$$

4. Inner product on $M(2n, p+q)$ regarded as a complex vector space

$$\begin{aligned}(X,Y) =& Tr(X_1^t\overline{Y_1} + X_2^t\overline{Y_2}) \\ =& Tr(X_{11}^t + iX_{21}^t)(Y_{11} - iY_{21}) + Tr(iX_{12}^t + X_{22}^t)(Y_{22} - iY_{12}) \\ =& Re(X,Y) + i\Omega(X,Y)\end{aligned} \tag{11.70}$$

We use τ to denote the action of $Sp_{2n}(\mathbb{R})$ on $Mat(2n, p+q, \mathbb{R})$.

1. The maximal split Abelian subgroup of $Sp_{2n}(\mathbb{R})$ consists of

$$\exp H = \begin{pmatrix} \exp\lambda & 0 \\ 0 & \exp -\lambda \end{pmatrix} \qquad (\lambda = diag(\lambda_1, \lambda_2, \dots, \lambda_n))$$

where

$$H = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$$

2. Let $g \in Sp_{2n}(\mathbb{R})$. Then

$$J \circ \tau(g) \circ J(X) = Wg(WXS_{p,q})S_{p,q}X = (WgW)X = \tau(WgW)X$$

In particular

$$J\tau(\exp H)J = \tau \begin{pmatrix} -\exp(-\lambda) & 0 \\ 0 & -\exp\lambda \end{pmatrix} = \tau(-\exp(-H))$$

3. We have

$$C_{\tau(\exp H)} = \frac{1}{2}(\tau(\exp H) - J\tau(\exp H)J) = \tau \begin{pmatrix} \cosh\lambda & 0 \\ 0 & \cosh\lambda \end{pmatrix}$$

We denote it by $\tau(\cosh H)$. We denote $\tau(\cosh H)^{-1}$ by $\tau(\mathrm{sech} H)$.

4. We have

$$A_{\tau(\exp H)} = \frac{1}{2}(\tau(\exp H) + J\tau(\exp H)J) = \tau \begin{pmatrix} \sinh\lambda & 0 \\ 0 & -\sinh\lambda \end{pmatrix}$$

We denote it by $\tau(\sinh H)$.

5. We have

$$Z_{\tau(\exp H)} = C^{-1}_{\tau(\exp H)} A_{\tau(\exp H)} = \tau \begin{pmatrix} \tanh(\lambda) & 0 \\ 0 & -\tanh(\lambda) \end{pmatrix}$$

We denote it by $\tau(\tanh H)$.

6. Now $k \in U_n \subseteq Sp_{2n}(\mathbb{R})$ can be represented by

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}; \qquad (A + iB \in U(n))$$

Now we have

$$\tau \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \begin{pmatrix} AX_{11} - BX_{21} & AX_{12} - BX_{22} \\ BX_{11} + AX_{21} & BX_{12} + AX_{22} \end{pmatrix}$$

Thus

$$\begin{aligned} \tau(\begin{pmatrix} A & -B \\ B & A \end{pmatrix})X_1 &= (AX_{11} - BX_{21}) + i(BX_{11} + AX_{21}) \\ &= (A + iB)X_{11} + i(A + iB)X_{21} \\ &= kX_1 \end{aligned} \tag{11.71}$$

$$\begin{aligned} \tau(\begin{pmatrix} A & -B \\ B & A \end{pmatrix})X_2 &= i(AX_{12} - BX_{22}) + (BX_{12} + AX_{22}) \\ &= (A - iB)(X_{22} + iX_{12}) \\ &= \overline{k}X_2 \end{aligned} \tag{11.72}$$

7. We have

$$C_{\tau(\exp H)}X = \begin{pmatrix} \cosh \lambda X_{11} & \cosh \lambda X_{12} \\ \cosh \lambda X_{21} & \cosh \lambda X_{22} \end{pmatrix}$$

Thus

$$C_{\tau(\exp H)}X_1 = \cosh \lambda X_1; \qquad C_{\tau(\exp H)}X_2 = \cosh \lambda X_2$$

8. We have

$$Z_{\tau(\exp H)}X = \begin{pmatrix} \tanh(\lambda)X_{11} & \tanh(\lambda)X_{12} \\ -\tanh(\lambda)X_{21} & -\tanh(\lambda)X_{22} \end{pmatrix}$$

Thus

$$Z_{\tau(\exp H)}X_1 = \tanh(\lambda)\overline{X_1}$$

$$Z_{\tau(\exp H)}X_2 = -\tanh(\lambda)X_{22} + i\tanh(\lambda)X_{12} = -\tanh(\lambda)\overline{X_2}$$

11.2 Integration kernel of $Sp_{2n}(\mathbb{R})$

Let $g \in Sp_{2n}(\mathbb{R})$ and $\tau(g) \in Sp(V, \Omega)$. We will ignore the τ, and regard g as element in $Sp(V, \Omega)$. Let $\widetilde{g} = (\xi, g)$ be a preimage of g under the metaplectic covering. Then the Bargmann-Segal kernel of the oscillator representation is given by

$$\xi \exp(\frac{1}{4}\mathcal{H}(g, z, w))$$

where

$$\mathcal{H}(g,z,w) = 2(\tau(\mathrm{sech}H)k_1^{-1}z, k_2w) + (k_1^{-1}z, \tau(\tanh H)k_1^{-1}z) - (\tau(\tanh H)k_2w, k_2w)$$

We shall make a remark here that $\mathcal{H}(g,z,w)$ is in fact $\mathcal{H}(\tau(g),z,w)$. This should not be confused with the $\mathcal{H}(g,z,w)$ for $g \in Sp_{2n}(\mathbb{R})$ and $z, w \in \mathbb{C}^{2n}$ in Theorem 6.7.

1. We have

$$\begin{aligned}(\tau(\mathrm{sech}H)k_1^{-1}z, k_2w) =& (\mathrm{sech}\lambda k_1^{-1}z_1, k_2w_1) + (\mathrm{sech}\lambda \overline{k_1}^{-1}z_2, \overline{k_2}w_2) \\ =& Tr(z_1^t\overline{k_1}\mathrm{sech}\lambda\overline{k_2}\overline{w_1}) + Tr(z_2^tk_1\mathrm{sech}\lambda k_2\overline{w_2})\end{aligned} \tag{11.73}$$

2. We have

$$\begin{aligned}(k_1^{-1}z, \tau(\tanh H)k_1^{-1}z) =& (k_1^{-1}z_1, \tanh(\lambda)\overline{k_1^{-1}z_1}) + (\overline{k_1}^{-1}z_2, -\tanh(\lambda)\overline{\overline{k_1}^{-1}z_2}) \\ =& (k_1^{-1}z_1, \tanh(\lambda)k_1^t\overline{z_1}) + (k_1^tz_2, -\tanh(\lambda)k_1^{-1}\overline{z_2}) \\ =& Tr(z_1^t\overline{k_1}\tanh(\lambda)k_1^{-1}z_1) - Tr(z_2^tk_1\tanh(\lambda)k_1^tz_2)\end{aligned} \tag{11.74}$$

3. We have

$$\begin{aligned}(\tau(\tanh H)k_2w, k_2w) =& \overline{(k_2w, \tanh(\lambda)k_2w)} \\ =& \overline{Tr(w_1^tk_2^t\tanh(\lambda)k_2w_1) - Tr(w_2^t\overline{k_2}^{-1}\tanh(\lambda)\overline{k_2}w_2)} \\ =& Tr(\overline{w_1^tk_2^{-1}}\tanh(\lambda)\overline{k_2}\overline{w_1}) - Tr(\overline{w_2^t\overline{k_2}^t}\tanh(\lambda)k_2\overline{w_2})\end{aligned} \tag{11.75}$$

4. Recall the definition of the compactification of $Sp_{2n}(\mathbb{R})$

$$\mathcal{H}(k_1\exp(H)k_2) = \begin{pmatrix} \overline{k_1} & 0 \\ 0 & k_2^t \end{pmatrix}\begin{pmatrix} \tanh(H_{\mathbb{C}}) & -i\mathrm{sech}(H_{\mathbb{C}}) \\ -i\mathrm{sech}(H_{\mathbb{C}}) & \tanh(H_{\mathbb{C}}) \end{pmatrix}\begin{pmatrix} k_1^{-1} & 0 \\ 0 & \overline{k_2} \end{pmatrix}$$

Therefore, we can simplify $\mathcal{H}(g,z,w)$ as follows.

$$\begin{aligned}\mathcal{H}(g,z,w) =& 2Tr(z_1^t\overline{k_1}\mathrm{sech}(\lambda)\overline{k_2}\overline{w_1}) + Tr(z_1^t\overline{k_1}\tanh(\lambda)k_1^{-1}z_1) - Tr(\overline{w_1^tk_2^{-1}}\tanh(\lambda)\overline{k_2}\overline{w_1}) \\ &+ 2Tr(z_2^tk_1\mathrm{sech}(\lambda)k_2\overline{w_2}) - Tr(z_2^tk_1\tanh(\lambda)k_1^tz_2) + Tr(\overline{w_2^t\overline{k_2}^t}\tanh(\lambda)k_2\overline{w_2}) \\ =& Tr\left((z_1^t, i\overline{w_1^t})\mathcal{H}(g)\begin{pmatrix} z_1 \\ i\overline{w_1}\end{pmatrix}\right) + Tr\left((-iz_2^t, \overline{w_2^t})\overline{\mathcal{H}(g)}\begin{pmatrix} -iz_2 \\ i\overline{w_2}\end{pmatrix}\right)\end{aligned} \tag{11.76}$$

11.3 On the Vanishing of Averaging Operator

Since the density theorems are a little cumbersome to deal with, we will first investigate the vanishing of the averaging operator and establish some equivalence relation between the density theorems and the nonvanishing theorems. The ideas and techniques are similar to the proof of Theorem 10.4. Let $G = Sp_{2n}(\mathbb{R})$ and $\mathcal{S} = \{UU^t \mid U \in U(2n)\}$.

Theorem 11.1 *We fix a group $O_{p,q}$ and an irreducible representation π in the semistable range of $(Sp_{2n}(\mathbb{R}), O_{p,q})$. Suppose that*

$$\mathcal{L}_{\widetilde{G}}(\mathcal{P}^c \otimes V_\pi) = 0$$

For $u, v \in V_\pi$ and $\widetilde{g} = (\xi, g) \in \widetilde{Sp_{2n}}(\mathbb{R})$ where $\xi^2 = \det_{\mathbb{C}}^{-1} C_{\tau(g)}$, let

$$F(\widetilde{g}) = \xi(v, \pi(\widetilde{g})u); \qquad (g = k_1 \exp H(\lambda) k_2)$$

Then F descends to a function on $Sp_{2n}(\mathbb{R})$. Let $\mathcal{H}$ be the analytic compactification of $Sp_{2n}(\mathbb{R})$, and let

$$\widetilde{F}(\mathcal{H}(g)) = F(g)(\frac{d_{U(2n)}\mathcal{H}(g)}{d_{Sp_{2n}} g})^{-1} \qquad (g \in Sp_{2n}(\mathbb{R}))$$

Then $\widetilde{F}$ is a smooth function defined on $\mathcal{H}(Sp_{2n})$ and

$$\int_S \widetilde{F}(s)(Tr(x^t sx))^l (Tr(y^t \overline{s} y))^m d_{U(2n)} s = 0; \qquad (x \in Mat(p, 2n, \mathbb{C}); y \in Mat(q, 2n, \mathbb{C}), l, m \in \mathbb{N})$$

Proof: We will just give an outline here, and skip the details. Essentially, the details were given in the proof of Theorem 10.4.

1. For each $u, v \in V_\pi$, $\alpha, \beta \in \mathbb{N}^{n(p+q)}$, we have

   $$\int_{\widetilde{G}} (z^\beta, \omega(\widetilde{g}) z^\alpha)(\pi(\widetilde{g})u, v) d\widetilde{g} = 0$$

   Taking conjugation we have

   $$\begin{aligned} 0 &= \int_{\widetilde{G}} (\omega(\widetilde{g}) z^\alpha, z^\beta)(v, \pi(\widetilde{g})u) d\widetilde{g} \\ &= \int_{\widetilde{G}} (\int_{\mathbb{C}^{n(p+q)} \times \mathbb{C}^{n(p+q)}} \xi \exp(\frac{1}{4}\mathcal{H}(g, z, w)) w^\alpha \overline{z}^\beta (v, \pi(\widetilde{g})u) d\mu(w) d\mu(z)) d\widetilde{g} \\ &= \int_{\widetilde{G}} F(\widetilde{g}) \int_{\mathbb{C}^{n(p+q)} \times \mathbb{C}^{n(p+q)}} \exp(\frac{1}{4}\mathcal{H}(g, z, w)) w^\alpha \overline{z}^\beta d\mu(w) d\mu(z)) d\widetilde{g} \end{aligned} \tag{11.77}$$

   Since $\epsilon(\xi, g) = (-\xi, g)$ and

   $$(v, \pi(\epsilon\widetilde{g})u) = -(v, \pi(\widetilde{g})u)$$

   $F(\widetilde{g}) = \xi(v, \pi(\widetilde{g})u)$ descends into a function of G. We may write all our integrals as integration over $Sp_{2n}(\mathbb{R})$.

2. Claim:

   $$\begin{aligned} &\int_G (\int_{\mathbb{C}^{n(p+q)} \times \mathbb{C}^{n(p+q)}} F(g) \exp(\frac{1}{4}\mathcal{H}(g, z, w)) w^\alpha \overline{z}^\beta d\mu(w) d\mu(z)) dg \\ = &\int_{\mathbb{C}^{n(p+q)} \times \mathbb{C}^{n(p+q)}} (\int_G F(g) \exp(\frac{1}{4}\mathcal{H}(g, z, w)) w^\alpha \overline{z}^\beta dg) d\mu(w) d\mu(z) \end{aligned} \tag{11.78}$$

It suffices to show that

$$F(g)\exp(\frac{1}{4}\mathcal{H}(g,z,w))w^{\alpha}\overline{z}^{\beta}\in L^1(dgd\mu(w)d\mu(z))$$

Recall that for $g=k_1\exp Hk_2$ ($H\in\mathfrak{a}^+$), we have

$$C_{\tau(\exp H)}X_1=\cosh\lambda X_1;\qquad C_{\tau(\exp H)}X_2=\cosh\lambda X_2$$

Therefore

$$|\det_{\mathbb{C}}(C_{\tau(g)})|=|\det_{\mathbb{C}}(C_{\tau(\exp H)})|=\prod_{i=1}^{n}(\cosh\lambda_i)^{p+q}$$

Hence

$$|\xi|=\prod_{i=1}^{n}(\cosh\lambda_i)^{-\frac{p+q}{2}}\leq C\prod_{i=1}^{n}\exp(-\frac{p+q}{2}\lambda_i)$$

From our assumption on semistable range,

$$F(g)\in L^1(dg)$$

On the other hand , we have

$$|\exp\frac{1}{4}\mathcal{H}(g,z,w)w^{\alpha}\overline{z}^{\beta}|\leq\exp(\frac{1}{4}(\|z_1\|^2+\|w_1\|^2+\|z_2\|^2+\|w_2\|^2))\|z\|^{\beta}\|w\|^{\alpha}\in L^1(d\mu(z)d\mu(w))$$

Thus

$$F(g)\exp(\frac{1}{4}\mathcal{H}(g,z,w))w^{\alpha}\overline{z}^{\beta}\in L^1(dgd\mu(w)d\mu(z))$$

Combined with 1), we have

$$\int_{\mathbb{C}^{n(p+q)}\times\mathbb{C}^{n(p+q)}}(\int_G F(g)\exp(\frac{1}{4}\mathcal{H}(g,z,w))w^{\alpha}\overline{z}^{\beta}dg)d\mu(w)d\mu(z)=0 \qquad (11.79)$$

3. The integral $\int F(g)\exp(\frac{1}{4}\mathcal{H}(g,z,w))dg$ is holomorphic with respect to z, and antiholomorphic with respect to w.
4. Combined with Equation 11.79, we have

$$\forall\ z,w;\qquad \int_G F(g)\exp(\frac{1}{4}\mathcal{H}(g,z,w))dg=0$$

5. Very similar to theorem 10.4, we have

$$\begin{aligned}0&=\int_{Sp_{2n}(\mathbb{R})}F(g)\exp(\frac{1}{4}\mathcal{H}(g,z,w))dg\\&=\int_S F(\mathcal{H}^{-1}(s))\exp(\frac{1}{4}(z_1^t,i\overline{w_1^t})s\begin{pmatrix}z_1\\ i\overline{w_1}\end{pmatrix})\exp(\frac{1}{4}(-iz_2^t,\overline{w_2^t})\overline{s}\begin{pmatrix}-iz_2\\ \overline{w_2}\end{pmatrix})\frac{d\mathcal{H}^{-1}(s)}{ds}d_{U(2n)}s\\&=\int_S\widetilde{F}(s)\exp(\frac{1}{4}(z_1^t,i\overline{w_1^t})s\begin{pmatrix}z_1\\ i\overline{w_1}\end{pmatrix})\exp(\frac{1}{4}(-iz_2^t,\overline{w_2^t})\overline{s}\begin{pmatrix}-iz_2\\ \overline{w_2}\end{pmatrix})d_{U(2n)}s\end{aligned} \qquad (11.80)$$

6. Finally, Taking $x = 0$ and $y = 0$, we obtain,

$$\widetilde{F}(s) \in L^1(\mathcal{S}, d_{U(2n)}s)$$

Since $\mathcal{S}$ is compact, it can be shown that $\forall\, m, n \in \mathbb{N}$,

$$\int_{\mathcal{S}} \widetilde{F}(s)(Tr(x^t sx))^m (Tr(y^t \bar{s} y))^n d_{U(2n)}s = 0; \qquad (x \in Mat(2n, p, \mathbb{C}); y \in Mat(2n, q, \mathbb{C}))$$

Q.E.D.

In the next section, we will prove that the linear span of

$$\{(Tr(x^t sx))^m (Tr(y^t \bar{s} y))^l \mid x \in Mat(2n, p, \mathbb{C}); y \in Mat(2n, q, \mathbb{C}); p + q = 2n; m, l \in \mathbb{N}\}$$

is dense in $C(Sp_{2n}(\mathbb{R}))$. Thus if $\mathcal{L}_{\widetilde{Sp_{2n}(\mathbb{R})}}(\mathcal{P}_{p,q} \otimes V_\pi)$ vanishes for every $p + q = 2n$, then

$$(\pi(\widetilde{g})u, v) = 0 \qquad (u, v \in V_\pi)$$

here $(\omega_{p,q}, \mathcal{P}_{p,q})$ is the Harish-Chandra module of the oscillator representation for dual pair $(Sp_{2n}(\mathbb{R}), O_{p,q})$.

11.4 Spherical Functions and Helgason's Theorems

Now let $\mathcal{O}_{\mathcal{S}}$ be the $U(2n)$-finite functions on $\mathcal{S}$. For $X \in Mat(2n, p, \mathcal{C})$, we write

$$F_X(s) = Tr(X^t sX) \qquad (s \in \mathcal{S})$$

Then $F_X(s) \in \mathcal{O}_{\mathcal{S}}$. Let R_p be the linear span of

$$\{F_X^i \mid X \in Mat(2n, p), i \in \mathbb{N}\}$$

and $\overline{R_p}$ be its conjugation. By a little multilinear algebra, we can show that

Theorem 11.2 *R_p is spanned by*

$$\{(X_1^t sX_1)^{i_1}(X_2^t sX_2)^{i_2} \dots (X_p^t sX_p)^{i_p} \mid X_1, \dots, X_p \in \mathbb{C}^{2n}, i_1 \dots i_p \in \mathbb{N}\}$$

Proof: Let $X = (X_1, X_2, \dots, X_p)$, and $X(t) = (t_1X_1, t_2X_2, \dots, t_pX_p)$, where $t \in \mathbb{C}^p$. Now

$$(F_{X(t)})^i = (\Sigma t_j^2 X_j^t sX_j)^i \in R_p$$

For $i_1 + i_2 + \dots i_p = i$, if we take the coefficient of $\Pi_{j=1}^p t_j^{2i_j}$ in the above expansion, we get

$$\Pi_1^p (X_j^t sX_j)^{i_j} \in R_p$$

On the other hand, every F_X^i can be written as a linear combination of $\Pi_{j=1}^p (X_j^t sX_j)^{i_j} (i_j \in \mathbb{N})$. Therefore R_p is spanned by

$$(X_1^t sX_1)^{i_1}(X_2^t sX_2)^{i_2} \dots (X_p^t sX_p)^{i_p};$$

Q.E.D.

The first theorem we state here is actually equivalent to the nonvanishing theorem for the stable range dual pair correspondence.

Theorem 11.3 *Suppose $p \geq 2n, q \geq 2n$. Then the multiplication*

$$m : R_p \otimes \overline{R_q} \to \mathcal{O}_S$$

is surjective.

Proof: This can be shown by using the Stone-Weierstrass Theorem. We skip the proof here. This theorem is just a special case of Theorem 11.11 which we are going to prove. Q.E.D.

Before we continue on to improve the theorem above, we want to give a description of $\mathcal{O}_S$ first. The ring of regular functions on a compact symmetric space was studied by Helgason. We will recall some definition and theorems here.

Definition 11.1 *Let (G, K) be a reductive symmetric pair. A finite dimensional representation π of G is called spherical if there exists a K-fixed vector $v \in \pi$. Such a K-fixed vector is called a spherical vector. We use $\hat{G}_K$ to denote the set of spherical representations.*

If the representation is irreducible, then the spherical vector is unique up to a scalar.

Theorem 11.4 (Helgason) *Let U be a compact connected Lie group, and K fixed point set of an involution σ. Let $\mathfrak{p}$ be the -1 eigenspace of $\mathfrak{g}$, and $\mathfrak{t}_{\mathfrak{p}}$ be a maximal abelian subspace in $\mathfrak{p}$. Let $T_{\mathfrak{p}}$ be the Lie group corresponding to $\mathfrak{t}_{\mathfrak{p}}$. Let $T \supseteq T_{\mathfrak{p}}$ be a Cartan subgroup of U, and $\mathfrak{t}$ be its Lie algebra. Let M be the centralizer of $\mathfrak{t}_{\mathfrak{p}}$ in G. Then $(T \cap M)_0$ is the maximal torus in M_0. Let π be an irreducible representation of highest weight vector v_0. Then π is spherical if and only if*

$$\pi(M)v_0 = v_0$$

A proof for semisimple groups can be found in [Helgason2] Theorem 5.4.1. The same proof applies for compact connected groups.

Theorem 11.5 (Helgason) *For a symmetric space $X = U/K$, let $\mathcal{O}_X$ be the space of U-finite functions on X. We have the following decomposition*

$$\mathcal{O}_X = \oplus_{\lambda \in \hat{U}_K} C_\lambda(X)$$

where $C_\lambda(X)$ is an irreducible spherical representation. Let δ be a K-fixed vector for V_λ^*. Then $C_\lambda(X)$ consists of functions

$$uK \to \delta(u^{-1}v) \qquad (v \in V_\lambda)$$

Let e be a spherical vector in V_λ. Then the spherical vector in $C_\lambda(X)$ is given by a multiple of

$$uK \to \delta(u^{-1}e)$$

This is proved in [Helgason2] Theorem 5.4.3.

Now let $U = U(2n)$, $K = O(2n)$.

$$\mathfrak{t}_\mathfrak{p} = \{diag(i\theta_1, i\theta_2, \ldots i\theta_{2n}) \mid \theta_i \in \mathbb{R}\}$$

The Weyl group $W(U, K)$ is simply the permutation group on $2n$ elements.

$$M = \{\epsilon = diag(\epsilon_1, \epsilon_2, \ldots, \epsilon_{2n}) \mid \epsilon_i = \pm 1\} \subseteq T_\mathfrak{p}$$

Let V_λ be the irreducible representation of $U(2n)$ with the highest weight λ. Of course we will have

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{2n}$$

Thus for v_0 as the highest weight vector of (π, V_λ),

$$\pi(\epsilon)v_0 = \Pi_{i=1}^{2n}(\epsilon_i)^{\lambda_i} v_0$$

Therefore V_λ is spherical if and only if λ is all *even*, i.e.,

$$\lambda_i \ even \ ; \qquad (i \in [1, 2n])$$

Therefore we obtain the following theorem from Helgason's theorem.

Theorem 11.6

$$\mathcal{O}_\mathcal{S} = \oplus_{\lambda \ even} C_\lambda(\mathcal{S})$$

where $C_\lambda(\mathcal{S})$ is the irreducible representation with highest weight

$$\lambda = (\lambda_1 \geq \lambda_2 \ldots \geq \lambda_{2n}) \qquad (\lambda_i \ even)$$

We denote the unique spherical vector associated with each λ by f_λ. Here we assume $f_\lambda(eK) = 1$.

11.5 Density Theorems and Nonvanishing Theorems

We will study R_1 first. Then from Theorem 11.2 R_p can be manipulated through the multiplication map

$$m : R_1 \otimes R_1 \otimes \ldots \otimes R_1 \to R_p$$

We have the following decomposition theorem for R_1

Theorem 11.7 *Let $\mathcal{S} = U(2n)/O_{2n}$, and $V = \mathbb{C}^{2n}$. R_1 can be decomposed as*

$$\oplus_{i \in \mathbb{N}} C_{(2i,0,\ldots,0)}(\mathcal{S})$$

Proof: We define (by abuse of notation)

$$Tr : \otimes^{2i} V \to \mathcal{O}_{\mathcal{S}}$$

as follows.

$$Tr(v_1 \otimes v_2, \dots \otimes v_{2i})(s) = \Pi_{j=1}^{i} Tr(v_j^t s v_{i+j}) \qquad (s \in \mathcal{S})$$

Now it is easy to see that

$$F^i : V \to \mathcal{O}_{\mathcal{S}}$$

defined by

$$F^i(X) = (Tr(X^t s X))^i \qquad (X \in V, s \in \mathcal{S})$$

is the composition of the diagonal map Δ^{2i} and Tr. In other words, F^i can be written as

$$F^i : V \xrightarrow{\Delta^{2i}} S^{2i}(V) \xrightarrow{Tr} \mathcal{O}_{\mathcal{S}}$$

All these maps are $U(2n)$-equivariant. It is well-known that

$$S^{2i}(V) \cong V_{(2i,0,\dots,0)}$$

is already irreducible. But $F^i(V) \neq 0$. Thus the image of F^i is isomorphic $V_{(2i,0,\dots,0)}$. Hence

$$R_1 \cong \oplus_i V_{(2i,0,\dots,0)}$$

From Helgason's theorem,

$$R_1 = \oplus_i C_{(2i,0,\dots,0)}(\mathcal{S})$$

Q.E.D.

In fact, the image $Tr(S^{2i}(V))$ is $C_{(2i,0,\dots,0)}(\mathcal{S})$. Next we want to compute the exact formula for the spherical functions in $C_{(2i,0,\dots,0)}(\mathcal{S})$.

Theorem 11.8 *Let*

$$f_i : g \to Tr(g^t g)^i \qquad (g \in U(2n))$$

Then f_i is a spherical vector in $C_{(2i,0,\dots,0)}(U(2n)/O_{2n})$.

Proof:

- We fix an i first. Since

$$f_i(kg) = Tr(g^t k^t k g)^i = Tr(g^t g) \qquad (g \in U(2n), K \in O_{2n})$$

  f_i is spherical. Now it suffices to show that f_i lies in $C_{(2i,0,\dots,0)}(U(2n)/O_{2n})$.

- If $i = 1$, one spherical vector in $S^2(V) \cong V_{(2,0,\dots,0)}$ is given by

  $$e_1^2 + e_2^2 + \dots + e_{2n}^2$$

  where $\{e_i\}$ is the standard orthonormal basis of $\mathbb{C}^{2n}$. On the other hand, let $\{\delta_i\}$ be the dual basis, i.e., $\{e_i\}$ itself. Then a spherical vector in $(C^{2n})^*$ is again given by

  $$e_1^2 + e_2^2 + \dots + e_{2n}^2$$

  Now according to Helgason's theorem, a spherical function can be constructed as follows.

  $$\begin{aligned} f_1(g) =&(e_1^2 + e_2^2 + \dots + e_{2n}^2, g^{-1}(e_1^2 + e_2^2 + \dots + e_{2n}^2)) \\ =&\sum_{i,j=1}^{2n} (e_i^2, (g^{-1}e_j)^2) \\ =&\sum_{i,j=1}^{2n} (e_i, g^{-1}e_j)^2 \\ =&\sum_{i,j=1}^{2n} (e_i^t\overline{g^{-1}}e_j)(e_i^t\overline{g^{-1}}e_j) \\ =&\sum_{i,j=1}^{2n} (e_i^t g^t e_j e_j^t g e_i) \\ =&\sum_{i=1}^{n} (e_i^t g^t g e_i) \\ =&Tr(g^t g) \end{aligned} \tag{11.81}$$

- Since $f_1 \in Tr(S^2(V))$, $f_i \in Tr(S^{2i}(V))$. Therefore,

  $$f_i \in C_{(2i,0,\dots,0)}(U(2n)/O_{2n})$$

Q.E.D.

In fact, all the theorems we have proved in this section hold for $\mathcal{S}_n = U(n)/O_n$. Next we will prove a theorem for $\mathcal{S}_n$. Just for the sake of the proof, we denote $\mathcal{S}_n$ by $\mathcal{S}$. For the rest of this thesis, $\mathcal{S}$ will still be $\mathcal{S}_{2n}$. Let $W = W(U, K)$ be the Weyl group. Let T be the fixed torus $T_\mathfrak{p}$ in U. Let $C(T)^W$ be the space of smooth W-invariant functions on T. Let $\mathcal{O}_T$ be the space of $T-$finite functions on T. Let $\mathcal{O}_T^W$ and $\mathcal{O}_T^M$ be the $W-$fixed and $M-$fixed functions of $\mathcal{O}_T$. Let $\mathcal{O}_T^{W,M} = \mathcal{O}_T^W \cap \mathcal{O}_T^M$. It is well-known that the restriction of $\mathcal{O}_\mathcal{S}^K$ onto T yields a bijection from $\mathcal{O}_\mathcal{S}^K$ onto $\mathcal{O}_T^{W,M}$.

Theorem 11.9 *Let $f_i(g) = Tr(g^t g)^i$ $(g \in U(n)/O_n)$. Then $\mathcal{O}_\mathcal{S}^K$ is spanned by*

$$f_{i_1} f_{i_2} \dots f_{i_k} \overline{f_{i_{k+1}}} \dots \overline{f_{i_n}} \qquad (i_j \in \mathbb{N})$$

Proof: It suffices to show that the restriction of

$$f_{i_1} f_{i_2} \dots f_{i_k} \overline{f_{i_{k+1}}} \dots \overline{f_{i_n}} \qquad (i_j \in \mathbb{N})$$

to T span $\mathcal{O}_T^{W,M}$. Notice that

$$\mathcal{O}_T = \oplus_{\alpha \in \mathbb{Z}^n} \mathbb{C} \exp i\alpha(\theta)$$

Thus

$$\mathcal{O}_T^M = \oplus_{\alpha \in \mathbb{Z}^n} \mathbb{C} \exp 2i\alpha(\theta)$$

Let $(x_1, x_2, \dots, x_n) = (\exp(2i\theta_1), \dots, \exp(2i\theta_n))$. Then we may identify $\mathcal{O}_T^{W,M}$ with

$$\mathcal{R} = P(x_1, x_2 \dots, x_n, x_1^{-1}, x_2^{-1}, \dots, x_n^{-1})^W$$

the space of symmetric Laurent polynomials. We define the signature of a monomial x^α in $\mathcal{R}$ to be a pair of integers (p, q) where p is the number of positive α_i's and q is the number of negative α_i's. We may write $sgn(\alpha)$ or $sgn(x^\alpha)$. The signature of α is invariant under the action of W. Let R_j be the symmetric Laurent polynomial such that each term contains at most j negative exponents. In other words,

$$\mathcal{R}_j = \mathcal{R} \cap (\oplus_{j' \leq j} \oplus_{i+j' \leq n} \oplus_{sgn(\alpha)=(i,j')} \mathbb{C} x^\alpha)$$

Observe that

$$f_j(\exp(i\theta)) = Tr(\exp(2i\theta))^j = Tr(\exp(2ij\theta)); \qquad (\theta \in \mathbb{R}^n, j \geq 0)$$

Thus in terms of elements in $\mathcal{R}$, f_j can be identified with $s_j = \sum_{k=1}^n x_k^j$, and $\overline{f_j}$ with $s_{-j} = \sum_{k=1}^n x_k^{-j}$.

Claim: $\mathcal{R}_0 = \oplus_{i_j \geq 0} \mathbb{C} \prod_{j=1}^n s_{i_j}$

Notice that $\mathcal{R}_0$ is simply the space of symmetric polynomials. Let σ_j be the classical symmetric functions on $(x_1 \dots, x_n)$, i.e.,

$$\sigma_j = \sum_{(i_1 \neq i_2 \dots \neq i_j) \subseteq [1,n]} \prod_{k=1}^{j} x_{i_k}$$

Then by the classical symmetric function theory

$$\mathcal{R}_0 = \oplus_{i_j \in \mathbb{N}} \mathbb{C} \sigma_1^{i_1} \sigma_2^{i_2} \dots \sigma_n^{i_n}$$

We may look at the following symmetric polynomials of degree l,

$$\{\prod_{j=1}^{n} s_{l_j} \mid \sum l_j = l, l_j \in \mathbb{N}\}$$

The number of such polynomials is given by the number of

$$\{0 \le l_1 \le l_2 \le \ldots \le l_n \mid \sum_{i=1}^{n} l_i = l\}$$

Also these polynomials are all linearly independent. On the other hand, the number of linearly independent symmetric polynomials of degree l is given by the number of

$$\{(i_1, i_2, \cdots, i_n) \mid \sum_{i=1}^{n} j i_j = l, i_j \in \mathbb{N}\}$$

Both numbers equal the number of partitions of l into a sum of no more than n positive integers. Thus we must have

$$\mathcal{R}_0 = \oplus_{i_j \in \mathbb{N}} \mathbb{C} \prod_{j=1}^{n} s_{i_j}$$

Similarly, for an arbitrary $\alpha \in \mathbb{Z}^n$, let $\overline{\alpha}$ be the ordered n-tuple, namely

$$(\overline{\alpha_1} \ge \overline{\alpha_2} \ge \ldots \ge \overline{\alpha_n})$$

with $sgn(\alpha) = (p, q)$ and we set all the middle $n - p - q$ to be zero. Let $\mathcal{R}^{i,j}_{p,q}$ with $i \ge 0$, $j \le 0$ be the linear span of

$$\{\sum_{\overline{\alpha} fixed} x^{\alpha} \mid sgn(\alpha) = (p, q), \sum_{k=1}^{p} \overline{\alpha_k} \le i, \sum_{k=n-q+1}^{n} \overline{\alpha_k} \ge j\}$$

Here i is simply the total positive degree of Laurent monomial, and j is the total negative degree of Laurent polynomial. Now we fix a nonnegative integer $p_0 \le n$, a nonnegative integer i and a nonpositive integer j.

Claim: $\oplus_{p \le p_0, q \le n-p_0} \mathcal{R}^{i,j}_{p,q}$ is spanned by

$$\{\prod_{k=1}^{p_0} s_{l_k} \prod_{r=p_0+1}^{n} s_{l_r} \mid l_1 \ge l_2 \ldots \ge l_{p_0} \ge 0; 0 \ge l_{p_0+1} \ge \ldots \ge l_n; \sum_{k=1}^{p_0} l_k \le i; \sum_{r=p_0+1}^{n} l_r \ge j\}$$

First of all, the set above is linear independent, since each one has a characteristic term, namely

$$x_1^{l_1} x_2^{l_2} \ldots x_{p_0}^{l_{p_0}} \ldots x_{p_0+1}^{l_{p_0+1}} \ldots x_n^{l_n}$$

Also the linear span of such set is contained in $\oplus_{p \le p_0, q \le n-p_0} \mathcal{R}^{i,j}_{p,q}$. Finally, the cardinality of such set equals

$$\sum_{0<k\le i} \#(partitions\ of\ k\ into\ less\ or\ equal\ to\ p_0\ parts)$$

multiplied by

$$\sum_{0<k\leq -j} \#(\textit{partitions of } k \textit{ into less or equal to } n-p_0 \textit{ parts})$$

This is exactly the dimension of $\oplus_{p\leq p_0, q\leq n-p_0}\mathcal{R}^{i,j}_{p,q}$. The claim is proved. Since every element of $\mathcal{R}$ is contained in some $\mathcal{R}^{i,j}_{p,q}$, $\mathcal{R}$ is spanned by

$$\{\prod_{k=1}^{n} s_{i_k} \mid i_k \in \mathbb{Z}\}$$

This implies that

$$(\mathcal{O}_T)^{W,M} = \oplus_{p\leq n, i_j\geq 0}\mathbb{C}(f_{i_1}f_{i_2}\cdots f_{i_p}\overline{f_{i_{p+1}}\cdots f_{i_n}})|_T$$

Q.E.D.

We will prove the following theorem:

Theorem 11.10 *Suppose* $p+q=l\geq 2n$. *Then*

$$m: \oplus_{p+q=l} R_p \otimes \overline{R_q} \to \mathcal{O}_S$$

is surjective.

Proof: It suffices to prove the theorem for $l=2n$. We write

$$R_{p,q} = m(R_p\otimes\overline{R_q}); \qquad R=\oplus_{p+q=m} m(R_p\otimes\overline{R_q})$$

These two spaces are subspaces of $\mathcal{O}_S$. Notice that if the spherical vector $f_\lambda\in R$, then $C_\lambda(\mathcal{S})\in R$. It suffices to show that $f_\lambda\in R$ for every $V_\lambda\in \widehat{U(2n)}_{O_{2n}}$. Since f_λ is $T-$finite and K-fixed, f_λ is contained in $\mathcal{O}_S^K$. From the last theorem, we see that $\mathcal{O}_S^K$ is spanned by

$$f_{i_1}f_{i_2}\cdots f_{i_k}\overline{f_{i_{k+1}}}\cdots\overline{f_{i_n}} \qquad (i_j\in\mathbb{N})$$

and according to Theorem 11.2 each of these is already in R. Thus $\mathcal{O}_S^K\subseteq R$. This implies $f_\lambda\in R$. Q.E.D.

From the same argument, a stronger statement can be proved.

Theorem 11.11

$$m:\oplus_{p=l-2n}^{2n}(R_p\otimes\overline{R_{l-p}})\to\mathcal{O}_S \qquad (l\geq 2n)$$

$$m: R_{2n}\otimes\overline{R_{2n}}\to\mathcal{O}_S$$

is surjective.

Now we have proved the following theorem concerning $\mathcal{O}_{S_n}$ for

$$\mathcal{S}_n=\{XX^t\mid X\in U(n)\}\cong U(n)/O_n$$

Theorem 11.12 (Density Theorem) *$\mathcal{O}_{\mathcal{S}_n}$ is spanned by*

$$Tr(X_1^t s X_1)^{i_1} \dots Tr(X_k^t s X_k)^{i_k} \overline{Tr(X_{k+1}^t s X_{k+1})^{i_{k+1}}} \dots \overline{Tr(X_n^t s X_n)^{i_n}} \qquad (i_j \in \mathbb{N}, s \in \mathcal{S}_n, X_j \in \mathbb{C}^n)$$

Now combined with Theorem 11.1, we have the following theorem.

Theorem 11.13 *Suppose $m \geq 2n$. Let $\omega_{p,q}$ be the dual pair correspondence of $(Sp_{2n}(\mathbb{R}), O_{p,q})$. Let $\mathcal{P}_{p,q}$ be the Harish-Chandra module of the oscillator representation of $Mp(V,\Omega)$ with respect to $(Sp_{2n}(\mathbb{R}), O_{p,q})$. Then*

$$\mathcal{R}_{ss}(Sp_{2n}(\mathbb{R}), m) \subseteq \sqcup_{p+q=m} \mathcal{R}(\widetilde{Sp_{2n}(\mathbb{R})}, \omega_{p,q})$$

We define

$$\omega : \mathcal{R}_{ss}(Sp_{2n}(\mathbb{R}), m) \to \oplus_{p+q=m} \widehat{\widetilde{O_{p,q}}}_{ad}$$

by letting

$$\omega_{p,q}(\pi) = \mathcal{L}_{\widetilde{Sp_{2n}(\mathbb{R})}}(\mathcal{P}^c_{p,q} \otimes V_\pi) \qquad (\pi \in \mathcal{R}(\widetilde{Sp_{2n}(\mathbb{R})}, \omega_{p,q}))$$

$$\omega_{p,q}(\pi) = 0 \qquad (\pi \notin \mathcal{R}(\widetilde{Sp_{2n}(\mathbb{R})}, \omega_{p,q}))$$

Then ω is injective.

We only sketch the proof here. Let $F(\widetilde{g}) = \xi(v, \pi(g)u)$ be a matrix coefficient of $\widetilde{Sp_{2n}(\mathbb{R})}$. Then $\widetilde{F}$ is an integrable function on $\mathcal{S}$. Suppose that

$$\mathcal{L}_{\widetilde{Sp_{2n}(\mathbb{R})}}(\mathcal{P}^c_{p,q} \otimes V_\pi) = 0 \qquad (\forall\, p+q=m)$$

From the density theorem and Theorem 11.1, the integration of F against any $\mathcal{O}_{\mathcal{S}}$ is zero. Therefore $\widetilde{F}$ must be almost everywhere zero on $\mathcal{S}$ and F must be almost everywhere zero on $Sp_{2n}(\mathbb{R})$. Then $(v, \pi(\widetilde{g})u)$ must be identically zero for all $\widetilde{g} \in \widetilde{Sp_{2n}(\mathbb{R})}$. This is a contradiction. Thus there exists $p_0 + q_0 = m$, such that

$$\mathcal{L}_{\widetilde{Sp_{2n}(\mathbb{R})}}(\mathcal{P}^c_{p_0,q_0} \otimes V_\pi) \neq 0$$

Q.E.D.

If $\pi \in \mathcal{R}_{ss}(Sp_{2n}(\mathbb{R}), m)$, then $\mathcal{L}_{\widetilde{Sp_{2n}(\mathbb{R})}}(\mathcal{P}^c_{p,q} \otimes V_\pi) \neq 0$ if and only if

$$\mathcal{L}_{\widetilde{Sp_{2n}(\mathbb{R})}}(\mathcal{P}^c_{q,p} \otimes V_\pi) \neq 0$$

In other words, $\pi \in \mathcal{R}(\widetilde{Sp_{2n}(\mathbb{R})}, \omega_{p,q})$ if and only if $\pi \in \mathcal{R}(\widetilde{Sp_{2n}(\mathbb{R})}, \omega_{q,p})$. Therefore, we can further show that

$$\omega : \mathcal{R}_{ss}(Sp_{2n}(\mathbb{R}), m) \to \oplus_{p+q=m, p \leq q} \widehat{\widetilde{O_{p,q}}}_{ad}$$

is injective.

We can further generalize Theorem 11.13.

Theorem 11.14 *Suppose $m \geq 2n$. Then*

$$\omega_1 : \mathcal{R}_{ss}(Sp_{2n}(\mathbb{R}), m) \to \oplus_{p=m-2n}^{l} \widehat{O_{p,m-p_{ad}}} \qquad (l = \min(m-2n, \lfloor \frac{m}{2} \rfloor))$$

is injective.

11.6 A conjecture on Unitarity

For a stable range dual pair (G, G') with G the smaller group, it is shown by Li that unitarity is preserved under the dual pair correspondence from $\mathcal{R}(\widetilde{G}, \omega)$ to $\mathcal{R}(\widetilde{G}', \omega)$. We formulate the following conjecture.

Conjecture 4 *Suppose (G, G') is a reductive dual pair and (π, V_π) is an irreducible unitary representation of G. If $\mathcal{L}_{\widetilde{G}}$ is well-defined and nonvanishing, then $\omega(\pi)$ is unitary.*

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