Homotopy Theory and Topoi

by

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A.B. Mathematics
Princeton University, 1993

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY
AT THE
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

JUNE 1998

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ABSTRACT

This work is concerned with functors from the category of topoi and geometric morphisms to homotopy model categories and adjoint functors in Quillen's sense. In the case when the model category is one of diagrams and the cofibrations are the monomorphisms, it is sufficient that the notion of weak equivalence be definable by a set of axioms of geometric logic, yielding a model category in Set. This is a corollary of the central result of this paper, a recognition theorem for locally presentable model categories with an accessible subcategory of weak equivalences. Various localization results follow, unifying and extending work of P. Goerss, J.F. Jardine, A. Joyal and S. Crans. Included is an internal construction of homotopy group objects of simplicial objects, valid in any elementary topos, and a detailed definition of geometric constructions (in the topos-theoretic sense).

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Introduction

> Les synthèses nouvelles par le rapprochement de disciplines mathématiques différentes constituent des événements remarquables dans l'histoire des mathématiques. Une telle synthèse semble émerger actuellement du rapprochement de:
>
> (1) La géométrie algébrique sous la forme élaborée par Grothendieck.
> (2) La logique formelle.
>
> Le point de contact s'est effectué aux environs de 1970 par W. Lawvere et M. Tierney et l'instrument de rapprochement a été la théorie des catégories, plus particulièrement la théorie des faisceaux. Depuis ce moment, une dialectique incessante imprime un mouvement dynamique à toute une série de recherches qui visent à rapprocher les méthodes suivantes:
>
> (1) Mathématique intuitionniste.
> (2) Forcing de Cohen et Robinson.
> (3) Logique algébrique.
> (4) Géométrie algébrique.
> (5) Géométrie différentielle et analytique.
> (6) Topologie algébrique: cohomologie, homotopie.
> (7) Théorie de Galois.
>
> Certains rapprochements sont dans un stade avancé, d'autres encore embryonnaires: $(6)\leftrightarrow(3)$.
>
> A. Boileau and A. Joyal [12]

Motivation. This research was prompted by two questions in abstract homotopy theory:

(i) Is there a recognition principle for cofibrantly generated homotopy model categories for which one does not have to supply the generating cofibration data *explicitly?*

To elaborate, in the existing attempts to organize the study of Quillen model categories around some central "recognition principle", the class of weak equivalences is (as in all of Quillen's axiomatics) an *a priori* existing class of morphisms with magical properties. But this is not so in practice, where weak equivalences do not arise through any implicit or non-constructive procedure (e.g. transfinite induction) but via some explicit, functorial definition (say, as a class of morphisms inducing isomorphisms on "naively" defined homotopy or homology groups). The recognition theorems for cofibrantly generated model categories found in Hirschhorn–Kan [20] and Goerss–Jardine [32] make the easy part of the proof automatic, but still seem to leave the tedious, set-theoretic conditions for a case-by-case analysis.

Is there a way to reverse this, and formulate a recognition principle that incorporates the *definition* of weak equivalences as part of the data?

(ii) Perhaps a decade after Kan pioneered the combinatorial-functorial approach to homotopy theory via simplicial sets, the need arose to repeat it for simplicial *sheaves* (Illusie [36]). "Repeat" is to be understood in the naivest possible sense here. Now is one reduced to intuition and experimentation when trying to define homotopy groups (group objects?), Postnikov towers... for simplicial sheaves, or is there a language common to every Grothendieck topos that is suitable for combinatorial homotopy theory in Kan's original sense?

Three surprising facts emerge. One is that (*i*) and (*ii*) can be made precise and answered within a logical syntax that is quite suited for practice. Another is that this syntax helps to understand and deepen Quillen's ideas [58] [59] on the interaction of cohomology of abstract algebras with sheaf cohomology. And the last is that this circle of ideas is known, and has been known for 25 to 10 years. It is rooted in work of Grothendieck's school on the classifying topos (Giraud [31]) and locally presentable categories (Gabriel–Ulmer [29]), in the highly formal study of categories of structured objects by Ehresmann's circle (Lair [48]), and in the *elementary topos theory* or *axiomatic sheaf theory* of Lawvere and Tierney [62]; more precisely, in the study of *geometric logic* and the *internal language* of a topos. (An attempt on the part of this writer to assign priority and credit to the contributors to this latter beautiful theory would surely result in omissions and disgraceful errors; see Section 0 for further literature and references.) The "circle of ideas" alluded to above can be collected as *categorical model theory* or the study of *accessible categories.*

Acknowledgments. I wish to thank Michael Makkai and Jiří Rosický for tutoring me on accessibility; Steve Awodey, Carsten Butz and Daniel Kan for reading and commenting on (versions of) this paper; Phil Hirschhorn, Paul Goerss and Rick Jardine for extended email-exchanges. Last but foremost, I am indebted to my advisor, Michael Hopkins, for mentioning the word *topos* to me for the first time and for never objecting to the logical idiosyncrasies of my mind.

A guide to this paper.

The rest of the introduction sets the stage with references, some standard and substandard terminology and a bit of category theory not fitting elsewhere. The section entitled *Accessibility* is a compendium of facts and figures intended mainly for orientation and reference. The thrust is that the category of accessible categories and accessible functors is where much of *workable* abstract homotopy takes place, and this provides explanations for such "visceral" differences between (say) topological spaces and simplicial sets, or abelian groups and the opposite category of abelian groups that every homotopist no doubt experienced when examining monomorphisms or notions of "small objects". (Of special interest in this respect is the closing subsection and its examples of non-accessibility.)

The second section of background material revolves on the notion of *geometric construction.* If category theory were homotopy theory, then a construction would be the building of a relative cell complex: start with some object (space); glue on a transfinite succession of cells, taking care of where the boundary goes and the ensuing topology at limit stages; and allow a retraction at the end, perhaps. Despite the suggestive (but *quite* partial) analogy[1], the adjective "geometric" owes its origin not to geometry, but — together with "geometric

[1]A relative cell complex is constructed via pushouts, or only colimits at any rate. The analogy fails to convey that a geometric construction also allows *finite limits* as steps.

logic" and "geometric morphism" — to the kind of adjoint functor between $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Y)$ that is induced by a continuous map from the space X to the space Y. The reader unwilling to ingest the syntactic details is advised to turn the page after the motivating examples: image factorizations, Verdier's coskeleton and Kan's Ex^∞ functor. Sketches, classifying topoi and geometric logic are examined from the perspective of geometric constructions. Topos theorists will not find new information here.

The next section is concerned with question (*ii*) above, and makes the heaviest use of logical formalism. Only some of its results are needed for model categories, however, and may be skipped by the reader so inclined.

Section 4 opens with yet another look at what must be one of the oldest transfinite factorization/reflection principles in category theory, the *small object argument* already present in Gabriel–Zisman [30] Proposition 5.5.1. The main result of this paper is Theorem 4.2, especially in light of Corollaries 5.1 and 5.8. The proof calls on most of the foundational material, but is otherwise well in line with existing model-categorical arguments. The next section reaps the benefits of the extra technology. We close with comparisons with existing results, and a discussion of limitations and hopes for extensions of the theory.

0. Preliminary

Set theory and generalities.
We work in an extension of ZFC with a sufficiently good calculus of classes, including the axiom of choice for proper classes. Gödel-Bernays (a fortiori, Morse-Kelley) set theory will do, the system ZFC/s of Feferman [24], or an additional Grothendieck universe $\mathcal{U}$ above the usual universe $\mathcal{V}$ of ZFC sets. Category theorists sometimes turn the tables and prescribe a sufficiently rich category of sets as foundations (see MacLane [50]). Either way, there are no instances of iterated Grothendieck universes in this paper.

We rely on vol. I-II of Borceux [13] and the less comprehensive MacLane [49] for general category theory, and Johnstone [40], MacLane and Moerdijk [51] and vol. III of Borceux [13] for topos theory (including geometric logic). The theory of classifying topoi is covered in all three of the latter references; see also Barr–Wells [9] and Makkai–Paré [52].

Terminology.
While "global" homotopy theory, for set-theoretic reasons, seems to be restricted to Grothendieck topoi, certain aspects of this account remain valid under less stringent conditions: in increasing order of specialization, for any elementary topos, any topos with a natural numbers object, and complete(=cocomplete) topoi. The context will make it clear where we're working. By default, *topos* means *elementary topos* till the end of Section 4, and *Grothendieck topos* thereafter.

Call a topos *Boolean* if it is equivalent to the category of sheaves on a complete Boolean algebra (in the canonical topology). A topos is *boolean* if its internal logic is classical, equivalently, if every subobject has a complement. Then a topos is Boolean iff it is boolean and localic. This terminology is a deviation from the literature where our Boolean topoi are named *AC* (for Axiom of Choice, ie. that epis split) and boolean topoi are called Boolean.[2] Our choice of orthography aims to increase consistency: Barr covers (surjective geometric morphisms with domain an AC topos) have traditionally been called Boolean covers, geometric morphisms with source an AC topos are *Boolean* (or *generalized*) points and AC topoi

[2]Or boolean, as a mere variant in spelling.

are bi-interpretable with models of ZFC known as *Boolean-valued* since the 60's (see [10]). As a final excuse, the weaker concept of boolean will not feature in this work.

Locally presentable and accessible categories.
As for the former, the definitive account is still Gabriel–Ulmer [29]; see [63] for the authors' English summary. The proof of the Limit theorem of Makkai and Paré, probably *the* central tool in our study of model categories and localization, is contained in [52]. That seminal study is likely to appeal to readers with background in logic and set theory besides category theory. The textbook of Adámek–Rosický [1] is probably the right mixture of depth and ease-of-read for a first study. Information on locally presentable and accessible categories, as well as their interaction with Grothendieck topoi, is scattered in various chapters of Borceux's encyclopedic monograph. Ehresmann's closely allied theory of *sketches* is covered in all three of the latter references, as well as in Barr–Wells [9].

Homotopy model categories.
We work exclusively in the context of Quillen's *closed model categories*; good introductions include Dwyer–Spaliński [22] and Goerss–Jardine [33]. For reference, Quillen's axioms follow:

M1: $\mathcal{E}$ has finite limits and colimits.

M2: If f and g are composable morphisms in $\mathcal{E}$, and if two of f, g and fg are weak equivalences, then so is the third.

M3: A retract (in the category of morphisms of $\mathcal{E}$) of a fibration, cofibration or weak equivalence is respectively a fibration, cofibration or weak equivalence.

M4: Given the commuting solid arrow diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \overset{l}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

with i a cofibration and p a fibration, if (i) p or (ii) i is a weak equivalence then a lifting l exists making both triangles commute. (One also says, "i has the left lifting property with respect to p" or "p has the right lifting property with respect to i" when an l exists in every commutative square of this type.)

M5: Every morphism can be factored as (i) an acyclic cofibration followed by a fibration, and also as (ii) a cofibration followed by an acyclic fibration.[3]

The classes defining a homotopy model category will always be listed in the order $\langle$cofibrations; weak equivalences; fibrations$\rangle$. The utterances

"$\langle$LLP; weak equivalences; fibrations$\rangle$ give a model structure"

"$\langle$cofibrations; weak equivalences; RLP$\rangle$ give a model structure"

mean, respectively, that cofibrations are defined as those maps having the left lifting property with respect to acyclic fibrations (fibrations as the maps having the right lifting property with respect to acyclic cofibrations).

[3](Co)fibrations that are also weak equivalences will be called *acyclic* (rather than *trivial* or *aspherical*).

The retract argument. For a category $\mathcal{C}$ and class Σ of morphisms of $\mathcal{C}$, let $\mathrm{RLP}(\Sigma)$ – resp. $\mathrm{LLP}(\Sigma)$ – denote the class of morphisms of $\mathcal{C}$ having the right – resp. left – lifting property with respect to every morphism in Σ. Though the following observation goes back to Quillen [58], we cite it as formulated by Hirschhorn [34].

Lemma 0.1. *Let g, p, i be morphisms in a category $\mathcal{E}$ such that $g = pi$.*

- *If $p \in \mathrm{RLP}(g)$, then g is a retract of i in the category $\mathrm{Mor}(\mathcal{E})$. Dually,*
- *If $i \in \mathrm{LLP}(g)$, then g is a retract of p.*

Transfinite composition vs. filtered colimits. The next proposition identifies a seemingly specialized notion with a much more common one. It is natural enough in the context of model categories, but does not appear to surface in their literature.

Definition 0.2. A diagram (ie. small category) is *filtered* if it contains a compatible cocone on each of its finite subdiagrams. For a cardinal κ, it is *κ-filtered* if it contains a compatible cocone on each of its subdiagrams of fewer than κ arrows. A diagram is *directed* if it is filtered and contains at most one arrow between any two objects (so it is a poset). A colimit $\mathrm{colim}_\alpha F$ is said to be a *transfinite composition* if it is indexed by a well-ordered set α, and has the following additional property: thinking of α as a (von Neumann) ordinal, for every $\beta \prec \alpha$ that is a limit ordinal, F restricted to the diagram $\{\gamma \preccurlyeq \beta\}$ is a colimiting cocone on F restricted to $\{\gamma \prec \beta\}$. (So F is "continuous".)

Let $\mathcal{C}$ be a category and Σ a subcategory of $\mathcal{C}$. Call Σ closed under colimits of type $\mathcal{D}$ in $\mathcal{C}$ (here $\mathcal{D}$ may run over some class of diagrams) if for any such diagram in Σ, every cocone on it which is colimiting in $\mathcal{C}$ lies entirely within Σ.

Proposition 0.3. *The following conditions are equivalent:*

(i) *Σ is closed under transfinite composition.*
(ii) *Σ is closed under directed colimits.*
(iii) *Σ is closed under filtered colimits.*

The proof given here is based on Adámek–Rosický [1].

Proof.
(i) $\Longrightarrow$ (ii): let $\mathcal{D}$ be a directed diagram. The proof is by transfinite induction on the cardinality α of objects of $\mathcal{D}$. When α is finite, $\mathcal{D}$ contains a terminal object, and the statement is trivial.

Now consider any directed diagram $\mathcal{D}$, and write D for its set of objects. By the assumption $\mathcal{D}$ is directed, there exists a function bd from the set of finite subsets of D to D with the property that $bd(J)$, $J \subseteq D$ finite, is an upper bound for J in $\mathcal{D}$. By iteration, define functions cl_i, $i \in \mathbb{N}$ from the power set of D to D:

$$\mathrm{cl}_0(I) := I \cup \bigcup_{\substack{J \subset I \\ J \text{ finite}}} bd(J)$$

$$\mathrm{cl}_{n+1}(I) := \mathrm{cl}_0(\mathrm{cl}_n(I)) \qquad n \in \mathbb{N}$$

and define yet another function $\mathsf{closure}$ from the power set of D to D by cases:

$$\mathsf{closure}(I) := \begin{cases} I \cup bd(I) & \text{if } I \text{ is finite} \\ \bigcup_{n \in \mathbb{N}} \mathrm{cl}_n(I) & \text{if } I \text{ is infinite.} \end{cases}$$

Observe

- $\mathsf{closure}(I)$ is finite if I is finite, and of the same cardinality as I if I is infinite
- for any $I \subseteq D$, the full subdiagram of $\mathcal{D}$ with objects $\mathsf{closure}(I)$ contains I, and is directed.

Assume now α infinite. Well-order D, ie. (using the axiom of choice) find a bijection $\alpha \xrightarrow{f} D$. By transfinite induction, define subsets S_β ($\beta \prec \alpha$) of D

$$S_0 := f(0)$$
$$S_{\beta+} := \mathsf{closure}(S_\beta \cup f(\beta))$$
$$S_\beta := \bigcup_{\gamma \prec \beta} S_\gamma \qquad \text{for limit ordinals } \beta$$

Let $\mathcal{D}_\beta$ be the full subdiagram of $\mathcal{D}$ with objects the S_β. Observe

(1) $\bigcup_{\beta \prec \alpha} \mathcal{D}_\beta = \mathcal{D}$
(2) $\mathcal{D}_{\beta_1} \hookrightarrow \mathcal{D}_{\beta_2}$ for $\beta_1 \prec \beta_2$
(3) $\mathcal{D}_\beta = \bigcup_{\gamma \prec \beta} \mathcal{D}_\gamma$ for a limit ordinal β
(4) S_β is finite if β is finite, and is at most the cardinality of β when β is infinite. So (since α is an infinite cardinal) the cardinality of the objects of $\mathcal{D}_\beta$ is less than α.

Now define a functor F from α to the underlying category $\mathcal{C}$; at $\beta \prec \alpha$, $F(\beta) := \operatorname{colim} \mathcal{D}_\beta$ and for $\beta_1 \prec \beta_2$, the arrow $F(\beta_1) \to F(\beta_2)$ is induced by (2). By the induction hypothesis and (4), the image of the diagram $F(\alpha)$ is entirely in Σ. By (3), it is a transfinite composition. By (1), it computes $\operatorname{colim} \mathcal{D}$. It follows that any cocone on $\mathcal{D}$ colimiting in $\mathcal{C}$ is also colimiting on a transfinite composition lying in Σ, completing the induction step.
(ii) $\Longrightarrow$ (iii): for every filtered diagram $\mathcal{C}$, there exist a directed $\mathcal{D}$ and a cofinal functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$. (This fact certainly goes back to Grothendieck-Verdier; see e.g. Adámek–Rosický [1] Theorem 1.5 for a proof).
(iii) $\Longrightarrow$ (i): evident. □

While for any regular cardinal κ and κ-filtered diagram there exists a cofinal *κ-directed* functor in it, the equivalence of (i) and (ii) breaks down for uncountable cardinals. That is, closure under transfinite composition of cofinality at least κ — for some fixed regular cardinal $\kappa \succ \aleph_0$ — does not imply closure under κ-filtered colimits. Thus Bousfield's notion of sequential smallness [14] is weaker than presentability (see Definition 1.2).

Colimits in the category of morphisms. For a category $\mathcal{C}$ and class of morphisms Σ, call Σ *closed under colimits of type $\mathcal{D}$ in* $\mathrm{Mor}(\mathcal{C})$, where $\mathcal{D}$ may run over a class of *filtered* diagrams now, if the full subcategory of the category of morphisms of $\mathcal{C}$ with objects the Σ is closed under colimits of type $\mathcal{D}$. The following trifle is only included for completeness.

Proposition 0.4. *If Σ is closed under colimits of type $\mathcal{D}$ in* $\mathrm{Mor}(\mathcal{C})$*, then Σ is closed under colimits of type $\mathcal{D}$ in $\mathcal{C}$.*

1. Accessibility

Other than a few specialized lemmas, the results in this section are in the literature; all unattributed propositions and definitions are due to Makkai and Paré [52] and/or Adámek–Rosický [1] (but see also their historical references).

Notation 1.1. Throughout this section, the variable κ is to range over the class of infinite regular cardinals.

This convention is merely to reduce eye-strain, since the phrases "there exists an infinite regular cardinal κ" and "for every infinite regular cardinal κ" are ubiquitous in the theory of accessibility.

Definition 1.2. An object X of a category $\mathcal{C}$ is κ-presentable if $\hom_{\mathcal{C}}(X, -)$ preserves κ-filtered colimits. It is *presentable* if it is κ-presentable for some κ.

The phrase that a functor *preserves (co)limits* of some type is used in this work in a sense slightly stronger than usual: it means both that the domain category possesses those type of (co)limits and that the functor takes (co)limiting (co)cones into (co)limiting ones.

A functor preserving κ-filtered colimits for some κ is said to have a *rank*. The rank (or *rank of presentability*) of an object is the least rank, if any, of the hom-functor it corepresents.

A colimit of size less than κ of κ-presentable objects is again κ-presentable, in any category; this follows from the fact that κ-filtered colimits commute with limits of size less than κ in *Set*.

Definition 1.3. A category $\mathcal{A}$ is *accessible* if it possesses

- all κ-filtered colimits, for some κ and
- a set of presentable objects G generating $\mathcal{A}$ under κ-filtered colimits; that is, every object of $\mathcal{A}$ is to be writable as a κ-filtered colimit of objects from G.

Proposition 1.4. *In an accessible category $\mathcal{A}$, every object has a rank. For any cardinal κ, there exists only a set of isomorphism types of κ-presentable objects in $\mathcal{A}$.*

Definition 1.5. A category $\mathcal{A}$ is *κ-accessible* if it has

- all κ-filtered colimits and
- a set of κ-presentable objects G generating $\mathcal{A}$ under κ-filtered colimits.

Definition 1.6. A set of objects G of a category $\mathcal{C}$ forms a *dense generator* if for every $X \in \operatorname{ob}\mathcal{C}$, $\operatorname{colim} \mathcal{G} = X$ where $\mathcal{G}$ is the canonical functor $G/X \to \mathcal{C}$ that forgets X; here G/X is the full subcategory of $\mathcal{C}/X$ whose objects are elements of G over X.

Proposition 1.7. *Let $\mathcal{A}$ be κ-accessible, and G a set of representatives of isomorphism types of κ-presentable objects. G is a dense generator. Write G^{op} for the full subcategory of $\mathcal{A}^{op}$ with objects the G; the contravariant hom-functor map $\mathcal{A} \to Set^{G^{op}}$ is full, faithful, and preserves κ-filtered colimits.*

Proposition 1.8. (raising the degree of accessibility)
For an accessible category $\mathcal{A}$, there exist arbitrary large regular cardinals κ such that $\mathcal{A}$ is κ-accessible.

The proof is substantial. In general, it is not the case that every regular κ (above a threshold) will do; see Borceux vol. II [13] p.267 for an explanation of the intricacy of transfinite combinatorics involved.

Definition 1.9. A functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is accessible if $\mathcal{A}$ is an accessible category, $\mathcal{B}$ is an accessible category, and F has a rank.

F is called *κ-accessible* if $\mathcal{A}$ and $\mathcal{B}$ are, and F preserves κ-filtered colimits.

Example 1.10. Let $\mathcal{A}$ be an accessible category. A functor $\mathcal{A} \to Set$ is accessible iff it is a colimit of a (small) diagram of covariant hom-functors.

Proposition 1.11. *Let $\mathcal{A}$, $\mathcal{B}$ be accessible categories. A functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is accessible iff for each $X \in \operatorname{ob}\mathcal{B}$, the functor $\mathcal{A} \to Set$ given by $\hom_{\mathcal{B}}(X, F(-))$ is accessible.*

Theorem 1.12. (the uniformization theorem of Makkai and Paré)
Given a set of accessible categories and a set of accessible functors between them, there exist arbitrary large regular cardinals κ such that simultaneously

- *Each of the categories is κ-accessible.*
- *Each of the functors preserves κ-filtered colimits (hence, is κ-accessible).*
- *For any functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ in the set and any object X κ-presentable in $\mathcal{A}$, $F(X)$ is κ-presentable in $\mathcal{B}$.*

In particular, accessible categories and accessible functors form a category, to be denoted ACC.

Proposition 1.13. (adjoints)

- *Every accessible functor satisfies Freyd's solution set condition.*
- *A functor between accessible categories that is an adjoint is accessible.*

The second item is news for right adjoints only, of course.

Theorem 1.14. (the Limit theorem of Makkai and Paré)
ACC *is closed under weighted bilimits.*[4]

It is hard to overestimate the power of the Limit theorem. Corollaries 1.16-1.32 all follow by judicious choices of the indexing diagrams.

Corollary 1.15. *Theorem 1.14 holds with*

- *pseudolimits,*
- *lax limits,*
- *op-lax limits*

in place of "bilimits".

All notions of Limit will do, so to say, *but* the naive one (ie. computed as in the category of small categories[5]). See Makkai–Paré [52] p.99 for an illustration. Roughly, the reason is that the data for ACC describe everything up to equivalence, while the 1-categorical limit construction is not equivalence-invariant.

Corollary 1.16. *Let $\mathcal{B}_i \xrightarrow{F_i} \mathcal{A}$ $(i = 1, 2)$ be accessible functors, $\mathcal{D}$ a diagram.*

- *The functor category $\mathcal{A}^{\mathcal{D}}$ is accessible. Maps between functor categories induced by precomposition are accessible.*

[4] The meaning of 2-categorical terms involved is carefully explained in [52]; note that usage may vary slightly from author to author.

[5] This is the reason behind the capitalization in "Limit".

- *Lawvere's "comma" category $F_1 \downarrow F_2$ is accessible, with the forgetful functors $F_1 \downarrow F_2 \to F_i$ being accessible. In particular:*
- *every over- and undercategory $\mathcal{A}/X$, $X/\mathcal{A}$, $X \in \operatorname{ob}\mathcal{A}$, is accessible.*

Corollary 1.17. *Let $\mathbb{T} = (T, \mu, \eta)$ be an* accessible monad *on $\mathcal{A}$, ie. a monad (triple) such that the underlying functor $T : \mathcal{A} \to \mathcal{A}$ is accessible. Then the category of algebras $\mathcal{A}^{\mathbb{T}}$ is accessible.*

Corollary 1.18. *Let $\mathbb{T} = (T, \mu, \eta)$ be an* accessible comonad *on $\mathcal{A}$, ie. a comonad (cotriple) such that the underlying functor $T : \mathcal{A} \to \mathcal{A}$ is accessible. Then the category of coalgebras $\mathcal{A}_{\mathbb{T}}$ is accessible.*

The temptation is to call a category $\mathcal{A}$ an accessible subcategory of $\mathcal{B}$ if the inclusion functor $\mathcal{A} \hookrightarrow \mathcal{B}$ is accessible. That would still allow, however, with respect to a fixed κ-filtered diagram $\mathcal{D} \to \mathcal{A}$ (κ here being greater than the rank of accessibility of the inclusion) for *some* of the cocones on $\mathcal{D}$ that are colimiting in $\mathcal{B}$ to fall into $\mathcal{A}$, and for other colimiting cocones not to be contained in $\mathcal{A}$. The failure for isomorphic (colimiting) cocones to exhibit identical behavior introduces wholly superfluous 2-categorical complications. Hence the stronger

Definition 1.19. An inclusion $\mathcal{A} \xrightarrow{i} \mathcal{B}$ is an *accessible subcategory* if

- $\mathcal{A}$ and $\mathcal{B}$ are accessible categories
- $\mathcal{A}$ is closed under κ-filtered colimits in $\mathcal{B}$. (As usual, it is assumed that κ is some infinite regular cardinal such that all κ-filtered colimits do exist in $\mathcal{B}$.)

Corollary 1.20. *An accessible subcategory $\mathcal{A} \hookrightarrow \mathcal{B}$ is isomorphism-closed, ie. if $X \xrightarrow{m} Y$ is an isomorphism in $\mathcal{B}$ and $X \in \operatorname{ob}\mathcal{A}$, then $m \in \operatorname{mor}\mathcal{A}$.*

Indeed, m is a colimiting cocone on the identity functor, which is κ-filtered.[6] □

Corollary 1.21. *A full accessible subcategory is closed under retracts.*

Proof. An idempotent morphism $A \xrightarrow{f} A$ can be thought of as a functor from the "idempotent loop" diagram

a diagram that is ∞-filtered, ie. κ-filtered for every κ. This idempotent splits iff the functor has a colimit. □

Corollary 1.22. (to Theorem 1.14)
The intersection of a set of accessible subcategories $\mathcal{A}_\lambda \hookrightarrow \mathcal{B}$ ($\lambda \in \Lambda$) is an accessible subcategory.

Proof. Consider the pseudo-pullback P of the diagram $\mathcal{A}_\lambda \hookrightarrow \mathcal{B}$, $\lambda \in \Lambda$. (The weighting is the functor to the terminal category, ie. is "none".) This pseudo-pullback is, by definition, the category with objects all $\Lambda+1+\Lambda$–sequences $\langle a_\lambda\,(\lambda \in \Lambda);\ b;\ m_\lambda\,(\lambda \in \Lambda)\rangle$ with $a_\lambda \in \mathcal{A}_\lambda$, $b \in \mathcal{B}$, $a_\lambda \xrightarrow{m_\lambda} b$ isomorphisms in $\mathcal{B}$, and as morphisms $\Lambda+1$–tuples of maps commuting

[6] Conceptually, it would seem cleaner to strengthen the definition of a functor *preserving colimits of some type* to demand that if a colimiting cocone has a pre-image, so does every cocone in its isomorphisms class (ie. initial cocone). This would leave the statements of the theory intact, and insofar as accessible category theory is only concerned with isomorphism classes, it does not seem to limit applications. We remain true to the large existing body of work on accessible categories, however.

with the respective isomorphisms. Let I be the intersection of the $\mathcal{A}_\lambda$; the obvious inclusion $I \hookrightarrow P$ is an equivalence of categories, since it is full, faithful and essentially surjective by 1.20. □

By analogous reasoning,

Corollary 1.23. *Given an accessible functor $\mathcal{X} \xrightarrow{F} \mathcal{A}$ and an accessible subcategory $\mathcal{B}$ of $\mathcal{A}$, the inverse image $F^{-1}(\mathcal{B})$ is an accessible subcategory of $\mathcal{X}$.*

Given that a category is accessible iff it is equivalent to a distinguished full subcategory of a *Set*-valued functor category (ie. presheaf topos), it is not surprising that *full* accessible subcategories are easier to characterize than arbitrary ones.

Proposition 1.24. *Let $F, G : \mathcal{A} \to \mathcal{B}$ be accessible functors, $F \xrightarrow{\xi} G$ a natural transformation. The full subcategory of $\mathcal{A}$ whose objects are those X such that ξ_X is an isomorphism, is an accessible subcategory.*

Proposition 1.25. *A full subcategory $\mathcal{A}$ of an accessible category $\mathcal{B}$ is an accessible subcategory iff it is closed under κ-filtered colimits in $\mathcal{B}$ (for some κ), and in addition the inclusion functor satisfies the solution set condition: for every $X \in \operatorname{ob}\mathcal{B}$ there exists a set $\{A_\lambda | \lambda \in \Lambda_X\}$ of objects of $\mathcal{A}$ such that any morphism $X \to A$ with $A \in \operatorname{ob}\mathcal{A}$ factors through a member of Λ_X.*

The previous statement seems best possible within the boundaries of ZFC set theory. The proposition that any full subcategory of a locally presentable category closed under κ-filtered colimits is an accessible subcategory is equivalent to the set-theoretic "large cardinal axiom" known as Vopěnka's Principle.

Proposition 1.26. *An accessible category is complete iff it is cocomplete.*

Definition 1.27. A locally presentable category is a (co)complete accessible category.

Being locally κ-presentable means being κ-accessible and (co)complete.

Proposition 1.28. (cf. 1.8)
Let $\mathcal{K}$ be a locally presentable category. There exists an infinite regular cardinal κ_0 such that K is κ-presentable iff $\kappa_0 \preccurlyeq \kappa$.

Proposition 1.29. (compare with 1.7)
$\mathcal{K}$ is locally κ-presentable iff it allows a full, reflective, κ-filtered colimit preserving embedding $\mathcal{K} \hookrightarrow \operatorname{Pre}(G)$ into a presheaf category.

Corollary 1.30. *In a locally κ-presentable category κ-filtered colimits commute with limits of size less than κ.*

Proposition 1.31. *A locally presentable category is well-powered and co-well-powered.*

Proposition 1.32. *Let $\mathcal{K}$ be locally presentable, and $\mathbb{T} = (T, \mu, \eta)$ an accessible monad on $\mathcal{K}$. The category of algebras $\mathcal{K}^{\mathbb{T}}$ is locally presentable.*

Note that the fact that $\mathcal{K}^{\mathbb{T}}$ has colimits is non-trivial. It follows from 1.17 and 1.26, since the forgetful functor creates limits of $\mathbb{T}$-algebras. But even without the assumption of accessibility, if $\mathcal{C}$ is complete and cocomplete and $\mathbb{T}$ is a monad with rank (ie. the functor part $T : \mathcal{C} \to \mathcal{C}$ preserves κ-filtered colimits for some κ), then $\mathcal{C}^{\mathbb{T}}$ is cocomplete (see Borceux [13] vol.II Prop. 4.3.6).

Among the many possible ways to introduce Grothendieck topoi, consider

Definition 1.33. (compare with 1.29)
A category $\mathcal{E}$ is a Grothendieck topos iff it is a full, reflective subcategory $\mathcal{E} \xrightarrow{i} \mathrm{Pre}(G)$ of a presheaf category such that the reflector ("sheafification") $\ell \dashv i$ preserves finite limits.

That $\mathcal{E}$ is an *accessible* subcategory follows from the prescribed property of the sheaf reflector. Moreover, there is a bijection between Grothendieck topologies on G and left exact (ie. finite limit-preserving) reflectors $\ell : \mathrm{Pre}(G) \to \mathcal{E}$.

Proposition 1.34. (interaction with elementary topoi)
An elementary topos is a Grothendieck topos iff it is locally presentable.

We end this list of positive results on accessible categories with two highly specific lemmas; they are the *sine qua non* of the proposed theory of homotopy model structures on locally presentable categories.

Proposition 1.35. *Let $\mathcal{K}$ be a locally presentable category, and* $\mathrm{Mono}(\mathcal{K})$ *the subcategory of $\mathcal{K}$ whose objects are the objects of $\mathcal{K}$ and whose morphisms are the monomorphisms of $\mathcal{K}$. Then* $\mathrm{Mono}(\mathcal{K})$ *is an accessible subcategory of $\mathcal{K}$.*

Proof. Let $\mathcal{K}$ be λ-presentable. Then its class of monomorphisms is closed under λ-filtered colimits in $\mathcal{K}$; moreover, for every compatible cocone of monomorphisms the induced morphism from the colimit is a monomorphism (Adámek–Rosický [1] Prop. 1.62). In other words, $\mathrm{Mono}(\mathcal{K})$ possesses λ-filtered colimits and is closed under them in $\mathcal{K}$.

Let G be a set of dense generators of $\mathcal{K}$, and H the set of (representatives of isomorphism types of) strong quotients of members of G. Each object of $\mathcal{K}$ can be written as a λ-directed colimit of monomorphisms with domains from the H (Adámek–Rosický [1] Prop. 1.70). Each member of H is λ'-presentable in $\mathcal{K}$ for some $\lambda \preccurlyeq \lambda'$; a fortiori, presentable in $\mathrm{Mono}(\mathcal{K})$. By Definition 1.3, $\mathrm{Mono}(\mathcal{K})$ is an accessible category. □

Proposition 1.36. *For a locally presentable category $\mathcal{A}$, let* $\mathrm{Cart}(\mathcal{A})$ *be the subcategory of* $\mathrm{Mor}(\mathcal{A})$ *whose objects are the same as those of* $\mathrm{Mor}(\mathcal{A})$*, and whose morphisms are those morphisms of* $\mathrm{Mor}(\mathcal{A})$ *which are cartesian squares in $\mathcal{A}$. Then* $\mathrm{Cart}(\mathcal{A})$ *is an accessible subcategory of* $\mathrm{Mor}(\mathcal{A})$.

Non-accessibility. [7]
Accessible categories occur in mathematical practice chiefly as categories of models in Grothendieck topoi of mathematical theories that can be axiomatized using a *set* of axioms of prescribed logical complexity. Slightly more precisely, one starts with a category whose objects are parameterized by a formal language, and passes to the full subcategory whose objects also verify a set of axioms. Morphisms are, then, arbitrary morphisms of the underlying category that commute with the operations. The permanence properties 1.16-1.32 reflect the fact that such categorical operations can be described in an appropriate logical language. Analyzing the languages involved would take us too far from the needs of abstract homotopy theory, but it may be of use to give examples of the *failure* of a category/functor to be accessible. The connection to the "logical intuition" given above is often remote.

[7]The term *inaccessible* has been reserved in set theory for certain large cardinal axioms since the 1930's. Unfortunately, the kinship in nomenclature is more a bug than a feature.

Failure to possess ∞-filtered colimits. A category that does not possess e.g. reflexive coequalizers and split idempotents cannot be accessible (see Corollary 1.21). It can be shown that a small category is accessible iff it has split idempotents (see Makkai–Paré [52] Theorem 2.2.2).

Being the opposite of a locally presentable category. That the theory of accessibility is far from self-dual is brought out dramatically by the following theorem of Gabriel and Ulmer:

Proposition 1.37. *If both a category $\mathcal{E}$ and its opposite $\mathcal{E}^{op}$ are locally presentable, then $\mathcal{E}$ is a poset. (Necessarily, then, it is a small complete lattice.)*

It is possible for both a non-trivial category and its opposite to be accessible. However, consider the opposite of a locally presentable category. It is (co)complete — being the dual of a (co)complete category — hence, if it were accessible, it would be locally presentable, and 1.37 would apply.

Example 1.38. Set^{op}, $\mathrm{Pre}(\mathcal{C})^{\mathrm{op}}$ (more generally $\mathcal{E}^{\mathrm{op}}$ for a Grothendieck topos $\mathcal{E}$), $\mathrm{Ab}^{\mathrm{op}}$ (more generally $(Set^{\mathbb{T}})^{\mathrm{op}}$ for a monad $\mathbb{T}$ with rank), functor, over, under... categories therein are not accessible.

Example 1.39. The category of topological spaces and continuous maps is not accessible; nor is the category of sober spaces and continuous maps.

The opposite category of either is what is known as a *quasi-variety of universal algebras*; such categories are locally presentable. See Barr–Pedicchio [7] [8] for equational descriptions of these duals. These are but two from an extensive family of topological-algebraic dualities in which the algebraic side is locally presentable, therefore its opposite is not accessible.

Example 1.40. The category of totally disconnected compact Hausdorff spaces is not accessible. (Its opposite category, by Stone duality, is the category of Boolean algebras.)

Example 1.41. The category of compact Hausdorff abelian groups is not accessible. (Its opposite category, by Pontryagin duality, is the category of abelian groups.)

Example 1.42. The category of locally compact Hausdorff spaces is not accessible. (Its opposite category, by Gelfand-Neimark duality, is the category of non-unital, commutative C^*-algebras, which is locally $\aleph_1$-presentable; cf. Fakir [23]. So is the category of C^*-algebras, so "noncommutative spaces" are not accessible either.)

Monads without rank. Let $\mathcal{A} \xrightarrow{F} \mathcal{A}$ be the functor part of a monad $\mathbb{T}$ on an accessible category $\mathcal{A}$. Suppose one establishes, by direct calculation, that F does not preserve κ-filtered colimits for any κ, ie. is not accessible. Then $\mathcal{A}^{\mathbb{T}}$ cannot be either. (It would contradict Prop. 1.13 since F, being the composite of two adjoints $\mathcal{A} \to \mathcal{A}^{\mathbb{T}} \to \mathcal{A}$, must be accessible if $\mathcal{A}^{\mathbb{T}}$ is so.)

Example 1.43. (after Borceux [13] vol. II Prop. 4.6.5)

The covariant power set functor $Set \xrightarrow{P} Set$ is not accessible. It underlies a monad whose algebras are complete lattices and lattice maps preserving all suprema. So the latter category is not accessible.

Example 1.44. The functor $Set \xrightarrow{F} Set$ that takes a set (considered as a discrete space) into the set underlying its Stone-Čech compactification has no rank. It underlies a monad whose algebras are compact Hausdorff spaces and continuous maps. So this topological category is not accessible either.

Categories that aren't concrete. Recall that any accessible category possesses a full embedding into a presheaf topos. (Categories with this property are sometimes called *bounded.*) Suppose that a category $\mathcal{E}$ possesses but a faithful functor into a presheaf topos $\mathrm{Pre}(\mathcal{C})$. Composing with the faithful $\mathrm{Pre}(\mathcal{C}) \to \mathit{Set}$ that sends a presheaf $\mathcal{C}^{\mathrm{op}} \xrightarrow{f} \mathit{Set}$ to $\coprod_{c \in \mathrm{ob}\, \mathcal{C}} f(c)$, one obtains a faithful $\mathcal{E} \to \mathit{Set}$. Categories possessing such representations are said to be *concrete.* Following work of J. Isbell, P. Freyd [25] found a necessary and sufficient condition for concreteness, and showed [26] [27] that the categories of CW-complexes and homotopy classes of maps, as well as that of small categories and *natural equivalence classes of functors* are not concrete; a fortiori, they are not accessible. I suspect the conclusion extends to many more "homotopy categories".

Subcategories of topological spaces convenient for homotopy theory are concrete, of course, and ultimately this is responsible for their possessing analogues of *rank* for gauging sizes of objects (suitably coupled to sizes of hom-sets); measures such as cardinality of the set underlying the space or size of the topology. These gauges are less tractable under 2-categorical operations such as passing to categories of algebras or – possibly non-full – subcategories, however.

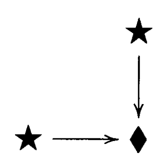
2. Geometric constructions

A *categorical construction with universal arrows* (*categorical construction* or just *construction* for short) has the following blueprint:

- Start with a diagram (ie. small category) $\mathcal{D}_0$ and functor $\mathcal{D}_0 \xrightarrow{F} \mathcal{E}$. (Assume $\mathcal{E}$ has enough limits and colimits.)
- Choose a functor $\mathcal{S} \xrightarrow{G} \mathcal{D}_0 \xrightarrow{F} \mathcal{E}$. (It is convenient to think of $\mathcal{S}$ as a subdiagram of $\mathcal{D}_0$.) There's a dual alternative now, of which we spell out the "co" case. Extend $\mathcal{D}_0$ and F by a colimiting cocone on the composite FG. Given any cocone in $\mathcal{D}_0$ on FG, add the induced arrow from the initial cocone. There results a larger diagram $\mathcal{D}_1$ and functor $\mathcal{D}_1 \xrightarrow{F} \mathcal{E}$ (still denoted by the same letter).
- Iterate. Iterate into the transfinite, if necessary, the process being continuous at limit ordinals.
- Stop at some ordinal β. Discard superfluous portions of the built-up diagram $\mathcal{D}_\beta \xrightarrow{F} \mathcal{E}$: that is, precompose with a fixed functor $\mathcal{C} \to \mathcal{D}_\beta$. (One may as well have *not* put in superfluous universal arrows, of course, but the blueprint as given here is more canonical.)

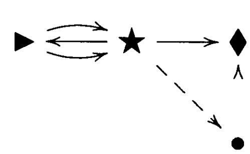
There results a $\mathcal{C}$-diagram in $\mathcal{E}$; but since the construction is natural, it yields in fact a functor $\mathcal{E}^{\mathcal{D}_0} \to \mathcal{E}^{\mathcal{C}}$.

Example 2.1. (image factorizations)
Let $\mathcal{D}_0$ be the diagram ★ $\longrightarrow$ ♦ (and suppress F and $\mathcal{E}$ from the pictures). Choose $\mathcal{S}$ to be the diagram



and G the "folding" functor that sends like objects to like ones. Attach a limit (ie. pullback) cone on FG, with vertex ►; $\mathcal{D}_1$ looks like

The identity on ★ induces canonically a common section of the parallel arrows. Let $\mathcal{S}_1$ be the diagram indexing a reflexive coequalizer (ie. two parallel arrows with a common section) and $\mathcal{S}_1 \xrightarrow{G_1} \mathcal{D}_1$ the inclusion. Attach a colimit (ie. reflexive coequalizer) cocone on FG_1, with vertex •. There's a canonical morphism from • to ♦; note that in addition to the dotted arrow, one has to add some composite arrows to obtain the diagram $\mathcal{D}_2$ which "looks like"

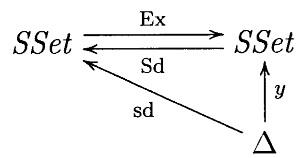


Let finally $\mathcal{C}$ be the diagram (i) ★ - - ➤ • and (ii) • ➤ ♦ respectively, to be sent into $\mathcal{D}_2$ as indicated by the typography. There result two geometric constructions of type $\mathcal{E}^{\mathcal{D}_0} \to \mathcal{E}^{\mathcal{C}}$, ie. $\mathrm{Mor}(\mathcal{E}) \to \mathrm{Mor}(\mathcal{E})$. ($i$) takes values in regular epimorphisms; in a regular category $\mathcal{E}$ — so certainly in a topos — (ii) takes values in monomorphisms.

That was a rather elaborate way of saying "coequalizer of the kernel pair" and "the induced morphism from the coequalizer of the kernel pair of a morphism into the codomain", but syntactic care pays off in the delicate analysis of naturality and transfinite accessibility. The next example is still combinatorial, though.

Example 2.2. (simplicial extension)
Let Δ be the cosimplicial indexing category. The n-simplex possesses a canonical first subdivision, which combine to a functor $\Delta \xrightarrow{\text{sd}} SSet$. (sd is in fact a [restriction of the] composite $Poset \to Poset \to SSet$ where the second arrow is the nerve functor and the first one associates the set of chains, ordered by inclusion, to any poset.) This gives rise, canonically again, to an adjoint situation $SSet \to SSet$ of the type that MacLane [51] calls "general hom-tensor adjunction":

(2.1)



where $\text{Ex}(X) := \text{Hom}_{SSet}(\text{sd}(-), X)$, Sd is the left Kan extension of sd along the Yoneda embedding y, Sd $\dashv$ Ex and the triangle commutes (canonically up to isomorphism, or on the nose by a suitable choice of Sd). It is in fact a geometric morphism from $SSet$ to itself, since the functor sd is what is known as *flat*.

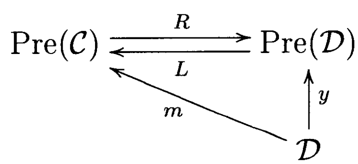
Since the natural level of generality is the following

Proposition 2.3. *Every functor between presheaf topoi* $\text{Pre}(\mathcal{C}) \xrightarrow{R} \text{Pre}(\mathcal{D})$ *that is a right adjoint can be written — up to isomorphism — as a categorical construction that uses limits only*

this is what we will prove. (The reader may wish to come back after Definition 2.8 to check that indeed a categorical construction will be described.)

Any such adjunction is isomorphic to one arising from a situation akin to (2.1):

(2.2)



where m is some *model* functor[8] and L and R are just as above.

For any $d \in \text{ob}\,\mathcal{D}$, write $\text{Elts}(d)$ for the comma category $y \downarrow m(d)$ having as objects morphisms $y(c) \to m(d)$ of $\text{Pre}(\mathcal{C})$ (here $c \in \text{ob}\,\mathcal{C}$) and as morphisms, commutative triangles. There is a forgetful $\text{Elts}(d) \xrightarrow{F_d} \mathcal{C}$. The colimit of the composite $G_d : \text{Elts}(d) \xrightarrow{F_d} \mathcal{C} \xrightarrow{y} \text{Pre}(\mathcal{C})$ is $m(d)$ — this is the fact that every presheaf is canonically a colimit of representables. $R(X) := \text{Hom}_{\text{Pre}(\mathcal{C})}(m(-), X) = \text{Hom}_{\text{Pre}(\mathcal{C})}(\text{colim}\, G_d, X) = \lim_{\text{Elts}(d)^{\text{op}}} \text{Hom}_{\text{Pre}(\mathcal{C})}(y(c), X) =$

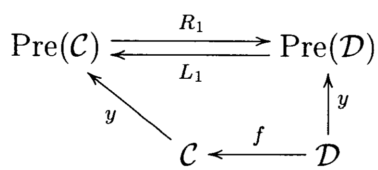
[8]The earliest instance in print of this terminology that I am aware of is Applegate and Tierney's [2]. The set-up itself is as old as adjoint functors: Gabriel and Zisman [30] leave it unnamed; Kan [44] (taking $\mathcal{D} = \Delta$ but aware of the general case) speaks simply of "functors involving c.s.s. complexes". The logician identifies the site $\mathcal{D}$ (lacking any topology in this example) with a syntactical theory, and thinks of m as giving a model in the logical sense. The homotopy theorist may well imagine $\mathcal{D}$ to be Δ and think of m as giving *models* of the affine simplices. (2.2) is far from being the most general set-up possible.

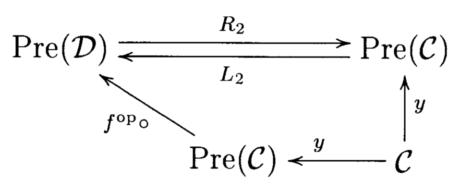
$\lim_{\mathrm{Elts}(d)^{\mathrm{op}}} X(c)$ which is indeed a limit diagram on $\mathcal{C}^{\mathrm{op}} \xrightarrow{X} Set$. So attach a limit cone onto $\mathcal{C}^{\mathrm{op}} \xrightarrow{X} Set$ for each composite XF_d^{op}. By naturality in d, there are induced morphisms among the limiting objects. Obtain a diagram $\mathcal{D}_+^{\mathrm{op}} \to Set$ from which a presheaf can be extracted by precomposition with the $\mathcal{D}^{\mathrm{op}} \xrightarrow{G} \mathcal{D}_+^{\mathrm{op}}$ that keeps only the limiting objects and induced $\mathcal{D}^{\mathrm{op}}$-diagram, omitting the projections from the cones.

Specializing to $\mathcal{D} := \Delta$ and $m := \mathrm{sd}$, note that $\mathrm{Elts}(d)$ will be finite. So one has a recipe for constructing Δ^{op}-diagrams from Δ^{op}-diagrams valid in any category with finite limits. This is what we mean by simplicial extension.

Remark 2.4. The argument does not dualize to the left Kan extension, though that is certainly given by a canonical colimit formula. (RX is given by a limit formula with constant "coefficients" $(y \downarrow m(d))^{\mathrm{op}}$ on the variable diagram X. LX is given by a colimit formula with variable coefficients $(y \downarrow X)$ on the constant diagram m. Our categorical constructions must have constant indexing diagrams.)

Specialize now to the case of a functor $\mathcal{D} \xrightarrow{f} \mathcal{C}$. It gives rise to two basic adjunction situations:

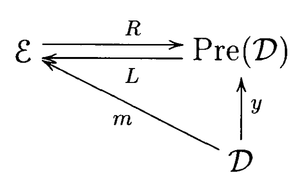




such that $L_1 \dashv R_1 \cong L_2 \dashv R_2$; better known as precomposition with f^{op} and its left and right adjoints (ie. Kan extensions).[9]

Example 2.5. (Verdier coskeleta)
Set $\mathcal{C} := \Delta$, $\mathcal{D} := \Delta|_0^n$ (the full subcategory of Δ containing the ordinals $[0], [1], \ldots, [n]$ as objects) and $\mathcal{D} \to \mathcal{C}$ the inclusion $\Delta|_0^n \xrightarrow{i_n} \Delta$. The composite $\mathrm{cosk}_n := (i_n)_*(i_n^{\mathrm{op}}) : SSet \to SSet$ is the n^{th} *coskeleton functor*. It is obviously a categorical construction with finite limits.

Remark 2.6. Prop. 2.3 stretches further. Consider a right adjoint between any two Grothendieck topoi: $\mathcal{E} \xrightarrow{R} \mathcal{F}$. Choosing some site $(\mathcal{D}, K)$ for $\mathcal{F}$, one knows that R arises as the co-restriction to $\mathrm{Sh}(\mathcal{D}, K)$ of a hom-tensor adjunction

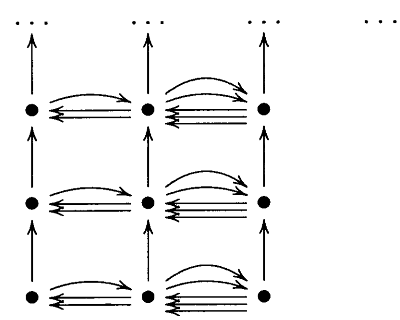


[9]The data combine to give what is called an *essential geometric morphism* $\mathrm{Pre}(\mathcal{D}) \to \mathrm{Pre}(\mathcal{C})$ with direct image $f_* = R_2$, inverse image $f^* = f^{\mathrm{op}}\circ$ and left adjoint to the inverse image $f_! = L_1$.

with $R(X) := \mathrm{Hom}_{\mathcal{E}}(m(-), X)$. Choose now a site of definition $(\mathcal{C}, J)$ for $\mathcal{E}$ as well; thinking of $m(-)$ and X as presheaves, $\mathrm{Hom}_{\mathcal{E}}(m(-), X) \cong \mathrm{Hom}_{\mathrm{Pre}(\mathcal{C})}(m(-), X)$ and from here the calculation is identical. So, in particular, every direct image is a subfunctor of a limit construction.

Example 2.7. (Kan's Ex^∞ functor)
In the generality of 2.2, there need not exist a natural transformation $\mathrm{Id}_{\mathrm{Pre}(\mathcal{D})} \to R$. Part of the subdivision data is a natural transformation $m \xrightarrow{\eta} y : \mathcal{D} \to \mathrm{Pre}(\mathcal{D})$, however. (When $\mathcal{D} = \Delta$ and $m = \mathrm{sd}$ is the nerve of the poset of chains on the finite ordinals that make up Δ, $\eta(d)(d')$ $(d, d' \in \mathrm{ob}\,\Delta)$ associates to $m(d)(d')$, which is in particular a $[d']$-ordered sequence of maps from finite ordinals to $[d]$, the *last* value in each mapping, thus getting a — non-decreasing — function $[d'] \to [d]$ which in turn is an element of $y(d)(d')$.) Returning to the generic presheaf case: η then yields, for any $d \in \mathrm{ob}\,\mathcal{D}$, a cone on the functor $\mathrm{Elts}(d)^{\mathrm{op}} \xrightarrow{F_d^{\mathrm{op}}} \mathcal{D}^{\mathrm{op}}$ with vertex d: associate to $y(d') \to m(d) \in \mathrm{ob}\,\mathrm{Elts}(d)$ the $d \to d' \in \mathrm{mor}\,\mathcal{D}^{\mathrm{op}}$ that corresponds to the composite $y(d') \to m(d) \xrightarrow{\eta_d} y(d)$ via the Yoneda bijection. Hence part of the enlarged diagram $\mathcal{D}_+^{\mathrm{op}}$ is an induced arrow giving a natural transformation between the inclusion $\mathcal{D}^{\mathrm{op}} \hookrightarrow \mathcal{D}_+^{\mathrm{op}} \to Set$ and the "new" presheaf $\mathcal{D}^{\mathrm{op}} \xrightarrow{G} \mathcal{D}_+^{\mathrm{op}} \to Set$.

Apply $\mathcal{D}_+^{\mathrm{op}}$ again to $\mathcal{D}^{\mathrm{op}} \xrightarrow{G} \mathcal{D}_+^{\mathrm{op}} \to Set$, and iterate countably many times. To specialize to simplicial extension for the sake of the following picture, there results a two-dimensional mesh of a diagram (from which the cone projections have been omitted for viewability):



Map the diagram ω (thought of as a poset) into each of the vertical columns, compute the colimit, and add the induced maps between them. Now keep only this "ω^{th}" horizontal line. This is Kan's $\mathcal{E}^{\Delta^{\mathrm{op}}} \xrightarrow{\mathsf{Ex}^\infty} \mathcal{E}^{\Delta^{\mathrm{op}}}$ construction; see Definition 2.12 for the origin of *geometric*.

Formalizing the notion.
The only point of caution is divorcing the enlargement of a diagram $\mathcal{D}$ from the enlargement of its *image* $F(\mathcal{D})$ in $\mathcal{E}$. To take an extreme example, let $\mathcal{D}$ be the diagram

$$A_1 \longrightarrow B_1$$

$$A_2 \longrightarrow B_2$$

and let $\mathcal{D} \xrightarrow{F} \mathcal{E}$ be such a functor that happens to identify A_1 with A_2, and B_1 with B_2 as well, in $\mathcal{E}$:

$$A_1 = A_2 \rightrightarrows B_1 = B_2$$

Taking the equalizer of this parallel pair will not do — a construction permitting such a step will not extend to a functor with domain $\mathcal{E}^{\mathcal{D}}$. In fact, the first half of our definition suppresses the underlying category $\mathcal{E}$ altogether, and is concerned with purely combinatorial manipulations of arrows, cones and cocones on the indexing diagrams.

A *graph* is a small category with all its condiments (set of objects, arrows; source, target, identity functions) *but* a composition law for arrows. A *graph with commutativity conditions* is a pair $\{G, C\}$ where G is a graph and C is a set of pairs of *paths* (finite chains of arrows) in G; within each pair, the chains are to begin and end at the same object. A morphism of graphs with commutativity conditions is a morphism of graphs (ie. two functions sending objects to objects, arrows to arrows, respecting source, target, identity) with the additional constraint that distinguished pairs of paths are to be taken to distinguished ones. Write CAT for the category of small categories, and GraphComm for the category of graphs with commutativity conditions. There is a functor $\mathsf{CAT} \xrightarrow{U}$ GraphComm that takes the graph underlying each category and adds as commutativity conditions all pairs of composable chains of morphisms that compose to the same morphism. U possesses a left adjoint L that allows an easy explicit description. Let now $\mathcal{D}$ be a small category, Arr a set of arrows with given sources and targets, and Comm a set of commutativity conditions on morphisms of $\mathcal{D}$ together with Arr. We are finally ready to state what *extending a diagram $\mathcal{D}$ by arrows* Arr *subject to commutativity conditions* Comm means: $L(U\mathcal{D} \cup \{\text{Arr}, \text{Comm}\})$.

A cone on a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ with vertex $X \in \operatorname{ob}\mathcal{D}$ is a natural transformation to F from the constant functor $\mathcal{C} \to X$. The *free cone* on F results by extending $\mathcal{D}$ by a new object (ie. identity arrow) X_F and new arrows $X_F \xrightarrow{\xi_c} F(c)$ for every $c \in \operatorname{ob}\mathcal{C}$. (This added cone is to be thought of as a "placeholder" for a limiting cone.) The commutativity conditions are $\xi_d = m\xi_c$ for every morphism $c \xrightarrow{m} d$ in $\mathcal{C}$. There is an obvious dual for cocones.

Fix $\mathcal{D}$. A *construction* requires a functor D_β from an ordinal β (thought of as an ordered set, thought of as a category) to CAT that permits definition by transfinite induction as follows:

- $\mathsf{D}_0 := \mathcal{D}$
- For a successor ordinal λ^+, freely add a cone with vertex X_λ and projections $\xi_{c,\lambda}$ ($c \in \operatorname{ob}\mathcal{C}_\lambda$) to D_λ for some chosen functor $\mathcal{C}_\lambda \xrightarrow{F_\lambda} \mathsf{D}_\lambda$. (We omit the dual case of a cocone.) Call the resulting diagram D_λ^+.
- Still at stage λ^+: consider the objects X_κ with $\kappa \preccurlyeq \lambda$. For every cone η with vertex, say, X on $\mathcal{C}_\kappa \xrightarrow{F_\kappa} \mathsf{D}_\kappa \hookrightarrow \mathsf{D}_\lambda \hookrightarrow \mathsf{D}_\lambda^+$ that exists in D_λ^+ add an arrow m_η from X to X_κ. This arrow is subject to the commutativity condition $\xi_{c,\kappa} m_\eta = \eta_c$ for every $c \in \operatorname{ob}\mathcal{C}_\kappa$ as well as to the condition $m_\eta = n$ for every morphism n in D_λ^+ that satisfies $\xi_{c,\kappa} m_\eta = \eta_c$ for every $c \in \operatorname{ob}\mathcal{C}_\kappa$. (Morally: m_η is a placeholder for the unique induced arrow into the limit. Observe that one should also place induced arrows into objects that were declared limiting at a stage κ *prior* to the stage λ when another cone η on the functor came into existence.) Dual construction applies to the cocones. This defines D_{λ^+}. Note that there is a natural inclusion $\mathsf{D}_\lambda \hookrightarrow \mathsf{D}_{\lambda^+}$.
- For a limit ordinal λ, $\mathsf{D}_\lambda := \operatorname{colim}_\alpha \mathsf{D}_\alpha$ for $\alpha \prec \lambda$. The morphisms into D_λ are the colimit inclusions.
- Stop having constructed D_β.

Note that $\mathsf{D}_{\lambda_1} \to \mathsf{D}_{\lambda_2}$ is the inclusion of a subcategory for $\lambda_1 \prec \lambda_2$. (It follows from the fact that two arrows distinct in D_{λ_1} will not get identified in D_{λ_2}. Indeed, all the arrows that get identified at a successor stage must contain a factor that was added only at that stage.)

Given a category $\mathcal{E}$ possessing the requisite (co)limits and functor $\mathcal{D} \xrightarrow{F} \mathcal{E}$, the "interpretation" of the above process in $\mathcal{E}$ is the tautologous one. Having fixed functors computing limits and colimits in $\mathcal{E}$, by transfinite induction define functors $\mathcal{E}^{\mathcal{D}} \xrightarrow{\mathsf{G}_\lambda} \mathcal{E}^{\mathsf{D}_\lambda}$:

- G_0 is the identity.
- For a successor ordinal λ^+, take any diagram $\mathsf{D}_\lambda \to \mathcal{E}$ to the diagram $\mathsf{D}_{\lambda^+} \to \mathcal{E}$ obtained by letting the added cone with vertex X_λ be a limiting cone on $\mathcal{C}_\lambda \xrightarrow{F_\lambda} \mathsf{D}_\lambda \to \mathcal{E}$, the morphisms m_η are the ones induced by the universal properties. (Dually for the cocone case.) This extends to a functor $\mathcal{E}^{\mathsf{D}_\lambda} \to \mathcal{E}^{\mathsf{D}_{\lambda^+}}$ as usual. G_{λ^+} is the composite $\mathcal{E}^{\mathcal{D}} \xrightarrow{\mathsf{G}_\lambda} \mathcal{E}^{\mathsf{D}_\lambda} \to \mathcal{E}^{\mathsf{D}_{\lambda^+}}$.
- For a limit ordinal λ, G_λ is the union of the chain G_α, $\alpha \prec \lambda$.

Definition 2.8. Let $\mathcal{D}, \mathcal{C}$ be diagrams, $\mathcal{E}$ a category with the requisite (co)limits. A functor $\mathcal{E}^{\mathcal{D}} \xrightarrow{U} \mathcal{E}^{\mathcal{C}}$ is a *categorical construction* if it is isomorphic to a functor of the form

$$\mathcal{E}^{\mathcal{D}} \xrightarrow{\mathsf{G}_\beta} \mathcal{E}^{\mathsf{D}_\beta} \xrightarrow{F\circ(-)} \mathcal{E}^{\mathcal{C}}$$

for some ordinal β, where the second arrow is induced by precomposition with a functor $\mathcal{C} \xrightarrow{F} \mathsf{D}_\beta$.

Proposition 2.9. *The composite of two categorical constructions is a categorical construction.*

Observe that saying "there is a categorical construction $\mathcal{E}^{\mathcal{D}} \xrightarrow{U} \mathcal{E}^{\mathcal{C}}$" implies the existence of a combinatorial recipe that makes sense starting with any functor $\mathcal{D} \xrightarrow{F} \mathcal{E}$ into a sufficiently (co)complete category; moreover, though possibly transfinite, the construction is bounded in length *independently of* F. The particular encoding D_λ chosen is non-canonical and may be redundant (e.g. in putting in *all* induced maps from (to) compatible (co)cones) but it should be clear what is meant by *the cones and cocones occurring in a construction.*

We turn to general invariance and accessibility properties of categorical constructions next.

Proposition 2.10. *If* $\mathcal{F} \xrightarrow{f} \mathcal{E}$ *is a functor preserving the (co)limits in the construction* U*,*

$$\begin{array}{ccc} \mathcal{F}^{\mathcal{D}} & \xrightarrow{U} & \mathcal{F}^{\mathcal{C}} \\ \downarrow f & & \downarrow f \\ \mathcal{E}^{\mathcal{D}} & \xrightarrow{U} & \mathcal{E}^{\mathcal{C}} \end{array}$$

commutes up to natural isomorphism.

An important case for toposophic purposes is when filtered colimits commute with finite limits in $\mathcal{E}$, $\mathcal{K}$ is a filtered diagram, $\mathcal{F}$ is $\mathcal{E}^{\mathcal{K}}$ and f is $\operatorname{colim}_{\mathcal{K}}$. (Notice in this connection that geometric constructions commute, up to natural isomorphism, with the formation of functor categories, since (co)limits in functor categories are computed "pointwise".)

Proposition 2.11. *Let* $\mathcal{E}$ *be a locally presentable category. Any categorical construction* $\mathcal{E}^{\mathcal{D}} \xrightarrow{U} \mathcal{E}^{\mathcal{C}}$ *is an accessible functor.*

Proof. Choose a κ exceeding the size of the largest limit employed by U, such that in addition $\mathcal{E}$ is locally κ-presentable (cf. 1.28). Then $\mathcal{E}^{\mathcal{D}}$ is locally κ-presentable as well, hence κ-filtered colimits commute with κ-limits in $\mathcal{E}^{\mathcal{D}}$. Apply 2.10 to deduce U preserves κ-filtered colimits. □

There are three natural ways to sort constructions into subtypes: by restricting the class of limits, the class of colimits, and the ordinal β employed.

Definition 2.12. A *geometric construction* is a categorical construction that employs only finite limits (while colimits may be arbitrary).

These are important for being invariant under inverse image parts of geometric morphisms. There is a type that is in a certain sense dual (or adjoint) to geometric constructions:

Definition 2.13. A *filtered construction* is a categorical construction that employs only filtered colimits (while limits may be arbitrary).

Definition 2.14. A *countable construction* is a categorical construction that employs countable diagrams only (this includes the initial and final diagrams and all cones and cocones attached in between) and terminates at a countable ordinal β.

Example 2.15. Ex^∞ is a countable, filtered geometric construction.

Remark 2.16. Since our definition of categorical construction was chosen to be a "one thing at a time" (and even well-ordered) process, building the infinite simplicial subdivision of a simplicial diagram takes $\beta := \omega \cdot \omega + \omega$ steps. At any rate, that is countable.

It is probably no exaggeration to say that without this fact the homotopy theory of simplicial objects would not exist in abstracto.

Geometric properties.
A *property of $\mathcal{D}$-diagrams in $\mathcal{E}$* is to mean the same as a subclass of the objects of $\mathcal{E}^{\mathcal{D}}$. Weak equivalences and cofibrations defined by geometric constructions play an essential role in our theory of model categories on structured objects in a topos, tying in with Quillen's axioms, localizations and the small object argument extremely well. Write $\mathrm{Mor}(\mathcal{E})$ for the category of morphisms of $\mathcal{E}$, ie. the functor category $\mathcal{E}^{\{\bullet\to\bullet\}}$.

Definition 2.17. P is a *categorical property* of $\mathcal{D}$-diagrams in $\mathcal{E}$ if there exists a set of constructions $\mathcal{E}^{\mathcal{D}} \xrightarrow{\mathsf{G}_\lambda} \mathrm{Mor}(\mathcal{E})$, $\lambda \in \Lambda$ such that $\mathcal{D} \xrightarrow{F} \mathcal{E}$ has property P iff $\mathsf{G}_\lambda(F)$ is an isomorphism for every $\lambda \in \Lambda$. Such a property is said to be *geometric*, *filtered*,... if each G_λ is such. A property is said to be *countably definable* if Λ can be taken to be a countable set, with each G_λ being a countable construction.

Obviously, categorical properties are isomorphism-invariant. When working in abelian categories or topoi that have Λ-indexed coproducts, the set of arrows $\mathsf{G}_\lambda(F)$ can be traded for a *single* arrow to be inverted, but for the sake of a general (co)complete category $\mathcal{E}$, it is better to keep Definition 2.17. One would hardly *introduce* some P literally so, yet the notion possesses considerable expressive strength in good categories. In view of the applications, we focus on geometric properties.

Proposition 2.18. *Let $\mathcal{E}^{\mathcal{D}} \xrightarrow{\mathsf{G}} \mathcal{E}^{\mathcal{C}}$ be a geometric construction, where $\mathcal{C}$ is a finite cone. "$\mathsf{G}(F)$ is a limit cone in $\mathcal{E}$" is a geometric property of $\mathcal{D}$-diagrams $\mathcal{D} \xrightarrow{F} \mathcal{E}$.*

Proposition 2.19. *Let $\mathcal{E}^{\mathcal{D}} \xrightarrow{\mathsf{G}} \mathcal{E}^{\mathcal{C}}$ be a geometric construction, where $\mathcal{C}$ is a cocone. "$\mathsf{G}(F)$ is a colimit cocone in $\mathcal{E}$" is a geometric property of $\mathcal{D}$-diagrams $\mathcal{D} \xrightarrow{F} \mathcal{E}$.*

If $\mathcal{E}$ has pushouts and pullbacks, respectively,

Proposition 2.20. *Let $\mathcal{E}^{\mathcal{D}} \xrightarrow{\mathsf{G}}$ Mor$(\mathcal{E})$ be a geometric construction. "$\mathsf{G}(F)$ is an epimorphism in $\mathcal{E}$" is a geometric property.*

Proposition 2.21. *Let $\mathcal{E}^{\mathcal{D}} \xrightarrow{\mathsf{G}}$ Mor$(\mathcal{E})$ be a geometric construction. "$\mathsf{G}(F)$ is a monomorphism in $\mathcal{E}$" is a geometric property.*

Logical aside. A conjunction of a set of geometric properties is obviously a geometric property again; so is a *disjunction* of finitely many geometric properties, it turns out, as long as one considers properties of diagrams in topoi or abelian categories. Negations $\neg\mathsf{P}$ and conditionals $\mathsf{P} \to \mathsf{Q}$ of geometric properties P, Q are not in general geometric. E.g. note that the property "identically false" cannot be constructed; a diagram made up entirely of isomorphisms will have every geometric property. "Identically true" is trivially geometric.

Observe that a set of functors $\mathcal{E}^{\mathcal{D}} \xrightarrow{\mathsf{G}_\lambda}$ Mor$(\mathcal{E})$, $\lambda \in \Lambda$ is the same datum as two functors from $\mathcal{E}^{\mathcal{D}}$ to $\mathcal{E}^{\Lambda}$, and a natural transformation between them. If $\mathcal{E}$ is accessible, so is $\mathcal{E}^{\Lambda}$ (directly or by the Limit theorem) and so are the "domain" and "codomain" functors Mor$(\mathcal{E}) \to \mathcal{E}$, so from 1.24 and 2.11 one obtains:

Proposition 2.22. *Let $\mathcal{E}$ be locally presentable, $\mathcal{D}$ a diagram, P a geometric property in $\mathcal{E}^{\mathcal{D}}$. The full subcategory of $\mathcal{E}^{\mathcal{D}}$ whose objects have property P is an accessible subcategory.*

Corollary 2.23. (to 2.10)
Suppose that filtered colimits commute with finite limits in $\mathcal{E}$. Let P be a geometric property of $\mathcal{D}$-diagrams. A filtered colimit in $\mathcal{E}^{\mathcal{D}}$ of diagrams with property P has property P.

Much more can be said when $\mathcal{E}$ is a topos; see 2.34.

Points.
A class P of points $Set \xrightarrow{p} \mathcal{E}$ of a topos $\mathcal{E}$ (assumed defined over *Set*) is called *sufficient* if it is collectively surjective (that is, the inverse images or "stalks" are collectively faithful): for parallel $f, g \in \operatorname{mor}\mathcal{E}$, if $p^*f = p^*g$ for every $p \in P$, then $f = g$. A topos has *enough points* if the class of all its points is sufficient.

Proposition 2.24. *Let $\mathcal{E}$ be a topos with a sufficient class P of points.*

- *$m \in \operatorname{mor}\mathcal{E}$ is an isomorphism iff all $p^*(m)$, $p \in P$, are isomorphisms.*
- *A finite cone-diagram $\mathcal{D}$ is limiting in $\mathcal{E}$ iff all $p^*(\mathcal{D})$, $p \in P$, are limit cones in Set.*
- *A cocone diagram $\mathcal{D}$ is colimiting in $\mathcal{E}$ iff all $p^*(\mathcal{D})$, $p \in P$, are colimit cocones in Set.*
- *For any geometric property P of $\mathcal{D}$-diagrams in $\mathcal{E}$, $\mathcal{D} \xrightarrow{F} \mathcal{E}$ has property P iff all $p^*(F)$, $p \in P$ do.*

There is a case of implication that geometric logic extends to:

Definition 2.25. A *geometric proposition* is a statement of the form "property P implies property Q" where P, Q are both geometric properties of $\mathcal{D}$-diagrams.

Corollary 2.26. (to 2.24)
If a geometric proposition holds in Set, it holds in every topos with enough points.

Proposition 2.27. *If a Grothendieck topos $\mathcal{E}$ has enough points, then it possesses a sufficient* set S *of points. Consequently, there exists a surjective geometric morphism* $Set^S \to \mathcal{E}$.

The preceding proposition is decidedly set-theoretic, ie. uses the fact that $\mathcal{E}$ is accessible in an essential way. It amounts to showing that $\mathrm{Pts}(\mathcal{E})$, the category of points of $\mathcal{E}$, is an accessible category too; hence such things as are detected by all points are already detected by a set of dense generators.

The topos Set^S can be thought of in several equivalent ways, each giving rise to different structural generalizations. It is Set/S, or a "slice topos" ($\mathcal{E}/X$ for $X \in \mathrm{ob}\,\mathcal{E}$, $\mathcal{E}$ a topos). It is the S-fold copower of Set in the category of Grothendieck topoi and geometric morphisms; so a Grothendieck topos has enough points iff it allows a surjection from a set-indexed copower of the terminal object (Set) in TOPOI. It is also $\mathrm{Sh}(\mathcal{P}(S))$, the category of sheaves (in the canonical topology) on the power set of S – which is a very special Boolean algebra: the complete atomic Boolean algebra, with atoms S.

Boolean points.
A *Boolean point* of a topos $\mathcal{E}$ is a geometric morphism to $\mathcal{E}$ from a topos $\mathrm{Sh}(\mathbb{B})$ where $\mathbb{B}$ is a complete Boolean algebra with its canonical Grothendieck topology. The next theorem follows from work of P. Freyd and R. Diaconescu or can be seen as an "artificially weakened" form of Barr's theorem. It is presented in an outline analogous to 2.24; but, in contrast to the situation when the domain has to be Set, the class of all Boolean points of a Grothendieck topos is always sufficient.

Theorem 2.28. *Let $\mathcal{E}$ be a Grothendieck topos. For parallel $f, g \in \mathrm{mor}\,\mathcal{E}$, if $p^*f = p^*g$ for every Boolean point* $\mathrm{Sh}(\mathbb{B}) \xrightarrow{p} \mathcal{E}$, *then $f = g$.*

Corollary 2.29. *Let $\mathcal{E}$ be a Grothendieck topos and B the class of its Boolean points.*

- *$m \in \mathrm{mor}\,\mathcal{E}$ is an isomorphism iff all $p^*(m)$, $p \in B$, are isomorphisms.*
- *For any geometric property* P *of $\mathcal{D}$-diagrams in $\mathcal{E}$, $\mathcal{D} \xrightarrow{F} \mathcal{E}$ has property* P *iff all $p^*(F)$, $p \in B$ do.*
- *If a geometric proposition holds in every Boolean topos, it holds in every Grothendieck topos.*

The next theorem bears a relation to 2.29 analogous to that of 2.27 to 2.24. It was so conjectured by Lawvere and proved by Michael Barr [4].

Theorem 2.30. (Barr covers)
For any Grothendieck topos $\mathcal{E}$, there exists a surjective geometric morphism $\mathrm{Sh}(\mathbb{B}) \to \mathcal{E}$.

For most homotopy-theoretic applications, 2.29 suffices; but the adjunction granted by 2.30 can be used in the standard construction (à la Godement) as well as in transferring, in certain cases, a model structure from $\mathrm{Sh}(\mathbb{B})$ to $\mathcal{E}$.

Sketches.
The transfinite tedium preceding Definition 2.8 can be replaced by a one-step process at the price of changing the objective: instead of describing a functor $\mathcal{E}^{\mathcal{C}} \to \mathcal{E}^{\mathcal{D}}$, one may be interested only in describing (up to equivalence) the full subcategory of diagrams satisfying some geometric property. One thus forfeits information such as Ex^∞ being a geometric construction, but gains the tremendous body of syntactic, categorical and set-theoretical knowledge

about models of *sketches* due to the school of C. Ehresmann [48] and (independently) several logicians and category theorists. Sketches also provide one of the intrinsic languages of accessible categories. Here we will have to be content with a

Definition 2.31. A *sketch* S is a triple $(\mathcal{C}, K, L)$ where $\mathcal{C}$ is a diagram, K is a set of cones and L is a set of cocones in $\mathcal{C}$: that is, for each $\kappa \in K$ there is given a functor $P_\kappa \to \mathcal{C}$ and for each $\lambda \in L$, a functor $Q_\lambda \to \mathcal{C}$ such that the diagram $P_\kappa \quad (Q_\lambda$, resp.) has the shape of a cone (cocone). Given a complete and cocomplete category $\mathcal{E}$, a *model* of a sketch S is a functor $\mathcal{C} \to \mathcal{E}$ with the property that for each designated cone (and cocone as well), the composite $P_\kappa \to \mathcal{C} \to \mathcal{E} \quad (Q_\lambda \to \mathcal{C} \to \mathcal{E})$ is a limiting cone (colimiting cocone, resp.) in $\mathcal{E}$. The category of models of S in $\mathcal{E}$, $\mathrm{Mod}_S(\mathcal{E})$, is the full subcategory of $\mathcal{E}^{\mathcal{C}}$ whose objects are models of S.

The sketch S is called *geometric* if all cones in K are finite (ie. contain finitely many arrows).

We restrict attention to geometric sketches since they possess classifying topoi (see below). Models of sketches are nearly always taken in topoi, or in locally presentable categories at least.

Proposition 2.32. *Let $\mathcal{E}$ be (co)complete,* P *a geometric property of $\mathcal{D}$-diagrams. There exists a geometric sketch $S = (\mathcal{C}, K, L)$ such that there is an equivalence of the full subcategory of $\mathcal{E}^{\mathcal{D}}$ whose objects have property* P *with* $\mathrm{Mod}_S(\mathcal{E})$*, models of the sketch S in the category $\mathcal{E}$. Here $\mathcal{C}$ is a diagram, equipped with an inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$; the equivalence is described one way by a geometric construction $\mathcal{E}^{\mathcal{D}} \to \mathcal{E}^{\mathcal{C}}$ and the other, by restriction of diagrams. The sketch-data are natural in the geometric construction defining* P.

Proposition 2.33. *Within the context of the previous proposition, a countably definable geometric property corresponds to a* separable *geometric sketch, that is, one with a countable diagram, a countable set of (finite) cones, and a countable set cocones, each with a countable domain.*

"Separable" is to be reminiscent of "separable topological space" or "separable Hilbert space": it is a cardinality constraint. (In fact, every Grothendieck topos is the classifying topos of some geometric sketch; the topos of sheaves $\mathrm{Sh}(X)$ on a topological space X classifies a separable sketch iff X is separable.) Sketches with higher cardinality constraints have been investigated (Makkai and Paré [52]). But separable sketches (and thus, countably definable geometric properties) stand apart by virtue of a property of their classifying topoi — Theorem 2.35 below — which is extremely handy for this work.

Classifying topoi.
One way to arrive at the following proposition is to replace the construction defining a geometric property by a sketch, and use the theory of classifying topoi for geometric sketches.

Proposition 2.34. *Let* P *be a geometric property of $\mathcal{D}$-diagrams. There exists a Grothendieck topos* $Set[\mathsf{P}]$ *such that for any Grothendieck topos $\mathcal{E}$ there is a natural equivalence between the full subcategory of $\mathcal{E}^{\mathcal{D}}$ whose objects have property* P *and* $\mathrm{GeoMor}(\mathcal{E}, Set[\mathsf{P}])$*, the category of geometric morphisms from $\mathcal{E}$ to* $Set[\mathsf{P}]$.

Words of explanation are in order. For topoi $\mathcal{E}, \mathcal{F}$, the objects of $\mathrm{GeoMor}(\mathcal{E}, \mathcal{F})$ are geometric morphisms, ie. adjoint pairs such that the left adjoint preserves finite limits, and morphisms are matching pairs of natural transformations. (One can impose the natural transformations solely on the direct or solely on the inverse image part without changing the equivalence type of the category GeoMor($\mathcal{E}, \mathcal{F}$).) The "*Set*" in the name $Set[\mathsf{P}]$ is to indicate that it is a classifying topos for ($\mathcal{D}$-diagrams such that P in) topoi that are defined and bounded over *Set*, ie. Grothendieck topoi. A site for $Set[\mathsf{P}]$ can be constructed from the syntactic data that specifies P. $Set[\mathsf{P}]$ is sometimes said to arise by formally adjoining a "free" or "generic" P-diagram; it comes equipped with a distinguished $\mathcal{D} \xrightarrow{\mathsf{gen}} Set[\mathsf{P}]$ that satisfies P.

The equivalence in 2.34 arises via inverse images of gen; that is, for every $F \in \mathrm{ob}\, \mathcal{E}^{\mathcal{D}}$ that satisfies P there exists a unique isomorphism class of geometric morphisms $\mathcal{E} \xrightarrow{f} Set[\mathsf{P}]$ such that $F \cong f^*(\mathsf{gen})$.

Theorem 2.35. (see Makkai–Reyes [53] or Barr–Makkai [6])
The classifying topos of a separable sketch has enough points.

The following corollary is practically a blessing.

Corollary 2.36. *Let $\mathcal{D}$ be a countable diagram, P, Q geometric properties of $\mathcal{D}$-diagrams. Suppose P is countably definable. If the geometric proposition "P implies Q" holds in Set, it holds in every Grothendieck topos.*

Proof. Let $\mathcal{E}$ be a Grothendieck topos, and suppose $\mathcal{D} \xrightarrow{F} \mathcal{E}$ has property P. It is classified by a geometric morphism $\mathcal{E} \xrightarrow{f} Set[\mathsf{P}]$. Via 2.33, 2.35 and 2.24, the generic diagram $\mathcal{D} \xrightarrow{\mathsf{gen}} Set[\mathsf{P}]$ satisfies Q. But geometric properties are invariant under inverse images of geometric morphisms, therefore so does $F \cong f^*(\mathsf{gen})$. $\square$

3. Internal homotopy theory

This section aims at motivating, in a logically transparent and uniform way, the well-known definitions of local fibrancy, "sheaves of homotopy groups", weak equivalence... for simplicial objects in a topos. The novelty is less in the results than in the methods, which are mixtures of Van Osdol's [64] simplicial tools and

Geometric logic.[10] Let $\mathcal{L}_\mathcal{E}$ denote the Mitchell-Benabou language of a topos $\mathcal{E}$. Its basic types correspond to objects of $\mathcal{E}$, type-forming operations being (cartesian) product and exponentiation. It has the propositional connectives $\wedge, \vee, \implies, \neg$; logical constants $\top$ "true" and $\bot$ "false"; quantifiers $\forall, \exists$; and relation symbols $=_X, \in_X$ for every type X. Term-forming operations include $\langle -, \dots, - \rangle$ (ie. "tuples"), suitable composition $\mathbf{f}(\tau)$ with constants $\mathbf{f}$ corresponding to morphisms of $\mathcal{E}$, and the "descriptor" $\{x|\tau\}$.

Write $\mathcal{L}_\mathcal{E}^{\mathrm{geo}}$ for the *geometric fragment* of $\mathcal{L}_\mathcal{E}$: its formulas are *sequents* $\phi \implies \psi$ where ϕ, ψ are *positive-existential formulas*, that is, built from atomic formulas using disjunctions, finite conjunctions, and $\exists$ only. For a diagram $\mathcal{D} \to \mathcal{E}$, let $\mathcal{L}_\mathcal{E}^{\mathrm{geo}}(\mathcal{D})$ denote those formulas of $\mathcal{L}_\mathcal{E}^{\mathrm{geo}}$ whose basic types are among objects in $\mathcal{D}$ (no exponentiation allowed), and whose constants $\mathbf{f}$ (if any) are morphisms in $\mathcal{D}$.

The *interpretation* of a sentence $\phi \in \mathcal{L}_\mathcal{E}$ is a morphism $\prod_{x_\lambda \in \mathsf{fv}(\phi)} X_\lambda \to \Omega$ whose domain is the product of types of free variables of ϕ. The *extension* $||\phi||$ of ϕ is the subobject classified by $|\phi|$; of course, this coincides with the interpretation of the descriptor $\{x_\lambda|\phi\}$ over all the free variables.

Conjunction is interpreted by intersection of subobjects (computable by pullback), disjunction by union of subobjects (computable by pullback followed by a pushout and an image factorization) and the existential quantifier by composition with projection, followed by an image factorization. So, by induction on the complexity,

Proposition 3.1. *For a positive-existential formula $\phi \in \mathcal{L}_\mathcal{E}^{geo}(\mathcal{D})$, extension is a geometric construction $\mathcal{E}^\mathcal{D} \xrightarrow{||-||} \mathrm{Mor}(\mathcal{E})$.*

Moreover, it takes values in monomorphisms. For syntactically more involved formulas — say, those containing universal quantifiers, multiple implications or negations — the inductive construction of $||\Phi||$ can be complicated. However, if $\Phi \in \mathcal{L}_\mathcal{E}^{\mathrm{geo}}(\mathcal{D})$, $\mathcal{E} \models \Phi$ translates to one subobject being contained in another, which means the invertibility of a pullback arrow (from the intersection of the subobjects to the lesser object). Thus, though 3.1 does not extend to geometric sequents, one still has

Proposition 3.2. *Let $\Phi \in \mathcal{L}_\mathcal{E}^{geo}(\mathcal{D})$. $\mathcal{E} \models \Phi$ is a geometric property of $\mathcal{D}$-diagrams.*

The approach to geometric logic via geometric constructions needs the language of diagrams only (hence works in a greater class of categories than topoi), but — depending on the application — may be more cumbersome.

The Kan condition.

Let $X_\bullet = \{X_n | \ n \in \mathbb{N}\}$ be a simplicial object in *Set*, with face maps δ_i^n and degeneracies s_i^n $(i = 0, \dots, n)$. D. Kan, in his first paper on the subject [45], defines $X_\bullet$ to satisfy the *extension condition* if for every collection of $n+1$ n-simplices $x_0, x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}$

[10] See Johnstone [40], MacLane and Moerdijk [51] or vol. III of Borceux [13] for a leisurely introduction to this theory, which cannot be given here.

which satisfy $\delta_i^n x_j = \delta_{j-1}^n x_i$ for $i < j, i \neq k, j \neq k$ there exists an $n+1$-simplex y such that $\delta_i^{n+1} y = x_i$ for $i \neq k$. But this is literally a two-parameter set of sentences of $\mathcal{L}_{Set}^{\text{geo}}(X_\bullet)$:

$$(\dagger) \qquad \bigwedge_{\substack{0 \leqslant i < j \leqslant n \\ i,j \neq k}} \delta_i^n x_j = \delta_{j-1}^n x_i \implies \exists y \Big(\bigwedge_{\substack{0 \leqslant i \leqslant n \\ i \neq k}} \delta_i^{n+1} y = x_i \Big)$$

where the x_i are variables of type X_n, y of type X_{n+1}. Denoting the sequent ($\dagger$) by $\kappa_{n,k}(X_\bullet)$, for any topos $\mathcal{E}$ now introduce

Definition 3.3. $X_\bullet \in \mathcal{E}^{\Delta^{\text{op}}}$ is an *internal Kan complex* if $\mathcal{E} \models \kappa_{n,k}(X_\bullet)$ for all $n \in \mathbb{N}$, $k = 0, 1, \ldots, n$.

By the generalities introduced above, being an internal Kan complex is a countably definable geometric property of simplicial diagrams; so it can be detected on points etc.

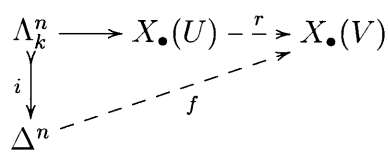
For a Grothendieck topos $\mathcal{E}$ given by an explicit site, the sheafified representables form a canonical set of generators which can be used in the Kripke-Joyal semantics to translate $\mathcal{E} \models \kappa_{n,k}(X_\bullet)$ into a statement involving functors, sets, functions and objects, morphisms and sieves of the site — the time-honored language of sheaf theory. This may permit case-by-case, site-dependent simplifications of the definitions.

Example 3.4. (equivariant – coarse)
A simplicial object $X_\bullet$ in the topos of G-sets (G being a discrete group) is an internal Kan complex iff it is a Kan (or "fibrant") object in *SSet* (having forgotten the action).

Example 3.5. (presheaf semantics)
$X_\bullet \in \text{Pre}(\mathcal{C})^{\Delta^{\text{op}}}$ is an internal Kan complex iff $X_\bullet(c)$ is a Kan complex in *SSet* for every $c \in \text{ob}\,\mathcal{C}$.

The next instance is the one occurring in Illusie [36].

Example 3.6. (topological semantics)
Take the topos of sheaves on a topological space. $X_\bullet \in \text{Sh}(T)^{\Delta^{\text{op}}}$ is an internal Kan complex iff for every open $U \subseteq T$, $n \in \mathbb{N}^+$ and every "(k,n)–horn" in $X_\bullet(U)$, that is simplicial morphism $\Lambda_k^n \xrightarrow{h} X_\bullet(U)$, every point $p \in U$ has a neighborhood $p \in V \subseteq U$ such that there exists a "filler" f



where i is the canonical inclusion and r is restriction.

Example 3.7. (Boolean semantics – axiom of choice)
A simplicial object $X_\bullet \in \text{Sh}(\mathbb{B})^{\Delta^{\text{op}}}$ is an internal Kan complex iff for every $n \in \mathbb{N}$ and $0 \leqslant k \leqslant n+1$ there exists a morphism $L_n^k \xrightarrow{\mathbf{m}} X_{n+1}$ such that

$$\text{Sh}(\mathbb{B}) \models \bigwedge_{\substack{0 \leqslant i \leqslant n \\ i \neq k}} \delta_i^{n+1} \mathbf{m}(\bar{x}) = \text{pr}_i(\bar{x})$$

where $\bar{x}$ is a variable of type L^k_n; here L^k_n and $\mathrm{pr}_i : L^k_n \to X_n$ are defined as follows: let the extension of the formula $\bigwedge\limits_{\substack{0\leqslant i<j\leqslant n\\ i,j\neq k}} \delta^n_i x_j = \delta^n_{j-1} x_i$ be $L^k_n \hookrightarrow \prod\limits_{\substack{0\leqslant i\leqslant n\\ i\neq k}} X_n$, and pr_i the composition with the projection on the i-indexed factor.

For the next example, recall that for any topological group $\mathbf{G}$, the category of sets with continuous $\mathbf{G}$-action (a set is thought of as a discrete space) is a topos, to be denoted $\mathcal{G}$. The fixed point set functor induces an equivalence between $\mathcal{G}$ and $\mathrm{Sh}(\mathrm{Op}(\mathbf{G}), J)$, where $\mathrm{Op}(\mathbf{G})$ is the full subcategory of $\mathcal{G}$ whose objects are right cosets $\mathbf{G}/U$, for all open subgroups of $\mathbf{G}$ – any cofinal set of open subgroups would do, in fact – and *any* nonempty sieve covers. For a subgroup U of $\mathbf{G}$ and $X \in \mathcal{G}$, let X^U denote the fixed point set.

Example 3.8. (atomic topology)
$X_\bullet \in \mathcal{G}^{\Delta^{\mathrm{op}}}$ is an internal Kan complex iff for every open subgroup U of $\mathbf{G}$ and every horn in $X^U_\bullet$, there exist an open subgroup V of $\mathbf{G}$, $j \in \mathbf{G}$ such that $V \subseteq j^{-1}Uj$ and a filler f

$$\begin{array}{ccccc} \Lambda^n_k & \longrightarrow & X^U_\bullet & \overset{j}{\dashrightarrow} & X^V_\bullet \\ {\scriptstyle i}\big\downarrow & & & \nearrow{\scriptstyle f} & \\ \Delta^n & & & & \end{array}$$

where the horizontal "j" is induced by $x \mapsto xj$.

Example 3.9. (generic site semantics)
$X_\bullet \in \mathrm{Sh}(\mathcal{C}, J)^{\Delta^{\mathrm{op}}}$ is an internal Kan complex iff for every $U \in \mathrm{ob}\,\mathcal{C}$ and (k,n)–horn in $X_\bullet(U)$ (cf. Example 3.6) there exists a covering $\{U_\lambda \xrightarrow{s_\lambda} U | \lambda \in \Lambda\}$ such that for every $\lambda \in \Lambda$ there is a filler f in

$$\begin{array}{ccccc} \Lambda^n_k & \longrightarrow & X_\bullet(U) & \overset{s_\lambda}{\dashrightarrow} & X_\bullet(V) \\ {\scriptstyle i}\big\downarrow & & & \nearrow{\scriptstyle f} & \\ \Delta^n & & & & \end{array}$$

with i as above.

This is precisely the classical notion of *local fibrancy* (see Jardine [39]). Note how unobvious it is that $X_\bullet$ being such is a function of the *topos* $\mathrm{Sh}(\mathcal{C}, J)$; that is to say, distinct sites $(\mathcal{C}_1, J_1)$ and $(\mathcal{C}_2, J_2)$ will give rise to the same sheaves being locally fibrant as long as $\mathrm{Sh}(\mathcal{C}_1, J_1)$ and $\mathrm{Sh}(\mathcal{C}_2, J_2)$ are equivalent categories.[11]

That all epis split is specific to 3.7, but site semantics specializes to all the other examples. Foregoing sites, however, one may look at the intrinsic interpretation of $\mathcal{E} \models \kappa_{n,k}(X_\bullet)$. It works out to the following. Consider the equalizer

$$L^k_n \to \prod_{\substack{0\leqslant i\leqslant n\\ i\neq k}} X_n \rightrightarrows \prod_{\substack{0\leqslant i<j\leqslant n\\ i,j\neq k}} X_{n-1}$$

[11] Jardine [39] allows $X_\bullet$ to be just a presheaf on the category $\mathcal{C}$. This notion too has a site-independent, intrinsic reformulation — an $X_\bullet \in \mathcal{E}^{\Delta^{\mathrm{op}}}$ being fibrant with respect to a Lawvere-Tierney topology $\Omega \xrightarrow{j} \Omega$ on $\mathcal{E}$ — but I am unaware of any applications.

where for the upper arrow, the j-indexed component of the map from the i-indexed X_n is δ_i^n and for the lower arrow, it is δ_{j-1}^n. There is a map

$$X_{n+1} \xrightarrow{p} L_n^k \tag{3.1}$$

with components the boundaries δ_i^{n+1}, since $\delta_i\delta_j = \delta_{j-1}\delta_i$ for $i < j$. $\mathcal{E} \models \kappa_{n,k}(X_\bullet)$ iff p is an epimorphism. $\mathcal{E}^{\Delta^{\mathrm{op}}}$ is a topos; in particular, it is cartesian closed. Via the global section functor $\mathcal{E}^{\Delta^{\mathrm{op}}} \to SSet$, it becomes enriched (and tensored and cotensored) over $SSet$. Recognizing the presentation of the horns in (‡), after some simplicial manipulation one obtains: $X_\bullet$ is internally fibrant iff

$$X_\bullet^{\Delta^n} \xrightarrow{p} X_\bullet^{\Lambda_k^n} \tag{$*$}$$

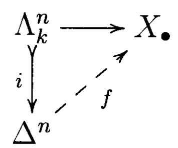
is an epimorphism for all n, k concerned, where p is induced by the inclusion of the (k, n)-horn into the n-simplex.

Combinatorial homotopy theory of internal Kan complexes.
Visibly, $(*)$ is the simplest and most conceptual definition of internal fibrancy. Why would one take a detour through the internal logic to do homotopy theory?

The reason is that it is seldom clear *which* of several *façon de parler* that are equivalent in *Set* becomes the "right" one within an arbitrary topos.

Example 3.10. The following define the same concept in *Set*: a simplicial set $X_\bullet$ such that

(1) given the solid arrows, there exists a broken arrow to make the diagram commute



(2) $X_\bullet^{\Delta^n} \xrightarrow{p} X_\bullet^{\Lambda_k^n}$ is epi
(3) $X_\bullet^{\Delta^n} \xrightarrow{p} X_\bullet^{\Lambda_k^n}$ is split epi
(4) the set function underlying $X_\bullet^{\Delta^n} \xrightarrow{p} X_\bullet^{\Lambda_k^n}$ is a surjection (the set underlying a simplicial set is defined as the disjoint union of the simplices).

All these definitions enjoy extensions to simplicial sheaves, but will no longer coincide in general. Some of them do appear (perhaps in disguise) in the homotopy theory of sheaves.

Now the internal logic of a topos is more than a symbolic calculus to endow objects with "virtual elements" and turn morphisms into "virtual functions" (and correspondingly, to allow one to specify objects and morphisms via this symbolic calculus). It is also a logical calculus for manipulating truth-values, making deductions and establishing theorems in a fashion akin to (but more restricted) than our "native" one in *Set*.[12] The segment of internal logic called "geometric" is, moreover, invariant under inverse image parts of geometric morphisms. Hence, when trying to extend a *Set*-based construction to arbitrary topoi, it is a good ansatz, even a priori, to try to describe it within geometric logic and use that as definition. (In the example above, the distinguished one is of course (2), being equivalent to Definition 3.3 with the eyes of the internal logic.) That way, one salvages, functorially within TOPOI, all the consequences of the definitions that are valid via geometric logic.

[12]The shibboleth is that the internal logic is *constructive* and *intuitionistically valid*.

That presupposes, of course, that this set of consequences is non-empty for the subject of interest. So one goes through e.g. the work of Kan [45] [46], Gabriel–Zisman [30] or May [54] to check. (The reason for citing the early papers in the genre is that they yield to syntactic analysis quite readily.[13]) The failures and successes of this strategy are both of interest; what is given below is only the very beginning.

Combinatorial weak equivalences. Let $X_\bullet \xrightarrow{\mathbf{f}} Y_\bullet \in SSet$ be a map between two simplicial sets satisfying the Kan extension condition. Saying that $\mathbf{f}$ is a weak equivalence is possible even without having defined the homotopy groups first; what it takes is for it to induce a bijection between homotopy classes of "singular n-spheres" (meaning n-simplices all of whose faces are at one and the same 0-simplex p). Under the assumption that $X_\bullet, Y_\bullet$ satisfy the extension condition, homotopy of simplices itself can be phrased without function complexes: it means respective faces are equal, together with the existence and face-matchings of some $n+1$-simplex h. To save some space, below we will write $s^n p$ for the unique n-dimensional degeneracy of a 0-simplex p (that is, s^n stands for $s_0^{n-1}s_0^{n-2}\dots s_0^1 s_0^0$); other than this, only the lower (face) index will appear on simplicial operators.

Proposition 3.11. *$X_\bullet \xrightarrow{f} Y_\bullet \in SSet$ is a weak equivalence iff*
(α_n) every singular simplex $y \in Y_n$ has, up to based homotopy $h \in Y_{n+1}$, a singular pre-image $x \in X_n$:

$$\Big(\bigwedge_{0\leqslant i\leqslant n} \delta_i y = s^{n-1}p\Big) \implies \exists x \exists h\Big(\{\bigwedge_{0\leqslant i\leqslant n} \delta_i x = s^{n-1}q\} \wedge \{\bigwedge_{0\leqslant i\leqslant n-1} \delta_i h = s^n p\} \wedge \delta_n h = \boldsymbol{f}(x) \wedge \delta_{n+1} h = y\Big)$$

where p ranges over Y_0, q over X_0, and in addition:
(β_n) if the $\boldsymbol{f}$-images of two singular simplices $x_1, x_2 \in X_n$ are based homotopic by $h \in Y_{n+1}$, they are based homotopic by some $H \in X_{n+1}$:

$$\Big(\{\bigwedge_{0\leqslant i\leqslant n} \delta_i x_1 = s^{n-1}q\} \wedge \{\bigwedge_{0\leqslant i\leqslant n} \delta_i x_2 = s^{n-1}q\} \wedge \{\bigwedge_{0\leqslant i\leqslant n-1} \delta_i h = s^n p\} \wedge \delta_n h = \boldsymbol{f}(x_1) \wedge \delta_{n+1} h = \boldsymbol{f}(x_2)\Big) \implies \exists H\Big(\{\bigwedge_{0\leqslant i\leqslant n-1} \delta_i H = s^n q\} \wedge \delta_n H = x_1 \wedge \delta_{n+1} H = x_2\Big)$$

with p ranging over Y_0, q over X_0.

Definition 3.12. Let $X_\bullet \xrightarrow{\mathbf{f}} Y_\bullet \in \mathcal{E}^{\Delta^{\mathrm{op}}}$ be a morphism of internally fibrant simplicial objects in a topos. Regard the expressions α_n and β_n as formulas of $\mathcal{L}_{\mathcal{E}}^{\mathrm{geo}}(X_\bullet \xrightarrow{\mathbf{f}} Y_\bullet)$ (with x being a variable of type X_n, y of Y_n, p of type Y_0, q of X_0 and so on). $\mathbf{f}$ is an *internal weak equivalence* if $\mathcal{E} \models \alpha_n(\mathbf{f})$ and $\mathcal{E} \models \beta_n(\mathbf{f})$ for all $n \in \mathbb{N}$.

Remark 3.13. For all their length, all that is of essence of α_n and β_n as regards topos-theoretic applications is

[13] Gabriel and Zisman use the rather unusual symbol $\mathcal{E}$ for the category *Set*. I cannot help wondering whether they were aware of the emerging topos theory of Grothendieck and his collaborators. I venture that their little monograph holds ideas unexplored even after these many years.

(1) each has the syntax of geometric logic: a universally quantified implication between two expressions, both of them being finitary compounds of relation symbols, existential quantifiers, conjunctions and disjunctions (no disjunctions are present here)
(2) it is a countable set of formulas
(3) together, they have the intended meaning in *Set*.

(1) implies that these properties are detected on stalks, faithfully on enough points, Barr covers... making the consistency with existing definitions evident. Also — granted that internal-fibrant replacement by geometric constructions is possible, see below — this permits the use of Props. 3.2 and 2.22, making the small object argument very convenient. (2), while not quintessential, allows an appeal to Prop. 2.36, simplifying many things. These remarks notwithstanding, we (re)construct some more homotopy theory internally — in shorthand where possible.

The bundle of homotopy groups. Think in *Set*. The set of singular simplices underlying the total space of the bundle $\begin{smallmatrix}\pi_n(X_\bullet)\\ \downarrow \\ X_0\end{smallmatrix}$ is $\{\langle p \in X_0, x \in X_n\rangle \mid \delta_0 x = \delta_1 x = \cdots = \delta_n x = s^{n-1}p\}$. (Recall we're solely concerned with fibrant $X_\bullet$ now.) That is, a subobject $S(n)$ of $X_0 \times X_n$: the equalizer of the $n+2$ arrows $(\delta_0 \operatorname{pr}_2, \delta_1 \operatorname{pr}_2, \dots \delta_n \operatorname{pr}_2, s^{n-1} \operatorname{pr}_1)$ to X_{n-1}, where pr_i are the projections.

Introduce $R(n)$, the set of based homotopies between singular simplices:

$$\{\langle p \in X_0, y \in X_{n+1}\rangle \mid \bigwedge_{0\leqslant i\leqslant n-1} \delta_i y = s^n p \ \wedge \bigwedge_{0\leqslant i\leqslant n} \delta_i\delta_n y = \delta_i \delta_{n+1} y = s^{n-1}p\}$$

which is again describable as a finite limit. The maps $\langle p, y\rangle \mapsto \langle p, \delta_n y\rangle$, $\langle p, y\rangle \mapsto \langle p, \delta_{n+1} y\rangle$ induce two morphisms $R(n) \overset{s}{\underset{t}{\rightrightarrows}} S(n)$. Let $P(n)$ be the coequalizer of s and t; it is the object underlying the homotopy group bundle $\pi_n(X_\bullet)$. A structure map $P(n) \to X_0$ is induced by $X_0 \times X_n \xrightarrow{\operatorname{pr}_1} X_0$, and a section $X_0 \xrightarrow{e} X_0 \times X_n$ of the structure map by $\langle \operatorname{id}_{X_0}, s^n\rangle$ (resp. $\langle \operatorname{id}_{X_0}, s^{n+1}\rangle$ to $X_0 \times X_n$ (resp. $X_0 \times X_{n+1}$).

The penultimate paragraph actually describes a geometric construction in the sense of Definition 2.8. It makes formal sense for any simplicial object in a category with finite limits and coequalizers; the assumptions enter in endowing $P(n) \in \operatorname{ob} \mathcal{E}/X_0$ with an algebraic structure. Let $n > 1$. Recall how it is done in *Set*. For brevity, introduce the abbreviation $\operatorname{comp}(x, y, z; p)$ for

$$\exists h \in X_{n+1}\Big(\delta_{n-1} h = x \ \wedge\ \delta_{n+1} h = y \ \wedge\ \delta_n h = z \ \wedge \bigwedge_{0\leqslant i\leqslant n-2} \delta_i h = s^n p\Big)$$

where the variables x, y, z range over X_n, p over X_0; of course, the intended meaning of this relation is "x and y are matching simplices to compose to z, with basepoint at p".

Proposition 3.14. *If $X_\bullet$ is an internal Kan complex in a topos $\mathcal{E}$, then the (internal translations) of the following statements hold in the internal logic of $\mathcal{E}$. "singular simplex" is to mean n-simplex having all faces $s^{n-1}p$; all homotopies are based at p.*

- *Homotopy is an equivalence relation on singular simplices.*
- *If x, y are singular simplices, there exists a singular simplex z such that* $\operatorname{comp}(x, y, z; p)$.
- *If x, y and x' are singular simplices, x and x' are homotopic,* $\operatorname{comp}(x, y, z; p)$ *and* $\operatorname{comp}(x', y, z'; p)$, *then z and z' are homotopic.*

- *If x, y and y' are singular simplices, y and y' are homotopic,* $\mathrm{comp}(x,y,z;p)$ *and* $\mathrm{comp}(x,y',z';p)$*, then z and z' are homotopic.*

Proof. Resolving the abbreviations, each statement marked by a bullet expands to a geometric sequent in the internal language of $\mathcal{E}$. The cheap way to finish now is to use — via Prop. 3.2 — Cor. 2.36. The less cheap method (and one that works in any elementary topos) is to observe that the proofs of the above propositions as given in e.g. May [54] I.3 and I.4 are valid in the internal logic. Indeed, the requisite homotopies are given by explicit horn-fillers, whose existence is guaranteed by the assumption. $\square$

The internal logic is convenient since it allows one to manipulate *elements*; its interpretation — wholly algorithmically — accounts for turning 3.14 into a statement involving objects, morphisms, limits and colimits, by taking the "extension" over all $p \in X_0$. By making this interpretation explicit, one observes that it makes sense in a class of categories much broader than topoi. "$\mathrm{comp}(x,y,z;p)$" is to be thought of as a subobject of $X_0 \times X_n \times X_n \times X_n \xrightarrow{\mathrm{pr}_1} X_0$ in $\mathcal{E}/X_0$. Write $(P \times P \times P)(n)$ for the cartesian third power of $P(n)$. It is an iterated pullback in $\mathcal{E}$, or a subquotient of $X_0 \times X_n \times X_n \times X_n \xrightarrow{\mathrm{pr}_1} X_0$ in $\mathcal{E}/X_0$. What Prop. 3.14 says is that the relation $\mathrm{comp}(x,y,z;p)$ descends to that subquotient to become the *graph of a morphism.* Recall that a subobject $\Gamma \overset{g}{\rightarrowtail} A \times B$ is said to be the graph of a morphism if the composite

$$(\dagger) \qquad \Gamma \overset{g}{\rightarrowtail} A \times B \xrightarrow{\mathrm{pr}_1} A$$

is an isomorphism. A morphism $A \xrightarrow{f} B$ has a graph $\Gamma(f)$ defined as an equalizer

$$\Gamma(f) \xrightarrow{e} A \times B \underset{\mathrm{pr}_2}{\overset{f\circ \mathrm{pr}_1}{\rightrightarrows}} B$$

(the inverse of $\mathrm{pr}_1 \circ e$ is $A \xrightarrow{\langle \mathrm{id}_A, f\rangle} A \times B$) and conversely, to a graph in the above sense corresponds the composite arrow $A \xrightarrow{(\mathrm{pr}_1 \circ g)^{-1}} \Gamma \overset{g}{\rightarrowtail} A \times B \xrightarrow{\mathrm{pr}_2} B$. In any category, morphisms biject, up to canonical isomorphism of subobjects, with their graphs. In a topos, the internal language allows a pretty reformulation via "virtual elements": thinking of $\Gamma \rightarrowtail A \times B$ as the extension of an expression $\gamma(a,b) \in \mathcal{L}_{\mathcal{E}}$ with a of type A, b of type B, one can write out condition $(\dagger)$:

Proposition 3.15. (see e.g. Borceux [13] vol. III Prop. 6.10.9 for a proof)
Provided

$$\mathcal{E} \models \forall a \exists b\ \gamma(a,b)$$
$$\mathcal{E} \models \forall a \forall b \forall c \big(\gamma(a,b) \wedge \gamma(a,c) \implies b = c\big)$$

there exists a unique $A \xrightarrow{f} B \in \mathcal{E}$ such that

$$\mathcal{E} \models \forall a\ \gamma\big(a, \boldsymbol{f}(a)\big).$$

To sum up: in any category, if we succeed in building the *graph* of the group operation ("internalizing" the composition as well), we succeed in constructing an actual morphism. By the proposition above, in a topos this follows from the internal definability of the graph. Hence, directly from the syntax of Prop. 3.14

Corollary 3.16. *When $\mathcal{E}$ is a topos, there exists a composition $P(n) \times P(n) \to P(n)$ in $\mathcal{E}/X_0$.*

One now has to inspect the syntax of the relation that gives rise to the operation of inverse, and the proofs of the associativity, commutativity and unit properties. These desired conclusions claim the equality of certain arrows, which can equivalently be stated via their graphs or via the internal logic. Their proofs given in e.g. May [54] I.4 are at the same level of logical complexity as the content of Prop. 3.14; that is, deduction of geometric sequents from a set of axioms of geometric logic, valid in any topos. We obtain

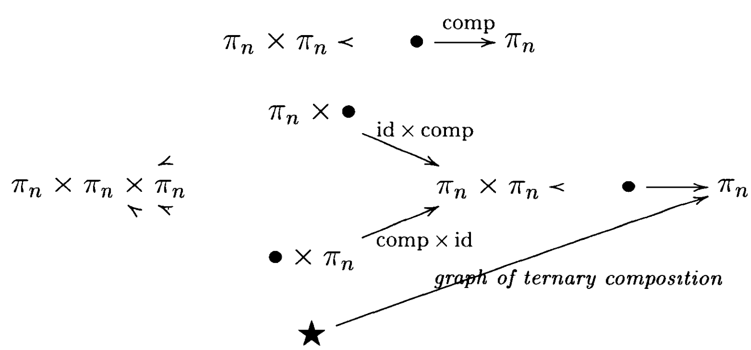
Corollary 3.17. *When $\mathcal{E}$ is a topos, $P(n)$ (with its section) underlies an abelian group object $\pi_n(X_\bullet)$ of $\mathcal{E}/X_0$ (with its unit).*

To phrase this functorially, observe that the following entity

"ordered triples, fibered over the base, of based homotopy equivalence classes of singular simplices such that the first two compose to the third"

can be defined in the geometric fragment of the internal language; hence constructed by a geometric construction. Similar remarks apply to the rest of the data. Recall also the correspondence between morphisms and their graphs, and the fact that being internal-fibrant is a geometric property, ie. equivalent to the invertibility of certain geometrically constructed arrows. It is here where that plays a role.

Proposition 3.18. *There exist a geometric construction (to be denoted π_n symbolically) from simplicial diagrams to morphisms-with-section, as well as geometric constructions from simplicial diagrams into diagrams in the category of morphisms of the shapes indicated below (for readability, the structure maps going to X_0 have been omitted from the pictures)*



and analogously for the commutativity property. When the underlying category is a topos and $X_\bullet$ is internally fibrant, the dotted arrows are isomorphisms, defining a functorial abelian structure on $\begin{smallmatrix}\pi_n(X_\bullet)\\ \downarrow\\ X_0\end{smallmatrix}$.

What properties does one really need of a category $\mathcal{E}$ for Prop. 3.18 to hold? There's a subtlety involving the conditional in the last sentence. Observe that in constructing the graph of the composition, one has to commute a mixed limit-colimit construction — $\mathrm{comp}(x, y, z; p)$ — past another: $P(n) \times P(n) \times P(n)$ is a *sub*quotient of $X_n \times X_n \times X_n$. To prove the result to be the graph of a morphism, one expects to need some "exactness condition" on $\mathcal{E}$. On the other hand, the list of options in Example 3.10 has to be extended when leaving topoi, since the meanings of plain, regular, strong, extremal, stable, effective, universally effective... epimorphisms will no longer coincide. There is a trade-off between choosing the strength of the fibrancy condition so that there'll be "enough fibrant objects" and demanding good behavior of the intrinsic singular homotopy relation thus arising.

Van Osdol's solution [64] is to assume $\mathcal{E}$ finitely complete and to satisfy the non-additive exactness property formulated by M. Barr [5]: equivalence relations are effective and universal. (See Borceux [13] vol. II for details. A category is abelian iff it is additive and Barr-exact; topoi and categories monadic over Set, or a Boolean topos more generally, are exact.) An epimorphism is *regular* if it arises as the coequalizer of some two arrows (equivalently, in an exact category, of its kernel pair, iff it is quotienting by an equivalence relation). Van Osdol defines $X_\bullet \in \mathcal{E}^{\Delta^{\mathrm{op}}}$ to be fibrant if the canonical morphism p of equation 3.1 is a regular epimorphism. For fibrant $X_\bullet$, then, $\pi_n(X_\bullet)$ can be formed via finite limits and quotients by equivalence relations.

These calculations can be tedious via diagrams, but any small Barr-exact category possesses an exact embedding into a Grothendieck topos (this is the non-additive analogue of the well-known abelian fact) which makes it possible, at the end of the day, to perform most diagram chases in the "universal example", *Set*.

The fundamental groupoid. Other than commutativity of composition, the above discussion generalizes to $n = 1$ to produce a group object $\pi_1(X_\bullet)$ in $\mathcal{E}/X_0$. One can also construct the fundamental groupoid object $P \underset{\delta_0}{\overset{\delta_1}{\rightrightarrows}} X_0$ in $\mathcal{E}$; the composition of paths, though a partial operation, has a domain that is internally definable. Finally, write C for the coequalizer of $X_1 \underset{\delta_0}{\overset{\delta_1}{\rightrightarrows}} X_0$. $\pi_0(X_\bullet) := X_0 \times C \xrightarrow{\mathrm{pr}_1} X_0$ is a pointed object of $\mathcal{E}/X_0$, the section being $\langle \mathrm{id}_{X_0}, s^1 \rangle$.

Effect of a morphism. $X_\bullet \xrightarrow{\mathbf{f}} Y_\bullet \in \mathcal{E}^{\Delta^{\mathrm{op}}}$ induces a morphism $\pi_n(X_\bullet) \to \pi_n(Y_\bullet)$ between the underlying $\mathcal{E}$-objects; equivalently, a morphism $\pi_n(X_\bullet) \to f^*\pi_n(Y_\bullet)$ in $\mathcal{E}/X_0$ where f^*, pullback along $\mathbf{f}$, is the inverse image part of the canonical geometric morphism $\mathcal{E}/X_0 \to \mathcal{E}/Y_0$ induced by $\mathbf{f}_0$.

Proposition 3.19. *Let $X_\bullet \xrightarrow{f} Y_\bullet \in \mathcal{E}^{\Delta^{op}}$ be a morphism of internally fibrant simplicial objects in a topos. f is an internal weak equivalence iff it induces isomorphisms $\pi_n(X_\bullet) \to f^*\pi_n(Y_\bullet)$ between the respective structures (pointed object, group object, abelian group object) in $\mathcal{E}/X_0$ for all $n \in \mathbb{N}$.*

Proof. Cor. 2.36 applies, so the claim only has to be checked in *Set*. It is certainly true there; things were *made* that way. □

On the level of underlying objects, Prop.3.19 means that the diagrams

$$\begin{array}{ccc} \pi_n(X_\bullet) & \longrightarrow & \pi_n(Y_\bullet) \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

are pullbacks. This was chosen as the definition of internal weak equivalence in Joyal [42], in which case Definition 3.12 would be the theorem. We prefer to reverse the order of implication, since the syntax of Definition 3.12 is easier to inspect.

Logical asides.

★ The group structure on $\pi_n(X_\bullet)$ comes into existence provided an arrow is invertible, since the canonical operations on homotopy classes are not induced by canonical operations on singular simplices (save the unit). This is not so when there exist functorial fillers for the requisite horns, e.g. for simplicial *group* objects.

★ There're two logical maneuvers to make the handling of internal base points in a topos completely elementary. One is to consistently think of expressions such as $\pi_n(X_\bullet, p)$ as sentences of $\mathcal{L}_\mathcal{E}$.[14] Upon instantiating the "virtual element" $p \in X_0$ by any *partial section* $U \xrightarrow{p} X_0$, the formulae give rise to actual objects and morphisms of $\mathcal{E}/U$. To allow "for every choice of basepoint", one has to permit U to run through a class (in the case of a Grothendieck topos, a set suffices) of generators.

The other maneuver, actually the universal (or degenerate) case of the above, is using the "generic basepoint" $X_0 \xrightarrow{\mathrm{id}_{X_0}} X_0$. Equivalently, there is a geometric morphism $\mathcal{E}/X_0 \to \mathcal{E}$ whose inverse image is cartesian product $X_0 \times (-)$, usually denoted X_0^* in this context. The zeroth space of $X_0^*(X_\bullet)$ has the tautologous global section $X_0 \xrightarrow{\mathrm{diag}} X_0 \times X_0$. The algebraic structure on $\pi_n(X_\bullet)$ (other than the fundamental groupoid) is simply the one corresponding to a globally pointed object in the topos $\mathcal{E}/X_0$.

★ Suppose one is interested solely in Grothendieck topoi, solely on explicitly given sites $(\mathcal{C}, J)$, and solely with sheaves of a specific type (e.g. representable), and one grants oneself all of classical mathematics over each section $X_\bullet(U)$, $U \in \mathrm{ob}\,\mathcal{C}$. Does that simplify the machine?

Yes, but not very much. The reason is that every time an existential quantifier is encountered, one "has to" allow for refinement along a sieve (ie. for local existence). The Kripke-Joyal semantics makes this translation completely algorithmic; see e.g. Jardine [39] for an explicit write-up under site semantics. One "has to", that is, in order to arrive at a notion that is detected on stalks, and more generally behaves well under inverse images of geometric morphisms. For algebraic notions such as "abelian group" or "category" (more generally, structures that are finite limit sketchable) formation of sheaves commutes with the defining relations, so the difference between "sheaves of abelian groups" and "abelian group objects in the category of sheaves" is invisible; but that is not so for more complex axioms. It may so happen that (to date) simplicial weak equivalence, with its intrinsic "there exists" has been the most involved notion describable in geometric logic that needed sheafification for mathematical practice. (Even so, one does not need to use *or* to define simplicial weak equivalences. That fact has consequences.)

All in all, exploring notions section-wise, in a site-dependent fashion, makes quite a bit of sense if one is not concerned with full functoriality in TOPOI. Within algebraic topology and algebraic geometry, one typically needs *some* functoriality, however; applications usually begin by constructing a functor from some ill-behaved category of structured spaces to TOPOI.

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[14]An algebraic topologist may take offense at the suggestion that homotopy groups are *formulas in a logical language.* But "For a topological space X and point $x \in X$, the group $\pi_n(X, x)$ is the quotient of the set of based maps from the pointed n-sphere by the equivalence relation generated by..." is a formula in a logical language. It is just not thought of that way.

★ The *combinatorial* aspect of homotopy theory, as initiated by work of Poincaré, J.H.C. Whitehead and D. Kan, is rooted in finitary manipulations of discrete entities. It is interesting to observe that combinatorics proper (I have in mind examples such as finite Ramsey-type theorems) proceeds at much greater logical levels; indeed, the very notion of *finite* does not have a single, robust translation in a topos. (The notion of a set being finite is not definable is first-order, let alone geometric, logic.) The point is that combinatorial homotopy theory is much more than just combinatorial; much of it survives by geometric logic. This is the fact that allows it to have a translation in e.g. any Barr-exact category. Similar remarks apply to parts of homological algebra.

Internal-fibrant replacement. Let $SSet_{\mathsf{fib}}$ be the full subcategory of $SSet$ with objects the Kan complexes. Kan [45] defines an adequate homotopy theory on $SSet$ to consist of a functor $SSet \xrightarrow{F} SSet_{\mathsf{fib}}$ and a natural transformation $\mathrm{Id}_{SSet} \xrightarrow{\eta} F$ satisfying certain properties. Strengthen those axioms to demand (cf. Remark 3.13)

(1) (F, η) arise via a filtered geometric construction from Δ^{op}-diagrams to ones of shape $\Delta^{\mathrm{op}} \times \{\bullet \to \bullet\}$.
(2) Moreover, this construction is *countable* and
(3) has the intended meaning in Set (as gauged by topological realization).

$X_\bullet \to \mathsf{Ex}^\infty(X_\bullet)$ fits the bill. Kan's [45] proof that $\mathsf{Ex}^\infty(X_\bullet)$ satisfies the extension property internalizes, in any topos with colimits over countable chains, to prove $\mathsf{Ex}^\infty(X_\bullet)$ internal-fibrant. It seems impossible either to replace or to improve Kan's original arguments in this context.

Corollary 3.20. *Being a weak equivalence in SSet is a countably definable, filtered geometric property.*

Proof. The countably many filtered geometric constructions are Ex^∞ followed by the interpretations of the geometric sequents in Definition 3.12. Note that the α_n and β_n only involve, besides the implication, conjunction and existential quantifiers; correspondingly, they are interpreted by finite limits and image factorizations, and the latter are computable by finite limits and *reflexive* coequalizers (see Example 2.1) which are filtered (in fact, ∞-filtered, ie. κ-filtered for any κ). □

Why isn't all homotopy theory internal?

The analysis on the preceding pages extends further, to relative homotopy, relative homotopy groups and their long exact sequence, short fibration sequences and the associated long exact sequence (and probably certain spectral sequences), "singular" homology and its basic relation to homotopy. It is worthwhile to point out two aspects where it fails. The first is that a simplicial object is isomorphic to the inverse limit of its Postnikov tower. (Note that this involves an *infinite* limit. The Postnikov sections themselves, their functoriality and effect on homotopy groups, internalize.)

The internal version of Theorem 12.5 of May [54]:

"Let K and L be connected Kan complexes. Then the following conditions are equivalent.

(i) K and L have the same homotopy type.
(ii) There exists a weak homotopy equivalence $f : K \to L$.
(iii) K and L have isomorphic minimal subcomplexes."

fails decidedly. A counterexample — (ii) holding and (i) failing — is provided already in *G-Set* by every contractible Kan complex with a G-action (G a discrete group) that is not equivariantly contractible. (Indeed, the forgetful functor *G-Set* $\to$ *Set* faithfully reflects the notion of weak equivalence intrinsic to *G-Set*, but not the existence of [equivariant] maps. Note that this phenomenon is responsible for there being such a thing as group cohomology!)

It is amusing to backtrack in the proof to identify the step where it departs from the internal logic. The implication (i)$\to$(ii) is of course valid, but the converse direction proceeds (ii)$\to$(iii)$\to$(i). The proof that every Kan complex has a minimal subcomplex, in Lemma 9.3, uses however the axiom of choice (the induction step *chooses* a set of representatives of homotopy classes of q-simplices all of whose faces are in the lower dimensional part already chosen!). This amounts to splitting an epimorphism. From the point of view of algebraic topology, the failure of sections to exist is cohomology; from the point of view of logic, it is departure from the axiom of choice.

Boolean homotopy theory.
For a complete Boolean algebra $\mathbb{B}$ equipped with its canonical topology, Sh($\mathbb{B}$) is the type of topos at once farthest from and closest to *Set*. Farthest in the sense that points of Sh($\mathbb{B}$) biject with atoms of $\mathbb{B}$: thus Sh($\mathbb{B}$) need have no points, so its structure need not be – even up to geometric logic, even "infinitesimally"! – homomorphic to *Set*. However, it is closest in that Sh($\mathbb{B}$) is globally a $\mathbb{B}$-valued model of ZFC while *Set* is a $\top, \bot$-valued one (this being the two-element Boolean algebra). The axioms modeled include the axiom of choice, and this characterizes Boolean topoi. Using a Boolean topos as a "back-prop" to homotopy theories on simplicial (pre)sheaves goes back to Van Osdol [64], Joyal [42] and Jardine [37]. But an explicit formulation may be useful:

Proposition 3.21. *Let $\mathcal{B}$ be a Boolean topos.*
$\langle$*monomorphisms*; *internal weak equivalences*; *internal fibrations*$\rangle$ *give a model structure on* $\mathcal{B}^{\Delta^{op}}$.[15]

From the logical point of view, this is thoroughly unsurprising; for the homotopy theorist, it gives the possibility of fibrant replacement via Godement's standard construction.

[15]It is obvious how internal fibrations are defined; they coincide with the classical "local" or "stalkwise" ones. Internal-injective maps in any topos are precisely the monomorphisms. So that is the "internal" notion of cofibration too.

Injective-factorization systems.
These bear a relation to *factorization systems* (sometimes called *orthogonal factorization systems*; see Freyd-Kelly [28] or Borceux [13] vol. I) analogous to that of "lifting property" to "orthogonality" or "weakly (co)limiting" to "(co)limiting": the result of weakening an existence-and-uniqueness condition to mere existence. The (simple-minded) definition is in (i) of Prop. 4.1 below; analogous ideas appear in Bousfield [14], Joyal [42] and Casacuberta [18].

Recall that for a category $\mathcal{C}$ and class Σ of morphisms of $\mathcal{C}$, $\mathrm{RLP}(\Sigma)$ – resp. $\mathrm{LLP}(\Sigma)$ – denotes the class of morphisms of $\mathcal{C}$ having the right – resp. left – lifting property with respect to every morphism in Σ. If $\mathcal{C}$ has a terminal object $\mathbf{1}$, $X \in \mathrm{ob}\,\mathcal{C}$ is said to be *injective* with respect to the class Σ if $X \to \mathbf{1} \in \mathrm{RLP}(\Sigma)$; obviously, this can be phrased in the absence of a terminal object too.

Proposition 4.1. (small injective-factorization systems in locally presentable categories) *Let $\mathcal{E}$ be a locally presentable category and Σ a* set *of morphisms of $\mathcal{E}$.*

(i) *Let $\mathsf{F}_\Sigma = \mathrm{RLP}(\Sigma)$ and $\mathsf{C}_\Sigma = \mathrm{LLP}(\mathsf{F}_\Sigma)$. Then $(\mathsf{C}_\Sigma, \mathsf{F}_\Sigma)$ is an injective-factorization system on $\mathcal{E}$. That is,*
  - $\mathsf{F}_\Sigma = \mathrm{RLP}(\mathsf{C}_\Sigma)$ *and* $\mathsf{C}_\Sigma = \mathrm{LLP}(\mathsf{F}_\Sigma)$.
  - *Every $f \in \mathrm{mor}\,\mathcal{E}$ can be factored as $f = me$ with $e \in \mathsf{C}_\Sigma$, $m \in \mathsf{F}_\Sigma$.*
  - *$\mathsf{C}_\Sigma \cap \mathsf{F}_\Sigma$ contains all isomorphisms.*

(ii) *Every member of C_Σ can be written as a retract of a transfinite composition of pushouts of morphisms from Σ.*

(iii) *C_Σ is the smallest subcategory of Σ containing all pushouts of elements of Σ that is closed under filtered colimits.*

(iv) *Let $\mathfrak{U}$ be the full subcategory of $\mathcal{E}$ whose objects are those injective w.r.t. Σ.*
  - *$\mathfrak{U}$ is a weakly reflective, accessible subcategory of $\mathcal{E}$. $\mathfrak{U}$ is weakly locally presentable (in particular, has weak colimits and all products).*
  - *The full subcategory of* $\mathrm{Mor}(\mathcal{E})$ *with objects the members of* $\mathrm{RLP}(\Sigma)$ *is an accessible subcategory.*
  - *The category whose objects are those of $\mathcal{E}$ and whose morphisms are the members of* $\mathrm{RLP}(\Sigma)$ *is accessible.*

(i) and (ii) follow from *much* weaker assumptions on $\mathcal{E}$; one can mimic, for example, the proof of Theorem 4.1 of Bousfield [14] modulo

Logical aside. The following "fact" in the proof of Lemma 4.3 of Bousfield [14] is erroneous:

"If the cardinality of J is less than β, then each set of objects of $\mathbf{Seq}[\beta]$ indexed by J has an upper bound in $\mathbf{Seq}[\beta]$"

where $\mathbf{Seq}[\beta]$ is defined as "the well-ordered set of ordinals less than $\mathrm{Ord}[\beta]$, regarded as a category in the usual way". Here β is some "infinite cardinal number" and $\mathrm{Ord}[\beta]$ is "the smallest ordinal number of cardinality β". (Let me remark that under the usual sense of von Neumann ordinals and cardinals, $\mathrm{Ord}[\beta]{=}\beta$ then.)

For a counterexample, take $\beta{=}\mathrm{Ord}[\beta]{=}\aleph_\omega$ which is an uncountable cardinal, and $J := \{\aleph_n | n \in \omega\}$ which is a countable set of cardinals, thought of as objects in $\mathbf{Seq}[\aleph_\omega]$. J has no upper bound in $\mathbf{Seq}[\aleph_\omega]$; indeed, the very definition of $\aleph_\omega$ is as the *least upper bound* or union of the ordinals $\{\aleph_n | n \in \omega\}$.

A cardinal with the property that it contains no subset of lesser cardinality cofinal with itself is called *regular*. There are arbitrary large *singular* (ie. not regular) cardinals, in fact cardinals of cofinality ω, such as $\aleph_\omega$. Being sequentially $\aleph_\omega$-small is the same as being sequentially ω-small.

The statement of Lemma 4.3 of Bousfield [14] is correct provided one reads "regular cardinal" for "cardinal". (This slight set-theoretical oversight of anomalous cofinalities marrs some of the older literature.)

The much more general "small object argument" of Hirschhorn [34] specializes to (i) and (ii) as well. (Recall that every object of an accessible category is presentable.) (iii) follows from Proposition 0.3. It is only (iv) that uses the accessibility of $\mathcal{E}$. The first statement is Theorem 4.11 of Adámek–Rosický [1]. The rest are unpublished theorems of Jiří Rosický included here for reference. They will not be needed in this paper.

A (closed) model category arises from the highly non-trivial interaction of two injective-factorization systems, (acyclic cofibrations,fibrations) and (cofibrations,acyclic fibrations). That the interaction is non-trivial is witnessed by the fact (among others) that those morphisms that can be written as an acyclic cofibration followed by an acyclic fibration, the weak equivalences that is, form a saturated class — ie. a class of morphisms inverted by a localization functor. (It seems impossible to get a handle on a model category based on this data, however.) The ingredient injective-factorization systems being *small* is precisely the same as the model structure being cofibrantly generated.

Presentable model categories.

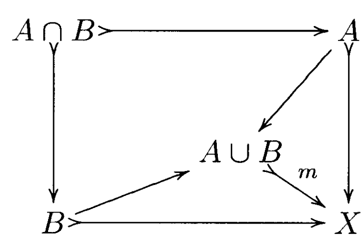
Let $\mathcal{E}$ be a category with a distinguished class of morphisms W. Write mono for the class of monomorphisms of $\mathcal{E}$.

Theorem 4.2. (presentable model categories)

Suppose $\mathcal{E}$ and W *satisfy the conditions:*

c0 (i) *$\mathcal{E}$ is a locally presentable category.*

(ii) *Subobjects have effective unions in $\mathcal{E}$. That is,*



given any two subobjects A, B of an object X, form their intersection $A \cap B = A \times_X B$ and their pushout $A \cup B$ over their intersection; the induced map m is to be a monomorphism (whence $A \cup B$ really is the supremum of A and B in the subobject lattice of X, ie. their union).

(iii) *A pushout of a monomorphism is a monomorphism.*

(iv) *Filtered colimits commute with finite limits in $\mathcal{E}$.*

c1 *The full subcategory of* $\mathrm{Mor}(\mathcal{E})$ *whose objects are the members of* W *is an accessible subcategory.*

c2 W *satisfies the 2-of-3 property (Quillen's axiom **M2**).*

c3 $\mathrm{RLP}(\mathit{mono}) \subseteq$ W.

c4 *A pushout of a morphism in mono $\cap$ W is in W.*

c5 *mono* $\cap$ W *is closed under filtered colimits in* $\mathcal{E}$.

Then $\langle$*mono*; W; RLP$\rangle$ *define a Quillen model structure on* $\mathcal{E}$.

The proof is modeled on Joyal's original argument [42] and on Jardine and Goerss' [32], on whose conditions **E1-E7** the above is a variant. The aggressive use of accessibility seems new. Note that **c2-c3-c4-c5** are necessary for the conclusion, while **c0** and **c1** suffice for the small object argument, including the existence of a *set* of dense generators for (acyclic) cofibrations.

Proof of Theorem 4.2. A locally presentable category is complete and cocomplete, verifying Quillen's axiom **M1**. That weak equivalences are closed under retracts follows from **c1**, Prop. 2.22 and the remark following Example 1.21. A retract of a monomorphism is a mono, and RLP(Σ) is closed under retracts for any class Σ; this verifies the rest of **M3**. **M4**(ii) is by fiat. The factorization axioms will be proved simultaneously, following two preliminary lemmas.

Corollary 4.3. *Let* CartMono($\mathcal{A}$) *be the subcategory of* Mor($\mathcal{A}$) *with the same objects as* Mor($\mathcal{A}$), *and as morphisms (between* $A \to B, X \to Y \in \mathrm{ob\,CartMono}(\mathcal{A})$*) those commutative squares*

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ B & \xrightarrow{g} & Y \end{array}$$

which are cartesian, with f *and* g *being monomorphisms in* $\mathcal{A}$. CartMono($\mathcal{A}$) *is an accessible subcategory of* Mor($\mathcal{A}$).

Indeed, it is the intersection of Cart($\mathcal{A}$) and Mono(Mor($\mathcal{A}$)), and Mor($\mathcal{A}$) is locally presentable if $\mathcal{A}$ is. Apply Prop. 1.35, Prop. 1.36 and Cor. 1.22. $\square$

The next lemma is central to this section. The class C of morphisms it uses can be thought of as the (acyclic) cofibrations; it will be verified in Prop. 4.5 and Prop. 4.6, respectively, that both do satisfy the assumptions.

Lemma 4.4. *Let the category* $\mathcal{E}$ *satisfy conditions* ***c0****(i)-(ii) of 4.2. Let* C *be a subcategory of* $\mathcal{E}$ *with the following properties:*

(i) *The full subcategory of* Mor($\mathcal{E}$) *whose objects are the members of* C *is an accessible subcategory.*
(ii) C $\subseteq$ *mono.*
(iii) C *is closed under filtered colimits (equivalently, under transfinite composition).*
(iv) C *is closed under pushout.*
(v) *If* $gf \in$ C, $f \in$ C *and* $g \in$ *mono*, *then* $g \in$ C.

Then (C,RLP(C)) *is a small injective-factorization system on* $\mathcal{E}$ *in the sense of 4.1.*

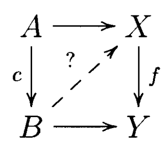
Proof. Consider the category CartMonoC whose objects are elements of C with a morphism from $a \in$ C to $x \in$ C being a cartesian square

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow a & & \downarrow x \\ B & \xrightarrow{g} & Y \end{array}$$

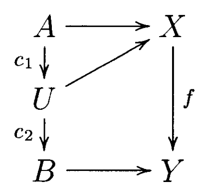
where f, g are monomorphisms in $\mathcal{E}$. CartMonoC is an accessible subcategory of Mor($\mathcal{E}$); indeed, it is the intersection of (i) of Lemma 4.4 and CartMono($\mathcal{E}$) of Corollary 4.3. The "codomain" functor Mor($\mathcal{E}$) $\to \mathcal{E}$ is accessible. Choose now a regular cardinal κ such that simultaneously

- CartMonoC is κ-accessible
- $\mathcal{E}$ is κ-accessible (thence, locally κ-presentable)
- The composite CartMonoC $\hookrightarrow$ Mor($\mathcal{E}$) $\to \mathcal{E}$ preserves κ-filtered colimits (ie. is κ-accessible).

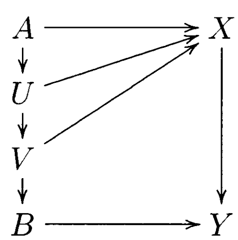
There is such a κ by the Uniformization Theorem. Let G be a set of κ-presentable generators for CartMonoC, and G_{cod} the set of their codomains. Let S be the set of subobjects — in $\mathcal{E}$ — of members of G_{cod}. (There is only a set, since a locally presentable category is well-powered.) Finally, let Σ be the intersection of S and C. As in Prop. 4.1, Σ induces a small-factorization system (C_Σ,F_Σ) on $\mathcal{E}$. We claim $\mathsf{C}_\Sigma \subseteq \mathsf{C}$ and $\mathsf{F}_\Sigma \subseteq$ RLP(C). Indeed, $\Sigma \subset \mathsf{C}$ and every member of C_Σ can be written as a retract of a transfinite composition of pushouts of morphisms from Σ. But C is closed under those operations. (Closure under retracts follows from (i) of Lemma 4.4 and the remark following Example 1.21.) $\mathsf{F}_\Sigma \subseteq$ RLP(C) is true by yet another instance of Bousfield's argument. Consider a lifting problem



with $c \in \mathsf{C}$, $f \in \mathsf{F}_\Sigma$. A *partial lift* is a commutative diagram

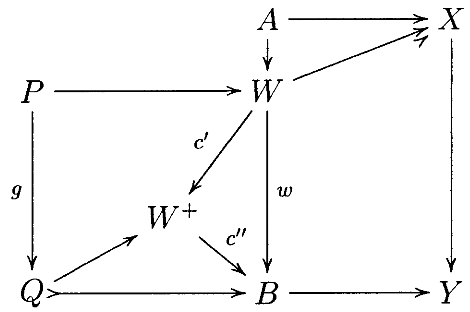


with $c_1, c_2 \in \mathsf{C}$, $c_2 c_1 = c$. There is only a set of (isomorphisms classes of) partial lifts, partially ordered by compatibility: $U \prec V$ if



commutes. The set of partial lifts is non-empty, thanks to the tautological partial "lift" with $U = A$. Any linearly ordered $\prec$-chain of lifts has an upper bound, its colimit; this uses assumptions (iii) and (v) of the Lemma. By Zorn's lemma, there exists a maximal lift W.

Now the assumption that W is a proper subobject of B leads to a contradiction:



Write $w \in \operatorname{ob} \mathrm{CartMonoC}$ as a κ-filtered colim $\mathcal{G}$ of generators. Now consider the composite $\mathcal{G}_{\mathrm{cod}} : \mathcal{G} \to \mathrm{CartMonoC} \hookrightarrow \mathrm{Mor}(\mathcal{E}) \xrightarrow{\text{codomain}} \mathcal{E}$. Since $\mathrm{CartMonoC} \hookrightarrow \mathrm{Mor}(\mathcal{E}) \xrightarrow{\text{cod}} \mathcal{E}$ was to preserve κ-filtered colimits, colim $\mathcal{G}_{\mathrm{cod}} = B$. It is impossible that every object of $\mathcal{G}_{\mathrm{cod}}$ (hence, the whole mono-diagram) factor through the proper subobject W. Choose a $P \xrightarrow{g} Q \in \mathcal{G}$ such that Q is not a subobject of W. As illustrated above, let W^+ be the union of W and Q, so the skew quadrangle is both a pushout and a pullback. (This is the only one, and crucial, place where condition **c0**(ii) is used.) There is a lift from Q to X since $f \in \mathsf{F}_\Sigma = \mathrm{RLP}(\Sigma)$. $c' \in \mathsf{C}$ by (iv) and $c'' \in \mathsf{C}$ by (v) of the assumptions. Then the induced (dotted) map from W^+ to X extends the one assumed maximal in our poset. This contradiction proves $\mathsf{F}_\Sigma \subseteq \mathrm{RLP}(\mathsf{C})$.

Since $\mathsf{C}_\Sigma \subseteq \mathsf{C}$, $\mathsf{F}_\Sigma = \mathrm{RLP}(\mathsf{C}_\Sigma) \supseteq \mathrm{RLP}(\mathsf{C})$. Take now any $m \in \mathsf{C}$ and factor it $m = fc$ with $f \in \mathsf{F}_\Sigma$, $c \in \mathsf{C}_\Sigma$. But $f \in \mathrm{RLP}(m)$ so a retract argument finishes the proof of the Lemma. □

Proposition 4.5. C = *cofibrations* = mono *satisfy the assumptions of Lemma 4.4.*

Being a monomorphism is a geometric property in $\mathrm{Mor}(\mathcal{E})$, so **c0**(iv) with 2.23 and 0.4 imply 4.4(iii). The only other one not explicitly part of the hypotheses of Theorem 4.2 is 4.4(i). Even for a category that is merely accessible, that is Proposition 6.2.1(ii) of Makkai and Paré (see [52] p.146). For a locally presentable $\mathcal{E}$, it also follows from the remark above and Prop. 2.22.

Thus (mono,RLP(mono)) factorizations exist, and are (cofibration,acyclic fibration) factorizations by **c3**. That together with **M2** and the retract argument establishes **M4**(i) as usual.

Proposition 4.6. C = *acyclic cofibrations* = mono $\cap$ W *satisfy the assumptions of Lemma 4.4.*

(i) follows from 4.5, **c1** and Proposition 1.22; (ii), (iii) and (iv) were axioms and (v) – as well as the fact it's a subcategory! – follow from **M2**.

Thus (acyclic cofibration,fibration) factorizations also exist, completing the proof of Theorem 4.2. □

5. From *Set* to TOPOI

We pursue the following objectives:

- providing the first examples of presentable model categories;
- giving conditions for the homotopy model structures constructed to exist functorially within every Grothendieck topos provided they exists on *Set* and
- transferring them along adjoints to produce more examples and localizations.

Proposition 5.1. (geometric homotopy model structures on diagrams)
Let $\mathcal{E}$ be a category, $\mathcal{D}$ some diagram and W *a geometric property of morphisms of $\mathcal{D}$-diagrams. Define weak equivalences as the class of morphisms of $\mathcal{E}^{\mathcal{D}}$ having property* W*. (We will denote this class by $\mathsf{W}(\mathcal{E}^{\mathcal{D}})$ or even* W*, if no confusion can arise.) Write* mono *for the class of monomorphisms of $\mathcal{E}^{\mathcal{D}}$. Suppose $\mathcal{E}$ satisfies* **c0***, and $\mathcal{E}^{\mathcal{D}}$ and* W *satisfy conditions* **c2***,* **c3***,* **c4** *of Theorem 4.2. Then $\langle \mathsf{mono}; \mathsf{W}(\mathcal{E}^{\mathcal{D}}); \mathrm{RLP} \rangle$ define a Quillen model structure on $\mathcal{E}^{\mathcal{D}}$.*

Proof. Indeed, if $\mathcal{E}$ satisfies **c0**, so does $\mathcal{E}^{\mathcal{D}}$. By assumption, being a weak equivalence is a geometric property of $\mathcal{D} \times \{\bullet \to \bullet\}$-diagrams, so Prop. 2.22 yields **c1**. **c0**(iv), Prop. 2.23 and 0.4 together imply that W is closed under filtered colimits, and (as already used in the proof of Theorem 4.2) being a monomorphism is a geometric property in $\mathrm{Mor}(\mathcal{E})$, so the same imply monos are closed under filtered colimits, giving **c5**. □

Proposition 5.2. (well-known)
Condition **c0** *is satisfied for $\mathcal{E}$ a Grothendieck topos.* □

Grothendieck abelian categories (ie. cocomplete “AB5” abelian categories with a generator) also satisfy **c0** save for part (i) (being locally presentable) although for abelian categories in practice, that condition is met too. The class of categories satisfying **c0** is closed under formation of functor categories, as well as over- and under-categories. So it is wider than just abelian categories and topoi; indeed, the category of globally pointed objects of a topos — being a pointed category — is never cartesian closed, let alone a topos, save the degenerate case. Within this section, we focus on the topos case, however.

A topos is locally presentable iff it is a Grothendieck topos. For this reason, within topoi we restrict attention to Grothendieck ones from now on, and consequently omit the adjective *Grothendieck*. If an occasional statement holds in more generality, that will be emphasized by the phrase *for every elementary topos....*

The first example of a geometric model structure on diagrams is unsurprising and fundamental: simplicial sets. All details save **c2**-**c3**-**c4** have been established; that those are satisfied is known *after the fact*, as it were, since they follow from *SSet* being a model category. Quillen [58] (see Goerss–Jardine [33] for a more leisurely write-up) does not prove that along the lines of Theorem 4.2. **c2** and **c3** are not hard to show and, ironically, one *knows*, thanks to an extension of Cor. 2.36, that there exists a proof of **c4** proceeding in geometric logic. (This validates Quillen's intuition that the use of minimal fibrations is not necessary for the argument.) The truly “elementary” proof, however, is still to be found.

Staying within the set-up of Prop. 5.1, if one is able to exhibit a geometric construction $\mathcal{E}^{\mathcal{D}} \to \mathcal{E}$ giving rise to the $\mathsf{G} : \mathrm{Mor}(\mathcal{E}^{\mathcal{D}}) \to \mathrm{Mor}(\mathcal{E})$ defining weak equivalences, then **c2** follows instantly. For the case of weak homotopy equivalence of simplicial objects, this doesn't seem possible, but it is so for homological localization:

Example 5.3. (Bousfield localization)
Let h_* be a homology theory on *SSet* that is representable in the sense of G. Whitehead, that is,

$$(\dagger) \qquad h_n(X) = \mathrm{colim}\ \pi_n(X_+ \wedge W_i)$$

where W_i is a "naive spectrum", ie. sequence of simplicial sets and connecting maps from the suspension of W_i to W_{i+1}. The data for $h_n(X)$ (the underlying set and the operations) arise via geometric constructions through the steps

- for a fixed $W \in \mathrm{ob}\, SSet$, the functor $X \mapsto X \times W$ is a geometric construction
- for pointed W, the functor $X \mapsto X_+ \wedge W$ is a geometric construction
- $\pi(-)$ is a geometric construction
- Kan's simplicial cone and suspension are geometric constructions
- the colimit of abelian groups in ($\dagger$) is a geometric construction.

The other conditions of Prop. 5.1 can be checked easily. Hence Bousfield's [15] homological localization of spaces is a geometric model category.

Model structures on diagrams arising via Prop. 5.1 are particularly easy to implement across categories. They have two independent parameters: the geometric constructions G_λ defining the weak equivalences, and the underlying category $\mathcal{E}$. Quite often, the axioms **c2-c4** are easier to check for distinguished $\mathcal{E}$, but follow formally for a wider class. We turn to proving that if a geometrically defined W satisfies the axioms in a restricted class of topoi, then it does so in every topos. The distinguished class will be Boolean topoi in general, though *Set* by itself will suffice under the additional assumption that W is countably definable.

Call a *morphism* injective if it has the right lifting property with respect to every monomorphism; write $\mathsf{inj}(\mathcal{E})$ for the class of injective morphisms of a category $\mathcal{E}$. Throughout Prop. 5.4-5.7, $\mathcal{D}$ is supposed to be a diagram, and W some geometric property of morphisms of $\mathcal{D}$-diagrams. Write, as usual, $\mathsf{W}(\mathcal{E}^\mathcal{D})$ for the class of morphisms of $\mathcal{E}^\mathcal{D}$ having that property.

Proposition 5.4. *If* $\mathsf{inj}(Set^\mathcal{D}) \subseteq \mathsf{W}(Set^\mathcal{D})$, *then* $\mathsf{inj}(\mathcal{E}^\mathcal{D}) \subseteq \mathsf{W}(\mathcal{E}^\mathcal{D})$ *for any topos* $\mathcal{E}$.

Proof. First, extend the conclusion from *Set* to presheaf topoi $\mathrm{Pre}(\mathcal{C})$. Choose any $c \in \mathrm{ob}\,\mathcal{C}$. The inclusion $\{c\} \hookrightarrow \mathcal{C}$ induces a geometric morphism $Set \xrightarrow{p} \mathrm{Pre}(\mathcal{C})$; the inverse image p^* is simply evaluation of a presheaf at c, and p^* possesses a left adjoint (the left Kan extension) $p_!$. One checks that $p_!$ preserves monomorphisms. It follows that in the adjunction $Set^\mathcal{D} \xrightarrow{p} \mathrm{Pre}(\mathcal{C})^\mathcal{D}$ (denoted by the same letter, slightly abusively) p^* takes injective morphisms to injective ones; in short, injective morphisms in $\mathrm{Pre}(\mathcal{C})^\mathcal{D}$ are $\mathcal{C}$-objectwise injective. Since categorical constructions act objectwise too in diagram categories, and $\mathrm{Pre}(\mathcal{C})^\mathcal{D} \cong (Set^\mathcal{D})^{\mathcal{C}^{\mathrm{op}}}$, $m \in \mathsf{W}(\mathrm{Pre}(\mathcal{C})^\mathcal{D})$ iff $m(c) \in \mathsf{W}(Set^\mathcal{D})$ for every $c \in \mathrm{ob}\,\mathcal{C}$. Thence $\mathsf{inj}(\mathrm{Pre}(\mathcal{C})^\mathcal{D}) \subseteq \mathsf{W}(\mathrm{Pre}(\mathcal{C})^\mathcal{D})$.

Consider now an arbitrary topos $\mathcal{E}$, and choose a site $(\mathcal{C}, J)$ of definition for $\mathcal{E}$. One has the canonical inclusion $\mathrm{Sh}(\mathcal{C}, J) \hookrightarrow \mathrm{Pre}(\mathcal{C})$ inducing a geometric morphism $\mathcal{E}^\mathcal{D} \underset{i}{\overset{\ell}{\leftrightarrows}} \mathrm{Pre}(\mathcal{C})^\mathcal{D}$. Take $m \in \mathsf{inj}(\mathcal{E}^\mathcal{D})$. Since direct image parts of geometric morphisms preserve injective morphisms, $i(m) \in \mathsf{inj}(\mathrm{Pre}(\mathcal{C})^\mathcal{D})$ whence $i(m) \in \mathsf{W}(\mathrm{Pre}(\mathcal{C})^\mathcal{D})$. But geometric properties are preserved by inverse image parts of geometric morphisms, so $m \cong \ell i(m) \in \mathsf{W}(\mathrm{Sh}(\mathcal{C}, J)^\mathcal{D})$. □

Proposition 5.5. *If* $\mathsf{W}(\mathcal{E}^\mathcal{D})$ *satisfies condition* **c4** *in every Boolean topos* $\mathcal{E}$, *it does so in every topos.*

Indeed, choose a diagram to be a commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \longrightarrow & Y \end{array}$$

of $\mathcal{D}$-diagrams, P to be "the diagram is a pushout and f is a monomorphism and $f \in \mathsf{W}$" and Q as "$g \in \mathsf{W}$". Apply Corollary 2.29. □

By the same reasoning and an appeal to 2.24:

Proposition 5.6. *If* $\mathsf{W}(\mathcal{E}^{\mathcal{D}})$ *satisfies condition* ***c4*** *for Set* $= \mathcal{E}$*, it does so in every topos with enough points.*

By the same reasoning and an appeal to 2.36:

Proposition 5.7. *If* W *is a countably definable property such that* $\mathsf{W}(\mathcal{E}^{\mathcal{D}})$ *satisfies* ***c4*** *for Set* $= \mathcal{E}$*, then it does for every topos* $\mathcal{E}$*.*

One can substitute "**c2**" for "**c4**" in each of Props. 5.5–5.7; for diagrams, **c2** is a geometric proposition too. To sum up, identifying a presentable homotopy model structure on diagrams with the geometric property W defining the weak equivalences as in Prop. 5.1:

Corollary 5.8. (to 5.4, 5.5, 5.6 and 5.7)

- *If* W *yields a model category in every Boolean topos, it does so in every topos.*
- *If* W *yields a model category in Set, it does so in every topos with enough points.*
- *If* W *is countably definable and yields a model category in Set, it does so in every topos.*

Bootstrapping from the classical model structure on *SSet*, Corollary 5.8 yields a closed model structure on simplicial objects in a topos with cofibrations being the monomorphisms, and weak equivalences the "local" or internal weak equivalences. This is due to Joyal [42], methods of whose proof inspired Theorem 4.2.

Example 5.3 gives rise to a model structure on simplicial objects in a topos with weak equivalences being local homology equivalences (and cofibrations the monomorphisms). The relation of this structure to the one constructed by Goerss and Jardine [32] remains to be understood. Note in this respect that the size of the geometric construction defining h_*, as given in Example 5.3, is $\max(\aleph_0, \kappa_{i,n})$ where $\kappa_{i,n}$ is the cardinality of the non-degenerate simplices of the i^{th} space in the spectrum of h_n. It would seem important to understand which generalized homology theories can be defined "economically", that is, by a countable construction over *Set*.

We close with two small observations; the first is of interest in view of the number of model structures on topoi (note that $\mathcal{E}^{\mathcal{D}}$ is a topos if $\mathcal{E}$ is one) with monomorphisms being the cofibrations.

Proposition 5.9. (see Borceux [13] vol. III Prop. 5.6.4)
An object of a topos is injective iff it is a retract of an object of the form Ω^X*. (Here* Ω *is the subobject classifier or "truth object" or "Lawvere element" of the topos.)*

Proposition 5.10. (functoriality in TOPOI)
Keep the assumptions of Prop. 5.1. Let $\mathcal{E} \xrightarrow{f} \mathcal{F}$ *be a geometric morphism, and suppose* W *gives a presentable homotopy model structure on* $\mathcal{D}$*-diagrams in both* $\mathcal{E}$ *and* $\mathcal{F}$*. Then* f *yields a Quillen adjoint pair* $\mathcal{E}^{\mathcal{D}} \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathcal{F}^{\mathcal{D}}$*.*

True since the left adjoint f^* preserves cofibrations (ie. monomorphisms) and weak equivalences (these being defined by a geometric property by assumption). □

Creating model categories via right adjoints.
The number of examples of geometric model structures on diagrams is infinite, simply because this class is closed under the formation of functor categories, as we shall shortly see. Nonetheless, there is a plethora of homotopy model structures on locally presentable categories that cannot arise via Theorem 4.2. They seem to fall into three types:

(i) The model structure, while not a presentable one in the sense of 4.2, is Quillen-equivalent to one, for the *same* class of weak equivalences in fact. Examples include Heller's "left" and "right" structures on simplicial presheaves and the two structures on simplicial objects in an abelian category due to Quillen. In these cases of competing structures, one's class of cofibrations includes the other's, hence the identity functor induces a Quillen equivalence between them. It is possible to concoct examples where the set of possible notions of cofibrations is arbitrary large.

(ii) While the underlying category satisfies **c0**, the model structure is not known to be equivalent to a presentable one — and perhaps can even be shown not to allow all monomorphisms as cofibrations while leaving weak equivalences unchanged. Examples abound: Thomason's [61] homotopy theory on the category of small categories, the "fine"equivariant structures on simplicial G-sets (G a discrete group), Moerdijk's [55] "diagonal" model structure on bisimplicial sets, the E^2 model structure on bisimplicial sets (Dwyer-Kan-Stover [21]).

(iii) The category, while locally presentable, fails some part of condition **c0**, most often (ii). (Note that if it fails (iii), it cannot have a Quillen model structure with monomorphisms being the cofibrations!) Examples include simplicial groups, and most simplicial T-algebras in fact.

What is common to many of these cases is the existence of a functor R that is a right adjoint and defines the weak equivalences and the fibrations from a target model category. M. Hopkins christened the set-up:

Terminology 5.11. (creating model structures via right adjoints)
Let $\mathcal{M}$ be a model category with data $\langle \mathsf{cof}; \mathsf{W}; \mathsf{fib} \rangle$, and $\mathcal{E} \underset{R}{\overset{L}{\leftrightarrows}} \mathcal{M}$ an adjunction. If $\langle \text{LLP}; R^{-1}(\mathsf{W}); R^{-1}(\mathsf{fib}) \rangle$ gives a model structure on $\mathcal{E}$, say that model structure is *created by* R from the one on $\mathcal{M}$. ($R^{-1}(-)$ here is simply the pre-image of a class of morphisms.)

Under these circumstances, R and its left adjoint become a Quillen adjoint pair between $\mathcal{E}$ and $\mathcal{M}$. Guessing a model structure in this form goes back to Quillen's seminal [58]. It is such a natural situation to consider that the following list of contributions to the question is certain to be incomplete: Blanc [11], Cabello-Garzón [17], Crans [19], Goerss-Jardine [33], Hirschhorn-Kan [20], Rezk [60]. The next observation has cognates in all these papers; cf. especially Crans [19] Theorem 3.3.

Proposition 5.12. *Let $\mathcal{M}$, with data $\langle \mathsf{cof}; \mathsf{W}; \mathsf{fib} \rangle$, be a cofibrantly generated model category and $\mathcal{E} \underset{R}{\overset{L}{\leftrightarrows}} \mathcal{M}$ an adjunction. Suppose*

(0) *$\mathcal{E}$ is a locally presentable category*
(1) *R preserves filtered colimits*

(2) *whenever f is an acyclic cofibration in $\mathcal{M}$, and g a pushout of $L(f)$ in $\mathcal{E}$, then $R(g) \in \mathsf{W}$.* *Then R creates a cofibrantly generated model structure on $\mathcal{E}$.*

Proof. **M1**, **M2**, **M3**, **M4**(i) are gratis. Let I be the set of generating cofibrations of $\mathcal{M}$. As in Prop. 4.1, $\Sigma := \{L(i) \mid i \in I\}$ induces a small-factorization system $(\mathsf{C}_\Sigma, \mathsf{F}_\Sigma)$ on $\mathcal{E}$. By adjunction, $L(f)$ has the left lifting property against g in $\mathcal{E}$ iff $R(g)$ has the right lifting property w.r.t. f in $\mathcal{M}$. Since every member of C_Σ arises as retract of a transfinite composition of pushouts of $L(i)$, $i \in I$, and an LLP class is closed under those operations, every member of C_Σ is a cofibration in $\mathcal{E}$. Adjointly, for $g \in \mathsf{F}_\Sigma$, $R(g)$ has the right lifting property w.r.t. every $c \in \operatorname{mor}\mathcal{M}$ such that $L(c)$ belongs to C_Σ. But by the assumption $\mathcal{M}$ is cofibrantly generated, that includes all $\mathcal{M}$-cofibrations. So $R(g)$ is an acyclic fibration in $\mathcal{M}$, and by definition g an acyclic fibration in $\mathcal{E}$. This gives **M5**(ii). Let J be the set of generating acyclic cofibrations in $\mathcal{M}$. Consider now the small-factorization system induced on $\mathcal{E}$ by $\Sigma := \{L(j) \mid j \in J\}$, denoted, say, $(\mathsf{C}'_\Sigma, \mathsf{F}'_\Sigma)$. If $g \in \mathsf{F}'_\Sigma$, then $R(g)$ has the RLP w.r.t. every acyclic cofibration in $\mathcal{M}$, so g is an $\mathcal{E}$-fibration. Any $f \in \mathsf{C}'_\Sigma$ is a retract of a transfinite composition of pushouts of $L(j)$, $j \in J$. Using Prop. 0.3 (a corollary of which is that a functor preserves filtered colimits iff it preserves transfinite compositions) or by direct transfinite induction, (1) and (2) imply that $R(f)$ is a filtered colimit of acyclic cofibrations, hence a weak equivalence in $\mathcal{M}$. Since f is an $\mathcal{E}$-cofibration by the argument used for $(\mathsf{C}_\Sigma, \mathsf{F}_\Sigma)$, we get **M5**(i). A retract argument now yields **M4**(ii). □

Weaker (if more awkward to state) conditions on $\mathcal{E}$ would also make the proof work; notably, it is enough for certain objects of $\mathcal{E}$ to be sequentially small, and for R to preserve *certain* filtered colimits.

Running assumption. For the rest of this section, fix a diagram $\mathcal{D}$ and a geometric property W satisfying Corollary 5.8, so that $\langle \mathsf{mono}; \mathsf{W}(\mathcal{E}^{\mathcal{D}}); \mathrm{RLP} \rangle$ is a geometric homotopy model structure on $\mathcal{E}^{\mathcal{D}}$, in the sense of 5.1, for every topos $\mathcal{E}$.[16] We give three applications of 5.12, essentially three ad hoc ways to get around the clumsy condition (2): cocontinuous direct images; functor categories; and coherent topologies.

Proposition 5.13. (cocontinuous direct images)
Let $\mathcal{E} \xrightarrow{f} \mathcal{F}$ be a geometric morphism. If f_* preserves colimits (equivalently, if it is a left adjoint) then it creates a model structure on $\mathcal{E}^{\mathcal{D}}$ from the W-structure on $\mathcal{F}^{\mathcal{D}}$.

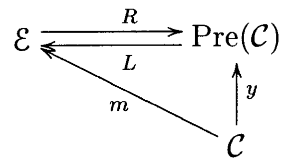
Proof. Apply Prop. 5.12. (0) and (1) are part of the assumptions. For (2), observe that being an acyclic cofibration in $\mathcal{F}^{\mathcal{D}}$ is a conjunction of geometric properties (being a monomorphism and being a weak equivalence — the latter being a geometric property by assumption) and is thus preserved by f^*. Since f_* commutes with pushouts, since it preserves geometric properties itself (being both a left and a right adjoint!), and since acyclic cofibrations are closed under pushouts in $\mathcal{F}^{\mathcal{D}}$ (by the assumption it's a model category) we are done. □

Direct images arising via *cofiltering precomposition* between functor categories satisfy the hypothesis (though they may lack homotopical significance). Recall that a functor $\mathcal{C} \xrightarrow{m} \mathcal{E}$

[16]Several corollaries continue to hold under the weaker assumption that $\langle \mathsf{mono}; \mathsf{W}(\mathcal{E}^{\mathcal{D}}); \mathrm{RLP} \rangle$ is a geometric homotopy model structure on $\mathcal{E}^{\mathcal{D}}$ for *that particular* topos $\mathcal{E}$, but I am unaware of geometric homotopy model structures that do work across TOPOI.

from a small category into a cocomplete elementary topos is *flat* iff the basic adjunction scheme

(5.1)



where y is the Yoneda embedding, L is the left Kan extension of m along y, and $R(X) := \mathrm{Hom}_{\mathcal{E}}(m(-), X)$, yields a geometric morphism $\mathcal{E} \to \mathrm{Pre}(\mathcal{C})$; which amounts to saying that L preserves finite limits.

Definition 5.14. Call a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ between diagrams *cofiltering* if the composite with the Yoneda embedding $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{y} \mathrm{Pre}(\mathcal{D})$ is flat.

Equivalently (where $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is called filtering iff $\mathcal{C}^{\mathrm{op}} \xrightarrow{F^{\mathrm{op}}} \mathcal{D}^{\mathrm{op}}$ is cofiltering, of course)[17]

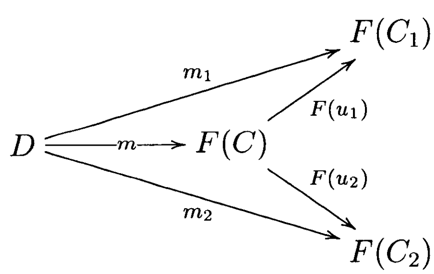
Corollary 5.15. *The left Kan extension of the functor $Set^{\mathcal{D}} \to Set^{\mathcal{C}}$ induced by precomposition with $\mathcal{C} \xrightarrow{F} \mathcal{D}$ preserves finite limits iff $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is filtering.*

Under these circumstances, $Set^{\mathcal{D}} \to Set^{\mathcal{C}}$ is the *direct* image of a geometric morphism f; the inverse image is the left Kan extension; and the direct image has an extra right adjoint, to wit, the right Kan extension. So that is the situation of Prop. 5.13.

The following two statements are easy corollaries of MacLane-Moerdijk [51] Theorem VII.10.1.

Proposition 5.16. *The functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is cofiltering iff the following three conditions are met:*

- *For any $D \in \mathrm{ob}\,\mathcal{D}$, there exist $C \in \mathrm{ob}\,\mathcal{C}$ and a morphism $D \xrightarrow{m} F(C)$ of $\mathcal{D}$.*
- *Let $C_1, C_2 \in \mathrm{ob}\,\mathcal{C}$, $D \in \mathrm{ob}\,\mathcal{D}$ and morphisms $D \xrightarrow{m_1} F(C_1)$, $D \xrightarrow{m_2} F(C_2)$. Then there exist $C \in \mathrm{ob}\,\mathcal{C}$ and morphisms $C \xrightarrow{u_1} C_1$, $C \xrightarrow{u_2} C_2$, $D \xrightarrow{m} F(C)$ such that*



[17] In MacLane-Moerdijk [51], the term *filtering* is applied for functors from diagrams into cocomplete elementary topoi as a synonym of *flat*. Note that this usage cannot possibly conflict with the case of a functor between diagrams, since a cocomplete topos is never small. Flat, on the other hand, is so fixed a term in commutative algebra that using it for the concept of Definition 5.14 – or its dual – seemed unwise. Elsewhere, *filtering* is sometimes a synonym of *filtered*. As a confusion to end all confusions, the senses of *filtered* and *cofiltered* diagrams oscillate (MacLane-Moerdijk [51] and Borceux [13] follow opposite conventions, for example) though there's hardly a question about what *filtered colimits* and *filtered limits* are to be. Under the proposed interpretation, at any rate: $\mathcal{C}$ is a cofiltered category iff the canonical functor $\mathcal{C} \xrightarrow{F} \{*\}$ to the terminal (one-morphism) category is cofiltering; $\mathrm{colim}_{\mathcal{D}}$ commutes with finite limits in topoi iff $\mathcal{D}$ is a filtered diagram.

commutes.[18]

- *Let $C, C' \in \operatorname{ob}\mathcal{C}$, u_1, u_2 morphisms from C to C', and $D \xrightarrow{m} F(C) \in \operatorname{mor}\mathcal{D}$ such that $F(u_1)m = F(u_2)m$. Then there exists $B \xrightarrow{w} C \in \operatorname{mor}\mathcal{C}$ and $D \xrightarrow{b} F(B)$ such that $F(w)b = m$ and $u_1 w = u_2 w$.*

$$D \xrightarrow{m} F(C) \underset{F(u_2)}{\overset{F(u_1)}{\rightrightarrows}} F(C')$$

$$B \xrightarrow{w} C \underset{u_2}{\overset{u_1}{\rightrightarrows}} C' \qquad\qquad D \xrightarrow{b} F(B) \xrightarrow{F(w)} F(C) = m$$

Proposition 5.17. *Corollary 5.15 continues to hold for presheaves valued in an arbitrary topos $\mathcal{E}$; that is, upon replacing Set by $\mathcal{E}$ in the statement.*

Returning to Prop. 5.13, denote the right adjoint of the direct image $\mathcal{E} \xrightarrow{f_*} \mathcal{F}$, when it exists, by f^+. The string of adjunctions $f^* \dashv f_* \dashv f^+$ can be viewed in many ways. It is an essential geometric morphism — with direct image f^+ — whose far left adjoint is left exact. Or, it encodes two geometric morphisms: one from $\mathcal{E}$ to $\mathcal{F}$ (with direct image f_* and inverse image f^*) and one from $\mathcal{F}$ to $\mathcal{E}$ (with direct image f^+ and inverse image f_*). The existence of the adjunction natural transformations $\mathrm{Id}_{\mathcal{F}} \to f_* f^*$ and $f^* f_* \to \mathrm{Id}_{\mathcal{E}}$, if thought of as "continuous deformations", means that $\mathcal{E}$ and $\mathcal{F}$ are "homotopy equivalent" (see Joyal–Wraith [43]).

Following SGA4, Johnstone and Moerdijk [41] call a geometric morphism $\mathcal{E} \xrightarrow{f} \mathcal{F}$ *local* if the direct image f_* possesses a right adjoint as above, *and* the inverse image f^* is full and faithful. The motivating example is the inclusion of petit sites in corresponding gros ones; see Johnstone–Moerdijk [41] for extensive discussion. In the local case $\mathrm{Id}_{\mathcal{F}} \cong f_* f^*$, so it is tempting to think of $\mathcal{F}$ as a (deformation) *retract* of $\mathcal{E}$ (see Moerdijk–Reyes [57]). Advances in the theory of classifying topoi (cf. especially Joyal–Wraith [43] and Moerdijk [56]) make it likely that the "correct" structural study of algebraic topological invariants of topoi will involve the 2-category TOPOI with its intrinsic homotopy classes.

Functor categories. Let $\langle \mathsf{mono}; \mathsf{W}(\mathcal{E}^{\mathcal{D}}); \mathrm{RLP}\,\rangle$ be a geometric homotopy model structure on $\mathcal{E}^{\mathcal{D}}$, and $\mathcal{C}$ a diagram. There are two model structures on $(\mathcal{E}^{\mathcal{D}})^{\mathcal{C}}$ whose weak equivalences are $\mathcal{C}$-objectwise; the identity functor induces a Quillen equivalence between them. One is $\langle \mathsf{mono}; \mathsf{W}((\mathcal{E}^{\mathcal{C}})^{\mathcal{D}}); \mathrm{RLP}\,\rangle$ (note $(\mathcal{E}^{\mathcal{D}})^{\mathcal{C}} \cong (\mathcal{E}^{\mathcal{C}})^{\mathcal{D}}$ and geometric constructions, hence geometric properties, are evaluated "objectwise" in functor categories) and the other, $\langle \mathrm{LLP}; \mathsf{W}((\mathcal{E}^{\mathcal{C}})^{\mathcal{D}}); \mathsf{fib}\rangle$ where fib is the class of morphisms of $(\mathcal{E}^{\mathcal{D}})^{\mathcal{C}}$ that are $\mathcal{C}$-objectwise fibrations in the underlying model structure on $\mathcal{E}^{\mathcal{D}}$, that is, $\langle \mathsf{mono}; \mathsf{W}(\mathcal{E}^{\mathcal{D}}); \mathrm{RLP}\,\rangle$.

Let $\mathcal{O}$ be the set of objects of $\mathcal{C}$, thought of as a category with identity morphisms only. The inclusion $\mathcal{O} \hookrightarrow \mathcal{C}$ induces an essential geometric morphism $(\mathcal{E}^{\mathcal{D}})^{\mathcal{O}} \xrightarrow{f} (\mathcal{E}^{\mathcal{D}})^{\mathcal{C}}$. Put the obvious homotopy model structure on the product category $(\mathcal{E}^{\mathcal{D}})^{\mathcal{O}}$. Though $\mathcal{O} \hookrightarrow \mathcal{C}$ is cofiltering only if $\mathcal{O} = \mathcal{C}$, the second model structure quoted is created by f^* (which is a right as well as left adjoint); one can check the criteria of Prop. 5.12 directly. It also follows from a very general theorem of Hirschhorn [34] asserting the existence of a cofibrantly generated model structure on a diagram category $\mathcal{M}^{\mathcal{C}}$ (of the "weak equivalences and fibrations are objectwise" type) solely on the assumption that there is a cofibrantly generated model structure on $\mathcal{M}$.

[18]The first two conditions imply — and are already stronger than — that $\mathcal{C}$ is cofinal in $\mathcal{D}$ via F.

Coherent topologies. Let $\Omega \xrightarrow{j} \Omega$ be a Lawvere-Tierney topology on the topos $\mathcal{E}$. For Prop. 5.12 to apply to the geometric morphism (inclusion) $\mathrm{Sh}(\mathcal{E}, j) \to \mathcal{E}$, it is necessary that the direct image preserve filtered colimits. But in the case of homotopy model categories defined by *filtered* geometric properties, that is also sufficient:

Proposition 5.18. *Strengthen the running assumption regarding $\mathcal{D}$ and W on page 48 to W being a filtered geometric property. If $\mathrm{Sh}(\mathcal{E}, j) \xrightarrow{i_*} \mathcal{E}$ preserves filtered colimits, then it creates a model structure on $\mathrm{Sh}(\mathcal{E}, j)^{\mathcal{D}}$ from the W-structure on $\mathcal{E}^{\mathcal{D}}$.*

Proof. (0) and (1) of 5.12 are gratis. As for (2), note that colimits in $\mathrm{Sh}(\mathcal{E}, j)^{\mathcal{D}}$ are computed by sheafifying the colimit in $\mathcal{E}^{\mathcal{D}}$. Acyclic cofibrations are preserved by pushouts in $\mathcal{E}^{\mathcal{D}}$, sheafification preserves geometric properties, and by assumption the inclusion preserves filtered geometric properties (as well as monos). □

When does the inclusion $\mathrm{Sh}(\mathcal{E}, j) \hookrightarrow \mathcal{E}$ preserve filtered colimits? Karazeris [47] investigates the problem in the much more general context of locally presentable categories. Under the assumption that $\mathcal{E}$ is locally *finitely* presentable, this is equivalent to $\mathrm{Sh}(\mathcal{E}, j) \hookrightarrow \mathcal{E}$ preserving filtered colimits of monomorphisms, and can be turned into the condition of "$\aleph_0$-compactness" of the topology j. The paradigm is: let $\mathcal{E}$ be a presheaf topos and j a *coherent* Grothendieck topology, ie. such that every covering sieve is generated by finitely many morphisms.

Accessible localization. A locally presentable model category such that weak equivalences (as objects) form an accessible full subcategory of the category of morphisms seems to be a powerful framework for localization arguments. We give three examples; the first simply casts Theorem 4.2 into a convenient mold.

Proposition 5.19. (accessible localization)
Let $\mathcal{E} \xrightarrow{F} \mathcal{A}$ be a functor, w a class of morphisms of $\mathcal{A}$. Define $\mathsf{W} \subseteq \mathrm{mor}\,\mathcal{E}$ as $F^{-1}(\mathsf{w}) := \{f \in \mathrm{mor}\,\mathcal{A} \mid F(f) \in \mathsf{w}\}$. Suppose

- *conditions **c0**, **c3**, **c4** of Theorem 4.2 hold*
- *$\mathcal{A}$ is an accessible category*
- *w satisfies the 2-of-3 property*
- *$\mathsf{w} \cap \mathsf{mono}$ is closed under filtered colimits in $\mathcal{A}$*
- *F preserves filtered colimits of monomorphisms.*

Then $\langle \mathsf{mono}; \mathsf{W}; \mathrm{RLP}\rangle$ define a presentable model structure on $\mathcal{E}$.

Use 1.23 to obtain **c1** of 4.2. □

The immediate example is Bousfield (homological) localization; here $\mathcal{E} = SSet$, $\mathcal{A}$ =graded abelian groups, F is a homology functor and w is the class of isomorphisms. (Unlike Example 5.3, this proof does not require the representability of homology, but does need a condition akin to Milnor's axiom.) The immediate *non-example* is cohomological localization; the opposite category of graded abelian groups is not accessible.

The next application generalizes to geometric model categories (we are reverting to the running assumption on page 48) a theorem of Goerss and Jardine [32] about simplicial weak equivalences of (pre)sheaves.

Corollary 5.20. (inverse image localization)
Let $\mathcal{F} \xrightarrow{f} \mathcal{E}$ *be a geometric morphism,* W *a geometric model structure on* $\mathcal{D}$*-diagrams. The* f^**-localization of* $\mathcal{E}^{\mathcal{D}}$ *in the sense of 5.19 (*$\mathcal{E} := \mathcal{E}^{\mathcal{D}}$*,* $\mathcal{A} := \mathcal{F}^{\mathcal{D}}$*,* $\mathsf{w} := \mathsf{W}(\mathcal{F}^{\mathcal{D}})$*,* $F := f^*$*) exists.*

Proof. **c3** follows from the fact that f^* preserves geometric properties; hence $(f^*)^{-1}(\mathsf{w}) \supseteq \mathsf{W}(\mathcal{F}^{\mathcal{D}}) \supseteq \mathrm{RLP}(\mathsf{mono})$. The rest of the criteria are clear. □

We can now produce further instances of behavior listed under (i) on page 47. For any homotopy model structure created via 5.13, there exists a Quillen-equivalent presentable model structure as well: use Corollary 5.20. (Note that the geometric homotopy model structure on the functor category $(\mathcal{E}^{\mathcal{D}})^{\mathcal{C}}$ also arises through localization along the inverse image (i.e. restriction) of the essential geometric morphism $(\mathcal{E}^{\mathcal{D}})^{\mathcal{O}} \xrightarrow{f} (\mathcal{E}^{\mathcal{D}})^{\mathcal{C}}$, where $\mathcal{O}$ is the set of objects of the diagram $\mathcal{C}$.) And if the inclusion of topoi $\mathrm{Sh}(\mathcal{E}, j) \xrightarrow{i_*} \mathcal{E}$ creates a model structure on $\mathrm{Sh}(\mathcal{E}, j)$ — cf. Prop. 5.18 — i_* can be shown to serve as a localizing functor (the F of Prop. 5.19) to generate a Quillen-equivalent presentable model structure.

As long as the f^* of Corollary 5.20 is sheafification, one only obtains new models for an old homotopy theory:

Proposition 5.21. *Let* $j : \Omega \to \Omega$ *be a Lawvere-Tierney topology on the topos* $\mathcal{F}$*, with associated geometric morphism (inclusion of topoi)* $\mathrm{Sh}(\mathcal{F}, j) \xrightarrow{i} \mathcal{F}$*. Consider the following two homotopy model structures on* $\mathrm{Sh}(\mathcal{F}, j)^{\mathcal{D}}$*:* $\langle \mathsf{mono}; \mathsf{W}(\mathrm{Sh}(\mathcal{F}, j)^{\mathcal{D}}); \mathrm{RLP}\,\rangle$ *(guaranteed by the running hypotheses on* $\mathcal{D}$ *and* W*) and* $\langle \mathsf{mono}; i^{-1}(\mathsf{W}_{\mathcal{F}}); \mathrm{RLP}\,\rangle$ *(guaranteed by Prop. 5.20). They are Quillen-equivalent.*

Proof. For brevity, write $\mathcal{E}$ for $\mathrm{Sh}(\mathcal{F}, j)$. The Quillen equivalence is induced by $i^* \dashv i_*$. i^* takes cofibrations to cofibrations and weak equivalences (by definition) to weak equivalences; hence it is a Quillen adjunction. It is a Quillen equivalence if for each cofibrant $A \in \mathrm{ob}\,\mathcal{F}^{\mathcal{D}}$ and fibrant $X \in \mathrm{ob}\,\mathcal{E}^{\mathcal{D}}$, a map $A \to i_*(X)$ is a weak equivalence in $\mathcal{F}^{\mathcal{D}}$ iff its adjoint $i^*(A) \to X$ is a weak equivalence in $\mathcal{E}^{\mathcal{D}}$. But that is tautologous, since $i^* i_* \cong \mathrm{Id}_{\mathcal{E}^{\mathcal{D}}}$. □

Again, for the simplicial case, with $\mathcal{E}$ being a presheaf topos, the discovery is due to Jardine; see [39] and [32].

The last proposition shows that if one can perform a set of accessible localizations, one can also perform them *simultaneously.*

Proposition 5.22. (intersection of homotopy theories)
Fix a category $\mathcal{E}$*. Suppose that* $\{\mathsf{W}_\lambda | \lambda \in \Lambda\}$ *is a* set *of presentable model structures on* $\mathcal{E}$*, that is, for each* $\lambda \in \Lambda$*,* $\langle \mathsf{mono}, \mathsf{W}_\lambda, \mathrm{RLP}\,\rangle$ *is a presentable model category in the sense of Theorem 4.2. Defining* $\mathsf{W} := \bigcap_{\lambda \in \Lambda} \mathsf{W}_\lambda$*,* $\langle \mathsf{mono}, \mathsf{W}, \mathrm{RLP}\,\rangle$ *is again a presentable model structure on* $\mathcal{E}$*.*

Proof. **c2**,**c3**,**c4**,**c5** follow λ-wise, and **c1** by Corollary 1.22 (intersection of accessible subcategories). □

Despite the deceptive simplicity of its proof (granted the deep Limit Theorem of Makkai and Paré!), Prop. 5.22 seems to be the demarcation line between model categories that are merely cofibrantly generated and model categories that are accessibly so. Such speculations, however, are best kept for follow-up work.

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