Initial Data to Vacuum Einstein Equations With Asymptotic Expansion

by

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ABSTRACT

We investigate the Vacuum Einstein Equations (E-V) for a globally hyperbolic and maximally time-foliated space-time (M, $\mathbf{g}$) such that M $\cong \mathbb{R}_t \times \mathbb{R}^3$. For such a space-time, it is well-known that solving (E-V) is equivalent to first finding (g, k) which satisfy the *constraint equations*, where g is a Riemannian metric on $\mathbb{R}^3$ and k is a symmetric 2-tensor on $\mathbb{R}^3$, and then, with such (g, k) as initial data, solving the *evolution equations.* We prove that there exist solutions to the constraint equations which have a complete asymptotic expansion at spatial infinity. In order to do this, we compactify the space ($\cong \mathbb{R}^3$) into a manifold with boundary ($\cong \mathbb{B}^3$), thereby bringing spatial infinity to $\partial\mathbb{B}^3$, and reformulate our problem in the b-setting, developed by R.B.Melrose. We then use analytic tools of the b-calculus to obtain the main result. We briefly discuss at the end how such solutions might evolve according to the evolution equations.

Thesis Supervisor: Richard Melrose
Title: Professor of Mathematics

To Mom and Dad

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Contents

Chapter 1

Introduction

According to Einstein, the large scale structure of a space-time (M, $\mathbf{g}$) where M is a differentiable 4-manifold and $\mathbf{g}$ is a Lorentzian metric on M, is governed by his famous field equations. These equations relate the bending of the space-time, measured in the Ricci curvature $\mathbf{R}_{\alpha\beta}$ of $\mathbf{g}$, to the gravitation, manifested through an energy-momentum tensor, $\mathbf{T}_{\alpha\beta}$, of a matter field as following :

$$\mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\mathbf{R} = 8\pi\mathbf{T}_{\alpha\beta}$$

where $\mathbf{R}$ is the scalar curvature of $\mathbf{g}$.

In a vacuum space-time, we have $\mathbf{T}_{\alpha\beta} = 0$ and the above reduces to the vacuum Einstein equations, henceforth denoted by (E-V) :

$$\mathbf{R}_{\alpha\beta} = 0$$

Thus, to solve the vacuum Einstein equations (E-V) is to find a pair (M, $\mathbf{g}$) such that $\mathbf{R}_{\alpha\beta} = 0$ on M.

In this thesis, we will assume that a space-time M is globally hyperbolic, i.e. there exists a hypersurface Σ in M which any causal curve intersects exactly at one point. Such a space-time is known to allow the existence of the globally defined differentiable function t whose gradient $\mathbf{D}t$ is time-like everywhere. We, then, can foliate M by the level surfaces of t (see [C-M]).

Since $\mathbf{D}t$ is normal to the level sets $t^{-1}(a), a \in \mathbb{R}$, if we define a set of coordinates $\{t, x^1, x^2, x^3\}$, where $\{x^1, x^2, x^3\}$ is a set of coordinates for $t^{-1}(0)$ and

$\mathbb{R}^4 \ni (s, x^1, x^2, x^3) \mapsto$ the point in M obtained by following the integral curve of $\mathbf{D}t$ from $(x^1, x^2, x^3) \in t^{-1}(0)$ till it intersects $t^{-1}(s)$

then the metric $\mathbf{g}$ can then be written as :

$$\mathbf{g} = -\phi^2(t, \mathbf{x})dt^2 + \sum_{i,j} g_{ij}(t, \mathbf{x})dx^i dx^j$$

where ϕ, called the lapse function, is defined to be $\phi = -\frac{1}{(<\mathbf{D}t,\mathbf{D}t>)^2}$. Moreover, $g_{ij}(t,x)$ is a Riemannian metric on Σ_t in these coordinates, and the extrinsic curvature k_{ij} of Σ_t is then given by :

$$k_{ij} = -(2\phi)^{-1}\partial_t g_{ij}$$

In this setting, the (E-V) for the 4-metric $\mathbf{g}$ can be re-written as equations for g, k, and ϕ. Indeed, if we apply the condition $\mathbf{R}_{\alpha\beta} = 0$ to the well-known structure equations of the foliations (where T is future-oriented unit normal) :

$$\begin{aligned}
\partial_t k_{ij} &= -\nabla_i\nabla_j\phi + \phi(\mathbf{R}_{iTjT} - k_{ia}k^a_j) \\
\nabla_i k_{jm} - \nabla_j k_{im} &= \mathbf{R}_{mTij} \\
R_{ij} - k_{ia}k^a_j + k_{ij}trk &= \mathbf{R}_{iTjT} + \mathbf{R}_{ij}
\end{aligned}$$

we get the following set of equations that relate g and k on each $t-$slice

$$\begin{aligned}
\nabla^j k_{ji} - \nabla_i trk &= 0 \\
R - |k|^2 + (trk)^2 &= 0
\end{aligned}$$

and another set of equations which shows how g and k evolve with respect to time t :

$$\begin{aligned}
\partial_t g_{ij} &= -2\phi k_{ij} \\
\partial_t k_{ij} &= -\nabla_i\nabla_j\phi + \phi(R_{ij} + trkk_{ij} - 2k_i ak^a_j)
\end{aligned}$$

We make two observations here. First, if the first set of equations are satisfied by $(g(0), k(0))$ on $(t = 0)-$slice, they are satisfied by $(g(t), k(t))$ on any $t-$slice, provided that $(g(t), k(t))$ evolve with respect to time t, due to the Bianchi-indentities. Secondly, we note that, in the second set of equations, there are 13 unknowns : (g_{ij}, k_{ij}, ϕ), but actually only 12 equations. To remove this indeterminancy, we add one more condition, often called the maximal foliation condition,

$$tr_g(k) = 0$$

and then we finally get the following *determined* system of equations :
Constraint Equations :

$$\begin{aligned}
\mathrm{tr}_g(k) &= 0 \\
\nabla^j k_{ji} &= 0 \\
R(g) &= |k|^2_g
\end{aligned}$$

Evolution Equations :

$$\begin{aligned}
\partial_t g_{ij} &= -2\phi k_{ij} \\
\partial_t k_{ij} &= -\nabla_i\nabla_j\phi + \phi(R_{ij} - 2k_{ia}k^a_j) \\
\Delta\phi &= |k|^2\phi
\end{aligned}$$

Thus, finally, solving the Einstein Field equations, in this case of maximally time-foliated vacuum space-time, is reduced to the following two steps :

Step 1 Find a pair (g, k) which satisfies the **Constraint Equations** on Σ_o.

Step 2 With such (g, k) as initial data, solve the **Evolution Equations**.

Now we state two important results proven in this direction, when $\Sigma = \mathbb{R}^3$. First, in [C-M], Christodoulou and O'murchadra proved that there indeed exist pairs (g, k) which satisfy the constraint equations, and moreover, which are asymptotically flat near infinity. More precisely, they proved :

Theorem 1.1 *Let g, a Riemannian metric on $\mathbb{R}^3$, be such that $g - e \in H_{s,\delta}$ and $R(g) \geq 0$, where e is the Euclidean metric on $\mathbb{R}^3$, $s \geq 4$, and $-3/2 < \delta < -1/2$. Then there exists $\tilde{g}$ in the conformal class of g such that $\tilde{g} - e \in H_{s,\delta}$, and a symmetric 2-cotensor, $\tilde{k} \in H_{s-1,\delta+1}$, such that $(\tilde{g}, \tilde{k})$ satisfy the constraint equations.*

Then, in [C-K], Christodoulou and Klainerman showed that

Theorem 1.2 *Such $(\tilde{g}, \tilde{k})$, which, in addition, satisfies a global smallness assumption, leads to a unique, globally hyperbolic, smooth, and geodesically complete solution of the evolution equations. Moreover, this solution is globally asymptotically flat in the sense that the curvature tensor tends to zero on any causal or spacelike geodesic, as the corresponding affine parameter approaches infinity.*

I started this thesis work as an attempt to refine these results in terms of the regularity of (g, k) near infinity. Of course, the work of [C-M] does say something about the behaviors of (g, k) near infinity : by knowing that $(g - e, k) \in H_{s,\delta}$, one knows that $(g - e, k)$ becomes small at infinity (for s and δ in our range). However, it would be better if we knew, in addition, exactly how $(g - e, k)$ becomes small at infinity – a property not quite captured by $H_{s,\delta}$. Thus, we look for solutions to the Einstein Vacuum equations with "*better*" control at infinity - i.e. $(g - e, k)$ that has a complete asymptotic expansion at infinity. (we will define this more precisely later.)

Why is having a complete asymptotic expansion at infinity important for g (and also for k)? For example, if (g, k) were a solution to the constraint equations and if we can write g as follows :

$$g_{ij} = (1 + C\frac{1}{r})\delta_{ij} + \text{higher order terms}$$

for some constant C, then it turns out that in fact

$$C = 2M$$

where M is what is called the A-D-M mass, which is, *roughly speaking*, the mass of the universe seen from spatial infinity. Scheon-Yau proved in [Schoen-Yau] that M was in fact non-negative, settling a long-standing Positive Mass Conjecture. It is also conjectured that the coefficients of the higher order terms and some functions of them might also carry some physical significance. In order to investigate this further, you obviously first have to know that g_{ij} indeed has expansions, in higher order terms, and most desirably a complete asymptotic expansion. Thus I asked the following questions :

Naturally, I asked the following two questions :

Question 1 Is there a solution (g, k) to the *constraint equations* which are *not only* asymptotically flat in the sense of [C-M] *but also* have a complete asymptotic expansion at infinity?

Question 2 If so, does the solution $(g(t), k(t))$ to the *evolution equations* also have such an expansion at least locally in time?

In order to answer these two questions. I first radially compactify $\mathbb{R}^3$ to $\mathbb{B}^3$, a manifold with boundary (see [Melrose 1]). Then the behaviors of (g, k) at infinity of $\mathbb{R}^3$ correspond to the behaviors at the boundary of $\mathbb{B}^3$, so that the above questions can be analyzed by $b-$calculus developed by my advisor, Richard Melrose. In this $b-$setting, $\mathcal{A}^{\infty}_{\mathcal{E}}(\mathbb{B}^3)$, the set of all functions (tensors defined similarly) with a complete asymptotic expansion at infinity, with respect to an index set $\mathcal{E}$ can be defined as follows :

$$\mathcal{A}^{\infty}_{\mathcal{E}}(\mathbb{B}^3) := \{f \in C^{\infty}(\mathbb{R}^3) \mid \forall r \in \mathbb{R}^+ \Rightarrow f(x,\theta) = \sum_{s,k\in\mathcal{E}}^{s<r} c_{s,k}(\theta) x^s (log x)^k + R_r\}$$

where x is a boundary defining function and $R_r \in x^r H^{\infty}_b(\mathbb{B}^3)$. We will discuss more in detail (in **Chapter 2**) about index sets and $b-$ *Sobolev* spaces, $x^r H^{\infty}_b(\mathbb{B}^3)$.

With this definition of a complete asymptotic expansion, I answer **Q1** *affirmatively.* This is the main result of this thesis, as stated in the following theorem (we will give a more precise $b-$version in **Chapter 2** :

Theorem 1.3 *Let g be as in the hypothesis of the theorem(1.1), and, moreover, have a complete asymptotic expansion at infinity. Then there exists $\tilde{g}$ in the conformal class of g such that ($\tilde{g} - e \in H_{s,\delta}$) and a symmetric 2-cotensor $\tilde{k} \in H_{s-1,\delta+1}$, so that $(\tilde{g}, \tilde{k})$ is a maximal initial data set with both $\tilde{g}$ and $\tilde{k}$ having a complete asymptotic expansion at infinity.*

In proving this theorem, I, show how to construct a pair with a complete asymptotic expansion. I do so by following the steps that Christodoulou and O'murchadra used in [C-M] to produce an asymptotically flat solution $(\tilde{g}, \tilde{k})$ to **constraint equations** by modifying any arbitrary asymptotically flat pair (g, k) with $R(g) \geq 0$. Likewise, I also start out with an asymptotically flat pair (g, k) with $R(g) \geq 0$, but this time, with a complete asymptotic expansion, in addition. Then I show that in each of the steps that they used, the property of having a complete asymptotic expansion is preserved, by proving the following three theorems about three equations that play a very crucial role in the modification process.

Assuming that

$$\begin{aligned} a &\in (0,1) \\ g-e &\in x^a H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \cap A_{\mathcal{G}}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \text{ for an index set } \mathcal{G} \end{aligned}$$

we prove :

Theorem 1.4 (York's Equation) *Suppose* $\tau \in x^{a+2} H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3}) \cap A_{\mathcal{E}}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3})$. *Then the unique solution* $\sigma \in x^a H_b^\infty(\mathbb{B}^3; T\overline{\mathbb{R}^3})$ *to*

$$\operatorname{div}_g \circ L_g(\sigma) = \tau$$

is in $A_{\mathcal{I}}^\infty(\mathbb{B}^3; T\overline{\mathbb{R}^3})$, *for an index set* $\mathcal{I}$ *which depends on* $\mathcal{G}$ *and* $\mathcal{E}$.

Theorem 1.5 (Scalar-Flat Equation) *Suppose* $f \in x^{a+2} H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\mathcal{F}}^\infty(\mathbb{B}^3)$ *and* $f \geq 0$. *If* $h \in x^{a+2} H_b^\infty(\mathbb{B}^3) \cap A_{\mathcal{E}}^\infty(\mathbb{B}^3)$, *then the unique solution* $u \in x^a H_b^\infty(\mathbb{B}^3)$ *to*

$$(\Delta_g - f)u = h$$

is in $A_{\mathcal{I}}^\infty(\mathbb{B}^3)$ *for an index set* $\mathcal{I}$ *which depends on* $\mathcal{G}$, $\mathcal{F}$, *and,* $\mathcal{E}$.

Theorem 1.6 (Lichnerowicz Equation) *Suppose* $M \in x^{a+2} H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\mathcal{M}}^\infty(\mathbb{B}^3)$ *and* $M \geq 0$. *Let,* ψ *be the unique solution to*

$$\Delta_g \psi + M\psi^{-7} = 0$$

such that $\psi - 1 \in x^a H_b^\infty(\mathbb{B}^3)$. *Then* $\psi - 1 \in \mathcal{A}_{\mathcal{I}}^\infty(\mathbb{B}^3)$, *for some index set* $\mathcal{I}$, *which depends on* $\mathcal{G}$ *and* $\mathcal{M}$.

The rest of this paper is devoted to the proof of the main theorem and the above three theorems. As alluded to before, my advisor's insight that what is happening at infinity can be analyzed right "*in front of our eyes*" by radially compactifying $\mathbb{R}^3$ to $\mathbb{B}^3$ and thus putting our problem in $b-$setting, laid the foundation of this work. In **Chapter 2**, we, therefore, review some aspects of $b-$calculus as they pertain to our problem and restate the main theorem (**Theorem 1.3**) in the $b-$language. Once this is done, we can use analytical tools of $b-$calculus to prove the main theorem and the theorems mentioned above. For instance, we will see in **Chapter 3**, how we can understand the asymptotic behaviors of solutions to the flat Laplacian Δ_o on $\mathbb{R}^3$ by viewing it as an elliptic $b-$differential operator acting on $b-$Sobolev spaces as follows :

$$\Delta_o : x^a H_b^{s+2}(\mathbb{B}^3) \to x^{a+2} H_b^s(\mathbb{B}^3)$$

We will first discuss the mapping properties of the above and then show how to use its indicial operator of Δ_o to conclude that if $h \in x^{a+2} H_b^s(\mathbb{B}^3)$ of

$$\Delta_o u = h$$

has a complete asymptotic expansion, then the solution $u \in x^a H_b^{s+2}(\mathbb{B}^3)$ does also. We will also precisely give the index set for u based on the index set of h.

The result of **Chapter 3** is important not only in its own right, but also for the fact that it can be used as a model to prove the above three theorems. We first tackle scalar-flat equation in **Chapter 4**, since it is closest to the model case. We will see that we can write

$$(\Delta_g - f)u = (\Delta_o + Q)u = h$$

where Q is an operator coming from the non-flat*ness* of the metric. We will see that near $\partial\mathbb{B}^3$, Δ_o dominates Q, since the metric is asymptotically flat, after all, and we can use what we know from Δ_o to understand the asymptotic behaviors of u given the asymptotic behaviors of h.

Next, we tackle the Lichnerowicz equation

$$\Delta_g \psi + M\psi^{-7} = 0$$

in **Chapter 5**. Though this is a non-linear equation, we will see that the non-linearity is rather mild (only in ψ^{-7} term) and moreover, becomes milder and milder near ∂B^3 so that almost all the arguments of the linear case of the previous chapter can be used to establish that the solution ψ has a complete asymptotic expansions of the solution ψ, given the same for M.

Lastly, we discuss the York's equation

$$\mathrm{div}_g \circ L_g(\sigma) = \tau$$

in **Chapter 6**. We will see that we can write

$$\mathrm{div}_g \circ L_g = Y_o + Q$$

where $Y_o \in \mathrm{Diff}_b^2(\mathbb{B}^3; T\overline{\mathbb{R}^3}; T^*\overline{\mathbb{R}^3})$ and moreover, $x^{-2}Y_o$ is $b-$elliptic. These facts will enable us to imitate the results of **Chapter 4** to establish that σ has a complete asymptotic expansions, given the same for τ.

Then finally, in **Chapter 7**, we put all these together to get an initial maximal pair $(\tilde{g}, \tilde{k})$ with a complete asymptotic expansion at infinity, thereby finally proving the main theorem.

As for **Question 2**, I have obtained a partial, but what I believe to be *pivotal*, result in answering **Question 2** affirmatively. In **Chapter 8**, we state this result and discuss the future directions in establishing the veracity of **Question 2**.

Chapter 2

The Setting

The b−calculus was developed by Melrose to study analytic problems on manifolds with boundary. A good introduction can be found in ([Melrose 2]). In this section, we mainly show how to put our problem in this setting and make a few relevant definitions.

2.1 $\mathbb{B}^3$, the radial compactification of $\mathbb{R}^3$

Since it is the behavior at spatial infinity of the initial data that we are mainly interested in, we help ourselves, psychologically speaking, by bringing the infinity right upto our eyes. Analytically, this means that we radially compactify ([Melrose 1]) $\mathbb{R}^3$ to the upper hemisphere $S^{3,+}$, which is topologically $\mathbb{B}^3$, a manifold with boundary. We will choose $x = \frac{1}{\phi(r)+r}$, to be the boundary defining function, where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the Euclidean distance and ϕ is a smooth cut-off function supported near $0 \in \mathbb{R}^3$. After this compactification, the spatial infinity corresponds to $\partial\mathbb{B}^3 \cong S^2$ and the interior of $\mathbb{B}^3$ corresponds to $\mathbb{R}^3$. And on a neighborhood O near $\partial\mathbb{B}^3$, say $O = \mathbb{B}^3 - (0 \in \mathbb{R}^3)$ for instance, we have the obvious diffeomorphism :

$$O \cong [0, \infty)_x \times S_\theta^2$$

2.2 Function Spaces and Asymptotic Expansions

Now that we talked about the structure of the manifold with boundary $\mathbb{B}^3$, we can define a few function spaces on it. We first define

$$\begin{aligned} C^\infty(\mathbb{B}^3) &= \{\text{smooth functions on } \mathbb{B}^3\} \\ V_b(\mathbb{B}^3) &= \{\text{smooth vector fields on } \mathbb{B}^3 \text{ that are tangent to } \partial\mathbb{B}^3\} \\ \text{Diff}_b^k(\mathbb{B}^3) &= \text{span}_{0\le s\le k}(V_b(\mathbb{B}^3))^s;\ V_b^0(\mathbb{B}^3) = C^\infty(\mathbb{B}^3) \end{aligned}$$

Now, we introduce a $b-$density, $d\mu$, on $\mathbb{B}^3$. In fact, we can easily write one down explicitly, such as :

$$d\mu = (\frac{1}{\phi(r) + r^3})\ r^2 dr d\theta$$

where $\phi(r)$ is again the cutoff function introduced in the previous section. With respect to this density, we can then define

$$\begin{aligned} L_b^2(\mathbb{B}^3, d\mu) &= \{f \in L^2_{loc}(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} |f|^2 d\mu < \infty\} \\ H_b^s(\mathbb{B}^3, d\mu) &= \{f \in H^s_{loc}(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} |P(f)|^2 d\mu < \infty;\ P \in \mathrm{Diff}_b^s(\mathbb{B}^3)\} \\ x^a H_b^s(\mathbb{B}^3, d\mu) &= \{f \in H^s_{loc}(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} |P(x^{-a} f)|^2 d\mu < \infty;\ P \in \mathrm{Diff}_b^s(\mathbb{B}^3)\} \end{aligned}$$

It's always assuring to know that these newly-defined spaces $H_b^s(\mathbb{B}^3; d\mu)$ satisfy some properties of the existing spaces $H^s(\mathbb{R}^3)$. We mention two that are of particular importance to us,

Lemma 2.1 (b-Sobolev Imbedding) *For $a \geq 0$ and for $0 \leq k < s - 3/2$, we have*

$$x^a H_b^s(\mathbb{B}^3, d\mu) \subset C^k(\mathbb{R}^3)$$

Proof : For any $f \in x^a H_b^s(\mathbb{B}^3, d\mu)$, we decompose it using the cutoff function ϕ as follows :

$$f = \phi \cdot f + (1 - \phi) \cdot f = f_+ + f_-$$

Now

$$f_+ \in H_c^s(\mathbb{R}^3) \subset C_c^k(\mathbb{R}^3)$$

by the regular Sobolev Imbedding theorem. As for f_-, we change coordinates as follows :

$$(0,1) \times S^2 \ni (x, \theta) \mapsto (\log x, \theta) = (t, \theta) \in (-\infty, 0)_t \times S^2$$

and under this coordinate-change, we can see that

$$f_-(t,\theta) \in H^s(\mathbb{R} \times S^2);\ f_- \equiv 0 \text{ for t large}$$

Again by the regular Sobolev Imbedding theorem, we have

$$H^s(\mathbb{R} \times S^2) \subset C^k(\mathbb{R} \times S^2)$$

and thus

$$f_-(t,\theta) \in C^k(\mathbb{R} \times S^2);\ f_- \equiv 0 \text{ for t large}$$

So, we can see for both f_+ and f_- that

$$f_+ \text{ and } f_- \in C_b^k$$

where we define

$$C_b^k(\mathbb{B}^3) = \{f \in C(\mathbb{B}^3) \mid P(f) \in C(\mathbb{B}^3);\ P \in \mathrm{Diff}_b^k(\mathbb{B}^3)\}$$

Therefore, we conclude that

$$x^a H_b^s(\mathbb{B}^3, d\mu) \subset C_b^k(\mathbb{B}^3)$$

Since we can easily show that $C_b^k(\mathbb{B}^3) \subset C^k(\mathbb{R}^3)$, we finally have

$$x^a H_b^s(\mathbb{B}^3, d\mu) \subset C^k(\mathbb{R}^3)$$

□

An immediate consequence of this theorem that we will use often in this chapter is

Corollary 2.2 *For* $s \geq 2$

$$x^a H_b^s(\mathbb{B}^3) \times x^c H_b^{s-2}(\mathbb{B}^3) \subset x^{a+c} H_b^{s-2}(\mathbb{B}^3)$$

Proof : For $f \in x^a H_b^s(\mathbb{B}^3)$ and $g \in x^a H_b^{s-2}(\mathbb{B}^3)$, we can apply the above lemma, repeatedly, to show that

$$P(fg) \in x^{a+c} L_b^2(\mathbb{B}^3);\ P \in \mathrm{Diff}_b^{s-2}(\mathbb{B}^3)$$

□

A slightly non-trivial extension of this corollary that we will also use often is

Lemma 2.3 (b-Multiplication Theorem) *For* $s, s' > 3/2$, *we have*

$$x^a H_b^s(\mathbb{B}^3) \times x^c H_b^{s'}(\mathbb{B}^3) \subset x^{a+c} H_b^{s_o}(\mathbb{B}^3)$$

where $s_o \leq s, s'$ *and* $s_o < s + s' - 3/2$

Proof : For $f \in x^a H_b^s(\mathbb{B}^3)$ and $g \in x^a H_b^{s-2}(\mathbb{B}^3)$, we decompose

$$fg = \phi \cdot fg + (1 - \phi) \cdot fg = (fg)_+ + (fg)_-$$

and apply the standard Gagliano-Nirenberg-type multiplication theorem of $H^s(\mathbb{R}^n)$ (see [Melrose-Ritter] for example) to each piece. □

Now in order to define what it means for a function to have a complete asymptotic expansions at $\partial\mathbb{B}^3 = S^2$, which corresponds to spatial infinity, we need to define an index $\mathcal{E}$, a set of powers that appear in an expansion, to be a discrete subset of $R^+ \times (\mathbb{N} \cup \{0\})$ with the following three conditions.

$$\begin{array}{lll} 1)\ (s,k) \in \mathcal{E} & \rightarrow & (s+l,k) \in \mathcal{E},\ \forall l \in \mathbb{N} \\ 2)\ (s,k) \in \mathcal{E} & \rightarrow & (s,l) \in \mathcal{E},\ 0 \leq l \leq k \\ 3)\ (s,k) \rightarrow \infty & \rightarrow & s \rightarrow \infty \end{array}$$

Then we define $\mathcal{A}_{\mathcal{E}}^{\infty}(\mathbb{B}^3)$ as following :

$$\mathcal{A}_{\mathcal{E}}^{\infty}(\mathbb{B}^3) := \{f \in C^{\infty}(\mathbb{R}^3) \quad | \quad (\forall r \in \mathbb{R}^+) f(x,\theta) = \sum_{(s,k)\in\mathcal{E}}^{s<r} c_{s,k}(\theta) x^s (log x)^k + R_r \ ; \\ R_r \in x^r H_b^{\infty}(\mathbb{B}^3)\}$$

The first condition makes $\mathcal{A}_{\mathcal{E}}^{\infty}$ a $C^{\infty}(\mathbb{B}^3)-$ module. The first and the second condition assures that $\mathcal{A}_{\mathcal{E}}^{\infty}$ is invariant under different choices of coordinates (x,θ). Finally, the last condition on $\mathcal{E}$ ensures that the summation is always a finite sum, and the more terms you have in the expansion, the smaller the remainder term becomes.

We remark than $\mathcal{A}_{E}^{\infty}(\mathbb{B}^3)$ can be given a complete locally convex topology by constructing seminorms from $||c_{s,k}(\theta)||_{C^{\infty}(S^2)}$ for $(s,k) \in \mathcal{E}$ and $||R_{r_n}||_{x^{r_n}H_b^{\infty}(\mathbb{B}^3)}$ for some sequence $\{r_n\} \subset \mathbb{R}^+$ such that $r_{\infty} = \infty$. It is straight-forward to see that such a topology on $\mathcal{A}_{E}^{\infty}(\mathbb{B}^3)$ is independent of the choice of the sequence $\{r_n\}$.

Now let $\mathcal{E}$ and $\mathcal{F}$ be index sets as defined above. We then prove :

Lemma 2.4

$$\begin{aligned} \mathcal{A}_{\mathcal{E}}^{\infty}(\mathbb{B}^3) + \mathcal{A}_{\mathcal{F}}^{\infty}(\mathbb{B}^3) &\subset \mathcal{A}_{\mathcal{E}\cup\mathcal{F}}^{\infty}(\mathbb{B}^3) \\ \mathcal{A}_{\mathcal{E}}^{\infty}(\mathbb{B}^3) \cdot \mathcal{A}_{\mathcal{F}}^{\infty}(\mathbb{B}^3) &\subset \mathcal{A}_{\mathcal{E}+\mathcal{F}}^{\infty}(\mathbb{B}^3) \end{aligned}$$

Proof : The lemma follows straight-forwardly from **Lemma 2.1** and **Lemma 2.3**. □

2.3 Extensions of Tensor Bundles

We first consider the simplest case, i.e. $T\mathbb{R}^3$. Let us choose on $\mathbb{R}^3$ a flat coordinates system $\{x^1, x^2, x^3\}$ (i.e. for which $g_{ij} = \delta_{ij}$), and trivialize $T\mathbb{R}^3$ using this coordinate system as following :

$$T\mathbb{R}^3 \overset{\Phi}{=} \mathbb{R}^3_{\{x^1,x^2,x^3\}} \times \mathbb{R}^3_{\{\partial_1,\partial_2,\partial_3\}}$$

Under this trivialization Φ, smooth vector fields correspond to smooth maps into a fixed vector space $\mathbb{R}^3$ with a fixed basis $\{\partial_1, \partial_2, \partial_3\}$ as :

$$\begin{array}{rcll} v = v_1(\mathbf{x})\partial_1 + v_2(\mathbf{x})\partial_2 + v_3(\mathbf{x})\partial_3 & \in & C^{\infty}(\mathbb{R}^3; T\mathbb{R}^3) & \Longleftrightarrow \\ (v_1, v_2, v_3) & \in & C^{\infty}(\mathbb{R}^3; \mathbb{R}^3) & \Longleftrightarrow \\ v_i & \in & C^{\infty}(\mathbb{R}^3; \mathbb{R}), \ 1 \le i \le 3 & \end{array}$$

L^2 and H^s sections of $T\mathbb{R}^3$ can be defined in a similar component-wise manner.

Now, in order to discuss the behaviors at infinity of the vector fields on $\mathbb{R}^3$, we extend $T\mathbb{R}^3$ to a bundle over $\mathbb{B}^3$. One way to do this is simply to radially compactify the base manifold $\mathbb{R}^3$ of $T\mathbb{R}^3$ and leave the fibers intact :

$$T\mathbb{R}^3 \overset{\Phi}{=} \mathbb{R}^3_{\{x^1,x^2,x^3\}} \times \mathbb{R}^3_{\{\partial_1,\partial_2,\partial_3\}} \overset{\Psi}{\hookrightarrow} \mathbb{B}^3 \times_\Phi \mathbb{R}^3_{\{\partial_1,\partial_2,\partial_3\}}$$

where

$$\Psi : \mathbb{R}^3 \times \mathbb{R}^3 \ni (\mathbf{x}, v) \mapsto (SP(\mathbf{x}), v) \in \mathbb{B}^3 \times_\Phi \mathbb{R}^3$$

and $\{\partial_1, \partial_2, \partial_3\}$ is now a basis for all the fibres over all $p \in \mathbb{B}^3$, simultaneously. Here we retain Φ in the notation $B^3 \times_\Phi \mathbb{R}^3$ to remind ourselves of the fact that a choice of trivialization of $T\mathbb{R}^3$, namely Φ, was made in the above definition.

Then, similar to what we did for $T\mathbb{R}^3$, we define C^∞-sections of $\mathbb{B}^3 \times_\Phi \mathbb{R}^3$ as follows :

$$\begin{array}{ccll} v = v_1\partial_1 + v_2\partial_2 + v_3\partial_3 & \in & C^\infty(\mathbb{B}^3; \mathbb{B}^3 \times_\Phi \mathbb{R}^3) & \Longleftrightarrow \\ (v_1, v_2, v_3) & \in & C^\infty(\mathbb{B}^3; \mathbb{R}^3) & \Longleftrightarrow \\ v_i & \in & C^\infty(\mathbb{B}^3; \mathbb{R}),\ 1 \le r \le 3 & \end{array}$$

Moreover, we can define L^2_b-, H^s_b-, and $x^a H^s_b-$ sections of $\mathbb{B}^3 \times_\Phi \mathbb{R}^3$ in a similar component-wise way. Thus in this way, the behaviors at infinity of the sections of $T\mathbb{R}^3$ are eventually reduced to the boundary regularity of its components, i.e. the boundary regularity of functions, which we defined in the previous section.

Similar to what we did for $\mathcal{A}^\infty_\mathcal{E}(\mathbb{B}^3)$, we define $\mathcal{A}^\infty_\mathcal{E}(\mathbb{B}^3; \mathbb{B}^3 \times_\Phi \mathbb{R}^3)$ to be a subset of $C^\infty(\mathbb{R}^3; T\mathbb{R}^3)$ which have a complete expansion, with respect to $\mathcal{E}$, at infinity. In other words, σ is in $\mathcal{A}^\infty_\mathcal{E}(\mathbb{B}^3; \mathbb{B}^3 \times_\Phi \mathbb{R}^3)$ if

$$\sigma \in C^\infty(\mathbb{R}^3; \mathbb{R}^3 \times_\Phi \mathbb{R}^3)$$

and for some neighborhood near $\partial\mathbb{B}^3$, say $U \cong [0,1)_x \times S^2$,

$$\forall r \in \mathbb{R}^+ \Rightarrow \sigma|_U = \sum_{(s,k)\in E}^{s<r} \pi^*(c_{s,k}) x^s (\log x)^k + R_r;\ R_r \in x^r H^s_b(U; (B^3 \times_\Phi \mathbb{R}^3)|_U)$$

where $c_{s,k} \in C^\infty(\partial\mathbb{B}^3; (\mathbb{B}^3 \times_\Phi \mathbb{R}^3)|_{\partial\mathbb{B}^3})$ and

$$(\mathbb{B}^3 \times_\Phi \mathbb{R}^3)|_U \cong \pi^*((\mathbb{B}^3 \times_\Phi \mathbb{R}^3)|_{\partial\mathbb{B}^3})$$

where $\pi : U \ni (x, \theta) \to \theta \in S^2$.

Now it is easy to check that the above, somewhat complicated looking, definition is equivalent to a component-wise definition of $\mathcal{A}^\infty_\mathcal{E}(\mathbb{B}^3; \mathbb{B}^3 \times_\Phi \mathbb{R}^3)$ as follows :

$$\sigma \in \mathcal{A}^\infty_\mathcal{E}(\mathbb{B}^3; \mathbb{B}^3 \times_\Phi \mathbb{R}^3) \Longleftrightarrow \sigma_i \in \mathcal{A}^\infty_\mathcal{E}(\mathbb{B}^3),\ 1 \le i \le 3$$

Since we used the trivialization Φ in defining the spaces above, we need to assert that they are all well-defined under a similar choice. We thus prove :

Lemma 2.5 *Let $\Phi' : T\mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ be another trivialization of $T\mathbb{R}^3$ such that the transition map $t_{ij} \in C^\infty(\mathbb{R}^3, Gl(3))$ is in $C^\infty(\mathbb{B}^3, Gl(3))$. Then, t_{ij} maps following spaces isomorphically.*

$$\begin{array}{rcrcl} t_{ij} & : & C^\infty(\mathbb{B}^3; \mathbb{B}^3 \times_\Phi \mathbb{R}^3) & \to & C^\infty(\mathbb{B}^3; \mathbb{B}^3 \times_{\Phi'} \mathbb{R}^3) \\ t_{ij} & : & x^a H_b^s(\mathbb{B}^3; \mathbb{B}^3 \times_\Phi \mathbb{R}^3) & \to & x^a H_b^s(\mathbb{B}^3; \mathbb{B}^3 \times_{\Phi'} \mathbb{R}^3) \\ t_{ij} & : & \mathcal{A}_\mathcal{E}^\infty(\mathbb{B}^3; \mathbb{B}^3 \times_\Phi \mathbb{R}^3) & \to & \mathcal{A}_\mathcal{E}^\infty(\mathbb{B}^3; \mathbb{B}^3 \times_{\Phi'} \mathbb{R}^3) \end{array}$$

Proof : This is a consequence of the algebraic properties of the function spaces $C^\infty(\mathbb{B}^3)$, $x^a H_b^s(\mathbb{B}^3)$, and $\mathcal{A}_\mathcal{E}^\infty(\mathbb{B}^3)$ □

Because of this lemma, from now on, we will stick with the trivialization Φ. And in fact, we will denote :

$$T\overline{R^3} = \mathbb{B}^3 \times_\Phi \mathbb{R}^3$$

Lastly, we remark that through a similar reasoning, we have

$$T^*\mathbb{R}^3 \overset{\Phi^*}{=} \mathbb{R}^3 \times \mathbb{R}^3_{\{dx^1, dx^2, dx^3\}} \overset{\Psi}{\hookrightarrow} \mathbb{B}^3 \times_{\Phi^*} \mathbb{R}^3, \text{ denoted by } T^*\overline{\mathbb{R}^3}$$

and

$$T^*\mathbb{R}^3 \otimes T^*\mathbb{R}^3 \overset{(\Phi^*)\otimes(\Phi^*)}{=} \mathbb{R}^3 \times (\mathbb{R}^3 \otimes \mathbb{R}^3) \overset{\Psi}{\hookrightarrow} \mathbb{B}^3 \times_{(\Phi^*)\otimes(\Phi^*)} (\mathbb{R}^3 \otimes \mathbb{R}^3), \text{ denoted by } T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}$$

$C^\infty-$, $x^a H_b^s-$, and $\mathcal{A}_\mathcal{E}^\infty-$ sections of $T^*\overline{\mathbb{R}^3}$ and $T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}$ can be defined in a similar way.

2.4 Back to $\mathbb{R}^3$

We now compare the function spaces that we defined on $\mathbb{B}^3$ and those that we defined on $\mathbb{R}^3$. First we prove :

Lemma 2.6

$$C^\infty(\mathbb{R}^3) \supset H_b^\infty(\mathbb{B}^3) \supset \mathcal{A}_\mathcal{E}^\infty(\mathbb{B}^3) \supset x^\epsilon C^\infty(\mathbb{B}^3)$$

for any index set $\mathcal{E}$ and for any positive ϵ such that $(\epsilon, l) \in \mathcal{E}$ for some $l \in \mathbb{N}$.

Proof : The first inclusion follows from $b-$Sobolev imbedding theorem. The second inclusion is straight-forward and the third inclusion follows from the fact that $C^\infty(\mathbb{B}^3) = \mathcal{A}_\mathcal{E}^\infty(\mathbb{B}^3)$ for $\mathcal{E} = \mathbb{N}$. □

Now, we prove the following lemma which translates the notation of ([C-M]) to this $b-$language.

Lemma 2.7 *The following equalities hold where the right hand side was defined in ([C-M]) :*

$$\begin{array}{rcl} H_{s,\delta}(\mathbb{R}^3) & = & x^{\delta+\frac{3}{2}} H_b^s(\mathbb{B}^3, \mu_b) \\ H_{s,\delta}(\mathbb{R}^3; E) & = & x^{\delta+\frac{3}{2}} H_b^s(\mathbb{B}^3; \overline{E}, \mu_b) \end{array}$$

Here, E is any tensor bundle (i.e. $T\mathbb{R}^3$, $T^*\mathbb{R}^3$, $T^*\mathbb{R}^3 \times T^*\mathbb{R}^3$, etc.) of $\mathbb{R}^3$ and $\overline{E}$ is its extension, (i.e. $T\overline{\mathbb{R}^3}$, $T^*\overline{\mathbb{R}^3}$, $T^*\overline{\mathbb{R}^3} \times T^*\overline{\mathbb{R}^3}$, etc)

Proof : We will just prove the first statement, for the second one follows easily from the first.

Recall the following definition from ([C-M])

$$H_{s,\delta} = \{u \in L^2_{loc}(\mathbb{R}^3) \mid (\sqrt{(1+|x|^2)})^{\delta+|\alpha|} D^\alpha u \in L^2(\mathbb{R}^3) \text{ for } |\alpha| \leq s\}$$

For $f \in H_{s,\delta}$, we write

$$f = f_+ + f_- = \phi f + (1-\phi) f$$

It is easy to check, for any δ and a

$$\phi \cdot H_{s,\delta}(\mathbb{R}^3) = \phi \cdot x^a H^s_b(\mathbb{B}^3)$$

because δ and a matter only at $infty$ (or $\partial\mathbb{B}^3$). we just need to show that

$$(1-\phi) \cdot H_{s,\delta}(\mathbb{R}^3) = (1-\phi) \cdot x^a H^s_b(\mathbb{B}^3)$$

Suppose $s = 0$ for now. Then we have

$$\begin{aligned}
& f_- \in H_{0,\delta}(\mathbb{R}^3);\ f \equiv 0 \ \text{ near } \ 0 \in \mathbb{R}^3. \\
\Leftrightarrow\ & \int_{\mathbb{R}^3} (\sqrt{(1+|x|^2)})^{2\delta} |f_-|^2 r^2 dr d\theta < \infty \\
\Leftrightarrow\ & \int_{\mathbb{R}^3} r^{2\delta} |f_-|^2 r^3 \frac{dr}{r} d\theta < \infty \\
\Leftrightarrow\ & \int_{\mathbb{R}^3} |r^{\delta+\frac{3}{2}} f_-|^2 \frac{dr}{r} d\theta < \infty \\
\Leftrightarrow\ & \int_{\mathbb{R}^3} |x^{-(\delta+\frac{3}{2})} f_-|^2 \mu_b < \infty;\ \frac{dr}{r} = \frac{dx}{x} = \mu_b \ \text{ near } \ \partial\mathbb{B}^3
\end{aligned}$$

For $s = 1$, we furthermore have

$$\begin{aligned}
& \int_{\mathbb{R}^3} (\sqrt{(1+|x|^2)})^{2+2\delta} |Df_-|^2 r^2 dr d\theta < \infty \\
\Leftrightarrow\ & \int_{\mathbb{R}^3} |r^{\delta+\frac{3}{2}} r\partial_i f_-|^2 \frac{dr}{r} d\theta < \infty, \ \text{ for } \ 1 \leq i \leq 3 \\
\Leftrightarrow\ & \int_{\mathbb{R}^3} |x^{-(\delta+\frac{3}{2})} P(f_-)|^2 \mu_b < \infty \ \text{ for } \ P \in \mathrm{Diff}^1_b(\mathbb{B}^3)
\end{aligned}$$

because $r\partial_i \in r \cdot x\mathrm{Diff}^1_b(\mathbb{B}^3) = \mathrm{Diff}^1_b(\mathbb{B}^3)$ for functions that are supported away from $0 \in \mathbb{R}^3$.

Now, the similar arguments work for all $s > 1$ and we conclude

$$(1-\phi) \cdot H_{s,\delta}(\mathbb{R}^3) = (1-\phi) \cdot x^a H^s_b(\mathbb{B}^3)$$

□

2.5 Restatement of Theorem 3.3

Now we are finally ready to give a precise version of our theorem, in $b-$category.

Theorem 2.8 *Let $0 < a < 1$. Suppose we have a pair (g,k), where g is a Riemannian metric on $\mathbb{R}^3$ such that $g - e \in x^a H_b^\infty(\mathbb{B}^3;\, T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \cap \mathcal{A}_{\mathcal{G}}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$ for an index set $\mathcal{G}$ and $R(g) \geq 0$; and k is a symmetric 2-tensor on $\mathbb{R}^3$ such that $k \in x^{a+1} H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \cap \mathcal{A}_{\mathcal{K}}^\infty(\mathbb{B}^3;\, T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$, for an index set $\mathcal{K}$.*

Then there exists a new pair $(\tilde{g},\tilde{k})$ that depends on (g,k) , where $\tilde{g}$ is in the conformal class of g and $\tilde{g} - e \in x^a H_b^\infty(\mathbb{B}^3;\, T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \cap \mathcal{A}_{\mathcal{I}_{\tilde{\mathcal{G}}}}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$; and $\tilde{k} \in x^{a+1} H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \cap \mathcal{A}_{\mathcal{I}_{\tilde{\mathcal{K}}}}^\infty(\mathbb{B}^3;\, T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$, for index sets $\mathcal{I}_{\tilde{\mathcal{G}}}$ and $\mathcal{I}_{\tilde{\mathcal{K}}}$ that depend on $\mathcal{G}$ and $\mathcal{K}$, such that $(\tilde{g},\tilde{k})$ satisfies the contraint equations.

Chapter 3

The Asymptotic Behaviors of Solutions to the Flat Laplacian on $\mathbb{R}^3$

In order to understand the asymptotic behaviors of the solutions of Δ_o, we revisit the known result of McOwen ([McOwen]) to first understand the mapping properties of Δ_o. McOwen proved, phrased here in the $b-$language :

Theorem 3.1 *The flat Laplacian*

$$\Delta_o : x^a H_b^s(\mathbb{B}^3) \rightarrow x^{a+2} H_b^{s-2}(\mathbb{B}^3)$$

is Fredholm for $a \notin \mathbb{Z}$. And for $a \notin \mathbb{Z}$, it is an injection if $a > 0$, a surjection if $a < 1$, and thus an isomorphism if $0 < a < 1$.

Since the theorem is phrased in the $b-$language, we will try to prove this theorem using the b-analysis which will help us in analyzing the asymptotic behaviors of solutions to $\Delta_o u = f$. With this in mind, we define a cutoff function ϕ to be

$$\begin{aligned} \phi &\in C_c^\infty(\mathbb{R}^3) \\ \phi &\equiv 1 \text{ near } 0 \in \mathbb{R}^3 \end{aligned}$$

and we actually prove the following equivalent statement instead :

Theorem 3.2

$$(\phi + r^2)\Delta_o : x^a H_b^s(\mathbb{B}^3) \rightarrow x^a H_b^{s-2}(\mathbb{B}^3)$$

is Fredholm for $a \notin \mathbb{Z}$. And for $a \notin \mathbb{Z}$, it is an injection if $a > 0$, a surjection if $a < 1$, and thus an isomorphism if $0 < a < 1$.

Recall that r here is again a Euclidean distance on $\mathbb{R}^3$, which again corresponds, via compactification, to the interior of $\mathbb{B}^3$

We will prove **Theorem 3.2** by constructing a parametrix to $(\phi+r^2)\Delta_o$. As we will see, the Fredholmness of $(\phi+r^2)\Delta_o$ depends much on mapping properties of $r^2\Delta_o$. This is not so surprising considering the fact that $(\phi+r^2)\Delta_o$ and $r^2\Delta$ only differ in supp$(\phi) \in \mathbb{R}^3$, and even in that case, only by a positive scalar except at the point $0 \in \mathbb{R}^3$. However, $r^2\Delta_o$ has the advantage of being its own indicial operator on $(0,\infty) \times S^2$. Thus, we first investigate $r^2\Delta_o$ in detail in **Section 3.1**, **Section 3.2**, and **Section 3.3**. Then in **Section 3.4**, we will show how to construct a parametrix to $(\phi+r^2)\Delta_o$, using $r^2\Delta_o$. Once we have a parametrix for $(\phi+r^2)\Delta_o$, we can easily deduce the mapping property of $(\phi+r^2)\Delta_o$, thereby proving **Theorem 3.2**. This is done in **Section 3.5**.

Once we establish the mapping properties of $(\phi+r^2)\Delta_o$, we use them to prove the main theorem of this chapter in **Section 3.6** as follows :

Theorem 3.3 *Suppose* $f \in x^{a+2}H_b^\infty(\mathbb{B}^3) \cap A_{\mathcal{E}}^\infty(\mathbb{B}^3)$ *for* $0 < a < 1$. *Then the unique solution* $u \in x^a H_b^\infty(\mathbb{B}^3)$ *to*

$$\Delta_o u = f$$

is in $A_{\mathcal{I}_{\mathcal{E}}}^\infty(\mathbb{B}^3)$ *for an index set* $\mathcal{I}_{\mathcal{E}}$ *where* $\mathcal{I}_{\mathcal{E}} = (\mathcal{E}-2)\overline{\cup}\mathbb{N}$.

In other words, if f has a complete asymptotic expansion, so does u.

3.1 $r^2\Delta_o$ as Operator On $[0,1] \times S^2$

We realize the following two facts about $r^2\Delta_o$:

1. $r^2\Delta_o$ is *not* elliptic at $0 \in \mathbb{B}^3$ because of a rather spurious degeneracy at $0 \in \mathbb{R}^3$ introduced by multiplying Δ_o by r^2.
2. $r^2\Delta_o$ is invariant under $(\mathbb{R}^+, \times)$-action on $\mathbb{R}^3$

Motivated by these two facts, we introduce $[0,1]\times S^2$, a manifold with boundary closely related to $\mathbb{B}^3$. $[0,1]\times S^2$ comes about first by blowing up the origin in $\mathbb{R}^3$ by introducing the polar coordinates near the origin. In other words, we have :

$$\mathbb{R}^3 - \{0\} \cong (0,\infty)_r \times S^2_\theta$$

by sending

$$\mathbf{x} \to (|\mathbf{x}|, \mathbf{x}/|\mathbf{x}|)$$

We then compacitfy $(0,\infty)\times S^2$ to $[0,1]\times S^2$ as following :

$$(0,\infty)\times S^2 \ni (r,\theta) \overset{\Psi}{\hookrightarrow} (r/(r+1),\theta) \in [0,1]\times S^2$$

Thus $(0,\infty)\times S^2$, under Ψ, corresponds to the interior of $[0,1]\times S^2$ and $[0,1]\times S^2$ has 2 boundary components :

$$\partial([0,1]\times S^2) = (\{0\}\times S^2) \amalg (\{1\}\times S^2)$$

Now let us write $r^2\Delta_o$ using the polar coordinates on $(0,\infty)\times S^2$,(r,θ) :

$$r^2\Delta_o = (r\partial_r)^2 + (r\partial_r)^2 + \Delta_{S^2}$$

Furthermore, near $r=\infty$, we can let $y = 1/r$ and then we have

$$\begin{aligned} r\partial_r &= -y\partial_y \in \mathrm{Diff}_b^2([0,1]\times S^2) \\ r^2\Delta_o &= (y\partial_y)^2 - (y\partial_y)^2 + \Delta_{S^2} \end{aligned}$$

Thus, it is clear that indeed :

$$r^2\Delta_o \in \mathrm{Diff}_b^2([0,1]\times S^2)$$

3.2 $b-$Sobolev Spaces on $[0,1]\times S^2$

Now in order to define b-Sobolev spaces on $[0,1]\times S^2$, we need to introduce a $b-$density, τ_b. There is an easy choice for τ_b, namely :

$$\tau_b = |\frac{dr}{r}d\theta|$$

Near the boundary component $r = 0$, we can see easily see that this is a $b-$density. On the other end, where $r=\infty$, i.e. $y = 1/r = 0$, we note that

$$|\frac{dr}{r}d\theta| = |\frac{dy}{y}d\theta|$$

which shows that τ_b behaves as a $b-$metric should near the other end $(r=\infty)$, as well as $(r=0)$.

Now using τ_b and again the fact that $(0,\infty)\times S^2$ corresponds to the interior of $[0,1]\times S^2$ under the compactification Ψ, we define :

$$r^cH_b^m([0,1]\times S^2,\tau_b) = \{f \in H_{loc}^m((0,\infty)\times S^2) \quad | \quad \int_0^\infty\int_{S^2} |P(r^{-c}f)|^2\tau_b < \infty; \\ P\in \mathrm{Diff}_b^m([0,1]\times S^2)\}$$

Note here that the weight "r" blows up on one end and vanishes on the other, depending on the sign of c.

Now, using the fact that

$$\mathrm{span}_{C^\infty(\mathbb{B}^3)}\{r\partial_r, \partial_\theta\} = V_b([0,1]\times S^2)$$

we conclude that

$$r^cH_b^m([0,1]\times S^2,\tau_b) = \{f = r^c\cdot\tilde{f} \mid \int_0^\infty\int_{S^2} |\sum_{0\le k+l\le m} (r\partial_r)^k(\partial_\theta)^l\tilde{f}(r,\theta)|^2 d\theta\frac{dr}{r} < \infty\}$$

Now let us introduce a variable t such that $r = e^t$. We then have

$$(0,\infty)_r \times S_\theta^2 \ni (r,\theta) \leftrightarrow (\log r,\theta) \in (-\infty,\infty)_t\times S_\theta^2$$

and

$$f \in r^c H_b^m(\mathbb{B}^3) \quad \Leftrightarrow \quad f = r^c \cdot \tilde{f}(r,\theta) = e^{ct}\tilde{f}(e^t,\theta) = e^{ct} f^\flat(t,\theta)$$
$$\int_0^\infty \int_{S^2} | \sum_{0\le k+l\le m} (\partial_t)^k (\partial_\theta)^l f^\flat(t,\theta)|^2 d\theta\, dt < \infty$$

In other words, $f^\flat \in H^m(\mathbb{R}\times S^2)$ and thus we concluded that under the coordinate change $r = e^t$:

$$r^c H_b^m([0,1]\times S^2) \leftrightarrow e^{ct} H^m(\mathbb{R}\times S^2)$$

Lastly, we describe how $e^{ct}H^m(\mathbb{R}\times S^2)$ behaves under taking the Fourier Transform on $t-$variable. (the reason for doing this will become clear in the forthcoming sections.) We first prove :

Lemma 3.4 *The Fourier transform gives the following isomorphism :*

$$h(t,\theta) \in H^m(\mathbb{R}\times S^2) \quad \longleftrightarrow \left\{ \begin{array}{ccc} \hat{h}(s,\theta) & \in & L^2(\mathbb{R}; H^m(S^2)) \\ (1+|s|^m)\hat{h}(s,\theta) & \in & L^2(\mathbb{R}; L^2(S^2)) \end{array} \right.$$

Proof : Suppose $h \in L^2(\mathbb{R}\times S^2)$. For $t \in (\mathbb{R}-\Sigma)$, where Σ is a set of measure 0, the following definition :

$$h_t(x) := h(t,x) \in L^2_{loc}(S^2)$$

makes sense. Moreover, since we have, by Fubini's theorem,

$$\int_{\mathbb{R}} (\int_{S^2} |h(t,\theta)|^2 d\theta)\, dt = \int\int_{\mathbb{R}\times S^2} |h(t,\theta)|^2 dt d\theta < \infty$$

we conclude that for $t \in \mathbb{R}-\Sigma-\Sigma'$ for possibly another set of measure 0 , Σ', we have

$$h_t \in L^2(S^2)$$

and

$$\int_{\mathbb{R}} ||h_t||^2_{L^2(S^2)}\, dt < \infty \Longleftrightarrow h \in L^2(\mathbb{R}; L^2(S^2))$$

Now if we take the Fourier transform in t-variable to get

$$\hat{h}(s,\theta) = \int_{\mathbb{R}} e^{-ist} h(t,\theta) dt$$

and we have

$$\begin{aligned} \int_{\mathbb{R}} ||\hat{h}_s||^2_{L^2(S^2)}\, ds &= \int_{\mathbb{R}}\int_{S^2} |\hat{h}(s,\theta)|^2 d\theta ds \\ &= \int_{S^2}\int_{\mathbb{R}} |\hat{h}(s,\theta)|^2 ds d\theta \quad \text{(by Fubini)} \end{aligned}$$

$$\begin{aligned} &= \int_{S^2}\int_{\mathbb{R}} |h(t,\theta)|^2 dt d\theta \quad \text{(by Plancheral)} \\ &= \int_{\mathbb{R}}\int_{S^2} |h(t,\theta)|^2 d\theta dt \quad \text{(by Fubini)} \\ &= \int_{\mathbb{R}} \|h_t\|^2_{L^2(S^2)} dt \end{aligned}$$

Thus

$$L^2(\mathbb{R}\times S^2) \cong L^2(\mathbb{R}_t; L^2(S^2)) \overset{\mathcal{F}}{\longleftrightarrow} L^2(\mathbb{R}_s; L^2(S^2)) \tag{3.1}$$

Now if $h \in H^m(\mathbb{R}\times S^2)$, then we have $(\partial_t)^k(\partial_\theta)^l h(t,\theta) \in L^2(\mathbb{R}\times S^2)$ for $0 \le k+l \le m$. In fact, by the well-known interpolation argument (which can be easily proved by using the Fourier transform), we have

$$\begin{aligned} & (\partial_t)^k(\partial_\theta)^l h(t,\theta) \in L^2(\mathbb{R}\times S^2),\ 0 \le k+l \le m \\ \Longleftrightarrow & \begin{cases} (\partial_\theta)^l h & \in \ L^2(\mathbb{R}\times S^2), \quad 0 \le l \le m \\ (\partial_t)^m h & \in \ L^2(\mathbb{R}\times S^2)), \end{cases} \\ \Longleftrightarrow & \begin{cases} h & \in \ L^2(\mathbb{R}; H^m(S^2)) \\ (\partial_t)^m h & \in \ L^2(\mathbb{R}; L^2(S^2)) \end{cases} \end{aligned}$$

By (3.1), we therefore see that

$$\begin{aligned} & \widehat{(\partial_t)^k(\partial_\theta)^l h} \in L^2(\mathbb{R}; L^2(S^2)),\ 0 \le k+l \le m \\ \Longleftrightarrow & \begin{cases} (\partial_\theta)^l \hat{h} & \in \ L^2(\mathbb{R}; L^2(S^2)), \quad 0 \le l \le m \\ (1+|s|^m)\hat{h} & \in \ L^2(\mathbb{R}; L^2(S^2)) \end{cases} \\ \Longleftrightarrow & \begin{cases} \hat{h} & \in \ L^2(\mathbb{R}; H^m(S^2)) \\ (1+|s|^m)\hat{h} & \in \ L^2(\mathbb{R}; L^2(S^2)) \end{cases} \end{aligned}$$

□

Thus we can finally define

$$\begin{aligned} & f \in r^c H_b^m([0,1]\times S^2) \\ \Longleftrightarrow & \begin{cases} \text{on} \quad int([0,1]\times S^2) & \cong \ (0,\infty)_r \times S^2 \ \cong \ (-\infty,\infty)_r \times S^2 \\ \qquad f & = \ r^c \tilde{f}(r,\theta) \ = \ e^{ct} f^\flat(t,\theta) \end{cases} \end{aligned}$$

where

$$\begin{aligned} \widehat{f^\flat}(s,\theta) & \in \ L^2(\mathbb{R}; H^m(S^2)) \\ (1+|s|^m)\widehat{f^\flat}(s,\theta) & \in \ L^2(\mathbb{R}; L^2(S^2)) \end{aligned}$$

From now on, we will denote $H_b^s([0,1]\times S^2, \tau_b)$ simply by $H_b^s([0,1]\times S^2)$, assuming that we are using the measure $\tau_b = |\frac{dr}{r} d\theta|$ on $[0,1]\times S^2$. We are now ready to state the main theorem for $r^2\Delta_o$.

3.3 Main theorem for $r^2\Delta_o$

We prove :

Theorem 3.5 *$(r^2\Delta_o) : r^cH_b^{s+2}([0,1]\times S^2) \to r^cH_b^s([0,1]\times S^2)$ is an isomorphism iff $c \notin \mathbb{Z}$.*

Proof : We will prove this by showing that the inverse

$$(r^2\Delta_o)^{-1} : r^cH_b^s([0,1]\times S^2) \to r^cH_b^{s+2}([0,1]\times S^2)$$

exists for $c \notin \mathbb{Z}$. We will consider the case when $s = 0$ (all the remaining cases are proven in a similar way). Thus, we start with

$$u \in r^cH_b^2([0,1]\times S^2)$$

and

$$f = (r^2\Delta_o)u \in r^cL_b^2([0,1]\times S^2)$$

Writing $r^2\Delta_o$, u, and f in (r,θ) coordinates on $(0,\infty)\times S^2$, we have :

$$((r\partial_r)^2 + (r\partial_r) + \Delta_\theta)u(r,\theta) = f(r,\theta)$$

The above equation becomes in (t,θ) :

$$((\partial_t)^2 + (\partial_t) + \Delta_\theta)u(t,\theta) = f(t,\theta)$$

Using the definition of $H_b^s([0,1]\times S^2)$ that we gave in the preceding section, we can write :

$$f = r^c\tilde{f}(r,\theta) = e^{ct}f^\flat(t,\theta);\ \ f^\flat \in L^2(\mathbb{R}\times S^2)$$

and

$$u = r^c\tilde{u}(r,\theta) = e^{ct}u^\flat(t,\theta);\ \ u^\flat \in H^2(\mathbb{R}\times S^2)$$

Then the above equation becomes :

$$(e^{-ct}\{(\partial_t)^2 + (\partial_t) + \Delta_\theta\}e^{ct})u^\flat(t,\theta) = f^\flat(t,\theta)$$

Therefore,

$$(r^2\Delta_o) : r^cH_b^{s+2}([0,1]\times S^2) \to r^cH_b^s([0,1]\times S^2)$$

can be equivalently written as

$$(e^{-ct}\{(\partial_t)^2 + (\partial_t) + \Delta_\theta\}e^{ct}) : H^{s+2}(\mathbb{R}\times S^2) \to H^s(\mathbb{R}\times S^2)$$

The advantage of writing as above is that the domain and the range are now fixed and dependency of the parameter "c" lies solely on the operator $(e^{-ct}\{(\partial_t)^2 + (\partial_t)^2 + \Delta_\theta\}e^{ct})$, where $r^2\Delta_o$ is conjugated by e^{ct}. Carrying out the conjugation we get :

$$((\partial_t + c)^2 + (\partial_t + c) + \Delta_\theta))u^\flat(t,\theta) = f^\flat(t,\theta)$$

Now we can take Fourier transform of both sides in $t-$variable to get :

$$((is + c)^2 + (is + c) + \Delta_\theta)\widehat{u^\flat}(s,\theta) = \widehat{f^\flat}(s,\theta)$$

We now state following well-known result from the Analytic Fredholm theory.

Lemma 3.6 *1.* $((is)^2 + (is) + \Delta_\theta)^{-1} = A(s) + \sum_{k\in Z} B_k/(s - ik)$, *where* $A(s)$ *is holomorphic with values in* $\Psi^{-2}(S^2)$, *thus holomorphic in* $\Psi^0(S^2)$ *as well, and* B_k *is a projection operator to the eigenspace of* Δ_{S^2} *with the eigenvalue* $k(k+1)$.

2. $||((is)^2 + (is) + \Delta_\theta)^{-1}||_{L^2(S^2)\to L^2(S^2)} < C/(1+|s|^2)$ *as* $|s| \to \infty$ *away from* $i\mathbb{Z}$.

3. $||((is)^2 + (is) + \Delta_\theta)^{-1}||_{L^2(S^2)\to H^2(S^2)} < C$ *away from* $i\mathbb{Z}$.

Proof : This is a result from the Analytic Fredholm theory ([Melrose 2]). We simply note

$$\begin{aligned} (is+c)^2 + (is+c) &\notin Spec(\Delta_\theta) \\ (is+c)^2 + (is+c) &\notin k(k+1), k \in \mathbb{Z}^+ \\ c &\notin \mathbb{Z} \end{aligned}$$

□

Thus, when $c \notin \mathbb{Z}$, the resolvent

$$\forall s \in \mathbb{R} \ : \ R_s = ((is+c)^2 + (is+c) + \Delta_\theta)^{-1} : L^2(S^2) \to H^2(S^2)$$

is well-defined and $u^\flat(t,\theta)$ can be written as following :

$$u^\flat(t,\theta) = \int_{\mathbb{R}} e^{ist}((is+a)^2 + (is+a) + \Delta_\theta)^{-1}\widehat{f^\flat}(s,\theta)ds = \int_{\mathbb{R}} e^{ist}R_s(\widehat{f^\flat})(s,\theta)ds$$

Now it remains to show that u, thus given by the above formulae, is indeed in $H^2(\mathbb{R}\times S^2)$, given $f \in L^2(\mathbb{R}\times S^2)$. By **Lemma 3.4**, this is equivalent to showing that

$$\begin{cases} \widehat{u^\flat}(s,\theta) & \in \ L^2(\mathbb{R}; H^m(S^2)) \\ (1+s^2)\widehat{u^\flat}(s,\theta) & \in \ L^2(\mathbb{R}; L^2(S^2)) \end{cases}$$

provided that $\widehat{f^\flat}_s \in L^2(\mathbb{R}; L^2(S^2))$.

By **(2)** of **Lemma3.6**, we know

$$||R_s||_{L^2(S^2)\to H^2(S^2)} < C'$$

Thus, we have

$$\begin{aligned} ||\widehat{u^\flat}||_{L^2(\mathbb{R};H^2(S^2))} &= ||R_s(\widehat{f^\flat})||^2_{L^2(\mathbb{R};H^2(S^2))} \\ &= \int_{\mathbb{R}} ||R_s(\widehat{f^\flat}_s)||^2_{H^2(S^2)}ds \\ &< \int_{\mathbb{R}} ||R_s||^2_{L^2(S^)\to H^2(S^2)}||\widehat{f^\flat}_s||^2_{L^2(S^2)}ds \\ &< C\cdot\int_{\mathbb{R}} ||\widehat{f^\flat}_s||^2_{L^2(S^2)}ds \\ &< C||\widehat{f^\flat}||^2_{L^2(\mathbb{R};H^2(S^2))} < \infty \end{aligned}$$

By **(3)** of **Lemma3.6**, we also know

$$||\tilde{R}_s||_{L^2(S^)\to L^2(S^2)} < C/(1+|s|^2)$$

Then, we can conclude that

$$\begin{aligned}
||(1+|s|^2)\widehat{u^\flat}||^2_{L^2(\mathbb{R};L^2(S^2))} &= ||(1+|s|^2)R_s(\widehat{f^\flat})||^2_{L^2(\mathbb{R};L^2(S^2))} \\
&= \int_{\mathbb{R}} |(1+s^2)|^2 ||\tilde{R}_s(\widehat{f^\flat}_s)||^2_{L^2(S^2)} ds \\
&< \int_{\mathbb{R}} |(1+s^2)|^2 ||\tilde{R}_s||^2 ||\widehat{f^\flat}||^2_{L^2(S^2)} ds \\
&< \int_{\mathbb{R}} |(1+s^2)|^2 \cdot \frac{(C')^2}{(1+|s|^2)^2} \cdot ||\widehat{f^\flat}||^2_{L^2(S^2)} ds \\
&< C' \; ||\widehat{f^\flat}||^2_{L^2(\mathbb{R};L^2(S^2))} < \infty
\end{aligned}$$

Therefore, we finally conclude that

$$\begin{aligned}
u^\flat(t,\theta) &\in H^2(\mathbb{R}\times S^2) &\Longleftrightarrow \\
\tilde{u}(r,\theta) &\in H^2((0,\infty)\times S^2)) &\Longleftrightarrow \\
u = r^c\tilde{u} &\in r^c H^2_b([0,1]\times S^2)
\end{aligned}$$

This shows that when $c \notin Z$, we have a well-defined inverse :

$$(r^2\Delta_o)^{-1} : r^c L^2_b([0,1]\times S^2) \to r^c H^2_b([0,1]\times S^2)$$

and similarly for all $s > 0$

$$(r^2\Delta_o)^{-1} : r^c H^s_b([0,1]\times S^2) \to r^c H^{s+2}_b([0,1]\times S^2)$$

is well defined for $c \notin Z$ □

Remark 3.7 *Another way to prove the above lemma is to keep the operator $r^2\Delta_o$ fixed, instead of conjugating it, and then investigate what the Fourier transform does to $e^{ct}L^2(\mathbb{R};L^2(S^2))$. We recall the well-known isomorphism :*

$$L^2(\mathbb{R}) \ni f \longleftrightarrow \widehat{f} \in L^2(\mathbb{R}) \tag{3.2}$$

Using this, we deduce the following 1-1 correspondence :

$$\begin{aligned}
\mathcal{F} \quad &: \quad e^{ct}L^2(\mathbb{R};L^2(S^2)) &\to \quad L^2(\mathbb{R}-ic;L^2(S^2)) \\
\mathcal{G}=\mathcal{F}^{-1} \quad &: \quad L^2(\mathbb{R}-ic;L^2(S^2)) &\to \quad e^{ct}L^2(\mathbb{R};L^2(S^2))
\end{aligned}$$

Going in ($\to$) direction, we have

$$\begin{aligned}
\mathcal{F}(f)(s,\theta) &= \int_{\mathbb{R}} e^{-ist} f(t,\theta)dt \\
&= \int_{\mathbb{R}} e^{-ist}e^{ct}\tilde{f}(t,\theta)dt \\
&= \int_{\mathbb{R}} e^{-i(s+ic)t}\tilde{f}(t,\theta)dt \\
&= \widehat{\tilde{f}}(s+ic,\theta)
\end{aligned}$$

and $\widehat{\tilde{f}}(s+ic,\theta)$ *is* L^2*, exactly when* $s=\sigma-ic$*,* $\sigma\in\mathbb{R}$*, because*

$$\widehat{\tilde{f}}(s+ic,\theta)=\widehat{\tilde{f}}(\sigma-ic+ic,\theta)=\widehat{\tilde{f}}(\sigma,\theta)\in L^2(\mathbb{R};L^2(S^2))$$

Going in the other direction, we have

$$\begin{aligned}\mathcal{G}(g)(t,\theta) &= \int_{\mathbb{R}-ic} e^{ist}g(s,\theta)ds\\ &= \int_{\mathbb{R}} e^{i(\sigma-ic)t}g(\sigma-ic,\theta)d\sigma\\ &= e^{ct}\cdot\int_{\mathbb{R}} e^{i\sigma t}g(\sigma-ic,\theta)d\sigma\end{aligned}$$

Again by (3.2), $\mathcal{G}(g)(t,\theta)\in e^{ct}L^2(\mathbb{R};L^2(S^2))$*.*

Going back to the main equation with $u\in r^cH^2_b([0,1]\times S^2)$ *and* $f\in r^cL^2_b([0,1]\times S^2)$*, we have,:*

$$((is)^2+(is)+\Delta_\theta)\widehat{u}(s,\theta)=\widehat{f}(s,\theta) \tag{3.3}$$

or after all the changes of variables, we have

$$((is)^2+(is)+\Delta_\theta)\widehat{\tilde{u}}(s+ic,\theta)=\widehat{\tilde{f}}(s+ic,\theta) \tag{3.4}$$

Now we have :

$$\begin{array}{ll} c\notin\mathbb{Z} & \Rightarrow\\ \forall s\in\mathbb{R}-ic, s\notin Spec(\Delta_\theta) & \Rightarrow\\ ((is)^2+(is)+\Delta_\theta)^{-1} \text{ is well-defined} & \Rightarrow\\ ((is)^2+(is)+\Delta_\theta)^{-1}\hat{\tilde{f}}(s+ic,\theta)\in L^2(\mathbb{R}-ic;L^2(S^2)) & \end{array}$$

And by the above 1-1 correspondence,

$$\begin{aligned}u(t,\theta) &= \int_{\mathbb{R}-ic} e^{ist}((is)^2+(is)+\Delta_\theta)^{-1}\hat{f}(s,\theta)ds\\ &= \int_{\mathbb{R}-ic} e^{ist}((is)^2+(is)+\Delta_\theta)^{-1}\hat{\tilde{f}}(s+ic,\theta)ds\\ &= \mathcal{G}((is)^2+(is)+\Delta_\theta)^{-1}\hat{\tilde{f}}(s+ic,\theta))\end{aligned}$$

is indeed in $e^{ct}L^2(\mathbb{R};L^2(S^2))$*. By the mapping properties of* R_s *and* $\tilde{R}_s$ *as discussed above,* $u(t,\theta)$ *is, actually, in* $e^{ct}H^2(\mathbb{R}\times S^2)$*.*

□

3.4 Construction of a Parametrix to $(\phi+r^2)\Delta_o$

We first need to define the following cutoff functions. Recall ϕ was such that

$$\phi\in C^\infty_c(\mathbb{R}^3);\quad \phi=\begin{cases}1 & \text{if } r\le 1\\ \in[0,1] & \text{if } 1<r<2\\ 0 & \text{if } r\ge 2\end{cases}$$

Similarly we define $\tilde{\phi}$, ψ, and $\tilde{\psi}$ to be

$$\tilde{\phi} \in C_c^\infty(\mathbb{R}^3); \qquad \tilde{\phi} = \begin{cases} 1 & \text{if } r \leq 7/3 \\ \in [0,1] & \text{if } 7/3 < r < 8/3 \\ 0 & \text{if } r \geq 8/3 \end{cases}$$

$$\psi \in C_c^\infty(\mathbb{R}^3); \qquad \psi = \begin{cases} 1 & \text{if } r \leq 3 \\ \in [0,1] & \text{if } 3 < r < 4 \\ 0 & \text{if } r \geq 4 \end{cases}$$

$$\tilde{\psi} \in C_c^\infty(\mathbb{R}^3); \qquad \tilde{\psi} = \begin{cases} 1 & \text{if } r \leq 5 \\ \in [0,1] & \text{if } 5 < r < 6 \\ 0 & \text{if } r \geq 6 \end{cases}$$

Using these cutoff functions, we define :

$$\begin{aligned} Q(f) &= \tilde{\psi}\,[\frac{1}{4\pi r} * \frac{\psi f}{(\phi + r^2)}] + (1-\tilde{\phi})(r^2\Delta_o)^{-1}(1-\psi)f \\ &= Q_1(f) + Q_2(f) \end{aligned}$$

We prove :

Lemma 3.8 *Q is a map on following spaces :*

$$Q : x^a H_b^s(\mathbb{B}^3) \to x^a H_b^{s+2}(\mathbb{B}^3)$$

for $a \notin \mathbb{Z}$.

Proof : Suppose $f \in x^a H_b^s(\mathbb{B}^3)$. Now look at the first term $Q_1(f) = \tilde{\psi}\,[\frac{1}{4\pi r} * \frac{\psi f}{(\phi+r^2)}]$. We note that $\frac{\psi}{\phi + r^2} \cdot f \in H_c^s(\mathbb{R}^3)$ and make the following claim :

Claim 3.9 *$(\tilde{\psi}\frac{1}{4\pi r} * \_\,)$ is a well-defined map on the following spaces :*

$$\tilde{\psi}\frac{1}{4\pi r} * \_ \; : H_c^s(\mathbb{R}^3) \to H_c^{s+2}(\mathbb{R}^3)$$

Proof : Take an element $f \in H_c^s(\mathbb{R}^3)$. We have

$$\widehat{\frac{1}{4\pi r} * f} = \widehat{\frac{1}{4\pi r}} \cdot \widehat{f} = \frac{1}{|\xi|^2} \cdot \widehat{f}$$

Now let us write

$$\widehat{f} = \phi\widehat{f} + (1-\phi)\widehat{f} = \widehat{f}^+ + \widehat{f}^-$$

where ϕ once again is a cutoff function defined above. Recall the following well-known correspondence :

$$h \in H^s(\mathbb{R}^3) \leftrightarrow (1 + |\xi|^s)\widehat{h} \in L^2(\mathbb{R}^3)$$

Thus, $(1+|\xi|^s)\widehat{f}^- \in L^2(\mathbb{R}^3)$ and since $\widehat{f}^- \equiv 0$ near $0 \in \mathbb{R}^3$,

$$(1 + |\xi|^{s+2})\frac{1}{|\xi|^2}\widehat{f}^- \in L^2(\mathbb{R}^3)$$

Thus $\mathcal{F}^{-1}(\frac{1}{|\xi|^2}\widehat{f}^-) \in H^{s+2}(\mathbb{R}^3)$ and $\tilde{\psi} \cdot \mathcal{F}^{-1}(\frac{1}{|\xi|^2}\widehat{f}^-) \in H_c^{s+2}(\mathbb{R}^3)$.

Now, since f is compactly supported, we see that $\widehat{f} \in C^\infty(\mathbb{R}^3)$ by Paley-Wiener, and thus $\widehat{f}^+$ is in $C_c^\infty(\mathbb{R}^3)$. We then have :

$$\frac{1}{|\xi|^2}\widehat{f}^+ \in L_c^1(\mathbb{R}^3) \subset \mathcal{D}_c'(\mathbb{R}^3)$$

Again by Paley-Wiener theorem, $\mathcal{F}^{-1}(\frac{1}{|\xi|^2}\widehat{f}^+) \in C^\infty(\mathbb{R}^3)$ and $\tilde{\psi} \cdot \mathcal{F}^{-1}(\frac{1}{|\xi|^2}\widehat{f}^+) \in C_c^\infty(\mathbb{R}^3) \subset H_c^{s+2}(\mathbb{R}^3)$.

□

Thus the above claim shows that

$$Q_1(f) \in H_c^{s+2}(\mathbb{R}^3) \in x^a H_b^{s+2}(\mathbb{B}^3)$$

Furthermore, since it is compactly supported, it is indeed in $x^a H_b^{s+2}(\mathbb{B}^3)$.

Now look at the second term

$$Q_2(f) = (1 - \tilde{\phi})(r^2\Delta_o)^{-1}(1 - \psi)f$$

Since $(1 - \psi)$ is supported away from $0 \in \mathbb{R}^3$ and $x^a = r^{-a}$ near ∞, we realize that

$$(1 - \psi)f \in r^{-a}H_b^s([0,1] \times S^2)$$

Since we showed that for $a \notin \mathbb{Z}$:

$$(r^2\Delta_o)^{-1} : r^{-a}H_b^s([0,1] \times S^2) \to r^{-a}H_b^{s+2}([0,1] \times S^2)$$

is well-defined,

$$(r^2\Delta_o)^{-1}(1 - \psi)f \in r^{-a}H_b^{s+2}([0,1] \times S^2)$$

Now cutting this off away from $0 \in \mathbb{R}^3$ by multiplying it by $(1 - \tilde{\phi})$, we thus conclude that

$$Q_2(f) = (1 - \tilde{\phi})(r^2\Delta_o)^{-1}(1 - \psi)f \in x^a H_b^{s+2}(\mathbb{B}^3)$$

Therefore,

$$Q(f) = Q_1(f) + Q_2(f) \in x^a H_b^{s+2}(\mathbb{B}^3)$$

□

In order to prove that Q is a parametrix to $(\phi+r^2)\Delta_o$ which makes $(\phi+r^2)\Delta_o$ Fredholm, we need to show

$$(\phi + r^2)\Delta_o \circ Q = I + R$$

and

$$R : x^a H_b^s(\mathbb{B}^3) \to x^a H_b^s(\mathbb{B}^3)$$

is a compact smoothing operator. Thus we prove

Lemma 3.10
$$(\phi + r^2)\Delta_o \circ Q_1(f) = \psi f + O_K(f)$$
where
$$O_K : x^a H_b^s(\mathbb{B}^3) \to x^a H_b^s(\mathbb{B}^3)$$
is a compact smoothing operator.

Proof : We have, in the sense of distributions,

$$\begin{aligned}
&(\phi + r^2)\Delta_o \circ Q_1(f) \\
&= (\phi + r^2)\Delta_o \circ \tilde{\psi}\, [\frac{1}{4\pi r} * \frac{\psi f}{(\phi + r^2)}] \\
&= (\phi + r^2)\{(\Delta_o\tilde{\psi})(\frac{1}{4\pi r} * \frac{\psi f}{(\phi + r^2)}) + 2(\nabla\tilde{\psi})(\nabla(\frac{1}{4\pi r} * \frac{\psi f}{(\phi + r^2)})) + \\
&\qquad (\tilde{\psi})(\Delta_o(\frac{1}{4\pi r} * \frac{\psi f}{(\phi + r^2)}))\}
\end{aligned}$$

Now, as for the rightmost term, we have :

$$\begin{aligned}
(\phi + r^2)(\tilde{\psi})(\Delta_o(\frac{1}{4\pi r} * \frac{\psi f}{(\phi + r^2)})) &= (\phi + r^2)\tilde{\psi}\frac{\psi f}{(\phi + r^2)} \\
&= \psi f
\end{aligned}$$

because $\tilde{\psi} \cdot \psi = \psi$. We assert that the remaining first two terms give a smooth error. From the definition of $\tilde{\psi}$, we can see that

$$\text{supp}(\Delta_o\tilde{\psi}) \subset \{p \in \mathbb{R}^3 \mid 3 < |p| < 4\} \to \text{supp}(\Delta_o\tilde{\psi}) \cap \text{supp}\psi = \emptyset$$

Now, with this in mind, we can write :

$$\begin{aligned}
&(\phi + r^2) \cdot (\Delta_o\tilde{\psi})(\frac{1}{4\pi r} * \frac{\psi f}{(\phi + r^2)})(\mathbf{x}) \\
&= \int_{\mathbb{R}^3} [(\phi(\mathbf{x}) + |\mathbf{x}|^2)\Delta_o\tilde{\psi}(\mathbf{x})][\frac{1}{4\pi|\mathbf{x} - \mathbf{y}|}][\frac{\psi(\mathbf{y})}{(\phi(\mathbf{y}) + |\mathbf{y}|^2)}]f(\mathbf{y})d\mathbf{y} \\
&= \int_{\mathbb{R}^3} K_1(\mathbf{x}, \mathbf{y})f(\mathbf{y})d\mathbf{y} \\
&= O_{K_1}(f)
\end{aligned}$$

We then assert the following claim :

Claim 3.11 K_1 *is in* $C_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ *and moreover*
$$O_{K_1} : C^{-\infty}(\mathbb{R}^3) \to C_c^\infty(\mathbb{R}^3)$$
is a compact smoothing operator.

Proof : $K_1 \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ is clear because

$$\mathrm{supp}([(\phi(\mathbf{x}) + |\mathbf{x}|^2)\Delta_o\tilde{\psi}(\mathbf{x})] \cdot [\frac{\psi(\mathbf{y})}{(\phi(\mathbf{y}) + |\mathbf{y}|^2)}]) \subset \mathrm{Diag}(\mathbb{R}^3 \times \mathbb{R}^3)^{\mathbf{c}}$$

and

$$\mathrm{singsupp}(\frac{1}{4\pi|\mathbf{x} - \mathbf{y}|}) = \mathrm{Diag}(\mathbb{R}^3 \times \mathbb{R}^3)$$

Thus O_{K_1} is in fact a smoothing operator. Now it remains to show that

$$O_{K_1} : x^a H_b^s(\mathbb{B}^3) \to x^a H_b^s(\mathbb{B}^3)$$

is a compact operator. By definition, this means that we need to prove that given a bounded sequence $\{f_n\}$, i.e. $\|f_n\|_{x^a H_b^s(\mathbb{B}^3)} < C$, we need to show that $\{O_{K_1}(f_n)\}$ has a convergent subsequence. Since

$$\Sigma = \mathrm{supp}[(\phi(\mathbf{x}) + |\mathbf{x}|^2)\Delta_o\tilde{\psi}(\mathbf{x})]$$

is a compact subset of $\mathbb{B}^3$ and $\mathrm{supp}(O_{K_1}(f_n)) \subset \Sigma$ we see that

$$\forall n,\ \mathrm{supp}(\{O_{K_1}(f_n)\}) \subset \Sigma$$

Furthermore, we can prove that $(\{O_{K_1}(f_n)\})$ is an *equicontinuous* family, as we can see from below :

$$\begin{aligned}
|O_{K_1}(f_n)(x) - O_{K_1}(f_n)(x')| &= |\int_{\mathbb{R}^3}(K_1(x,y) - K_1(x',y))f_n(y)dy| \\
&= |\int_{\Sigma'}(K_1(x,y) - K_1(x',y))f_n(y)dy| \\
&< (\int_{\Sigma'}|K_1(x,y) - K_1(x',y)|^2dy)^{\frac{1}{2}} \cdot (\int_{\Sigma'}|f_n|^2dy)^{\frac{1}{2}} \\
&< C \cdot (\int_{\Sigma'}|K_1(x,y) - K_1(x',y)|^2dy)^{\frac{1}{2}}
\end{aligned}$$

where

$$\Sigma' = \mathrm{supp}(\frac{\psi(\mathbf{y})}{(\phi(\mathbf{y}) + |\mathbf{y}|^2)})$$

Now since $K_1(x,y) \in C_c^0(\mathbb{R}^3 \times \mathbb{R}^3)$ (in fact $C_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$), we have

$$(\forall\epsilon > 0)(\exists\delta > 0)(|(x,y) - (x',y')| < \epsilon \to |K_1(x,y) - K_1(x',y')| < \frac{\epsilon}{\sqrt{C \cdot (\int_{\sigma'}\ 1\ dy)^{\frac{1}{2}}}})$$

Letting $y = y'$, we then have

$$(\forall\epsilon > 0)(\exists\delta > 0)(|(x,x'| < \epsilon \to |K_1(x,y) - K_1(x',y)| < \frac{\epsilon}{\sqrt{C \cdot (\int_{\sigma'}\ 1\ dy)^{\frac{1}{2}}}})$$

Thus,

$$\begin{aligned}(\forall\epsilon>0)(\forall n)(\exists\delta>0)(|(x,x'|<\epsilon \quad \rightarrow \quad & |O_{K_1}(f_n)(x)-O_{K_1}(f_n)(x')| \\ & < C\cdot(\int_{\Sigma'}|K_1(x,y)-K_1(x',y)|^2dy)^{\frac{1}{2}} \\ & < C\cdot(\frac{\epsilon}{C\cdot(\int_{\sigma'}dy)^{\frac{1}{2}}})(\int_{\sigma'}1\ dy)^{\frac{1}{2}} \\ & < \epsilon\end{aligned}$$

and thus the equicontinuity of $O_{K_1}(f_n)(x)$ is established.

Similarly, we have

$$\begin{aligned}|O_{K_1}(f_n)(x)| &< (\int_{\Sigma'}|K_1(x,y)|^2dy)^{\frac{1}{2}}(\int_{\Sigma'}|f_n(y)|^2dy)^{\frac{1}{2}} \\ &< C'\cdot C\end{aligned}$$

where $C'=max_{x\in\Sigma}(\int_{\Sigma'}|K_1(x,y)|^2dy)^{\frac{1}{2}}$. Now, the following computation

$$\begin{aligned}(\int_{\mathbb{R}^3}|x^{-a}O_{K_1}f_n(x)|^2dy) &= (\int_{\Sigma'}|x^{-a}O_{K_1}f_n(x)|^2dy) \\ &< A\cdot(\int_{\Sigma'}|O_{K_1}f_n(x)|^2dy) \text{ for some constant A} \\ &< A\cdot C^2\cdot C'^2\cdot(\int_{\Sigma'}1\ dy)\end{aligned}$$

shows that

$$||O_{K_1}(f_n)||_{x^aL^2_b(\mathbb{B}^3)}<L$$

Now putting all the three above facts together, i.e.

1. $\{O_{K_1}f_n\}$ is supported on a compact set.
2. $\{O_{K_1}f_n\}$ is equicontinuous.
3. $\{O_{K_1}f_n\}$ is bounded.

we can apply Azela-Ascoli theorem to prove that there exists a subsequence of $\{O_{K_1}(f_n)\}$ that converges to $f^0\in x^aL^2_b(\mathbb{B}^3)$. Now we can repeat the argument for all $k\le s$ to get subsequences that converge to f^k. I.e. We get a convergent subsequence in $x^aH^2_b(\mathbb{B}^3)$. □

Similarly, we can define

$$\begin{aligned}O_{K_2}(f) &= (\phi+r^2)\cdot(2\nabla\tilde{\psi})(\frac{1}{4\pi r}*\nabla(\frac{\psi f}{(\phi+r^2)})) \\ &= (\phi+r^2)\cdot(2\nabla\tilde{\psi})(\frac{1}{4\pi r}*\nabla(\frac{\psi}{(\phi+r^2)})f)+(\phi+r^2)\cdot(2\nabla\tilde{\psi})(\frac{1}{4\pi r}*\frac{f}{(\phi+r^2)}\nabla(\psi))\end{aligned}$$

and conclude that

$$O_{K_2} : x^a H^s(\mathbb{B}^3) \to x^a H^s(\mathbb{B}^3)$$

is again a compact smoothing operator because

$$\begin{aligned} \text{supp}(2(\nabla\tilde{\psi})) \subset \{p \in \mathbb{R}^3 \mid 3 < |p| < 4\} &\to \text{supp}(2(\nabla\tilde{\psi})) \cap \text{supp}\nabla\psi = \emptyset \\ &\to \text{supp}(2(\nabla\tilde{\psi})) \cap \text{supp}\psi = \emptyset \end{aligned}$$

Thus, we finally conclude that

$$\begin{aligned} (\phi + r^2)\Delta_o \circ Q_1(f) &= \psi f + O_{K_1}(f) + O_{K_2}(f) \\ &= \psi f + O_K(f) \end{aligned}$$

□

Now for $(\phi + r^2)\Delta_o \circ Q_2(f)$, we prove a similar lemma :

Lemma 3.12

$$(\phi + r^2)\Delta_o \circ Q_2(f) = \psi f + O_{K'}(f)$$

where

$$O_{K'} : x^a H_b^s(\mathbb{B}^3) \to x^a H_b^s(\mathbb{B}^3)$$

is a compact smoothing operator.

Proof : As in the preceding lemma, we compute

$$\begin{aligned} &(\phi + r^2)\Delta_o \circ Q_2(f) \\ &= (\phi + r^2)\Delta_o \circ ((1-\tilde{\phi})(r^2\Delta_o)^{-1}(1-\psi)f \\ &= (\phi + r^2)\{(\Delta_o(1-\tilde{\phi}))((r^2\Delta_o)^{-1}(1-\psi)f) + \\ &\qquad 2(\nabla(1-\tilde{\phi}))(\nabla((r^2\Delta_o)^{-1}(1-\psi)f)) + \\ &\qquad (1-\tilde{\phi})\Delta_o(r^2\Delta_o)^{-1}(1-\psi)f\} \end{aligned}$$

For the rightmost term, we again see

$$\begin{aligned} (\phi + r^2)(1-\tilde{\phi})\Delta_o(r^2\Delta_o)^{-1}(1-\psi)f) &= (1-\tilde{\phi})(r^2\Delta_o)(r^2\Delta_o)^{-1}((1-\psi)f) + \\ &\quad \phi(1-\tilde{\phi})\Delta_o(r^2\Delta_o)^{-1}(1-\psi)f \\ &= (1-\psi)f \end{aligned}$$

because we chose $\tilde{\phi}$ and ψ such that

$$\begin{aligned} (1-\tilde{\phi})(1-\psi)f &= (1-\psi)f \\ \phi(1-\tilde{\phi}) &= 0 \end{aligned}$$

As for the remaining first two terms, by noting, as we did for Q_1, that

$$\begin{aligned} \text{supp}(\Delta_o(1-\tilde{\phi}) &\cap \text{supp}(1-\psi) = \emptyset \\ \text{supp}(2\nabla(1-\tilde{\phi})) &\cap \text{supp}\nabla(1-\psi) = \emptyset \\ \text{supp}(2\nabla(1-\tilde{\phi})) &\cap \text{supp}(1-\psi) = \emptyset \end{aligned}$$

we again conclude that

$$(\phi + r^2)\Delta_o \circ Q_2(f) = (1-\psi)f + O_{K'}(f)$$

where

$$O_{K'} : x^a H_b^s(\mathbb{B}^3) \to x^a H_b^s(\mathbb{B}^3)$$

is again a compact and smoothing operator. □

Therefore, finally, we have :

$$\begin{aligned}(\phi + r^2)\Delta_o \circ Q(f) &= (\phi + r^2)\Delta_o \circ (Q_1(f) + Q_2(f)) \\ &= \psi f + (1-\psi)f + (O_K + O_{K'})(f) \\ &= f + R(f)\end{aligned}$$

where

$$R : x^a H_b^s(\mathbb{B}^3) \to x^a H_b^s(\mathbb{B}^3)$$

is a compact and smoothing operator.

3.5 Mapping properties of $(\phi + r^2)\Delta_o$

In the previous section we saw that when $a \notin \mathbb{Z}$, the map

$$(\phi + r^2)\Delta_o^a : x^a H_b^s(\mathbb{B}^3) \to x^a H_b^{s-2}(\mathbb{B}^3)$$

had a parametrix Q_a such that

$$(\phi + r^2)\Delta_o^a \circ Q_a = Id + R_a$$

where R_a is a compact and smoothing operator. This proves, by standard arguments, that $(\phi + r^2)\Delta_o^a$ is Fredholm if $a \notin Z$. Moreover, $a \notin \mathbb{Z}$ is indeed a necessary and sufficient condition for $(\phi + r^2)\Delta_o^a$ to be Fredholm for because of the following lemma :

Lemma 3.13

$$(\phi + r^2)\Delta_o^a : x^a H_b^s(\mathbb{B}^3) \to x^a H_b^{s-2}(\mathbb{B}^3)$$

is not closed if $a \in \mathbb{Z}$.

Proof : We construct a counter-example. We refer to Paul Loya's thesis ([Loya]). □

Now when $a \notin \mathbb{Z}$, i.e. when $(\phi + r^2)\Delta_o^a$ is Fredholm, we can talk about the dimensions of $\ker((\phi + r^2)\Delta_o)$ and $\operatorname{coker}((\phi + r^2)\Delta_o)$. We first note the obvious fact that

$$\ker((\phi + r^2)\Delta_o) = \ker(\Delta_o)$$

Now suppose $f \in \mathcal{S}'$, (i.e. f is a tempered distribution) then

$$\begin{aligned}(\Delta_o)f = 0 \quad &\Leftrightarrow \quad |\xi|^2 \widehat{f} = 0 \\ &\Leftrightarrow \quad \text{supp}(\widehat{f}) \subset 0 \in \mathbb{R}^3 \\ &\Leftrightarrow \quad \widehat{f} = \sum_{\alpha=1}^{k} c_\alpha D^\alpha \delta_0, \text{ for some } k \in \mathbb{N} \\ &\Leftrightarrow \quad f \text{ is a polynomial of degree k}\end{aligned}$$

Thus if $f \in \mathcal{S}'$ and $\Delta_o f = 0$, then f is indeed a harmonic polynomial. Since

$$x^a H_b^s(\mathbb{B}^3) \subset \mathcal{S}', \ \forall a \in \mathbb{R}$$

we therefore conclude that

$$\ker(\Delta_o^a) \subset \ \{\text{Harmonic polynomials}\}$$

Since there is no polynomial on $\mathbb{R}^3$ that vanishes at ∞, we see that

$$a \geq 0 \to \ker((\phi + r^2)\Delta_o^a) = \emptyset \Leftrightarrow ((\phi + r^2)\Delta_o^a) \text{ injective}$$

Furthermore,

$$\begin{array}{lcccccc} -1 < a < 0 & \to & \{1\} & = & x^a H_b^s(\mathbb{B}^3) \cap \ker((\phi + r^2)\Delta_o^a) & \Leftrightarrow & \dim(\ker) = 1 \\ -2 < a < -1 & \to & \{1, x^1, x^2, x^3\} & = & x^a H_b^s(\mathbb{B}^3) \cap \ker((\phi + r^2)\Delta_o^a) & \Leftrightarrow & \dim(\ker) = 4 \end{array}$$

and in general

$$-(k+1) < a < -k \to \ker((\phi + r^2)\Delta_o^a) = \mathcal{H}^k$$

where $\mathcal{H}^k$ is a set of harmonic polynomials of degree k.

On the other hand, if a is too large, $(\phi + r^2)\Delta_o^a$ fails to be surjective. For instance, we can prove

Lemma 3.14 *For $a > 1$, $1/(\phi + r^2)$ defines a linear map on $x^a H_b^s(\mathbb{B}^3)$ by*

$$x^a H_b^s(\mathbb{B}^3) \ni f \to \int_{\mathbb{R}^3} \frac{1}{(\phi + r^2)} \ f \ vol$$

where vol is the standard Euclidean measure.

Proof : Setting $f = x^a \tilde{f}$ for $\tilde{f} \in H_b^s(\mathbb{B}^3)$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{1}{(\phi + r^2)} \ f \text{ vol} \quad &= \quad \int_{\mathbb{R}^3} \frac{1}{(\phi + r^2)} \ x^a \tilde{f} \ r^2 dr d\theta \\ &= \quad \int_{\mathbb{R}^3} (\frac{r^3}{(\phi + r^2)} x^a)(\tilde{f}) \ \frac{dr}{r} d\theta \\ &< \quad (\int_{\mathbb{R}^3} |\frac{r^3}{(\phi + r^2)} x^a|^2 \frac{dr}{r} d\theta)^{\frac{1}{2}} (\int_{\mathbb{R}^3} |\tilde{f}|^2 \frac{dr}{r} d\theta)^{\frac{1}{2}} \\ &= \quad (\int_{\mathbb{R}^3} |\frac{r^3}{(\phi + r^2)} x^a|^2 \frac{dr}{r} d\theta)^{\frac{1}{2}} \ \|\tilde{f}\|_{L_b^2(\mathbb{B}^3)}\end{aligned}$$

where the inequality is justified by the Cauchy Schawrtz when

$$(\int_{\mathbb{R}^3} |\frac{r^3}{(\phi+r^2)}x^a|^2 \frac{dr}{r} d\theta) < \infty$$

which is true when we have

$$2 - 2a < \epsilon \Leftrightarrow a > 1 + \epsilon/2$$

□

Using this lemma, we conclude that for $a > 1$

$$\text{Im}\{(\phi + r^2)\Delta_o^a : x^a H_b^s(\mathbb{B}^3) \to x^a H_b^{s-2}(\mathbb{B}^3)\} \subset \ker\{\frac{1}{(\phi+r^2)} : x^a H_b^s(\mathbb{B}^3) \to \mathbb{R}\}$$

because

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{1}{(\phi+r^2)} (\phi + r^2)\Delta_o^a f \text{ vol} &= \int_{\mathbb{R}^3} 1 \cdot \Delta_o^a f \text{ vol} \\ &= \int_{\mathbb{R}^3} \Delta_o(1) \cdot f \text{ vol} \\ &= 0 \end{aligned}$$

and furthermore we can show that indeed we have

$$\begin{aligned} 1 < a < 2 \quad &: \quad \text{Im}((\phi + r^2)\Delta_o^a) \subset \ker\{\tfrac{1}{(\phi+r^2)} : x^a H_b^s(\mathbb{B}^3) \to \mathbb{R}\} \\ 2 < a < 3 \quad &: \quad \text{Im}((\phi + r^2)\Delta_o^a) \subset \ker\{\tfrac{1}{(\phi+r^2)}, \tfrac{x^1}{(\phi+r^2)}, \tfrac{x^2}{(\phi+r^2)}, \tfrac{x^3}{(\phi+r^2)} : x^a H_b^s(\mathbb{B}^3) \to \mathbb{R}\} \end{aligned}$$

This pattern suggests a relationship between the kernel and cokernel of $(\phi + r^2)\Delta_o$. In fact, we have

Lemma 3.15

$$1 \le k < a < k+1 \ : \ \textit{coker}((\phi + r^2)\Delta_o^a) = \frac{1}{(\phi+r^2)} \cdot \mathcal{H}^k$$

Proof : We refer to Paul Loya's thesis ([Loya]). □

Putting this all together, we finally see that for

$$(\phi + r^2)\Delta_0 : x^a H_b^s(\mathbb{B}^3) \to x^a H_b^{s-2}(\mathbb{B}^3)$$

is

$$\begin{cases} \text{Fredholm} & \text{iff} \quad a \notin \mathbb{Z} \\ \text{surjection} & \text{if} \quad a < 1 \\ \text{injection} & \text{if} \quad a > 0 \\ \text{isomorphism} & \text{if} \quad 0 < a < 1 \end{cases}$$

3.6 Asymptotic Behaviors to Solutions to Δ_o

Now, we are ready to prove the main theorem of this chapter :

Theorem 3.16 *Suppose* $f \in x^{a+2}H_b^\infty(\mathbb{B}^3) \cap A_{\mathcal{E}}^\infty(\mathbb{B}^3)$ *for* $0 < a < 1$. *Then the unique solution* $u \in x^a H_b^\infty(\mathbb{B}^3)$ *to*

$$\Delta_o u = f$$

such that $u \in A_{\mathcal{I}_\mathcal{E}}^\infty(\mathbb{B}^3)$ *for an index set* $\mathcal{I}_\mathcal{E}$ *where* $\mathcal{I}_\mathcal{E} = (\mathcal{E} - 2)\overline{\cup}\mathbb{N}$.

We first prove the following important lemma :

Lemma 3.17 *Let* $a \in (0,1)$, $f \in x^{a+2}H_b^s(\mathbb{B}^3)$ *and* u *be the unique solution to*

$$\Delta_o u = f$$

such that $u \in x^a H_b^{s+2}(\mathbb{B}^3)$. *Suppose* f *were in fact in* $x^{a+2+\gamma}H_b^s(\mathbb{B}^3)$ *(i.e. decays faster by a factor of* x^γ, $\gamma > 0$*). Then* u *has the following asymptotic expansion near infinity :*

$$u = \sum_{1 \le j < a+\gamma} c_j(\theta)x^j + x^{a+\gamma}H_b^{s+2}(\mathbb{B}^3)$$

Proof :

Remark 3.18 *Throughout this proof, we will state several claims. The proofs of these claims will be postponed till the Appendix at the end of the section to allow the proof of the theorem to flow smoothly.*

We write, using the cutoff function ϕ supported near 0,

$$u = \phi u + (1-\phi)u = u_- + u_+$$
$$f = \phi f + (1-\phi)f = f_- + f_+$$

By multiplying both sides of $\Delta_o u = f$ by r^2, we get

$$(r^2\Delta_o)u_+ = r^2 f_+ + r^2 f_- - (r^2\Delta_o)u_-$$

We then claim :

Claim 3.19 *If* $c \ge 3/2$, *then*

$$x^c H_b^m(\mathbb{B}^3; d\mu) \subset r^{-c}H_b^m([0,1] \times S^2, \tau_b)$$

□

Since $2 < a+2 < 3$, the **Claim 3.19** shows

$$\begin{aligned} f_-,\ f_+,\ \Delta_o u+ &\in x^{a+2}H_b^s(\mathbb{B}^3) \subset r^{-(a+2)}H_b^s([0,1] \times S^2, \tau_b) \Rightarrow \\ r^2 f_+ + r^2 f_- - (r^2\Delta_o)u_- &\in r^{-a}H_b^s([0,1] \times S^2, \tau_b) \end{aligned}$$

Moreover

$$u_+ \equiv 0 \text{ near } 0 \in \mathbb{R}^3 \to u_+ \in r^{-a} H_b^{s+2}([0,1] \times S^2, \tau_b)$$

Recall that we proved the following isomorphism for $a \notin \mathbb{Z}$ in **Theorem 3.5**

$$r^2 \Delta_o : r^{-a} H_b^{s+2}([0,1] \times S^2, \tau_b) \to r^{-a} H_b^{s}([0,1] \times S^2, \tau_b)$$

or letting $r = e^t$

$$e^{2t} \Delta_o : e^{-at} H^{s+2}(\mathbb{R} \times S^2) \to e^{-at} H^{s}(\mathbb{R} \times S^2)$$

Thus, if we let $U(t,\theta) = u(e^t,\theta)$ and $F(t,\theta) = f(e^t,\theta)$, U_+ can be given as following :

$$U_+(t,\theta) = \int_{\mathbb{R}-ia} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}(\widehat{e^{2t}F_+} + \widehat{e^{2t}F_-} - \widehat{(e^{2t}\Delta_o)U_-})(s,\theta)ds$$

We now look at the righthand side of the above, more in detail. We first look at

$$\int_{\mathbb{R}-ia} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}\widehat{e^{2t}F_+}(s,\theta)ds$$

We note the following properties of u

$$e^{2t}F_+ \in e^{-at}H^s(\mathbb{R} \times S^2), \ \text{supp}(e^{2t}F_+) \subset \mathbb{R}^+ \times S^2$$

which can be equivalently written as, using **Lemma 3.4**,

$$\begin{array}{llll} (\dagger) & e^{2t}F_+ & \in & e^{-at}L^2(\mathbb{R}; H^s(S^2)), \ \text{supp}(e^{2t}F_+) \subset \mathbb{R}^+ \times S^2 \\ (\ddagger) & e^{2t}F_+ & \in & e^{-at}H^s(\mathbb{R}; L^2(S^2)), \ \text{supp}(e^{2t}F_+) \subset \mathbb{R}^+ \times S^2 \end{array}$$

We now prove the following lemma of Paley-Wiener type which characterizes functions with above properties :

Claim 3.20 *Let $\mathcal{H}$ be some Hilbert space with a norm $||\cdot||_{\mathcal{H}}$ given by its inner product. Then, there is a following 1-1 correspondence :*

$$h \in e^{ct}H^m(\mathbb{R};\mathcal{H}) \quad \textit{and} \quad \textit{supp}(h) \subset \mathbb{R}^+$$

$$\Longleftrightarrow \begin{cases} (1) \ \textit{For} \ \ 0 \le k \le m, (is)^k \hat{h}(s) \ \ \textit{is holomorphic, with values in } \mathcal{H} \textit{ on } \ \{s \in \mathbb{C} \mid Im(s) < -c\}. \\ (2) \ ||\hat{h}(s)||_{\mathcal{H}} < \frac{||h||_{H^m(\mathbb{R};\mathcal{H})}}{(1+|s|^m)\sqrt{2(Im(s)+c)}} \quad \textit{on} \ \ \{s \in \mathbb{C} \mid Im(s) < -c\}. \\ (3) \ \forall \eta \le -c, \widehat{h_\eta}(\sigma) := \hat{h}(\sigma + i\eta) \to \begin{cases} \hat{h_\eta}(\sigma) \in H^m(\mathbb{R};\mathcal{H}) \\ ||\hat{h_\eta}||_{H^m(\mathbb{R};\mathcal{H})} \le ||\hat{h}||_{H^m(\mathbb{R};\mathcal{H})} \\ (-\infty,-c] \ni \eta \mapsto \hat{h_\eta} \in H^m(\mathbb{R};\mathcal{H}) \ \textit{continuously} \end{cases} \end{cases}$$

□

Applying **Claim 3.20** to (†) and (‡), we see that

(†) $\widehat{e^{2t}F_+}$ is holomorphic, with values in $H^s(S^2)$ for $\{s \in \mathbb{C} \mid Im(s) < a\}$

(‡) $\widehat{e^{2t}F_+}$ is holomorphic, with values in $L^2(S^2)$ for $\{s \in \mathbb{C} \mid Im(s) < a\}$ with such decay properties as specified in the lemma.

It is the second notion of holomorphy that we use for what follows.

Let us finally use our assumption of f. Since f actually has an extra decay near ∞, i.e. $f \in r^{a+2+\gamma}H_b^s(\mathbb{B}^3)$, we in fact have

$$e^{2t}F_+ \in e^{-(a+\gamma)t}H(\mathbb{R} \times S^2),\ \text{supp}(e^{2t}F_+) \subset \mathbb{R}^+ \times S^2$$

and thus $\widehat{e^{2t}F_+}$ is actually holomorphic, again with the values in $L^2(S^2)$ using (‡), in $Im(s) < a + \gamma$. This motivates us to make the following claim :

Claim 3.21

$$\int_{\mathbb{R}+ia} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}\widehat{e^{2t}F_+}ds = \sum_{a<k\in\mathbb{Z}<a+\gamma} C^+(\theta)e^{-kt} + R_+(t,\theta);$$
$$R_+(t,\theta) \in e^{-(a+\gamma)t}H^{m+2}(\mathbb{R} \times S^2)$$

□

Now, as for the remaining terms $r^2f_- - r^2\Delta_o u_-$, we can proceed in a very similar manner as it was for $r^2 f_+$. We first prove the following claim, which follows easily from **Claim 3.19**

Claim 3.22 *If $c \geq 3/2$, then*

$$3/2 \leq k \leq c \to x^c H_b^m(\mathbb{B}^3; d\mu) \subset r^{-k}H_b^m([0,1] \times S^2, \tau_b).$$

□

Thus, we see that

$$\begin{aligned} f_- - \Delta_o u_- &\in x^{a+2}H_b^m(\mathbb{B}^3) \subset r^{-2}H_b^m([0,1] \times S^2, \tau_b) \to \\ r^2 f_- - (r^2\Delta_o)u_- &\in H_b^m([0,1] \times S^2, \tau_b) \end{aligned}$$

and after the coordinate-change of $(r,\theta) \to (t,\theta)$

$$(\dagger')\quad e^{2t}F_- - (e^{2t}\Delta_o)U_- \in e^{-at}L^2(\mathbb{R}; H^s(S^2)),\ \text{supp}(e^{2t}F_- - (e^{2t}\Delta_o)U_-) \subset \{t \leq \log 2\} \times S^2$$
$$(\ddagger')\quad e^{2t}F_- - (e^{2t}\Delta_o)U_- \in e^{-at}H^s(\mathbb{R}; L^2(S^2)),\ \text{supp}((e^{2t}F_- - (e^{2t}\Delta_o)U_-) \subset \{t \leq \log 2\} \times S^2$$

Now to this, we can apply the following claim : (again of the Paley-Winer type but with different holomorphy region)

Claim 3.23 *Let $\mathcal{H}$ be some Hilbert space with a norm $\|\cdot\|_{\mathcal{H}}$ given by its inner product. Then, there is a following 1-1 correspondence :*

$$h \in H^m(\mathbb{R};\mathcal{H}) \quad and \quad supp(h) \subset \mathbb{R}^{\leq a} \quad for\ some \quad a \in \mathbb{R}$$

$$\Longleftrightarrow \begin{cases} (1)\ For\ 0 \leq k \leq m,\ (is)^k \hat{h}(s)\ is\ holomorphic\ on\ \{s \in \mathbb{C} \mid Im(s) > 0\}. \\ (2)\ |\hat{h}(s)| < \frac{\|h\|_{H^m}}{(1+|s|^m)\sqrt{2Im(s)}} \quad on \quad \{s \in \mathbb{C} \mid Im(s) > 0\} \\ (3)\ \forall \eta \geq 0, \widehat{h_\eta}(\sigma) := \hat{h}(\sigma + i\eta) \to \begin{cases} \widehat{h_\eta}(\sigma) \in H^m(\mathbb{R};\mathcal{H}) \\ \|\hat{h}_\eta\|_{H^m(\mathbb{R})} \leq \|\hat{h}\|_{H^m(\mathbb{R};\mathcal{H})} \\ (0,\infty) \ni \eta \mapsto \hat{h}_\eta \in H^m(\mathbb{R};\mathcal{H})\ continuously \end{cases} \end{cases}$$

□

to conclude that $\widehat{e^{2t}F_-} - (\widehat{e^{2t}\Delta_o})U$ is holomorphic for $Im(s) > 0$. We can again use the Cauchy's theorem to assert

Claim 3.24

$$\begin{aligned} \int_{\mathbb{R}+ia} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}(\widehat{e^{2t}F_-} + \widehat{e^{2t}U_-})ds &= \sum_{a<k\in\mathbb{Z}<a+\gamma} C^-(\theta)e^{-kt} + R_-(t,\theta); \\ & \quad R_-(t,\theta) \in e^{-(a+\gamma)t}H^{m+2}(\mathbb{R}\times S^2) \end{aligned}$$

□

Thus finally, we have

$$\begin{aligned} U_+(t,\theta) &= \int_{\mathbb{R}+ia} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}(\widehat{e^{2t}F_+} + \widehat{e^{2t}F_-} - (\widehat{e^{2t}\Delta_o})U_-)(s,\theta)ds \\ &= \int_{\mathbb{R}+ia} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}\widehat{e^{2t}F_+}(s,\theta)ds + \\ & \quad \int_{\mathbb{R}+ia} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}(\widehat{e^{2t}F_-} - (\widehat{e^{2t}\Delta_o})U_-)(s,\theta)ds \\ &= \sum_{a<k\in\mathbb{Z}<a+\gamma} C_j^+(\theta)e^{-kt} + R_+(t,\theta) + \sum_{a<k\in\mathbb{Z}<a+\gamma} C_j^-(\theta)e^{-kt} + R_-(t,\theta) \\ &= \sum_{a<k\in\mathbb{Z}<a+\gamma} C_j(\theta)e^{-kt} + R(t,\theta) \end{aligned}$$

and going back to the variable $r = e^t$, we have

$$u_+(r,\theta) = \sum_{a<k\in\mathbb{Z}<a+\gamma} C_j(\theta)r^{-j} + \widetilde{R}(r,\theta)$$

Note that the RHS, though seemingly singular at the origin because of r^{-j} terms, does vanish near at the origin, since u_+ does. Therefore, we can finally conclude

$$u = \sum_{1\leq j\in\mathbb{Z}<a+\gamma} C_j(\theta)x^j + x^{a+\gamma}H_b^{s+2}(\mathbb{B}^3)$$

This ends the proof of **Lemma 3.17**. □

Now with this lemma, the proof of **Theorem 3.16** follows almost immediately. Here is how.

Proof : Since $f \in x^{a+2}H_b^\infty(\mathbb{B}^3) \cap A_{\mathcal{E}}^\infty(\mathbb{B}^3)$, we can first write :

$$(\forall p > 0) \ : \ f(r,\theta) = h(r) \sum_{(q,k)\in\mathcal{E}}^{q<p} c_{q,k}(\theta) r^{-q}(\log r)^k + f_p^o(r,\theta)$$

where $f_p^o(r,\theta) \in x^p H_b^\infty(\mathbb{B}^3)$. Here $h(r) \in C^\infty(\mathbb{R})$ is a cutoff function such that

$$h(r) \equiv 0, \text{for } r < 1; \ h(r) \equiv 1, \ \text{for } r > 2$$

Again writing $u = \phi u + (1-\phi)u = u_+ + u_-$, we have

$$u_+ \in r^{-a}H_b^\infty([0,1]\times S^2) \to U_+(t,\theta) \in e^{-at}H^\infty(\mathbb{R}\times S^2)$$

and taking $p \geq a+2$ and using **Claim 3.22**

$$\begin{aligned} (-r^2\Delta_o)u_- + r^2 f_p^o &\in r^{-(p-2)}H_b^\infty([0,1]\times S^2) \Rightarrow \\ (e^{-2t}\Delta_o)U_- + e^{2t}F_p^o &\in e^{-(p-2)t}H^\infty(\mathbb{R}\times S^2) \end{aligned}$$

and

$$h(r) \to H(t); \ \text{supp}(H) \in \mathbb{R}^+$$

We then write

$$\begin{aligned} U_+(t,\theta) &= \int_{\mathbb{R}+ia} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}\big(\sum_{(q,k)\in\mathcal{E}}^{q\leq p} (c_{s,k}(\theta)\widehat{H(t)e^{-(q-2)t}t^k})(s,\theta)\big)ds + \\ &\int_{\mathbb{R}+ia} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}((\widehat{e^{2t}\Delta)U^-} + (\widehat{e^{2t}\Delta)F_p^o})(s,\theta)ds \end{aligned}$$

To the second term of the above, we can apply the approaches in the proof of **Lemma 3.17** to see that

$$\begin{aligned} &\int_{\mathbb{R}+ia} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}((\widehat{e^{2t}\Delta)U^-} + (\widehat{e^{2t}\Delta)F_p^o})(s,\theta)ds \\ &= \sum_{1\leq j\in\mathbb{Z}<p-2} C_j(\theta)e^{-jt} + R_p(t,\theta); \ \ R_p(t,\theta) \in e^{-(p-2)t}H^\infty(\mathbb{R}\times S^2) \end{aligned}$$

As for the first term, we first make the following claim :

Claim 3.25 $\widehat{He^{s_o t}t^l}(s)$ *is meromorphic in* $\mathbb{C}$ *with a pole of order* l *at* $s = -is_o$ *such that for all* $k, N \in \mathbb{N}$, $\exists\ C_{k,N}$ *such that*

$$|(\frac{d}{ds})^k((s+is_o)\widehat{He^{s_o t}t^l}(s))| < C_{k,N}e^{Im(s)}(1+|s|)^N$$

□

Thus, $H(t)c_{s_o,k}\widehat{(\theta)e^{-(q-2)t}t^k}(s)$ is a Schwartz function on each horizontal line away from its poles, which implies

$$\lim_{A\to\infty}\int_{C_A+C_{-A}} e^{ist}((is)^2+(is)+\Delta_\theta)^{-1}(\sum_{(q,k)\in\mathcal{E}}^{q\leq p} H(t)c_{q,k}\widehat{(\theta)e^{-(q-2)t}t^k})(s,\theta)ds = 0$$

and

$$\int_{\mathbb{R}+i(p-2)} e^{ist}((is)^2+(is)+\Delta_\theta)^{-1}(\sum_{(q,k)\in\mathcal{E}}^{q\leq p} H(t)c_{q,k}\widehat{(\theta)e^{-(q-2)t}t^k})(s,\theta)ds$$
$$\in e^{-(p-2)t}S(\mathbb{R};H^\infty(S^2))$$

Thus, using Cauchy's theorem, we conclude that

$$\int_{\mathbb{R}+ia} e^{ist}((is)^2+(is)+\Delta_\theta)^{-1}(\sum_{(q,k)\in\mathcal{E}}^{q<p} H(t)c_{q,k}\widehat{(\theta)e^{-(q-2)t}t^k})(s,\theta)ds$$

$$= \sum_{z\in\{poles\}}^{a<Im(z)<p-2} \text{Residue at } z + R_p'(t,\theta);\ \ R_p'(t,\theta)\in e^{-(p-2)t}S(\mathbb{R};H^\infty(S^2))$$

Now, we only have to say what the poles look like between $Im(s)=a$ and $Im(s)=p-2$. From **Lemma 3.6**, we see that

$$\{\text{poles of } ((is)^2+(is)+\Delta_\theta)^{-1}\} = \{\text{simple poles along } i\mathbb{Z}\}$$

and from **Claim 3.25**

$\{\text{poles of } (\sum_{(q,k)\in\mathcal{E}}^{q\leq p} H(t)c_{q,k}\widehat{(\theta)e^{-(q-2)t}t^k})(s,\theta)\} = \{\text{poles of order } k \text{ at } -i(q-2),\ q\leq p\}$

Therefore, if we denote a pole at $is\in\mathbb{C}$ of order l_s by (s,l_s), we see that

$$\begin{aligned}(is,l_s) \quad &\in \quad \{\text{ poles of } \ ((is)^2+(is)+\Delta_\theta)^{-1}(\sum_{(q,k)\in\mathcal{E}}^{q\leq p} H(t)c_{q,k}\widehat{(\theta)e^{-(q-2)t}t^k})(s,\theta)\}\\ &\Leftrightarrow \quad s<p-2 \ \text{ and } \ (s,l)\in\mathcal{E}'\end{aligned}$$

where we define

$$\mathcal{E}' := \{(q',k') \mid \left\{\begin{array}{lll} q' & \in & \mathbb{N}\cup\{q-2\mid(q,k)\in\mathcal{E}\} \\ k' & = & \left\{\begin{array}{ll} 1 & \text{if } \ q'\in\mathbb{N}-\{q-2\mid(q,k)\in\mathcal{E}\} \\ k & \text{if } \ q'\in\{q-2\mid(q,k)\in\mathcal{E}\}\text{-}\mathbb{N} \\ k,k+1 & \text{if } \ q'\in\mathbb{N}\cap\{q-2\mid(q,k)\in\mathcal{E}\}\end{array}\right\}\end{array}\right\}\}$$

We now compute the residue at these poles, and finally conclude that

$$\int_{\mathbb{R}+ia} e^{ist}((is)^2+(is)+\Delta_\theta)^{-1}(\sum_{(q,k)\in\mathcal{E}}^{q\leq p} H(t)c_{q,k}\widehat{(\theta)e^{-(q-2)t}t^k})(s,\theta)ds$$

$$= \sum_{(q',k')\in\mathcal{E}'}^{q'\leq p-2} c'_{q',k'}(\theta)e^{-q't}t^{k'} + R'_p(t,\theta); \quad R'_p(t,\theta)e^{-(p-2)t}S(\mathbb{R}; H^\infty(S^2))$$

Thus finally

$$\begin{aligned} U_+(t,\theta) &= \int_{\mathbb{R}+ia} e^{ist}((is)^2+(is)+\Delta_\theta)^{-1}(\sum_{(q,k)\in\mathcal{E}}^{q\leq p} H(t)c_{q,k}\widehat{(\theta)e^{-(q-2)t}t^k})ds + \\ & \int_{\mathbb{R}+ia} e^{ist}((is)^2+(is)+\Delta_\theta)^{-1}((\widehat{e^{2t}\Delta)U^-} + (\widehat{e^{2t}\Delta)F_o})(s,\theta)ds \\ &= \sum_{(q',k')\in\mathcal{E}'}^{q'<p-2} c'_{q',k'}(\theta)e^{-q't}t^{k'} + R'_p(t,\theta) + \sum_{1\leq j\in\mathbb{Z}<p-2} c_j(\theta)e^{-jt} + R_p(t,\theta) \end{aligned}$$

Since $\mathcal{E}$ is an index set, it is not hard to see that the smallest index $\mathcal{I}_\mathcal{E}$ set containing $E'\cup(\mathbb{N}\times\{0\})$ is

$$\mathcal{I}_\mathcal{E} = (\mathcal{E}-2)\overline{\cup}\mathbb{N} \overset{defined}{=} (\mathcal{E}-(2,0))\cup(\mathbb{N}\times 0)\cup\{(s,l) \mid s\in(\mathcal{E}-(2,0)) \text{ and } s\in\mathbb{N}\}$$

Thus, combining $c(\theta)$ and $c'(\theta)$, and R_p and R'_p and switching back to the variable $r=e^t$, we have for all $p>a+2$:

$$u_+(r,\theta) = \sum_{(\tilde{q},\tilde{k})\in\mathcal{I}_\mathcal{E}}^{\tilde{q}<p-2} C_{\tilde{q},\tilde{k}}(\theta)r^{-\tilde{q}}(logr)^{\tilde{k}} + \tilde{R}_{p-2}(r,\theta); \quad \tilde{R}_p(t,\theta)r^{-(p-2)}H_b^\infty([0,1]\times S^2)$$

which, finally, implies

$$(\forall p>a): u = \sum_{(\tilde{q},\tilde{k})\in\mathcal{I}_\mathcal{E}}^{\tilde{q}<p} C_{\tilde{q},\tilde{k}}(\theta)x^{\tilde{q}}(logx)^{\tilde{k}} + \tilde{R}_p(x,\theta); \quad \tilde{R}_p(x,\theta)x^pH_b^\infty(\mathbb{B}^3)$$

This finishes, **finally** the proof of **Theorem 3.16** □

Appendix : Proofs of Claims

Proof of Claim 3.19

Proof : Recall that on (interior of $\mathbb{B}^3$) $\cong\mathbb{R}^3$ we have

$$d\mu = (\frac{1}{\phi(r)+r^3})\ r^2drd\theta$$

where ϕ is a cutoff function supported near 0. Also recall that on $\mathbb{R}^+\times S^2$, we have

$$\tau_b = \frac{dr}{r}d\theta$$

For $f \in x^c H_b^m(\mathbb{B}^3; d\mu)$, let us write

$$f = \phi f + (1-\phi) f = f_+ + f_-$$

Since we have, as $r \to \infty$, $d\mu = \tau_b$ and $x = 1/r$, we then see that

$$(1-\phi)\cdot x^c H_b^m(\mathbb{B}^3; d\mu) = (1-\phi)\cdot r^{-c} H_b^m([0,1]\times S^2, \tau_b)$$

So the question boils down to :

$$\text{Is } \ \phi\cdot x^c H_b^m(\mathbb{B}^3; d\mu) \subset \phi\cdot r^{-c} H_b^m([0,1]\times S^2, \tau_b) \ ?$$

Let B be a ball in $\mathbb{R}^3$ and $\text{supp}(\phi) \subset B$. Since x is smooth and nonvanishing on B,

$$f_+ \in \phi\cdot x^c H_b^m(\mathbb{B}^3; d\mu) \Leftrightarrow \sum_{0\le i+j+k\le m} \int_B |\partial_{x^1}^i \partial_{x^2}^j \partial_{x^3}^k f_+|^2 r^2 dr d\theta$$

Now let us first look when $i = j = k = 0$. Since $c \geq 3/2$, we have

$$\int_B |f_+|^2 r^2 dr d\theta < \infty \to \int_B |r^c f_+|^2 \frac{dr}{r} d\theta < \infty$$

because

$$\begin{aligned} \int_B |r^c f_+|^2 \frac{dr}{r} d\theta &= \int_B |f_+|^2 r^{2c-3} r^2 dr d\theta \\ &\leq (\text{radius of B})^{2c-3}\cdot \int_B |f_+|^2 r^2 dr d\theta \end{aligned}$$

Now for the case when $i+j+k = 1$, it helps to first recall that we can write for $1 \leq j \leq 3$

$$\partial_{x_j} = c_1^j(\theta)\partial_r + c_2^j(\theta)\frac{1}{r}\partial_\theta$$

Thus we see that

$$\int_B |\partial_{x^i} f_+|^2 r^2 dr d\theta < \infty \Leftrightarrow \begin{cases} \int_B |\partial_r f_+|^2 r^2 dr d\theta & < \infty \\ \int_B |\partial_\theta f_+|^2 dr d\theta & < \infty \end{cases}$$

which implies that

$$\begin{aligned} \int_B |r\partial_r(r^c f_+)|^2 \frac{dr}{r} d\theta &= \int_B |r^c(cf_+ + r\partial_r f_+)|^2 \frac{dr}{r} d\theta \\ &= \int_B c^2 |f_+|^2 r^{2c-3} r^2 dr d\theta + \int_B |\partial_r f_+|^2 r^{2c-1} r^2 dr d\theta \\ &< c^2 (\text{radius of B})^{2c-3} \int_B |f_+|^2 r^2 dr d\theta \ + \\ &\quad (\text{radius of B})^{2c-1} \int_B |\partial_r f_+|^2 r^2 dr d\theta \\ &< \infty \end{aligned}$$

and

$$\begin{aligned}\int_B |\partial_\theta (r^c f_+)|^2 \frac{dr}{r} d\theta &= \int_B |\partial_\theta f_+|^2 r^{2c-1} dr d\theta \\ &< (\text{radius of B})^{2c-1} \int_B |\partial_\theta f_+|^2 r^{2c-1} dr d\theta \\ &< \infty\end{aligned}$$

All the remaining cases for the higher derivatives follow the same pattern as above and we can thus conclude that

$$f_+ \in \phi \cdot x^c H_b^m(\mathbb{B}^3; d\mu) \to f_+ \in \phi \cdot r^{-c} H_b^m([0,1] \times S^2; \tau_b)$$

and putting this with what we discussed about about f_- we finally conclude that if $c > 3/2$

$$f \in x^c H_b^m(\mathbb{B}^3; d\mu) \to f \in r^{-c} H_b^m([0,1] \times S^2; \tau_b)$$

□

Proof of Claim 3.20

Proof : This claim can be easily deduced from the following subclaim of the scalar case :

SubClaim 3.26 *There is a following 1-1 correspondence :*

$$h \in L^2(\mathbb{R}) \quad and \quad supp(h) \subset \mathbb{R}^+$$

$$\Longleftrightarrow \left\{ \begin{array}{l} (1)\ \hat{h}(s) \ \text{ is holomorphic on } \ \{s \in \mathbb{C} \mid Im(s) < 0\} \\ (2)\ |\hat{h}(s)| < \frac{\|h\|_{L^2}}{\sqrt{2Im(s)}} \ \text{ on } \ \{s \in \mathbb{C} \mid Im(s) < 0\} \\ (3)\ \forall \eta \leq 0, \hat{h_\eta}(\sigma) := \hat{h}(\sigma + i\eta) \to \left\{ \begin{array}{l} \hat{h_\eta}(\sigma) \in L^2(\mathbb{R}) \\ \|\hat{h_\eta}\|_{L^2(\mathbb{R})} \leq \|\hat{h}\|_{L^2(\mathbb{R})} \\ (-\infty, 0] \ni \eta \mapsto \hat{h_\eta} \in L^2(\mathbb{R}) \ \text{continuously} \end{array} \right. \end{array} \right.$$

Proof : ($\Rightarrow$)

We first start with the definition, with $h(t) \in L^2(\mathbb{R})$

$$L^2(\mathbb{R}) \ni h(t) \leftrightarrow h(s) = \int_{\mathbb{R}} e^{-ist} h(t) dt \in L^2(\mathbb{R})$$

which is *a priori* defined for $s \in \mathbb{R}$. Now, we can try to extend this definition for $s = \sigma + i\eta \in \mathbb{C}$

$$\begin{aligned}\hat{h}(s) &= \int_{\mathbb{R}} e^{-ist} h(t) dt \\ &= \int_{\mathbb{R}} e^{-i\sigma t} e^{\eta t} h(t) dt\end{aligned}$$

Now, if $\eta < 0$, we see that

$$\forall t \in \mathbb{R} \ : \ |e^{\eta t} h(t)| < |h(t)|$$

because supp$(h) \in \mathbb{R}^+$. Thus

$$\hat{h_\eta}(\sigma) = \hat{h}(\sigma + i\eta) = \int_{\mathbb{R}} e^{-i\sigma t} e^{\eta t} h(t) dt$$

is in $L^2(\mathbb{R})$ and it is also easy to show that

$$||\hat{h_\eta}||_{L^2} \leq ||\hat{h}||_{L^2} = ||h||_{L^2}$$

Moreover, since

$$(-\infty, 0] \ni \eta \mapsto e^{-\eta t} h \in L^2(\mathbb{R})$$

is continuous

$$(-\infty, 0] \ni \eta \mapsto \hat{h_\eta} \in L^2(\mathbb{R})$$

is also continuous since the Fourier transform maps $L^2(\mathbb{R}_t)$ to $L^2(\mathbb{R}_s)$ continuously.

We now note that, for $\eta < 0$, $h(s)$ is actually well-defined for all $\{s \in \mathbb{C} \mid Im(s) < 0\}$ because

$$\begin{aligned}
|h(s)| &= |\int_{\mathbb{R}} e^{\eta t} e^{-i\sigma t} h(t) dt| \\
&< (\int_{\mathbb{R}} e^{2\eta t} dt)^{\frac{1}{2}} (\int_{\mathbb{R}} |h|^2 dt)^{\frac{1}{2}} \\
&< (\int_0^\infty e^{2\eta t} dt)^{\frac{1}{2}} (\int_0^\infty |h|^2 dt)^{\frac{1}{2}} \quad (supp(h) \subset R^+) \\
&< \frac{||h||_{L^2}}{\sqrt{-2\eta}} = \frac{||h||_{L^2}}{\sqrt{2|Im(s)|}}
\end{aligned}$$

Since a similar estimate can be found for $(\frac{d}{ds})^k h(s)$ as

$$\begin{aligned}
|(\frac{d}{ds})^k h(s)| &= |\int_{\mathbb{R}} (\frac{d}{ds})^k e^{-ist} h(t) dt| \\
&= |\int_{\mathbb{R}} (it)^k e^{-ist} h(t) dt| \\
&< (\int_{\mathbb{R}} |(it)^k e^{\eta t}|^2 dt)^{\frac{1}{2}} (\int_{\mathbb{R}} |h|^2 dt)^{\frac{1}{2}} \\
&< (\int_0^\infty |(it)^k e^{\eta t}|^2 dt)^{\frac{1}{2}} (\int_0^\infty |h|^2 dt)^{\frac{1}{2}} \\
&< C_{k,\eta} ||h||_{L^2}
\end{aligned}$$

where

$$C_{k,\eta} := (\int_0^\infty |t^k e^{\eta t}|^2 dt)^{\frac{1}{2}}$$

is a constant depending only on η and k that is well defined on $\eta < 0$ and blows up at $\eta = 0$. Thus we conclude that h is in fact holomorphic on $Im(s) < 0$.

($\Leftarrow$)

Since

$$(-\infty, 0] \ni \eta \to \hat{f}_\eta \in L^2(\mathbb{R})$$

is continuous, if we define :

$$\hat{f}(\sigma) = \lim_{\eta \to 0} \hat{f}_\eta(\sigma) = \lim_{\eta \to 0} f(\sigma - i\eta)$$

then $\hat{f}(\sigma)$ is also in $L^2(\mathbb{R})$. Then, simply define $f(t)$ to be the inverse Fourier transform of $\hat{f}(s)$.

Now it remains to show that $supp(f(t)) \subset \mathbb{R}^+$. Actually, we will equivalently show that for any small $\epsilon > 0$, $\tilde{f} = e^{-\epsilon t} f$ will be supported in $\mathbb{R}^+$. Then

$$\begin{aligned}
\widehat{\tilde{f}}(s) &= \int_{\mathbb{R}} e^{-ist} e^{-\epsilon t} f(t) dt \\
&= \int_{\mathbb{R}} e^{-i(s - i\epsilon)t} f(t) dt \\
&= \hat{f}(s - i\epsilon)
\end{aligned}$$

The assumptions on $\widehat{f}$, then, imply the following on $\widehat{\tilde{f}}$

$$\begin{cases}
\widehat{\tilde{f}}(s) \text{ is holomorphic on } Im(s) < i\epsilon \\
|\widehat{\tilde{f}}(s)| < \frac{\|f\|_{L^2}}{\sqrt{2(Im(s) - \epsilon)}} \\
\widehat{\tilde{f}}(s) \text{ is } L^2 \text{ on each horizontal line } Im(s) = \eta \text{ and} \\
\forall \eta \leq \epsilon, \ \|\widehat{\tilde{f}}(\sigma + i\eta)\|_{L^2} \leq \|f\|_{L^2}
\end{cases}$$

Now let $\phi \in C_c^\infty(\mathbb{R})$ be a test function such that

$$supp(\phi) \subset (-b, -a) \subset R^-$$

Then, by the Paley-Winer theorem (see [?]). ϕ is an entire function. Now we have

$$\begin{aligned}
\int_{\mathbb{R}} \tilde{f}(t)\phi(t) dt &= \int_{\mathbb{R}} \tilde{f}(t)\overline{\phi(t)} dt \quad \text{(because } \forall t \in \mathbb{R}, \phi(t) \in \mathbb{R}) \\
&= \int_{\mathbb{R}} \widehat{\tilde{f}}(s)\overline{\hat{\phi}(s)} ds \quad \text{(Plancheral's formula)} \\
&= \int_{\mathbb{R}} \widehat{\tilde{f}}(s)\overline{\hat{\phi}(\overline{s})} ds \ (s \in \mathbb{R} \text{ and } \overline{\hat{\phi}(\overline{s})} \text{ is holomorphic since } \hat{\phi} \text{ is.)} \\
&= \int_{\mathbb{R} - i\eta} \widehat{\tilde{f}}(s)\overline{\hat{\phi}(\overline{s})} ds \ + \\
&\quad \lim_{A \to \infty} \left(\int_{C_{\eta, A}} \hat{\tilde{f}}(s)\overline{\hat{\phi}(\overline{s})} ds + \int_{C_{\eta, -A}} \hat{\tilde{f}}(s)\overline{\hat{\phi}(\overline{s})} ds \right)
\end{aligned}$$

where the last equality is justified, for any aribitrary $\eta > 0$, by the Cauchy's theorem since $\widehat{f}(s)\overline{\widehat{\phi}(\overline{s})}$ is holomorphic on $\{s \in \mathbb{C} \mid Im(s) < i\epsilon\}$. Here, $C_{\eta,A}$ is a segment given by

$$t \in [0,1] \mapsto t(A - i0) + (1-t)(A - i\eta)$$

We first assert that

$$\lim_{A\to\infty} (\int_{C_{\eta,A}} \hat{\tilde{f}}(s)\overline{\hat{\phi}(\overline{s})}ds + \int_{C_{\eta,-A}} \hat{\tilde{f}}(s)\overline{\hat{\phi}(\overline{s})}ds) = 0$$

We see that

$$\begin{aligned}
|\widehat{\phi}(s)| &= |\int_{\mathbb{R}} e^{-ist}\phi(t)dt| \\
&= |\int_{\mathbb{R}} e^{Im(s)t}e^{-\imath Re(s)t}\phi(t)dt| \\
&< e^{-aIm(s)} \int_{\mathbb{R}} |\phi'(t)|dt \quad (\text{since } supp(\phi) \in (-b,-a) \subset \mathbb{R}^-) \\
&< C_\phi \cdot e^{-aIm(s)} \text{ (where } C_\phi \text{ is a constant depending on } \phi)
\end{aligned}$$

and moreover

$$\begin{aligned}
|s\widehat{\phi}(s)| &= |\int_{\mathbb{R}} e^{-ist}\frac{d}{dt}\phi(t)dt| \\
&= |\int_{\mathbb{R}} e^{Im(s)t}e^{-iRe(s)t}|\frac{d}{dt}\phi(t)dt| \\
&< e^{-aIm(s)} \int_{\mathbb{R}} |\phi(t)|dt \\
&< C_{\phi'} \cdot e^{-aIm(s)} \text{ (where } C'_\phi \text{ is a constant depending on } \phi')
\end{aligned}$$

Thus in fact

$$|\widehat{\phi}(s)| < \frac{C_{\phi,\phi'}}{1+|s|} \cdot e^{-aIm(s)}$$

Then,

$$\begin{aligned}
|\int_{C_{\eta,A}} \hat{\tilde{f}}(s)\overline{\hat{\phi}(\overline{s})}ds| &< \frac{1}{1+A}\int_0^\eta (C_{\phi,\phi'}e^{-aq})(\frac{\|f\|_{L^2}}{\sqrt{2(q-\epsilon)}})dq \\
&< \frac{1}{1+A} \cdot \tilde{C}_{\phi,\phi'} \ (\tilde{C}_{\phi,\phi'} \text{ independent of } \eta)
\end{aligned}$$

Therefore,

$$\lim_{A\to\infty} \int_{C_{\eta,A}} \hat{\tilde{f}}(s)\overline{\hat{\phi}(\overline{s})}ds = 0$$

Similarly,

$$\lim_{A\to\infty} \int_{C_{\eta,-A}} \hat{\tilde{f}}(s)\overline{\hat{\phi}(\overline{s})}ds = 0$$

Thus we see that $\forall\eta > 0$, we have

$$\int_{\mathbb{R}} \tilde{f}(t)\phi(t)dt = \int_{\mathbb{R}-i\eta} \hat{\tilde{f}}(s)\overline{\hat{\phi}(\overline{s})}ds$$

Now

$$\begin{aligned} |\int_{\mathbb{R}-i\eta} \hat{\tilde{f}}(s)\overline{\hat{\phi}(\overline{s})}ds| &= |\int_{\mathbb{R}} \hat{\tilde{f}}(\sigma - i\eta)\overline{\hat{\phi}}(\sigma + \imath\eta)d\sigma| \\ &< (\frac{C_{\phi,\phi'}e^{-a\eta}}{1+|\eta|})(\frac{\|f\|_{L^2}}{\sqrt{2(\eta-\epsilon)}}) \end{aligned}$$

Thus, for any arbitrary $\eta > 0$

$$\begin{aligned} |\int_{\mathbb{R}} \tilde{f}(t)\phi(t)dt| &< (\frac{C_{\phi,\phi'}e^{-a\eta}}{1+|\eta|})(\frac{\|f\|_{L^2}}{\sqrt{2(\eta-\epsilon)}}) \\ &\to 0 \text{ as } \eta \to \infty \end{aligned}$$

Thus $\int_{\mathbb{R}} f(t)\phi(t)dt = 0$ for any test function ϕ that is supported in $\mathbb{R}^-$, and this implies that $supp(f) \subset \mathbb{R}^+$. □
□

Proof of Claim 3.21

Proof : We start with the following subclaim :

SubClaim 3.27 *For any $t \in \mathbb{R}$, we have*

$$\int_{\mathbb{R}+ia} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}\widehat{e^{2t}F_+}ds =$$

$$\sum_{a<k\in\mathbb{Z}<a+\gamma} \textit{Residue at } k + \int_{\mathbb{R}+i(a+(\gamma-\epsilon))} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}\widehat{e^{2t}F_+}ds$$

for any small $\epsilon > 0$.

Proof : We use the following conclusions of the **Lemma3.6** :

1. $((is)^2 + (is) + \Delta_\theta)^{-1} = A(s) + \sum_{k\in Z} B_k/(z - ik)$, where $A(s)$ is holomorphic with values in $\Psi^{-2}(S^2)$, thus holomorphic in $\Psi^0(S^2)$, and B_k is a projection operator to the eigenspace with the eigenvalue $k(k+1)$.
2. $\|((is)^2 + (is) + \Delta_\theta)^{-1}\|_{L^2(S^2)\to L^2(S^2)} < C/(1+|s|^2)$ as $|s| \to \infty$ away from $i\mathbb{Z}$.
3. $\|((is)^2 + (is) + \Delta_\theta)^{-1}\|_{L^2(S^2)\to H^2(S^2)} < C$ away from $i\mathbb{Z}$.

By (1), we can use the Cauchy integral formula to get, for any small $\epsilon > 0$:

$$\int_{\mathbb{R}+ia} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}\widehat{e^{2t}F_+}ds$$

$$\begin{aligned}
= & \sum_{a<k\in\mathbb{Z}<a+\gamma} \text{Residue at } k + \\
& \int_{\mathbb{R}+i(a+(\gamma-\epsilon))} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}\widehat{e^{2t}F_+}ds + \\
& \lim_{A\to\infty} \int_{C_{-A}+C_A} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}\widehat{e^{2t}F_+}ds
\end{aligned}$$

where C_{-A} is a contour from $-A + i(a+\gamma)$ to $-A + ia$ and C_A is a contour from $A + ia$ to $A + i(a+\gamma)$.

Remark 3.28 *We cannot make the $Im(s) = a+\gamma$ a part of the contour, quite yet, because the integrand is holomorphic only for $Im(s)$ strictly less than $a+\gamma$.*

Now we must show that

$$\lim_{A\to\infty} \int_{C_A+C_{-A}} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}\widehat{e^{2t}F_+}ds = 0$$

This follows because of (2). For $Im(s) < a+\gamma$, we have :

$$||e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}\widehat{e^{2t}F_+}||_{L^2(S^2)}$$

$$\begin{aligned}
= & \; e^{-Im(s)t} \cdot ||((is)^2 + (is) + \Delta_\theta)^{-1}||_{L^2(S^2)\to L^2(S^2)} \cdot ||\widehat{e^{2t}F_+}||_{L^2(S^2)} \\
< & \; e^{-Im(s)t} \cdot \frac{C}{(1+|s|^2)} \cdot \frac{||e^{-(a+\gamma)t}e^{2t}F_+||_{H^m(\mathbb{R};L^2(S^2))}}{(1+|s|^m)} \\
\to & \; 0 \text{ as } |Re(s)| \to \infty
\end{aligned}$$

□

Now, we can easily compute the residues as follows :

$$\sum Res_{a<k\in\mathbb{Z}<a+\gamma} = \sum_{k\in\mathbb{Z},\ a<k<a+\gamma} e^{-kt}C_k^+(\theta);$$

Thus, by the above subclaim, we have, for any $t \in \mathbb{R}$,

$$\int_{\mathbb{R}+ia} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}\widehat{e^{2t}F_+}ds =$$

$$\sum_{a<k\in\mathbb{Z}<a+\gamma} \text{Residue at } k + \int_{\mathbb{R}+i(a+(\gamma-\epsilon))} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}\widehat{e^{2t}F_+}ds$$

where $C_k^+(\theta)$ in the eigen-space of Δ_θ with the eigenvalue $k(k+1)$. Now we note here that the LHS and the residue term does **NOT** depend on ϵ **!!!** Since the above were true for any small $\epsilon > 0$, we can thus take the limit of the last term as $\epsilon \to 0$. More precisely, we write :

$$\int_{\mathbb{R}+i(a+(\gamma-\epsilon))} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}\widehat{e^{2t}F_+}ds = e^{-(a+\gamma-\epsilon)t}\int_{\mathbb{R}} e^{i\sigma t}\Phi_\epsilon(s)d\sigma$$

where we define :

$$\Phi_\epsilon(s) := ((i(\sigma+i(a+\gamma-\epsilon))^2 + i(\sigma+i(a+\gamma-\epsilon)) + \Delta_\theta)^{-1}\widehat{e^{2t}F_+}(\sigma+i(a+\gamma-\epsilon))$$

then

$$[0,1) \ni \epsilon \mapsto \Phi_\epsilon(s) \in L^2(\mathbb{R}; L^2(S^2))$$

is continuous because

$$\mathbb{C} - i\mathbb{Z} \ni s \mapsto ((is)^2 + (is) + \Delta_\theta)^{-1} \in B(L^2(S^2); L^2(S^2))$$

and

$$(-\infty, a+\gamma] \ni \eta \to \widehat{e^{2t}F_+}(\sigma+\eta)$$

are continuous. Now since the inverse Fourier transform is continuous on $L^2(\mathbb{R}; L^2(S^2))$, we finally conclude that

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\mathbb{R}+i(a+(\gamma-\epsilon))} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}\widehat{e^{2t}F_+}ds \\
&= e^{-(a+\gamma)t}\int_{\mathbb{R}} e^{i\sigma t}\Phi_0(s)d\sigma \\
&= \int_{\mathbb{R}+i(a+\gamma)} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}\widehat{e^{2t}F_+}ds \\
&\in e^{-(a+\gamma)t}H^{m+2}(\mathbb{R}; L^2(S^2))
\end{aligned}$$

which is in $e^{-(a+\gamma)t}H^{m+2}(\mathbb{R}; S^2)$. Once we established this, we use the fact that

$$||((is)^2 + (is) + \Delta_\theta)^{-1}||_{L^2(S^2)\to H^2(S^2)} < C \quad \text{(a constant independent of } s)$$

to conclude that indeed

$$\int_{\mathbb{R}+i(a+\gamma)} e^{ist}((is)^2 + (is) + \Delta_\theta)^{-1}\widehat{e^{2t}F_+}ds \in e^{-(a+\gamma)t}H^{m+2}(\mathbb{R}\times S^2)$$

□

Proof of Claim 3.22

Proof : We have for $3/2 \le k \le c$,

$$(1-\phi)\cdot r^{-k} H_b^m([0,1]\times S^2, \tau_b) \subset (1-\phi)\cdot r^{-c} H_b^m([0,1]\times S^2, \tau_b)$$

For $k \ge 3/2$, we also have

$$\phi \cdot H_b^m(\mathbb{B}^3, \tau_b) \subset \phi \cdot r^{-c} H_b^m([0,1]\times S^2, \tau_b)$$

□

Proof of Claim 3.23

Proof : This claim can also be easily deduced from the following **SubClaim 3.26.** □

Proof of Claim 3.24

Proof : Using **Claim3.22** and **Claim 3.23**, instead of **Claim3.19** and **Claim 3.20**, we can imitate the proof of **Claim 3.21** □

Proof of Claim 3.25

Proof : In order to see this, we first make the following two subclaims, all of which are simple applications of the Paley-Winer theorem.

SubClaim 3.29 *$\widehat{H}(s)$ is meromorphic in $\mathbb{C}$ with a simple pole at $s = 0$ such that for all $k, N \in \mathbb{N}$, $\exists\ C_{k,N}$ such that*

$$|(\frac{d}{ds})^k(s\widehat{H}(s))| < C_{k,N} e^{Im(s)}(1+|s|)^N$$

Proof : We first see that

$$\widehat{\frac{d}{dt}H} = s\widehat{H(s)}$$

Since $\frac{d}{dt}H(t) \in C_o^\infty(\mathbb{R})$, Paley-Winer theorem asserts that $\widehat{\frac{d}{dt}H}(s)$ is an entire function with

$$|\partial_s^k(\widehat{\frac{d}{dt}H})| < C_{k,N} e^{Im(s)}(1+|s|)^N$$

for all $k, N \in \mathbb{N}$. □

SubClaim 3.30 *$\widehat{He^{s_o t}}(s)$ is meromorphic in $\mathbb{C}$ with a simple pole at $s = -is_o$ such that for all $k, N \in \mathbb{N}$, $\exists\ C_{k,N}$ such that*

$$|(\frac{d}{ds})^k((s+is_o)\widehat{He^{s_o t}}(s))| < C_{k,N} e^{Im(s)}(1+|s|)^N$$

Proof : Simply, we note

$$\widehat{He^{s_o t}}(s) = \widehat{H}(s + is_o)$$

□

Now to prove our claim, we simply, we note

$$\begin{aligned} \widehat{He^{s_o t}t^l}(s) &= (-i)^l (\frac{d}{ds})^l \widehat{e^{s_o t}H}(s) \\ &= (\frac{d}{ds})^l \widehat{H}(s + is_o) \end{aligned}$$

and use the above subclaims. □

Chapter 4

Asymptotic Behaviors of Solutions to the Laplacian on the Asymptotically Flat $\mathbb{R}^3$

In the previous section, we investigated how the behavior, as $r \to \infty$, of a solution u to

$$\Delta_o u = h$$

depends on the behavior of h near infinity of $\mathbb{R}^3$. In this section, we generalize the results of the previous section to the case when g is an asymptotically flat metric on $\mathbb{R}^3$. We will see that the asymptotic behaviors of u, in this case, depends not only on that of the function f but also on the asymptotic behavior of the metric g. However, we will see that on the asymptotically flat $\mathbb{R}^3$, Δ_o will eventually dominate, near the infinity of $\mathbb{R}^3$, the error terms that arise from the non-flatness of the metric. As we will see, we can recursively apply what we know of Δ_o, to iterate away errors that arise from non-flatness and in so doing, we can understand the asymptotic behavior of u.

In order to see this, as in the case of the previous section, we first need to understand the mapping properties of Δ_g. In this direction, Christodoulou and O'Munchadra proved in ([C-M]) :

Theorem 4.1 *Let g be a Riemannian metric and f a non-negative function on $\mathbb{R}^3$ such that $g - e \in H_{s',\delta'}$ and $h \in H_{s'-2,\delta'+2}$ with $s' > 3/2 + 1, \delta' > -3/2$. Then the operator $\Delta_g - h$ (acting on scalar function on $\mathbb{R}^3$) is an isomorphism*

$$H_{s,\delta} \to H_{s-2,\delta+2}$$

for each $2 \leq s \leq s'$, $-3/2 < \delta < -1/2$.

We rewrite this in the $b-$setting.

Theorem 4.2 *Let g, a Riemannian metric on* $\mathbb{R}^3$, *be such that* $g-e \in x^\epsilon H_b^s(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$ *for* $\epsilon > 0$. *Also let* $f \in x^{2+\epsilon} H_b^{s-2}(\mathbb{B}^3)$ *be a non-negative function on* $\mathbb{R}^3$. *Then,*

$$\Delta_g : x^a H_b^{s'}(\mathbb{B}^3) \to x^{a+2} H_b^{s'-2}(\mathbb{B}^3)$$

is an isomorphism for $2 \leq s' \leq s, \ 0 < a < 1$.

Once re-written in this $b-$setting, we believe that the proof of the theorem becomes much more transparent than otherwise. A major ingredient in the b-setting proof is the fact that

$$\Delta_g = \Delta_o + Q, \text{ where } Q \in x^{2+\epsilon} \cdot H_b^{s-1}(\mathbb{B}^3) \cdot \mathrm{Diff}_b^2(\mathbb{B}^3)$$

Though **Theorem 4.2** is a mere restatement of **Theorem 4.1**, we discuss its proof in **Section 4.1**, hoping that the $b-$setting proof, especially using the above fact, is more lucid.

Once we understand the mapping properties of $\Delta_g - f$, we then turn our attention to its action on the asymptotic behaviors of function. In **Section 4.2**, we therefore prove :

Theorem 4.3 *Let g, a Riemannian metric on* $\mathbb{R}^3$, *be such that* $g-e \in x^\epsilon H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \cap \mathcal{A}_\mathcal{G}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$ *for* $\epsilon > 0$, *and* $f \in x^{2+\epsilon} H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_\mathcal{F}^\infty(\mathbb{B}^3)$ *be a non-negative function on* $\mathbb{R}^3$. *Suppose* $h \in x^{a+2} H_b^\infty(\mathbb{B}^3) \cap A_\mathcal{E}^\infty(\mathbb{B}^3)$, *where* $0 < a < 1$. *Then, the unique solution* $u \in x^a H_b^\infty(\mathbb{B}^3)$ *to*

$$(\Delta_g - f)u = h$$

is also in $A_\mathcal{I}^\infty(\mathbb{B}^3)$ *for an index set* $\mathcal{I}$ *which depends on* $\mathcal{G}$, $\mathcal{F}$, *and,* $\mathcal{E}$. *(*$\mathcal{I}$ *is given precisely in the proof.)*

4.1 Mapping Properties of $\Delta_g - f$

As alluded to above, we first prove :

Lemma 4.4 *Let g, a Riemannian metric on* $\mathbb{R}^3$, *be such that* $g-e \in x^\epsilon H_b^s(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$ *for* $s > 3/2 + 1, \epsilon > 0$. *Then*

$$\Delta_g = \Delta_o + Q$$

where $Q \in x^{2+\epsilon} \cdot H_b^{s-1}(\mathbb{B}^3) \cdot \mathrm{Diff}_b^2(\mathbb{B}^3)$.

Proof : By the assumption, we can write g in coordinates (see **Lemma 2.3**)

$$g_{ij} = \delta_{ij} + x^\epsilon h_{ij}; \ h_{ij} \in H_b^s(\mathbb{B}^3)$$

Then, we can write g^{-1} as

$$g^{ij} = \delta^{ij} + x^\epsilon H^{ij};$$

where $H^{ij} \in H_b^s(\mathbb{B}^3)$ by Sobolev Imbedding (**Lemma 2.1**) and the multiplication theorem(**Lemma 2.3**). Next, we have

$$G = det(g) = 1 + x^\epsilon u;\ u \in H_b^s(\mathbb{B}^3)$$

which implies

$$\begin{aligned} \sqrt{G} &= 1 + x^\epsilon v;\ v \in H_b^s(\mathbb{B}^3) \\ \frac{1}{\sqrt{G}} &= 1 + x^\epsilon w;\ w \in H_b^s(\mathbb{B}^3) \end{aligned}$$

Again the fact that $u, v, w \in H_b^s(\mathbb{B}^3)$ because of **Lemma 2.3**. We did the above computations just so that we can compute Δ_g as

$$\Delta_g = \frac{1}{\sqrt{G}} \partial_i \sqrt{G} g^{ij} \partial_j$$

Now we recall that on some neighborhood U of $\partial\mathbb{B}^3$, we have

$$U \cong [0,1)_x \times S_\theta^2;\ x = 1/r$$

and

$$\begin{aligned} 1 \le i \le 3\ :\ \partial_i &= \alpha_i(\theta)(x\partial_x) + \beta_i(\theta)\partial_\theta;\quad \alpha_i(\theta), \beta_i(\theta) \in C^\infty(S^2) \\ &= xP_i\quad P_i \in \mathrm{Diff}_b^1(\mathbb{B}^3) \end{aligned}$$

and

$$\begin{aligned} \partial_i\partial_j &= x^2\{\alpha_i(\theta)\alpha_j(\theta)(x\partial_x)^2 + (\alpha_i(\theta)\beta_j(\theta) - \beta_i(\theta)\alpha_j(\theta))(x\partial_x)(\partial_\theta) + \beta_i(\theta)\beta_j(\theta)\partial_\theta^2 + \\ &\quad (\alpha_i(\theta)\alpha_j(\theta) - \beta_i(\theta)\alpha_j'(\theta))(x\partial_x) + (\alpha_i(\theta)\beta_j(\theta) - \beta_i(\theta)\beta_j'(\theta))(\partial_\theta)\} \\ &= x^2 Q_{ij};\ Q_{ij} \in \mathrm{Diff}_b^2(\mathbb{B}^3) \end{aligned}$$

With this in mind, let us compute :

$$\begin{aligned} \frac{1}{\sqrt{G}}\partial_i(\sqrt{G}g^{ij}\partial_j) &= \frac{1}{\sqrt{G}}\partial_i(\sqrt{G}(\delta^{ij} + x^\epsilon H^{ij})\partial_j) \\ &= \frac{1}{\sqrt{G}}\{\partial_i(\sqrt{G}\delta^{ij}\partial_j) + \sqrt{G}x^\epsilon H^{ij}\partial_i\partial_j + \partial_i(x^\epsilon\sqrt{G}H^{ij})\partial_j\} \\ &= \Delta_o + x^\epsilon H^{ij}\partial_i\partial_j + \delta^{ij}\frac{\partial_i(\sqrt{G})}{\sqrt{G}}\partial_j + \frac{\partial_i(x^\epsilon\sqrt{G}H^{ij})}{\sqrt{G}}\partial_j \end{aligned}$$

Moreover, since $\partial_i = xP_i \in x\mathrm{Diff}_b^1(\mathbb{B}^3)$

$$1 \le j \le 3 \to \partial_j : x^\epsilon H_b^m(\mathbb{B}^3) \to x^{1+\epsilon}H_b^{m-1}(\mathbb{B}^3)$$

which implies

$$\delta^{ij}\frac{\partial_i(\sqrt{G})}{\sqrt{G}}\partial_j = \sum_{i=1}^{3}(1 + x^\epsilon w)\partial_i(x^\epsilon v)\partial_i$$

$$
\begin{aligned}
&= \sum_{i=1}^{3}(1+x^{\epsilon}w)(x^{1+\epsilon}c_1^i(\theta)v + x^{1+\epsilon}P_i v)\partial_i \\
&= \sum_{i=1}^{3}(1+x^{\epsilon}w)(x^{1+\epsilon}\tilde{v})\partial_i; \quad \tilde{v}^i = c_1^i(\theta)v + P_i(v) \in H_b^{s-1}(\mathbb{B}^3) \\
&= \sum_{i=1}^{3} x^{2+\epsilon}\tilde{v}^i(1+x^{\epsilon}w)P_i, \quad \tilde{v}^i(1+x^{\epsilon}w) \in H_b^{s-1}(\mathbb{B}^3);
\end{aligned}
$$

and by **Lemma 2.3** (note $s-1 > 3/2$)

$$
\left\{ \begin{array}{lcl} \tilde{v} & \in & H_b^{s-1}(\mathbb{B}^3) \\ (1+x^{\epsilon}w) & \in & H_b^{s}(\mathbb{B}^3) \end{array} \right\} \Rightarrow \tilde{v}(1+x^{\epsilon}w) \in H_b^{s-1}(\mathbb{B}^3)
$$

Similarly,

$$
\frac{\partial_i(x^{\epsilon}\sqrt{G}H^{ij})}{\sqrt{G}}\partial_j = \sum_{1\le i,j\le 3} x^{2+\epsilon}\alpha_{ij}T^{ij}; \alpha_{ij} \in H_b^{s-1}(\mathbb{B}^3), \quad T^{ij} \in \mathrm{Diff}_b^1(\mathbb{B}^3)
$$

Putting these all together, we conclude

$$
\begin{aligned}
\Delta_g = \Delta_o \quad &+ \quad x^{2+\epsilon}(H^{ij}Q_1 + \sum_{i=1}^{3}\tilde{v}(1+x^{\epsilon}w)P_i + \sum_{i,j\le 3}\alpha_{ij}T^{ij}) \Longleftrightarrow \\
\Delta_g - \Delta_o \quad &\in \quad x^{2+\epsilon}\cdot H_b^{s-1}(\mathbb{B}^3)\cdot \mathrm{Diff}_b^2(\mathbb{B}^3)
\end{aligned}
$$

□

One immediate application of this lemma is the following result about the kernel of Δ_g. We prove

Lemma 4.5 *If $0 < a < 1$ and $u \in x^a H_b^s(\mathbb{B}^3)$ is in the kernel of $(\Delta_g - f)$, then actually $u \in x^{\rho} H_b^s(\mathbb{B}^3)$ for any $0 < \rho < 1$.*

Proof : Since $u \in \ker(\Delta_g - f)$, we can write

$$
\begin{aligned}
\Delta_g u - fu &= 0 \\
(\Delta_o + Q)u &= fu \\
\Delta_o u &= -Q(u) + fu
\end{aligned}
$$

Now, we see, by (**Corollary 2.2**) that

$$
\left\{ \begin{array}{lcl} u & \in & x^a H_b^s(\mathbb{B}^3) \\ f & \in & x^{2+\epsilon}H_b^{s-2}(\mathbb{B}^3) \end{array} \right\} \to fu \in x^{2+\epsilon+a}H_b^{s-2}(\mathbb{B}^3)
$$

Since $Q \in x^{2+\epsilon}\cdot H_b^{s-1}(\mathbb{B}^3)\cdot \mathrm{Diff}_b^2(\mathbb{B}^3)$,

$$
Q(u) = x^{2+\epsilon}\cdot h\cdot P(u); \; P(u) \in x^a H_b^{s-2}(\mathbb{B}^3) \text{ and } h \in H_b^{s-1}(\mathbb{B}^3)
$$

Since, by **Lemma 2.3**

$$H_b^{s-1}(\mathbb{B}^3) \times x^a H_b^{s-2}(\mathbb{B}^3) \subset x^a H_b^{s-2}(\mathbb{B}^3) \text{ because } s-1 > 3/2$$

we see that Q is well-defined as the following map :

$$Q : x^a H_b^s(\mathbb{B}^3) \to x^{2+\epsilon+a} H_b^{s-2}(\mathbb{B}^3)$$

Now in the previous chapter, we proved that

$$\Delta_o : x^a H_b^s(\mathbb{B}^3) \to x^{a+2} H_b^{s-2}(\mathbb{B}^3)$$

is an isomorphism for $0 < a < 1$. Since we have

$$\Delta_o u = -Q(u) + fu \in x^{2+\epsilon+a} H_b^{s-2}(\mathbb{B}^3)$$

we can conclude that u, originally in $x^a H_b^s(\mathbb{B}^3)$, is in fact in $x^{a+\epsilon} H_b^{s-2}(\mathbb{B}^3)$. Moreover, now that u is in $x^{a+\epsilon} H_b^s(\mathbb{B}^3)$, we can repeat the argument to assert that u is in $x^{a+2\epsilon} H_b^s(\mathbb{B}^3)$, and so on, to conclude that

$$u \in x^{a+m\epsilon} H_b^s(\mathbb{B}^3)$$

as long as $a + m\epsilon < 1$. □

We are now ready to the prove of **Theorem 4.2**, which again is a restatement of **Theorem 4.1** of Christodoulou and O'Munchrada ([C-M]). We mostly follow their arguments, but put each step in b−setting. Near the end of the proof, we will use the preceding lemma.

Proof : For $t \in [0, 1]$, let us define :

$$g_t := \delta_{ij} + t \cdot h_{ij}$$

Note that $t \cdot h_{ij}$ is still in $x^\epsilon H_b^s(\mathbb{B}^3)$ and that $t \cdot f$ is still non-negative. Now define :

$$P_t := \Delta_{g_t} - t \cdot f = x^a H_b^{s'-2}(\mathbb{B}^3) \to x^{a+2} H_b^{s'}(\mathbb{B}^3)$$

We see that

$$P_0 = \Delta_o; \quad P_1 = \Delta_g - f$$

Moreover, we prove

Claim 4.6 *$\forall t \in [0, 1]$, P_t is Fredholm.*

Proof : We only give a sketch of proof here.

We can construct a parametrix Q_{g_t} for $(\phi + r^2)(\Delta_{g_t} - tf)$ imitating the steps for the parametrix for $(\phi + r^2)\Delta_o$ as done in **Section 3.4**. We define

$$\begin{aligned} Q_{g_t}(f) &= \tilde{\psi}\,[A_{g_t} \frac{\psi f}{(\phi + r^2)}] + (1 - \tilde{\phi})(r^2(\Delta_{g_t} - tf))^{-1}(1 - \psi)f \\ &= Q_1(f) + Q_2(f) \end{aligned}$$

where in the interior, we can define A_{g_t} such that

$$(r^2(\Delta_{g_t} - tf)) \circ A_{g_t} = Id$$

Near the boundary, we can write

$$(r^2(\Delta_{g_t} - tf))\phi = \sum_{0 \leq k \leq 2} a_k(x)(x\partial_x)^k$$

where

$$a_k(x) = \sum a_{kl}(x,\theta)(\partial_\theta)^l;\ a_{kl}(x,\theta) \in C^\infty([0,\infty] \times S^2) + x^\epsilon H_b^\infty([0,1] \times S^2)$$

Thus we can define :

$$\begin{aligned} (r^2(\Delta_g - tf))^{-1} \quad &: \quad r^a H_b^s([0,1] \times S^2) \to r^a H_b^{s+2}([0,1] \times S^2) \\ (r^2(\Delta_g - tf))^{-1}(\phi)(t,\theta) \quad &= \quad \int e^{its}(\sum_{0 \leq k \leq 2} a_k(x)(is)^k)^{-1}\hat{\phi}(s,\theta)ds \end{aligned}$$

and then show that this is well defined for $a \notin \mathbb{Z}$.

Finally we show that

$$(\phi + r^2)P_t \circ Q_{g_t} = Id + R_{g_t}$$

where $R_{g_t} \in x^\delta H_b^\infty(\mathbb{B}^3) \cdot \Psi DO_b^{-1}(\mathbb{B}^3)$. By standard argument for ΨDO_b^*, we indeed have

$$(\phi + r^2)P_t \circ Q'_{g_t} = Id + R'_{g_t}$$

where $R'_{g_t} \in x^\delta H_b^\infty(\mathbb{B}^3) \cdot \Psi DO_b^{-1}(\mathbb{B}^3)$ □

Thus P_t is a continuous family of Fredholm operators and by the continuity of index for Fredholm families, :

$$\begin{aligned} index(P_1) \quad &= \quad index(P_0) = index(\Delta_o) \\ &= \quad 0, \text{ for } 0 < a < 1 \end{aligned}$$

Thus it suffices to show that

$$\ker(P_1) = \ker(\Delta_g - f) = \emptyset$$

Suppress $u \in x^a H_b^{s'}(\mathbb{B}^3)$ were in $\ker(\Delta_g - f)$, i.e.

$$\Delta_g u - fu = 0$$

Multiplying both sides by u, we get

$$u\Delta_g u - fu^2 = 0$$

At this point, one is tempted to integrate both sides over $\mathbb{R}^3$ However, it is not *a priori* clear that you can do so because

$$\left.\begin{cases} u & \in \quad x^a H_b^{s'}(\mathbb{B}^3) \\ (\Delta_g - f)u & \in \quad x^{a+2} H_b^{s'-2}(\mathbb{B}^3) \end{cases}\right\} \to u \cdot (\Delta_g u - fu) \in x^{2+2a} H_b^{s'-2}(\mathbb{B}^3)$$

but,

$$x^\beta H_b^s(\mathbb{B}^3) \subset L^1(\mathbb{R}^3) \text{ only if } \beta > 3$$

However, **Lemma 4.5** asserts that

$$\begin{aligned} u \in \ker(r^2\Delta_g) \quad &\to \quad u \in x^\beta H_b^s(\mathbb{B}^3) \text{ for } \frac{1}{2} < \beta < 1 \\ &\to \quad u(\Delta_g u - fu) \in x^{2+2\beta} H_b^s(\mathbb{B}^3) \subset L^1(\mathbb{R}^3) \end{aligned}$$

Thus, we can indeed integrate both sides over $\mathbb{R}^3$ to get

$$\begin{aligned} u\Delta_g u - fu^2 &= 0 \\ \int_{\mathbb{R}^3} |\nabla u|_g d\mathbf{x} + \int_{\mathbb{R}^3} fu^2 d\mathbf{x} &= 0 \end{aligned}$$

which implies $u = 0$. □

4.2 Action of $\Delta_g - f$ on Asymptotic Behaviors of Functions

As we did for the case of Δ_o, we first see what happens when h is small :

Lemma 4.7 *Let g, a Riemannian metric on $\mathbb{R}^3$ and f, a non-negative function on $\mathbb{R}^3$, be as in* **Theorem 4.2**. *Furthermore, Let $a \in (0,1)$, $h \in x^{a+2}H_b^{s'-2}(\mathbb{B}^3)$, and $u \in x^a H_b^{s'}(\mathbb{B}^3)$ be a unique solution to*

$$(\Delta_g - f)u = h$$

for $2 \leq s' \leq s$. Suppose h were in fact in $x^{a+2+\gamma}H_b^{s'-2}(\mathbb{B}^3)$ (i.e. decays faster by a factor of x^γ, $\gamma > 0$) and that $a + \gamma > 1$. Then u has the following asymptotic expansion near infinity :

$$u = \sum_{1 \leq j \in \mathbb{Z} < \rho} C_j(\theta) x^j + x^\rho H_b^{s'}(\mathbb{B}^3)$$

where $\rho = \min(a+\gamma, a+m\epsilon)$ where m is the smallest integer such that $a+m\epsilon > 1$.

Remark 4.8 *Notice that this lemma is very reminiscent of* **Lemma 3.17**, *except that in this case, no matter how large γ is, the asymptotic expansion stops at $a + m\epsilon$ instead of at $a + \gamma$. We will see why this is the case in the ensuing proof.*

Proof : Again using

$$\Delta_g = \Delta_o + Q$$

we have

$$\begin{aligned}(\Delta_g - f)u &= h \\ \Delta_o u &= fu - Q(u) + h\end{aligned}$$

Since $f \in x^{2+\epsilon} H_b^{s'-2}(\mathbb{B}^3)$ and $Q \in x^{2+\epsilon} \cdot H_b^{s'-1}(\mathbb{B}^3) \cdot \mathrm{Diff}_b^2(\mathbb{B}^3)$, we can see, as in the proof of **Theorem 4.2** that

$$fu - Q(u) \in x^{2+a+\epsilon} H_b^{s'-2}(\mathbb{B}^3)$$

Recall also that h is in $x^{a+2+\gamma} H_b^{s'-2}(\mathbb{B}^3)$, with $a + \gamma > 1$. If $a + \epsilon < 1$, then we see that

$$fu - Q(u) + h \in x^{2+a+\epsilon} H_b^{s'-2}(\mathbb{B}^3)$$

which, by the fact that Δ_o is an isomorphism for $0 < a < 1$, implies that u, which *a priori* is in $x^a H_b^{s'}(\mathbb{B}^3)$, indeed in $x^{a+\epsilon} H_b^{s'}(\mathbb{B}^3)$. However, this fact in turn implies that $u \in x^{a+2\epsilon} H_b^{s'}(\mathbb{B}^3)$ and so forth, which leads to the conclusion that

$$u \in x^{a+n\epsilon} H_b^{s'}(\mathbb{B}^3)$$

as long as $n\epsilon < 1$. Suppose $m \in \mathbb{N}$ be the first integer such that $a + m\epsilon > 1$. Then $u \in x^{a+(m-1)\epsilon} H_b^{s'}(\mathbb{B}^3)$ which implies that

$$fu - Q(u) + h \in x^{2+\rho} H_b^{s'-2}(\mathbb{B}^3)$$

where $\rho = \min(a + m\epsilon, a + \gamma)$. Now we can use **Lemma 3.17** of the previous section to conclude that

$$u = \sum_{1 \le j < \rho} C_j(\theta) x^j + x^\rho H_b^{s'}(\mathbb{B}^3)$$

□

Notice that, unlike in **Lemma 3.17**, we cannot get *a priori* any more terms in the expansion, even if we assume that γ is really large. Here is why :

u, by the conclusion of the lemma, now belongs to $x^{1-\delta} H_b^s(\mathbb{B}^3)$ which re-plugged into the equation implies that RHS($= fu + Q(u) + h$) is in $x^{2+\epsilon+(1-\delta)} H_b^s(\mathbb{B}^3)$ (since we assumed that γ is sufficiently large). This implies that

$$u = \sum_{1 \le j \in \mathbb{Z} < \rho} C_j(\theta) x^j + x^{\epsilon+(1-\delta)} H_b^{s'}(\mathbb{B}^3)$$

which is a slight improvement of the conclusion of the lemma, in so far as the remainder term is concerned. However, u is again in $x^{1-\delta} H_b^s(\mathbb{B}^3)$, and there is *a priori* no more improvement of the remainder term in the expansion or any more terms in the expansion - *unless* we assume some more regularity on the metric g and on the non-negative function f. In fact, the content of **Theorem 4.3** is that when we do assume more on g and h, we do get more information on u. However, before we get to the theorem, we first need to prove a variant of **Lemma 4.4**.

Lemma 4.9 *Let g, a Riemannian metric on $\mathbb{R}^3$, be such that $g-e \in x^\epsilon H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \cap \mathcal{A}_{\mathcal{G}}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$ for $\epsilon > 0$. If we let $\tilde{\mathcal{G}} = \cup_{n=1}^\infty n \cdot \mathcal{G} + (2,0)$, then Δ_g can be written as*

$$\Delta_g = \Delta_o + Q$$

where $Q \in (x^{2+\epsilon} H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\tilde{\mathcal{G}}}^\infty(\mathbb{B}^3)) \cdot \mathrm{Diff}_b^2(\mathbb{B}^3)$.

Proof : Similar to what we did in the proof of **Lemma 4.4**, we can write g in coordinates (see **Lemma 2.3**)

$$g_{ij} = \delta_{ij} + k_{ij};\ k_{ij} \in x^\epsilon H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\mathcal{G}}^\infty(\mathbb{B}^3)$$

Now let

$$\hat{\mathcal{G}} = \cup_{n=1}^\infty n \cdot \mathcal{G}$$

We then can write g^{-1} using **Lemma 2.1**, **Lemma 2.3**, and **Lemma 2.4** as following :

$$g^{ij} = \delta^{ij} + K^{ij};\ K^{ij} \in x^\epsilon H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\hat{\mathcal{G}}}^\infty(\mathbb{B}^3)$$

and we also have

$$G = det(g) = 1 + u;\ u \in x^\epsilon H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\hat{\mathcal{G}}}^\infty(\mathbb{B}^3)$$

which implies

$$\begin{aligned} \sqrt{G} &= 1 + v;\ v \in x^\epsilon H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\hat{\mathcal{G}}}^\infty(\mathbb{B}^3) \\ \frac{1}{\sqrt{G}} &= 1 + w;\ w \in x^\epsilon H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\hat{\mathcal{G}}}^\infty(\mathbb{B}^3) \end{aligned}$$

We compute again :

$$\begin{aligned} \frac{1}{\sqrt{G}}\partial_i(\sqrt{G}g^{ij}\partial_j) &= \frac{1}{\sqrt{G}}\partial_i(\sqrt{G}(\delta^{ij} + K^{ij})\partial_j) \\ &= \frac{1}{\sqrt{G}}\{\partial_i(\sqrt{G}\delta^{ij}\partial_j) + \sqrt{G}K^{ij}\partial_i\partial_j + \partial_i(\sqrt{G}K^{ij})\partial_j\} \\ &= \Delta_o + K^{ij}\partial_i\partial_j + \delta^{ij}\frac{\partial_i(\sqrt{G})}{\sqrt{G}}\partial_j + \frac{\partial_i(\sqrt{G}K^{ij})}{\sqrt{G}}\partial_j \end{aligned}$$

Because

$$\partial_i\partial_j = x^2 Q_{ij};\ Q_{ij} \in \mathrm{Diff}_b^2(\mathbb{B}^3)$$

it is clear that

$$K^{ij}\partial_i\partial_j \in (x^{2+\epsilon} H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\hat{\mathcal{G}}+(2,0)}^\infty(\mathbb{B}^3)) \cdot \mathrm{Diff}_b^2(\mathbb{B}^3)$$

Moreover, since $\partial_i = xP_i \in x\mathrm{Diff}_b^1(\mathbb{B}^3)$

$$1 \le j \le 3 \to \partial_j : x^\epsilon H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\hat{\mathcal{G}}}^\infty(\mathbb{B}^3) \to x^{1+\epsilon} H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\hat{\mathcal{G}}+(1,0)}^\infty(\mathbb{B}^3)$$

we have

$$\begin{aligned} \delta^{ij} \frac{\partial_i(\sqrt{G})}{\sqrt{G}} \partial_j &= \sum_{i=1}^{3} (1+w)x^2 P_i(v) P_i \\ &= \sum_{i=1}^{3} (1+w)\tilde{v} P_i;\ (1+w)\tilde{v} \in x^{2+\epsilon} H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\hat{\mathcal{G}}+(2,0)}^\infty(\mathbb{B}^3) \end{aligned}$$

Similarly,

$$\frac{\partial_i(\sqrt{G}K^{ij})}{\sqrt{G}} \partial_j = \sum_{i,j \le 3} \alpha_{ij} T^{ij}; \alpha_{ij} \in x^{2+\epsilon} H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\hat{\mathcal{G}}+(2,0)}^\infty(\mathbb{B}^3),\ T^{ij} \in \mathrm{Diff}_b^1(\mathbb{B}^3)$$

Letting

$$\tilde{\mathcal{G}} = \hat{\mathcal{G}} + (2,0)$$

Putting this all together, we conclude

$$\begin{aligned} \Delta_g = \Delta_o &+ (K^{ij} Q_1 + \sum_{i=1}^{3} \tilde{v}(1+w) P_i + \sum_{i,j \le 3} \alpha_{ij} T^{ij} \longleftrightarrow \\ \Delta_g - \Delta_o &\in (x^{2+\epsilon} H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\tilde{\mathcal{G}}}^\infty(\mathbb{B}^3)) \cdot \mathrm{Diff}_b^2(\mathbb{B}^3) \end{aligned}$$

□

Now we are in position to prove the main theorem of this section, namely **Theorem 4.3**.

Proof : First, let us assume, as we did for **Lemma 4.7**, that h is in fact in $x^{a+2+\gamma} H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\mathcal{E}}^\infty(\mathbb{B}^3)$ (i.e. decays faster by a factor of x^γ, $\gamma > 0$) and that $a + \gamma > 1$. Then **Lemma 4.7** shows that

$$u = \sum_{1 \le j \in \mathbb{Z} < \rho} C_j^u(\theta) x^j + R_\rho^u(x,\theta); R_\rho^u(x,\theta) \in x^\rho H_b^\infty(\mathbb{B}^3)$$

where $\rho = \min(a+\gamma, a+m\epsilon)$ where m is the smallest integer such that $a+m\epsilon > 1$.

As a matter of fact, using the assumption that h does have a full expansion at the boundary, we can indeed show a similar result even when we don't assume an extra decay on h. We claim that

Claim 4.10 *If $h \in \mathcal{A}_{\mathcal{E}}^\infty(\mathbb{B}^3)$ and the first term of the expansion of h as in*

$$h = C(\theta) x^{k_0+2} (\log x)^{l_0} + R_{2+\beta}^h(x,\theta);\quad R_{2+\beta}^h(x,\theta) \in x^{2+\beta} H_b^\infty(\mathbb{B}^3);\quad \beta > k_0$$

is such that $k_0 \le 1$, then the unique solution $u \in x^{(k_0-\delta)} H_b^\infty(\mathbb{B}^3)$ ($\delta > 0$ arbitrarily small) to

$$(\Delta_g - f)u = h$$

has the following expansion :

$$u = \tilde{C}(\theta)x^{k_0}(\log x)^{l'_0} + \sum_{k_0<j\in\mathbb{Z}\leq\rho} C^u_j(\theta)x^j + R^u_\rho(x,\theta); \quad R^u_\rho(x,\theta) \in x^\rho H^\infty_b(\mathbb{B}^3)$$

where $\rho = \min((k_0-\delta)+\epsilon, \beta)$ *and* $l'_0 = l_0$ *if* $k_0 < 1$ *and* $l'_0 = l_0 + 1$ *if* $k_0 = 1$.

Proof : Plugging into the equation, we have

$$\Delta_o u = C(\theta)x^{k_1+2}(\log x)^{l_0} - Q(u) + fu + R^h(x,\theta)$$

Since $u \in x^{(k_0-\delta)}H^\infty_b(\mathbb{B}^3)$,

$$-Q(u) + fu \in x^{(k_0-\delta)+2+\epsilon}H^\infty_b(\mathbb{B}^3)$$

and thus

$$-Q(u) + fu + R^h(x,\theta) \in x^{2+\rho}H^\infty_b(\mathbb{B}^3)$$

where $\rho = \min((k_0-\delta)+\epsilon, \beta)$. Then the claim follows by the main result of the previous section, **Theorem 3.16**. □

Thus, regardless of the size of h, we can conclude right away that u has some partial expansion at $\partial\mathbb{B}^3$. Let us define $\mathcal{I}_0$ to be the set of powers in this finite expansion, i.e.

$$\mathcal{I}_0 = \cup_{1\leq k<\rho}\{(k,0)\};$$

or

$$\mathcal{I}_0 = (k_0, l'_0) \cup_{k_0<k<\rho} \{(k,0)\}$$

depending on the size of h, as we can see from the above claim and the paragraph preceding the claim.

In fact, we can combine this into one expression. Noting that $\tilde{G} = \hat{G} + (2,0)$, we write

$$\begin{aligned} \mathcal{E} &= \mathcal{E}^- + (2,0) \\ \mathcal{F} &= \mathcal{F}^- + (2,0) \end{aligned}$$

Now if k_0 is the smallest among $\{(k,l) \in \mathcal{E}\}$ and

$$h = C(\theta)x^{k_0+2}(\log x)^{l_0} + R^h_{r_0}(x,\theta); \quad R^h_{r_0}(x,\theta) \in x^{2+r_0}H^\infty_b(\mathbb{B}^3); \quad r_0 > k_0$$

then we have

$$u = \sum_{(k,l)\in\mathcal{I}_0} C^u_{k,l}(\theta)x^k(\log x)^l + R^u_{\rho_0}(x,\theta); \; R^u_{\rho_0}(x,\theta) \in x^{\rho_0}H^\infty_b(\mathbb{B}^3)$$

where

$$\begin{aligned} \mathcal{I}_0 &= ((\mathcal{E}-(2,0))\overline{\cup}\mathbb{N}) \cap \{(k,l) \mid k < \rho_0\} \\ &= (\mathcal{E}^- \cup \; \mathbb{N}) \cap \{(k,l) \mid k < \rho_0\} \end{aligned}$$

and

$$\rho_0 = min(k_0 - \delta + \epsilon, r_0)$$

Now once we conclude that u has this finite expansion, it's easy to see that we, in fact, u has a full asymptotic expansion at infinity of $\mathbb{R}^3$. We can see this iteratively as follows : First plug this back into

$$\Delta_o u = -Q(u) + fu + h$$

By the above **Lemma 4.9** :

$$\begin{array}{rcl} Q : \sum_{(k,l)\in\mathcal{I}_0} C^u_{k,l}(\theta) x^k (\log x)^l & \longrightarrow & \mathcal{A}^\infty_{\tilde{\mathcal{G}}+\mathcal{I}_0}(\mathbb{B}^3) \\ Q : x^\rho H^\infty_b(\mathbb{B}^3) & \longrightarrow & x^{2+\epsilon+\rho} H^\infty_b(\mathbb{B}^3) \end{array}$$

Moreover, since $f \in \mathcal{A}^\infty_{\mathcal{F}}(\mathbb{B}^3)$

$$\begin{array}{rcl} C^u_1(\theta) \sum_{(k,l)\in\mathcal{I}_0} C^u_{k,l}(\theta) x^k (\log x)^l & \xrightarrow{\times f} & \mathcal{A}^\infty_{\mathcal{F}+\mathcal{I}_0}(\mathbb{B}^3) \\ x^\rho H^\infty_b(\mathbb{B}^3) & \xrightarrow{\times f} & x^{2+\epsilon+\rho} H^\infty_b(\mathbb{B}^3) \end{array}$$

Now using the assumption that $h \in x^{a+2} H^\infty_b(\mathbb{B}^3) \cap \mathcal{A}^\infty_{\mathcal{E}}(\mathbb{B}^3)$, we can write

$$h = \sum_{(m,n)\in\mathcal{E}}^{m<2+\rho+\epsilon} C^h_{m,n} x^m (logx)^n + R^h_{2+\rho_1}(x,\theta);\ R^h_{2+\rho_1}(x,\theta) \in x^{2+\rho_1} H^\infty_b(\mathbb{B}^3);$$

where we define

$$\rho_1 = \rho_0 + 1 \cdot \epsilon$$

Thus we have

$$\begin{aligned} \Delta_o u \quad = \quad & Q\Big(\sum_{(k,l)\in\mathcal{I}_0} x^k (\log x)^l\Big) + f \cdot \sum_{(k,l)\in\mathcal{I}_0} x^k (\log x)^l + \sum_{(m,n)\in\mathcal{E}}^{m<2+\rho_1} C^h_{m,n} x^m (\log x)^n \\ & + Q(R^u_{\rho_0}) + f \cdot R^u_{\rho_0} + R^h_{2+\rho_1} \end{aligned}$$

The first three terms of LHS can be written as

$$\sum_{(i,j)\in\mathcal{T}_1}^{i<2+\rho_1} C^L_{i,j} x^i (\log x)^j + R_{\rho_1};\ R_{\rho_1} \in x^{\rho_1} H^\infty_b(\mathbb{B}^3)$$

where

$$\mathcal{T}_1 = ((\tilde{\mathcal{G}} \cup \mathcal{F}) + \mathcal{I}_0) \cup \mathcal{E}$$

As for the next three terms of LHS

$$Q(R^u_\rho) + f \cdot R^u_\rho + R^h_{\rho_1} \in x^{\rho_1} H^\infty_b(\mathbb{B}^3)$$

Putting this all together, we thus finally we see that

$$\Delta_o u = \sum_{(i,j)\in\mathcal{T}_1}^{i<2+\rho_1} C^L_{i,j} x^i (\log x)^j + R^L_{\rho_1};\ R^L_{\rho_1} \in x^{\rho_1} H^\infty_b(\mathbb{B}^3)$$

Then we can use the main result of the previous section **Theorem 3.16** to conclude that

$$u = \sum_{(k,l)\in\mathcal{I}_1} C^u_{k,l} x^k (\log x)^l + R^u_{\rho_1};\ R^u_{\rho_1} \in x^{\rho_1} H^\infty_b(\mathbb{B}^3)$$

where

$$\begin{aligned} \mathcal{I}_1 &= ((((\tilde{\mathcal{G}} \cup \mathcal{F}) + \mathcal{I}_0) \cup \mathcal{E}) - 2)\ \overline{\cup}\mathbb{N} \cap \{(k,l) \mid k < \rho_1\} \\ &= (((\hat{\mathcal{G}} \cup \mathcal{F}^-) + \mathcal{I}_0) \cup \mathcal{E}^-)\overline{\cup}\mathbb{N} \cap \{(k,l) \mid k < \rho_1\} \end{aligned}$$

and

$$\rho_1 = \rho_0 + \epsilon$$

Iterating the above argument, we can see that

$$\forall n \in \mathbb{N} : u = \sum_{(k,l)\in\mathcal{I}_n} C^u_{k,l} x^k (\log x)^l + R^u_{\rho_n};\ R^u_{\rho_n} \in x^{\rho_n} H^\infty_b(\mathbb{B}^3)$$

where

$$\rho_n = \rho_0 + n \cdot \epsilon$$

and

$$\mathcal{I}_{n+1} = (((\hat{\mathcal{G}} \cup \mathcal{F}^-) + \mathcal{I}_n) \cup \mathcal{E}^-)\overline{\cup}\mathbb{N} \cap \{(k,l) \mid k < \rho_{n+1}\}$$

Thus, if the following claim is true

Claim 4.11 *The following direct limit*

$$\mathcal{I} = \lim_{n\to\infty} \mathcal{I}_n$$

exists and, moreover, $\mathcal{I}$ is an index set.

then we can finally see that

$$u \in x^a H^\infty_b(\mathbb{B}^3) \cap \mathcal{A}^\infty_{\mathcal{I}}(\mathbb{B}^3)$$

Thus it only remains to prove the above claim.

Proof : of **Claim 4.11** :

We first show that the direct limit exits. I.e. we need to show

$$\mathcal{I}_n \subset \mathcal{I}_{n+1}$$

We prove this by induction. It is easy to see

$$\mathcal{I}_0 \subset \mathcal{I}_1$$

Now, suppose $\mathcal{I}_{n-1} \subset \mathcal{I}_n$. From

$$\begin{aligned} \mathcal{I}_n &= (((\hat{\mathcal{G}} \cup \mathcal{F}^-) + \mathcal{I}_{n-1}) \cup \mathcal{E}^-)\overline{\cup}\mathbb{N} \cap \{(k,l) \mid k < \rho_n\} \\ \mathcal{I}_{n+1} &= (((\hat{\mathcal{G}} \cup \mathcal{F}^-) + \mathcal{I}_n) \cup \mathcal{E}^-)\overline{\cup}\mathbb{N} \cap \{(k,l) \mid k < \rho_{n+1}\} \end{aligned}$$

we can easily see that

$$\mathcal{I}_n \subset \mathcal{I}_{n+1}$$

Now that the direct limit $\mathcal{I}$ is well-defined, we need to show that it indeed is an index set. i.e. we must show that

$$\forall r > 0 \ : \ \#\{(k,l) \in \mathcal{I} \mid k < r\} < \infty$$

In order to see this, let us write $\mathcal{I}_n$ in the following way

$$\mathcal{I}_{n+1} = [((\hat{\mathcal{G}} \cup \mathcal{F}^-) + \mathcal{I}_n)\overline{\cup}\mathbb{N} \cup \mathcal{E}^-\overline{\cup}\mathbb{N}] \cap \{(k,l) \mid k < \rho_{n+1}\}$$

Since $\mathcal{E}^-\overline{\cup}\mathbb{N}$ does not change during iteration, we only need to worry about

$$(\hat{\mathcal{G}} \cup \mathcal{F}^-) + \mathcal{I}_n$$

Now let i_0 = smallest k, where $(k,l) \in \mathcal{I}_0$. We know that $i_0 > 0$ and i_0 is the smallest k for all other $\mathcal{I}_n$, $n > 0$.

Now suppose $(a,b) \in \hat{\mathcal{G}} \cup \mathcal{F}^-$ and $a < r$. If we look at how (a,b) moves (say to (a',b')) in (k,l)−plane under each iteration of $(\hat{\mathcal{G}} \cup \mathcal{F}^-) + \mathcal{I}_n$, it always moves to the right at least by i_0. Thus even though we may acquire new points - one additional point $(a', b'+1)$ whenever $a' \in \mathbb{Z}$ - the number of the new points is bounded because after finitely many iterations, the new point (a',b') will eventually be on the right of the line $k = r$.

More precisely speaking we have :

$$\begin{aligned} \#\{(k,l) \in \mathcal{I} \mid k < r\} &< \frac{r - i_0}{i_0} \cdot \#\{(k,l) \in \hat{\mathcal{G}} \cup \mathcal{F}^- \mid k < r\} + \#\{(k,l) \in \mathcal{E}^-\overline{\cup}\mathbb{N} \mid k < r\} \\ &< \infty \end{aligned}$$

□
□

Chapter 5

Non-linear Constraint

Based on what we learned about how the linear operator $(\Delta_g - f)$ act on functions with complete asymptotic expansions, we will study, in this section, the asymptotic behavior of solutions to the following non-linear equation :

$$\Delta_g \psi + M\psi^{-7} = 0 \qquad (\dagger)$$

This is a reduced form of what is referred to as Lichnerowicz equation in [C-K]. We remark that the non-linearity of the equation is not so severe in the sense that it appears only at the $M\psi^{-7}$ term. The main result that we want to establish is the following :

Theorem 5.1 (Lichnerowicz Equation) *Let g, a Riemannian metric on $\mathbb{R}^3$, be such that $g - e \in x^\epsilon H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \cap \mathcal{A}_\mathcal{G}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$ for $\epsilon > 0$, and M, a non-negative function, be in $x^{a+2} H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_\mathcal{M}^\infty(\mathbb{B}^3)$. Then, a solution ψ to*

$$\Delta_g \psi + M\psi^{-7} = 0$$

which exists uniquely so that $\psi - 1 \in x^a H_b^\infty(\mathbb{B}^3)$ is also such that $\mathcal{A}_\mathcal{I}^\infty(\mathbb{B}^3)$, for some index set $\mathcal{I}$, which depends on $\mathcal{G}$ and $\mathcal{M}$. ($\mathcal{I}$ is given precisely in the proof.)

Once again, the reason why one would expect such a result lies in the fact that it is eventually the linear term Δ_g that dominates over the non-linear term $M\psi^{-7}$ near the infinity of $\mathbb{R}^3$. Furthermore, the properties of Δ_g can be deduced, as we saw in the previous section, from the properties of the model operator Δ_o. The main points of the proof of the above theorem basically lie in trying to reduce the non-linear case all the way down to the model case Δ_o, in a systematic way.

Our starting point for the proof of the above theorem is, as in the linear case, some theorem which gives existence (and *uniqueness*) of solutions to this nonlinear equation ($\dagger$). In this direction, Christodoulou and O'munchadra had again gone before me and proved :

Theorem 5.2 *Let g be a Riemannian metric and M, a non-negative function on $\mathbb{R}^3$ such that $g - e \in H_{s,\delta}$ and $M \in H_{s-2,\delta+2}$ with $s >\geq 4$ and $-3/2 < \delta < -1/2$. Then the semilinear elliptic differential equation*

$$\Delta_g \psi + M\psi^{-7} = 0$$

has one and only one positive solution ψ such that $\psi - 1 \in H_{s,\delta}$. Furthermore, $\psi \geq 1$.

We again rewrite this in $b-$setting.

Theorem 5.3 *Let g be a Riemannian metric such that $g - e \in x^\epsilon H_b^s(\mathbb{B}^3)$ for $s \geq 4$ and $\epsilon > 0$ and M, a non negative function on $\mathbb{R}^3$, be $\in x^{a+2} H_b^{s-2}(\mathbb{B}^3)$ with and $0 < a < 1$. Then the semilinear elliptic differential equation*

$$\Delta_g \psi + M\psi^{-7} = 0$$

has one and only one positive solution ψ such that $\psi - 1 \in x^a H_b^s(\mathbb{B}^3)$. Furthermore, $\psi \geq 1$.

Remark 5.4 *Note that $g - e \in x^\epsilon H_b^s(\mathbb{B}^3)$ for $\epsilon > 0$ is slightly more general than $g - e \in H_{s,\delta}$ for $-3/2 < \delta < -1/2$ in* **Theorem 5.2**. *However, it is not hard to see that the main steps of this proof - the continuity argument - as found in [Cantor] go through in the slightly more general setting.*

Now using the above **Theorem 5.3** for the case of $s = \infty$, we can begin the proof of the main theorem of this section **Theorem 5.1**

Proof : Let us first write

$$\psi = 1 + u; \quad u \in x^a H_b^\infty(\mathbb{B}^3), \ u \geq 0$$

Now, from **Lemma 4.9** we know

$$\Delta_g = \Delta_o + Q; \text{ where } Q \in (x^{2+\epsilon} H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\mathcal{G}}^\infty(\mathbb{B}^3)) \cdot \mathrm{Diff}_b^2(\mathbb{B}^3)$$

thus, we can see that our main equation (†) can be written as

$$\Delta_o u = -Q(u) - M(1+u)^{-7}$$

Since $u \in x^a H_b^\infty(\mathbb{B}^3) \subset C^\infty(\mathbb{R}^3)$ and $u \geq 0$, we can Taylor-expand $(1+u)^{-7}$ near $u = 0$ to get

$$\begin{aligned} \Delta_o u &= -Q(u) - M(1 - 7u + K(u)u^2); \quad K \in C^\infty((-1,\infty)) \\ &= -Q(u) + 7Mu - M - MK(u)u^2 \end{aligned}$$

Compare the above with the situation that we had for the linear case in the previous section :

$$\Delta_o u = -Q(u) + fu + h$$

Notice that the only difference between the two is, *as one would expect*, the non-linear term, $MK(u)u^2$. Our strategy now is to show that this non-linear term is *innocuous* in the sense that we can still use most of the arguments of the linear case proof with a slight modification to tame the non-linear term. With this in mind, we prove :

Lemma 5.5 *If $u \in x^a H_b^s(\mathbb{B}^3)$ for $a > 0$ and if K is a C^∞ function on the range of u, then $K(u)$ is bounded and $\mathcal{V}(K(u)) \in x^a H_b^{s-1}(\mathbb{B}^3)$, for all $\mathcal{V} \in \mathrm{Diff}_b^1(\mathbb{B}^3)$.*

Proof : The fact that $K(u)$ is bounded is immediate. Now we want to show :

$$\mathcal{V}(K(u)) = K'(u) \cdot \mathcal{V}(u) \in x^a H_b^{s-1}(\mathbb{B}^3)$$

Since $K'(u)$ is bounded, we see that

$$\begin{aligned} & K'(u) \cdot \mathcal{V}(u) \in x^{a+1} H_b^{s-1}(\mathbb{B}^3) \\ \Longleftrightarrow \quad & K''(u)\mathcal{W}(u)\mathcal{V}(u) + K'(u)\mathcal{W}(\mathcal{V}(u)) \in x^{a+1} H_b^{s-2}(\mathbb{B}^3) \end{aligned}$$

for $\mathcal{V}, \mathcal{W} \in \mathrm{Diff}_b^1(\mathbb{B}^3)$. Reapting this argument,

$$\mathcal{V}(K(u)) \in x^a H_b^{s-1}(\mathbb{B}^3)$$

if and only if the following expression :

$$\begin{aligned} & K^{(s)}(u)\mathcal{V}_1(u) \cdot \mathcal{V}_2(u) \cdots \mathcal{V}_k(u) \\ + \quad & K^{(s-1)}(u) \sum_{1 \le i \ne j \le s} (\mathcal{V}_i \cdot \mathcal{V}_j)(u) \cdot \mathcal{V}_1(u) \cdots \widehat{\mathcal{V}_i(u)} \cdots \widehat{\mathcal{V}_j(u)} \cdots \mathcal{V}_k(u) \\ + \quad & K'(u)(\mathcal{V}_1 \cdots \mathcal{V}_s)(u) \end{aligned}$$

is in $x^a L_b^2(\mathbb{B}^3)$. Now it's easy to check that $u \in x^a H_b^s(\mathbb{B}^3)$ implies that each term in the above sum is in $x^a L_b^2(\mathbb{B}^3)$. □

Now, as an application of this lemma, we prove :

Lemma 5.6 *Suppose f is bounded and that $\mathcal{V}(f) \in x^\epsilon H_b^{s-1}(\mathbb{B}^3)$, for $\epsilon > 0$, $s > 2 + 3/2$, and $\mathcal{V} \in \mathrm{Diff}_b^1(\mathbb{B}^3)$. Now for any $g \in x^\beta H_b^s(\mathbb{B}^3)$, $fg \in x^\beta H_b^s(\mathbb{B}^3)$.*

Proof : Since f is bounded, fg is in $x^\beta L_b^2(\mathbb{B}^3)$, and thus it suffices to show that

$$\mathcal{V}(fg) = \mathcal{V}(f) \cdot g + f \cdot \mathcal{V}(g) \in x^\beta H_b^{s-1}(\mathbb{B}^3)$$

But this follows easily because the assumption on s is just so that the multiplication lemma **Lemma 2.3** can be applied to this case. □

Now let us apply these two lemmas to the last term $MK(u)u^2$ on the LHS of

$$\Delta_o u = -Q(u) + 7Mu - M - MK(u)u^2$$

If $M \in x^{2+a}H_b^\infty(\mathbb{B}^3)$, then $u \in x^a H_b^\infty(\mathbb{B}^3)$, which implies that

$$\begin{aligned} Q(u) &\in x^{2+a+\epsilon}H_b^\infty(\mathbb{B}^3) \\ 7Mu &\in x^{2+2a}H_b^\infty(\mathbb{B}^3) \\ MK(u)u^2 &\in x^{2+3a}H_b^\infty(\mathbb{B}^3) \end{aligned}$$

We see that $MK(u)u^2$ is the smallest term on the LHS. Thus, we can assert that u has some finite expansion to begin with, just in the same way as we did in the previous section, as follows :

$$u = \sum_{(k,l)\in\mathcal{I}_0} C^u_{k,l}(\theta)x^k(\log x)^l + R^u_{\rho_0}(x,\theta);\ R^u_{\rho_0}(x,\theta) \in x^{\rho_0}H_b^\infty(\mathbb{B}^3)$$

where

$$\begin{aligned} \mathcal{I}_0 &= ((\mathcal{M}-2)\overline{\cup}\mathbb{N}) \cap \{(k,l) \mid k < \rho_0\} \\ &= (\mathcal{M}^-\overline{\cup}\ \mathbb{N}) \cap \{(k,l) \mid k < \rho_0\} \end{aligned}$$

and

$$\rho_0 = min(k_0 - \delta + \epsilon, 2(k_0 - \delta), r_0)$$

Here, let us note that ρ_0 is slightly different form ρ_0 of the previous section, because $M \in x^{(k_0-\delta)}H_b^\infty(\mathbb{B}^3)$.

Now before we can crank out the expansions as we did in the previous section, we must pay attention to $K(u)$-term that we didn't have before. We first do some rearranging :

$$\begin{aligned} \Delta_o u &= -Q(u) + 7Mu - M - MK(u)u^2 \\ &= -Q(u) + M(7u - K(u)u^2) - M \end{aligned}$$

Then we prove :

Lemma 5.7 *Suppose v has the following expansion near infinity :*

$$v = \sum_{(k,l)\in\mathcal{I}}^{k<\beta} C_{k,l}(\theta)x^k(\log x)^l + R_\beta(x,\theta);\ R_\beta(x,\theta) \in x^\beta H_b^\infty(\mathbb{B}^3)$$

Furthermore, suppose $\Phi \in C^\infty$ and $\Phi(0) = 0$. Then, $\Phi(v)$ also has a finite expansion, given below, with a remainder term of the same size as the remainder term of v :

$$\Phi(v) = \sum_{(k',l')^{k'\le\beta}\in\mathcal{I}'} C_{k',l'}(\theta)x^{k'}(\log x)^{l'} + R'_\beta(x,\theta);\ R'_\beta(x,\theta) \in x^\beta H_b^\infty(\mathbb{B}^3)$$

where $\mathcal{I}' = \cup_{k=1}^{m-1} k\cdot\mathcal{I}$ and m is the smallest integer such that $mk_o > \beta$ (here k_o is the smallest k such that $(k,l) \in \mathcal{I}_0$)

Proof : This is a simple application of the Taylor's theorem. Since $\Phi(0) = 0$,

$$\Phi(u) = \sum_{1 \le j \le m-1} \frac{\Phi^{(j)}(0)}{j!} u^j + R_m(u) u^m$$

where R_m is again a smooth function on the range of u. Then, we can just plug in the partial expansion of u and use the multiplication lemma **Lemma 2.3** to multiply out the terms in the expansion to get new terms with powers in $\mathcal{I}' = \cup_{k=1}^{m-1} k \cdot \mathcal{I}$, where $mk_o > \beta$. For the remainder term, we can use **Lemma 5.5** and **Lemma 5.6** to conclude that

$$R_m(u)u^m \in x^{mk_o} H_b^\infty(\mathbb{B}^3)$$

Since $mk_o > \beta$

$$R_m(u)u^m \in x^\beta H_b^\infty(\mathbb{B}^3)$$

□

Now let us apply this lemma when $\Phi(u) = 7u - K(u)u^2$. Then, we have

$$\Delta_o u = -Q(u) + M\Phi(u) - M$$

and the exact same arguments of the previous section go through to deduce that

$$\forall n \in \mathbb{N} : u = \sum_{(k,l) \in \mathcal{I}_n} C^u_{k,l} x^k (\log x)^l + R^u_{\rho_n};\ R^u_{\rho_n} \in x^{\rho_n} H_b^\infty(\mathbb{B}^3)$$

where

$$\rho_n = \rho_0 + n \cdot min(\epsilon, k_0 - \delta)$$

and

$$m_n = \text{ smallest integer such that } m_n \cdot i_o > \rho_n$$

and

$$\mathcal{I}_{n+1} = (((\hat{\mathcal{G}} + \mathcal{I}_n) \cup (\mathcal{M}^- + \cup_{k=1}^{m_n - 1} k \cdot \mathcal{I}_n)) \cup \mathcal{M}^-)\overline{\cup}\mathbb{N} \cap \{(k,l) \mid k < \rho_{n+1}\}$$

where $\mathcal{M} = \mathcal{M}^- + (2,0)$. Then, by the claim

Claim 5.8 *The direct limit*

$$\mathcal{I} = \lim_{n \to \infty} \mathcal{I}_n$$

exists and it is an index set.

Proof : The same arguments of the previous section go through. We simply note that $m_{n+1} > m_n$ □

we can finally conclude that

$$u \in \mathcal{A}^\infty_\mathcal{I}(\mathbb{B}^3)$$

□

Chapter 6

York's Equation

In this section, we will concern ourselves with the following operator :

$$\operatorname{div}_g \circ L_g : x^a H_b^s(\mathbb{B}^3; T\overline{\mathbb{R}^3}) \to x^{a+2} H_b^{s-2}(\mathbb{B}^3; T^*\overline{\mathbb{R}^3})$$

where L_g is the conformal killing operator that York introduced, which maps vector fields into trace-free 2-covariant symmetric tensor-fields.

$$L_g : x^a H_b^s(\mathbb{B}^3; T\overline{\mathbb{R}^3}) \ni X \to L_X g - \frac{2}{3}(\operatorname{div}_g X) g \in x^{a+1} H_b^{s-1}(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$$

We delay discussing the motivation for such the above operators till the next section. Christodoulou and O'munchadra proved the following theorem regarding the mapping property of $\operatorname{div}_g \circ L_g$, written here in b–language :

Theorem 6.1 *Suppose* $g - e \in x^\epsilon H_b^s(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$ *for* $\epsilon > 0$ *and* $s \geq 3/2 + 2$. *Then*

$$\operatorname{div}_g \circ L_g : x^a H_b^s(\mathbb{B}^3; T\overline{\mathbb{R}^3}) \to x^{a+2} H_b^{s-2}(\mathbb{B}^3; T^*\overline{\mathbb{R}^3})$$

is an isomorphism for $0 < a < 1$.

Based on this, we want to prove the following theorem, which is the main result of this section :

Theorem 6.2 *Suppose* $g - e \in x^\epsilon H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \cap A_{\mathcal{G}}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$, *for* $\epsilon > 0$ *and an index set* $\mathcal{G}$. *Moreover, let* $\tau \in x^{a+2} H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3}) \cap A_{\mathcal{E}}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3})$, *for another index set* $\mathcal{E}$. *A unique solution* $\sigma \in x^a H_b^\infty(\mathbb{B}^3; T\overline{\mathbb{R}^3})$ *to*

$$\operatorname{div}_g \circ L_g(\sigma) = \tau$$

is also in $A_{\mathcal{I}}^\infty(\mathbb{B}^3; T\overline{\mathbb{R}^3})$, *for some index set* $\mathcal{I}$ *which depends on* $\mathcal{G}$ *and* $\mathcal{E}$. *($\mathcal{I}$ will be given precisely in the proof)*

Proof : Similar to what we did in the scalar case, we will decompose

$$\operatorname{div}_g \circ L_g : x^a H_b^s(\mathbb{B}^3; T\overline{\mathbb{R}^3}) \to x^{a+2} H_b^{s-2}(\mathbb{B}^3; T^*\overline{\mathbb{R}^3})$$

as following :

$$\operatorname{div}_g \circ L_g = Y_o + Q$$

where Y_o can be thought of as a model operator and Q is a perturbation term from the model operator. There is an obvious choice for this model operator, namely when $g = e$. Thus we let,

$$Y_o = \operatorname{div}_e \circ L_e$$

Now, using the coordinates that we chose above,

$$\operatorname{div}_g \circ L_g = \nabla^i(\nabla_i X_j + \nabla_j X_i - \frac{2}{3} g_{ij} \nabla^l X_l)$$

Correspondingly, Y_o looks as following in coordinates :

$$Y_o = \begin{pmatrix} \Delta_o & 0 & 0 \\ 0 & \Delta_o & 0 \\ 0 & 0 & \Delta_o \end{pmatrix} + \frac{1}{3} \begin{pmatrix} \partial_1^2 & \partial_1\partial_2 & \partial_1\partial_3 \\ \partial_1\partial_2 & \partial_2^2 & \partial_2\partial_3 \\ \partial_3\partial_1 & \partial_3\partial_2 & \partial_3^2 \end{pmatrix}$$

Of course, what is most important about the model operator Y_o are the facts of the following 2 lemmas.

Lemma 6.3 *The model operator Y_o is*

$$Y_o = x^2 \tilde{Y_o}$$

where $\tilde{Y_o} \in \operatorname{Diff}_b^2(\mathbb{B}^3; T\overline{\mathbb{R}^3}; T^*\overline{\mathbb{R}^3})$ and moreover $\tilde{Y_o}$ is $b-$elliptic.

If we can, furthermore, prove that

Lemma 6.4

$$Q \in (x^{2+\epsilon} H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\tilde{\mathcal{G}}}^\infty(\mathbb{B}^3)) \cdot \operatorname{Diff}_b^2(\mathbb{B}^3)(\mathbb{B}^3; T\overline{\mathbb{R}^3}; T^*\overline{\mathbb{R}^3})$$

where $\tilde{\mathcal{G}} = (\cup_{n=1}^\infty n \cdot \mathcal{G}) + (2,0)$.

If the above lemmas are true, then the proof of **Theorem 4.3** can be employed to show the veracity of the conclusions of **Theorem 6.2**. This is because in the proof of **Theorem 4.3**, we never used the fact that the operator was scalar-valued, and thus it is easy to see that the arguments will go through just as well for bundle-valued operators, which we have in this case.

Thus, if we assume **Lemma 6.3** and **Lemma 6.4**, for the time being, we can proceed as follows :

$$div_g \circ L_g(\sigma) = \tau$$

can be re-written as

$$Y_o(\sigma) = Q(\sigma) + \tau$$

Since Y_o is $b-$elliptic, Proposition 5.3 of [Melrose 2] asserts that

$$\begin{aligned}\mathcal{Y}_o &= \{ \text{ indicial roots of } \hat{Y_o}(s)\} \\ &= \{(y, l_y) \in \mathbb{C} \times \mathbb{N} \mid y \text{ is a pole of order } l_y \text{ of } \hat{Y_o}^{-1}(s)\}\end{aligned}$$

is a discrete subset of $\mathbb{C} \times \mathbb{N}$, where $\tilde{Y_o}$ is the indicial family of Y_o.

Thus the exact same arguments of **Theorem 4.3** can be used to conclude that the solution

$$\sigma \in x^a H_b^\infty(\mathbb{B}^3; T\overline{\mathbb{R}^3}) \cap A_{\mathcal{I}}^\infty(\mathbb{B}^3; T\overline{\mathbb{R}^3})$$

where $\mathcal{I} = \lim_{n\to\infty} \mathcal{I}_n$ and $\mathcal{I}_n$ is recursively given as

$$\begin{aligned}\mathcal{I}_{n+1} &= ((\hat{\mathcal{G}} + \mathcal{I}_n) \cup \mathcal{E}^-)\overline{\cup}\mathcal{Y}_o \\ &= ((\hat{\mathcal{G}} + \mathcal{I}_n) \cup \mathcal{E}^-) \cup \mathcal{Y}_o \cup \\ &\quad \{(k, j+m) \mid (k,j) \in (\hat{\mathcal{G}} + \mathcal{I}_n) \cup \mathcal{E}^- \text{and } (\exists m \in \mathbb{N})((k,m) \in \mathcal{Y}_o)\}\end{aligned}$$

This finishes the proof of our main theorem of this section, **Theorem 6.2.** □

Now we will prove the two aforementioned lemmas :

Proof of Lemma 6.3 :

We will deduce the lemma from the following two claims.

Claim 6.5

$$Y_o : C^\infty(\mathbb{R}^3; T\mathbb{R}^3) \to C^\infty(\mathbb{R}^3; T^*\mathbb{R}^3)$$

is a second order elliptic-differential operator.

Proof : By the trivialization

$$T^*\mathbb{R}^3 = \mathbb{R}^3_{\{x^1,x^2,x^3\}} \times \mathbb{R}^3_{\{dx^1,dx^2,dx^3\}}$$

we can compute the top symbol of Y_o as follows :

$$\sigma(Y_o) = \begin{pmatrix} |\xi|^2 & 0 & 0 \\ 0 & |\xi|^2 & 0 \\ 0 & 0 & |\xi|^2 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & \xi_2^2 & \xi_2\xi_3 \\ \xi_3\xi_1 & \xi_3\xi_2 & \xi_3^2 \end{pmatrix}$$

Now

$$\begin{aligned}\det(\sigma(Y_o)) &= (|\xi|^2 + \frac{1}{3}\xi_1^2)(|\xi|^4 + \frac{1}{3}(\xi_2^2|\xi|^2 + \xi_3^2) + \frac{1}{9}\xi_2^2\xi_3^2 - \frac{1}{9}\xi_2^2\xi_3^2) - \\ &\quad (\frac{1}{3}\xi_1\xi_2)(\frac{1}{3}\xi_1\xi_2(|\xi|^2 + \frac{1}{3}\xi_3^2) - \frac{1}{9}\xi_1\xi_2\xi_3^2) + \\ &\quad (\frac{1}{3}\xi_1\xi_3)(\frac{1}{9}\xi_1\xi_2^2\xi_3 - \frac{1}{3}\xi_1\xi_3(|\xi|^2 + \frac{1}{3}\xi_2^2))\end{aligned}$$

$$\begin{aligned}
&= |\xi|^2(|\xi|^2 + \frac{1}{3}\xi_1^2)(|\xi|^2 + \frac{1}{3}(\xi_2^2 + \xi_3^2)) - \frac{1}{9}\xi_1^2\xi_2^2|\xi|^2 - \frac{1}{9}\xi_1^2\xi_3^2|\xi|^2 \\
&= |\xi|^2(|\xi|^2(|\xi|^2 + \frac{1}{3}(\xi_2^2 + \xi_3^2) + \frac{1}{3}\xi_1^2|\xi|^2) \\
&= |\xi|^2(|\xi|^2(|\xi|^2 + \frac{1}{3}(\xi_1^2 + \xi_2^2 + \xi_3^2))) \\
&= \frac{4}{3}|\xi|^6 \\
&> 0 \qquad \text{for } \xi \neq 0
\end{aligned}$$

This proves that Y_o is elliptic. That it is a second order differential operator is evident from its formula given above in coordinates. □

Claim 6.6

$$Y_o : x^a H_b^\infty(\mathbb{B}^3; T\overline{\mathbb{R}^3}) \to x^{a+2} H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3})$$

is in $x^2\mathrm{Diff}_b^2(\mathbb{B}^3; T\overline{\mathbb{R}^3}; T^*\overline{\mathbb{R}^3})$.

Proof : We recall that near $\partial\mathbb{B}^3$

$$1 \le i \le 3 \ : \ \partial_i = \alpha_i(\theta)(x\partial_x) + \beta_i(\theta)\partial_\theta; \quad \alpha_i(\theta), \beta_i(\theta) \in C^\infty(S^2)$$

and

$$\begin{aligned}
\partial_i\partial_j &= x^2\{\alpha_i(\theta)\alpha_j(\theta)(x\partial_x)^2 + (\alpha_i(\theta)\beta_j(\theta) - \beta_i(\theta)\alpha_j(\theta))(x\partial_x)(\partial_\theta) + \beta_i(\theta)\beta_j(\theta)\partial_\theta^2 + \\
&\quad (\alpha_i(\theta)\alpha_j(\theta) - \beta_i(\theta)\alpha_j'(\theta))(x\partial_x) + (\alpha_i(\theta)\beta_j(\theta) - \beta_i(\theta)\beta_j'(\theta))(\partial_\theta)\} \\
&= x^2 Q_{ij}; \ Q_{ij} \in \mathrm{Diff}_b^2(\mathbb{B}^3)
\end{aligned}$$

Thus, near $\partial\mathbb{B}^3$, we have

$$\begin{aligned}
Y_o &= \begin{pmatrix} \Delta_o & 0 & 0 \\ 0 & \Delta_o & 0 \\ 0 & 0 & \Delta_o \end{pmatrix} + \frac{1}{3}\begin{pmatrix} \partial_1^2 & \partial_1\partial_2 & \partial_1\partial_3 \\ \partial_1\partial_2 & \partial_2^2 & \partial_2\partial_3 \\ \partial_3\partial_1 & \partial_3\partial_2 & \partial_3^2 \end{pmatrix} \\
&= x^2 \begin{pmatrix} (x\partial_x)^2 + (x\partial_x) + \Delta_\theta & 0 & 0 \\ 0 & (x\partial_x)^2 + (x\partial_x) + \Delta_\theta & 0 \\ 0 & 0 & (x\partial_x)^2 + (x\partial_x) + \Delta_\theta \end{pmatrix} + \\
&\quad \frac{x^2}{3}\begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix} \\
&= x^2(\mathrm{I}_{3\times3}((x\partial_x)^2 + (x\partial_x) + \Delta_\theta) + \frac{1}{3}Q_{ij}) \\
&= x^2\tilde{Y}_o; \quad \tilde{Y}_o \in \mathrm{Diff}_b^2(\mathbb{B}^3; T\overline{\mathbb{R}^3}; T^*\overline{\mathbb{R}^3})
\end{aligned}$$

This proves that $Y_o \in x^2\mathrm{Diff}_b^2(\mathbb{B}^3; T\overline{\mathbb{R}^3}; T^*\overline{\mathbb{R}^3})$. □

Now, using these two claims, we will show that $\tilde{Y_o}$ is $b-$**elliptic**, i.e. elliptic as a $b-$differential operator. This means that if we make the following substitutions

$$\begin{array}{rcl} \zeta_1 & \leftrightarrow & x\partial_x \\ \zeta_\theta & \leftrightarrow & \partial_\theta \end{array}$$

where $\{\zeta_1, \zeta_\theta\}$ is a basis for ${}^bT^*\mathbb{B}^3$, then we want

$$\begin{aligned} \det({}^b\sigma_2(\tilde{Y_o})) &= \det\left(\begin{pmatrix} |\zeta|^2 & 0 & 0 \\ 0 & |\zeta|^2 & 0 \\ 0 & 0 & |\zeta|^2 \end{pmatrix} \right. \\ &\qquad \left. +\frac{1}{3} \begin{pmatrix} q_{11}(\zeta_1,\zeta_\theta,\theta) & q_{12}(\zeta_1,\zeta_\theta,\theta) & q_{13}(\zeta_1,\zeta_\theta,\theta) \\ q_{12}(\zeta_1,\zeta_\theta,\theta) & q_{22}(\zeta_1,\zeta_\theta,\theta) & q_{23}(\zeta_1,\zeta_\theta,\theta) \\ q_{31}(\zeta_1,\zeta_\theta,\theta) & q_{32}(\zeta_1,\zeta_\theta,\theta) & q_{33}(\zeta_1,\zeta_\theta,\theta) \end{pmatrix} \right) \\ &= \Sigma(\zeta_1,\zeta_\theta,\theta) \\ &> 0 \text{ for } \zeta \neq 0 \end{aligned}$$

where

$$q_{ij}(\zeta_1,\zeta_\theta,\theta) = \alpha_i(\theta)\alpha_j(\theta)(\zeta_1)^2 + (\alpha_i(\theta)\beta_j(\theta) - \beta_i(\theta)\alpha_j(\theta))(\zeta_1)(\zeta_\theta) + \beta_i(\theta)\beta_j(\theta)\zeta_\theta^2$$

Now what is most important in the above computation is ironically what is *missing*. Notice that in

$$\det({}^b\sigma_2(\tilde{Y_o})) = \Sigma(\zeta_1,\zeta_\theta,\theta)$$

there is **no** x dependance. Now, on $\mathbb{R}^3$, i.e. the interior of $\mathbb{B}^3$, we have

$$\det(\sigma(Y_o)) = x^6 \cdot \det(Jac(x,\theta)) \cdot \det({}^b\sigma_2(\tilde{Y_o}))$$

Claim 6.5 asserts that

$$\det(\sigma(Y_o))(x,\xi) > 0, \text{ for } \xi \neq 0$$

Since $x > 0$ and $\det(Jac(x,\theta)) > 0$, for $x = 1/r > 0$

$$\begin{aligned} \det({}^b\sigma_2(\tilde{Y_o})) &= \frac{1}{x^6} \cdot \frac{1}{\det(Jac(x,\theta))} \cdot \det(\sigma(Y_o)) \\ &> 0 \text{ for } \zeta \neq 0 \end{aligned}$$

Moreover, since $\det({}^b\sigma_2(\tilde{Y_o}))$ is independent of x,

$$\det({}^b\sigma_2(\tilde{Y_o})) > 0 \text{ for } \zeta \neq 0$$

even when $x = 0$. This proves that $\tilde{Y_o}$ is a $b-$elliptic operator. □

Proof of Lemma 6.4 :

Recall, in coordinates,

$$\mathrm{div}_g \circ L_g = \nabla^i(\nabla_i X_j + \nabla_j X_i - \frac{2}{3} g_{ij} \nabla^l X_l)$$

Now we can write this in a matrix form as follows :

$$\begin{pmatrix} \Delta_g & 0 & 0 \\ 0 & \Delta_g & 0 \\ 0 & 0 & \Delta_g \end{pmatrix} + \begin{pmatrix} \nabla^1\nabla_1 - \frac{2}{3}\partial_1\nabla^1 & \nabla^2\nabla_1 - \frac{2}{3}\partial_1\nabla^2 & \nabla^3\nabla_1 - \frac{2}{3}\partial_1\nabla^3 \\ \nabla^1\nabla_2 - \frac{2}{3}\partial_2\nabla^1 & \nabla^2\nabla_2 - \frac{2}{3}\partial_2\nabla^2 & \nabla^3\nabla_2 - \frac{2}{3}\partial_2\nabla^3 \\ \nabla^1\nabla_3 - \frac{2}{3}\partial_3\nabla^1 & \nabla^2\nabla_3 - \frac{2}{3}\partial_3\nabla^2 & \nabla^3\nabla_3 - \frac{2}{3}\partial_3\nabla^3 \end{pmatrix}$$

Since we assumed that

$$g - e \in x^\epsilon H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \cap \mathcal{A}_{\mathcal{G}}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$$

we know, from **Lemma 4.9** that

$$\Delta_g = \Delta_o + Q$$

where $Q \in (x^{2+\epsilon} H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\tilde{\mathcal{G}}}^\infty(\mathbb{B}^3)) \cdot \mathrm{Diff}_b^2(\mathbb{B}^3)$ and $\tilde{\mathcal{G}} = \hat{\mathcal{G}} + (2,0)$, where

$$\hat{\mathcal{G}} = \cup_{n=1}^\infty n \cdot \mathcal{G}$$

Thus, we will be done if we can prove

$$\nabla^j \nabla_i - \frac{2}{3}\partial_i \nabla^j = \frac{1}{3}\partial_i\partial_j + S_{ij}$$

where S_{ij} is also in $(x^{2+\epsilon} H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\tilde{\mathcal{G}}}^\infty(\mathbb{B}^3)) \cdot \mathrm{Diff}_b^2(\mathbb{B}^3)$.

This can be shown by brute-force computation. We first recall

$$\begin{aligned} g_{ij} &= \delta_{ij} + k_{ij};\ k_{ij} \in x^\epsilon H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\mathcal{G}}^\infty(\mathbb{B}^3) \\ g^{ij} &= \delta^{ij} + K^{ij};\ K^{ij} \in x^\epsilon H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\hat{\mathcal{G}}}^\infty(\mathbb{B}^3) \end{aligned}$$

Now

$$\nabla^j \nabla_i X_k = g^{jc}\{\partial_c(\partial_i X_k - \Gamma_{ik}^l X_l) - \Gamma_{ci}^m(\partial_m X_k - \Gamma_{mk}^l X_l) - \Gamma_{ck}^m(\partial_i X_m - \Gamma_{im}^l X_l)\}$$

We pick apart the above formula term by term.

$$\begin{aligned} g^{jc}\partial_c\partial_i X_k &= (\delta^{jc} + K^{jc})\partial_c\partial_i X_k \\ &= \partial_j\partial_i X_k + K^{jc}\partial_c\partial_i X_k \end{aligned}$$

Since

$$\partial_c\partial_i = x^2 Q_{cj};\ \ Q_{cj} \in \mathrm{Diff}_b^2(\mathbb{B}^3)$$

we see that

$$K^{jc}\partial_c\partial_i \in (x^{2+\epsilon} H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}_{\tilde{\mathcal{G}}}^\infty(\mathbb{B}^3)) \cdot \mathrm{Diff}_b^2(\mathbb{B}^3)$$

Now for the rest of the computation, the following claim is crucial :

Claim 6.7
$$\Gamma^i_{jk} \in x^{1+\epsilon} H^\infty_b(\mathbb{B}^3) \cap \mathcal{A}^\infty_{\mathcal{G}^\sharp}(\mathbb{B}^3)$$
where $\mathcal{G}^\sharp = \hat{\mathcal{G}} + (1,0)$.

Proof :

$$\begin{aligned}
\Gamma^i_{jl} &= \frac{1}{2} g^{im}(\partial_l g_{mj} + \partial_j g_{ml} - \partial_m g_{jl}) \\
&= \frac{1}{2}(\delta^{im} + K^{im})(\partial_l k_{mj} + \partial_j k_{ml} - \partial_m k_{jl}) \\
&= \frac{1}{2}(\partial_l k_{ij} + \partial_j k_{il} - \partial_i k_{jl}) + \frac{1}{2} K^{im}(\partial_l k_{mj} + \partial_j k_{ml} - \partial_m k_{jl})
\end{aligned}$$

Since
$$\partial_k = x P_k : x^\epsilon H^\infty_b(\mathbb{B}^3) \cap \mathcal{A}^\infty_{\mathcal{G}}(\mathbb{B}^3) \to x^{1+\epsilon} H^\infty_b(\mathbb{B}^3) \cap \mathcal{A}^\infty_{\mathcal{G}+(1,0)}(\mathbb{B}^3)$$

we see that
$$\Gamma^i_{jl} \in x^{1+\epsilon} H^\infty_b(\mathbb{B}^3) \cap \mathcal{A}^\infty_{\mathcal{G}^\sharp}(\mathbb{B}^3)$$

where
$$\begin{aligned}
\mathcal{G}^\sharp &= (\mathcal{G} + (1,0)) \cup (\mathcal{G} + (1,0) + \hat{\mathcal{G}}) \\
&= (\mathcal{G} \cup (\mathcal{G} + \hat{\mathcal{G}})) + (1,0) \\
&= (\mathcal{G} \cup (\mathcal{G} + \cup_{n=1}^\infty n \cdot \mathcal{G})) + (1,0) \\
&= (\mathcal{G} \cup (\cup_{n=2}^\infty n \cdot \mathcal{G})) + (1,0) \\
&= (\cup_{n=1}^\infty n \cdot \mathcal{G})) + (1,0) \\
&= \hat{\mathcal{G}} + (1,0)
\end{aligned}$$

□

Thus, using this claim, we can tackle the second term as follows :

$$\begin{aligned}
g^{jc}\partial_c(\Gamma^l_{ik} X_l) &= (\delta^{jc} + K^{jc})((\partial_c \Gamma^l_{ik})X_l + \Gamma^l_{ik}\partial_c X_l) \\
&= (\partial_j \Gamma^l_{ik})X_l + \Gamma^l_{ik}(\partial_j X_l) + K^{jc}(\partial_j \Gamma^l_{ik})X_l + K^{jc}\Gamma^l_{ik}(\partial_j X_l)
\end{aligned}$$

Since

$$\begin{aligned}
\partial_j \Gamma^l_{ik} &\in x^{2+\epsilon} H^\infty_b(\mathbb{B}^3) \cap \mathcal{A}^\infty_{\mathcal{G}^\sharp+(1,0)}(\mathbb{B}^3) \\
\Gamma^l_{ik}\partial_i &\in (x^{2+\epsilon} H^\infty_b(\mathbb{B}^3) \cap \mathcal{A}^\infty_{\mathcal{G}^\sharp+(1,0)}(\mathbb{B}^3)) \cdot \mathrm{Diff}^1_b(\mathbb{B}^3)
\end{aligned}$$

we see that

$$\begin{aligned}
\partial_j \Gamma^l_{ik} &\in (x^{2+\epsilon} H^\infty_b(\mathbb{B}^3) \cap \mathcal{A}^\infty_{\mathcal{G}^\sharp+(1,0)}(\mathbb{B}^3)) \cdot \mathrm{Diff}^0_b(\mathbb{B}^3) \\
K^{jc}\partial_j \Gamma^l_{ik} &\in (x^{2+\epsilon} H^\infty_b(\mathbb{B}^3) \cap \mathcal{A}^\infty_{\hat{\mathcal{G}}+\mathcal{G}^\sharp+(1,0)}(\mathbb{B}^3)) \cdot \mathrm{Diff}^0_b(\mathbb{B}^3) \\
\Gamma^l_{ik}\partial_j &\in (x^{2+\epsilon} H^\infty_b(\mathbb{B}^3) \cap \mathcal{A}^\infty_{\mathcal{G}^\sharp+(1,0)}(\mathbb{B}^3)) \cdot \mathrm{Diff}^1_b(\mathbb{B}^3) \\
K^{jc}\Gamma^l_{ik}\partial_j &\in (x^{2+\epsilon} H^\infty_b(\mathbb{B}^3) \cap \mathcal{A}^\infty_{\hat{\mathcal{G}}+\mathcal{G}^\sharp+(1,0)}(\mathbb{B}^3)) \cdot \mathrm{Diff}^1_b(\mathbb{B}^3)
\end{aligned}$$

we see that

$$g^{jc}\partial_c(\Gamma^l_{ik}\cdot\_\,)\in(x^{2+\epsilon}H_b^\infty(\mathbb{B}^3)\cap\mathcal{A}^\infty_{\mathcal{G}_2})\cdot\mathrm{Diff}^1_b(\mathbb{B}^3)$$

where

$$\begin{aligned}
\mathcal{G}_2 &= (\mathcal{G}^\sharp+(1,0))\cup(\hat{\mathcal{G}}+\mathcal{G}^\sharp+(1,0))\\
&= (\hat{\mathcal{G}}+(1,0)+(1,0))\cup(\hat{\mathcal{G}}+(\hat{\mathcal{G}}+(1,0))+(1,0))\\
&= (\hat{\mathcal{G}}+(2,0))\cup(2\hat{\mathcal{G}}+(2,0))\\
&= (\hat{\mathcal{G}}+(2,0)) \text{ because } 2\hat{\mathcal{G}}\subset\hat{\mathcal{G}}\\
&= \tilde{\mathcal{G}}
\end{aligned}$$

Thus, we finally conclude that

$$g^{jc}\partial_c(\Gamma^l_{ik}\cdot\_\,)\in(x^{2+\epsilon}H_b^\infty(\mathbb{B}^3)\cap\mathcal{A}^\infty_{\tilde{\mathcal{G}}})\cdot\mathrm{Diff}^1_b(\mathbb{B}^3)$$

Next we tackle :

$$\begin{aligned}
g^{jc}\Gamma^m_{ci}(\partial_m X_k-\Gamma^l_{mk}X_l) &= (\delta^{jc}+K^{jc})(\Gamma^m_{ci}(\partial_m X_k-\Gamma^l_{mk}X_l)\\
&= \Gamma^m_{ji}(\partial_m X_k)+K^{jc}\Gamma^m_{ci}(\partial_m X_k)-\Gamma^m_{ji}\Gamma^l_{mk}X_l-K^{jc}\Gamma^m_{ci}\Gamma^l_{mk}X_l
\end{aligned}$$

Now, since

$$\begin{aligned}
\Gamma^m_{ji}(\partial_m\cdot_\,) &\in (x^{2+\epsilon}H_b^\infty(\mathbb{B}^3)\cap\mathcal{A}^\infty_{\mathcal{G}^\sharp+(1,0)})\cdot\mathrm{Diff}^1_b(\mathbb{B}^3)\\
K^{jc}\Gamma^m_{ci}(\partial_m\cdot_\,) &\in (x^{2+\epsilon}H_b^\infty(\mathbb{B}^3)\cap\mathcal{A}^\infty_{\hat{\mathcal{G}}+\mathcal{G}^\sharp+(1,0)})\cdot\mathrm{Diff}^1_b(\mathbb{B}^3)\\
\Gamma^m_{ji}\Gamma^l_{mk}\cdot_ &\in (x^{2+\epsilon}H_b^\infty(\mathbb{B}^3)\cap\mathcal{A}^\infty_{2\mathcal{G}^\sharp})\cdot\mathrm{Diff}^0_b(\mathbb{B}^3)\\
K^{jc}\Gamma^m_{ci}\Gamma^l_{mk}\cdot_ &\in (x^{2+\epsilon}H_b^\infty(\mathbb{B}^3)\cap\mathcal{A}^\infty_{2\mathcal{G}^\sharp+\hat{\mathcal{G}}})\cdot\mathrm{Diff}^0_b(\mathbb{B}^3)
\end{aligned}$$

we see that

$$g^{jc}\Gamma^m_{ci}(\partial_m\cdot\_\;-\Gamma^l_{mk}\cdot\_\,)\in(x^{2+\epsilon}H_b^\infty(\mathbb{B}^3)\cap\mathcal{A}^\infty_{\mathcal{G}_3})\cdot\mathrm{Diff}^1_b(\mathbb{B}^3)$$

where

$$\begin{aligned}
\mathcal{G}_3 &= (\mathcal{G}^\sharp+(1,0))\cup(\hat{\mathcal{G}}+\mathcal{G}^\sharp+(1,0))\cup 2\mathcal{G}^\sharp\cup(2\mathcal{G}^\sharp+\hat{\mathcal{G}})\\
&= ((\hat{\mathcal{G}}+(1,0))+(1,0))\cup(\hat{\mathcal{G}}+(\hat{\mathcal{G}}+(1,0))+(1,0))\cup\\
&\quad\; 2(\hat{\mathcal{G}}+(1,0))\cup(2(\hat{\mathcal{G}}+(1,0))+\hat{\mathcal{G}})\\
&= (\hat{\mathcal{G}}+(2,0))\cup(2\hat{\mathcal{G}}+(2,0))\cup(2\hat{\mathcal{G}}+(2,0))\cup(3\hat{\mathcal{G}}+(2,0))\\
&= (\hat{\mathcal{G}}+(2,0)) \text{ because } 2\hat{\mathcal{G}},3\hat{\mathcal{G}}\subset\hat{\mathcal{G}}\\
&= \tilde{\mathcal{G}}
\end{aligned}$$

Thus we can conclude that

$$g^{jc}\Gamma^m_{ci}(\partial_m\cdot\_\;-\Gamma^l_{mk}\cdot\_\,)\in(x^{2+\epsilon}H_b^\infty(\mathbb{B}^3)\cap\mathcal{A}^\infty_{\tilde{\mathcal{G}}})\cdot\mathrm{Diff}^1_b(\mathbb{B}^3)$$

Similarly the last term

$$\Gamma^m_{ck}(\partial_i \cdot \_ \; - \Gamma^l_{im} \cdot \_\,) \in (x^{2+\epsilon} H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}^\infty_{\mathcal{G}}) \cdot \mathrm{Diff}^1_b(\mathbb{B}^3)$$

Putting these all together, we can finally conclude that

$$\nabla^j \nabla_i = \partial_i \partial_j \; + S'_{ij};\;\; S'_{ij} \in (x^{2+\epsilon} H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}^\infty_{\mathcal{G}}) \cdot \mathrm{Diff}^2_b(\mathbb{B}^3)$$

Now in a similar manner, we can carry out the computation of

$$\begin{aligned}
\partial_i(\nabla^j X_k) &= \partial_i(g^{jl}(\partial_l X_k - \Gamma^m_{lk} X_m)) \\
&= \partial_i((\delta^{jl} + K^{jl})(\partial_l X_k - \Gamma^m_{lk} X_m)) \\
&= \partial_i(\partial_j X_k + K^{jl}\partial_l X_k - \Gamma^m_{jk} X_m - K^{jl}\Gamma^m_{lk} X_m) \\
&= \partial_i \partial_j X_k + \partial_i(K^{jl}\partial_l X_k) - \partial_i(\Gamma^m_{jk} X_m) - \partial_i(K^{jl}\Gamma^m_{lk} X_m)
\end{aligned}$$

to conclude that

$$\partial_i \nabla^j \;\; = \partial_i \partial_j + S''_{ij};\;\; S''_{ij} \in (x^{2+\epsilon} H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}^\infty_{\mathcal{G}}) \cdot \mathrm{Diff}^2_b(\mathbb{B}^3)$$

Letting $S_{ij} = S'_{ij} + S''_{ij}$, we can finally conclude that

$$\nabla^j \nabla_i - \frac{2}{3}\partial_i \nabla^j = \frac{1}{3}\partial_i \partial_j + S_{ij};\;\; S_{ij} \in (x^{2+\epsilon} H_b^\infty(\mathbb{B}^3) \cap \mathcal{A}^\infty_{\mathcal{G}}) \cdot \mathrm{Diff}^2_b(\mathbb{B}^3)$$

□

Chapter 7

Construction of Initial Data

Now using all the ingredients of the last few section, we finally show how to construct initial data to Einstein's vacuum equation with complete asymptotic expansions, thereby proving the main theorem of this thesis, **Theorem 2.8**. Let us first start with g, a Riemannian metric on $\mathbb{R}^3$, such that

$$g - e \in x^a H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \cap A_{\mathcal{G}}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$$

for $0 < a < 1$ and a index set $\mathcal{G}$, as is assumed for **Theorem 2.8**. Now, we need to come up with a symmetric 2-tensor k_{ij} such that

$$k_{ij} \in x^{a+1} H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \cap A_{\mathcal{K}}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$$

for some index set $\mathcal{K}$ and

$$\begin{aligned} \mathrm{tr}_g(k) &= 0 \\ \mathrm{div}_g(k) &= 0 \\ R(g) &= |k|_g^2 \end{aligned}$$

In the following sections, we will show that given any element k_{ij} in $x^{a+1} H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \cap A_{\mathcal{K}}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$, for any index set $\mathcal{K}$, we can modify (g, k) to $(\tilde{g}, \tilde{k})$ such that $(\tilde{g}, \tilde{k})$ indeed satisfies the above constraint equations.

7.1 Trace-Free Condition

The first constraint equation

$$\mathrm{tr}_g(k) = 0$$

is rather easy to satisfy. We define

$$\begin{aligned} k'_{ij} &= k_{ij} - \frac{1}{3}\mathrm{tr}_g(k_{ij})k_{ij} \\ &= k_{ij} - \frac{1}{3} g^{lmj} k_{lm} k_{ij} \\ &= k_{ij} - \frac{1}{3}(\delta^{lm} + \Omega^{lm}) k_{lm} k_{ij} \end{aligned}$$

We recall from the previous sections that

$$\Omega^{ij} \in x^a H_b^\infty(\mathbb{B}^3) \cap A_{\hat{\mathcal{G}}}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}); \text{ where } \hat{\mathcal{G}} = \cup_{n=1}^\infty n \cdot \mathcal{G}$$

Thus

$$k'_{ij} \in x^{a+1} H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \cap A_{\mathcal{K}'}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$$

where

$$\begin{aligned} \mathcal{K}' &= \mathcal{K} \cup (2\mathcal{K} \cup (2\mathcal{K} + \hat{\mathcal{G}})) \\ &= \mathcal{K} \cup (2\mathcal{K} + \hat{\mathcal{G}}) \end{aligned}$$

and (g, k') satisfies the first of the constraint equations.

7.2 Divergence-Free Condition

Now that k'_{ij} is trace-free, we now try to modify it such that it will satisfy the second constraint equation :

$$\text{div}_g(k) = 0$$

Following York's approach, we try to find a symmetric 2-cotensor p_{ij} that we exactly need to subtract from k_{ij} such that

$$\begin{aligned} k''_{ij} &= k'_{ij} - p_{ij} \\ \text{div}_g(k'') &= 0 \end{aligned}$$

In other words, we want to find p_{ij} such that

$$\text{div}_g(p_{ij}) = \text{div}_g(k'_{ij}) \quad (\dagger)$$

Such p_{ij}, of course, exists. For instance, one can simply take $p_{ij} = k'_{ij}$, in which case, however, $k''_{ij} = 0$. To get a more nontrivial outcome, York introduced L_g (see [C-M]), the conformal killing operator, which maps vector fields into *trace-free* 2-covariant symmetric tensorfields.

$$L_g : C^\infty(\mathbb{R}^3; T\mathbb{R}^3) \ni X \to L_X g - \frac{2}{3}(\text{div}_g X) g \in C^\infty(\mathbb{R}^3; Sym^2 T^*\mathbb{R}^3)$$

suggested that one should look for p_{ij} in $\text{Ran}(L_g) \subset C^\infty(\mathbb{R}^3; Sym^2 T^*\mathbb{R}^3)$. Given our assumption on g and a, **Theorem 6.1** of Christodoulou and O'Munchadra asserts that

$$\text{div}_g \circ L_g : x^a H_b^\infty(\mathbb{B}^3; T\overline{\mathbb{R}^3}) \to x^{a+2} H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3})$$

is an isomorphism, we can indeed find the unique $\sigma \in x^a H_b^\infty(\mathbb{B}^3; T\overline{\mathbb{R}^3})$ such that

$$\begin{aligned} p_{ij} &= L_g(\sigma) \\ \text{div}_g(p_{ij}) &= \text{div}_g(k'_{ij}) \end{aligned}$$

Now by the following claim, which we will prove at the end of this chapter,

Claim 7.1 *Given our assumption on g, we see that*

$$\begin{aligned}\text{div}_g \quad : \quad & x^c H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3} \cap A_{\mathcal{L}}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \longrightarrow \\ & x^{c+1} H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3} \cap A_{\tilde{\mathcal{L}}}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3})\end{aligned}$$

where

$$\tilde{\mathcal{L}} = (\mathcal{L} + (1,0)) \cup (\mathcal{L} + (1,0) + \hat{\mathcal{G}})$$

□

we see that

$$div_g(k') \in x^{a+2} H_b^\infty(\mathbb{B}^3) \cap A_{\tilde{\mathcal{K}'}}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3})$$

and thus $(g, div_g(k'))$ satisfies the assumptions of **Theorem 6.2**, and we can apply the theorem to obtain σ such that

$$\begin{aligned}\text{div}_\circ L_g(\sigma) &= \text{div}_g(k') \\ \sigma &\in x^a H_b^\infty(\mathbb{B}^3) \cap A_{\mathcal{I}_{\mathcal{G},\tilde{\mathcal{K}'}}}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3})\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_{\mathcal{G},\tilde{\mathcal{K}'}} &= \lim_{n\to\infty} \mathcal{I}_n \\ \mathcal{I}_{n+1} &= ((\hat{\mathcal{G}} + \mathcal{I}_n) \cup \tilde{\mathcal{K}'}^-)\overline{\cup}\mathcal{Y}_o\end{aligned}$$

Once σ is obtained, we let

$$p_{ij} = L_g(\sigma)$$

and using the following claim

Claim 7.2 *Given our assumptions of g,*

$$L_g : x^c H_b^\infty(\mathbb{B}^3) \cap A_{\mathcal{L}}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3}) \to x^{c+1} H_b^\infty(\mathbb{B}^3) \cap A_{\tilde{\mathcal{L}}}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$$

where

$$\tilde{\mathcal{L}} = (\mathcal{L} + (1,0)) \cup (\mathcal{L} + (1,0) + \hat{\mathcal{G}})$$

□

we can conclude that

$$\begin{aligned}p_{ij} &\in x^{a+1} H_b^\infty(\mathbb{B}^3) \cap A_{\mathcal{I}_\mathcal{P}}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \\ \mathcal{I}_\mathcal{P} &= (\mathcal{I}_{\mathcal{G},\tilde{\mathcal{K}'}} + (1,0)) \cup (\mathcal{I}_{\mathcal{G},\tilde{\mathcal{K}'}} + (1,0) + \hat{\mathcal{G}})\end{aligned}$$

Thus, finally we have

$$\begin{aligned}k''_{ij} &= k'_{ij} - p_{ij} \\ k''_{ij} &\in x^{a+1} H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \cap A_{\mathcal{K}''}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})\end{aligned}$$

where

$$\mathcal{K}'' = \mathcal{K}' \cup \mathcal{I}_\mathcal{P}$$

and (g_{ij}, k''_{ij}) now satisfies the first two of the constraint equations.

7.3 Non-linear Constraint

Now it remains to satisfy the last constraint equation which, unlike the first two, is non-linear. The key observation York ([York]) made is that if (g, k'') satisfies the first two constraint equations, so does $(\Phi^4 g, \Phi^{-2}k'')$ for any conformal factor Φ. Moreover, York observed that if we choose Φ, just so that it satisfies the Licnerowicz equation :

$$\Delta_g \Phi - \frac{1}{8}R(g)\Phi + \frac{1}{8}|k''|_g^2 \Phi^{-7} = 0$$

then $(\tilde{g}, \tilde{k}) = (\Phi^4 g, \Phi^{-2}k'')$ will indeed satisfy the final constraint

$$R(\tilde{g}) = |\tilde{k}|_{\tilde{g}}^2$$

in addition to the first two constraint equations.

Now as it is done in practice ([Cantor], [C-M]), we look for the solution of the Lichnerwicz equations in the form of two conformal factors.

$$\Phi = \chi\Psi$$

The conformal factor χ is a solution to

$$\Delta_g \chi - \frac{1}{8}R(g)\chi = 0 \quad (\dagger)$$

Then $(\overline{g}, \overline{k''}) = (\chi^4 g, \chi^{-2}k'')$ is such that

$$R(\chi^4 g) = 0$$

Now, we then can look for a solution Ψ to

$$\Delta_{\overline{g}} \Psi + \frac{1}{8}|\overline{k''}|_{\overline{g}}^2 \Psi^{-7} = 0 \quad (\ddagger)$$

Let us first look at $(\dagger)$. We first establish the following claim:

Claim 7.3 *If*

$$g - e \in x^a H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \cap A_{\mathcal{G}}^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$$

then

$$R(g) \in x^{a+2} H_b^\infty(\mathbb{B}^3) \cap A_{\hat{\mathcal{G}}+(2,0)}^\infty(\mathbb{B}^3)$$

where $\hat{\mathcal{G}} = \cup_{n=1}^\infty n \cdot \mathcal{G}$. □

Since we want our conformal factor χ to approach 1 at spatial infinity, we let $\chi = 1 + \overline{\chi}$. Then $\overline{\chi}$ satisfies

$$\Delta_g \overline{\chi} - \frac{1}{8}R(g)\overline{\chi} = \frac{1}{8}R(g) \quad (\dagger')$$

Now because of the preceding claim, $(g, R(g))$ satisfies the assumptions of **Theorem4.3**, so we can apply it to $(\dagger')$ to get :

$$\overline{\chi} \in x^a H_b^\infty(\mathbb{B}^3) \cap A^\infty_{\mathcal{I}_{\overline{\chi}}}(\mathbb{B}^3)$$

where $\mathcal{I}_{\overline{\chi}} = \lim_{n\to\infty} \mathcal{I}_n$, and

$$\mathcal{I}_{n+1} = ((\hat{\mathcal{G}} + \mathcal{I}_n) \cup \hat{\mathcal{G}})\overline{\cup}\mathbb{N}$$

Then $(\overline{g}, \overline{k''})$ is such that

$$\overline{g} - e \in x^a H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \cap A^\infty_{\mathcal{I}_{\overline{\mathcal{G}}}}(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$$

where

$$\mathcal{I}_{\overline{\mathcal{G}}} = (\cup_{i=1}^4 i \cdot \mathcal{I}_{\overline{\chi}}) \cup \mathcal{G} \cup ((\cup_{i=1}^4 i \cdot \mathcal{I}_{\overline{\chi}}) + \mathcal{G})$$

and similarly

$$\overline{k''}_{ij} \in x^{a+1} H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \cap A^\infty_{\mathcal{I}''_{\mathcal{K}}}(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$$

where

$$\mathcal{I}_{\overline{\mathcal{K}''}} = (\cup_{i=1}^4 i \cdot \mathcal{I}_{\overline{\chi}}) \cup \mathcal{K}'' \cup ((\cup_{i=1}^4 i \cdot \mathcal{I}_{\overline{\chi}}) + \mathcal{K}'')$$

We finally turn to the $(\ddagger)$. We first claim

Claim 7.4

$$|\overline{k''}|_{\overline{g}}^2 \in x^{a+1} H_b^\infty(\mathbb{B}^3) \cap A^\infty_{\mathcal{M}}(\mathbb{B}^3)$$

for

$$\mathcal{M} = 2\mathcal{I}_{\overline{\mathcal{K}''}} \cup (2\mathcal{I}_{\overline{\mathcal{K}''}} + \hat{G})$$

□

Then since $(\overline{g}, |\overline{k''}|_{\overline{g}})$ satisfies the assumptions of **Theorem 5.1**, we can use the theorem to $(\ddagger)$ to conclude that

$$\Psi \in x^a H_b^\infty(\mathbb{B}^3) \cap A^\infty_{\mathcal{I}_\Psi}(\mathbb{B}^3)$$

where $\mathcal{I}_\Psi = \lim_{n\to\infty} \mathcal{I}_n$, and

$$\mathcal{I}_{n+1} = ((\hat{\overline{\mathcal{G}}} + \mathcal{I}_n) \cup (\mathcal{M} + \cup_{i=1}^{m_n - 1} i \cdot \mathcal{I}_n) \cup \mathcal{M})\overline{\cup}\mathbb{N}$$

and finally we have

$$\begin{aligned} (\tilde{g}, \tilde{k}) &= (\Psi^4 \overline{g}, \Psi^{-2}\overline{k''}) \\ &= (\Psi^4 \xi^4 g, \Psi^{-2}\xi^{-2} k'') \\ &= (\Phi^4 g, \Phi^{-2} k'') \end{aligned}$$

and

$$\tilde{g} - e \in x^a H_b^\infty(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \cap A^\infty_{\mathcal{I}_{\tilde{\mathcal{G}}}}(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$$

where

$$\mathcal{I}_{\tilde{\mathcal{G}}} = (\cup_{i=1}^{4}\mathcal{I}_{\Psi}) \cup \mathcal{G} \cup ((\cup_{i=1}^{4}\mathcal{I}_{\Psi}) + \mathcal{G})$$

and similarly

$$\tilde{k}_{ij} \in x^{a+1}H_b^{\infty}(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3}) \cap A^{\infty}_{\mathcal{I}_{\tilde{\mathcal{K}}}}(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$$

where

$$\mathcal{I}_{\tilde{\mathcal{K}}} = (\cup_{i=1}^{4}\mathcal{I}_{\Psi}) \cup \overline{\mathcal{K}''} \cup ((\cup_{i=1}^{4}\mathcal{I}_{\Psi}) + \mathcal{I}_{\overline{\mathcal{K}''}})$$

This concludes the proof of the main result of this thesis, **Theorem 2.8**. We finish with proofs of the claims that were made in this chapter.

□

Proof of Claim 7.1 If

$$k'_{ij} \in x^{a+1}H_b^{\infty}(\mathbb{B}^3) \cap A^{\infty}_{\mathcal{K}'}(\mathbb{B}^3; T^*\overline{\mathbb{R}^3} \otimes T^*\overline{\mathbb{R}^3})$$

then we can compute

$$\begin{aligned}
\mathrm{div}_g(k') &= g^{ci}\nabla_c k'_{ij} \\
&= (\delta^{ci} + \Omega^{ci})(\partial_c k'_{ij} - \Gamma^m_{ci}k'_{mj} - \Gamma^m_{cj}k'_{im}) \\
&= \partial_i k'_{ij} + \Omega^{ci}\partial_c k'_{ij} - \Gamma^m_{ii}k'_{mj} - \Omega^{ci}\Gamma^m_{ci}k'_{mj} - \Gamma^m_{ij}k'_{im} - \Omega^{ci}\Gamma^m_{cj}k'_{im}
\end{aligned}$$

Now

$$\begin{aligned}
\partial_i k'_{ij} &\in x^{a+2}H_b^{\infty}(\mathbb{B}^3) \cap A^{\infty}_{\mathcal{K}'+(1,0)}(\mathbb{B}^3; T^*\overline{\mathbb{R}^3}) \\
\Omega^{ci}\partial_c k'_{ij} &\in x^{2+2a}H_b^{\infty}(\mathbb{B}^3) \cap A^{\infty}_{\mathcal{K}'+\hat{\mathcal{G}}+(1,0)}(\mathbb{B}^3; T^*\overline{\mathbb{R}^3}) \\
\Gamma^m_{ii}k'_{mj} - \Gamma^m_{ij}k'_{im} &\in x^{2+2a}H_b^{\infty}(\mathbb{B}^3) \cap A^{\infty}_{\mathcal{K}'+\mathcal{G}^\sharp}(\mathbb{B}^3; T^*\overline{\mathbb{R}^3}) \\
\Omega^{ci}\Gamma^m_{ci}k'_{mj} - \Omega^{ci}\Gamma^m_{cj}k'_{im} &\in x^{2+3a}H_b^{\infty}(\mathbb{B}^3) \cap A^{\infty}_{\mathcal{K}'+\mathcal{G}^\sharp+\hat{\mathcal{G}}}(\mathbb{B}^3; T^*\overline{\mathbb{R}^3})
\end{aligned}$$

Thus we see that

$$\mathrm{div}_g(k') \in x^{a+2}H_b^{\infty}(\mathbb{B}^3) \cap A^{\infty}_{\widetilde{\mathcal{K}'}}(\mathbb{B}^3; T^*\overline{\mathbb{R}^3})$$

where

$$\begin{aligned}
\widetilde{\mathcal{K}'} &= (\mathcal{K}' + (1,0)) \cup (\mathcal{K}' + \hat{\mathcal{G}} + (1,0)) \cup (\mathcal{K}' + \mathcal{G}^\sharp) \cup (\mathcal{K}' + \mathcal{G}^\sharp + \hat{\mathcal{G}}) \\
&= (\mathcal{K}' + (1,0)) \cup (\mathcal{K}' + \mathcal{G}^\sharp) \cup (\mathcal{K}' + \mathcal{G}^\sharp) \cup (\mathcal{K}' + \mathcal{G}^\sharp + \hat{\mathcal{G}}) \\
&= (\mathcal{K}' + (1,0)) \cup (\mathcal{K}' + \mathcal{G}^\sharp) \cup (\mathcal{K}' + \mathcal{G}^\sharp + \hat{\mathcal{G}}) \\
&= (\mathcal{K}' + (1,0)) \cup (\mathcal{K}' + \mathcal{G}^\sharp) \\
&= (\mathcal{K}' + (1,0)) \cup (\mathcal{K}' + (1,0) + \hat{\mathcal{G}})
\end{aligned}$$

□

Proof of Claim 7.2
This is a straightfoward computation as in the preceding proof.
□

Proof of Claim 7.3 We simply compute

$$\begin{aligned} R &= g^{ik} g^{jl} R_{ijkl} \\ &= g^{ik} g^{jl} g_{ic} R^c_{jkl} \\ &= g^{ik} g^{jl} g_{ic} (\partial_k \Gamma^c_{jl} - \partial_l \Gamma^c_{jk}) \\ &= g^{jl} \delta^k_c (\partial_k \Gamma^c_{jl} - \partial_l \Gamma^c_{jk}) \\ &= g^{jl} (\partial_k \Gamma^k_{jl} - \partial_l \Gamma^k_{jk}) \\ &= (\delta^{jl} + \Omega^{jl})(\partial_k \Gamma^k_{jl} - \partial_l \Gamma^k_{jk}) \end{aligned}$$

Thus, we see that

$$\begin{aligned} R &\in x^{a+2} H_b^\infty(\mathbb{B}^3) \cap A^\infty_{(\mathcal{G}^\sharp+(1,0)) \cup (\hat{\mathcal{G}}+\mathcal{G}^\sharp+(1,0))}(\mathbb{B}^3) \\ R &\in x^{a+2} H_b^\infty(\mathbb{B}^3) \cap A^\infty_{(\hat{\mathcal{G}}+(2,0))}(\mathbb{B}^3) \end{aligned}$$

□

Proof of Claim 7.4 This is immediate from

$$\begin{aligned} |\overline{k''}|^2_{\overline{g}} &= g^{ij} g^{mn} k_{im} k_{jn} \\ &= (\delta^{ij} + \Omega^{ij})(\delta^{mn} + \Omega^{mn}) k_{im} k_{jn} \end{aligned}$$

□

Chapter 8

Future Direction

By the work of the previous seven chapters, we now know that there are pairs (g,k) which satisfy the **Constraint Equations**

$$\begin{aligned} \mathrm{tr}_g(k) &= 0 \\ \mathrm{div}_g(k) &= 0 \\ R(g) &= |k|_g^2 \end{aligned}$$

and have a complete asymptotic expansion, i.e.

$$\begin{aligned} g-e &\in x^a H_b^\infty(\mathbb{B}^3;T^*\overline{\mathbb{R}^3}\otimes T^*\overline{\mathbb{R}^3})\cap A_{\mathcal{G}}^\infty(\mathbb{B}^3;T^*\overline{\mathbb{R}^3}\otimes T^*\overline{\mathbb{R}^3}) \\ k_{ij} &\in x^{a+1} H_b^\infty(\mathbb{B}^3;T^*\overline{\mathbb{R}^3}\otimes T^*\overline{\mathbb{R}^3})\cap A_{\mathcal{K}}^\infty(\mathbb{B}^3;T^*\overline{\mathbb{R}^3}\otimes T^*\overline{\mathbb{R}^3}) \end{aligned}$$

for $0<a<1$ and index sets $\mathcal{G}$ and $\mathcal{K}$

Now, with (g,k) as initial data, let us suppose $(g(t),k(t))$ is a solution to the **Evolution Equations** that were introduced in **Chapter 1** :

$$\begin{aligned} \partial_t g_{ij} &= -2\phi k_{ij} \\ \partial_t k_{ij} &= -\nabla_i\nabla_j\phi+\phi(R_{ij}-2k_{ia}k_j^a) \\ \Delta\phi &= |k|^2\phi \end{aligned}$$

Recall that we asked the following question :

Question 2 : Does $(g(t),k(t))$ also have a complete asymptotic expansion for each t, at least for t small?

I believe the answer to the above question is **Yes**. I have not yet quite established this, but for the remainder of this chapter, I want to briefly describe where I am now and how I would proceed in the future in order to answer **Question 2** affirmatively.

In order to make use of the well-known local existence result of Choquet-Bruhat, we first re-write **Evolution Equations** as follows (see Lemma 10.2.1

of [C-K] for more detail), :

$$\begin{array}{rcl} -(\phi^{-1}\partial_t)^2 k_{ij} + \Delta k_{ij} & = & N_{ij} \\ -\phi^{-1}\partial_t g_{ij} & = & 2k_{ij} \\ \Delta\phi & = & |k|^2\phi \end{array}$$

where N_{ij} is a non-linear expression in g, k, ϕ, and their covariant derivatives. We would like to show that if an initial pair (g, k) has a complete asymptotic expansion at infinity on $(t = 0) - slice$, so does the local solution $(g(t), k(t))$ to the above elliptic-hyperbolic system on each $t - slice$, for t small.

As what I believe a *major* step in affirming this, I consider the following second order linear hyperbolic operator :

$$P = \Box_{\mathbf{g}} + \mathbf{g}(\mathbf{a}, \nabla) + b$$

where

$$\mathbf{g}_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta};\ h_{\alpha\beta} \in x^{\epsilon} H_b^{\infty}(\overline{M}; Sym^2 T^*\overline{M}) \cap \mathcal{A}_{\mathcal{H}}^{\infty}(\overline{M}; Sym^2 T^*\overline{M})$$

and $\overline{M} \cong \mathbb{R}_t \times \mathbb{B}^3$, $\epsilon > 0$, and η_{ij} is the Minkowski metric. Moreover, a vector field $\mathbf{a} \in \mathcal{A}_{\mathcal{E}}^{\infty}(\overline{M}; T\overline{M})$ and a function $b \in \mathcal{A}_{\mathcal{B}}^{\infty}(\overline{M})$. If we now denote D^+(M) to be the distributions of M with a past compact support, we prove :

Theorem 8.1 *Suppose $f \in D^+(M)$ is in $\mathcal{A}_{\mathcal{F}}^{\infty}(\overline{M})$. Then $u \in D^+(M)$, the unique solution to $Pu = f$, is also in $\mathcal{A}_{\mathcal{I}}^{\infty}(\overline{M})$, for an index set $\mathcal{I}$ which depends on $\mathcal{F}$, $\mathcal{E}$, and $\mathcal{B}$*

Proof : I first explicitly write down the solution u, using the Hardarmard's construction as following :

$$u(q) = \frac{1}{2\pi}\int_{C^-(q)} U(p,q) f(p)\mu_\Gamma(p) + \frac{1}{2\pi}\int V^-(p,q) f(p)\mu(p)$$

By analyzing how the mapping properties of the kernel $U(p, q)$ and $V(p, q)$ depend on the regularity of $\mathbf{g}_{\alpha\beta}$, we can actually prove

Lemma 8.2 *Let P be as above. Suppose $f \in D^+(M)$ is in $H_b^{\infty}(\overline{M})$. Then $u \in D^+(M)$, the unique solution to $Pu = f$, is also in $H_b^{\infty}(\overline{M})$.*

Using the lemma and the observation that the above theorem is true for $(-\partial_t)^2 + \Delta_o$, the flat wave-operator, (this can be easily deduced for we can explicitly write down solutions of the flat wave operator using the fundamental solutions), we can crank out the terms in the expansion for u, given that f has a complete asymptotic expansion. $\Box$

With this theorem at our disposal, let us look at the re-written **Evoultion Equations** above. Using the theorem and the work of the previous chapters, I have proven, so far, that the solutions of the each of the three re-written evolution equations, considered independently from each other, have asymptotic expansions if the metric, coefficients and the RHS do. However, since these three equations are coupled together, I need to understand the interplay between the elliptic and the hyperbolic equations. My current idea is to set up an *iteration* argument which produces more and more terms in the asymptotic expansion of (g, k) at each iteration, similar to the original iteration argument of Choquet-Bruhat on short-time existence of a hyperbolic system.

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