CERN-PH-TH/2006-112

# Phase transitions, double–scaling limit, and topological strings

Nicola Caporaso[a], Luca Griguolo[b], Marcos Mariño[c1], Sara Pasquetti[b] and Domenico Seminara[d]

[a]Center for Theoretical Physics, MIT, Cambridge, MA 02139 USA
caporaso@fi.infn.it

[b] Dipartimento di Fisica, Università di Parma, INFN Gruppo Collegato di Parma,
Parco Area delle Scienze 7/A, 43100 Parma, Italy
griguolo@fis.unipr.it, pasquetti@fis.unipr.it

[c]Department of Physics, Theory Division, CERN, Geneva 23, CH-1211 Switzerland
marcos@mail.cern.ch

[d] Dipartimento di Fisica, Polo Scientifico Università di Firenze,
INFN Sezione di Firenze Via G. Sansone 1, 50019 Sesto Fiorentino, Italy
seminara@fi.infn.it



**Abstract**

Topological strings on Calabi–Yau manifolds are known to undergo phase transitions at small distances. We study this issue in the case of perturbative topological strings on local Calabi–Yau threefolds given by a bundle over a two-sphere. This theory can be regarded as a q–deformation of Hurwitz theory, and it has a conjectural nonperturbative description in terms of q–deformed 2d Yang–Mills theory. We solve the planar model and find a phase transition at small radius in the universality class of 2d gravity. We give strong evidence that there is a double–scaled theory at the critical point whose all genus free energy is governed by the Painlevé I equation. We compare the critical behavior of the perturbative theory to the critical behavior of its nonperturbative description, which belongs to the universality class of 2d supergravity, and we comment on possible implications for nonperturbative 2d gravity. We also give evidence for a new open/closed duality relating these Calabi–Yau backgrounds to open strings with framing.

[1]Also at Departamento de Matemática, IST, Lisboa, Portugal.

# Contents

# 1 Introduction

It has been recognized for a long time that string theory provides in a natural way a deformation of classical geometry. The deformation parameter is given by

$$\frac{\ell_s}{R} \tag{1.1}$$

where $\ell_s$ is the lenght of the string and $R$ is the characteristic size of the target space. When this parameter is very small, the target geometry can be regarded as a classical background corrected by stringy effects. However, when this parameter is big, classical geometric intuition breaks down and one is forced to use some notion of stringy or quantum geometry.

These phenomena have been studied in detail in the context of type A topological string theory on Calabi–Yau manifolds, where the characteristic size $R$ of the target space is set by the Kähler moduli (see [1, 2] for reviews and a list of references). When these sizes are small, so that the deformation parameter (1.1) is big, the breakdown of classical geometry can be made precise quantitatively by looking at the behavior of topological string amplitudes. Typically, the large radius expansion of the amplitudes becomes divergent at a critical value of the Kähler moduli, and this divergence signals the onset of the stringy regime. We will refer to this change of regime as a phase transition. In known cases (like the quintic), this transition exhibits a universal behavior which has been identified with the $c = 1$ string at the self–dual radius [3, 4].

In this paper we will analyze the critical behavior of type A topological string theory in the local Calabi–Yau manifold

$$X_p = \mathcal{O}(p-2) \oplus \mathcal{O}(-p) \to \mathbb{P}^1. \tag{1.2}$$

This theory has been studied from a mathematical point of view in [5] and it is closely related to Hurwitz theory. From a physical point of view, it was proposed in [6] that the model could be defined non–perturbatively by using a q–deformed version of 2d Yang–Mills theory. Although perturbative topological string theory on the background $X_p$ shares many known properties with other local Calabi–Yau manifolds (like for example Gopakumar–Vafa integrality), we will show in this paper that for $p > 2$ it exhibits a phase transition at small radius which is not in the same universality class as the examples considered so far, i.e. the $c = 1$ string. Rather, the critical behavior is in the universality class of pure two–dimensional gravity (i.e. the $(2, 3)$ model) for all $p > 2$.

It turns out that, as in the case of matrix models, the critical behavior of higher genus amplitudes makes it possible to define a double–scaled theory at the transition point. The double–scaling limit erases a lot of information about the original topological string theory, since it only keeps the leading singularity in the genus $g$ free energies, but it is interesting for at least three reasons:

(1) It gives a compact characterization of the critical behavior at every order in string perturbation theory.

(2) It makes possible to study the free energy at all genus, and provides information about the nonperturbative structure of the theory.

(3) As argued in [7, 8] in the context of unitary matrix models describing thermal AdS, the phase transition might be an artifact of perturbation theory, and the double–scaling limit provides a way to smooth out the critical behavior.

We give convincing evidence that the double–scaled free energy is described for all $p > 2$ by the Painlevé I equation which characterizes 2d gravity [9, 10, 11]. The resulting scaled free energy is known to be non–Borel summable as a genus expansion, and this confirms that perturbative topological string theory on these backgrounds lacks fundamental non–perturbative input. Interestingly, the non–perturbative definition of the model proposed in [6] also exhibits critical behavior [12, 13, 14], but this time in the universality class of the Gross–Witten–Wadia unitary model [15, 16] described by Painlevé II [17, 18], which is also the universality class of 2d supergravity [19]. The Painlevé II equation is known to have a well–defined unique solution with the appropriate boundary conditions [20], and this can be regarded as further support for the proposal of [6] as a non–perturbative definition of the theory.

Conversely, our result shows that 2d gravity and 2d supergravity can be respectively embdedded in a one–family of topological string theories and their nonperturbative holographic completions. The relation that emerges between 2d gravity and 2d supergravity can be better understood by using the results of [64, 65], and we speculate that the proposal of [6], together with our results on the critical behavior, give further support to the nonperturbative solution of 2d gravity presented in [64].

Mathematically, perturbative string theory on $X_p$ can be regarded as a q–deformation of Hurwitz theory, and the Gromov–Witten invariants of $X_p$ promote simple Hurwitz numbers to polynomials. This ingredient is very important in our study, and indeed the critical behavior we describe for $p > 2$ is already present in the Hurwitz case. This was noted at the planar level in [21], and it is implicit in the results of [22] for higher genus Hurwitz theory. Our results are also probably related to the description of topological gravity in terms of the asymptotics of Hurwitz numbers due to Okounkov and Pandharipande [23]. In fact, we can reach higher critical points by perturbing the model with higher Casimir operators. This perturbed model is an analog of the general one–matrix model, while the theory of Hurwitz numbers should be an analog of the Gaussian model with general D–brane insertions. Both are related by an open–closed duality in the spirit of [24, 25].

Double–scaling limits of topological string theory have been studied before, mostly in the context of the Dijkgraaf–Vafa correspondence (see [26] for a recent example with references to the literature)[2]. In that case one studies type B topological string theory

[2] [27] considers the double–scaling limit of Chern–Simons theory on the sphere, and recovers the $c = 1$ behavior at the conifold.

on Calabi–Yau backgrounds related to the deformed conifold. The topological string amplitudes are given by matrix integrals, which can then be analyzed with the standard technology developed in the study of two–dimensional gravity. In contrast, we look at type A topological strings in backgrounds related to the resolved conifold, where the amplitudes are given by sums over partitions, and the connection to 2d gravity is more surprising (although it could have been suspected with some hindsight, due to the results of [21, 22, 23]).

The organization of this paper is as follows. In section 2 we review some results on phase transitions and phases of topological string theory. In section 3 we introduce perturbative topological string theory on $X_p$, write down its partition function, and relate it to various close cousins, in particular, we show that the model can be regarded in a precise sense as a q–deformation of Hurwitz theory. In section 4 we solve the model at genus zero (i.e. the planar theory) by using standard matrix model techniques. In section 5 we extract the critical behavior of the planar theory and we show that including higher Casimirs leads to multicritical points a la Kazakov (albeit we do the detailed analysis only in the undeformed case, i.e. in Hurwitz theory). In section 6 we analyze the higher genus theory. To do this we make an ansatz for $F_g(t)$ which generalizes the results of Goulden, Jackson and Vakil for Hurwitz theory [22, 28] to the deformed case. We also define the double–scaled theory and give strong evidence that it is described by 2d gravity. In section 7 we compare the structure found in the perturbative setting with the structure that one finds in the non–perturbative definition given by q–deformed 2d YM, and we speculate on the possible consequences for 2d gravity. In section 8 we point out evidence for a new open/closed duality generalizing the results of Aganagic and Vafa in [29]. In Appendix A we recall the very useful Lagrange inversion formula and apply it to various examples of the paper. In Appendix B we give some useful integrals for the saddle–point analysis of section 4.

# 2 Phase transitions in topological string theory

For simplicity, we will assume that the Calabi–Yau $X$ has a single Kähler parameter $t$, i.e. $h^{1,1}(X) = 1$. This is usually taken as a complexified parameter of the form

$$t = r + \mathrm{i}\theta, \tag{2.1}$$

but since we will be interested in questions regarding the convergence of the prepotential as a function of $\mathrm{e}^{-t}$, the $\theta$ parameter is not relevant and we will set it to zero.

## 2.1 Phase transitions in the planar free energy

As we mentioned in the introduction, when $t$ is large (in the so–called large radius regime) the geometry probed by string theory can be regarded as a classical geometry together with stringy corrections. This is well reflected in the structure of the prepotential $F_0(t)$ or genus zero topological string amplitude, which in the large radius regime is of the form

$$F_0(t) = \frac{C}{6}t^3 + \sum_{k=1}^{\infty} N_{0,k} \mathrm{e}^{-kt}. \tag{2.2}$$

In this equation, $C$ is the classical intersection number for the two-cycle whose size is measured by $t$. The infinite sum in the r.h.s. is given by worldsheet instanton corrections, which are obtained by "counting" (in an appropriate sense) holomorphic maps from $\mathbb{P}^1$ to $X$. The instanton counting numbers $N_{0,k}$ are genus zero Gromov–Witten invariants, and we have chosen units in which $\ell_s = \sqrt{2\pi}$.

The series of worldsheet instanton corrections, regarded as a power series in $\mathrm{e}^{-t}$, has in general a finite radius of convergence $t_c$ which can be obtained by looking at the asymptotic growth with $k$ of the numbers $N_{0,k}$. We will characterize this asymptotic growth by $t_c$ and by a critical exponent $\gamma$:

$$N_{0,k} \sim k^{\gamma-3} \mathrm{e}^{k t_c}, \quad k \to \infty. \tag{2.3}$$

When this holds, the prepotential behaves near $t_c$ as

$$F_0(t) \sim (\mathrm{e}^{-t_c} - \mathrm{e}^{-t})^{2-\gamma}. \tag{2.4}$$

It turns out that typical Gromov–Witten invariants of Calabi–Yau manifolds behave asymptotically as

$$N_{0,k} \sim \frac{\mathrm{e}^{k t_c}}{k^3 \log^2 k}, \quad k \to \infty. \tag{2.5}$$

This is of the form (2.3), with critical exponent

$$\gamma = 0 \tag{2.6}$$

and subleading log corrections. This behavior was first established in [30] in the example of the quintic, and since then it has been verified in other examples, like for example in local $\mathbb{P}^2$, where the critical radius is given by [31, 32]

$$t_c = \frac{1}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)} \mathrm{Re}\, G\left(\frac{1}{3}, \frac{2}{3}, 1; 1\right) \sim 2.90759 \tag{2.7}$$

and $G$ is the Meijer function.

The subleading log in (2.5) leads to log corrections near the critical point (also referred to as scaling violations) of the form

$$F_0(t) \sim (\mathrm{e}^{-t_c} - \mathrm{e}^{-t})^2 \log(\mathrm{e}^{-t_c} - \mathrm{e}^{-t}). \tag{2.8}$$

This is the genus zero free energy of the $c = 1$ string at the self–dual radius, once the scaling variable $e^{-t_c} - e^{-t}$ is identified with the cosmological constant [3, 4].

The behavior of the prepotential gives a precise quantitative meaning to the distinction between classical and quantum geometry. We will refer to the divergence of the large radius expansion at $t = t_c$ as a *phase transition* with a critical exponent $\gamma$ defined in (2.5). The phase with

$$t > t_c \tag{2.9}$$

where the expansion (2.2) is convergent, is called the large radius or Calabi–Yau phase, where classical geometry makes sense (albeit it is corrected by worldsheet instantons). When $t \leq t_c$, the nonlinear sigma model approach is not well defined, and classical geometric intuition is misleading.

There are two possible approaches to this problem. In [31, 33], Aspinwall, Greene and Morrison proposed to study this small area phase by using mirror symmetry. One finds that the point $t = t_c$ separating the two phases corresponds to the discriminant locus of the mirror moduli space (the conifold point). For $t < t_c$, mirror symmetry makes possible to compute the prepotential by analytic continuation. The second approach is due to Witten, who proposed in [34] the linear $\sigma$ model description of $\mathcal{N} = 2$ sigma models on Calabi–Yau manifolds largely as a way to understand this phase transition. In this formalism, one recovers the mirror symmetric description, and in addition the stringy or non–geometric phase can be described in a precise way. It is typically a Landau–Ginzburg orbifold perturbed by twist operators.

## 2.2 Higher genus

In order to describe the phase structure of the model we have relied on the behavior of the prepotential, i.e. the planar free energy. It is natural to ask what happens when higher genus topological string amplitudes are taken into account. These amplitudes, which we denote by $F_g(t)$, can be expressed in the large radius limit in terms of genus $g$ Gromov–Witten invariants $N_{g,k}$:

$$F_g(t) = \sum_{k=1}^{\infty} N_{g,k} e^{-kt}. \tag{2.10}$$

Here we have omitted contributions from degree zero maps as well as classical pieces for $g = 1$, which will not be important in the discussion. It turns out that the Gromov–Witten invariants have the asymptotic behavior [3]

$$N_{g,k} \sim k^{(\gamma-2)(1-g)-1} e^{kt_c}, \quad k \to \infty, \tag{2.11}$$

where $t_c$ is the critical radius obtained at genus zero and it is common to all $g$, and $\gamma$ is the critical exponent that appears in (2.3). This is equivalent to the following behavior near the critical point

$$
\begin{aligned}
&F_1(t) \sim c_1 \log\left(\mathrm{e}^{-t_c} - \mathrm{e}^{-t}\right), \\
&F_g(t) \sim c_g(\mathrm{e}^{-t_c} - \mathrm{e}^{-t})^{(1-g)(2-\gamma)}, \quad g \geq 2.
\end{aligned}
\tag{2.12}
$$

In conventional topological string theory, as we have mentioned, $\gamma = 0$, but the more general form we have written above will be useful later.

We then see that the phase transition at $t = t_c$ is common for all $F_g(t)$, and the critical exponent changes with the genus in the way prescribed by (2.11). This sort of coherent behavior in the genus expansion is not obvious, but seems to characterize a wide variety of systems that admit a genus expansion (like for example matrix models, see [35] for a review). When this is the case, one can define a *double–scaling limit* [9, 10, 11] as follows. Let us consider the total free energy $F$ as a perturbative expansion in powers of the string coupling constant $g_s$:

$$
F(g_s, t) = \sum_{g=0}^{\infty} F_g(t) g_s^{2g-2}. \tag{2.13}
$$

We define the double–scaled string coupling as

$$
\kappa = a g_s (\mathrm{e}^{-t_c} - \mathrm{e}^{-t})^{\gamma/2-1}, \tag{2.14}
$$

where $a$ is an appropriate constant. We can then consider the limit

$$
t \to t_c, \qquad g_s \to 0, \qquad \kappa \text{ fixed}. \tag{2.15}
$$

In this limit, only the most singular part of $F_g(t)$ survives at each genus, and the total free energy becomes the *double-scaled free energy*

$$
F_{\mathrm{ds}}(\kappa) = f_0 \kappa^{-2} + f_1 \log\, \kappa + \sum_{g \geq 2} f_g \kappa^{2g-2}, \tag{2.16}
$$

where $f_g = a^{2-2g} c_g$. It is also customary to express the double–scaled free energy in terms of the scaling variable $z = \kappa^{2/(\gamma-2)}$.

It turns out that, in some cases, one can determine the coefficients $f_g$ in closed form. In the double–scaling limit of matrix models, they are governed by a differential equation of the Painlevé type [35]. In the case of topological string theory on Calabi–Yau manifolds, it was conjectured in [4] that, in terms of a natural coordinate

$$
\mu \sim \mathrm{e}^{-t_c} - \mathrm{e}^{-t} \tag{2.17}
$$

which in the mirror model measures the distance to the conifold point $\mu = 0$, the double-scaled free energy is universal and reads

$$
F_{ds}(\mu) = \frac{1}{2}\mu^2 \log\, \mu - \frac{1}{12} \log\, \mu + \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} \mu^{2-2g}. \tag{2.18}
$$

This is exactly the all genus free energy of the $c = 1$ string at the self–dual radius (for a review, see [36]). This behavior has been checked in many examples [32, 37].

# 3 Topological string theory on bundles over $\mathbb{P}^1$

## 3.1 The model and its partition function

In this paper we study perturbative topological string theory on a Calabi–Yau threefold given by the total space of a rank two holomorphic complex bundle over $\mathbb{P}^1$. It is a well–known result that these bundles split into a direct sum of line bundles, therefore these spaces are of the form

$$\mathcal{O}(p-2) \oplus \mathcal{O}(-p) \to \mathbb{P}^1, \tag{3.1}$$

where $p \in \mathbb{Z}$. There is an obvious symmetry

$$p \to -p+2 \tag{3.2}$$

and we can restrict to $p > 0$.

The topological string partition function on the space $X_p$ is the exponential of the total free energy (2.13)

$$Z_{X_p} = \exp\, F_{X_p}(g_s, t) \tag{3.3}$$

where

$$F_{X_p}(g_s, t) = \sum_{g=0}^{\infty} g_s^{2g-2} F_g^{X_p}(t) = \sum_{g=0}^{\infty}\sum_{d=1}^{\infty} N_{g,d}(p) \mathrm{e}^{-dt} g_s^{2g-2}, \tag{3.4}$$

and $N_{d,g}(p)$ are the Gromov–Witten invariants. This partition function can be computed by using the topological vertex [38], and the same result is obtained from the local Gromov–Witten theory of curves of [5] (see section 2 of [6] for a useful summary). In order to write down the result, we recall some necessary ingredients. First of all, we define the q–number $[n]$ as

$$[n] = q^{n/2} - q^{-n/2}, \qquad q = \mathrm{e}^{-g_s}. \tag{3.5}$$

A representation $R$ of $U(\infty)$ can be represented by a Young tableau, labeled by the lengths of its rows $\{l_i\}$. The quantity

$$\ell(R) = \sum_i l_i \tag{3.6}$$

is the total number of boxes of the tableau. Notice that a tableau $R$ with $\ell(R)$ boxes can be regarded as a representation of the symmetric group $S_{\ell(R)}$ of $\ell(R)$ elements. Another important quantity associated to a tableau is

$$\kappa_R = \sum_i l_i(l_i - 2i + 1), \tag{3.7}$$

which is closely related to the second Casimir of $R$ regarded as a $U(N)$ representation. Finally, we introduce the quantity

$$W_R = q^{-\kappa_R/4} \prod_{\square \in R} \frac{1}{[\mathrm{hook}(\square)]}. \tag{3.8}$$

This is in fact a specialization of the topological vertex. The product in the r.h.s. is over all the boxes in the tableau, and hook(□) denotes the hook length of a given box in the tableau.

The topological string partition function on $X_p$ is given by

$$Z_{X_p} = \sum_R W_R W_{R^t} q^{(p-1)\kappa_R/2} (-1)^{\ell(R)p} \mathrm{e}^{-\ell(R)t} \tag{3.9}$$

where $R^t$ denotes the transpose tableau (i.e. the tableau where we exchange rows and columns). The above expression has to be understood, as in [38], as a power series in $\mathrm{e}^{-t}$. At every degree it gives the all–genus answer. One can easily compute the first few terms in the expansion:

$$\begin{aligned} F_0^{X_p}(t) &= (-1)^p \mathrm{e}^{-t} + \frac{1}{8}(2\,p^2 - 4\,p + 1)\mathrm{e}^{-2t} + \frac{(-1)^p}{54}(1 - 6\,p + 3\,p^2)(2 - 6\,p + 3\,p^2)\mathrm{e}^{-3t} \\ &\quad + \mathcal{O}(\mathrm{e}^{-4t}), \\ F_1^{X_p}(t) &= -\frac{(-1)^p}{12}\mathrm{e}^{-t} + \frac{1}{48}(p^4 - 4\,p^3 + p^2 + 6\,p - 2)\mathrm{e}^{-2t} \\ &\quad + \frac{(-1)^p}{72}(-2 + 14\,p - 19\,p^2 - 20\,p^3 + 45\,p^4 - 24\,p^5 + 4\,p^6)\mathrm{e}^{-3t} + \mathcal{O}(\mathrm{e}^{-4t}), \end{aligned} \tag{3.10}$$

and so on. The Gromov–Witten invariants $N_{g,d}(p)$ are polynomials in $p$ of degree $2d-2+2g$ with rational coefficients:

$$N_{g,d}(p) = \sum_{i=0}^{2d-2+2g} N_{g,d}^i \, p^i. \tag{3.11}$$

As we will see, the coefficient of the highest power in (3.11) has a simple geometric interpretation in terms of classical Hurwitz theory. Note that, due to the definition we are using of $g_s$, the Gromov–Witten invariants $N_{d,g}$ differ in a sign $(-1)^{g-1}$ from the standard ones. This is of no consequence for most of our discussion, but the extra sign should be taken into account when extracting integer Gopakumar–Vafa invariants $n_{g,d}(p)$ [39] from the above expressions.

The partition function (3.9) on $X_p$ is calculated, as explained in [5], by considering the equivariant Gromov–Witten theory on $X_p$ with respect to scalings of the line bundles over $\mathbb{P}^1$, and by using the antidiagonal action. This is, therefore, an equivariant partition function, and leads to a topological string theory which is different from the standard ones. For $p = 1$ this theory specializes to (standard) topological string theory on the resolved conifold. For $p = 2$, the geometry is that of $\mathbb{C} \times A_1$. The standard topological string theory has a trivial partition function on this space, while the equivariant theory considered here has $Z_{X_2} = Z_{X_1}^{-1}$. See [40] for more details on the equivariant theory from the point of view of mirror symmetry.

Another way to understand the geometric content of this non–standard theory is to consider the equivariant version of topological strings presented in section 7.6 of [41]. Let us look at equivariant topological string theory on local $\mathbb{P}^2$. This model contains, on top of the Kähler parameter $t$, three equivariant parameters $x_i$, $i = 1, 2, 3$, related to twisted masses in the sigma model. The partition function depends now on the $x_i$,

$$Z_{\mathbb{P}^2}(x_1, x_2, x_3), \tag{3.12}$$

and can be computed again by using the topological vertex (explicit formulae are given in [41]). It is completely symmetric under permutations of the $x_i$ and, if we set $x_i = 1$ we recover the standard local $\mathbb{P}^2$ partition function. One can deduce from the explicit formulae in [41] that

$$Z_{\mathbb{P}^2}(1, 0, 0) = Z_{X_3}. \tag{3.13}$$

This is easy to understand, since the hyperplane class of $\mathbb{P}^2$ in the total geometry is indeed a $\mathbb{P}^1$ with normal bundle $\mathcal{O}(-3) \oplus \mathcal{O}(1)$. By taking $x_2 = x_3 = 0$ we are considering the local geometry of this curve inside $\mathcal{O}(-3) \to \mathbb{P}^2$. For $p > 3$ we can regard $Z_{X_p}$ as a specialization of equivariant topological string theory on more complicated toric backgrounds.

Finally, we point out that (3.9) can be interpreted in terms of representations of Hecke algebras by using the results of [42].

## 3.2 Relation to Hurwitz theory

Before analyzing this partition sum in detail, let us relate it to other generating functionals. The quantity $W_R$ turns out to be a q–deformation of the dimension $d_R$ of the representation $R$ of $S_{\ell(R)}$. As $g_s \to 0$, one has that

$$W_R \to g_s^{-\ell(R)} \frac{d_R}{|\ell(R)|!}. \tag{3.14}$$

This suggests taking the following limit,

$$g_s \to 0, \quad t \to \infty, \quad p \to \infty, \tag{3.15}$$

in such a way that

$$p g_s = \tau_2/N, \quad (-1)^p \mathrm{e}^{-t} = (g_s N)^2 \mathrm{e}^{-\tau_1}, \tag{3.16}$$

and $\tau_1$, $\tau_2$ and $N$ are new parameters that are kept fixed. It will be convenient in what follows to introduce a 't Hooft parameter as

$$T = g_s N. \tag{3.17}$$

In the limit (3.15)–(3.16) one has $T \to 0$. The partition function becomes:

$$Z_{X_p} \to Z_{\text{Hurwitz}} = \sum_R \left( \frac{d_R}{|\ell(R)|!} \right)^2 N^{2\ell(R)} \mathrm{e}^{-\tau_2 \kappa_R / 2N} \mathrm{e}^{-\tau_1 \ell(R)}. \tag{3.18}$$

This is the generating functional of simple Hurwitz numbers of $\mathbb{P}^1$ at all genus and degrees. In order to see this, it is useful to recall some basic ingredients of Hurwitz theory.

Hurwitz theory studies branched covers of Riemann surfaces. The structure of the covering will be labeled by a vector $\vec{k}$ of nonnegative entries, and we define

$$\ell(\vec{k}) = \sum_j j k_j, \quad |\vec{k}| = \sum_j k_j, \quad z_{\vec{k}} = \prod_j j^{k_j} k_j!. \tag{3.19}$$

If a cover has degree $d$, one necessarily has

$$\ell(\vec{k}) = d. \tag{3.20}$$

Given $m$ branched points on $\mathbb{P}^1$, their branching structure can be specified by $m$ vectors

$$\vec{k}^{(1)}, \cdots, \vec{k}^{(m)}. \tag{3.21}$$

The Riemann–Hurwitz formula relates the entries of these vectors, the degree of the cover and the genus $g$ of the covering surface as follows:

$$2g - 2 + 2d = \sum_i (\ell(\vec{k}^{(i)}) - |\vec{k}^{(i)}|). \tag{3.22}$$

The number of Hurwitz covers at degree $d$ is given by the classical formula

$$H_d^{\mathbb{P}^1}(\vec{k}_1, \cdots, \vec{k}_m) = \sum_{\ell(R)=d} \left( \frac{d_R}{\ell(R)!} \right)^2 \prod_{i=1}^m f_R(\vec{k}^{(i)}). \tag{3.23}$$

In this equation,

$$f_R(\vec{k}) = \frac{\ell(R)!}{d_R} \frac{\chi_R(C(\vec{k}))}{z_{\vec{k}}}, \tag{3.24}$$

where $C(\vec{k})$ is the conjugacy class associated to the vector $\vec{k}$ in the symmetric group of $\ell(\vec{k})$ elements, and $\chi_R(C(\vec{k}))$ is its character.

A branched point whose branching vector is of the form

$$\vec{k}_i = (d-2, 1, \cdots) \equiv (2) \tag{3.25}$$

is called a simple branched point. A particularly important example of Hurwitz theory occurs when we have one non–simple branched point specified by a vector $\vec{k}$, and $m(g, \vec{k})$ simple branched points. By Riemann–Hurwitz, the number of simple branched points is given by

$$m(g, \vec{k}) = 2g - 2 + |\vec{k}| + \ell(\vec{k}), \tag{3.26}$$

where we used that $d = \ell(\vec{k})$. Since

$$f_R(d-2, 1, 0, \cdots) = \frac{1}{2} \kappa_R, \tag{3.27}$$

The Hurwitz number for this configuration is given by

$$H_{g,d}^{\mathbb{P}^1}(\vec{k}) = \sum_{\ell(R)=d} \left(\frac{d_R}{\ell(R)!}\right)^2 f_R(\vec{k})(\kappa_R/2)^{m(g,\vec{k})}. \tag{3.28}$$

The case where there are only $m$ simple branched points is a particular case of the above. It is obtained when the branching structure is trivial,

$$\vec{k} = (d, 0, \cdots) \equiv 1^d, \tag{3.29}$$

and

$$m = 2g - 2 + 2d. \tag{3.30}$$

The resulting Hurwitz number is called a *simple Hurwitz number*, and it is given at genus $g$ and degree $d$ by

$$H_{g,d}^{\mathbb{P}^1}(1^d) = \sum_{\ell(R)=d} \left(\frac{d_R}{\ell(R)!}\right)^2 (\kappa_R/2)^{2g-2+2d}, \tag{3.31}$$

where the sum is over representations $R$ with fixed number of boxes equal to the degree $d$.

We can now rewrite (3.18) as

$$\begin{aligned} Z_{\text{Hurwitz}} &= \sum_{d,m} N^{2d-m} \mathrm{e}^{-\tau_1 d} \sum_{\ell(R)=d} \left(\frac{d_R}{\ell(R)!}\right)^2 \frac{(-\tau_2)^m}{m!} (\kappa_R/2)^m \\ &= \sum_{g\geq 0} N^{2-2g} \sum_{d\geq 0} \mathrm{e}^{-\tau_1 d} H_{g,d}^{\mathbb{P}^1}(1^d) \frac{\tau_2^{2g-2+2d}}{(2g-2+2d)!}, \end{aligned} \tag{3.32}$$

where in the second line we have traded the sum over $m$ by a sum over $g$, and we used (3.30). The model described by (3.18) has been studied in detail due to its connection to Hurwitz theory. From the physical point of view, it was analyzed in [21, 43, 44], and in the mathematical literature it has been studied for example in [22, 28].

The free energy of $Z_{\text{Hurwitz}}$ describes *connected*, simple Hurwitz numbers $H_{g,d}^{\mathbb{P}^1}(1^d)^\bullet$:

$$F_{\text{Hurwitz}} = \log Z_{\text{Hurwitz}} = \sum_{g\geq 0} N^{2-2g} \sum_{d\geq 0} \mathrm{e}^{-\tau_1 d} H_{g,d}^{\mathbb{P}^1}(1^d)^\bullet \frac{\tau_2^{2g-2+2d}}{(2g-2+2d)!} \tag{3.33}$$

If we compare this to the total free energy $F_{X_p}$ written in (3.4) in terms of Gromov–Witten invariants, and take the limit (3.15)–(3.16), we find

$$\lim_{p\to\infty} p^{2-2g-2d}(-1)^p N_{g,d}(p) = \frac{H_{g,d}^{\mathbb{P}^1}(1^d)^\bullet}{(2g-2+2d)!}. \tag{3.34}$$

The l.h.s. is precisely the coefficient of the highest power in $p$ of (3.11)[3]. We can therefore interpret the Gromov–Witten invariants of this model as q–deformed connected, simple Hurwitz numbers, since they promote $H^{\mathbb{P}^1}_{g,d}(1^d)^\bullet$ to polynomials of degree $2g-2+2d$ (which is equal to the number of simple branch points).

We also point out that a closely related model to the partition function $Z_{X_p}$ and its Hurwitz limit is $U(1)$, four–dimensional $\mathcal{N}=2$ supersymmetric gauge theory with Casimir operators, considered in [45] from the point of view of Nekrasov's instanton counting [46]. When only the first and the second Casimir operators are turned on, this model is equivalent to the Hurwitz model (3.18), and via the Hurwitz/Gromov–Witten correspondence of [48], it is related to topological string theory on $\mathbb{P}^1$. The q–deformed model $Z_{X_p}$ we are considering here is in turn related to the five–dimensional or K–theoretic version of the gauge theory (see for example [49]).

## 3.3 Relation to open Gromov–Witten invariants

There is a generating functional in *open* Gromov–Witten theory which is also closely related to Hurwitz numbers. Consider a Lagrangian submanifold with the topology $\mathbb{C}\times\mathbb{S}^1$ in $\mathbb{C}^3$, and with framing $f$. The generating functional of open Gromov–Witten invariants can be computed with the Chern–Simons invariants of the unknot [50] or with the topological vertex [38, 51], and reads:

$$Z(V)=\sum_R W_R q^{f\kappa_R/2}(-1)^{\ell(R)f}\,\mathrm{Tr}_R\,V, \tag{3.35}$$

where $V$ is a $U(\infty)$ matrix source of open string moduli that takes into account the open string sectors associated to different winding numbers. Let us now rescale $V\to \mathrm{e}^{-t}V$ and take the limit (appropriate for the open sector)

$$g_s\to 0,\quad t\to\infty,\quad f\to\infty, \tag{3.36}$$

keeping fixed

$$fg_s=\tau_2/N,\quad (-1)^f\mathrm{e}^{-t}=g_sN\mathrm{e}^{-\tau_1}. \tag{3.37}$$

The generating functional (3.35) becomes

$$\begin{aligned} Z_{\mathrm{Hurwitz}}(V)&=\sum_R N^{\ell(R)}\left(\frac{d_R}{\ell(R)!}\right)\mathrm{e}^{-\tau_2\kappa_R/2N-\tau_1\ell(R)}\mathrm{Tr}_R V\\ &=1+\sum_{\vec{k}}\sum_{g\geq 0}N^{2-2g-|\vec{k}|}\mathrm{e}^{-\tau_1\ell(\vec{k})}H^{\mathbb{P}^1}_{g,\ell(\vec{k})}(\vec{k})\frac{(-\tau_2)^{2g-2+|\vec{k}|+\ell(\vec{k})}}{(2g-2+|\vec{k}|+\ell(\vec{k}))!}\Upsilon_{\vec{k}}(V), \end{aligned} \tag{3.38}$$

[3]This result, for the genus zero case, was already derived in [47]

where

$$\Upsilon_{\vec{k}}(V) = \prod_j \left(\mathrm{Tr}\, V^j\right)^{k_j}. \tag{3.39}$$

(3.38) is a generating functional for the more general Hurwitz numbers (3.28). If we now consider the free energy associated to the open Gromov–Witten functional, and we write them in terms of open Gromov–Witten invariants $F_{g,\vec{k}}(f)$,

$$F(V) = \sum_{\vec{k}} \sum_{g=0}^{\infty} F_{g,\vec{k}}(f) g_s^{2g-2+|\vec{k}|} \Upsilon_{\vec{k}}(V), \tag{3.40}$$

we find that

$$\lim_{f\to\infty} f^{2-2g-|\vec{k}|-\ell(\vec{k})}(-1)^f F_{g,\vec{k}}(f) = \frac{H^{\mathbb{P}^1}_{g,\ell(\vec{k})}(\vec{k})^{\bullet}}{(2g-2+|\vec{k}|+\ell(\vec{k}))!}. \tag{3.41}$$

This expresses the more general Hurwitz numbers (3.28) as limits of open Gromov–Witten invariants. We point out that the open Gromov–Witten invariants $F_{g,\vec{k}}(f)$ can be expressed in terms of triple Hodge integrals [52, 53], while the Hurwitz numbers (3.28) can be written in terms of simple Hodge integrals [54], and the relation (3.41) has been noted in [55, 56] by considering their integral expression.

We now relate the closed Gromov–Witten invariants $N_{g,d}(p)$ in the background $X_p$ to the open Gromov–Witten invariants $F_{g,\vec{k}}(f)$. It is clear that (3.35) reduces to (3.9) if we set $f = p-1$ and if we give a value to the source $V$ in such a way that

$$\mathrm{Tr}_R\, V = (-1)^{\ell(R)} \mathrm{e}^{-\ell(R)t} W_{R^t}. \tag{3.42}$$

There is indeed a choice of the (infinite–dimensional) matrix $V$ which produces this, with eigenvalues $x_i = q^{i-1/2}\mathrm{e}^{-t}$. With this choice, one has that

$$\Upsilon_{\vec{k}}(V) = (-1)^{|\vec{k}|} \mathrm{e}^{-\ell(\vec{k})t} \prod_j \frac{1}{(q^{\frac{j}{2}} - q^{-\frac{j}{2}})^{k_j}} \tag{3.43}$$

If we expand this in powers of $g_s$,

$$\Upsilon_{\vec{k}}(V) = \mathrm{e}^{-\ell(\vec{k})t} \sum_{g=0}^{\infty} g_s^{-|\vec{k}|+2g} c_{g,\vec{k}}, \tag{3.44}$$

where $c_{g,\vec{k}}$ are rational numbers, we find

$$N_{g,d}(p) = \sum_{\ell(\vec{k})=d} \sum_{g'=0}^{g} F_{g',\vec{k}}(p) c_{g-g',\vec{k}}, \tag{3.45}$$

which can be used to express the q–deformed Hurwitz number $N_{g,d}(p)$ in terms of triple Hodge integrals.

There is a more elegant way to relate the closed and the open invariants, by using the underlying integer invariants. It was shown in section 5 of [41] that, when closed string functionals are obtained from open string functionals by "condensing" the source term as in (3.42), the Gopakumar–Vafa invariants $n_{g,d}$ of the closed geometry can be written in terms of the open BPS invariants introduced in [50, 57]. In the case of framed branes in $\mathbb{C}^3$, the integer invariants $N_{R,g}(f)$ depend only on the genus $g$ and a representation $R$ of $U(\infty)$. It is easy to obtain from the results in [41] that

$$n_{g,d}(p) = N_{\underbrace{\square\cdots\square}_{d\,\text{boxes}},g}(p-1). \tag{3.46}$$

The tableau in the r.h.s. corresponds to the trivial representation of $S_d$, and in deriving this relation we have used the conventions for open BPS invariants in [53, 57].

# 4 The planar limit

The explicit expression (3.9) gives the total partition function but it is not useful in extracting the functions $F_g^{X_p}(t)$ in a compact way. If we are interested for example in the asymptotic behavior of the Gromov–Witten invariants for large degree, we would like to have closed expressions for the free energies at fixed genus but at all degrees. These are the kind of expressions that can be obtained by using mirror symmetry. Unfortunately, there is no complete mirror description of the $X_p$ backgrounds that can be used efficiently to compute $F_g^{X_p}(t)$. Some results concerning this description have been recently obtained in [40], but most of the general ingredients (like Picard–Fuchs equations) are still lacking.

In these circumstances, we have to extract the $F_g^{X_p}(t)$ directly from the sum over partitions (3.9). In this section we solve the planar limit, i.e. we find the genus zero contribution or prepotential, by doing a saddle–point analysis of (3.9). Some aspects of our analysis were worked out in [14].

## 4.1 Saddle–point analysis

The sum over partitions that defines $Z_{X_p}$ in (3.9) can be rewritten as

$$Z_{X_p} = \sum_R W_R^2\, q^{(p-2)\kappa_R/2}\, \mathrm{e}^{-\ell(R)t}. \tag{4.1}$$

where for simplicity we have absorbed the sign $(-1)^{\ell(R)p}$ in the Kähler parameter $t$. To analyze this in the saddle–point limit, we will adopt a strategy used by Kostov, Staudacher and Wynter in [21] to analyze the partition function of Hurwitz theory (3.18). We will introduce a “fake” $N$ dependence in the theory, which allows for standard large $N$ analysis, and we will then extract the genus zero result.

To see this, we first notice that (4.1) admits an *evocative* representation in terms of a q-deformed group theoretical quantity of $U(N)$. Let $\{l_i\}$ be the lengths of rows in a Young tableau, and let $h_i = l_i + N - i$. Consider the q-deformed quantity

$$ {}_q\Omega_R = \prod_{i=1}^{N} \frac{[h_i]!}{[N-i]!}, \tag{4.2} $$

and the familiar quantum dimension of an irreducible representation $R$

$$ \dim_q R = \prod_{1\le i<j\le N} \frac{[l_i - l_j + j - i]}{[j-i]}. \tag{4.3} $$

The brackets denote as usual q–numbers as in (3.5). Using the equality

$$ \frac{\dim_q R}{{}_q\Omega_R} = q^{-\kappa_R/4} W_R, \tag{4.4} $$

we can write

$$ Z_{X_p} = \sum_R \left( \frac{\dim_q R}{{}_q\Omega_R} \right)^2 q^{(p-1)\kappa_R/2} \mathrm{e}^{-t\ell(R)}. \tag{4.5} $$

However, the actual equality in (4.5) holds only when the r.h.s is expanded as an asymptotic series in $1/N$: this immediately suggests to exploit the general large $N$ techniques in investigating (4.5).

The planar theory can be in fact immediately analyzed along the usual strategy originally introduced by Douglas and Kazakov for $\mathrm{QCD}_2$ [58]. We introduce the auxiliary ‘t Hooft parameter $T = Ng_s$ as in (3.17) and continuous variables in the standard way:

$$ \frac{h_i}{N} = \frac{l_i}{N} - \frac{i}{N} + 1 \to \ell(x) - x + 1 = h(x), \tag{4.6} $$

The delicate point is to evaluate the large $N$ limit of the deformed measure. The numerator of ${}_q\Omega_R$ leads to

$$ \log \prod_{i=1}^{N} \prod_{j=1}^{h_i} (q^{\frac{h_i-j}{2}} - q^{-\frac{h_i-j}{2}})^2 = 2\sum_{i=1}^{N}\sum_{j=1}^{h_i} \log\, 2\sinh g_s \frac{h_i - j}{2}, \tag{4.7} $$

which becomes in the large $N$ limit

$$ \frac{2N^2}{T}\int_0^1 dx \int_0^{h(x)} dy \log 2\sinh T\frac{h(x)-y}{2} = \frac{2N^2}{T}\int_0^1 dx \left( \frac{T^2h^2}{4} - \frac{\pi^2}{6} + \mathrm{Li}_2(\mathrm{e}^{-Th}) \right). \tag{4.8} $$

The denominator of ${}_q\Omega_R$ cancels against the denominator of $(\dim_q R)^2$. The numerator of the quantum dimension leads to a $\sinh(x-y)$ interaction, as explained in [12, 13, 14] in a related context. Then we can write the effective action controlling the leading large $N$ contribution as follows

$$\begin{aligned} S = &-\int_0^1\int_0^1 dxdy \log\Big|2\sinh\frac{T}{2}(h(x)-h(y))\Big| + \frac{2}{T}\int_0^1 dx \mathrm{Li}_2(\mathrm{e}^{-Th}) \\ &+\int_0^1 dx h(x)(t-(p-1)T) + \frac{pT}{2}\int_0^1 dx h^2(x) + (p-1)\frac{T}{3} - \frac{\pi^2}{3T} - \frac{1}{2}t. \end{aligned} \tag{4.9}$$

The planar theory can be, thus, understood as coming from a matrix model: we notice, in fact, that the effective action can be derived from a Chern–Simons–like matrix model [59] with a potential $V(h)$ of the form

$$V(h) = \frac{2}{T}\mathrm{Li}_2(\mathrm{e}^{-Th}) + (t-(p-1)T)h + \frac{pT}{2}h^2, \tag{4.10}$$

and the saddle–point equation is simply

$$\int dh'\rho(h')\coth\frac{T}{2}(h-h') = ph + \frac{2}{T}\log(1-\mathrm{e}^{-Th}) + \frac{t}{T} - (p-1), \tag{4.11}$$

where the density $\rho(h)$ is defined in terms of the inverse function $x(h)$ as follows

$$\rho(h) = -\frac{dx(h)}{dh}. \tag{4.12}$$

Because of the positivity constraint

$$h_1 > h_2 > \cdots h_N \geq 0 \Rightarrow h(x) \geq 0, \tag{4.13}$$

which the Young tableaux variables $h_i$ must satisfy, the support of $\rho(h)$ will be chosen in the interval $[0, a]$.

The above equation can be related to a standard Riemann-Hilbert problem: to this aim we first introduce $x = 1 - hT$ and then we pass to exponential variables

$$s = \mathrm{e}^x, \quad y = \mathrm{e}^z. \tag{4.14}$$

In terms of these variables the saddle–point equation reads

$$\int_{\mathrm{e}^{-\beta}}^{\mathrm{e}} \frac{dy}{y}\rho(y)\frac{s+y}{s-y} = p\log s - (t+1) + (p-1)(T-1) - 2\log(1-\mathrm{e}^{-1}s), \tag{4.15}$$

with $-\beta = 1 - Ta$. The normalization of $\rho$ is now

$$\int_{\mathrm{e}^{-\beta}}^{\mathrm{e}} dy\frac{\rho(y)}{y} = T. \tag{4.16}$$

The support of the density $\rho(s)$ comes from the original tableau variables $h$, i.e. $[0, a]$.

To solve the saddle point equation (4.15) we need a further ingredient, namely we have to choose an ansatz for the density $\rho(s)$. In order to recover the large radius expansion in $\mathrm{e}^{-t}$ the analogy with $\mathrm{QCD}_2$ [44] suggests to choose a chiral, one–cut ansatz: for $x \in [-\beta, -\gamma]$ $(-\gamma < 1)$ the $\rho(s)$ is arbitrary, while for $x \in [-\gamma, 1]$ we require $\rho$ to be equal to 1. With this assumption, we easily arrive at our final form for the saddle-point equation

$$\int_{\mathrm{e}^{-\beta}}^{\mathrm{e}^{-\gamma}} \frac{dy}{y}\frac{\rho(y)}{s-y} = \frac{p}{2s}\log s - \frac{t - p(T-1)}{2s} - \frac{1}{s}\log(1 - \mathrm{e}^{\gamma}s). \tag{4.17}$$

Notice that in principle this equation and its solution depend on the parameter $T = Ng_s$, therefore the genus zero free energy of this model $F_0(T, p, t)$ depends on $T$, as well as on $p$ and $t$. However, the parameter $N$ was introduced by hand and does not appear in the original model. Consistency of our procedure requires then

$$F_0(T, p, t) = \frac{1}{T^2}F_0(T = 1, p, t) = \frac{1}{T^2}F_0^{X_p}(p, t). \tag{4.18}$$

The reason for this is that $N^2F_0(T, p, t)$ should equal $g_s^{-2}F_0^{X_p}(p, t)$. We can indeed verify this in detail as follows. If we perform the changes of variable

$$s \mapsto z = s\mathrm{e}^{T-1} \quad \text{and} \quad y \mapsto v = y\mathrm{e}^{T-1}, \tag{4.19}$$

accompanied with the redefinitions

$$\hat{\gamma} = \gamma + 1 - T, \quad \hat{\beta} = \beta + 1 - T \quad \text{and} \quad \hat{\rho}(v) = \rho(v\mathrm{e}^{1-T}), \tag{4.20}$$

the saddle point equation takes the form

$$\int_{\mathrm{e}^{-\hat{\beta}}}^{\mathrm{e}^{-\hat{\gamma}}} \frac{dv}{v}\frac{\hat{\rho}(v)}{z-v} = \frac{p}{2z}\log z - \frac{t}{2z} - \frac{1}{z}\log(1 - \mathrm{e}^{\hat{\gamma}}z). \tag{4.21}$$

The normalization condition instead reduces to

$$\int_{\mathrm{e}^{-\hat{\beta}}}^{\mathrm{e}^{-\hat{\gamma}}} \frac{dv}{v}\hat{\rho}(v) = -\gamma. \tag{4.22}$$

We can now verify (4.18) by computing the $T-$dependence of the derivative of $F_0$ with respect to $t$. In terms of the original Young tableaux variables, we have

$$-\frac{\partial F_0}{\partial t} = \int_0^a \rho(h)hdh - \frac{1}{2}. \tag{4.23}$$

Now if we set $h = 1 - \log(ve^{1-T})/T$, with the help of the redefinitions (4.19) and (4.20), the above equation can be cast in the following form

$$\frac{\partial F_0}{\partial t} = -\frac{1}{T^2}\left(\frac{T^2}{2} + \frac{\hat{\gamma}^2}{2} - \int_{\mathrm{e}^{-\hat{\beta}}}^{\mathrm{e}^{-\hat{\gamma}}} \rho(v)\log(v)\right) + \frac{1}{2} = \frac{1}{T^2}\frac{\partial F_0^{\mathrm{T}=1}}{\partial t}, \tag{4.24}$$

as needed. In the following we shall set $T = 1$, since in this way the planar free energy of the above matrix model equals the prepotential $F_0^{X_p}(p,t)$; the $T$ dependence is eventually recovered through the relations (4.19), (4.20) and (4.24).

## 4.2 Solving the saddle–point equation

The solution to the integral equation (4.17) can be written in terms of the *effective* resolvent function

$$\omega(z) := \int_{\mathrm{e}^{-\beta}}^{\mathrm{e}^{-\gamma}} \frac{dv}{v} \frac{\rho(v)}{z-v}, \tag{4.25}$$

which is given by Muskhelishvili–Migdal's formula

$$\omega(z) = \frac{1}{2\pi i}\sqrt{(z-\mathrm{e}^{-\beta})(z-\mathrm{e}^{-\gamma})} \oint_C \frac{dv}{(z-v)v} \frac{\frac{p}{2}\log v - \frac{t}{2} - \log\left(1 - ve^{\gamma}\right)}{\sqrt{(v-\mathrm{e}^{-\beta})(v-\mathrm{e}^{-\gamma})}} \tag{4.26}$$

where the closed contour $C$ encircles the support $[\mathrm{e}^{-\beta}, \mathrm{e}^{-\gamma}]$ of the distribution $\rho(v)$ with counterclockwise orientation in the complex $z$-plane. If we choose the square root and logarithmic[4] branch cuts in (4.26) as indicated in Fig. 1, then since the integrand decays

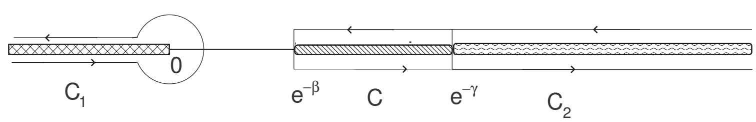


Figure 1: The contour $C_1$ surrounds the branch cut $(-\infty, 0]$ of $\log(z)$, $C_2$ encircles the cut $[\mathrm{e}^{-\gamma}, \infty)$ of $\log\left(1 - e^{\gamma}v\right)$, while $C$ encloses the physical branch cut $[\mathrm{e}^{-\beta}, \mathrm{e}^{-\gamma}]$ .

as $v^{-3}$ at $|v| \to \infty$, we can deform the contour of integration $C$ so that it encircles the cuts of the two logarithms. This deformation picks up an additional contribution from

[4]We have always defined the logarithm function as having a branch cut along the negative axis. This choice implies that $\log(1 - e^{\gamma}w)$ has a cut for $w \geq e^{-\gamma}$.

the pole at $w = z$ and we find

$$\omega(z) = \left[\frac{p}{2z}\log\left(\frac{\left(e^{-\frac{\beta}{2}}\sqrt{z-e^{-\gamma}}+e^{-\frac{\gamma}{2}}\sqrt{z-e^{-\beta}}\right)^2}{\left(\sqrt{z-e^{-\gamma}}+\sqrt{z-e^{-\beta}}\right)^2}\right)+\frac{1}{z}\log\left(\frac{\left(1+\frac{\sqrt{z-e^{-\beta}}}{\sqrt{z-e^{-\gamma}}}\right)^2}{(e^{-\gamma}-e^{-\beta})}\right)\right.$$
$$\left.-\frac{e^{\frac{\beta+\gamma}{2}}}{z}\sqrt{(z-\mathrm{e}^{-\gamma})(z-\mathrm{e}^{-\beta})}\left(p\log\left(\frac{e^{\frac{\beta}{2}}+e^{\frac{\gamma}{2}}}{2}\right)-\log\left(\frac{e^{-\frac{\gamma}{2}}+e^{-\frac{\beta}{2}}}{e^{-\frac{\gamma}{2}}-e^{-\frac{\beta}{2}}}\right)\right)\right]$$
$$-\frac{\gamma}{z}-\frac{t}{2z}-\frac{t}{2z}\mathrm{e}^{(\beta+\gamma)/2}\sqrt{(z-\mathrm{e}^{-\gamma})(z-\mathrm{e}^{-\beta})}. \tag{4.27}$$

Because of the chiral ansatz, the boundary condition for large $z$ explicitly depends on the extremes of the interval and it imposes that $\omega(z) \sim -\gamma/z$. Then the vanishing of the constant term in (4.27) at infinity implies that

$$(1-p)\log\left(\frac{e^{\frac{\beta}{2}}+e^{\frac{\gamma}{2}}}{2}\right)-\log\left(\frac{e^{\frac{\beta}{2}}-e^{\frac{\gamma}{2}}}{2}\right)=\frac{t}{2}. \tag{4.28}$$

The matching of the subleading term in the expansion ($z^{-1}$) instead imposes the following endpoint equation

$$(p-1)\log\left(\frac{e^{-\frac{\beta}{2}}+e^{-\frac{\gamma}{2}}}{2}\right)-\log\left(\frac{e^{-\frac{\gamma}{2}}-e^{-\frac{\beta}{2}}}{2}\right)=\frac{t}{2}. \tag{4.29}$$

Our goal, in the following, is to solve these two equations in a closed form. The first step is to show that (4.28) and (4.29) can be reduced to a polynomial equation for an auxiliary unknown $w$. First, we introduce two intermediate variables $x$ and $y$ defined as

$$x=\frac{e^{-\frac{\beta}{2}}+e^{-\frac{\gamma}{2}}}{2} \qquad y=\frac{e^{-\frac{\gamma}{2}}-e^{-\frac{\beta}{2}}}{2}, \tag{4.30}$$

which, because of their definition, have to obey the following inequalities: $x > 0,\ y \geq 0,\ x - y = e^{-\frac{\beta}{2}} > 0$. The saddle point equations (4.28) and (4.29) now read

$$\frac{e^{-t/2}x^{1-p}\left(x^2-y^2\right)^p}{y}-1=0 \qquad \frac{e^{-t/2}x^{p-1}}{y}-1=0. \tag{4.31}$$

The second equation can be easily solved with respect to $y$ ($y = e^{-t/2}x^{p-1}$) and, after a trivial algebraic manipulation, we are left with

$$x^{-2\frac{(p-1)^2}{p}}\left(x^{2/p}-1\right)=e^{-t}. \tag{4.32}$$

The form of (4.32) suggests that the natural variable for our problem is

$$w = x^{-\frac{2}{p}}, \tag{4.33}$$

in terms of which (4.32) becomes a polynomial equation

$$\mathrm{e}^{-t} = w^{(p-1)^2-1} - w^{(p-1)^2} \equiv f(w) \quad . \tag{4.34}$$

Because of the inequalities constraining $x$ and $y$, we are interested in the solutions of (4.34) that satisfy $w > e^{-\frac{t}{p(p-2)}}$. Once we have solved (4.34), the endpoints are then recovered through the relation

$$\gamma = -2\log\left(w^{-p/2} + e^{-t/2}w^{-\frac{1}{2}(p-1)p}\right) \text{ and } \beta = -2\log\left(w^{-p/2} - e^{-t/2}w^{-\frac{1}{2}(p-1)p}\right). \tag{4.35}$$

The existence and the properties of such solutions can be investigated by studying the behavior of the r.h.s. of (4.34) and, in particular, it is sufficient to focus on the region $0 \leq w \leq 1$, since $e^{-t}$ belongs to the interval $(0,1]$ when $t \geq 0$. In this region, for $p > 2$, the derivative of the r.h.s. of (4.34), *i.e.*

$$f'(w) = w^{(p-1)^2-2}((p-2)p - (p-1)^2 w), \tag{4.36}$$

has a unique zero given by

$$w_c = \frac{p(p-2)}{(p-1)^2} < 1. \tag{4.37}$$

Moreover it is positive for $0 < w < w_c$ and negative for $w > w_c$.

Then solutions of (4.34) exist if and only if $e^{-t}$ is less than the maximum located at $w = w_c$. In other words, we must have $\mathrm{e}^{-t} \leq f(w_c)$. This inequality is saturated by a minimum value of $t$ given by

$$t_c = \log\left((p(p-2))^{p(2-p)}(p-1)^{2(p-1)^2}\right). \tag{4.38}$$

Above this critical value, there are two solutions: the former is smoothly connected to $w = 1$ while the latter is smoothly connected to $w = 0$, as fig. 2 shows.
Because of the bound $w > e^{-\frac{t}{p(p-2)}}$, the solution which appears to describe the regime for large $t$ is the one near $w = 1$. This solution can be constructed as a series in $e^{-t}$:

$$w = 1 + \sum_{n=1}^{\infty} x_n \mathrm{e}^{-nt}.$$

The coefficients $x_n$ are determined, in a closed form, by means of the Lagrange inversion formula applied to (4.34) (see appendix A). We obtain, after reinserting the sign factor through $\mathrm{e}^{-t} \to (-1)^p \mathrm{e}^{-t}$,

$$w = 1 + \sum_{n=1}^{\infty} \prod_{k=1}^{n-1} \left(k + 1 - n(p-1)^2\right) \frac{(-1)^{n(p+1)}}{n!} \mathrm{e}^{-nt}. \tag{4.39}$$

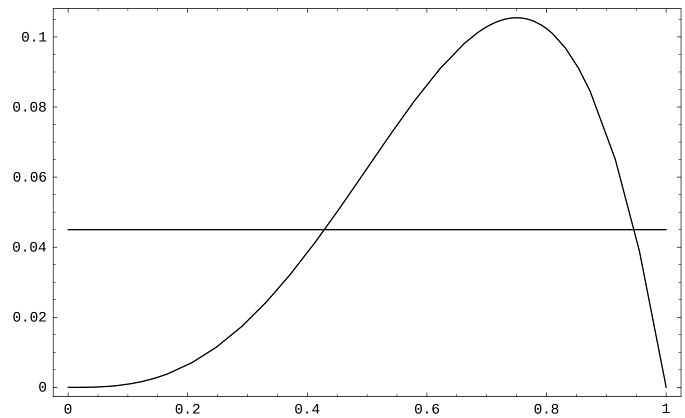


Figure 2: Plot of the r.h.s. of (4.34) in the region $0 < w < 1$. The straight line represents the constant value $e^{-t}$.

An intriguing feature of this expansion, even if it is not evident, is that its coefficients are *integer numbers* for integer $p$. This suggests that the relation between $w$ and $\mathrm{e}^{-t}$ is a mirror map, which is indeed characterized (for yet unknwon reasons) by integer coefficients.
It is worth noticing that the above series can be also summed. Our equation belongs, in fact, to a special class of algebraic equations known as *trinomial equations*. These are equations of the form

$$w^n - aw^s + b = 0, \qquad \text{with } n > s, \tag{4.40}$$

and it was shown by Birkeland (see Appendix A in [60]) that they can be solved in terms of generalized hypergeometric functions. In our case, we find

$$w = \frac{1}{n} + \frac{n-1}{n} \ {}_{n-1}F_{n-2}\left( \begin{array}{cccc} \frac{n-2}{n} & \cdots & \frac{1}{n} & -\frac{1}{n} \\ \frac{n-2}{n-1} & \cdots & \frac{1}{n-1} & \end{array} \middle| \ \zeta \right), \tag{4.41}$$

with $n = (p-1)^2$ and $\zeta = e^{-(t-t_c)}$.
It is instructive to investigate the radius of convergence of the series (4.39). The simplest way of finding it is to look directly at (4.41). From the theory of hypergeometric series, we know that our series converges only if $\zeta \leq 1$, namely $t \geq t_c$: it starts diverging for the same value $t_c$ for which the solutions to the endpoints equations stop existing. The same result can be reached by an estimate of the asymptotic behavior of the $x_n$ through the Stirling formula. As we will see in the next section, this is also the radius of convergence of the prepotential.

In this analysis, we have neglected the cases $p = 1, 2$, which deserve special attention. In these two cases, solutions exist for any value of the parameter $t$ greater than zero and the solution can be given in terms of elementary functions: (4.34) is, in fact, of the first order in $w$ both for $p = 1$ and $p = 2$.

At this point we should remark that the above solution can be potentially shadowed by the breaking of the chiral ansatz. In fact, the endpoint $\gamma$ may reach $-1$ ($h = 0$) for a value $t_C \geq t_c$. However, there are various reasons why this transition is not relevant for our analysis. First of all, this transition depends on the value of $T$: it exists only for small value of $T$ and when $T \gtrsim 1.68$, the transition disappears completely. This indicates that the transition is not intrinsic to our original model (4.1), but rather an artifact of the matrix model description. Second, this transition is related to the Gross–Witten–Wadia transition [15, 16], therefore it does not characterize the expansion (4.5) in itself. The transition appears only when this expansion is regarded as the strong coupling expansion of a unitary matrix model.

### 4.3 The planar free energy

When the $(\beta, \gamma)$ are chosen so that equations (4.28) and (4.29) are satisfied, the form of the resolvent greatly simplifies. In fact, only the logarithmic parts survive and we are left with

$$
\begin{aligned}
\omega(z) = & -\frac{\gamma}{z} + \frac{t}{2z} - \\
& - \left[ \frac{p}{2z} \log\left( \frac{\left(e^{-\frac{\beta}{2}}\sqrt{z - e^{-\gamma}} + e^{-\frac{\gamma}{2}}\sqrt{z - e^{-\beta}}\right)^2}{\left(\sqrt{z - e^{-\gamma}} + \sqrt{z - e^{-\beta}}\right)^2} \right) + \frac{1}{z} \log \left( \frac{\left(1 + \frac{\sqrt{z - e^{-\beta}}}{\sqrt{z - e^{-\gamma}}}\right)^2}{(e^{-\gamma} - e^{-\beta})} \right) \right].
\end{aligned}
\tag{4.42}
$$

Its jump around the cut $(\mathrm{e}^{-\beta}, \mathrm{e}^{-\gamma})$ determines our genus zero density function $\rho$. We get

$$
\rho(s) = \frac{p}{\pi} \arctan\left( \sqrt{\frac{e^{-\gamma} - s}{s - e^{-\beta}}} \right) + \frac{2}{\pi} \arctan\left( \sqrt{\frac{s - e^{-\beta}}{e^{-\gamma} - s}} \right) - \frac{p}{\pi} \arctan\left( e^{\frac{\gamma}{2} - \frac{\beta}{2}} \sqrt{\frac{e^{-\gamma} - s}{s - e^{-\beta}}} \right).
\tag{4.43}
$$

We now possess all the necessary ingredients for computing the partition function at genus zero. We proceed as usual and we first evaluate its derivative with respect to $t$. It is given by

$$
\mathcal{I} = \frac{\partial(F_0(t, p))}{\partial t} = \int_{e^{-\beta}}^{e^{-\gamma}} \frac{\mathrm{d}s}{s} \, \log s \, \rho(s) - \frac{\gamma^2}{2} = -\frac{1}{2} \int_{e^{-\beta}}^{e^{-\gamma}} \mathrm{d}s \, (\log s)^2 \, (\rho(s))',
\tag{4.44}
$$

where the contribution $-\gamma^2/2$ comes from the region where the original density is constant. The last integral in (4.44) can be computed in a closed form and only in terms of $w$. To this purpose, we shall first change the integration variable from $s$ to $\phi$

$$
s = \frac{1}{2}(e^{-\gamma} + e^{-\beta} - (e^{-\gamma} - e^{-\beta}) \cos \phi).
\tag{4.45}
$$

The integral then reduces to

$$\mathcal{I} = \int_0^\pi \mathrm{d}\phi \left[\frac{p-2}{4\pi} - \frac{e^{-\frac{\beta+\gamma}{2}}p}{2\pi\left(\left(e^{-\beta}-e^{-\gamma}\right)\cos\phi+e^{-\beta}+e^{-\gamma}\right)}\right]\log^2\left(\frac{e^{-\beta}+e^{-\gamma}-\left(e^{-\gamma}-e^{-\beta}\right)\cos\phi}{2}\right) \tag{4.46}$$

and it can be exactly performed as shown in appendix B and the result in terms of the two endpoints is

$$\mathcal{I} \;=\; -\mathrm{Li}_2\left(\left(\frac{e^{\beta/2}-e^{\gamma/2}}{e^{\beta/2}+e^{\gamma/2}}\right)^2\right) + (p-2)\log^2\left(\frac{e^{-\beta/2}+e^{-\gamma/2}}{2}\right) - p\log^2\left(\frac{e^{\beta/2}+e^{\gamma/2}}{2}\right).$$

By employing the expression of $\beta$ and $\gamma$ in terms of $w$ (4.35), we obtain

$$\mathcal{I} = \frac{1}{2}p(p-2)\log^2(w) - \mathrm{Li}_2(1-w). \tag{4.47}$$

It is interesting to notice that taking another derivative of the partition function with respect to $t$ produces an even simpler result. Since (4.34) fixes the derivative with respect to $t$ of $w$ to be

$$w'(t) = -\frac{(w-1)w}{(p-1)^2w-(p-2)p}, \tag{4.48}$$

an easy computation shows that

$$\frac{d^2F_0}{dt^2} = -\log(w). \tag{4.49}$$

To obtain the actual free energy of genus zero, we have to integrate (4.47) over $t$. This integration, however, can be transformed in an integration over $w$ with help of (4.48). We can write

$$\frac{dF_0}{dw}(w,p) = -\left[\frac{p(p-2)}{w} + \frac{1}{w-1}\right]\left(\frac{1}{2}p(p-2)\log^2(w) - \mathrm{Li}_2(1-w)\right), \tag{4.50}$$

which, integrated over $w$, gives

$$\begin{aligned} F_0(w(t),p) = &-\frac{p^2(p-2)^2}{6}\log^3(w) + \mathrm{Li}_3(1-w) + p(p-2)\log(w)\mathrm{Li}_2(1-w) + \\ &+ \frac{(p-2)p}{2}\left(\log(1-w)\log^2(w) + 2\mathrm{Li}_2(w)\log(w) - 2\mathrm{Li}_3(w) + 2\zeta(3)\right), \end{aligned} \tag{4.51}$$

up to $p-$dependent constant. With the help of the polylogarithmic identities that connect polylogarithms of different arguments (see appendix B), we can reduce the partition function to its final form

$$F_0(w(t),p) = p(p-2)\mathrm{Li}_3\left(1-\frac{1}{w}\right) + (p-1)^2\mathrm{Li}_3(1-w) - \frac{p}{6}(p-2)(p-1)^2\log^3(w). \tag{4.52}$$

We can also provide a closed expansion for the prepotential $F_0$ as a series in $\mathrm{e}^{-t}$. This is better done by working out the expansion of $\log w$ through Lagrange inversion and integrating (4.49) twice. In this way we obtain

$$F_0^{X_p}(t) = \sum_{d=1}^{\infty} \frac{1}{d!} \frac{1}{d^2} \frac{((p-1)^2 d - 1)!}{(((p-1)^2 - 1)d)!} (-1)^{dt} \mathrm{e}^{-dt}. \tag{4.53}$$

It is now easy to check the consistency of this result with the direct expansion of free-energy in (3.10), justifying therefore *a posteriori* the choice of the chiral ansatz. We notice that the partition function for $p = 1, 2$ greatly simplifies and we obtain, in both cases, the partition function of the resolved conifold, as expected, and up to a global sign due to our choice of $g_s$,

$$F_0^{X_1}(t) = -\mathrm{Li}_3\left(\mathrm{e}^{-t}\right), \qquad F_0^{X_2}(t) = \mathrm{Li}_3(\mathrm{e}^{-t}). \tag{4.54}$$

Finally, we note that the exact expression (4.53) only depends on $p$ through $(p-1)^2$, in accord with the symmetry (3.2).

## 4.4 Comparison with Hurwitz theory

As we mentioned in section 3, the model (4.1) can be regarded as a q–deformation of Hurwitz theory, which should be recovered in the limit (3.16). This limit can be taken order by order in the genus expansion, as one can easily see by writing it in terms of the 't Hooft parameter $T$. We have to take

$$T \to 0, \quad t \to +\infty, \quad p \to \infty, \tag{4.55}$$

in such a way that

$$pT = \tau_2, \quad (-1)^p \mathrm{e}^{-t} t = T^2 \mathrm{e}^{-\tau_1}, \tag{4.56}$$

are kept fixed. If we write the free energy of Hurwitz theory (3.33) as

$$F_{\mathrm{Hurwitz}} = \sum_{g=0}^{\infty} \left(\frac{N}{\tau_2}\right)^{2-2g} F_g^{\mathrm{Hurwitz}}(\mu), \tag{4.57}$$

where

$$\mu = \tau_2^2 \mathrm{e}^{-\tau_1}, \tag{4.58}$$

then in the limit (4.56) one has

$$p^{2-2g} F_g^{X_p}(t) \to F_g^{\mathrm{Hurwitz}}(\mu). \tag{4.59}$$

The planar limit of the Hurwitz model was analyzed in [21] by using matrix model techniques. It is easy to see that all of their results can be recovered from the planar solution of the deformed model. In the solution of [21], a crucial role is played by an auxiliary variable $\chi$, which is related to the endpoints $[b, a]$ of the Young tableaux density through

$$\chi = \tau_1 + 2\log\frac{a-b}{4}. \tag{4.60}$$

The endpoint equations of [21] lead to an equation relating $\chi$ to the parameter $\mu$

$$\mu = \chi e^{-\chi}. \tag{4.61}$$

The solution $\chi(\mu)$ to this equation is provided by Lambert's $W$ function

$$\chi = -W(-\mu) = \sum_{k=1}^{\infty}\frac{k^{k-1}}{k!}\mu^k, \tag{4.62}$$

which has convergence radius $\mu_c = e^{-1}$. It follows from the results of [21] that the prepotential $F_0^{\rm Hurwitz}(\mu)$ of Hurwitz theory satisfies

$$\left(\mu\frac{\partial}{\partial\mu}\right)^2 F_0^{\rm Hurwitz}(\mu) = \chi, \tag{4.63}$$

and from here one can find the power series expansion for the free-energy

$$F_0^{\rm Hurwitz}(\mu) = \sum_{k=1}^{\infty}\frac{k^{k-3}}{k!}\mu^k. \tag{4.64}$$

Comparing the undeformed case with the deformed one, we realize that the variable $\chi$ is playing here the role of the variable $w$ introduced in (4.33). In fact in the limit (4.55)-(4.56) one has

$$w - 1 \to -\frac{\chi}{p^2}. \tag{4.65}$$

Also, the equation (4.64) is the limit of the equation (4.49), and using this fact or the explicit expansions (4.53), (4.63) we can verify (4.59) for $g = 0$.

We finish the section remarking that the connection with this model will provide a useful tool in understanding the relation between Hurwitz numbers and Gromov-Witten invariants, also *beyond* genus zero, as presented in the Section 6.

# 5 Critical behavior in the planar limit

In this section we will show, by looking at the genus zero solution for the Gromov–Witten invariants obtained in the previous section, that there is a phase transition for $p > 2$ with the critical exponent typical of 2d gravity.

## 5.1 Critical properties of the planar free energy

Since we have an exact expression for the genus zero Gromov–Witten invariants at all degrees, we can analyze the critical behavior by simply studying their asymptotic growth. We found,

$$N_{0,k} = \frac{1}{k!k^2} \frac{((p-1)^2 k - 1)!}{(((p-1)^2 - 1)k)!}, \tag{5.1}$$

up to a sign $(-1)^{pk}$. By using Stirling's formula, we obtain

$$N_{0,k} \sim \mathrm{e}^{kt_c} k^{-7/2}, \qquad k \to \infty, \tag{5.2}$$

where $t_c$ is given in (4.38). Comparing with (2.3) we deduce that $t_c$ gives indeed the convergence radius of the expansion of the prepotential around $t = \infty$, and we also deduce that

$$\gamma = -\frac{1}{2}. \tag{5.3}$$

The above results are valid for $p > 2$. For $p = 1, 2$ the series is convergent for all $t > 0$.

We will now analyze the behavior of the prepotential near the critical point. By using the explicit relation between $t$ and $w$ (4.39) we find

$$w - w_c = A(\mathrm{e}^{-t_c} - \mathrm{e}^{-t})^{1/2} + \cdots \tag{5.4}$$

where $w_c$ is given in (4.37) and

$$A = \sqrt{2} \frac{(p(p-2))^{1-(p-1)^2/2}}{(p-1)^{3-(p-1)^2}} = \sqrt{2} \frac{w_c^{1-(p-1)^2/2}}{p-1}. \tag{5.5}$$

In order to extract the most singular part of the prepotential, we use (4.49), which leads to the expansion

$$\frac{d^2 F_0^{X_p}}{dt^2} = -\log w_c - \frac{1}{w_c}(w - w_c) + \cdots \sim -\frac{A}{w_c}(\mathrm{e}^{-t_c} - \mathrm{e}^{-t})^{\frac{1}{2}}. \tag{5.6}$$

Integrating this equation, we find

$$F_0^{X_p}(t) \sim -\frac{4}{15} \frac{(p-1)^8}{4w_c^3}(w - w_c)^5 \sim -\frac{4}{15} \frac{(p-1)^8}{4w_c^3} A^5 (\mathrm{e}^{-t_c} - \mathrm{e}^{-t})^{\frac{5}{2}}. \tag{5.7}$$

which confirms the expected behavior (2.4) and also gives us a precise value for the coefficient of the most singular piece. Again, the above holds only for $p > 2$, and for $p = 1, 2$ the prepotential has a conifold–like behavior at $t = 0$.

If we describe the saddle–point by the density $\rho(h)$ given in (4.43), the critical behavior can be understood exactly as in the case of matrix models (see for example [35]). This was pointed out for the limiting case of Hurwitz theory in [21], where $F_0^{\mathrm{Hurwitz}}(\mu)$ undergoes the same kind of phase transition at $\mu_c = \mathrm{e}^{-1}$. For $t > t_c$, the density $\rho(h)$ behaves near the endpoint $h = b$ as

$$\rho(h) - 1 \sim (b-h)^{\frac{1}{2}}, \tag{5.8}$$

where $-\gamma = 1 - b$ and $-\beta = 1 - a$. Indeed, by expanding around this point we find

$$\rho(h) - 1 = \sum_{k=1}^{\infty} \alpha_k(t,p)(b-h)^{k-\frac{1}{2}}, \tag{5.9}$$

where

$$\alpha_1(t,p) = \frac{1}{\sqrt{a-b}}\Big(p - 2 - p\sqrt{\frac{a}{b}}\Big) \tag{5.10}$$

If we now use the explicit expressions for the endpoints given in (4.35) we find that, at the critical point,

$$\alpha_1(t_c,p) = 0, \quad \alpha_2(t_c,p) \neq 0. \tag{5.11}$$

Therefore, criticality means, at the level of the density, that the leading branch cut singularity near $b$ is enhanced to

$$\rho(h) - 1 \sim (b-h)^{\frac{3}{2}}. \tag{5.12}$$

The picture which emerges from this analysis is the following. In the large area phase $t > t_c$, the planar model is described by the one–cut ansatz of the previous section, and the large radius expansion of the prepotential converges. At $t = t_c$ there is a phase transition controlled by the critical exponent $\gamma = -1/2$, and not by the conventional one $\gamma = 0$ which is found in the quintic, in local $\mathbb{P}^2$, and in other Calabi–Yau manifolds, as we reviewed in section 2. This is probably due to the fact that $Z_{X_p}$ is an equivariant partition function, and our result suggests that general equivariant partition functions like (3.12) will exhibit different critical behaviors for different choices of the equivariant parameters. On the other hand, in the small area phase $t < t_c$ the one–cut ansatz is no longer a valid solution of the planar model, since the equation (4.34) does not have real solutions for $w$. This phase should be described by a chiral *two–cut* ansatz, as argued in [21] in the case of Hurwitz theory. It would be very interesting to obtain an explicit solution for this phase and to understand its geometric meaning.

## 5.2 Multicritical behavior

We have seen that, for $p > 2$, $Z_{X_p}$ exhibits critical behavior with $\gamma = -1/2$. In particular, if we regard this model as a deformation of the Hurwitz model, we have found that the critical behavior found in [21] at $p \to \infty$ persists for all $p > 2$.

It is then natural to ask if one can find *multicritical behavior* a la Kazakov [61], and obtain the critical behavior of the $(2, 2m-1)$ models, with $m \geq 3$. The answer is yes, provided we turn on higher Casimir operators. This does not have a clear interpretation in the context of topological string theory on toric Calabi–Yau manifolds, but it is natural to do if we view $Z_{X_p}$ as the partition function of 5d, $\mathcal{N}=2$ Abelian gauge theory (the 5d version of [45]). We will however perform a precise analysis only in the undeformed model (3.18) and at the planar level. It is clear that the deformed model will have the same multicritical behavior, but it is more difficult to solve in an explicit form.

Let us then turn on higher Casimir operators in the Hurwitz model and consider the partition function

$$Z_{\rm Hurwitz}(\tau_k) = \sum_R \left(\frac{d_R}{|\ell(R)|!}\right)^2 N^{2\ell(R)} {\rm e}^{-\tau_1 \ell(R) - \tau_2 \kappa_R/2N - \sum_{k\geq 3} \frac{\tau_k}{kN^{k-1}}\mathcal{C}_k}, \tag{5.13}$$

where

$$\mathcal{C}_k = \sum_{i=1}^N h_i^k. \tag{5.14}$$

Other definitions for the operators are possible. For example, using the operators $\mathbf{p}_k$ defined in [48] we obtain the partition function of topological string theory on $\mathbb{P}^1$ with descendants of the Kähler class, and by using the operators defined in [45] we obtain the partition function of $U(1)$, four–dimensional $\mathcal{N}=2$ supersymmetric gauge theory perturbed with Casimir operators. In any case, with the above modification, the saddle point equation becomes

$$\int_b^a dh' \frac{\rho(h')}{h-h'} = \frac{1}{2}W'(h) + \log(h-b), \tag{5.15}$$

where

$$W'(h) = \tau_2(h-1) + \tau_1 + \sum_{k\geq 2} \tau_k h^k. \tag{5.16}$$

The resolvent is

$$\omega_0(h) = \frac{1}{2}W'(h) - \frac{1}{2}M(h)\sqrt{(h-a)(h-b)} + \log(a-b) + \log\, h - 2\log\left[\sqrt{h-a}+\sqrt{h-b}\right], \tag{5.17}$$

where

$$M(h) = \oint_0 \frac{dz}{2\pi {\rm i}} \frac{W'(1/z)}{1-hz} \frac{1}{\sqrt{(1-az)(1-bz)}}. \tag{5.18}$$

The density of eigenvalues is

$$\rho(h) = \frac{1}{2\pi} M(h)(h-b)^{\frac{1}{2}}(a-b)^{\frac{1}{2}}\sqrt{1-\frac{h-b}{a-b}} + \frac{2}{\pi}\cos^{-1}\left(\frac{h-b}{a-b}\right)^{\frac{1}{2}}. \tag{5.19}$$

We can expand it around the endpoint $h=b$, and we find as before

$$\rho(h) - 1 = \sum_{k=1}^\infty \alpha_k(\tau_k, p)(b-h)^{k-\frac{1}{2}}. \tag{5.20}$$

The $m$–th critical point is achieved when

$$\rho(h) - 1 \sim (b-h)^{m-\frac{1}{2}}, \qquad m \geq 2, \tag{5.21}$$

i.e. we have to fine–tune the $\tau_k$ in such a way that

$$\alpha_k(\tau_k, p) = 0, \quad 1 \leq k \leq m-1. \tag{5.22}$$

When this is the case, we get a critical model with

$$\gamma = -\frac{1}{m}. \tag{5.23}$$

To guarantee this, $M(h)$ has to be a polynomial of degree at least $m-2$ in $h$, which we denote by $P_m(h)$,

$$M(h) = P_m(h). \tag{5.24}$$

This is simply obtained by taking the first $m-2$ powers of $h-b$ in the series

$$P_m(h) = \left[\left(1 - \frac{h-b}{a-b}\right)^{-\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; \frac{h-b}{a-b}\right)\right]_{m-2}. \tag{5.25}$$

We have used in the above that

$$\frac{\pi}{2} - \cos^{-1} z = zF\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; z^2\right) = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} z^{2k+1}. \tag{5.26}$$

It is now easy to extract the derivative $W_m'(h)$ of the $m$–th multicritical potential as

$$W_m'(h) = 4\Big[P_m(h)\frac{h}{a-b}\sqrt{\left(1-\frac{a}{h}\right)\left(1-\frac{b}{h}\right)}\Big]_+, \tag{5.27}$$

where the subscript $+$ means as usual that we take the positive powers of $h$ in the expansion of the above functions. We now list the first few critical polynomials. Denoting

$$z = \frac{h-b}{a-b} \tag{5.28}$$

we have the simplified formula

$$W_m'(z) = 4\Big[P_m(z)z\sqrt{\left(1-\frac{1}{z}\right)}\Big]_+. \tag{5.29}$$

For example, one has

$$
\begin{aligned}
W_2'(z) &= 4z, \\
W_3'(z) &= \frac{8}{3}z + \frac{8}{3}z^2, \\
W_4'(z) &= \frac{12}{5}z + \frac{8}{5}z^2 + \frac{32}{15}z^3, \\
W_5'(z) &= \frac{16}{7}z + \frac{48}{35}z^2 + \frac{128}{105}z^3 + \frac{64}{35}z^4.
\end{aligned}
\tag{5.30}
$$

We then see that, in what concerns critical behavior, and at least at the planar level, the general partition function (5.13) is equivalent to the one matrix model, since one can reach all the $(2, 2m-1)$ points. As we will see in the next section, the double–scaling limit of the critical model where the first and the second Casimirs have been turned on is the $(2, 3)$ model (i.e. pure gravity). It is natural to conjecture that the double–scaling limit of (5.13) is equivalent to the double–scaling limit of the one matrix model, and one can use it to describe the general $(2, 2m-1)$ models coupled to 2d gravity. This should also hold for the deformed model (3.18) with higher Casimirs.

On the other hand, Okounkov and Pandharipande have shown in [23] that the asymptotics of the Hurwitz model (3.38) is equivalent to the asymptotics of the edge–of –the–spectrum matrix model, which in turn is equivalent to topological gravity in two dimensions. It is easy to see that the partition function (5.13), when expanded in powers of the couplings $\tau_k$, can be also expressed in terms of Hurwitz numbers. Indeed, it is closely related to (3.38) by a nontrivial map relating $\tau_k$ and $\mathrm{Tr}\, V^k$. In this sense, we can regard the Hurwitz model as an analog of the Gaussian matrix model with brane insertions, while the model (5.13) is an analog of the one matrix model. The relation between them and their equivalence in the double–scaling limit should be interpreted as some sort of open/closed duality for Hurwitz theory, in the spirit of [24, 25].

# 6 Higher genus analysis and double–scaling limit

## 6.1 Explicit expressions at higher genus

We have seen that the planar free energy of topological string theory on $X_p$ undergoes a phase transition at small volume, with the same critical exponent as 2d gravity. We would like to understand now the behavior at higher genus. First we have to see if the $F_g^{X_p}(t)$ have a critical point at the same $t_c$, and if the critical exponent depends on the genus as in (2.12). If this is the case, we can define a double–scaled theory at the transition point capturing the all–genus behavior, and we can try to extract the coefficients of the leading singularities at every genus in a unified way.

We have then to compute $F_g^{X_p}(t)$ from the partition function (3.9). Unfortunately, to the best of our knowledge, there is no systematic way to compute corrections to the saddle–point from sums over partitions. We have then to use a different approach. We will in fact proceed in reverse: we will present an ansatz for the structure of $F_g^{X_p}(t)$ which manifestly has the critical behavior in (2.12). Then, we will give evidence for the ansatz. The main evidence comes from the $p \to \infty$ limit. In this case, and as we explained in section 3, our model is equivalent to Hurwitz theory, and the ansatz holds thanks to the results of [22, 28], which were the source of inspiration for the conjectural expressions we will present.

Our ansatz gives closed expressions for $F_1^{X_p}(t)$ in terms of a simple logarithmic expression,

$$F_1^{X_p} = -\frac{1}{24}\log(w-w_c) - \frac{1}{12}\log(p-1) + \frac{1}{24}(p^2-2p+3)\log\, w, \tag{6.1}$$

and for the higher genus $F_g^{X_p}(t)$ as rational functions of the variable $w$

$$F_g^{X_p} = \frac{\mathcal{P}_g(w,p)}{(w-w_c)^{5(g-1)}}, \quad \mathcal{P}_g(w,p) = \sum_{i=1}^{5(g-1)} a_{g,i}(p)(w-1)^i. \tag{6.2}$$

The $a_{g,i}(p)$ have the form

$$a_{g,i}(p) = \frac{b_{g,i}(p)}{(p-1)^n}, \tag{6.3}$$

where $n$ is a positive integer and $b_{g,i}(p)$ are polynomials in $p$ with rational coefficients. We have verified this ansatz by direct computation up to genus 4. In this ansatz the $F_g^{X_p}(t)$ are determined by $5(g-1)$ coefficients $a_{g,i}(p)$, which can be uniquely determined from the genus $g$ Gromov–Witten invariants up to degree $5(g-1)$. These can be computed from the explicit expression (3.9) for $Z_{X_p}$. One then verifies that the expression (6.2) obtained in this way reproduces correctly the Gromov–Witten invariants of higher degree.

For genus 2, for example, one finds in this way:

$$\begin{aligned}
a_{2,5}(p) &= \frac{1}{2880}\frac{p(p-2)}{(p-1)^2}, \\
a_{2,4}(p) &= -\frac{1}{2880}\frac{12-14p+7p^2}{(p-1)^4}, \\
a_{2,3}(p) &= -\frac{1}{2880}\frac{36-106\,p+161\,p^2-204\,p^3+171\,p^4-72\,p^5+12\,p^6}{(p-1)^8}, \\
a_{2,2}(p) &= -\frac{1}{2880}\frac{36-90\,p+121\,p^2-60\,p^3-5\,p^4+12\,p^5-2\,p^6}{(p-1)^{10}}, \\
a_{2,1}(p) &= -\frac{1}{240}\frac{1}{(p-1)^{10}}.
\end{aligned} \tag{6.4}$$

Once the coefficients $a_{g,i}(p)$ have been obtained, one can deduce closed formulae for the Gromov–Witten invariants $N_{g,d}$ for all $d$ by using Lagrange inversion. We give the results for $g = 1$ in Appendix A.

It is interesting to notice that the ansatz (6.2) is very similar to the holomorphic ambiguity in the standard B–model topological string [3]. There, one finds that $F_g(t)$ is given by a piece which is completely determined in a recursive way by special geometry data and amplitudes at lower genera, plus an undetermined, holomorphic piece of the form

$$f_g(z) = \frac{p_g(z)}{\Delta(z)^{2g-2}}, \tag{6.5}$$

where $z$ is a natural coordinate in the complex structure moduli space, $\Delta(z)$ is the discriminant locus, and $p_g(z)$ is a polynomial in $z$ with unknown coefficients. The exponent $2-2g$ in the discriminant is related to the critical exponent $\gamma = 0$ characteristic of the $c = 1$ behavior (2.18). In our model, the $F_g^{X_p}(t)$ is given entirely by an analog of the holomorphic ambiguity, with a different singularity due to the different critical exponent. This also confirms the role of $w$ as a natural mirror coordinate, and that the relation between $w$ and $\mathrm{e}^{-t}$ is a mirror map.

To further justify this ansatz, let us first recall the results of [22, 28] for the generating functionals of simple Hurwtiz numbers:

$$\begin{aligned} F_1^{\text{Hurwitz}}(\mu) &= -\frac{1}{24}\Big(\log(1-\chi) + \chi\Big), \\ F_g^{\text{Hurwitz}}(\mu) &= \frac{P_g(\chi)}{(1-\chi)^{5(g-1)}}, \quad P_g(\chi) = \sum_{i=1}^{3g-3} c_{g,i}\, \chi^i. \end{aligned} \tag{6.6}$$

Here, $\chi$ is the variable introduced in (4.60), which is related to $\mu$ by (4.62). Moreover, there are explicit expressions for $c_{g,i}$ in terms of Hodge integrals. In particular, one has that

$$P_g(1) = \frac{1}{(3g-3)!}\langle \sigma_2^{3g-3}\rangle_g, \tag{6.7}$$

which is a correlation function at genus $g$ in 2d topological gravity (see for example [35] for a review). This result will be useful later in this section.

It is easy to see that, in the limit (3.15)–(3.16), (6.1) becomes $F_1^{\text{Hurwitz}}(\mu)$, and that the ansatz (6.2) leads to expressions which are compatible with (6.6), provided the polynomials $\mathcal{P}(w,p)$ satisfy certain conditions. These conditions can be written as limiting conditions on the coefficients $a_{g,i}(p)$:

$$\lim_{p\to\infty} p^{8(g-1)-2i} a_{g,i}(p) = (-1)^i c_{g,i}. \tag{6.8}$$

Using (4.65) as well as the result

$$w - w_c \to \frac{1-\chi}{p^2} \tag{6.9}$$

in the limit (3.15)–(3.16), it is easy to see that, if (6.8) holds, then (4.59) is obtained. For example, by using the above expressions for genus two, one obtains in the $p \to \infty$ limit,

$$\frac{1}{p^2} F_2^{X_p}(t) \to \frac{1}{(1-\chi)^5} \left( \frac{12}{2880} \chi^3 + \frac{2}{2880} \chi^2 \right), \tag{6.10}$$

which is the expression found in [22, 28].

## 6.2 Double–scaling limit and Painlevé I

Using the above results for higher genus, it is immediate to see that the ansatz (6.2) gives the critical behavior we are looking for. This is a consequence of (5.4). We want to analyze now the coefficients of the leading singularity. The behavior of the singular part of the planar free energy (5.7) suggests defining a scaled string coupling $z$ as

$$z^{5/2} = g_s^{-2} \frac{(p-1)^8}{4 w_c^3} (w - w_c)^5 = g_s^{-2} \frac{(p-1)^8}{4 w_c^3} A^5 (\mathrm{e}^{-t_c} - \mathrm{e}^{-t})^{\frac{5}{2}}. \tag{6.11}$$

We can now consider the double–scaled theory in which we take the limit

$$t \to t_c, \quad g_s \to 0, \quad z \text{ fixed}. \tag{6.12}$$

In this limit, the total free energy of the model becomes the double–scaled free energy $F_{\rm ds}(z)$,

$$F_{X_p} \to F_{\rm ds}(z). \tag{6.13}$$

Taking into account (5.7) and the expression (6.1) for $F_1^{X_p}(t)$, we find that up to genus one

$$F_{\rm ds}(z) = -\frac{4}{15} z^{5/2} - \frac{1}{48} \log z + \cdots. \tag{6.14}$$

This is, up to this order, the perturbative expansion of the free energy of 2d gravity, $F_{(2,3)}(z)$. We recall (see [35]) that $F_{(2,3)}(z)$ is determined as a function of $z$ by the following equation,

$$F''_{(2,3)}(z) = -u(z), \tag{6.15}$$

where $u(z)$, the specific heat, is a solution of the Painlevé I equation

$$u^2 - \frac{1}{6} u'' = z \tag{6.16}$$

with the asymptotics at weak string coupling

$$u(z) = z^{\frac{1}{2}} + \cdots, \quad z \to \infty. \tag{6.17}$$

This leads to the asymptotic expansion at large $z$,

$$F_{(2,3)}(z) = -\frac{4}{15}z^{5/2} - \frac{1}{48}\log z + \sum_{g\geq 2} a_g z^{-5(g-1)/2}, \tag{6.18}$$

with

$$a_2 = \frac{7}{5560}, \quad a_3 = \frac{245}{331776}, \quad a_4 = \frac{259553}{159252480}, \tag{6.19}$$

an so on.

In view of the above results for genus $g = 0, 1$, it is natural to conjecture that the double–scaled free energy of topological string theory on $X_p$ equals the free energy of 2d gravity, at least in the genus expansion

$$F_{\rm ds}(z) = F_{(2,3)}(z). \tag{6.20}$$

This means that the coefficient of the most singular part of $F_g^{X_p}(t)$ is identical to the coefficient $a_g$ in (6.18), when expressed in terms of the variable $z$ defined in (6.11). In terms of the ansatz (6.2), in order to verify the conjecture (6.20) we have to verify that the polynomials $\mathcal{P}_g(w,p)$ have the following value at the critical point $w = w_c$

$$\mathcal{P}_g(w_c, p) = \left(4\frac{w_c^3}{(p-1)^8}\right)^{g-1} a_g, \quad g \geq 2. \tag{6.21}$$

It is easy to check with (6.4) that this is indeed the case for $g = 2$, and we have verified (6.21) up to genus four. We can indeed provide evidence for (6.21) at all genus in the limit $p \to \infty$. Taking into account the overall factors of $p$, one easily finds that in this limit the conjectured equality (6.21) becomes

$$P_g(1) = 4^{g-1}a_g, \quad g \geq 2, \tag{6.22}$$

where $P_g(\chi)$ are the polynomials that appear in the Goulden–Jackson–Vakil expressions for $F_g^{\rm Hurwitz}$ in (6.6). But due to (6.7) this reduces to checking the following equality for 2d topological gravity correlators,

$$\frac{1}{(3g-3)!}\langle \sigma_2^{3g-3}\rangle_g = 4^{g-1}a_g, \quad g \geq 2, \tag{6.23}$$

which indeed was shown to be true in [62]. This verifies our conjecture in the limiting case $p \to \infty$ at all genus.

To conclude this section, we have provided strong evidence that the critical behavior of topological string theory on $X_p$ at small radius is controlled by the Painlevé I equation for all $p > 2$ (the dependence on $p$ only enters in the normalization factor relating $z$ to the "bare" couplings $t, g_s$). Therefore the double–scaled theory at the transition is 2d gravity (i.e. the $(2,3)$ model). This is in contrast to conventional topological strings, which undergo a phase transition at the conifold point controlled by $c = 1$ string theory at the self–dual radius. The possibility that different kinds of singularities in Calabi–Yau manifolds lead to different universality classes was pointed out at the end of [4]. Here we have found a concrete realization of this possibility.

# 7 Non–perturbative proposal and 2d gravity

## 7.1 The non–perturbative proposal and its critical properties

Topological string theories, as well as ordinary string theories, are defined in principle only perturbatively, in a genus by genus expansion, and it is natural to ask whether a nonperturbative definition is possible. In the case of the backgrounds that we are considering, i.e. Calabi–Yau manifolds of the type $X_p$, it was proposed in [6] based on previous works [63, 67] that such a definition can be given in terms of a q–deformed version of two–dimensional Yang–Mills theory with gauge group $U(N)$. The strong coupling expansion of this partition function is given by the expression

$$Z_{\rm qYM} = \sum_R \left(\dim_q R\right)^2 q^{pC_2(R)/2} e^{i\theta\ell(R)}. \tag{7.1}$$

Here, $R$ is a tableau representing an irreducible representation of $U(N)$, $C_2(R) = \kappa_R + N\ell(R)$, and the quantum dimension of $R$ is given by

$$\dim_q R = \prod_{1\le i<j\le N} \frac{[l_i - l_j + j - i]}{[j-i]}. \tag{7.2}$$

In [6] it was shown that the asymptotic expansion of (7.1) can be written as

$$Z_{\rm qYM} = \sum_{l\in\mathbb{Z}} \sum_{R_1,R_2} Z^+_{R_1R_2}(t + pg_s l) Z^-_{R_1R_2}(\bar{t} - pg_s l). \tag{7.3}$$

In this expression, $t$ is given in terms of the gauge theory data as

$$t = \frac{1}{2}(p-2)Ng_s - {\rm i}\theta. \tag{7.4}$$

The amplitude $Z^{\pm}_{R_1R_2}$ is a perturbative *open* string amplitude on the $X_p$ Calabi–Yau and in the presence of two stacks of D–branes, labeled by the representations $R_1, R_2$. An explicit expression for $Z^{\pm}_{R_1R_2}$, written in terms of the topological vertex, can be found in [6]. When $R_1 = R_2 = \bullet$ is the trivial representation, one recovers the perturbative *closed* string amplitude

$$Z^{\pm}_{\bullet\bullet} = Z_{X_p}, \tag{7.5}$$

up to minor overall factors that do not affect the worldsheet instanton expansion. Therefore, the nonperturbative completion (7.1) gives the square of the perturbative answer (as expected from [63]), plus some extra insertions of branes.

Later on, it was noticed in [12, 13, 14] that, at large $N$, the partition function (7.1) (or more precisely, its zero charge sector $l = 0$) has a third–order phase transition for $p > 2$ which is qualitatively similar to the Douglas–Kazakov transition of 2d Yang–Mills theory. In terms of the parameter $t$ which is identified with the Kähler parameter of the perturbative theory, and for $\theta = 0$, the critical point is given by

$$t_{\rm np}(p) = \frac{1}{2}p(p-2)\log\left(1+\tan^2\left(\frac{\pi}{p}\right)\right). \tag{7.6}$$

Based on the analysis of [43], we expect that this critical value gives the radius of convergence of the strong coupling expansion of (7.1), in the same way that (4.38) gives the radius of convergence of the perturbative theory.

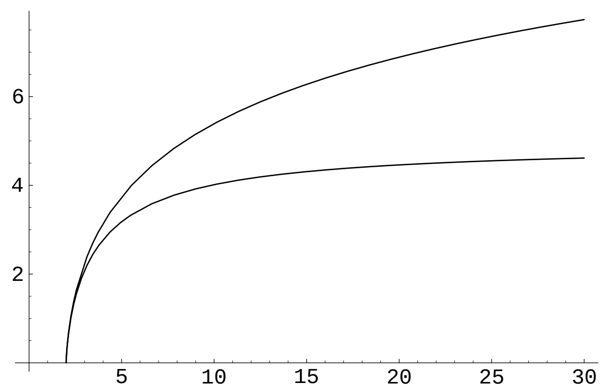


Figure 3: The curve in the top shows the critical value of the Kähler parameter in the perturbative theory, which goes as $2\log p$ for large $p$. The curve in the bottom shows the critical value in the q–deformed 2d Yang–Mills theory, which asymptotes $\pi^2/2$ as $p$ goes to infinity.

Therefore, *both* the perturbative theory, defined by $Z_{X_p}$, and the nonpertubative definition proposed in [6], undergo a phase transition at small radius for $p > 2$. It is interesting to compare the critical behaviors as a further probe of the proposal of [63, 6]. The first things to compare are the radii of convergence $t_c(p)$ and $t_{\rm np}(p)$, as a function of $p$. In Fig. 3 we plot the curves (4.38) and (7.6) for $2 < p < 30$. In the perturbative theory, one has

$$t_c \to 2\log p, \quad p \to \infty, \tag{7.7}$$

while

$$t_{\rm np} \to \frac{\pi^2}{2}, \quad p \to \infty. \tag{7.8}$$

Therefore, at large $p$, the proposed nonperturbative completion has better convergence properties as a function of $t$ at small radius.

Another interesting difference between both models is the universality class of the theory at the transition. We have seen in the previous section that one can define a double–scaled theory at the critical point, and further analysis shows that this theory is 2d gravity and it is governed by the Painlevé I equation. A similar study in the case of q–deformed 2d Yang–Mills theory is still lacking, but the *undeformed* theory (which appears as $p \to \infty$, as explained in [12, 13, 14]) has been analyzed from that point of view in [18]. It turns out that there is a well–defined double–scaled theory as well, but this time it is in the universality class of the Gross–Witten–Wadia unitary matrix model [15, 16]. The double–scaled free energy $F(z)$ is given by $F''(z) = v(z)^2/4$, where $v(z)$ satisfies the Painlevé II equation [17],

$$2v'' - v^3 + zv = 0. \tag{7.9}$$

As argued in [19], this also describes the universality class of pure 2d supergravity. As in the case of the perturbative theory, where the universality class is the same for all $p > 2$, we expect that q–deformed 2d YM has a double–scaled theory described by Painlevé II for all $p > 2$. Some evidence for this was given in [13], based on the behavior of the instanton suppression factor. It was shown by Crnkovic, Douglas and Moore [20] that the Painlevé II equation (7.9) has a unique real, pole–free solution which gives a non–perturbative solution of the double–scaled Gross–Witten–Wadia unitary matrix model and of 2d supergravity [19]. This solution, which we will call $v_{\rm CDM}$, is also the correct nonperturbative description of the double–scaled theory at the critical point of q–deformed 2d Yang–Mills.

We therefore reach the conclusion that perturbative topological string theory on $X_p$ exhibits a phase transition in the universality class of 2d gravity, while the proposed non-perturbative completion exhibits a phase transition in the universality class of 2d supergravity. In the remaining of this section, we will present some comments and speculations on the possible implications of this fact for 2d gravity theories[5].

## 7.2 Implications for 2d gravity

It is well–known (see again [35]) that the Painlevé I equation does not define 2d gravity beyond the perturbation regime, since the resulting series for the specific heat is not Borel summable. There have been various proposals to provide a nonperturbative definition of 2d gravity which agrees with the perturbative expansion (6.18) at large $z$. A particularly interesting proposal was made by Dalley, Johnson and Morris in [64]. By requiring that the KdV flows of 2d gravity hold nonperturbatively they found a modified string equation,

$$u\mathcal{R}^2 - \frac{1}{2}\mathcal{R}\mathcal{R}'' + \frac{1}{4}(\mathcal{R}')^2 = 0, \quad \mathcal{R} = u^2 - u''/3 - z. \tag{7.10}$$

[5]We would like to thank Clifford Johnson for pointing out to us the relevance of [64, 65] to our work, and for an illuminating email exchange. He informed us that he had previously envisaged the possibility that the results of [64, 65] might be relevant in topological string theory.

Notice that $\mathcal{R} = 0$ is just the Painlevé I equation, with a different normalization than (6.16). In this normalization, the free energy computed from $u$ is actually twice the free energy of pure gravity (see for example [35]). It can be shown that the above equation has a smooth, real solution $u_{\rm DJM}$, which at $z \to \infty$ reproduces the genus expansion obtained from Painlevé I, and vanishes as $1/z^2$ when $z \to -\infty$. This solution was proposed in [64] as a nonperturbative definition of 2d gravity.

The equation (7.10) can be generalized by including an extra parameter $\Gamma$, as follows:

$$u\mathcal{R}^2 - \frac{1}{2}\mathcal{R}\mathcal{R}'' + \frac{1}{4}(\mathcal{R}')^2 = \Gamma^2. \tag{7.11}$$

It was argued in [65] that this parameter corresponds to the introduction of an open string sector. This equation has solutions that are open string generalizations of $u_{\rm DJM}$ and which we denote by $u^{\Gamma}_{\rm DJM}$. For $z \to -\infty$ they behave as

$$u^{\Gamma}_{\rm DJM}(z) \sim \frac{\Gamma^2 - \frac{1}{4}}{z^2}. \tag{7.12}$$

It was shown in [65] that any solution $v$ to the Painlevé II equation (7.9) can be mapped to a solution $u$ of (7.11) with $\Gamma = \pm 1/2$, through the so–called Miura transformation (see also [66]). In particular, the solution $v_{\rm CDM}$ maps under the Miura transformation to the solutions $u^{\Gamma=\pm 1/2}_{\rm DJM}$. Moreover, if we denote by $Z_{\rm PII}$ and $Z_\Gamma$ the partition functions associated to the solutions related by Miura transformation, we have the relation

$$Z_{\rm PII} = Z_{\Gamma=\frac{1}{2}} Z_{\Gamma=-\frac{1}{2}}. \tag{7.13}$$

Notice that this is very similar to (7.3), and in both cases one needs to introduce open string sectors (represented in (7.3) by the D–branes and here by the $\Gamma$ parameter). This seems to indicate that the relation (7.3) between the perturbative topological string amplitude and its nonperturbative completion is inherited as the relation (7.13) between their critical counterparts.

Recall that the proposal of [6, 63, 67] is to regard q–deformed Yang–Mills as a nonperturbative completion of topological string theory on $X_p$. If we focus on the critical points of the corresponding theories, the proposal indicates that the nonperturbative completion of the 2d gravity perturbative expansion (6.14) should be associated to the solution $v_{\rm CDM}$ of Painlevé II. In other words, the connection to Painlevé II should provide the extra nonperturbative information needed in 2d gravity. As we have just reviewed, $v_{\rm CDM}$ is related to the solutions $u^{\Gamma=\pm 1/2}_{\rm DJM}$ with an open string sector, as encoded in (7.13). However, the perturbative theory is obtained properly speaking by "removing" the open sector, as in (7.5). We conclude that the embedding of 2d gravity in perturbative topological string theory on $X_p$, together with the proposal of [6, 63, 67], suggest that the natural nonperturbative completion of 2d gravity is indeed the solution presented in [64] $u^{\Gamma=0}_{\rm DJM} = u_{\rm DJM}$.

We emphasize that, although we find this line of argumentation suggestive and compelling, it should be put on a much firmer ground by a better understanding of the nonperturbative proposal of [6, 63, 67] and of the embedding of the critical theories.

# 8 An open–closed duality

In section 3 we saw that the partition function $Z_{X_p}$ is related to a configuration involving a topological D–brane in $\mathbb{C}^3$ with framing $f = p - 1$. In this section, we will point out a different relation which holds at genus zero and generalizes an open–closed duality proposed in [29] in the context of M–theory.

Consider then the open string background given by a D6 brane wrapping a Lagrangian submanifold with topology $\mathbb{S}^1 \times \mathbb{R}^2$ in $\mathbb{C}^3$ and with trivial framing $f = 0$. As explained in [68, 69], this leads to a superpotential $W$ in four dimensions which is given by

$$W_{f=0} = \sum_{d=1}^{\infty} \frac{\mathrm{e}^{du}}{d^2}. \tag{8.1}$$

where $u$ is an open string modulus. From the point of view of the open string amplitudes $F_{g,\vec{k}}$ which appeared in section 3, this superpotential is the generating functional of disk amplitudes with $g = 0$ and a vector $\vec{k}$ with one single entry $k_d = 1$. As pointed out in [29], this superpotential is related to the free energy of the resolved conifold by

$$W_{f=0} = -\frac{dF_0^{X_1}(t)}{dt}, \tag{8.2}$$

after identifying $t = -u$. This relation between superpotential and prepotential is typical from open/closed dualities, where the presence of the brane in the open side is traded by the presence of flux in the closed side.

Consider now the same brane configuration but with arbitrary framing $f$. The superpotential has been computed in [69] and reads

$$W_f = \sum_{k=1}^{\infty} W_{f,k} \mathrm{e}^{ku}, \quad W_{f,k} = \frac{(-1)^{k(f+1)}}{k\, k!} \prod_{j=1}^{k-1} (kf - j). \tag{8.3}$$

Comparing this to our prepotential $F_0^{X_p}(t)$ given in (4.54), we find

$$W_f = -\frac{dF_0^{X_p}}{dt} \tag{8.4}$$

with the choice of framing

$$f = (p-1)^2. \tag{8.5}$$

In particular, the genus zero Gopakumar–Vafa invariants $n_{0,d}(p)$ are given by

$$n_{0,d}(p) = \frac{1}{d} N_d(f), \tag{8.6}$$

where $N_d(f)$ are the open BPS invariants associated to disk instantons. Notice that this relation is very different from (3.46). The invariants $N_d(f)$ can be written as linear combinations of BPS invariants $N_{R,g=0}(f)$ involving hook tableaux of $d$ boxes. But the open string backgrounds involved in (3.46) and in (8.4) are different, since in the first case one has framing $f = p - 1$ while in the second case one has $f = (p-1)^2$.

The result (8.4) indicates the existence of an open–closed duality between closed string theory on $X_p$ and open string theory on $\mathbb{C}^3$ in the presence of a framed D6 brane with framing (8.5), generalizing in this way the relation (8.2) for the resolved conifold $p = 1$. In the $p = 1$ case, the open/closed duality can be derived by lifting both configurations to M–theory on a manifold with $G_2$ holonomy and with topology $\mathbb{S}^3 \times \mathbb{R}^4$ [29]. For general $p$, one should be able to lift the D6 brane with framing to M–theory and relate it there to closed string theory on $X_p$, explaining in this way the equality (8.4). This would produce a new and interesting class of M–theory dualities.

# 9 Conclusions and open problems

The main conclusions of our analysis are the following:

- We have shown that, in some backgrounds, topological string theory exhibits a critical behavior which is not in the universality class of the $c = 1$ string, and leads to a double–scaled theory which is identical to pure 2d gravity (in perturbation theory).

- From a more methodological point of view, we have shown that double–scaling limits can be useful to characterize critical string theories, and they allow us to partially capture their all–genus behavior. This is a useful strategy to address questions related to their nonperturbative behavior, and we have argued in this paper that the double–scaled theory reflects aspects of the nonperturbative structure of the original theory. In particular, we have seen that the proposal for a nonperturbative completion of topological string theory on $X_p$ leads to a double–scaled theory with a well–defined nonperturbative description in terms of the solution $v_{\rm CDM}$ to Painlevé II.

- Conversely, we suggested that this result gives further support to the proposal of [64] for a nonperturbative completion of 2d gravity.

Our work also leaves many open questions, and we list some of them here:

- From a technical point of view, our analysis has some gaps which should be filled. For example, it would be interesting to give a first principles derivation of the higher genus ansatz (6.2), as well as a constructive way of computing the coefficients $a_{g,i}(p)$. In order to do that, one should develop general techniques to compute higher genus corrections for sums over partitions, which would find applications in many other problems related to topological strings and instanton counting. A systematic method to treat higher genus corrections would also make possible to derive the precise properties of the double–scaled theory, as in the case of matrix models.

- We have not determined the nature of the theory in the phase with $t < t_c$. In principle, this corresponds to a two–cut solution of the saddle–point equations, but finding it is technically challenging. In this respect, it would be very interesting to develop mirror symmetry techniques to analyze this model (both at the planar level and at higher genus). Some preliminary steps in this direction have already been given in [40], but much remains to be done. This would also shed light on the B–model analysis of equivariant topological strings introduced in [41].

- We have found evidence for an open–closed duality involving a lift to M–theory, and generalizing the observations of [29]. It would be very interesting to use $G_2$ holonomy manifolds and justify the relation between the D6 brane background with framing $f = (p-1)^2$ and the closed $X_p$ background.

- There have been proposals for nonperturbative formulations of topological string theory on other toric manifolds, like local $\mathbb{P}^2$ [70]. It would be very interesting to analyze the critical behavior of these models and their universality class. Based on the results of this paper, we would expect a "well–behaved" critical theory, probably related to doubly–scaled unitary matrix models.

**Note added**: After this paper was submitted, the work [71] appeared, which studies the B–model realization of topological string theory on $X_p$. They derive the following explicit expression for the mirror map,

$$-q\frac{dt}{dq} = \frac{1-(-1)^p(p-1)^2 q}{1-(-1)^p q}. \tag{9.1}$$

It can be easily seen that after setting $(-1)^{p-1}q = w - 1$ the above expression becomes precisely our equation (4.34). This confirms our conjecture that the relation between $w$ and $t$ is indeed the mirror map of this model. By using B–model techniques, [71] also derives an expression for $F_1$ which agrees precisely with (6.1).

# Acknowledgments

We would like to thank Mina Aganagic, Luis Álvarez–Gaumé, Renzo Cavalieri, Gianni Cicuta, Ian Goulden, Antonella Grassi, Sergei Gukov, David Jackson, Clifford Johnson, Albrecht Klemm, Davesh Maulik, Hirosi Ooguri, Rahul Pandharipande, and Ravi Vakil, for useful conversations, comments, and correspondence. M.M. would like to thank the string theory group at Caltech and the MSRI at Berkeley for their warm hospitality while this work was in progress. L.G. and S.P. would like to thank the Theory Unit at CERN for hospitality. The work of N.C. is supported by a fellowship of the "Angelo della Riccia" foundation.

# A Lagrange inversion

The Lagrange inversion method makes possible to explicitly invert a relation between two variables in power series form. A good exposition can be found for example in [72].

Let $z = f(w)$ be a relation that implicitly defines $w$ as a function of $z$ around $w = 0$. We can assume that $f(0) = 0$. We can invert this relation and find the inverse function

$$w = f^{-1}(z) \tag{A.2}$$

as follows

$$w = \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Res}\left( \frac{1}{f(w)^n} \right) z^n, \tag{A.3}$$

where the residue is computed around $w = 0$. A slight modification of this result makes possible to compute the function

$$g(w) = g(f^{-1}(z)) \tag{A.4}$$

as

$$g(w) = \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Res}\left( \frac{g'(w)}{f(w)^n} \right) z^n. \tag{A.5}$$

We assumed again that $g(0) = 0$.
Let us now apply this procedure to the equation (4.34) relating $t$ and the variable $w$

$$q = w^{(p-1)^2}(w^{-1} - 1), \tag{A.6}$$

where $q = \mathrm{e}^{-t}$. Set $f = (p-1)^2$, $x = w - 1$. The equation defining $x$ is

$$q = -x(1+x)^{f-1}, \tag{A.7}$$

We want to find now $x$ (or $w = 1 + x$) as a function of $q$. By (A.3), we have to compute

$$\frac{(-1)^n}{n} \mathrm{Res}\, x^{-n} \frac{1}{(1+x)^{n(f-1)}} = \frac{(-1)^n}{n} \mathrm{Res} \sum_{k=0}^{\infty} \frac{(n(f-1)+k-1)!}{k!(n(f-1)-1)!} (-1)^k x^{k-n}, \tag{A.8}$$

which equals

$$\frac{(-1)^n}{n!} \prod_{k=2}^{n} (k - nf). \tag{A.9}$$

Therefore,

$$w = 1 + \sum_{n=1}^{\infty} \prod_{k=1}^{n-1} \Big( k + 1 - n(p-1)^2 \Big) \frac{(-1)^n}{n!} q^n. \tag{A.10}$$

Lagrange inversion can be also used to find explicit expressions for the Gromov–Witten invariants of $X_p$ from (6.2). Let us calculate the genus one invariants. In terms of $x = w - 1$, we have

$$F_1 = -\frac{1}{24}\log\Big[1+(p-1)^2 x\Big] + \frac{1}{24}(p^2-2p+3)\log(1+x). \tag{A.11}$$

Setting again $f = (p-1)^2$ we have to compute the residue:

$$-\frac{(-1)^n}{24n}\mathrm{Res}\left(\frac{1}{x^n}\frac{f}{1+fx}\frac{1}{(1+x)^{n(f-1)}} - \frac{1}{x^n}\frac{f+2}{(1+x)^{n(f-1)+1}}\right). \tag{A.12}$$

and we easily obtain

$$N_{1,k} = \frac{1}{24k}\sum_{\ell=0}^{k-1}\frac{f^{k-\ell}}{\ell!}\prod_{j=1}^{\ell}(k(f-1)+j-1) - \frac{1}{24}\frac{(kf-1)!}{k!(k(f-1))!}(f+2). \tag{A.13}$$

We can check the limiting behavior (3.34),

$$\lim_{p\to\infty}\frac{N_{1,k}}{p^{2k}} = \frac{1}{24k}\sum_{\ell=0}^{k-2}\frac{k^\ell}{\ell!}. \tag{A.14}$$

which is indeed the expected answer.

# B Useful Integrals

In this appendix we shall summarize some integrals that have been used to perform the large $N$ analysis in sec. 4.
All the the integrals appearing in the computation of the resolvent $\omega(z)$ can be reduced to the following elementary indefinite integral

$$\begin{aligned}&\int\frac{\mathrm{d}w}{w-s}\,\frac{1}{\sqrt{(w-\mathrm{e}^{-\gamma})(w-\mathrm{e}^{-\beta})}} \\ &\quad = -\frac{1}{\sqrt{(s-\mathrm{e}^{-\gamma})(s-\mathrm{e}^{-\beta})}}\log\left(\frac{\left(\sqrt{(w-\mathrm{e}^{-\beta})(s-\mathrm{e}^{-\gamma})}+\sqrt{(s-\mathrm{e}^{-\beta})(w-\mathrm{e}^{-\gamma})}\right)^2}{(s-w)\sqrt{(s-\mathrm{e}^{-\beta})(s-\mathrm{e}^{-\gamma})}}\right).\end{aligned} \tag{B.15}$$

The computation of the planar free energy involves instead the following families of definite integrals. First we have to consider

$$
\begin{aligned}
\mathcal{I}_1 = & \frac{1}{2}\int_{-\pi}^{\pi} \mathrm{d}\phi \ \log^2\left(\frac{e^{-\beta}+e^{-\gamma}-\left(e^{-\gamma}-e^{-\beta}\right)\cos\phi}{2}\right) = \\
= & \frac{1}{2}\int_{-\pi}^{\pi} \mathrm{d}\phi \ \left[\log\frac{\left(e^{-\beta/2}+e^{-\gamma/2}\right)^2}{4} + \log\left(1-\frac{e^{\beta/2}-e^{\gamma/2}}{e^{\beta/2}+e^{\gamma/2}}e^{-i\phi}\right) + \right. \\
& \left. + \log\left(1-\frac{e^{\beta/2}-e^{\gamma/2}}{e^{\beta/2}+e^{\gamma/2}}e^{i\phi}\right)\right]^2 = 4\pi\log^2\frac{e^{-\beta/2}+e^{-\gamma/2}}{2} + 2\pi\mathrm{Li}_2\left(\left(\frac{e^{\beta/2}-e^{\gamma/2}}{e^{\beta/2}+e^{\gamma/2}}\right)^2\right)
\end{aligned}
\tag{B.16}
$$

This integral is actually computed by expanding the square, then by expanding in series the logarithms and finally integrating the series. The final series can be summed in terms of polylogarithms and logarithms .

With the same technique, we can also show that

$$
\begin{aligned}
\mathcal{I}_2 = & \frac{1}{2}\int_{-\pi}^{\pi} \mathrm{d}\phi \ \log^2\left(1-Ce^{i\phi}\right)\log\left(1-Ce^{-i\phi}\right) = \\
= & -\pi\left(\log\left(\mathrm{C}^2\right)\log^2\left(1-\mathrm{C}^2\right) + 2\mathrm{Li}_2\left(1-\mathrm{C}^2\right)\log\left(1-\mathrm{C}^2\right) - 2\mathrm{Li}_3\left(1-\mathrm{C}^2\right) + 2\zeta(3)\right)
\end{aligned}
\tag{B.17}
$$

and

$$
\begin{aligned}
\mathcal{I}_3 = & \frac{1}{2}\int_{-\pi}^{\pi} \mathrm{d}\phi \ \log^3\left(A-B\cos\phi\right) = \pi\log^3\left(\frac{A}{2}+\frac{1}{2}\sqrt{A^2-B^2}\right) + \\
& + 6\pi\mathrm{Li}_2\left(\left(\frac{B}{A+\sqrt{A^2-B^2}}\right)^2\right)\log\left(\frac{A}{2}+\frac{1}{2}\sqrt{A^2-B^2}\right) + 6\mathcal{I}_2\left(\left(\frac{B}{A+\sqrt{A^2-B^2}}\right)^2\right)
\end{aligned}
\tag{B.18}
$$

Using the result for $\mathcal{I}_3$, we can obtain

$$
\begin{aligned}
\mathcal{I}_4 = & \frac{1}{2}\int_{-\pi}^{\pi} \mathrm{d}\phi \ \log^2\left(A-B\cos\phi\right)/(A-B\cos(\theta)) = \frac{1}{6}\frac{\partial}{\partial A}\int_{-\pi}^{\pi} \mathrm{d}\phi \ \log^3\left(A-B\cos\phi\right) = \\
= & \frac{\pi\left(\log^2\left(\frac{2\left(B^2-A\left(A+\sqrt{A^2-B^2}\right)\right)^2}{\left(A+\sqrt{A^2-B^2}\right)^3}\right) + 2\mathrm{Li}_2\left(\frac{B^2}{\left(A+\sqrt{A^2-B^2}\right)^2}\right)\right)}{\sqrt{(A-B)(A+B)}},
\end{aligned}
\tag{B.19}
$$

which can be used to evaluate the following integral appearing in our computation of the planar free energy

$$
\begin{aligned}
& \int_0^{\pi} \mathrm{d}\phi \ \frac{e^{-\frac{\beta+\gamma}{2}}}{\left(\left(e^{-\beta}-e^{-\gamma}\right)\cos\phi+e^{-\beta}+e^{-\gamma}\right)}\log^2\left(\frac{e^{-\beta}+e^{-\gamma}-\left(e^{-\gamma}-e^{-\beta}\right)\cos\phi}{2}\right) = \\
& = \pi\left(2\log^2\left(\frac{\left(e^{\beta/2}+e^{\gamma/2}\right)}{2}\right) + \mathrm{Li}_2\left(\tanh^2\left(\frac{\beta-\gamma}{4}\right)\right)\right)
\end{aligned}
\tag{B.20}
$$

The results of these integrals can be also presented in an equivalent and more useful way, if we use the identities

$$\log(1-x)\log(x) + \mathrm{Li}_2(1-x) + \mathrm{Li}_2(x) - \frac{\pi^2}{6} = 0. \tag{B.21}$$

and

$$\mathrm{Li}_3(w) = \frac{1}{6}\log^3(w) - \frac{1}{2}\log(1-w)\log^2(w) + \frac{\pi^2}{6}\log(w) - \mathrm{Li}_3(1-w) - \mathrm{Li}_3\left(\frac{w-1}{w}\right) + \zeta(3). \tag{B.22}$$

# References


[1] B. R. Greene, "String theory on Calabi-Yau manifolds," [arXiv:hep-th/9702155.]

[2] P. S. Aspinwall, "The Moduli space of N=2 superconformal field theories," [arXiv:hep-th/9412115.]

[3] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, "Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes," Commun. Math. Phys. **165**, 311 (1994) [arXiv:hep-th/9309140].

[4] D. Ghoshal and C. Vafa, "c= 1 string as the topological theory of the conifold," Nucl. Phys. B **453**, 121 (1995) [arXiv:hep-th/9506122].

[5] J. Bryan and R. Pandharipande, "The local Gromov-Witten theory of curves," [arXiv:math.AG/0411037.]

[6] M. Aganagic, H. Ooguri, N. Saulina and C. Vafa, "Black holes, q-deformed 2d Yang-Mills, and non-perturbative topological strings," Nucl. Phys. B **715**, 304 (2005) [arXiv:hep-th/0411280].

[7] H. Liu, "Fine structure of Hagedorn transitions," [arXiv:hep-th/0408001.]

[8] L. Álvarez-Gaumé, C. Gómez, H. Liu and S. Wadia, "Finite temperature effective action, AdS(5) black holes, and 1/N expansion," Phys. Rev. D **71**, 124023 (2005) [arXiv:hep-th/0502227].

[9] D. J. Gross and A. A. Migdal, "Nonperturbative Two-Dimensional Quantum Gravity," Phys. Rev. Lett. **64**, 127 (1990).

[10] E. Brézin and V. A. Kazakov, "Exactly Solvable Field Theories Of Closed Strings," Phys. Lett. B **236**, 144 (1990) .

[11] M. R. Douglas and S. H. Shenker, "Strings In Less Than One-Dimension," Nucl. Phys. B **335**, 635 (1990).

[12] N. Caporaso, M. Cirafici, L. Griguolo, S. Pasquetti, D. Seminara and R. J. Szabo, "Topological strings and large N phase transitions. I: Nonchiral expansion of q-deformed Yang-Mills theory," JHEP **0601**, 035 (2006) [arXiv:hep-th/0509041].

[13] X. Arsiwalla, R. Boels, M. Mariño and A. Sinkovics, "Phase transitions in q-deformed 2d Yang-Mills theory and topological strings," Phys. Rev. D **73**, 026005 (2006) [arXiv:hep-th/0509002].

[14] D. Jafferis and J. Marsano, "A Douglas-Kazakov phase transition in $q$-deformed Yang-Mills on $\mathbf{S}^2$ and topological strings," [ arXiv:hep-th/0509004.]

[15] D. J. Gross and E. Witten, "Possible Third Order Phase Transition In The Large N Lattice Gauge Theory," Phys. Rev. D **21**, 446 (1980).

[16] S. R. Wadia, "N = Infinity Phase Transition In A Class Of Exactly Soluble Model Lattice Gauge Theories," Phys. Lett. B **93**, 403 (1980).

[17] V. Periwal and D. Shevitz, "Exactly Solvable Unitary Matrix Models: Multicritical Potentials And Correlations," Nucl. Phys. B **344**, 731 (1990).

[18] D. J. Gross and A. Matytsin, "Instanton induced large N phase transitions in two-dimensional and four-dimensional QCD," Nucl. Phys. B **429**, 50 (1994) [arXiv:hep-th/9404004].

[19] I. R. Klebanov, J. Maldacena and N. Seiberg, "Unitary and complex matrix models as 1-d type 0 strings," Commun. Math. Phys. **252**, 275 (2004) [arXiv:hep-th/0309168].

[20] C. Crnkovic, M. R. Douglas and G. W. Moore, "Physical Solutions For Unitary Matrix Models," Nucl. Phys. B **360**, 507 (1991).

[21] I. K. Kostov, M. Staudacher and T. Wynter, "Complex matrix models and statistics of branched coverings of 2D surfaces," Commun. Math. Phys. **191**, 283 (1998) [arXiv:hep-th/9703189].

[22] I. P. Goulden, D.M. Jackson and R. Vakil, "The Gromov-Witten potential of a point, Hurwitz numbers, and Hodge integrals," Proc. Lond. Math. Soc., III. Ser. 83, No.3, 563-581 (2001) [arXiv:math.AG/9910004].

[23] A. Okounkov and R. Pandharipande, "Gromov-Witten theory, Hurwitz numbers, and matrix models, I" [math.AG/0101147.]

[24] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Mariño and C. Vafa, "Topological strings and integrable hierarchies," Commun. Math. Phys. **261** (2006) 451 [arXiv:hep-th/0312085].

[25] J. M. Maldacena, G. W. Moore, N. Seiberg and D. Shih, "Exact vs. semiclassical target space of the minimal string," JHEP **0410** (2004) 020 [arXiv:hep-th/0408039].

[26] G. Bertoldi, T. J. Hollowood and J. L. Miramontes, "Double scaling limits in gauge theories and matrix models," [arXiv:hep-th/0603122.]

[27] S. Das and C. Gómez, "Open–closed duality and double scaling," JHEP **0410**, 026 (2004) [arXiv:hep-th/0408051].

[28] I. P. Goulden and D.M. Jackson, "The number of ramified coverings of the sphere by the double torus, and a general form for higher genera," [math.AG/9902011.]

[29] M. Aganagic and C. Vafa, "Mirror symmetry and a G(2) flop," JHEP **0305**, 061 (2003) [arXiv:hep-th/0105225].

[30] P. Candelas, X. C. De La Ossa, P. S. Green and L. Parkes, "A Pair Of Calabi-Yau Manifolds As An Exactly Soluble Superconformal Theory," Nucl. Phys. B **359**, 21 (1991).

[31] P. S. Aspinwall, B. R. Greene and D. R. Morrison, "Measuring small distances in N=2 sigma models," Nucl. Phys. B **420**, 184 (1994) [arXiv:hep-th/9311042].

[32] A. Klemm and E. Zaslow, "Local mirror symmetry at higher genus," [arXiv:hep-th/9906046.]

[33] P. S. Aspinwall, B. R. Greene and D. R. Morrison, "Calabi-Yau moduli space, mirror manifolds and spacetime topology change in string theory," Nucl. Phys. B **416**, 414 (1994) [arXiv:hep-th/9309097].

[34] E. Witten, "Phases of N = 2 theories in two dimensions," Nucl. Phys. B **403**, 159 (1993) [arXiv:hep-th/9301042].

[35] P. Di Francesco, P. Ginsparg and J. Zinn-Justin, "2-D Gravity and random matrices," Phys. Rept. **254**, 1 (1995) [arXiv:hep-th/9306153].

[36] I. R. Klebanov, "String theory in two-dimensions," [arXiv:hep-th/9108019.]

[37] S. Katz, A. Klemm and C. Vafa, "M-theory, topological strings and spinning black holes," Adv. Theor. Math. Phys. **3**, 1445 (1999) [arXiv:hep-th/9910181].

[38] M. Aganagic, A. Klemm, M. Mariño and C. Vafa, "The topological vertex," Commun. Math. Phys. **254**, 425 (2005) [arXiv:hep-th/0305132].

[39] R. Gopakumar and C. Vafa, "M-theory and topological strings. I& II," [arXiv:hep-th/9809187] and [arXiv:hep-th/9812127].

[40] B. Forbes and M. Jinzenji, "J functions, non-nef toric varieties and equivariant local mirror symmetry of curves," [arXiv:math.ag/0603728.]

[41] M. Aganagic, M. Mariño and C. Vafa, "All loop topological string amplitudes from Chern-Simons theory," Commun. Math. Phys. **247**, 467 (2004) [arXiv:hep-th/0206164].

[42] S. de Haro, S. Ramgoolam and A. Torrielli, "Large N expansion of q-deformed two-dimensional Yang-Mills theory and Hecke algebras," arXiv:hep-th/0603056.

[43] W. Taylor, "Counting strings and phase transitions in 2-D QCD," [arXiv:hep-th/9404175.]

[44] M. J. Crescimanno and W. Taylor, "Large N phases of chiral QCD in two-dimensions," Nucl. Phys. B **437**, 3 (1995) [arXiv:hep-th/9408115].

[45] A. S. Losev, A. Marshakov and N. A. Nekrasov, "Small instantons, little strings and free fermions," [arXiv:hep-th/0302191.]

[46] N. A. Nekrasov, "Seiberg-Witten prepotential from instanton counting," Adv. Theor. Math. Phys. **7**, 831 (2004) [arXiv:hep-th/0206161].

[47] N. Caporaso, M. Cirafici, L. Griguolo, S. Pasquetti, D. Seminara and R. J. Szabo, "Topological strings and large N phase transitions. II: Chiral expansion of q-deformed Yang-Mills theory," JHEP **0601** (2006) 036 [arXiv:hep-th/0511043].

[48] A. Okounkov and R. Pandharipande, "Gromov–Witten theory, Hurwitz theory, and completed cycles" [math.AG/0204305.]

[49] T. Maeda, T. Nakatsu, K. Takasaki and T. Tamakoshi, "Five-dimensional supersymmetric Yang-Mills theories and random plane partitions," JHEP **0503**, 056 (2005) [arXiv:hep-th/0412327].

[50] H. Ooguri and C. Vafa, "Knot invariants and topological strings," Nucl. Phys. B **577**, 419 (2000) [arXiv:hep-th/9912123].

[51] M. Mariño, "Chern-Simons theory and topological strings," Rev. Mod. Phys. **77** (2005) 675 [arXiv:hep-th/0406005]. *Chern–Simons theory, matrix models, and topological strings*, Oxford University Press, 2005.

[52] S. Katz and C.-C. M. Liu, "Enumerative Geometry of Stable Maps with Lagrangian Boundary Conditions and Multiple Covers of the Disc," Adv. Theor. Math. Phys. **5**, 1 (2002) [arXiv:math.ag/0103074].

[53] M. Mariño and C. Vafa, "Framed knots at large N," [arXiv:hep-th/0108064.]

[54] T. Ekedahl, S. Lando, M. Shapiro and A. Vainshtein, "Hurwitz numbers and intersections on moduli spaces of curves," Invent. Math. **146**, 297 (2001) [math.AG/0004096].

[55] A. Okounkov and R. Pandharipande, "Hodge integrals and invariants of the unkot," Geometry and Topology **8**, 675 (2004) [math.AG/0307209 ]

[56] C. C. Liu, K. Liu and J. Zhou, "Mariño-Vafa Formula and Hodge Integral Identities," [arXiv:math.AG/0308015.]

[57] J. M. F. Labastida, M. Mariño and C. Vafa, "Knots, links and branes at large N," JHEP **0011** (2000) 007 [arXiv:hep-th/0010102].

[58] M. R. Douglas and V. A. Kazakov, "Large N phase transition in continuum QCD in two-dimensions," Phys. Lett. B **319**, 219 (1993) [arXiv:hep-th/9305047].

[59] M. Mariño, "Chern–Simons theory, matrix integrals, and perturbative three–manifold invariants," Commun. Math. Phys. **253**, 25 (2004) [arXiv:hep-th/0207096]; "Les Houches lectures on matrix models and topological strings," arXiv:hep-th/0410165.

[60] M. M. Akbar and G. W. Gibbons, "Ricci-flat Metrics with U(1) Action and the Dirichlet Boundary-value Problem in Riemannian Quantum Gravity and Isoperimetric Inequalities," arXiv:hep-th/0301026.

[61] V. A. Kazakov, "The Appearance Of Matter Fields From Quantum Fluctuations Of 2-D Gravity," Mod. Phys. Lett. A **4**, 2125 (1989).

[62] C. Itzykson and J. B. Zuber, "Combinatorics of the modular group. 2. The Kontsevich integrals," Int. J. Mod. Phys. A **7** (1992) 5661 [arXiv:hep-th/9201001].

[63] H. Ooguri, A. Strominger and C. Vafa, "Black hole attractors and the topological string," Phys. Rev. D **70**, 106007 (2004) [arXiv:hep-th/0405146].

[64] S. Dalley, C. V. Johnson and T. R. Morris, "Nonperturbative two-dimensional quantum gravity," Nucl. Phys. B **368**, 655 (1992).

[65] S. Dalley, C. V. Johnson, T. R. Morris and A. Watterstam, "Unitary matrix models and 2-D quantum gravity," Mod. Phys. Lett. A **7**, 2753 (1992) [arXiv:hep-th/9206060].

[66] T. J. Hollowood, L. Miramontes, A. Pasquinucci and C. Nappi, "Hermitian Versus Anti-Hermitian One Matrix Models And Their Hierarchies," Nucl. Phys. B **373**, 247 (1992) [arXiv:hep-th/9109046].

[67] C. Vafa, "Two dimensional Yang-Mills, black holes and topological strings," [arXiv:hep-th/0406058.]

[68] M. Aganagic and C. Vafa, "Mirror symmetry, D-branes and counting holomorphic discs," [arXiv:hep-th/0012041.]

[69] M. Aganagic, A. Klemm and C. Vafa, "Disk instantons, mirror symmetry and the duality web," Z. Naturforsch. A **57**, 1 (2002) [arXiv:hep-th/0105045].

[70] M. Aganagic, D. Jafferis and N. Saulina, "Branes, black holes and topological strings on toric Calabi-Yau manifolds," [arXiv:hep-th/0512245.]

[71] B. Forbes and M. Jinzenji, "Local mirror symmetry of curves: Yukawa couplings and genus 1," math.AG/0609016.

[72] P. Henrici, *Applied and computational complex analysis*, vol. I, Wiley, 1974.