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Orbits and Attractors for $N = 2$ Maxwell-Einstein Supergravity Theories in Five Dimensions

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Abstract

BPS and non-BPS orbits for extremal black-holes in $N = 2$ Maxwell-Einstein supergravity theories (MESGT) in five dimensions were classified long ago by the present authors for the case of symmetric scalar manifolds. Motivated by these results and some recent work on non-supersymmetric attractors we show that attractor equations in N=2 MESGTs in $d = 5$ do indeed possess the distinct families of solutions with finite Bekenstein-Hawking entropy. The new non-BPS solutions have non-vanishing central charge and matter charge which is invariant under the maximal compact subgroup $\tilde{K}$ of the stabilizer $\tilde{H}$ of the non-BPS orbit. Our analysis covers all symmetric space theories G/H such that G is a symmetry of the action. These theories are in one-to-one correspondence with (Euclidean) Jordan algebras of degree three. In the particular case of $N = 2$ MESGT with scalar manifold $SU^*(6)/USp(6)$ a duality of the two solutions with regard to $N = 2$ and $N = 6$ supergravity is also considered.

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1 Introduction

Extremal black hole solutions of supergravity theories exhibit an attractor mechanism" [1] in their evolution towards the horizon, in which the scalar fields move from their (arbitrary) asymptotic value ϕ_∞ toward a critical value

$$r \to r_H \Rightarrow \phi(r) \to \phi_c(q)$$

which is determined by the critical points of the black hole potential function [2, 3] $V(\phi, q)$ such that $\phi_c(q)$ is a solution of

$$\partial_i V \equiv \frac{\partial}{\partial \phi^i} V = 0$$

The Bekenstein-Hawking area entropy S is then given by

$$S \sim V \mid_{\partial_i V = 0} \qquad (1 - 1)$$

in $d = 4$ and

$$S \sim V^{3/4} \mid_{\partial_i V = 0} \qquad (1 - 2)$$

in $d = 5$.

For the case of scalar fields described by symmetric spaces of the corresponding geometry of vector multiplets, the value of the entropy is actually related to some invariants (cubic in $d = 5$ and quartic in $d = 4$) of the representation R of the charge vector of the black hole (B-H) charges [13, 12, 4, 8]. For fixed non-vanishing values of these invariants, the charge vectors describe a $(dimR - 1)$ dimensional orbits whose nature is strictly related to the supersymmetry properties of the critical point. It was pointed out in [4] that different orbits of charge vectors correspond to different BPS and non-BPS configurations. Such non-BPS configurations have been recently found in $d = 4$ in some particular cases [14, 16, 15, 18, 17, 3, 19, 20], and this has prompted further study in this direction. For example, in a recent work [8] it was shown that the $N = 8$ attractors have in $d = 4$ two solutions, of "maximal" symmetry, the 1/8 BPS attractor with $SU(2) \times SU(6)$ symmetry and the non-BPS attractor with $USp(8)$ symmetry. These symmetry groups are the maximal compact subgroups of the stabilizers $E_{6(2)}$ and $E_{6(6)}$ of the two orbits [3]

$$\frac{E_{7(7)}}{E_{6(2)}}$$

[3] We use the standard mathematical notation for labelling non-compact real forms of Lie groups. The bracket in the subscript is the difference between the number of non-compact generators and compact generators.

and

$$\frac{E_{7(7)}}{E_{6(6)}}$$

found in [4]. They correspond to a positive and a negative value of the quartic invariant I_4 in the 56 dimensional representation of the charge vector of the $N = 8$ theory. The analysis of the four dimensional $N = 2$ MESGTs and their BPS and non-BPS orbits and attractors is treated in [5].

In this paper we will study the five dimensional supergravity theories. For the $N = 8$ theory there is only one orbit with non-vanishing entropy which is [4]

$$\frac{E_{6(6)}}{F_{4(4)}}$$

The attractor nature of this orbit was derived in [10] where a solution of the attractor equation was shown to have $USp(6) \times USp(2)$ symmetry which is the maximal compact subgroup of $F_{4(4)}$ [4]. This corresponds to a 1/8 BPS attractor of the $N = 8$ theory in $d = 5$. Contrary to the four dimensional case no other solution exists for the attractor equation, in accordance with the analysis of [4]. However, in [4], it was shown that there exist two classes of orbits for the $N = 2$ MESGTs with symmetric scalar manifolds in $d = 5$. These orbits correspond to extremal BPS and non-BPS attractors in $d = 5$. In this paper we find the explicit solutions to the attractor equations corresponding to these orbits.

In section 2 we shall first review the real special geometry of $N = 2$ MESGTs as first formulated by Günaydin, Sierra and Townsend (GST). In section 3 we reproduce the classification of the orbits with non-vanishing entropy using the theory of Jordan algebras. The section 4 contains our main results on the solutions of the attractor equations corresponding to the orbits classified in [4]. In section 5 we discuss the scalar mass spectrum of the solutions. In section 6 we consider the attractor equations for self-dual strings in $d = 6$ and in section 7 a concluding summary is given.

2 Geometry and symmetries of $N = 2$ Maxwell-Einstein supergravity theories in five dimensions

In this section we will review symmetry groups of $N = 2$ MESGT's in five and four dimensions whose scalar manifolds are symmetric spaces. The

[4] We should note that the orbits of maximal supergravities in D dimensions were also studied in [11]

MESGT's describe the coupling of an arbitrary number n of (Abelian) vector multiplets to $N=2$ supergravity and five dimensional MESGT's were constructed in [6]. The bosonic part of the Lagrangian can be written as [6]

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{bosonic}} &= -\frac{1}{2}R - \frac{1}{4}\overset{\circ}{a}_{IJ}F^I_{\mu\nu}F^{J\mu\nu} - \frac{1}{2}g_{xy}(\partial_\mu\varphi^x)(\partial^\mu\varphi^y) \\ &\quad + \frac{e^{-1}}{6\sqrt{6}}C_{IJK}\varepsilon^{\mu\nu\rho\sigma\lambda}F^I_{\mu\nu}F^J_{\rho\sigma}A^K_\lambda, \end{aligned} \tag{2 - 1}$$

where e and R denote the fünfbein determinant and the scalar curvature in $d=5$, respectively. $F^I_{\mu\nu}$ are the field strengths of the Abelian vector fields A^I_μ, $(I=0,1,2\cdots,n)$ with A^0_μ denoting the "bare" graviphoton. The metric, g_{xy}, of the scalar manifold $\mathcal{M}$ and the "metric" $\overset{\circ}{a}_{IJ}$ of the kinetic energy term of the vector fields both depend on the scalar fields φ^x ($x,y,..=1,2,..,n$). The n-dimensional scalar manifold can be identified with the $\mathcal{V}=1$ hypersurface of an $n+1$ dimensional ambient space with coordinates h^I and the metric

$$a_{IJ}(h) := -\frac{1}{3}\frac{\partial}{\partial h^I}\frac{\partial}{\partial h^J}\ln\mathcal{V}(h)\ . \tag{2 - 2}$$

where

$$\mathcal{V}(h) := C_{IJK}\,h^I h^J h^K\ . \tag{2 - 3}$$

where $(I=0,1,\ldots,n)$. We shall denote the flat indices on the scalar manifold with lower case Latin indices $a,b,..=1,2,..,n$.

The metric g_{xy} of the scalar manifold is simply the pull-back of (2 - 2) to $\mathcal{M}$

$$g_{xy} = h^I_x h^J_y \overset{\circ}{a}_{IJ} \tag{2 - 4}$$

where

$$h^I_x = -\sqrt{\frac{3}{2}}\frac{\partial}{\partial\phi^x}h^I \tag{2 - 5}$$

and $\overset{\circ}{a}_{IJ}$ is the ambient metric evaluated at the hypersurface:

$$\overset{\circ}{a}_{IJ}(\varphi) = a_{IJ}|_{\mathcal{V}=1}\ . \tag{2 - 6}$$

Supersymmetry implies further the algebraic constraints

$$\begin{aligned} \overset{\circ}{a}_{IJ} &= h_I h_J + h^a_I h^a_J \\ h^I h_I &= 1 \\ h^I_a h_I &= h^a_I h^I = 0 \\ h^I_a h^J_b \overset{\circ}{a}_{IJ} &= h^I_a h_{Ib} = \delta_{ab} \end{aligned} \tag{2 - 7}$$

as well as the differential constraints

$$\begin{aligned} h_{I,x} &= \beta h_{Ix} \\ h^I_{,x} &= -\beta h^I_x \\ h_{Ix;y} &= \beta(g_{xy}h_I + T_{xyz}h^z_I) \\ h^I_{x;y} &= -\beta(g_{xy}h^I + T_{xyz}h^{Iz}) \end{aligned} \qquad (2 - 8)$$

where $\beta = \sqrt{\frac{2}{3}}$.

The Riemann curvature of the scalar manifold has the simple form

$$K_{xyzu} = \frac{4}{3}\left(g_{x[u}g_{z]y} + T_{x[u}{}^{w}T_{z]yw}\right) \qquad (2 - 9)$$

where T_{xyz} is the symmetric tensor

$$T_{xyz} = h^I_x h^J_y h^K_z C_{IJK} \qquad (2 - 10)$$

The full symmetry group of $N = 2$ MESGT in $d = 5$ is simply $G \times \mathrm{SU}(2)_R$ where $\mathrm{SU}(2)_R$ denotes the local R-symmetry group of the $N = 2$ supersymmetry algebra and G denotes the symmetry group of the tensor C_{IJK}. Now the covariant constancy of T_{xyz} implies the covariant constancy of K_{xyzu}:

$$T_{xyz;w} = 0 \Rightarrow K_{xyzu;w} = 0 \qquad (2 - 11)$$

Hence the scalar manifolds $\mathcal{M}_5$ with covariantly constant T tensor are locally symmetric spaces. If $\mathcal{M}_5$ is a homogeneous space the covariant constancy of T_{xyz} is equivalent to the following identity:

$$C^{IJK}C_{J(MN}C_{PQ)K} = \delta^I{}_{(M}C_{NPQ)} \qquad (2 - 12)$$

where the indices are raised by $\overset{\circ}{a}{}^{IJ}$. For proof of this equivalence an expression for constants C_{IJK} in terms of scalar field dependent quantities was used

$$C_{IJK} = \frac{5}{2}h_I h_K h_K - \frac{3}{2}\overset{\circ}{a}_{(IJ}h_{K)} + T_{xyz}h^x_I h^y_J h^z_K \qquad (2 - 13)$$

as well as algebraic constraints $h_I h^I = 1$ and $h^I_x h_I = 0$ that follow from supersymmetry [6]. Using this "adjoint identity" , GST [6] proved that the cubic forms defined by C_{IJK} of $N = 2$ MESGTs with symmetric target spaces $\mathcal{M}_5$ (with $n \geq 2$) and covariantly constant T tensors are in one-to-one correspondence with the norm forms of Euclidean (formally real) Jordan algebras J of degree 3. The corresponding symmetric spaces are of the form

$$\mathcal{M} = \frac{\mathrm{Str}_0\left(J\right)}{\mathrm{Aut}\left(J\right)} \tag{2 - 14}$$

where $\mathrm{Str}_0\left(J\right)$ is the invariance group of the norm (reduced structure group) and $\mathrm{Aut}\left(J\right)$ is the automorphism group of the Jordan algebra J respectively.

Following Schafers [21], GST [6] listed the allowed cubic forms, which we reproduce below:

1. $J = \mathbb{R}$ and $\mathcal{V}(x) = x^3$. This case corresponds to pure $d = 5$ supergravity.

2. $J = \mathbb{R} \oplus \Gamma$, where Γ is a simple Jordan algebra with identity, which we denote as $\mathbf{e}_2$, and quadratic norm $Q\left(\mathbf{x}\right)$, for $\mathbf{x} \in \Gamma$, such that $Q\left(\mathbf{e}_2\right) = 1$. The norm is given as $\mathcal{V}\left(x\right) = aQ\left(\mathbf{x}\right)$, with $x = \left(a, \mathbf{x}\right) \in J$. This includes two special cases

   (a) $\Gamma = \mathbb{R}$ and $Q = b^2$, with $\mathcal{V} = ab^2$. This is applicable to $n = 1$.

   (b) $\Gamma = \mathbb{R} \oplus \mathbb{R}$ and $Q = bc$, and $\mathcal{V} = abc$ and is applicable to $n = 2$.

   For these special cases the norm is completely factorized, so that $\mathcal{M}$ is flat. For $n > 2$, $\mathcal{V}$ is still factorized into a linear and quadratic parts. The positive definiteness of the kinetic energy terms requires that Q has Lorentzian signature $(+, -, -, \ldots, -)$. The invariance group of the norm is

   $$\mathrm{Str}_0\left(J\right) = \mathrm{SO}\left(n-1,1\right) \times \mathrm{SO}\left(1,1\right) \tag{2 - 15}$$

   where the $\mathrm{SO}\left(1,1\right)$ factor arises from the invariance of $\mathcal{V}$ under the dilatation $\left(a, \mathbf{x}\right) \to \left(e^{-2\lambda}a, e^{\lambda}\mathbf{x}\right)$ for $\lambda \in \mathbb{R}$, and that $\mathrm{SO}\left(n-1\right)$ is $\mathrm{Aut}\left(J\right)$. Hence

   $$\mathcal{M} = \frac{\mathrm{SO}\left(n-1,1\right)}{\mathrm{SO}\left(n-1\right)} \times \mathrm{SO}\left(1,1\right) \tag{2 - 16}$$

   This infinite family is referred to as the generic Jordan family of MESGTs.

3. Simple Euclidean Jordan algebras $J = J_3^{\mathbb{A}}$ generated by 3×3 Hermitian matrices over the four division algebras $\mathbb{A} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$. An element $x \in J_3^{\mathbb{A}}$ can be written as

   $$x = \begin{pmatrix} \alpha_1 & a_3 & \bar{a}_2 \\ \bar{a}_3 & \alpha_2 & a_1 \\ a_2 & \bar{a}_1 & \alpha_3 \end{pmatrix} \tag{2 - 17}$$

where $\alpha_k \in \mathbb{R}$ and $a_k \in \mathbb{A}$ with $^-$ indicating the conjugation in the underlying division algebra. The cubic norm $\mathcal{V}$ is given by

$$\mathcal{V}(x) = \alpha_1\alpha_2\alpha_3 - \alpha_1 |a_1|^2 - \alpha_2 |a_2|^2 - \alpha_3 |a_3|^2 + a_1a_2a_3 + \overline{(a_1a_2a_3)} \tag{2 - 18}$$

The corresponding spaces $\mathcal{M}$ are irreducible of dimension $3(1+\dim\mathbb{A})-1$, which we list below:

$$\mathcal{M}(J_3^{\mathbb{R}}) = \frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(3)} \qquad \mathcal{M}(J_3^{\mathbb{H}}) = \frac{\mathrm{SU}^*(6)}{\mathrm{USp}(6)}$$
$$\mathcal{M}(J_3^{\mathbb{C}}) = \frac{\mathrm{SL}(3,\mathbb{C})}{\mathrm{SU}(3)} \qquad \mathcal{M}(J_3^{\mathbb{O}}) = \frac{\mathrm{E}_{6(-26)}}{\mathrm{F}_4} \tag{2 - 19}$$

The "magical" supergravity theories described by simple Jordan algebras $J_3^{\mathbb{A}}$ (A = R, C, H or O) can be truncated to theories belonging to the generic Jordan family. This is achieved by restricting the elements of $J_3^{\mathbb{A}}$

$$\begin{pmatrix} \alpha_1 & a_3 & \overline{a}_2 \\ \overline{a}_3 & \alpha_2 & a_1 \\ a_2 & \overline{a}_1 & \alpha_3 \end{pmatrix} \tag{2 - 20}$$

to their subalgebra $J = \mathbb{R} \oplus J_2^{\mathbb{A}}$ by setting $a_1 = a_2 = 0$. Their symmetry groups are as follows:

$$\begin{aligned} J &= \mathbb{R} \oplus J_2^{\mathbb{R}} : \mathrm{SO}(1,1) \times \mathrm{SO}(2,1) \subset \mathrm{SL}(3,\mathbb{R}) \\ J &= \mathbb{R} \oplus J_2^{\mathbb{C}} : \mathrm{SO}(1,1) \times \mathrm{SO}(3,1) \subset \mathrm{SL}(3,\mathbb{C}) \\ J &= \mathbb{R} \oplus J_2^{\mathbb{H}} : \mathrm{SO}(1,1) \times \mathrm{SO}(5,1) \subset \mathrm{SU}^*(6) \\ J &= \mathbb{R} \oplus J_2^{\mathbb{O}} : \mathrm{SO}(1,1) \times \mathrm{SO}(9,1) \subset \mathrm{E}_{6(-26)} \end{aligned} \tag{2 - 21}$$

3 Orbits of U-duality groups and Jordan algebras

Jordan algebras are commutative and non-associative algebras with a symmetric Jordan product $\circ$

$$X \circ Y = Y \circ X \tag{3 - 1}$$

that satisfies the Jordan identity [22, 23]

$$X \circ (Y \circ X^2) = (X \circ Y) \circ X^2 \tag{3 - 2}$$

Automorphism group of a Jordan algebra J is formed by linear transformations A that preserve the products in J:

$$X \circ Y = Z \Rightarrow (AX) \circ (AY) = (AZ) \qquad (3 - 3)$$

The structure group $Str(J)$ of J is formed by linear transformations S that preserve the norm form $\mathcal{V}$ up to an overall scale factor λ :

$$Str(J) : X \to S(X) \Rightarrow \mathcal{V}(S(X)) = \lambda \mathcal{V}(X) \qquad (3 - 4)$$

and the reduced structure group $Str_0(J)$ of J is the subgroup of $Str(J)$ that leaves the norm form invariant, i.e. those transformations S for which $\lambda = 1$.

The Lie algebra $\mathfrak{aut}(J)$ of the automorphism group $Aut(J)$ is generated by derivations D that satisfy the Leibniz rule:

$$D(X \circ Y) = (DX) \circ Y + X \circ (DY)$$

It is easy to verify that by exponentiating derivations one obtains automorphisms:

$$e^D(X \circ Y) = (e^D X) \circ (e^D Y)$$

Hence it is customary to refer to $\mathfrak{aut}(J)$ as the derivation algebra $Der(J)$ of J. Every derivation D of J can be written in the form [23]

$$D_{X,Y} \equiv [L_X, L_Y] \qquad , X, Y \in J \qquad (3 - 5)$$

where L_X denotes multiplication by X i.e.

$$D_{X,Y} Z = X \circ (Y \circ Z) - Y \circ (X \circ Z)$$

Structure algebra $\mathfrak{str}(J)$ of J is generated by derivations and multiplication by elements of J:

$$\mathfrak{str}(J) = Der(J) \oplus L_J$$

whose commutation relations are very simple

$$\begin{aligned} [L_X, L_Y] &= D_{X,Y} \qquad (3 - 6) \\ [D_{X,Y}, L_Z] &= L_{(D_{X,Y} Z)} \\ [D_{X,Y}, D_{Z,W}] &= D_{(D_{X,Y} W),Z} + D_{W,(D_{X,Y} Z)} \end{aligned}$$

Multiplication by the identity element of J commutes will all the elements of $\mathfrak{str}(J)$ and acts like a central charge. The reduced structure algebra $\mathfrak{str}_0(J)$

is generated by derivations and multiplications by traceless elements of J. The automorphism group $Aut(J)$ leaves the identity element $\mathfrak{e}$ of J invariant or equivalently derivations annihilate the identity element

$$D\mathfrak{e} = 0$$

As mentioned in the previous section, there exist four simple (Euclidean) Jordan algebras of degree three $J_3^{\mathbb{A}}$ of 3×3 Hermitian matrices over the four division algebras $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$. We shall denote a general element X of $J_3^{\mathbb{A}}$ as

$$X = \begin{pmatrix} \alpha_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \alpha_2 & x_1 \\ x_2 & \bar{x}_1 & \alpha_3 \end{pmatrix} \equiv \sum_{i=1}^{3} \alpha_i E_i + (x_3)_{12} + (x_2)_{31} + (x_1)_{23} \qquad (3 - 7)$$

where E_i ($i = 1, 2, 3$) are the 3 irreducible idempotents of $J_3^{\mathbb{A}}$ and $x_i \in \mathbb{A}$ and the bar denotes conjugation in $\mathbb{A}$. It is well-known that an element X of the algebra $J_3^{\mathbb{A}}$ can be diagonalized by the action of the automorphism group. For the exceptional Jordan algebra $J_3^{\mathbb{O}}$ this was shown explicitly in [24]. Thus

$$Aut(J_3^{\mathbb{A}}) : \qquad X \Rightarrow (\lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3) \qquad (3 - 8)$$

where λ_i are the eigenvalues of X. Norm of X

$$\mathcal{V}(X) = \alpha_1\alpha_2\alpha_3 - \alpha_1|x_1|^2 - \alpha_2|x_2|^2 - \alpha_3|x_3|^2 + 2Re(x_1x_2x_3)$$

is simply the "determinant" and , hence, is equal to $\lambda_1\lambda_2\lambda_3$.

Invariance group of the identity element $\mathfrak{1}$ is the automorphism group $Aut(J)$. The subgroup of the automorphism group that leaves an irreducible idempotent invariant is generated by derivations that annihilate that idempotent. For , say, the irreducible idempotent E_3 the corresponding derivations are:

$$\begin{aligned} D_{(x_3)_{12},(y_3)_{12}} \\ D_{(x_3)_{12},(E_1-E_2)} \end{aligned} \qquad (3 - 9)$$

We list , in Table 1, simple Jordan algebras of degree three and invariance groups K of their irreducible idempotents that are subgroups of $Aut(J)$.

The subgroup K of the automorphism group that leaves the idempotent E_3 invariant leaves also the element

$$\mathfrak{b} = (-E_1 - E_2 + E_3)$$

J	$K \subset Aut(J)$
$J_3^{\mathbb{R}}$	$SO(2) \subset SO(3)$
$J_3^{\mathbb{C}}$	$SU(2) \times U(1) \subset SU(3)$
$J_3^{\mathbb{H}}$	$USp(4) \times USp(2) \subset USp(6)$
$J_3^{\mathbb{O}}$	$SO(9) \subset F_4$

Table 1: Above we list the subgroups K of the automorphism groups of simple Euclidean Jordan algebras of degree three that leave an irreducible idempotent invariant.

J	$\tilde{H} \subset Str_0(J)$
$J_3^{\mathbb{R}}$	$SO(2,1)) \subset SL(3,\mathbb{R})$
$J_3^{\mathbb{C}}$	$SU(2,1) \subset SL(3,\mathbb{C})$
$J_3^{\mathbb{H}}$	$USp(4,2) \subset SU^*(6)$
$J_3^{\mathbb{O}}$	$F_{4(-20)} \subset E_{6(-26)}$
$\mathbb{R} \oplus \Gamma_n$	$SO(n-2,1) \subset SO(n-1,1) \times SO(1,1)$

Table 2: Above we list the subgroups $\tilde{H}$ of the reduced structure groups of Euclidean Jordan algebras of degree three that leave the element $\mathfrak{b} = -E_1 - E_2 + E_3$ invariant.

with unit norm invariant. The little group of $\mathfrak{b}$, defined as the subgroup of $Str_0(J)$ that leaves it invariant, is generated by the above derivations plus the multiplications by the traceless elements $(x_2)_{31}$ and $(x_1)_{23}$

$$L_{(x_2)_{31}+(x_1)_{23}} = L_{(x_2)_{31}} + L_{(x_1)_{23}}$$

which are non-compact generators. The little groups $\tilde{H}$ of $\mathfrak{b}$ for different Jordan algebras of degree 3 are listed in Table 2.

As for the generic Jordan family $J = \mathbb{R} \oplus \Gamma_n$, where Γ_n is a Jordan algebra of degree two whose quadratic norm has Lorentzian signature, we shall denote their elements in the form:

$$X = (\alpha; \beta_0, \vec{\beta})$$

where

$$(\beta_0 \mathbf{1} + \vec{\beta} \cdot \vec{\sigma}) \in \Gamma_n$$

The cubic norm of X is simply

$$\mathcal{V} = \alpha(\beta_0^2 - \vec{\beta} \cdot \vec{\beta}) \qquad (3 - 10)$$

The three irreducible idempotents are

$$\begin{aligned} E_1 &= (1,0,\vec{0}) \\ E_2 &= (0;\frac{1}{2},\frac{1}{2},0,...,0) \\ E_3 &= (0;\frac{1}{2},-\frac{1}{2},0,..,0) \end{aligned} \tag{3 - 11}$$

with the identity element $\imath$ given by

$$\imath = E_1 + E_2 + E_3 = (1:1,\vec{0})$$

Automorphism group of $J = \mathbb{R} \oplus \Gamma_n$ is $SO(n-1)$ and its reduced structure group is $SO(n-1,1) \times SO(1,1)$. The identity element $\imath$ is manifestly invariant under the automorphism group and the little group of the element $\mathfrak{b} = -E_1 - E_2 + E_3 = (-1;0,-1,0,..,0)$ is $SO(n-2,1)$ [5].

Let us now show that an element of $J_3^{\mathbb{A}}$ with positive norm $\mathcal{V}(X) = \kappa^3$ ($\kappa > 0$) can be brought to the form

$$\begin{pmatrix} \kappa & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \kappa \end{pmatrix} \tag{3 - 12}$$

by the action of its reduced structure group if all its eigenvalues are positive, or to the form

$$\begin{pmatrix} -\kappa & 0 & 0 \\ 0 & -\kappa & 0 \\ 0 & 0 & \kappa \end{pmatrix} \tag{3 - 13}$$

if two of its eigenvalues are negative. The global action of the reduced structure group $Str_0(J)$ is generated by automorphisms and by quadratic action U_A by elements A whose norm squared is one. The quadratic operator U_A acts on J via:

$$U_A X = \{AXA\} \equiv 2(A \circ X) \circ A - A^2 \circ X \tag{3 - 14}$$

and satisfies the property

$$\mathcal{V}(U_A X) = [\mathcal{V}(A)]^2 \mathcal{V}(X) \tag{3 - 15}$$

Thus if $\mathcal{V}(A) = \pm 1$ then

$$\mathcal{V}(U_A X) = \mathcal{V}(X)$$

[5] Note that the invariance group of the idempotent E_1 is $SO(n-1,1)$, while the invariance group of the idempotents E_2 and E_3 is SO(n-2).

J	Aut(J)$\subset Str_0(J)$
$J_3^{\mathbb{R}}$	$SO(3)) \subset SL(3,\mathbb{R})$
$J_3^{\mathbb{C}}$	$SU(3) \subset SL(3,\mathbb{C})$
$J_3^{\mathbb{H}}$	$USp(6) \subset SU^*(6)$
$J_3^{\mathbb{O}}$	$F_{4(-52)} \subset E_{6(-26)}$
$\mathbb{R}\oplus\Gamma_n$	$SO(n-1) \subset SO(n-1,1)\times SO(1,1)$

Table 3: Above we list the subgroups $Aut(J)$ of the reduced structure groups $Str_0(J)$ of Euclidean Jordan algebras of degree three that leave the identity element invariant.

Using the automorphism group one can bring an element X to the diagonal form :

$$Aut(J) : X \Rightarrow (\lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3) \qquad (3 - 16)$$

The quadratic action by U_A that preserves the diagonal form must involve A which is also diagonal

$$A = A_1 E_1 + A_2 E_2 + A_3 E_3$$

such that $(A_1 A_2 A_3)^2 = 1$. Then

$$U_A(\lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3) \Rightarrow \lambda_1 A_1^2 E_1 + \lambda_2 A_2^2 E_2 + \lambda_3 A_3^2 E_3 \qquad (3 - 17)$$

This shows that one can rescale the eigenvalues by a positive number such that the norm is preserved. Hence an element with all positive eigenvalues can be brought to a positive multiple of the identity $\kappa\mathbf{1}$ where $\kappa^3 = \lambda_1\lambda_2\lambda_3$. Consequently, the orbit of a timelike (positive norm) element X with all positive eigenvalues is

$$\frac{Str_0(J)}{Aut(J)} \qquad (3 - 18)$$

The automorphism groups of Jordan algebras of degree three and their reduced structure groups are reproduced in Table 3.

Similarly a timelike element with two negative eigenvalues and one positive can be brought to the form (modulo the permutation of the diagonal entries)

$$(-\kappa E_1 - \kappa E_2 + \kappa E_3)$$

where $\kappa^3 = \lambda_1\lambda_2\lambda_3$ and the corresponding orbit is

$$Str_0(J)/\tilde{H}$$

where $\tilde{H}$ is a non-compact real form of $Aut(J)$ listed in table 2.

Extension to the generic Jordan family is straightforward and one needs only to use the standard knowledge of the orbits of the Lorentz group in various dimensions. We should also note that orbits with negative norm are isomorphic to the above orbits , depending on whether all or one of the eigenvalues are negative.

4 Attractor equations for $N = 2$ MESGTs and their solutions in $d = 5$

We shall now consider the attractor mechanism in the framework of 5d , $N = 2$ MESGTs in an extremal B-H background described by a $(n + 1)$ dimensional charge vector

$$q_I = \int_{S^3} H_I = \int_{S^3} \overset{\circ}{a}_{IJ} *F^J \qquad (I = 0, 1, ...n)$$

The corresponding B-H potential , described in [2, 3], is elegantly written in the framework of GST real special geometry as follows:

$$V(\phi, q) = q_I \overset{\circ}{a}{}^{IJ} q_J \qquad (4 - 1)$$

where $\overset{\circ}{a}{}^{IJ}$ is the inverse of the metric $\overset{\circ}{a}_{IJ}$ of the kinetic energy term of the vector fields. The metric $\overset{\circ}{a}_{IJ}$ is related to the metric g_{xy} of the scalar manifold via

$$\overset{\circ}{a}_{IJ} = h_I h_J + \frac{3}{2} h_{I,x} h_{J,y} g^{xy} \qquad (4 - 2)$$

$$\overset{\circ}{a}{}^{IJ} = h^I h^J + \frac{3}{2} h^I_{,x} h^J_{,y} g^{xy}$$

or conversely

$$g_{xy} = \frac{3}{2} h_{I,x} h_{J,y} \overset{\circ}{a}{}^{IJ} \qquad (4 - 3)$$

Introducing the quantity

$$Z = q_I h^I$$

we can write the potential as

$$V(q, \phi) = Z^2 + \frac{3}{2} g^{xy} \partial_x Z \partial_y Z \qquad (4 - 4)$$

where

$$\partial_x Z = q_I h^I_{,x}$$

We should also note the identities

$$T_{xyz} = (3/2)^{\frac{3}{2}} h^I_{,x} h^J_{,y} h^K_{,z} C_{IJK} \qquad (4 - 5)$$

and

$$h^I_{,x;y} = \frac{2}{3}(g_{xy}h^I - \sqrt{\frac{3}{2}} T_{xyz} g^{zw} h^I_{,w}) \qquad (4 - 6)$$

The critical points of the potential are given by the solutions of the equation

$$\partial_x V = 2(2Z\partial_x Z - \sqrt{3/2} T_{xyz} g^{yy'} g^{zz'} \partial_{y'} Z \partial_{z'} Z) = 0 \qquad (4 - 7)$$

For BPS critical points we have

$$\partial_x Z = 0 \qquad (4 - 8)$$

and for non-BPS critical points

$$2Z\partial_x Z = \sqrt{\frac{3}{2}} T_{xyz} \partial^y Z \partial^z Z \qquad (4 - 9)$$

where

$$\partial^x Z \equiv g^{xx'} \partial_{x'} Z$$

The equation 4 - 9 can be inverted using the relation

$$q_I = h_I Z - \frac{3}{2} h_{I,x} \partial^x Z \qquad (4 - 10)$$

which follows from equation 4 - 2. For $\partial_x Z = 0$ this gives

$$q_I = h_I Z \qquad (4 - 11)$$

and for $\partial_x Z \neq 0$ we get

$$q_I = h_I Z - (3/2)^{3/2} \frac{1}{2Z} h_{I,x} T^{xyz} \partial_y Z \partial_z Z \qquad (4 - 12)$$

Let us first remark that the attractor solution of the BPS orbit with non-vanishing entropy given by [8, 7]

$$\partial_x Z = 0$$

is invariant under the stability group $Aut(J)$ of the orbit

$$Str_0(J)/Aut(J)$$

J	$\tilde{H}$	$\tilde{K}$
$\mathbb{R} \oplus \Gamma$	$SO(n-2,1)$	$SO(n-2)$
$J_3^{\mathbb{R}}$	$SL(2,\mathbf{R})$	SO(2)
$J_3^{\mathbb{C}}$	$SU(2,1)$	$SU(2) \times U(1)$
$J_3^{\mathbb{H}}$	$USp(4,2)$	$USp(4) \times USp(2)$
$J_3^{\mathbb{O}}$	$F_{4(-20)}$	$SO(9)$

Table 4: Above we list the stability groups $\tilde{H}$ of the non-BPS orbits of the $N = 2$ MESGT's with non-vanishing entropy in $d = 5$. The first column lists the Jordan algebras of degree 3 that define these theories. The third column lists the maximal compact subgroups $\tilde{K}$ of $\tilde{H}$.

listed in column 1 of table 1 of [4] (see table 3 above). If we now consider the second class of orbits $G/\tilde{H}$ with non-vanishing entropy listed in column 2 of table 1 of [4] , we can solve the attractor equation by considering $\partial_x Z$ invariant under the maximal compact subgroup K of $\tilde{H}$. The list of the stability groups $\tilde{H}$ and their maximal compact subgroups $\tilde{K}$ is given in Table 4.

For the MESGTs defined by Jordan algebras of degree 3, the tensor C_{IJK} is an invariant tensor. Similarly the tensor T_{abc} is an invariant tensor of the maximal compact subgroup H. Going to flat indices the attractor equation becomes:

$$2Z\partial_a Z = \sqrt{3/2}T_{abc}\partial^b Z\partial^c Z \qquad (4 - 13)$$

The BPS solution is $\partial_a Z = 0$, which then gives

$$V_{BPS} = Z^2 \qquad (4 - 14)$$

, using equation 4 - 4. If $\partial_a Z \neq 0$, by squaring equation 4 - 13 we get

$$4Z^2\partial_a Z\partial_a Z = \frac{3}{2}T_{abc}T_{ab'c'}\partial_b Z\partial_c Z\partial_b' Z\partial_c' Z \qquad (4 - 15)$$

Then using the identity

$$T_{a(bc}T^a_{b'c')} = \frac{1}{2}g_{(bc}g_{b'c')}$$

valid only for the MESGTs defined by Jordan algebras of degree three we get

$$\partial_a Z\partial_a Z = \frac{16}{3}Z^2 \qquad (4 - 16)$$

Hence at the non-BPS attractor point the potential becomes

$$V = Z^2 + \frac{3}{2}\partial_a Z \partial_a Z = 9Z^2 \tag{4 - 17}$$

which agrees with the formula

$$V = (d-2)^2 Z^2 \tag{4 - 18}$$

valid for $d = 4, 5$ dimensions [8, 5].

Let us consider the example of the exceptional supergravity defined by the exceptional Jordan algebra $J_3^{\mathbb{O}}$ whose U-duality group is $E_{6(-26)}$. The two extremal orbits are classified by two different stabilizers , which are the compact $F_{4(-52)}$ and the noncompact $F_{4(-20)}$ with the maximal compact subgroup $SO(9)$. For the non-BPS orbit we have the decompositions

$$\begin{aligned} E_{6(-26)} \supset F_{4(-20)} &\supset SO(9) \\ 27 &\to 26 + 1 \\ 26 &= 16 + 9 + 1 \end{aligned} \tag{4 - 19}$$

Furthermore,

$$T_{abc} \Rightarrow (\bar{26})^3 = (\bar{16}\bar{16}\bar{9}) + (\bar{16}\bar{16}\bar{1}) + (\bar{9}\bar{9}\bar{1}) + (\bar{1}\bar{1}\bar{1}) \tag{4 - 20}$$

It is easy to see that the solution

$$\partial_a Z = (\partial_{16} Z = \partial_9 Z = 0, \partial_1 Z \neq 0) \tag{4 - 21}$$

is a solution of equation 4 - 13 provided Z and $\partial_1 Z$ satisfy the following algebraic equation

$$Z = \frac{1}{2}\sqrt{\frac{3}{2}} T_{111} \partial_1 Z \tag{4 - 22}$$

where we are using flat coordinates. The entropy then becomes renormalized as in $d = 4$ [9]

$$S^{4/3} = V|_{\partial_a V = 0} = Z^2 (1 + \frac{4}{T_{111}^2}) \tag{4 - 23}$$

with the attractor point $T_{111}^2 = \frac{1}{2}$ for symmetric spaces defined by Jordan algebras.

The scalar manifold $\frac{SU^*(6)}{USp(6)}$ of the MESGT defined by $J_3^{\mathbb{H}}$ is the same as the scalar manifold of $N = 6$ supergravity theory , whose attractor equation was studied in [10]. Interestingly, the 1/6 BPS solution of $N = 6$ supergravity in $d = 5$ correspond precisely to the orbit whose stabilizer is $USp(4, 2)$

with maximal compact subgroup $USp(4) \times USp(2)$. Indeed it was shown there that the 6×6 symplectic traceless matrix Z^{AB} which represent the the $N=2$ charges in the parent $N=6$ theory lead to the following vacuum solution

$$Z^{AB} = \begin{pmatrix} e_1\epsilon & 0 & 0 \\ 0 & e_2\epsilon & 0 \\ 0 & 0 & e_3\epsilon \end{pmatrix} \tag{4 - 24}$$

,where $e_1+e_2+e_3=0$ and ϵ is the 2×2 symplectic matrix. At the attractor point we have

$$e_2 = e_3$$

i.e $e_1 = -2e_2$ at that point. The symplectic traceless matrix Z^{AB} , which generically has a $USp(2)^3$ symmetry , has an enhancement to $USp(2) \times USp(4)$ symmetry at the attractor point. This is the non-BPS orbit of the $N=2$ subtheory , which is instead 1/6 BPS in the $N=6$ theory. At the critical point the singlet X of the $N=6$ supergravity takes the value

$$e_1 = \frac{8}{3}X$$

and the entropy [6]

$$V = \frac{1}{2}Z^{AB}Z_{AB} + \frac{4}{3}X^2 \tag{4 - 25}$$

becomes

$$V = e_1^2 + e_2^2 + (e_1+e_2)^2 + \frac{4}{3}X^2 = 12X^2 \tag{4 - 26}$$

Therefore at the attractor point

$$V_{NBPS}(X) = 12X^2_{NBPS} \tag{4 - 27}$$

while at the BPS attractor point

$$V_{BPS} = \frac{4}{3}X^2_{BPS} \tag{4 - 28}$$

It is easy to check this result with the cubic invariant as given in [10]

$$I_3 = -\frac{1}{6}Tr(ZC)^3 - \frac{1}{6}Tr(ZC)^2X + \frac{8}{27}X^3 \tag{4 - 29}$$

[6] To compare with the formula of $5d$ $N=2$ geometry one should take into account the normalization of the terms in 4 - 4 compared to 4 - 25.

where C is the $USp(6)$ invariant symplectic metric. At the BPS attractor point $Z = 0$

$$I_3 = \frac{8}{27} X^3 \qquad (4 - 30)$$

with the entropy

$$S^{4/3} = V|_{BPS} = 3|I_3|^{2/3} = \frac{4}{3} X^2_{BPS} \qquad (4 - 31)$$

and at the non-BPS point

$$-I_3 = \frac{216}{27} X^3 = 8 X^3_{NBPS} \qquad (4 - 32)$$

with the entropy

$$S^{4/3} = 3|I|^{2/3} = 12 X^2_{NBPS}$$

The $N = 2$ derivation of the above result is through formula 4 - 17 of the real special geometry of Gunaydin, Sierra and Townsend.

5 Scalar masses at the attractor points

One can give general results on the quadratic fluctuations of the B-H potential V around its BPS and non-BPS critical points. The general form of V is given in equation 4 - 4 . By further differentiating equation 4 - 7 we get a general expression for the Hessian of the potential:

$$\begin{aligned} \frac{1}{4} D_x \partial_y V &= \frac{2}{3} g_{xy} Z^2 + \partial_x Z \partial_y Z - 2\sqrt{\frac{2}{3}} T_{xyz} g^{zw} \partial_w Z Z \qquad (5 - 1) \\ &\quad + T_{xpq} T_{yzs} g^{pz} g^{qq'} g^{ss'} \partial_{q'} Z \partial_{s'} Z \\ &= \frac{2}{3} (g_{xz} Z - \sqrt{\frac{3}{2}} T_{xzp} \partial^p Z)(g_{yz} Z - \sqrt{\frac{3}{2}} T_{yzq} \partial^q Z) + \partial_x Z \partial_y Z \end{aligned}$$

From the above equation we obtain the Hessian at the BPS critical point $\partial_x Z = 0$

$$\partial_x \partial_y V = \frac{8}{3} g_{xy} Z^2 \qquad (5 - 2)$$

which is the same result as in $d = 4$ [3]. Note that equation 5 - 2 implies that the scalar fluctuations have positive square mass which shows the attractor nature of the BPS critical points [3, 14]-[20]. At the non-BPS critical point we can split the index $x = (p, 1)$ where 1 is the singlet direction along the subgroup K of the stabilizer of the orbit. The using flat coordinates the attractor condition 4 - 7 becomes

$$\begin{aligned} \partial_1 Z &= \frac{4}{\sqrt{3}} Z \\ \partial_p Z &= 0 \\ (T_{111} &= \frac{1}{\sqrt{2}}) \end{aligned} \tag{5 - 3}$$

which when inserted in 5 - 1 gives

$$\frac{1}{4}\partial_p\partial_q V = \frac{2}{3}(\mathbb{I}-T)^2_{pq} Z^2 = \frac{2}{27}(\mathbb{I}-T)^2_{pq} V_{NBPS} \tag{5 - 4}$$

where $(\mathbb{I}-T)_{pq} = \delta_{pq} - T_{pq}$ and $T_{pq} = 2\sqrt{2}T_{pq1}$. For the singlet mode , $T_{111} = \frac{1}{\sqrt{2}}$ and then

$$\frac{1}{4}V_{11} = (\frac{2}{3} + \frac{16}{3})Z^2 = 6Z^2 = \frac{2}{3}V_{NBPS} \tag{5 - 5}$$

We can further split the indices (p,q) into $(i,j),(\alpha,\beta)$ where (i,α) refer to the two representations with non-vanishing T-tensor given by

$$T_{\alpha\beta 1}, T_{ij1}, T_{\alpha\beta i} \tag{5 - 6}$$

From the identities

$$T^1_{(\alpha\beta}T^1_{\gamma\delta)} + T^i_{(\alpha\beta}T^i_{\gamma\delta)} = \frac{1}{2}\delta_{(\alpha\beta}\delta_{\gamma\delta)} \tag{5 - 7}$$

$$T^1_{(ij}T^1_{lm)} = \frac{1}{2}\delta_{(ij}\delta_{lm)} \tag{5 - 8}$$

We have $T^1_{ij} = \frac{1}{\sqrt{2}}\delta_{ij}$ and thus

$$\frac{1}{4}\partial_i\partial_j V = \frac{2}{27}V_{NBPS}\delta_{ij} \tag{5 - 9}$$

From 5 - 7 we also have [7]

$$T^1_{\alpha\beta} = \lambda\delta_{\alpha\beta} \tag{5 - 10}$$

[7]Here we used the identity $\gamma_{\mu(ij}\gamma^{\mu}_{kl)} = \delta_{(ij}\delta_{kl)}$ which follows from the fact that $SO(9)$ Clifford algebra is isomorphic to $J_2^{\mathbb{O}}$, and which can also be proven using more traditional methods [29].

with

$$0 < \lambda < \frac{1}{\sqrt{2}} \tag{5 - 11}$$

If $\lambda = \frac{1}{2\sqrt{2}}$ so that $T_{\alpha\beta} = \delta_{\alpha\beta}$ we have

$$\partial_\alpha \partial_\beta V = 0 \tag{5 - 12}$$

For the J_3^H model this would agree with the splitting of the 14 scalars into (5+1) massive vector multiplets and 2 massless hypermultiplets according to the $N = 6$ interpretation of its moduli space. Equation 5 - 4 implies that for the non-BPS critical points the scalar square-mass matrix is semi-positive definite. For the massless fluctuations, attractor nature of the solution depends on third or higher derivatives [16]-[20]. We leave this problem to a future investigation.

6 Attractors in six dimensions

The analysis of the previous section can be extended to six dimensions which is the maximal dimension where theories with 8 supercharges exist. These are the (1,0) theories describing the coupling of n_T tensor multiplets to supergravity [26, 27]. In this case, as discussed in [28], the string tension plays the role of central charge and it depends on the tensor multiplet scalars through the coset representative X_I of the

$$\frac{SO(1, n_T)}{SO(n_T)}$$

σ model [26, 27]

$$\begin{aligned} Z &= X^\Lambda q_\Lambda \\ X^\Lambda \eta_{\Lambda\Sigma} X^\Sigma &= 1 \end{aligned} \tag{6 - 1}$$

where $\eta_{\Lambda\Sigma}$ is the $(1, n_T)$ Lorentzian metric $(\Lambda, \Sigma = 0, 1, .., n)$. The matter charges are $X_{\Lambda I}$ $(I = 1, ..., n_T)$ with the property

$$\begin{aligned} X_\Lambda X_\Sigma - X_{I\Lambda} X_{I\Sigma} &= \eta_{\Lambda\Sigma} \\ X_\Lambda X_\Sigma + X_{I\Lambda} X_{I\Sigma} = \mathcal{N}_{\Lambda\Sigma} &= 2 X_\Lambda X_\Sigma - \eta_{\Lambda\Sigma} \end{aligned} \tag{6 - 2}$$

and $\mathcal{N}_{\Lambda\Sigma}$ is the metric of the kinetic energy of the self-dual tensor fields. As shown in [25] if one defines a black string potential energy as

$$V = q^\Lambda \mathcal{N}_{\Lambda\Sigma} q^\Sigma = q_\Lambda \mathcal{N}^{\Lambda\Sigma} q_\Sigma = Z^2 + Z_I^2 \tag{6 - 3}$$

where $Z_I = X_{I\Lambda} q^\Lambda$ and the inverse formula , analogue of 4-10, holds

$$q^\Lambda = X^\Lambda Z - X^\Lambda_I Z_I$$

it is easy to show that the attractor condition $\partial_I V = 0$ implies [25]

$$Z Z_I = 0 \tag{6 - 4}$$

with the solution $Z \neq 0$, $Z_I = 0$ which is the BPS case and the other , $Z_I \neq 0$, $Z = 0$, is the non-BPS case and corresponds to "tensionless" strings. However, in both cases the string energy is non-vanishing and, moreover, the two cases correspond to time-like and space-like configurations of the charges since

$$q^\Lambda \eta_{\Lambda\Sigma} q^\Sigma = Z^2 - Z_I^2 \tag{6 - 5}$$

is an invariant. Therefore , as expected, the two classes of orbits are

$$\frac{SO(1, n_T)}{SO(n_T)} \qquad BPS \tag{6 - 6}$$

$$\frac{SO(1, n_T)}{SO(1, n_T - 1)} \qquad non - BPS \tag{6 - 7}$$

7 Conclusions

In this paper we have examined the nature of attractor equations for 5D extremal black holes based on the geometry $N = 2$ vector multiplet scalars which are symmetric spaces G/H such that G is a symmetry of the MESGT [6].

There are two generic classes of attractors, one BPS and one non-BPS, described by two classes of charge orbits previously found by the same authors. The value of the B-H potential at the non-BPS attractor point is

$$V|_{\partial_x V = 0} = (3 Z_{NBPS})^2 = |I_3|^{2/3}$$

so that the Bekenstein-Hawking entropy is given, in terms of the central charge, by

$$S_{NBPS} = (3 Z_{NBPS})^{3/2} = \sqrt{|I_3|}$$

This is to be compared with the supersymmetric attractors where [8, 10, 7]

$$V|_{\partial_x V = 0} = Z^2 = I_3^{2/3}$$

$$S_{BPS} = (Z)^{3/2}_{BPS} = \sqrt{I_3}$$

For non-symmetric spaces the analysis may change because the derivation of equation $V = 9Z^2$ will no longer be valid. It would be interesting to consider the case of Calabi-Yau compactifications [30] in which case the C_{IJK} coefficients are related to the C-Y intersection numbers. From the general validity of the equations 4-50-4.53, in the one modulus case, the renormalized formula

$$S^{4/3} = V|_{\partial V=0} = Z^2(1 + \frac{4}{T^2_{111}})$$

will still hold in analogy with a similar situation in $d = 4$ [9].

We have also analyzed the black self-dual string potential in the case of $d = 6$ $(1, 0)$ theories. In that case, due to the nature of the tensor moduli space, there are just two orbits with non-vanishing entropy, a time-like one corresponding to BPS attractors and a space-like one corresponding to non-BPS attractors which are tensionless strings (zero central charge). The $d = 4$ situation is much richer and will be considered elsewhere [5].

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