Aspects of Spin-Charge Separation in high T_c Cuprates

by

Don H. Kim

AB Physics
Harvard University 1994

Submitted to the Department of Physics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 1998

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August 19, 1998

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Abstract

High T_c cuprates have a number of features that are anomalous from the point of view of the conventional theory of metals, *i.e.* the Fermi liquid theory. These include the high T_c itself, the linear resistivity in optimally doped cuprates, the "spin gap" phenomena, and antiferromagnetic fluctuations. Some of the basic features of the high T_c phase diagram can be readily understood from slave boson mean field theories of the t-J model that yield a variety of phases like the uRVB phase (strange metal), d-wave RVB phase (spin gap), and the superconducting phase. This thesis is concerned with improving this picture by considering important fluctuations around the mean fields. This leads to certain gauge theories in which neutral spin 1/2 fermions ("spinons") and charged spinless bosons ("holons") are coupled to gauge fields. If the gauge field is not "confining", the spin and charge degrees of freedom are separated to some extent, and an unconventional picture for transport and magnetic properties is expected.

Following an introduction and overview, the first half of the main body of this thesis analyzes a simple model of a degenerate two-dimensional Bose liquid interacting with a fluctuating gauge field, with the goal of studying the charge degree of freedom in the cuprates. It is shown that the fluctuating gauge field efficiently destroys superfluidity even in the Bose degenerate regime. The nature of the resulting normal state is discussed in terms of the geometric properties of the imaginary-time paths of the bosons. Charge response functions are studied numerically (by path integral Monte Carlo methods); it is found that the transport scattering rate behaves as $\hbar/\tau_{\rm tr} \sim 2k_{\rm B}T$, consistent with the experiments on the cuprates in the normal state, and that the density correlations of our model resemble the charge correlations of the t-J model.

The second half considers the magnetic properties of cuprates from the point of view of a gauge theory of spinons, with emphasis on the underdoped regime. Despite the spin gap, there is a substantial antiferromagnetic correlation in the underdoped cuprates, as evidenced by the q-space scan of Neutron scattering cross section and different temperature dependences of the Copper and Oxygen site relaxation rates, features which are not captured well by mean field theories. As a concrete illustration of the gauge-fluctuation restoration of the antiferromangetic correlation and the feasibility of the $1/N$ perturbation theory, a U(1) gauge theory of 1d Heisenberg spin chain is worked out exactly, and then perturbatively. The difference between the behavior of uniform and staggerd spin correlation functions is discussed in terms of conserved and nonconserved currents. The $1/N$-perturbative treatment of the gauge theory in 2+1D (which can be motivated from the mean field Flux phase of the Heisnberg model) leads to a dynamical mass generation corresponding to an antiferromagnetic ordering, but it is argued that in a similar gauge theory with an additional coupling to a Bose (holon) field, symmetry breaking does not occur, but antiferromagnetic correlations are improved, which is the situation in the underdoped cuprates.

Thesis Supervisor: Patrick A. Lee
Title: William and Emma Rogers Professor of Physics

TO MY PARENTS

Acknowledgments

First and foremost, I would like to express my sincere gratitude to my advisor Patrick Lee, who took me on as a student of his upon my arrival at MIT, and has provided every encouragement and support all these years. The last four years that I spent under his tutelage have been a wonderful period of my life during which I could concentrate on my studies without cares and worries of the world. He would let me go my way, explore my curiosities, and pursue other interests. His sense of decency and humility, his gentle concern for people around him, and his quiet warmth underlied a happy, "strongly-correlated" community within which life was pleasant and physics was fun; I feel privileged to have been a part of it.

Scientifically, I am very grateful that Patrick entrusted me with the problem of high T_c cuprates. It has been a humbling, frustrating, yet wonderful challenge. I feel fortunate to be given an opportunity to work on a subject that concerned many giants in the the frontier of condensed matter physics. I doubt whether I measured up to Patrick's expectations, but in any case, he had kindness not to show his disappointments. In studying aspects of the cuprate physics, I benefitted enormously from Patrick's sharp intuition, uncommon common sense, and trenchant observations. Many of the ideas in this thesis are his, and I hope that my presentations are something more than a regurgitation of his writings and talks. Of course I should say at the very outset that any possible errors in this thesis are my responsibility alone. Patrick also had a big influence on my understanding of physics and scholarship in general. More than anyone else, he convinced me from early on that physics is not a potpourri of pretty calculations.

In addition to Patrick, I would like to thank Derek K. K. Lee and Xiao-Gang Wen for the most pleasant collaborations. I fondly look back at my early years at MIT when Derek Lee and I struggled together to find bugs in our computer code. I thoroughly enjoyed working with Derek. I remember his quick mind, optimism, and gentle demeanor. Throughout the past years, Xiao-Gang kindly and generously took his time to discuss physics and share his ideas with me, often on very short notices. I have benefitted a lot from his keen insights and patient explanations. Xiao-Gang has also kindly agreed to serve on the thesis committee, together with Patrick and Krishna Rajagopal; I thank them all.

During the last four years, I have enjoyed the company of "St. Patrick's disciples". I remember many pleasant dinners with Manfred Sigrist, Akira Furusaki, Y.B. Kim, and K. Kuboki, with occasional participation of D.K.K. Lee, K. Matveev, E. Westerberg, and C. Mudry. Particular thanks are due to Manfred for organizing those dinners and weekend lunches. For three summers I had a good fortune to overlap with Naoto Nagaosa who was always happy to discuss physics and willing to try new culinary adventures. I am also indebted to Patrick's later disciples, including Olav Syljuasen, Jan Brinkmann, Catherine Pepin, and Steve Simon, for pleasant interactions.

J.K. Jain's visit to MIT during the 1996-1997 academic year was accompanied by two of his good-natured students R.K. Kamilla and Kwon Park, with whom I shared quite a few enjoyable moments.

My present and erstwhile officemates Hyunwoo Lee, Dima Ivanov, and Andrey Shytov, have created a congenial atmosphere to discuss physics, life, and many other things, and I am very grateful to them. Dima, in particular, have pleasantly shared with me our office for the last three years, and tolerated my bohemian ways and the visual and musical abuse of the space.

I am also grateful to my Korean friends, in particular Ickjin Park and Kyongjae Cho, for their companionship, and many others, including K.S. Joo and K.W. Ahn, for their hospitality.

To come this far would not have been possible without the help and encouragements of the mentors from my formative years. During my early years in the U.S. as a high school student faced with some science requirement, I had to avoid English-ridden biology at all cost; all those years (till today) I have managed not to learn biology (At least, I am not proud of it!). Somewhere along the way I discovered physics — I would like to thank my teachers Joe Wesney, Victor Mazmanian, and Tony Nicholson, for introducing me to the world of physics. Among my mentors from the college years, I would like to thank particularly Henry Ehrenreich, Howard Stone, and Erich Runge. H. Ehrenreich got me initiated in physics research, and gave me a chance to work as a teaching assistant for his course on quantum theory of solids.

Lastly I would like to thank my parents and sisters for their moral support and encouragements.

addendum: My views and opinions expressed in this thesis have gone through changes and revisions till the very last minute. I would like to thank A. Furusaki, P. A. Lee, C. Mudry, N. Nagaosa, K. Rajagopal, and X. -G. Wen for a number of helpful inputs and suggestions on previous versions of this thesis. I would also like to thank the physics graduate administrator Peggy Berkovitz for her patience and flexibility in the scheduling of the thesis defense and the submission of the thesis. The outline for finishing touches to the thesis was largely envisioned during my trip to Greece and Switzerland this summer, and I would like to thank my parents and Mr. & Mrs. Max Roth who made my trip so relaxing, inspiring, and enjoyable.

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Chapter 1

INTRODUCTION AND OVERVIEW

Give me a condor's quill! Give me Vesuvius' crater for an inkstand! Friends, hold my arms! For the mere thought of penning my thoughts of this Leviathan, they weary me, and make me faint with their out-reaching comprehensiveness of sweep, as if to include the whole circle of the sciences,$\cdots$. To produce a mighty book, you must choose a mighty theme.

Herman Melville, *Moby-Dick*

The fair breeze blew, the white foam flew,
The furrow followed free;
We were the first that ever burst
Into that silent sea.

Samuel Taylor Coleridge, *The Rime of the Ancient Mariner*

The discovery of superconductivity in the layered perovskite compound LBCO by Bednorz and Müller in 1986[1] was something new and BIG by any measure, and was immediately recognized as such. It was quickly realized that the unusually high T_c was somehow due to the presence of copper oxide planes (hence the name cuprates), and the flurry of activities during the following couple of years have discovered cuprates with substantially higher T_cs, including the Yttrium compound $YBa_2Cu_3O_7$ (T_c=92K) and the Thallium compound $Tl_2Ba_2Ca_2Cu_3O_{10}$ (T_c=125K). By now a variety of superconducting cuprates are known, with T_cs as high as 164K[2]. Though the initial excitement about the technological and materials-science aspects seems to have subsided to some extent, in the intervening decade the physics of the cuprates would remain as *the problem* to many condensed matter physicists. Every now and then, there have been new experiments that repeatedly reminded us that the high superconducting transition temperature is only a part of the surprising saga. In the mean time a number of theories rose, and waxed, waned, or fell. Currently there is still little consensus on the mechanism of superconductivity or on the theory of the metallic states in the superconducting cuprates. One might even get the impression of an "optimistic resignation" that the problem won't be solved within near future.

Of course this does not mean to deny considerable progress toward the understanding of the physics of the cuprates over the years. Array of ingenious experiments, improvements in the resolution of experimental probes, and the availability of large single crystals and crystals with clean surface have settled some important controversies and thrown new lights on these fascinating materials. Compared to earlier days of the high T_c , we now have a lot better idea of what experiments are important, what needs to be explained, and what more experiments are needed, although the relative importance attached to certain experiments might depend strongly on different theoretical viewpoints, and uncertainties remain as for the future direction of the high T_c research; after all, we have been constantly surprised and directed by the experimentalists over the past decade. At this point it may be worthwhile to review what may be regarded as the key points of the cuprate physics.

1.1 Notable Features of the Cuprates

these extracts are solely valuable or entertaining, as affording a glancing bird's eye view of what has been promiscuously said, thought, fancied, and sung of Leviathan, by many nations and generations including our own.

Herman Melville, *Moby-Dick*

1.1.1 symmetry of the order parameter

In early days of the high T_c superconductivity, the NMR in the superconducting state of the cuprates showed power-law-like behaviors of temperature dependences, indicating the existence of nodes in the gap function as in the case of d-wave pairing, while the penetration depth did not seem to obey the d-wave prediction. This discrepancy was explained (in favor of d-wave) in terms of localized states in the gap[3]. More refined penetration depth measurements now show $\Delta\lambda(T)/\lambda(0) \propto T$, consistent with d-wave pairing. Strong support for the d-wave pairing also came from the angle resolved photoemission spectroscopy (ARPES) experiment which found maximum gap at $(\pi, 0)$[4]. Soon, half integer flux quantum effects in a superconducting loop containing Josephson junctions have shown unambiguously that the orbital symmetry of the superconducting order parameter is of the $d_{x^2-y^2}$ type[5].

1.1.2 quasiparticles in the superconducting state

Thermal conductivity measurement in $YBa_2Cu_3O_7$[6] indicates that the superconducting state has long lived quasiparticles with mean free path that exceeds 500 lattice spacings. Also, these quasiparticles can be "seen" in photoemission[7, 8] as a sharp peak whose spectral weight is about x (doping).

1.1.3 unusual superconducting state

The d-wave quasiparticles in themselves might not constitute something abnormal. P.W. Anderson had earlier on remarked that the superconducting state of the cuprates may be more "normal" than the normal state. However, some recent experimental and theoretical investigations have strongly questioned the standard BCS paradigm for the superconducting state[9]. For example, the photoemission experiments have found that the superconducting gap does not depend on doping, while it is well known that for not-overdoped cuprates T_c is roughly proportional to doping. This is hard to reconcile with the BCS theory which gives $\Delta/T_c = const$ for both weak and strong coupling because superconducting transition in that case has to do with self-consistent closing of the gap. The superfluid density (inferred from penetration depth and specific heat) is notably small ($\sim x$) and has a temperature dependence characteristic of a d-wave quasiparticle $\rho_s(T) \approx x - \alpha T$, but value of the coefficient of the T-linear term does not agree with simple theories[9].

1.1.4 transport anomalies in the normal state

Many of the normal state transport properties in the cuprates are fairly universal, and also highly anomalous. For example, optimally doped cuprates such as $YBa_2Cu_3O_7$, $La_{1.85}Sr_{0.15}CuO_4$, and BI2212 have a linear temperature dependence of in-plane resistivity $\rho(T) \approx aT$, with the slope a that is nearly uniform throughout these compounds despite differences in phonon spectrum, spin fluctuation properties, etc. This means that a common mechanism of unfamiliar origin is at work in these materials, scattering current carriers very strongly at low temperatures (Recall that the Fermi liquid theory can only give $\rho(T) \propto T^n$ with $n \geq 2$, n=2 for electron-electron interaction, and n=4 for electron-phonon interaction.) Also, the dc conductivity ($1/\rho$) is seen to scale roughly with doping x (rather than $1-x$), although the photoemission indicates the existence in the optimally doped cuprates of a large Fermi surface with area $1-x$. At least, the dc results are consistent with measured optical conductivity $\sigma_{ab}(\omega)$ that show a Drude-like peak of width $2T$ with spectral weight $\propto x$. The Drude weight actually measures the combined quantity n/m^*, (n =charge density) therefore one may argue for $n \sim 1-x$ and $m^* \sim 2m_e(1-x)/x$, but such dependence of m^* is inconsistent with other experiments like NMR.

The Hall effect is also anomalous. The Hall coefficient R_H, which in the Fermi liquid theory is given by a temperature independent expression $R_H = 1/nec$, has an anomalous temperature dependence (roughly $\sim T^{-1}$) in the cuprates like YBCO, though this is not as universal. The Hall effect in the cuprates is also quite sensitive to disorder; small Zinc doping in YBCO substantially weakens the temperature dependence into one similar to that of LSCO[10], possibly implying that the LSCO may be subject to some intrinsic disorder effects. The sign of R_H is *positive* (for doping $x <\sim 0.3$), indicating that "holes" are charge carriers. Another remarkable feature is that the relaxation time inferred from the Hall angle θ_H ($= \tan^{-1}\sigma_{xy}/\sigma_{xx}$) has a temperature dependence $\tau_H \propto T^{-2}$ in clear disagreement with the usual transport relaxation time $\tau \propto T^{-1}$ deduced from the resistivity.

1.1.5 spin gap in underdoped cuprates

NMR experiments including the Knight shift, the Cu site relaxation rate, and the O site relaxation rate in $YBa_2Cu_3O_{6+x}$ first gave an indication of a "gap" for spin excitations existing already in the normal state in the underdoped cuprates. For example, the Copper site relaxation rate $1/T_1T$ in $YBa_2Cu_3O_{6.63}$ drops with decreasing temperature below about 150K, far above $T_c \approx 60$K. Later, the ARPES[11] has revealed that a gap for quasiparticle-like excitations opens up in the normal state, leaving only "segments" of Fermi surface near $(\pm\pi/2, \pm\pi/2)$. The c-axis optical conductivity and the specific heat have also indicated the existence of gaplike features. The most important point is that the gaplike features (loss of low energy spectral weight) are seen only in probes that involve spin degrees of freedom. For example, the spin gap does not show up in the in-plane optical conductivity $\sigma_{ab}(\omega)$. Because of the reduced scattering of charge carriers, $\sigma_{ab}(\omega)$ of the underdoped cuprates has a sharper drudelike peak than the optimally doped cuprates (this also causes the deviation from linear in T resistivity), but the spectral weight is still there. The basic quasiparticle excitations of the Fermi liquid theory carry both spin and charge, therefore the gaplike features appearing only in spin-related responses are a strong violation of the conventional paradigm.

1.1.6 antiferromagnetic correlations

Neutron scattering experiments and numerical studies have established that the high T_c parent compounds such as La_2CuO_4, $YBa_2Cu_3O_6$ are a quantum antiferromagnet with an ordered ground state. Their low temperature behaviors are well described by the "renormalized classical" regime of the the nonlinear sigma model[12]. With about 2% doping, the antiferromagnetic order is rapidly destroyed, but certain short range antiferromagnetic correlations persist into the doping range where superconductivity is observed.

The magnetic properties of the superconducting cuprates are not as universal as the transport properties. In the normal state of (optimally doped) $YBa_2Cu_3O_7$, inelastic neutron scattering (INS) has failed to detect any magnetic scattering that would reveal itself above the background, while in the underdoped $YBa_2Cu_3O_{6+x}$ and $La_{1.85}Sr_{0.15}CuO_4$ (believed to be slightly underdoped from optimal) inelastic neutron scattering at or near the antiferromagnetic wavevector (π, π) displays a magnetic scattering for a wide range of frequencies, with a broad peak around 20-30 meV. In $YBa_2Cu_3O_{6+x}$ and LSCO, the peaks are also observed in the superconducting state, being somewhat sharper than the normal state counterparts. In the superconducting state of $YBa_2Cu_3O_7$, the INS shows a sharp peak at 41 meV, with little scattering below. This peak can be regarded to evolve (and broaden) into the aforementioned peak of the $YBa_2Cu_3O_{6+x}$ as the doping is reduced.

Evidence for antiferromagnetic correlations in $YBa_2Cu_3O_7$ comes from the NMR which shows the Copper site $1/T_1T$ increasing with decreasing T, in contrast to the Oxygen site $1/T_1T$ that seems to follow the Korringa behavior (i.e. temperature independent). The NMR reveals stronger antiferromagnetic fluctuations in the 214 system; in fact even in the doping concentration which is believed to be underdoped, LSCO Cu $1/T_1T$ shows no sign of spin gap. This and the relatively low T_c of LSCO system are suspected to be due to intrinsic disorder (due to Strontium doping).

1.2 Other Correlated Electron Systems

> *"Is there any point to which you would wish to draw my attention?"*
> *"To the curious incident of the dog in the night-time."*
> *"The dog did nothing in the night-time."*
> *"That was the curious incident," remarked Sherlock Holmes.*
>
> Arthur Conan Doyle, *Silver Blaze*

To appreciate just how unusual the properties of the cuprates are, it helps to compare and contrast the cuprates with other correlated electron systems. We are quite lucky in this regard, since the years following the discovery of high T_c have have enjoyed burst of activities in a number of other correlated electron systems. These include old (in the sense of pre-high T_c) systems like heavy fermion superconductors, organic superconductors, Mott insulators (like V_2O_3), and (literally) new ones that came into being following an enormous surge of interest in the synthesis of novel electronic systems.

1.2.1 heavy fermions

Heavy fermion systems, which may include $UPt_3, CeCu_6$, have rare earth or actinide atoms with partially filled f-shell arranged in a periodic lattice supplemented by other atoms that provide conduction electrons. The f-electrons behave like well-localized moments at high temperatures, while they are strongly coupled to the conduction electrons at low temperatures, and have to be counted as a part of the Fermi surface. Below some temperature scale usually associated with Kondo screening, the system behaves as a Fermi liquid with an enormously large effective mass (hence the name heavy fermions), as can be seen from the Pauli susceptibility and specific heat coefficient that are two to three orders of magnitude greater than the usual metal. The resistivity of this "almost localized" Fermi liquid has the standard form $\rho(T) = \rho_0 + aT^2$, with a very large a.

The antiferromagnetic order seen in some heavy fermion materials (like U_2Z_{17}) is usually understood in terms of the RKKY interaction between the local moments (rather than nesting or SDW effects of heavy quasiparticles), although the system seems to retain a substantial itinerant character in the ordered phase[13]. The superconductivity in heavy fermion systems, first found in 1979[14], is still poorly understood. In view of the fact that the order parameter seems to have a complicated symmetry and the fact the local moments are destructive to the superconductivity in the standard the BCS scenario, the superconductivity is regarded to be "unconventional." However, the normal state is essentially conventional in the sense of being a Fermi liquid, albeit one with a very heavy mass.

1.2.2 organic conductors and superconductors

Organics were proposed as a candidate for a "high T_c" superconductor as early as in 1964 by W. A. Little[15], but the superconductivity in organic compounds was not discovered until 1980. The first discovered organic superconductor, tetramethyltetraselenafulvalene $(TMTSF)_2PF_6$, is a quasi-one dimensional compound with a rather low T_c of $\sim$ 1K. Currently, organic compounds with T_c as high as 12K are known, e.g. the family $\kappa-(\mathrm{BEDT}-\mathrm{TTF})_2\mathrm{X}$, but the mechanism of superconductivity in these materials is not very well understood. P.W. Anderson said of them in 1992, "these are still almost a complete mystery"[16]. Strong electron-electron interaction accentuated by reduced dimensionality is an important characteristic of these materials; as in the cuprates the superconductivity seems to compete with antiferromagnetic or SDW instability. The higher T_c organics (BEDT class) seem to share titillating similarities with the cuprates[17], such as the layered (quasi 2d) structure, unusual transport properties, and gaplike features in the normal state, though these materials are not as well characterized as the cuprates.

1.2.3 Sr_2RuO_4

The layered perovskite material Sr_2RuO_4 may turn out to be the closest Galilean non-invariant analogue of He3. It has a lattice structure identical to that of the high T_c parent compound La_2CuO_4, but electronic properties are very different. Sr_2RuO_4 is a Fermi liquid with fairly large Fermi liquid parameters (F_1^s, etc.); mass enhancement is about $3 \sim 4$. It is also a prime example of an anisotropic Fermi liquid, having a metallic c-axis resistivity that are about 3 orders of magnitude greater than the in-plane resistivity. Recently[18] superconductivity with T_c of $\sim$ 1K was discovered in this system; it has been suggested that the pairing symmetry is p-wave[19]. Sr_2RuO_4 proves two important points: 1) interaction effects in a Fermi liquid do not produce high T_c , and 2) a complicated chemical formula with rather obscure elements is not a sufficient condition for a non-Fermi liquid behavior.

1.2.4 other (doped) Mott insulators

In the cuprates, as we dope away from the insulating parent compound, we quickly go into a metallic regime with carrier density $\sim x$. While the notion of metal insulator transition (MIT) caused by carrier density $\to 0$ might appear quite innocent now (a decade after the discovery of the high T_c), this is not the "usual" way a Mott-type metal-insulator transition occurs. The usual scenario due to Brinkman and Rice[83] is that the carriers become infinitely heavy as we approach the MIT from the metallic side, i.e. $m^* \propto 1/|y - y_c|$ where y is a parameter that can be tuned to the transition. In other words, in the conventional scenario, as the MIT is approached the carriers get localized, rather than disappear.

A typical example is the V_2O_3 system[21] which can undergo a MIT by changing pressure. A more spectacular example is the recently discovered $LaTiO_3$ compound[22], which is an antiferromagnetic insulator and a 3D analogue of the high T_c parent compound La_2CuO_4. $LaTiO_3$ *can be doped* with Sr ($La_{1-x}Sr_xTiO_3$; just like $La_{2-x}Sr_xCuO_4$!), and the system quickly becomes metallic. So far, so good, but the measurements of the Pauli susceptibility (χ), the specific heat coefficient (γ), and the Hall coefficient (R_H) show clear difference from the cuprate case: R_H is negative, and carrier density *increases* as the transition is approached; at the same time diverging effective mass is seen in χ and γ. The magnetic properties are also in stark contrast with the superconducting cuprates. NMR in the metallic state[23] gives little sign of antiferromagnetic correlation. In fact, the system seems nearly ferromagnetic, consistent with the view that ferromagnetic spin fluctuations go along with incipient localization[24, 25, 26, 27]. Ferromagnetic spin fluctuations in the viscinity of an antiferromagnetic insulating phase sounds strange, but so does the Nagaoka's theorem[28] which indicates that the Hubbard model with a very large repulsive interaction U is a ferromagnet in the limit of small doping.

1.2.5 novel spin systems

Ever since Anderson's 1987 suggestion[29] that the high T_c has to do with doping a liquid of spin singlet pairs (Resonating Valence Bonds: RVB), there has been considerable interest in the possibility of spin systems with "quantum" disordered ground state with no broken symmetry (quantum spin liquids). Among quantum spin liquids, we might make further distinction between critical and noncritical spin liquids.

1)critical spin liquids: These were an inspiration for Anderson's RVB theory. In Anderson's own words, "[I] grouped toward a connection with the "resonating valence bond" liquid of singlet pairs of which Bethe's linear spin chain was the only physical exemplar,..." Spin one-half (or more generally half integer) Heisenberg chain is a critical system with spin-spin correlation $\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle \sim (-1)^{i-j} \ln^{1/2}(i-j)/|i-j|$, and has gapless excitations, in accordance with the Lieb-Schultz-Mattis theorem. It is one of the best understood many body problems, filled with elegant methods like the Bethe ansatz and conformal field theories, and theoretical predictions are in excellent accordance with experiments on systems like Sr_2CuO_3[30] and $KCuF_3$[31].

2)noncritical spin liquids: These are more generic spin liquids, and the list of this class is longer. They have a gap to the lowest spin (triplet) excitations, and has only short range spin correlations. Although both the 2d square lattice and the triangular lattice of the original Anderson proposal[32, 29] turned out to have a Néel ordered ground state (in the case of triangular lattice, see R.R.P. Singh and D.A. Huse[33]), some 2d examples of spin liquids are known, including CaV_4O_9. In 1d or quasi-1d, there are a number of examples, including integer spin chains (Haldane gap materials like NENP), two-leg spin ladders ($SrCu_2O_3$), and spin-Peierls system ($CuGeO_3$).

- CaV_4O_9: 1/5 depleted 2d system CaV_4O_9 is, at present, probably the closest realization of Anderson's 2d RVB idea (but not quite, since Anderson wanted to have gapless excitations [Fermi surface]). It has been suggested that the disordered ground state of CaV_4O_9 can be understood in terms of the "plaquette RVB"[34]. There is a strong evidence that the frustration (n.n.n interaction) plays an important role; without it the system *might* have a Néel order. Inelastic neutron scattering[35] shows a sharp spin-triplet excitations, with finite minimum gap at $\mathbf{Q}$=(0,0), rather than at (π,π).

- Spin ladders like $SrCu_2O_3$ are a 1d example of RVB system with only a short range antiferromagnetic correlations[36, 37]. The neutron scattering[38] displays a sharp peak corresponding to a spin triplet excitation and a clear gap. Qualitatively similar behaviors are seen in the INS of the Haldane gap (integer spin chain) systems $AgVP_2P_6$[39] and NENP[40], and the inorganic spin-Peierls system $CuGeO_3$[41]. It should be noted that $CuGeO_3$ system is not a full quantum spin liquid in that it has a dimerized ground state which breaks translational symmetry and thus has nonvanishing correlator $\langle(\mathbf{S}_{2i}\cdot\mathbf{S}_{2i+1})(\mathbf{S}_{2j}\cdot\mathbf{S}_{2j+1})\rangle$, but B.S. Shastry and W. Sutherland[42] who studied the Majumdar Ghosh model[43] first applied (half a decade before high T_c) the terminology "quantum spin liquid" to such a state.

At present, all known examples of the generic spin liquids are nonconducting. The spin ladder compounds have been successfully doped. NMR measurements, made in the doping region where the material has a semi-conductor type resistivity, indicate that the spin gap remains in lighted doped spin ladders, though the size of the gap is reduced. Recently superconductivity was discovered a doped spin ladder system under very high pressures, but magnetic excitation system of that system is not well known.

1.3 Theoretical Considerations

What song the Sirens sang, or what name Achilles assumed when he hid himself among women, though puzzling questions, are not beyond all conjecture.

Sir Thomas Browne, *Urn-Burial*

We have reviewed the characteristics of the superconducting cuprates and other correlated electron systems. Some questions naturally emerge.

1.3.1 questions (beyond all conjecture?)

- How can we reconcile the presence of a large Fermi surface (in the optimally doped case) of area $(1-x)$ with the small number of "holelike" (positive-sign) charge carriers with density $\approx x$?[44].

- How can we understand the strong scattering of charge carriers, as evidenced in the linear in T resistivity? What kind of low lying fluctuations are responsible? In other correlated electron systems, we have seen that at low enough temperatures, the standard Fermi liquid form $\rho(T)\approx\rho_0+aT^2$ holds.

- How can we have a metal without a Fermi surface? In the 1960s, A. MacKintosh said of metals (as quoted in Kittel's solid state physics book), "*Few people* *would define a metal as 'a solid with a Fermi surface.' This may nevertheless be the most meaningful definition of a metal one can give today; it represents a profound advance in the understanding of why metals behaves as they do.*" To the extent that the Fermi surface is understood to be a closed locus (or loci) of points in the reciprocal space at which excitations with arbitrary small energy can occur, the underdoped cuprates are a metal without a Fermi surface.

- How can we understand the antiferromagnetic correlations in the superconducting cuprates? In the heavy fermion systems, the antiferromagnetic correlation has a lot to do with extra degrees of freedom (the f-electrons), which doesn't seem to be the case with cuprates. As indicated earlier, the notion of an "antiferromagetic Fermi liquid" is an illusory one (for a strong criticism of the "nearly antiferromagnetic Fermi liquid" scenario, see Ref.[45]). There are always spin-density-wave (SDW) systems like Chromium or systems near SDW transition, but the basic phenomenology of the cuprates seems inconsistent with the SDW ideas.

- Is there a relation between antiferromagnetism and superconductivity? To what extent is the proximity of the antiferomagnetic state and the superconducting state in the phase diagram of certain heavy fermions and organic compounds (e.g. BEDT-TTF) similar to that of the cuprates?

- In the underdoped cuprates why is there a gap for spin excitations, and not for charge excitations? How similar is this "spin gap" to the spin gaps seen in the bona fide quantum spin liquids, like the ladder systems and CaV_4O_9?

1.3.2 conjecture: spin-charge separation

Though there is still a lot of controversy surrounding these issues and other issues not discussed here, it wouldn't be too presumptuous to say there is a broad consensus that the essential physics of the cuprates can be explained by an electronic mechanism, and that the strong electron-electron interaction and the layered structure (2d) are the chief culprit for the bewildering array of anomalies.

Furthermore, there is certain optimism (not shared by everybody) that the basic physics of the cuprates is "simple". Simple, not in the sense having to do with triviality, but in the sense typified by Anderson's 2 dogmas[46] (He has many more, but we shall withhold judgement on other ones): *"All the relevant carriers of both spin and electricity reside in the* CuO_2 *planes and derive from the hybridized* $O_{2p} - d_{x^2-y^2}$ *orbital which dominates the bonding in these compounds*", and *"Magnetism and high* T_c *superconductivity are closely related, in a very specific sense: i.e., the electrons which exhibit magnetism are the same as the charge carriers."* That the basic physics of cuprate plane can be understood in terms of an electron at the copper site in each unit cell that can hop from cell to cell has been also convincingly argued by F.C. Zhang and T.M. Rice[47].

Accepting this point of view, we can approach the cuprate problem with some confidence (and hope) in terms of the one-band Hubbard model with on-site repulsive interaction U:

$$H = -t \sum_{<ij>} (c^{\dagger}_{i\sigma} c_{j\sigma} + \text{h.c.}) + U \sum_{i} n_i n_i \tag{1.1}$$

or the t-J model

$$H = \sum_{<ij>} -t\mathcal{P}(c^{\dagger}_{i\sigma} c_{j\sigma} + \text{h.c.})\mathcal{P} + J(\mathbf{S}_i \cdot \mathbf{S}_j - \frac{1}{4} n_i n_j) \tag{1.2}$$

where $n_i = c^{\dagger}_{i\alpha} c_{i\alpha}$, $\mathbf{S}_i = \frac{1}{2} c^{\dagger}_{i\alpha} \sigma_{\alpha\beta} c_{i\beta}$, and $\mathcal{P}$ is a projection operator that projects out double occupation. While the t-J model, the simpler of the two, can be understood to derive from the Hubbard model in the large U limit (second order perturbation would give $J = 4t^2/U$), the value of J deduced from magnetic Raman scattering is quite large ($J \approx 1500K \approx t/3$), so it is more reasonable to take the t-J model as a basic model in itself and explore the consequences.

Within the t-J model, the basic physics is seen to be the competition between the energy gain ($\sim xt$) due to the mobile holes and the cost of exchange energy ($\sim J$) resulting from the disruption of AF order. If J is small, the cost of exchange energy is overcome by delocalization, so we have the usual Fermi liquid, which seems to be the case of the aforementioned doped Mott insulator $La_{1-x}Sr_xTiO_3$. However, for the cuprates J is large ($J >\sim xt$), and we can expect different physics. Indeed, Nature seems to solve this problem in a unique way: spins on nearby sites form into "resonating" singlet pairs to retain some exchange energy, and have sufficiently liquid-like character so that the "holes" can propagate through them coherently and superconduct at low temperatures. This is the crux of P.W. Anderson's 1987 paper on the resonating valence bond (RVB) theory of the high T_c superconductivity. The AF order of the parent compound does not invalidate the essence of the argument; it is quite sensible that in the doped case, mobile holes frustrate the tendency for the spins to order and stablize the singlet liquid phase. It should be also pointed out that this scenario already points to a strong deviation from the usual metallic behavior: it is hard to believe that this picture can be reconciled with the usual Fermi liquid quasiparticles propagating with both spin and charge.

A rigorous mathematical theory that realizes above RVB picture has been lacking, and this is not for the lack of trying. Nevertheless, a trick applied to the t-J model, so called the slave boson method, supplemented by some heuristics and physical considerations, can be argued to be a right step forward. Within the slave boson method, one can decompose the electron operator into a neutral spin-1/2 fermion operator and a charge-e spinless boson operator ($c^{\dagger}_{i\sigma} = f^{\dagger}_{i\sigma} b_i$), and enforce the no-double-occupancy constraint by a Lagrange multiplier. In other words, a "slave" boson operator that keeps track of moving hole (vacancy) has been introduced. The t-J Lagrangian then becomes

$$L = -t \sum_{<ij>\sigma} (f^{\dagger}_{i\sigma} b_i b^{\dagger}_j f_{j\sigma} + \text{h.c.}) + J/4 \sum_{<ij>} f^{\dagger}_{i\alpha} \tau_{\alpha\beta} f_{i\beta} \cdot f^{\dagger}_{j\alpha} \tau_{\alpha\beta} f_{j\beta} + \sum_i \lambda_i (f^{\dagger}_{i\sigma} f_{i\sigma} + b^{\dagger}_i b_i - 1). \tag{1.3}$$

At this point the situation seems to have gotten more complicated, but now mean field

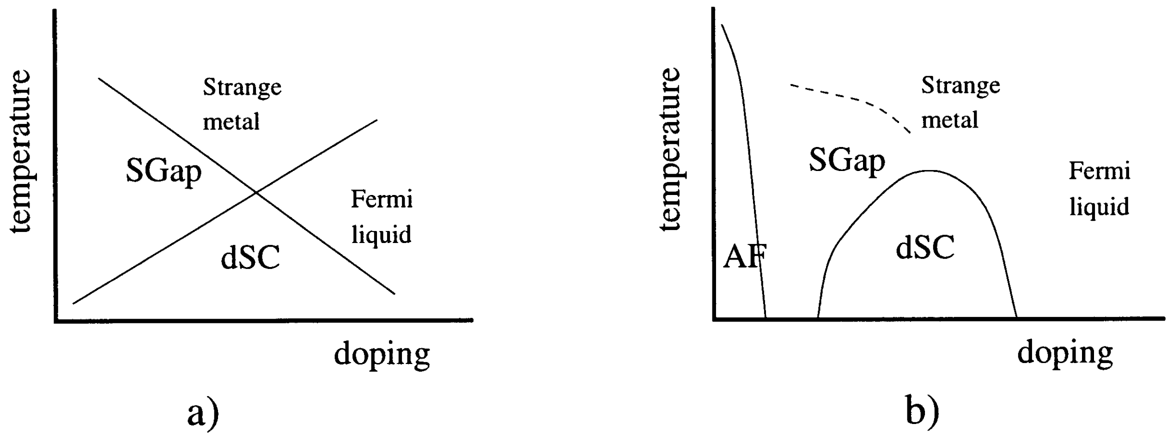

Figure 1-1: Cuprate phase diagrams: a) slave boson mean field theories (the SGap (spin gap) phase is identified with the d-wave RVB phase (or the sFlux phase of the SU(2) mean field theory to be discussed in Chap. 3); the "strange metal" phase with a large Fermi surface corresponds to the uRVB phase; "dSC" stands for d-wave superconducting phase), b) the "real" phase diagram.

analysis (similar to the original BCS theory) can be applied. There are quite a few possible mean field parameters, including $\Delta_{ij} \equiv \langle f_{i\uparrow}f_{j\downarrow} - f_{i\downarrow}f_{j\uparrow}\rangle$, $\chi_{ij} \equiv \langle f^\dagger_{i\sigma}f_{j\sigma}\rangle$, and $\langle b\rangle$. The slave boson mean field studies of Fukuyama and coworkers[48], and Kotliar and Liu[49] have found a phase diagram consisting of the d-wave superconducting phase ($\langle b\rangle \neq 0, \Delta_{i,i+x} = -\Delta_{i,i+y} \neq 0$), the "spin-gap phase" ($\langle b\rangle = 0, \Delta_{ij} \neq 0$), the uniform RVB (uRVB) phase ($\langle b\rangle = 0, \Delta_{ij} = 0$), and a Fermi liquid like phase ($\langle b\rangle \neq 0, \Delta_{ij} = 0$). In the superconducting phase, the bosons are condensed and the fermions are paired. The spin gap phase is a normal metallic phase since the bosons have not condensed, but the fermions are paired so that there would be a gap for magnetic excitations. Remarkably, this simple approach captures rough qualitative features of high T_c phase diagram (See Fig.1-1). It is even more remarkable, when we note that at the time there was little evidence for the d-wave symmetry of the order parameter or the spin gap phenomena.

The key feature of the mean field theory is that the spin carrying fermions (spinons) and charge carrying bosons (holons) are decoupled. In other words, we have a full *spin-charge separation.* This can explain two of the most puzzling feature of the cuprates: 1) "spin gap" and no "charge gap" in underdoped cuprates [The state with nonzero $\langle b\rangle, \Delta_{ij}$ would correspond to such a state], and 2) the large Fermi surface with area $1-x$ and charge density x near optimal doping (large Fermi surface can be understood to be that of the spinons in, say, uRVB phase, while the charge density is that of the holons).

1.3.3 spinons, holons, and gauge fields

Above slave-boson approach is subject to a number of criticisms. We know that such an approach would be a poor (or unnatural) description of most metals that we know.

Also, why not write instead the electron operator as $c_{i\sigma}^\dagger = b_{i\sigma}^\dagger f_i$ and proceed similarly (Schwinger boson method)? The point is that the slave boson (or Schwinger boson) mean field decompositions are not justified *a priori*. However, physics can more or less dictate theories; in our problem of doped Mott insulator with $J > xt$ it stands to reason that the slave boson mean field theory is a good starting point, because by construction the mean field parameters (Δ_{ij}, χ_{ij}) embody a lot of singlet correlations central to Anderson's RVB theory.

Granted that the mean field theory is "reasonable", we still have a host of problems to grapple with: 1) The mean field theory does not capture energy scales accurately. In particular, the Bose-Einstein temperature at which holons acquire macroscopic coherence ($\langle b \rangle \neq 0$) comes out too high (thus the superconducting T_c comes out too high), 2) there are fictitious phase transitions between certain mean field phases which should actually be crossovers (e.g. between spin gap phase and uRVB phase), 3) the mean field theory does not explain why the normal state is a poor metal. 4) the mean field theory loses most of the antiferromagnetic correlations.

The rest of this thesis will describe the efforts to remedy some of these problems. The key is the role of fluctuations around the RVB mean fields. It is quite natural to expect that the incorporation of fluctuations ignored in the mean field theory would improve the picture. We will focus mostly on the gauge fields that correspond the phase fluctuations of RVB order parameters and the fluctuation of Lagrange muliplier enforcing no double occupancy condition. The gauge field reflects the fact that the t-J model is invariant under the local transformation $f_i \to e^{i\theta_i} f_i, b_i \to e^{i\theta_i} b_i$ since $c_{i\sigma} = b_i^\dagger f_{i\sigma}$ is obviously a gauge singlet.

With the inclusion of gauge fields, the theory takes the following schematic form in the continuum

$$Z = \int D\mathbf{a} Da_0 Df^* Df Db^* Db e^{-\int d^2x d\tau \mathcal{L}} \tag{1.4}$$

$$\mathcal{L} = \mathcal{L}_F + \mathcal{L}_B - i\mathbf{a} \cdot (\mathbf{j}_F + \mathbf{j}_B) - ia_0 \cdot (n_f + n_b - 1). \tag{1.5}$$

where $\mathcal{L}_F, \mathcal{L}_B$ are mean field spinon and holon lagrangians. The main difference between this lagrangian and the more familiar QED lagrangians is that here the "kinetic term" for the gauge fields $(F_{\mu\nu}^2)$ is absent, so the apparent gauge coupling is infinitely strong. This enforces the no double occupancy constraint and the constraint that the boson current is cancelled by fermion backflow, as can be seen from integrating out the gauge field first in the functional integral of Eq1.5.

The gauge field acquires dynamics from the fermions and bosons. Integrating out the matter fields (fermions and bosons) in Eq.1.5, we see that the gauge propagator is given (in the Coulomb gauge) by

$$\begin{aligned} \langle a_i(q) a_j(q) \rangle &= (\delta_{ij} - q_i q_j/\mathbf{q}^2)(\Pi_F^\perp + \Pi_B^\perp)^{-1} \\ \langle a_0(q) a_0(q) \rangle &= (\Pi_F^{00} + \Pi_B^{00})^{-1} \end{aligned} \tag{1.6}$$

where $\Pi_{F,B}^\perp$ and $\Pi_{F,B}^{00}$ are transverse and longitudinal polarization functions of fermions and bosons. Therefore, the dynamics of gauge field depends on the mean field ground states and excitations of spinons and holons, and in turn the gauge field affects the dynamics of the matter fields.

This is a complicated problem; a fully self-consistent treatment of effects of the gauge field and the matter fields on one another seems out of question. Moreover, because of the gauge field, the fermions and the bosons are no longer decoupled. In this sense we do not have a full spin-charge separation. Yet, given the physical motivation, we can expect that the basic picture, "quasi-spin-charge separation," survives, in which spinons (holons) would feel the effects of holons (spinons) and themselves by a gauge field interaction. Within this picture, the magnetic properties of the cuprates can be understood mainly in terms of spinons interacting with a gauge field, while the transport properties can be understood mainly in terms of holons interacting with a gauge field (although in such a difficult transport case like the Hall effect, the spinon might play a more visible role, as suggested by Anderson). Of course, in the case of the ARPES where a physical electron is taken out of a system, spinons and holons would appear on a more or less equal footing.

Chapter 2 will discuss the transport and other charge response properties of the cuprates within the holon-gauge field picture. Fermions do not enter the picture directly, but they are believed to dominate the gauge field dynamics. The problem is not an easy one, because there is no "Bose surface" for Bose liquids, and "free bosons" is a poor limit to perturb around (Because of no double occupancy condition, our bosons are hard core bosons). Therefore, most of the results of this chapter are based on numerical techniques, in particular the path integral Monte Carlo method first used by Ceperley and Pollock to study superfluid transition in He4[50, 51].

The results of the Monte Carlo simulation are quite positive. It is found that the too high T_c found in the mean field theory is strongly suppressed by gauge field fluctuations. The conductivity obtained from the Kubo formalism (non-Boltzmann!) has a Drude like peak with width $\sim 2T$, indicating that the low lying gauge fluctuations, which can be interpreted as spin-chirality fluctuations[52], are the anomalous scattering mechanism whose existence was suspected earlier from the linear resistivity data. The density correlation function of the holon-gauge field theory has a fair agreement with the numerics on the full t-J model, supporting the spin-charge separation. The gauge field propagator of chapter 2 has in mind a spinon spectrum with a large Fermi surface, as in the uRVB phase (optimally doped normal state). If the spinon spectrum has a gap (like the spin gap phase), the gauge fluctuation might not scatter the holons as strongly, resulting in a sharper Drude peak and a reduced resistivity at low temperatures. The material of chapter 2 is pretty much self contained. It is actually an outcome of a thoroughly enjoyable collaboration with D.K.K. Lee and P.A. Lee (Phys. Rev. B **55**, 591, ©American Physical Society, 1997).

Chaper 3 discusses the magnetic properties of the cuprates within the spinon–gauge field picture. The magnetism in the cuprates is terribly interesting in its own right, but it is also at the heart of the high T_c debate. The emphasis in this chapter is given to the physics of underdoped cuprates and its connection to the Néel state. As noted earlier, the spin gap phenomena in the underdoped cuprates can be regarded as a confirmation of the spin-charge separation. H. Fukuyama[53], who advocates a similar point of view, nevertheless points out the following problems that the slave boson mean field theories face: a) the origin of very different temperature dependence of the Copper $(T_1T)^{-1}$ and the Knight shift, and b) the effects of the fluctuations (gauge field) on the mean field solutions.

Chapter 3 attempts to address these problems. Here, we look at them as different facets of a single problem: it will be argued that the issue of gauge fluctuation is essential to understanding the antiferromagetic spin dynamics probed by neutron scattering, as well as the question why the Oxygen and Copper site NMR relaxation rates differ markedly. The starting point is the mean field spinons with a gap (not a full gap, but rather a Dirac spectrum). A comparison is made of the spin gap seen in the cuprates with those of the more generally accepted quantum spin liquids (see Sec. 1.2.5), and the notion of deconfined (quasi free) spinons is discussed in this context. The inadequacy of the mean field theory is pointed out, and the bulk of the chapter examines how the inclusion of gauge fluctuation affects the mean field picture. The contents of this chapter represent a long collaboration with P.A. Lee, with additional participation of X.-G. Wen, and include an already published work (Don H. Kim, P. A. Lee, and X. -G. Wen, Phys. Rev. Lett. **79**, 2109, ©American Physical Society 1997).

Before concluding the overview, it is only fair to mention briefly the main limitations and omissions in this thesis. This thesis focuses mostly on the study of "abnormal" normal state properties to advocate that the explanation of these requires a new theory involving a spin-charge separation, more specifically the "RVB mechanism". To some extent the unusual normal state implies that the superconducting state has to be also unusual, and that this seems to be so was indicated early (Sec. 1.1.3). The point of view adopted here (the slave boson theory) naturally gives the maximum superfluid density $\rho_s(T=0) \approx x$. The long-lived quasiparticles inferred from experiments suggest that the finite temperature reduction of superfluid density is mainly due to the d-wave quasiparticle excitations. This gives the correct linear temperature dependence of $\rho_s(0) - \rho_s(T) \approx aT$, but the coefficient a deduced from simple theories does not agree well with the experiments. Eventually, one has to face the question: Can we understand the quasiparticles in the superconducting state via the "recombination" of spinons and holons through some modellable interaction, or is there a more subtle and intrinsically nonperturbative effect at work? (see, for example, Ref.[54]) Opinions on these matters are not discussed here in any depth.

Also, some well known experimental features are not discussed, notably the incommensurate nature of the antiferromagnetic spin excitations. In the case of LSCO, it is well documented: the **Q**-space scan at a fixed frequency shows peaks displaced by $(\delta, 0)$ or $(0, \delta)$ from the antiferromagnetic wavevector. In YBCO the spin excitation is nearly commensurate; some signatures of incommensurate behavior is observed at low temperatures (superconducting phase)[55]. *Static* "stripes" are seen in LSCO around 1/8 doping and its close relatives, and in a number of CMR (manganese) materials. The incommensurability issue has been taken seriously by those who advocate the "stripe" mechanism[56]. At least for me, it is difficult to reconcile the presence of quasiparticles of a convincingly two-dimensional character and a mean free path of $\sim$ 500 lattice spacings with quasi-1d mechanisms like stripes. I shall not attempt to explain the origin of the incommensurate features: currently I am of the opinion that a solid theoretical framework has not been established even for basic issues like the origin of antiferromagnetic correlation in metallic cuprates; the incommensurate features are then certainly beyond the scope. Of course, implicit here is my view that the incommensurability, though interesting, is not a central issue (if it is, then we will need to begin anew from an entirely different starting point). In 12 years since the discovery of the high T_c , a large set of strange (too strange to be a coincidence) features that are believed to be "universal" to all cuprate compounds has been collected; at the moment I am not sure whether (and how) the "stripes" fits into it.

Finally, I should like to emphasize once more, with a little bit of philosophizing, why I have adopted and pursued the slave boson and gauge theory approaches[57]. I believe that a theory (or even *the* theory) of high T_c cuprates *must* begin from the recognition that the superconductivity occurs near an insulating phase with a large charge gap, and that the charge carriers are positive and have a small density $\sim x$. Slave boson theories are quite *natural* in that respect, despite the necessity of dealing with cumbersome gauge fields. In high energy physics, the gauge fields (photons, gluons, etc.) are "god-given" — they are simply there, just like electrons and quarks. The gauge fields in our problem, or those arising in other condensed matter examples, may be regarded to reflect the inadequacy of our description of certain many body systems in the conventional langauge of commutative and anticommutative algebra of bosons and fermions. Fermi liquids are systems describable by Slater determinants (anticommuting algebra of fermions) or systems continuously (or adiabatically) deformable from Slater determinants. It may be difficult to define precisely what is meant by "continuously deformable", but it seems easier to give an example of what is not: for example, the Laughlin 1/3 state, for which a simple description in terms of wave functions exist[58], while an attempt to discuss the problem in terms of a commutative algebra (bosons) has to pay a price — the introduction of a (Chern-Simons) gauge field[59]. The "quantum spin liquid idea" of the high T_c cuprates envisions something quite different from Slater determinants. It may be a pessimistic streak in me to view that it would be (at least) *very difficult* to find a new algebra that realizes a more natural description of the materials than the known algebras. At the moment, the gauge theory approach stands as a partly charming and partly homely theoretical description in terms of currently available notions.

Chapter 2

HOLONS AND TRANSPORT PROPERTIES

[BOSONS, GAUGE FIELDS, AND HIGH T_c CUPRATES] — The following is a work done in collaboration with D. K. K. Lee and P. A. Lee, and has appeared in Physical Review B 55, 591 (1997).

We study the low-temperature behavior of repulsive bosons in a spatially fluctuating gauge field in two dimensions. This is motivated by the gauge theories of the t-J model for the cuprate superconductors, where low-energy charge excitations are described by bosonic degrees of freedom. The internal gauge field of this model suppresses superfluidity in the Bose liquid, even below the Bose degeneracy temperature when there is significant exchange among the bosons. We can study the imaginary-time trajectories of the bosons in the path-integral representation of this model. We see that the boson world-lines retrace themselves in the presence of strong gauge fluctuations, giving rise to interesting dynamics in this degenerate but metallic Bose liquid.

We have studied this metallic state using quantum Monte Carlo techniques. We find that this model does indeed capture some of the long-wavelength charge properties which are common to the cuprate superconductors. This includes a linear temperature dependence of the transport scattering rate $1/\tau_{\rm tr}$, as deduced from a Drude-like optical conductivity from our model. This is consistent with experimental data on the cuprate superconductors near optimal doping. We also find that the density excitations in our model are qualitatively similar to those in the full t-J model, by comparing our results with diagonalization results in the literature. A brief account of this work has already appeared [60].

2.1 Motivation

The normal metallic state of the superconducting cuprates displays many non-Fermi-liquid properties. For instance, the in-plane resistivity of $La_{2-x}Sr_xCuO_4$ has a power-law temperature dependence of the form $\rho \propto T^\alpha$ where α increases from 1 to 1.5 with increasing hole doping [61]. In particular, near optimal doping, the resistivity is linear in temperature up to 1000K. This linear-T dependence is found in many of the cuprate superconductors with similar values of $d\rho/dT$ ($1.2\mu\Omega\text{cm/K} \pm 20\%$) [62]. This should be contrasted with the quadratic temperature dependence of Fermi-liquid theory. Similarly, the transport relaxation rate appears to be universal among optimally-doped compounds: $1/\tau_{\text{tr}} \simeq 2k_{\text{B}}T$ (from a two-component-model analysis of the optical conductivity in YBCO[63], LSCO[64], Bi2212[65], Bi2201[65]). Transport in a magnetic field is also anomalous. The Hall coefficient indicates the existence of hole-like carriers in the doping range where superconductivity occurs. The Hall coefficient R_{H} increases with decreasing temperature, but it remains smaller than the classical value of $1/n_h ec$ for a hole density of n_h for a wide range of temperatures down to the superconducting transition. These compounds also have a small positive magnetoresistance with a temperature dependence[66] different from conventional theory using τ_{tr}.

The transport properties of these compounds appear to have common features in spite of considerable differences in the transition temperature and spin fluctuation properties among these compounds. This indicates that a common mechanism is responsible for the scattering of charge carriers in these materials. One might hope that this scattering mechanism can be understood in terms of a low-energy theory with a minimum number of microscopic parameters. In this paper, we study a Bose liquid in a fluctuating gauge field as a possible candidate for such an effective theory.

The anomalous transport behavior, together with other unusual features such as temperature-dependent magnetic susceptibility and non-Korringa behavior of the nuclear magnetic relaxation time, leads to the conclusion that the metallic state of the cuprates cannot be described in a simple Fermi-liquid scenario. It has been postulated that "spin-charge separation" is responsible for these anomalies[29]. For instance, such a scenario might reconcile the apparent low density and hole-like character of the charge carriers with the observation of a large, electron-like Fermi surface in photoemission. Numerical studies of the t-J model, which is believed to be a low-energy model of the cuprates, also provide some support for spin-charge separation [67, 68, 69], such as different energy scales for the spin and charge excitations, and the suppression of $2k_F$-scattering in the charge spectrum.

A model of spin-charge separation is a gauge theory where neutral spin-half fermions ("spinons") and charge-e bosons ("holons") interact *via* an internal U(1) gauge field [70, 52]. Physically, the transverse part of the gauge field is related to "spin chirality" fluctuations [52]. In this picture, the charge properties of the system should be dominated by the behavior of the holons. We will study the holon subsystem in this paper, treating the spinon subsystem simply as a medium through which the gauge field propagates. To be more precise, we study a model of bosons with on-site repulsion in the presence of a spatially fluctuating magnetic field with short-range correlations. The repulsive interaction is necessary for the stability of the system, which means that one cannot treat this problem perturbatively starting from an ideal Bose gas. Previous studies [71, 72, 73, 74, 75] have implicitly studied the non-degenerate regime of low density or high temperature, whereas the regime relevant to the cuprates is the degenerate regime where the thermal deBroglie wavelength of the bosons is greater than the mean particle spacing. A concern from earlier studies of the gauge model is that degenerate bosons would have strong diamagnetic response to the internal gauge field and hence effectively Bose-condense at a relatively high temperature ($k_B T_{BE} \sim 4\pi n_h t \sim 1000\text{K}$). This would in fact restore Fermi-liquid behavior to the system. We shall show here that gauge fluctuations suppress this diamagnetic response and the bosons remain normal without strong diamagnetism at all finite temperatures. Furthermore, our numerical results indicate that the resistivity of this Bose metallic phase has a linear temperature dependence which is consistent with experiments.

It should be noted that we will work exclusively in the "slave-boson" scheme where the holons are bosonic and the spinons are fermionic. One may also obtain a "slave-fermion" gauge theory where the statistics of the holons and spinons are interchanged. Although these two approaches are equivalent in principle, they do not produce the same results in the approximate treatments. We believe that, at the mean-field level, the slave-boson approach describes the cuprates near optimal doping (for instance, the observation of a large Fermi surface in photoemission), while the slave-fermion mean field theory is more suitable in the underdoped regime. The physics of the underdoped regime, such as the presence of a spin gap, is beyond the scope of this U(1) theory (see Ref.[113]).

Besides the possible relevance to the transport in the cuprate superconductors, the model we consider is of intrinsic theoretical interest. The model is a Bose version of the problem of a quantum particle in a random magnetic flux, which has received considerable attention in recent years. It is also related to frustrated spin systems and vortex glasses. However, since we deal exclusively with annealed averaging in this paper (see later), we cannot draw any direct conclusions about these problems with quenched disorder.

The rest of the paper is organized as follows. In section II, we review the connection between the gauge theory of the t-J model and our boson model. In section III, we discuss the path-integral formalism which provides a convenient framework to visualize physical processes in terms of the imaginary-time paths of the bosons. In section IV, we look at the effects of the gauge field on the world-line geometry of the bosons. We will see that the partition function of the system is dominated by self-retracing world-line configurations. We will also argue that superfluidity is destroyed by the fluctuating gauge field, giving rise to a degenerate Bose metal. In the subsequent sections, we present the results of a quantum Monte Carlo study of this metallic phase. We will discuss the transport properties and the density correlations in this boson model.

2.2 A Boson Gauge Model

In this section, we provide the motivation for studying an effective boson model from the gauge theory of the t-J model, which describes the motion of vacancies in a doped

Mott insulator:

$$\mathcal{H} = -t_0 \sum_{\langle ij \rangle \sigma} (c^\dagger_{i\sigma} c_{j\sigma} + \text{h.c.}) + J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j \tag{2.1}$$

with the constraint of no double occupancy. Experimentally, $J \simeq 1500\text{K}$ and $t_0/J \simeq 3$.

The constraint of no double occupancy allows us to write the creation of a physical hole in terms of the creation of a charged hard-core boson (holon) and the annihilation of a spin-half fermion (spinon): $c_{i\sigma} = f_{i\sigma} b^\dagger_i$. In terms of these slave bosons and fermions, the Hamiltonian of the t-J model can be written as:

$$\begin{aligned} \mathcal{H} &= -t_0 \sum_{\langle ij \rangle \sigma} (f^\dagger_{i\sigma} b_i b^\dagger_j f_{j\sigma} + \text{h.c.}) + J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j \\ &+ \sum_i a_{0i} (f^\dagger_{i\sigma} f_{i\sigma} + b^\dagger_i b_i - 1) \end{aligned} \tag{2.2}$$

where $\mathbf{S}_i = f^\dagger_{i\alpha} \sigma_{\alpha\beta} f_{i\beta}$, The a_{0i}-field is a Lagrange multiplier enforcing the local occupancy constraint, and acts as a fluctuating scalar potential for the spinons and holons.

Among the mean field theories proposed to decouple the quartic terms in Eq. (2.2), a candidate for the normal state near optimal doping is the the uniform resonating-valence-bond (RVB) ansatz: $\sum_\sigma \langle f^\dagger_{i\sigma} f_{j\sigma} \rangle = \xi e^{i a_{ij}}$. This incorporates short-range antiferromagnetic correlations without any long-range Néel order. The Lagrangian of this RVB phase can be written as:

$$\begin{aligned} \mathcal{L} &= \sum_{i,\sigma} f^*_{i\sigma} (\partial_\tau - \mu_{\text{F}} + a_{0i}) f_{i\sigma} + \sum_i b^*_i (\partial_\tau - \mu_{\text{B}} + a_{0i}) b_i \\ &\quad - \frac{J}{2} \xi \sum_{\langle ij \rangle} (e^{i a_{ij}} f^*_{i\sigma} f_{j\sigma} + \text{h.c.}) \\ &\quad - t_0 \xi \sum_{\langle ij \rangle} (e^{i a_{ij}} b^*_i b_j + \text{h.c.}) \end{aligned} \tag{2.3}$$

The vector potential a_{ij} arises from the fluctuations in the phase of the RVB order parameter. Longitudinal fluctuations of the gauge field a_{ij} do not affect the Lagrangian due to an internal U(1) gauge symmetry:

$$\begin{aligned} f_i &\longrightarrow f_i e^{i\theta_i} \\ b_i &\longrightarrow b_i e^{i\theta_i} \\ a_{ij} &\longrightarrow a_{ij} - \theta_i + \theta_j \end{aligned} \tag{2.4}$$

We will therefore work in a fixed gauge, such as the Coulomb gauge, and consider only the fluctuations in the transverse part of the gauge field a_{ij}. In other words, we will consider only fluctuations in the internal magnetic and electric fields which are gauge-invariant quantities.

Since we are interested in the charge degrees of freedom, we wish to consider an effective theory with bosons only, and regard the spinon fluid as a medium through which the gauge field propagates. The gauge field has no dynamics *in vacuo*. The response of the spinon fluid to the gauge field is responsible for the dynamics of the gauge field as seen by the holons. More specifically, we can obtain the Gaussian fluctuations of the a-fields by treating the spinon response in the random-phase approximation. The effective gauge-field propagator is:

$$\begin{aligned} \mathcal{S}_{\mathrm{G}} &= \frac{1}{2\beta L^2} \sum_{\mathbf{k},\omega_n} \Pi^{00}(\mathbf{k},\omega_n) a_0^*(\mathbf{k},\omega_n) a_0(\mathbf{k},\omega_n) \\ &+ \frac{1}{2\beta L^2} \sum_{\mathbf{k},\omega_n} \Pi^{\perp}(\mathbf{k},\omega_n) a_\perp^*(\mathbf{k},\omega_n) a_\perp(\mathbf{k},\omega_n), \end{aligned} \tag{2.5}$$

where $\beta = 1/T$, $\omega_n = 2\pi n T$, L is the linear size of the system, and $a_\perp$ is the transverse part of the gauge field. (We use units where distance is measured in terms of the lattice spacing and $k_{\mathrm{B}} = \hbar = e = 1$.) Here, for small k and ω_n, $\Pi^{00} \simeq \rho_{\mathrm{F}}$, the spinon density of states at the Fermi level. This describes the Thomas-Fermi screening of internal electric fields by the fermions. The effective interaction mediated by the screened a_0-field is a repulsion between the bosons (of range $\propto \rho_{\mathrm{F}}^{-1/2}$), consistent with the original hard-core requirement for the bosons. We will model this with an on-site repulsion energy, U. On the other hand, the magnetic fields due to fluctuations in a_{ij} are not effectively screened out by the fermions[76]. The gauge-field fluctuations as experienced by the holons are therefore strong. More specifically, the Gaussian fluctuations have the correlation function $D(\mathbf{k},\omega_n) = \langle a_\perp^*(\mathbf{k},\omega_n) a_\perp(\mathbf{k},\omega_n)\rangle$, given (in the continuum limit) by:

$$D(\mathbf{k},\omega_n) = \frac{1}{\Pi_\perp(\mathbf{k},\omega_n)} = \frac{1}{\gamma|\omega_n|/k + \chi k^2} \tag{2.6}$$

where χ is the orbital susceptibility of the spinon fluid and γ is a Landau damping coefficient. These gauge-field fluctuations cause profuse forward scattering of the bosons. We believe that this is the dominant scattering mechanism in this problem. Since it is overdamped at long wavelengths with a relaxation rate which diverges as $1/k^3$, we will ignore the slow relaxation and work in a "quasistatic" limit for the gauge fields:

$$D(\mathbf{k},\omega_n) \to D(\mathbf{k},\omega_n = 0)\, \delta_{n,0} = \delta_{n,0}/\chi k^2. \tag{2.7}$$

(On a square lattice, k^2 is replaced by $1 - (\cos k_x - \cos k_y)/2$.) This quasistatic approximation is justified when the gauge field relaxes on a time scale longer than $1/T$. Since the typical scattering wavevector of interest is the inverse deBroglie wavelength of the bosons, the relevant relaxation time scales as $1/k^3 \sim 1/T^{3/2}$. One might therefore expect[77] that this approximation is valid at a sufficiently low temperature.

One might object that arguments above are based on a weak-coupling theory of the response of the spinons to the gauge fields. However, we believe that the essential features remain correct in general, namely a separation of times scales between the relaxation of the gauge fields and the boson dynamics, as well as the magnitude of the gauge fluctuations being controlled by the spinon diamagnetic susceptibility χ.

The gauge-field correlator (2.7) corresponds to a spatially uncorrelated flux distribution with the correlation function:

$$\langle \Phi_{\mathbf{r}} \Phi_{\mathbf{r}'} \rangle = \frac{T}{\chi} \delta_{\mathbf{r},\mathbf{r}'} \tag{2.8}$$

where $\Phi_{\mathbf{r}} = (\Phi_0/2\pi) \sum_{\Box} a_{ij}$ (oriented sum around the links of plaquette $\mathbf{r}$) is the flux through plaquette $\mathbf{r}$. ($\Phi_0 = hc/e$ is the flux quantum.) Since we are treating the thermodynamics for the gauge field classically, we have a thermal factor of T in Eq. (2.8) for the flux variance $\langle \Phi^2 \rangle$. Given that the fermion orbital susceptibility is roughly constant at low temperatures, we might expect the flux variance to have a linear temperature dependence. However, a lattice calculation by Hlubina *et al.*[78] has indicated that the Gaussian fluctuations are sufficiently strong that the flux through a plaquette is of the order of the flux quantum Φ_0: $\langle \Phi^2 \rangle^{1/2} \geq 0.5\Phi_0$ down to a temperature of $0.4J$. Since the experimental superconducting T_c is of the order of $0.1J$, we expect that this regime of strong random flux is relevant to the normal state of the cuprates until one approaches the superconducting transition. In this regime, the precise value of $\langle \Phi^2 \rangle$ should not affect the behavior of the bosons, and we will focus on a *large and temperature-independent* flux variance when we study the transport and correlation functions of our boson system.

Another factor leading to the reduction of the flux variance at low temperature is one that has not been discussed so far, namely that the magnitude of the gauge field should also be affected by the diamagnetic response of the holons as well as the spinons, *i.e.*, $\langle \Phi^2 \rangle = T/(\chi_{\text{spinon}}(T) + \chi_{\text{holon}}(T))$. The holon contribution dominates near an instability to Bose condensation where χ_{holon} diverges and the bosons develop a Meissner response to expel the gauge field from the system altogether. However, we will see in this paper that Bose condensation and the holon diamagnetism are strongly suppressed even below the boson degeneracy temperature. Therefore, in a wide range of temperatures above the superconducting T_c, we are justified in neglecting this feedback effect of the holons on the magnitude of the gauge field fluctuations.

We can now define more precisely the effective model which we study in the rest of the paper. It is a model of lattice bosons interacting with a quasistatic gauge field, described by effective action $S = S_{\text{B}} + S_{\text{G}}$:

$$S_{\text{B}} = \int_0^{\beta} \left(\sum_i b_i^* (\partial_\tau - \mu_{\text{B}}) b_i - H_{\text{B}}(\tau) \right) d\tau$$

$$S_{\text{G}} = \frac{1}{2\beta L^2} \sum_{\mathbf{k}} D^{-1}(\mathbf{k},0) |a_{\perp}(\mathbf{k},0)|^2 \tag{2.9}$$

$$= \sum_{\mathbf{r}} \frac{\Phi_{\mathbf{r}}^2}{2\langle \Phi^2 \rangle} \tag{2.10}$$

with the boson Hamiltonian

$$H_{\text{B}} = -t \sum_{\langle ij \rangle} (e^{i a_{ij}} b_i^\dagger b_j + \text{h.c.}) + \frac{U}{2} \sum_i n_i (n_i - 1), \tag{2.11}$$

where $t = t_0\xi \sim t_0$, L is the linear size in units of lattice spacing and $U \gg t$. Note that, on performing the average over the gauge field, we average over static configurations only, *i.e.*, $\mathbf{a}(\mathbf{k}, \omega_n \neq 0) = 0$.

We cannot say that we have rigorously derived above effective action from the slave-boson mean field theory of the t-J model. Many approximations have been introduced to obtain this simple model with few adjustable parameters. For example, we have neglected the temperature dependence of the RVB order parameter ξ and also the gauge-field correlations of higher order[79]. We take the point of view that we are studying a "minimal" low-energy theory which hopefully captures many of the generic features of more complicated models.

2.3 Path Integral Representation

It is convenient to study our boson model in a first-quantized formulation. The partition function Z for a system with N bosons in the canonical ensemble can be written in terms of a Feynman path integral[80] over the boson trajectories $\{\mathbf{x}_\alpha(\tau)\}$ $(\alpha = 1, \ldots, N)$:

$$\begin{aligned} Z &= \frac{1}{N!}\sum_P \int^{\mathbf{x}(0)=P(\mathbf{x}(\beta))} \mathcal{D}[\mathbf{x}_1, \ldots, \mathbf{x}_N] \times \\ & \int \mathcal{D}\mathbf{a}\, \delta(\nabla \cdot \mathbf{a}) e^{-S_\mathrm{G}(\mathbf{a}) - i\sum_\alpha \int_0^\beta \mathbf{a}\cdot\dot{\mathbf{x}}_\alpha d\tau - S_\mathrm{B}^0(\{\mathbf{x}\})} \end{aligned} \tag{2.12}$$

where S_B^0 is the action for bosons in the absence of magnetic fields:

$$S_\mathrm{B}^0 = \int_0^\beta d\tau \left(\sum_i b_i^* \partial_\tau b_i - H_\mathrm{B}^0 \right) \tag{2.13}$$

where H_B^0 is given by (2.11) with $a_{ij} = 0$. In this section, we will discuss the model in the continuum limit for notational convenience. In the continuum, one would have:

$$S_\mathrm{B}^0 = \int_0^\beta d\tau \left[\sum_{\alpha=1}^{N} \frac{m}{2}\dot{\mathbf{x}}_\alpha^2 + \sum_{\alpha>\gamma} U\delta(\mathbf{x}_\alpha(\tau) - \mathbf{x}_\gamma(\tau)) \right]. \tag{2.14}$$

Particle identity is taken into account by performing the path integral over all trajectories where the set of final boson coordinates at $\{\mathbf{x}_1(\beta), \ldots, \mathbf{x}_N(\beta)\}$ is some permutation of the initial boson coordinates $\{\mathbf{x}_1(0), \ldots, \mathbf{x}_N(0)\}$. Any such permutations can be broken down to cycles. Each cycle forms a closed loop when the imaginary-time trajectories (world lines) of a many-boson configuration are projected onto real space. At high temperatures, cycles of length 1 dominate the partition function and the system is in a non-degenerate classical regime. At temperatures below the degeneracy temperature of the bosons, particles can travel large distances in the imaginary time, forming many ring exchanges (see Fig. 2-1).

In this formulation, we may integrate out the Gaussian fluctuations of the gauge field in (2.12). Thus, we arrive at a boson-only effective theory which we study

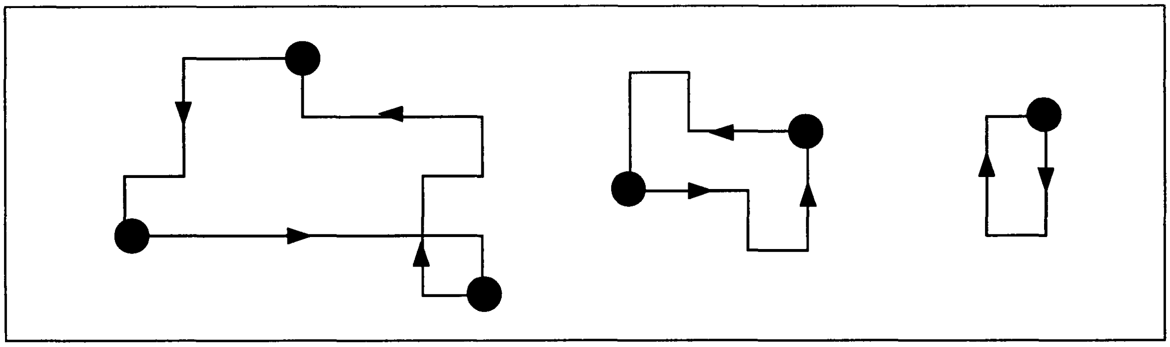

Figure 2-1: A schematic configuration for 6 bosons after projecting the imaginary-time paths onto the xy-plane. There are a total of 3 cycles: 1 cycle of one particle, 1 cycle of two particles, and 1 cycle of three particles. Solid circles denote particle positions at $\tau = 0$ and β.

numerically in this work. The system is described by the partition function $Z = \int \mathcal{D}\mathbf{x} e^{-S_{\text{eff}}}$ where the effective action is given by:

$$S_{\text{eff}} = S_{\text{B}}^{0} + S_2 \tag{2.15}$$

with

$$S_2 = \tfrac{1}{2} \sum_{\alpha\alpha'} \int_0^\beta \int_0^\beta \tilde{D}(\mathbf{x}_\alpha(\tau) - \mathbf{x}_{\alpha'}(\tau'))\, \dot{\mathbf{x}}_\alpha \cdot \dot{\mathbf{x}}_{\alpha'}\, d\tau\, d\tau'. \tag{2.16}$$

where $\tilde{D}(\mathbf{x}) = (1/\beta L^2) \sum_{\mathbf{k} \neq 0} D(\mathbf{k}, 0) e^{-i\mathbf{k}\cdot\mathbf{x}}$. Note that the $\mathbf{k} = 0$ contribution has been excluded in the sum over $\mathbf{k}$, corresponding to a gauge choice where the $\mathbf{k} = 0$ part of $\mathbf{a}$ is zero. This is one way to fix the remaining degree of gauge freedom which is not determined by the condition of $\nabla \cdot \mathbf{a} = 0$. If we consider a system with periodic boundary conditions in space, another scheme would be to fix the line integral of the gauge field around a specified path which wraps around the boundary. However, the latter scheme is inconvenient for our purposes because it breaks translational invariance explicitly.

The current interaction $D(\mathbf{x})$ mediated by the gauge field is logarithmic at large distances, and is attractive between opposite currents. Due to the quasistatic nature of the gauge fields, the interaction is also infinitely retarded in time. We will see in the next section that this encourages world lines to retrace themselves, with important consequences for the boson dynamics.

Before proceeding to discuss the physical consequences of the current interaction S_2, some remarks about our averaging procedure for the gauge fields are in order. We have performed an "annealed" average over the gauge fields, rather than a "quenched" average. Annealed averaging is necessary in our case because our gauge field $\mathbf{a}$ is an internal thermodynamic variable. Formally, we evaluate observables $\langle \mathcal{O} \rangle$ as:

$$\frac{1}{Z} \int \mathcal{D}\mathbf{x}\, \mathcal{O} e^{-S_{\text{eff}}} = \frac{\int \mathcal{D}\mathbf{x} \mathcal{D}\mathbf{a}\, P[\mathbf{a}]\, \mathcal{O}\, e^{-S_{\text{B}}^{0} - i \int \mathbf{a} \cdot d\mathbf{x}}}{\int \mathcal{D}\mathbf{x} \mathcal{D}\mathbf{a}\, P[\mathbf{a}]\, e^{-S_{\text{B}}^{0} - i \int \mathbf{a} \cdot d\mathbf{x}}} \tag{2.17}$$

where $P[\mathbf{a}] = \mathcal{N}^{-1}\delta(\nabla \cdot \mathbf{a})e^{-S_G[\mathbf{a}]}$ is the probability distribution for the gauge field, and $\mathcal{N}$ is a suitable normalization factor. This is different from quenched averaging which would be appropriate if we dealt with a system with frozen impurities, such as a vortex glass. Quenched averaging requires the evaluation of:

$$\int \mathcal{D}\mathbf{a}\, P[\mathbf{a}] \left[\frac{\int \mathcal{D}\mathbf{x}\, \mathcal{O}\, e^{-S_{\mathrm{B}}^0 - i\int \mathbf{a}\cdot d\mathbf{x}}}{\int \mathcal{D}\mathbf{x}\, e^{-S_{\mathrm{B}}^0 - i\int \mathbf{a}\cdot d\mathbf{x}}} \right]. \tag{2.18}$$

The differences between quenched and annealed averaging from the point of view of perturbation (diagrammatic) theory has been addressed elsewhere[74, 166].

From the point of view of the path integral Monte Carlo method, our ability to perform the annealed averaging means that we would not have to perform extensive averages over different frozen realizations of the random flux. Moreover, note that the effective action (2.15) is manifestly real, and so we avoid the sign problem which occurs numerically when performing a quenched average over the gauge fields. We have studied boson densities between $n_b = 1/4$ and $1/6$. We choose an on-site interaction strength $U \geq 4t$. We follow the Monte Carlo methods of Ceperley and Pollock[50] and Trivedi[82]. Each Monte Carlo step involves the reconstruction of the world lines, $\{\mathbf{x}_\alpha(\tau)\}$, for all N particles using the ideal boson propagator in a short interval in imaginary time. The on-site interaction and the current interaction S_2 are taken into account using Metropolis tests. To ensure quantum exchange, we may insist that each accepted configuration differs from the previous one by a pair exchange. This can be incorporated, without loss of detailed balance, as a Metropolis test. We refer readers to the original references[50, 82] for further details. (In evaluating the gauge field contribution S_2, we have also made use of a geometrical interpretation of S_2 which we discuss in the next section.) In the discretization of the imaginary time, we have used a small $\Delta\tau = \beta/M \leq 0.1/t$, so as to minimize the systematic error and to allow the reliable use of maximum entropy techniques to perform analytic continuation on our imaginary-time data to obtain the dynamical quantities of interest. This sets the lowest accessible temperature to $T \sim 0.1t$ for lattice sizes considered here. For studies on dynamic response to be discussed later, we have restricted ourselves to lattices of sizes up to 6×6, due to the need to obtain imaginary time correlation functions to a high accuracy. For the calculation of static properties, we have studied lattices as large as 10×10.

To summarize, we have obtained an effective theory of bosons with current interactions which are long-ranged in space and time. This model can be studied using path integral Monte Carlo methods. In the next section, we will discuss how these interactions affect the geometry of the boson world lines and hence the physical properties of the system.

2.4 Effect of Gauge fields on World Line Geometry

2.4.1 "Brinkman-Rice bosons"

In this section, we will discuss how the current interaction S_2 mediated by the gauge field affects world-line geometry. On the infinite plane, there is a simple geometrical interpretation of this interaction in terms of the winding numbers of the boson world lines. The winding number $w_\mathbf{r}$ around a plaquette $\mathbf{r}$ is the number of times the imaginary-time world lines of all the bosons wind around the plaquette. Consider the partition function before averaging over the gauge field. The effect of the gauge field enters the partition function as the phase factor $\exp[-i\sum_\alpha \int \mathbf{a}\cdot d\mathbf{x}_\alpha]$ in Eq. (2.12) over the gauge field. This phase factor can be written in terms of $w_\mathbf{r}$: $\sum_\alpha \int \mathbf{a}\cdot d\mathbf{x}_\alpha = \sum_\mathbf{r} w_\mathbf{r}\Phi_\mathbf{r}$. We can now perform the average directly over the Gaussian flux distribution (2.10), instead of the gauge field distribution (2.9). We will be working with periodic boundary conditions (*i.e.*, on a torus). This will be well-defined if we impose a constraint of zero total flux through the system. On averaging, the phase factor becomes:

$$\begin{aligned}
&\int d\lambda \int \prod_\mathbf{r} d\Phi_\mathbf{r}\; e^{-\sum_\mathbf{r}\frac{\Phi_\mathbf{r}^2}{2\langle\Phi^2\rangle}-\frac{2\pi i}{\Phi_0}\sum_\mathbf{r} w_\mathbf{r}\Phi_\mathbf{r}+i\lambda\sum_\mathbf{r}\Phi_\mathbf{r}} \\
&\propto \int d\lambda e^{-2\pi^2\frac{\langle\Phi^2\rangle}{\Phi_0^2}\sum_\mathbf{r}(w_\mathbf{r}+\lambda)^2} \quad \propto \quad e^{-S_2} \\
S_2 &= 2\pi^2\frac{\langle\Phi^2\rangle}{\Phi_0^2}\left[\sum_\mathbf{r} w_\mathbf{r}^2-\frac{1}{L^2}\left(\sum_\mathbf{r} w_\mathbf{r}\right)^2\right]
\end{aligned} \tag{2.19}$$

Thus we see that the action cost due to the current interaction is proportional to a geometrical property of the world lines, similar to an unoriented area, which has been termed the "Amperean area" [72]:

$$\mathcal{A}_a = \left[\sum_\mathbf{r} w_\mathbf{r}^2-\frac{1}{L^2}\left(\sum_\mathbf{r} w_\mathbf{r}\right)^2\right] \tag{2.20}$$

This geometrical interpretation of S_2 is particularly useful in the numerical evaluation of this quantity.

If we are working with periodic boundary conditions, the geometrical definition of $w_\mathbf{r}$ given above will not work because there is an ambiguity in identifying which plaquettes are inside or outside a loop on a torus. Nevertheless, we can still use the above analysis for paths which do not wrap around the boundaries. (We will discuss wrapping paths in the next section.) The only modification is that we need a definition of the winding numbers which preserves Stokes' theorem: $\oint \mathbf{a}(\mathbf{x})\cdot d\mathbf{x} = \sum_\mathbf{r} w_\mathbf{r}\Phi_\mathbf{r}$. In the case of zero total flux, a suitable definition is: $w_\mathbf{r} = \tilde{\Phi}^{-1}\oint[\mathbf{a}_\mathbf{r}^0(\mathbf{x})-\mathbf{a}_\mathbf{R}^0(\mathbf{x})]\cdot d\mathbf{x}$, where $\mathbf{a}_\mathbf{r}^0(\mathbf{x})$ is the vector potential at $\mathbf{x}$ due to a test flux $\tilde{\Phi}$ placed at plaquette $\mathbf{r}$, and $\mathbf{R}$ is an arbitrary reference plaquette. Geometrically, this picks $\mathbf{R}$ to be on the "outside" of any loop on the torus. The Amperean area as defined above is independent of the choice of this plaquette, because different choices amount to global changes in the winding numbers (*e.g.*, $w_{\mathbf{r}} \to w_{\mathbf{r}} + 1$) and the above definition is invariant under such changes.

The effect of the gauge field on the particles is now clear. The action S_2 suppresses world-line loops with large winding numbers. Indeed, since S_2 is non-negative, it excludes all configurations with finite Amperean area in the limit of infinite $\langle \Phi^2 \rangle$. This suppression can be related to the original problem of holes moving in a spin liquid with a slowly varying spin quantization axis. A hole moving in a loop comes back with a random phase due to the locally fluctuating spin chiralities of the spin background[52]. The random phase can be interpreted as arising from a fictitious random flux. World-line loops that enclose large areas are strongly suppressed when averaged over random flux distribution due to the destructive interference of the random phases. Therefore, we expect that, in the presence of strong random flux, the dominant contribution to the partition function comes from a special kind of paths that do not "see" the random flux, *i.e.*, paths where $\int \mathbf{a} \cdot d\mathbf{x} = 0$. These are "retracing paths" where each traversal of a link on the lattice is retraced in the opposite direction at some point in time [83, 84], and such paths have zero Amperean area.

A similar picture of retracing paths has been studied by Brinkman and Rice[83] who studied a single hole in a Mott insulator where the spins are treated classically. Indeed, studies of a single particle in a strong random flux have yielded a density of states nearly identical to that of the Brinkman-Rice problem[85, 86, 87]. The Brinkman-Rice model gives a linear-T resistivity at high temperatures ($T > t$) but a constant scattering rate of order t. Although we might expect this to be applicable to our model far above the degeneracy temperature of the bosons, this behavior does not extend down to the degenerate regime relevant to the present problem.

At boson densities of interest here and at low temperatures, Bose statistics and particle exchange are important; they can give rise to behavior different from the single-particle Brinkman-Rice result. We shall look at the effect of the gauge field on the quantum exchanges among bosons more carefully in section IV-c. For now, we point out that, even in the presence of strong gauge-field fluctuations, the bosonic nature of the particles cannot be ignored because the particles can form long exchange cycles that retraces themselves so that an individual boson does not have to retrace its own path. This is an important consideration at low temperatures where the imaginary-time paths are long allowing for a strong degree of particle exchange. Although the system can be highly degenerate at low temperatures, we shall now argue that these "Brinkman-Rice bosons" remain normal at all finite temperatures, due to interactions with the fluctuating gauge fields.

2.4.2 destruction of superfluidity

We will now discuss the effect of the gauge field fluctuations on the superfluidity of the Bose system. We will see, as in the previous section, that this can be understood in terms of the geometrical properties of the boson world lines.

A neutral Bose system with short-range interaction in two dimensions is a superfluid below Kosterlitz-Thouless temperature $T_{\rm KT}$. The onset of superfluidity at $T_{\rm KT}$ is caused by the binding of vortex-antivortex pairs in the Bose fluid so that vortex motion does not cause phase slips across the system. An essential ingredient of the existence of the superfluid phase is a long-range logarithmic attraction between the vortices and the antivortices. A single vortex costs infinite energy in an infinite system $E_v = (\pi\rho_s/m)\log(L/a)$ where a is a short distance cutoff ($\sim$ vortex core radius) and ρ_s is the superfluid density. Therefore single vortices cannot exist at low temperatures. Nevertheless, the proliferation of free vortices is possible above $T_{\rm KT}$ because this provides a gain in entropy which also scales as $\log L$. However, in a charged Bose system, screening currents causes the vortex interaction to be short-ranged. In our problem, the vortex interaction becomes exponentially weak at distances beyond $\lambda_P = [T/2\rho_s t\langle\Phi^2\rangle]^{\frac{1}{2}}\Phi_0$, which can be interpreted as a penetration depth of the Bose fluid. Now, the creation of a single vortex costs a finite amount of energy[88, 89] $E_v = (\pi\rho_s/m)\log(\lambda_P/a)$. This no longer compensates the entropic gain from vortex-antivortex unbinding, and so we do not expect to see a sharp phase transition of the Kosterlitz-Thouless type at finite temperatures.

One might still expect that there is a crossover temperature scale below which the vortex density will be sufficiently low that the Bose system would have strong diamagnetic response. A rough estimate of this temperature scale using a Boltzmann weight for the vortex density gives a large value for this crossover temperature[89]. However, we will see later that, in the presence of strong gauge fluctuations, the diamagnetic response of the bosons remains small.

To understand the suppression of superfluidity specifically in our model, we turn to the path-integral formulation of the problem with periodic boundary conditions (*i.e.*, on a torus). Ceperley and Pollock[50] have shown that superfluidity is associated with the existence of long world-line cycles which wrap around the torus. The superfluid density is given by

$$n_s = \frac{\langle \mathbf{W}^2 \rangle}{4\beta t} \tag{2.21}$$

where W_x (W_y) is the number of times the boson world lines wrap around the torus in the x (y) direction. In other words, $\mathbf{W} = \sum_{\alpha=1}^{N}\int_0^\beta d\tau\, \dot{\mathbf{x}}_\alpha/L$. In the presence of gauge fields, superfluidity is destroyed by the same mechanism that causes the Brinkman-Rice behavior: wrapping configurations pick up random phases, and should be suppressed by destructive interference on averaging over the gauge field. The number of plaquettes whose random fluxes contribute to the phase picked up by a wrapping path should increase with increasing system size. For a large enough system, one might expect this phase to be totally random. We therefore expect this suppression to be very strong. For instance, one can evaluate S_2 for a straight-line path which wraps around the torus in the y-direction. To do so, we use Eq. (2.16) instead of Eq. (2.19) because the geometrical interpretation of S_2 in terms of winding numbers is not applicable for wrapping paths. We find that such a path gives

$$S_2 = \frac{W_y^2}{2\beta}\sum_{k_x\neq 0, k_y=0} D(\mathbf{k},0) \simeq 2\pi^2 W_y^2 L^2 \frac{\langle\Phi^2\rangle}{\Phi_0^2}. \tag{2.22}$$

To compute S_2 for a more general path with wrapping W_y, one can break it down into a wrapping path with the same wrapping number and a non-wrapping path (Fig. 2-2). (S_2 will consist of the contributions of the wrapping paths and non-wrapping paths separately, as well as a cross-term between the two paths.) We argue that S_2 diverges for all wrapping paths in the thermodynamic limit, and so superfluidity is destroyed at all finite temperatures.

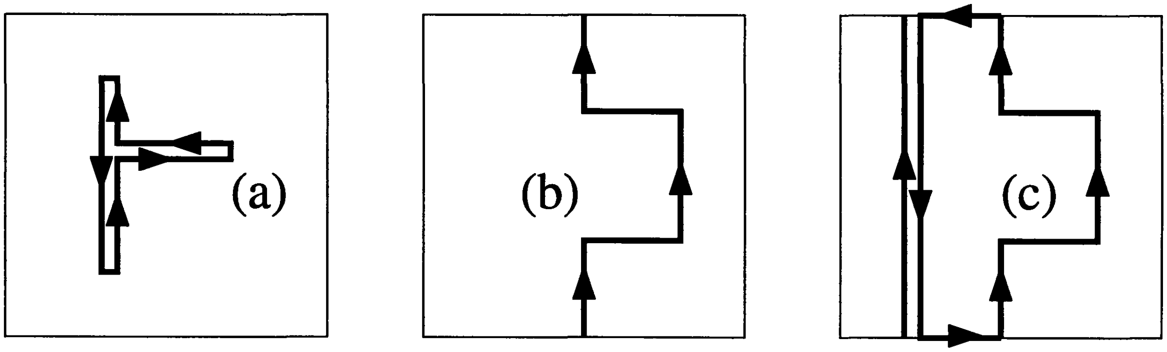

Figure 2-2: (a) Projection of a world line onto the xy plane shows a retracing path. (b) A wrapping path. (c) Decomposition of (b) into a reference path and a non-wrapping path.

It should be noted that we are studying a gauge model where the uniform part of **a** is set to zero. We may alternatively work with a model without this gauge fixing. With periodic boundary conditions, this model allows an arbitrary Aharonov-Bohm (AB) flux through the torus. This flux is related to the phase of the product of RVB parameters (ξ_{ij}) along a (Wilson) loop which wraps around the torus. If we average over this AB flux assuming a uniform distribution, we would find that all wrapping paths are strictly prohibited and $n_s = 0$ at *all* temperatures even for samples of finite size. We will not impose such a drastic condition on the wrapping paths in this work.

We can also ask whether long-range order exists in the Green's function for the bosons. The Green's function itself $\langle b^\dagger(\mathbf{r})b(0)\rangle$ is not gauge invariant, and would vanish on averaging over different gauges. However, we can study the Green's function in a fixed gauge, for example the transverse gauge $\nabla \cdot \mathbf{a} = 0$. In fact, one can write a gauge-invariant analogue correlation function which coincides with the Green's function in the transverse gauge[90]:

$$\begin{aligned} G(\mathbf{r}) &= \langle b^\dagger(\mathbf{r})b(0)\rangle_{\nabla\cdot\mathbf{a}=0} \\ &= \left\langle b^\dagger(\mathbf{r})b(0)e^{-i\int d^2\mathbf{r}' f(\mathbf{r}')\nabla\cdot\mathbf{a}(\mathbf{r}')}\right\rangle \end{aligned} \tag{2.23}$$

where $\nabla^2_{\mathbf{r}'} f(\mathbf{r}') = \delta(\mathbf{r}' - \mathbf{r}) - \delta(\mathbf{r}')$. In the path-integral representation, the evaluation of G involves a world line originating at site $\mathbf{r}$ and a world line terminating at site 0 at the same point in imaginary time. Note that this quantity coincides with the Green's function in the Coulomb gauge. Consider now the phase factor $\sum_\alpha \int \mathbf{a} \cdot d\mathbf{x}_\alpha$ picked up by the world lines $\{\mathbf{x}_\alpha\}$ in the evaluation of the Green's function $G(\mathbf{r})$ in this gauge. The random flux $\Phi_{\mathbf{R}}$ at a distant plaquette $\mathbf{R}$ (with $R \gg r$) has a contribution of magnitude $\Phi_{\mathbf{R}}/R$ to the vector potential at a point Q near 0 and $\mathbf{r}$. The sum of the contributions to the vector potential at Q due to the random fluxes at radius R from the origin is a random vector with a mean squared magnitude of $2\pi R \times (\langle\Phi^2\rangle/R^2) \sim \langle\Phi^2\rangle/R$. This analysis is valid for all fluxes which are at a distance $R > r$. Integrating over the contribution of such fluxes, the variance of the magnitude of the vector potential at Q scales as $\langle\Phi^2\rangle \log(L/r)$. Summing over all Q near 0 and $\mathbf{r}$, we obtain a random phase with a divergent variance: $\langle\Phi^2\rangle r^2 \log(L/r)$. Thus, averaging over the distant fluxes for these sites, one obtains a suppression factor of $\exp[-\langle\Phi^2\rangle r^2 g(r/L)]$ where $g(x) \sim \log 1/x$ for small x. This can be interpreted as a binding potential for the end points of $G(\mathbf{r})$. We therefore do not find long-range order in this quantity because of the destructive interference of the random phases due to distant fluxes.

We will now present numerical evidence for the suppression of superfluidity below the degeneracy temperature T_{D0} of the system. A measure of the degeneracy temperature is the Kosterlitz-Thouless temperature of the system at zero flux. We make use of the observation of Ceperley and Pollock[51] that the the probability of bosons to participate in the multi-particle exchange is about $\frac{1}{2}$ at Kosterlitz-Thouless transition. In other words, the probability P_1 that a boson is in an exchange cycle of length 1 is about $\frac{1}{2}$. We estimate that, for our lattice bosons with density $n_b = 0.25$ and on-site interaction $U = 4t$, the degeneracy temperature $T_{\mathrm{D0}} = 1.1t$. (For strong on-site repulsion, T_{D0} is not particularly sensitive to the value of U, *e.g.*, $T_{\mathrm{D0}} = 0.9t$ for $U = 16t$.)

We have measured, using Eq. (2.21), the superfluid fraction n_s/n_b at $T = t/6$ with $U = 4t$ for a range of flux variances and for systems up to 8×8 in size (Fig. 2-3). We see that the superfluid fraction decreases with increasing system size. In fact, the superfluid fraction as a function of $\langle\Phi^2\rangle L$ collapses onto a single curve (Fig. (2-3 inset), indicating that $n_s(L, \beta, \langle\Phi^2\rangle) = f(L\langle\Phi^2\rangle, \beta)$. Since $f(x, \beta) \to 0$ as $x \to \infty$, we see that an arbitrarily small random magnetic flux would destroy superfluidity in the thermodynamic limit. In the language of the renormalization group, this shows that the scattering by gauge fields is a relevant perturbation at finite temperature.

2.4.3 world line geometry in the normal phase

Having established that our system remains normal at low temperatures, we will now examine the geometry of the world lines in this normal phase in the presence of strong gauge fluctuations. In particular, we will look at the effect of the gauge fields on quantum exchange and imaginary-time diffusion. These are mutually related: imaginary-time diffusion over large distances aids quantum exchange among particles and quantum exchange facilitates imaginary-time diffusion. For example, in a dissipative model of bosons coupled to external heat bath, a slow logarithmic imaginary-time diffusion is expected to suppress quantum exchange very strongly, resulting in an incoherent liquid even at zero temperature[71, 91]. In our case, the bosons are elastically scattered by the gauge fields. We find that the gauge fields have less dramatic effects on quantum exchange and imaginary-time diffusion.

We have shown that the world lines retrace themselves in the presence of random

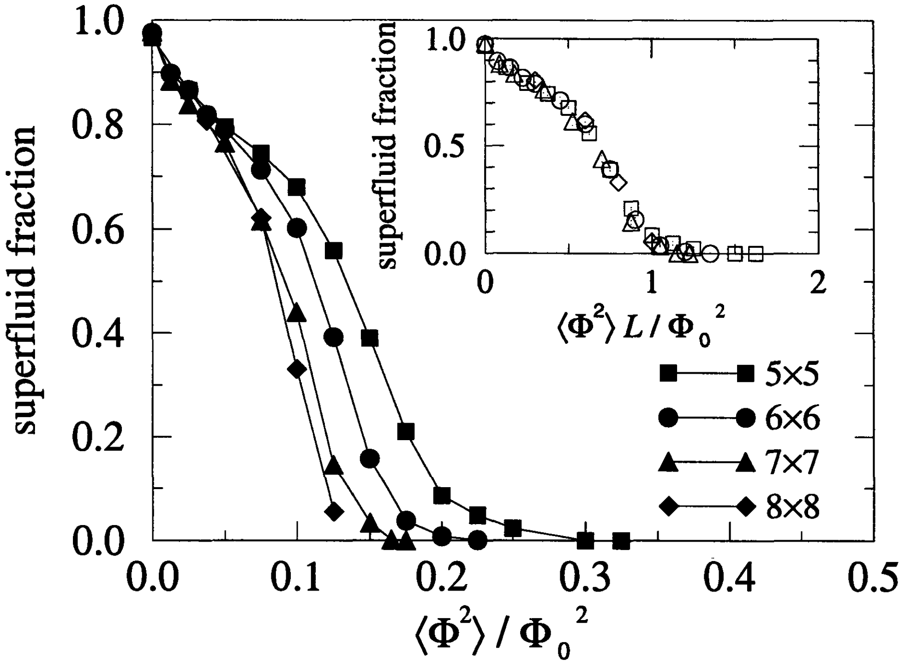

Figure 2-3: Superfluid density *vs.* $\langle \Phi^2 \rangle$ for different system sizes at $\beta t = 6$. Inset: a scaling plot suggests that superfluidity vanishes at $\langle \Phi^2 \rangle_c \sim 1/L$.

flux. One might expect that, compared with the case of zero flux, this would reduce the distance traveled by the particles in the imaginary-time interval β before their paths must return to some permutation of their starting positions. This should slow down the imaginary-time motion of the bosons as well as reduce the probabilities for exchange. We find that this is indeed the case.

We first look at the exchange probabilities P_i of a particle participating in an exchange cycle of i bosons. As before, we may deduce a degeneracy temperature T_{D} from the probability $(1 - P_1)$ for a particle to be involved in particle exchange[92]. This degeneracy temperature is reduced compared to the case of zero flux. For $U = 4t$ at quarter filling, we find that the zero-flux degeneracy temperature $T_{\mathrm{D}0} = 1.1t$ is reduced to $T_{\mathrm{D}} = 0.5t$ at $\langle \Phi^2 \rangle = 0.5\Phi_0^2$. At $\frac{1}{6}$-filling, it is reduced from $T_{\mathrm{D}0} = 0.8t$ to $T_{\mathrm{D}} = 0.34t$. A finite T_{D} does not imply Bose condensation at a finite temperature. Indeed, one cannot deduce a superfluid transition by examining the exchange probabilities. Remarkably, in the degenerate regime below T_{D}, the exchange probabilities for the cases of $\langle \Phi^2 \rangle = 0$ and $0.5\Phi_0^2$ are nearly identical (see Table 2.1). In this temperature regime, a particle is equally likely to participate in an exchange cycle of any size: $P_1 \simeq P_2 \simeq \ldots \simeq P_N \simeq 1/N$.

We can gain a qualitative understanding of the low-temperature exchange probabilities, by examining how the suppression of Amperean area by S_2 affects the geometry of the world-line configurations. When there is significant quantum exchange, individual bosons do not have to retrace their own paths in order to minimize the total Amperean area of the world-line configuration of all the bosons. Instead, one

Table 2.1: One, two, three, and four- boson exchange probability for various T, $\langle\Phi^2\rangle$, and U at quarter-filling.

T	U	$\langle\Phi^2\rangle/\Phi_0^2$	P_1	P_2	P_3	P_4
$0.5t$	$4t$	0.5	0.51	0.23	0.13	0.07
$0.5t$	$4t$	0	0.20	0.12	0.11	0.11
$0.25t$	$4t$	0	0.12	0.11	0.11	0.11
$0.25t$	$4t$	0.5	0.26	0.16	0.13	0.12
$0.25t$	$16t$	0.5	0.41	0.21	0.13	0.10
$0.11t$	$4t$	0	0.11	0.11	0.11	0.11
$0.11t$	$4t$	0.5	0.12	0.11	0.11	0.11
$0.11t$	$16t$	0.5	0.12	0.11	0.11	0.11

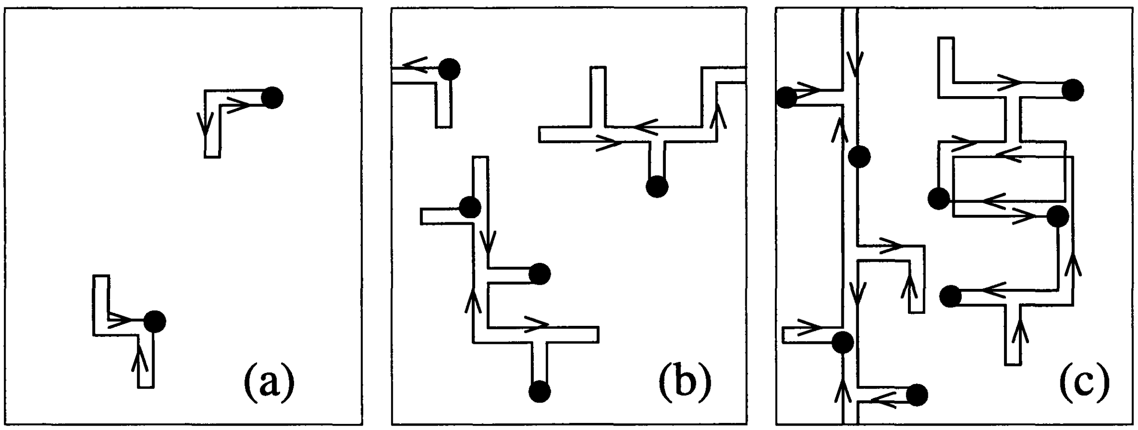

Figure 2-4: Schematic world-line cycles which retrace when projected onto the xy-plane. Solid circles denote boson positions at $\tau = 0$. (a) Each boson retraces its own path; (b) Exchange cycles with more than one boson retrace their own paths; (c) Two exchange cycles can retrace each others paths, and two wrapping paths can retrace each other to give zero total wrapping around the boundaries.

might minimize the Amperean area of each world-line loop formed by several bosons in the same exchange cycle. We find that this is not the entire situation at sufficiently low temperatures. Below T_{D}, the different world-line loops have strong overlap. We find that different cycles retrace each others' paths. (See Fig. 2-4.) Thus, although the gauge fields have a drastic effect on the *total* area enclosed by all the boson world lines, individual world-line cycles may enclose large areas. One might therefore expect that some aspects of the world-line geometry, which are insensitive to the total area, may indeed be very similar to the case of zero flux.

The observation that individual particles do not have to retrace their own paths suggests that they could diffuse a greater distance than in the single-particle case. One should see a reduction in the kinetic energy $\langle K\rangle$ of the particles compared to the Brinkman-Rice theory[83]. This is indeed the case. (See Appendix for a discussion of the measurement of the kinetic energy.) A single particle with retracing paths has a band edge at $-2\sqrt{3}t$ rather than $-4t$. In our system, the kinetic energy per particle goes below the Brinkman-Rice band edge at low temperatures, approaching $-4t$ roughly linearly in temperature (Fig. 2-5). Thus, we see that the strong gauge fluctuations do not have a large effect on some aspects of the world-line geometry (*e.g.*, exchange probabilities) while having a dramatic influence on others (*e.g.*, superfluidity).

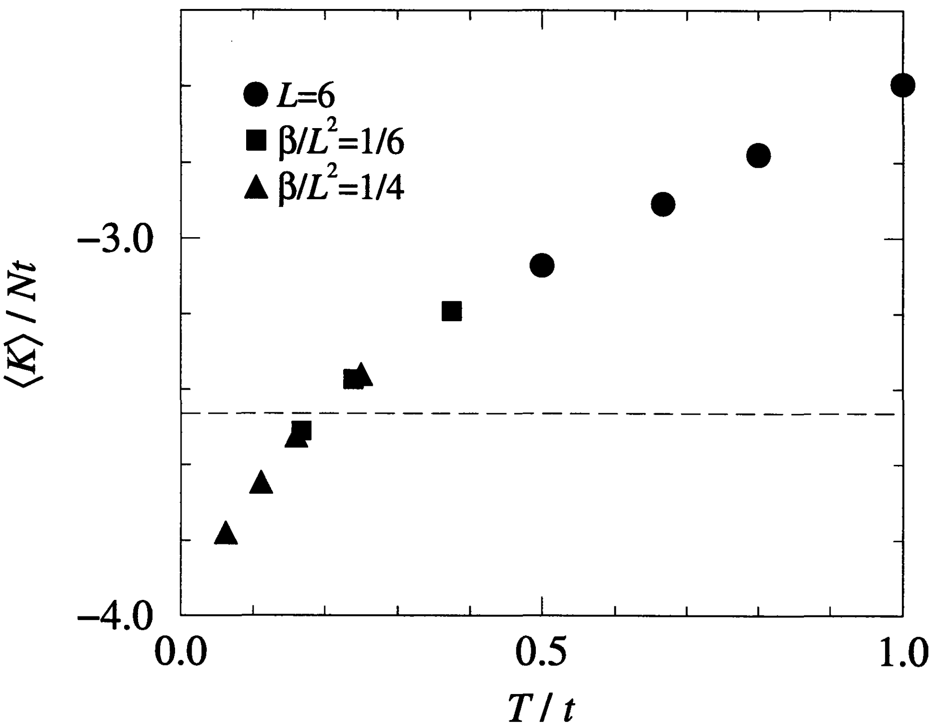

Figure 2-5: Kinetic energy per particle as a function of temperature. Dashed line marks the Brinkman-Rice band edge for the single-particle problem. $\langle\Phi^2\rangle = 0.5\Phi_0^2$ and $U = 4t$.

Let us now examine the imaginary-time motion of the particles in more detail. Ideal bosons are diffusive in imaginary time at all temperatures, *i.e.*, the mean-squared displacement of particle α is linear in imaginary time τ: $R^2(\tau) = \langle[\mathbf{x}_\alpha(\tau) - \mathbf{x}_\alpha(0)]^2\rangle = 4t\tau$ for $0 < \tau < \beta/2$. With repulsive interactions, there is an increase in the effective mass of the particle, e.g., for $U = 4t$ at quarter filling, we find $t \to t^* = 0.95t$. In the presence of random magnetic flux, the imaginary-time diffusion is slowed down, and the mean-squared displacement $R^2(\tau)$ is no longer linear in τ at all temperatures. Fig. 2-6 shows our results for the (superfluid) zero-flux case at temperature $\beta t = 9$ and the case of strong random flux at $\beta t = 4, 6, 9$. Since we are working with periodic boundary conditions, we have used the definition: $R^2(\tau) = \langle[\int_0^{\beta/2}\dot{\mathbf{x}}_\alpha(\tau)d\tau]^2\rangle$. We can see that, whereas $R^2(\tau)$ has significant downward curvature at $\beta t = 2$, it becomes closer to diffusive behavior as the temperature is lowered. However, we are unable to reach the asymptotic regime where the particle has traveled far on the scale of the interparticle spacing over a time period of $\beta/2$ (see Fig. 2-6 inset).

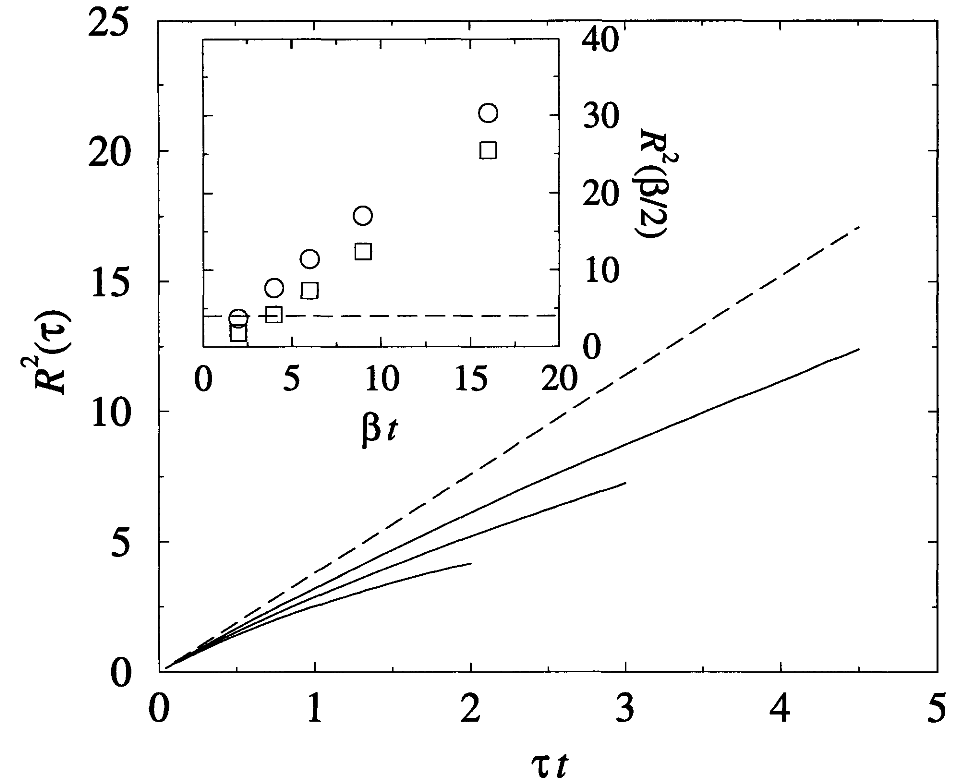

Figure 2-6: Single-particle diffusion $R^2(\tau) = \langle[\mathbf{x}(\tau) - \mathbf{x}(0)]^2\rangle$ in imaginary time for $0 < \tau < \beta/2$. Solid lines: strong random flux with $\langle\Phi^2\rangle = 0.5\Phi_0^2$ at $\beta t = 4, 6, 9$. Dashed line: zero flux at $\beta t = 9$. Inset: $R^2(\beta/2)$ for zero flux ($\circ$) and $\langle\Phi^2\rangle = 0.5\Phi_0^2$ ($\square$); dashed line marks the squared interparticle spacing.

In order to study the long-time behavior, we can examine the size of the world-line exchange cycles. A cycle where the world lines of l particles $\{\mathbf{x}_1, \ldots, \mathbf{x}_l\}$ form a loop can be roughly regarded as a particle traveling over a time interval of $l\beta$. Thus, the possibility of exchange means that a world-line cycle can travel large distances compared with an individual boson. In a system with periodic boundaries, the size R_l of the cycle is defined by:

$$
\begin{aligned}
R_l^2 &= \left\langle \left[\int_0^{\beta/2} \dot{\mathbf{x}}_{\frac{l+1}{2}}\, d\tau + \sum_{\alpha=1}^{(l-1)/2} \int_0^{\beta} \dot{\mathbf{x}}_\alpha\, d\tau \right]^2 \right\rangle \qquad l \text{ odd}, \\
&= \left\langle \left[\sum_{\alpha=1}^{l/2} \int_0^{\beta} \dot{\mathbf{x}}_\alpha\, d\tau \right]^2 \right\rangle \qquad l \text{ even}.
\end{aligned}
\tag{2.24}
$$

For ideal bosons, R_l^2 should equal $R^2(\tau = l\beta/2)$ at inverse temperature $l\beta$, and therefore should scale linearly with l. Fig. 2-7 shows R_l^2 for a 4×4 lattice with 9 particles. We have measured only cycles which do not have a net wrapping number around the periodic boundaries so that we do not have contributions from cycles with different topologies. We see that R_l^2 is linear in l for the cases of zero flux and strong random flux, although the slope of the case with the strong random flux is reduced substantially. This demonstrates that the imaginary-time motion of the bosons is diffusive at long distances.

These results indicate that we are probing an unconventional phase of a Bose liquid. Although the system remains normal, many aspects of the imaginary-time motion of the particles in the degenerate regime resemble that of a neutral Bose liquid which is a superfluid in such temperatures. In subsequent sections, we shall study the physical properties of this "strange metal" and discuss the relevance to the normal state of the cuprate superconductors.

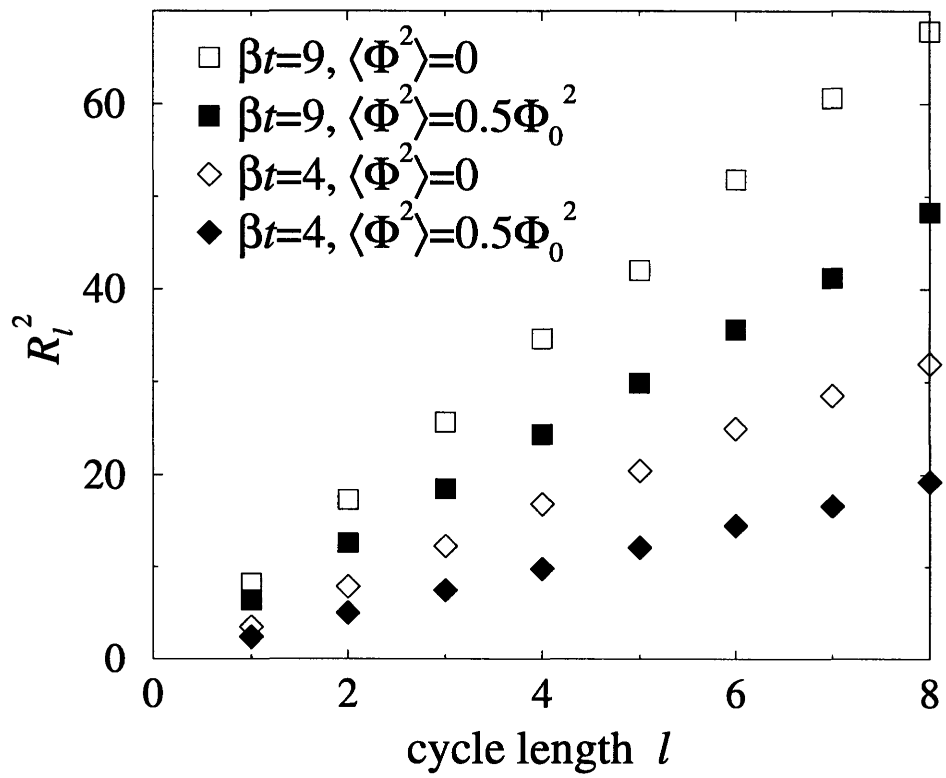

Figure 2-7: Cycle sizes R_l^2 as a function of cycle length l for a 6×6 lattice with 9 particles.

2.5 Transport and Optical Conductivity

In this section, we will present our quantum Monte Carlo (QMC) results on longitudinal transport for this strange Bose metal. To obtain the conductivity of the system, we measure its imaginary-time analogue $\sigma_{\alpha\beta}(i\omega_n)$ in our quantum Monte

Carlo simulation:

$$\sigma_{\alpha\beta}(i\omega_n) = \frac{1}{|\omega_n|}\Pi_{\alpha\beta}(i\omega_n) \tag{2.25}$$

$$\Pi_{\alpha\beta}(i\omega_n) = \int_0^\beta \langle j^\alpha_{\mathbf{q}=0}(\tau) j^\beta_{\mathbf{q}=0}(0)\rangle e^{i\omega_n\tau} d\tau, \tag{2.26}$$

where $\mathbf{j_q}(\tau) = \sum_\mathbf{r} \mathbf{j_r}(\tau)e^{i\mathbf{q}\cdot\mathbf{r}}$ and $\mathbf{j_r}(\tau) = \sum_\alpha \delta(\mathbf{r} - \mathbf{x}_\alpha(\tau))\frac{d\mathbf{x}_\alpha}{d\tau}$ is the gauge-invariant current (Fig. 2-8).

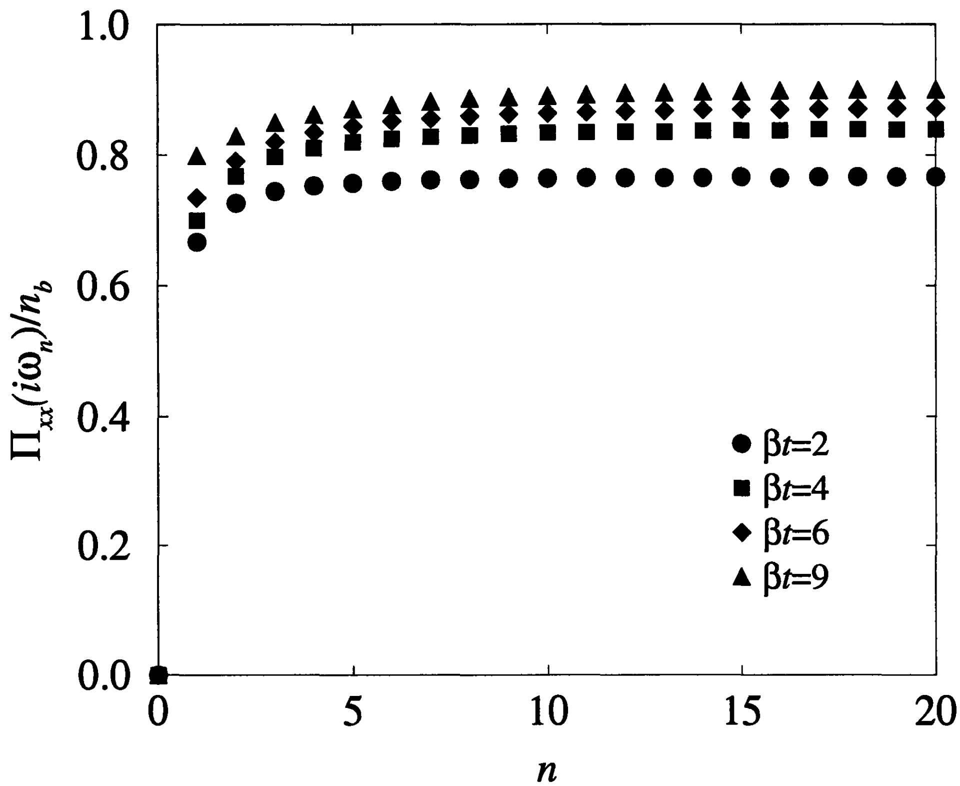

Figure 2-8: Current correlation function $\Pi_{xx}(i\omega_n)$ for a 6×6 lattice with 9 bosons with $\langle\Phi^2\rangle = 0.5\Phi_0^2$ and $U = 4t$.

The imaginary-time measurements are related to the real-time conductivity $\sigma(\omega) \equiv \sigma_{xx}(\omega)$ by:

$$-\frac{1}{2L^2}\langle \mathbf{j}_{\mathbf{q}=0}(\tau)\cdot \mathbf{j}_{\mathbf{q}=0}(0)\rangle = \int_{-\infty}^{\infty} \frac{\omega e^{-\omega\tau}\sigma(\omega)}{1-e^{-\beta\omega}}\frac{d\omega}{\pi}. \tag{2.27}$$

Deducing dynamical properties (such as conductivity) from imaginary-time data is in general an ill-posed problem. Several approximate methods are often used in the context of QMC studies. A simple method, which has been used in the study of the superfluid-insulator transition[93, 94], is to fit $\sigma(i\omega_n)$ to a simple functional form, such as the Drude form $\sigma(i\omega_n) = \sigma_0/(1+|\omega_n|\tau_{\text{tr}})$. More generally, one can use a Padé approximant to fit an arbitrary number of poles and zeroes:

$$\sigma(z) = \frac{a_0 + a_1 z + \dots a_{N_n} z^{N_n}}{b_0 + b_1 z + \dots b_{N_d} z^{N_d}}. \tag{2.28}$$

This approach is particularly suitable if the scattering rate $1/\tau_{\rm tr}$ (or the position of the pole closest to the origin in (2.28)) is large compared to the temperature at low temperatures. This is however not the case in our problem. In our system, $\Pi_{xx}(i\omega_n)$ is nearly constant as a function of n for finite n even at low temperatures, suggesting that $1/\tau_{\rm tr}$ is proportional to T. (Note that $\Pi_{xx}(n = 0) = 0$ in the limit of strong random flux because paths which wrap around the torus are strongly suppressed.)

We have calculated the conductivity by numerical analytic continuation using the maximum-entropy (MaxEnt) method[95, 96]. Eq. (2.27) takes the form of a linear integral equation:

$$d(\tau) = \int K(\tau,\omega) r(\omega)\, d\omega, \tag{2.29}$$

where $K(\tau, \omega)$ is the kernel relating the imaginary-time data $d(\tau)$ to the response function $r(\omega)$. In our QMC simulations, $d(\tau)$ is measured at discrete points $\tau_l = l\Delta\tau$ with mean $\overline{d}_l$. The errors for the time points l and m are correlated with a covariance matrix $C_{lm} = \langle (d_l - \overline{d}_l)(d_m - \overline{d}_m)\rangle$. The MaxEnt method finds an estimate of $r(\omega)$ as the function $\hat{r}(\omega)$ which maximizes the functional: $\phi[\hat{r}(\omega); \alpha] = -\chi^2/2 + \alpha S$. The goodness of fit is $\chi^2 = \sum_{l,m}(D_l - \overline{d}_l)[C^{-1}]_{lm}(D_m - \overline{d}_m)$ where $D_l = \int d\omega K(\tau_l, \omega)\hat{r}(\omega)$. The "entropy" S is:

$$S = \int d\omega \left[\hat{r}(\omega) - m(\omega) - \hat{r}(\omega) \log \frac{\hat{r}(\omega)}{m(\omega)}\right]. \tag{2.30}$$

where $m(\omega)$ is a default model (or measure). We have chosen $m(\omega)$ to be a constant in order not to build in any bias. Our results are not sensitive to this choice. The variable α controls the competition between the smoothness and the goodness of the fit, and ϕ is also maximized with respect to it[97]. Details of the MaxEnt method are given in Refs.[95, 96, 97].

One can check the results of the MaxEnt inversion using relevant sum rules. In the case of conductivity, we have used the sum rule

$$\int_0^\infty \sigma(\omega) d\omega = -\frac{\pi}{4}\frac{\langle K\rangle}{L^2}. \tag{2.31}$$

which is the lattice version of the more familiar form in the continuum: $\int_0^\infty \sigma(\omega) d\omega = \pi n_b/2m$. In our MaxEnt results, this sum rule is obeyed to within 3% error. In order to obtain reliable data for the MaxEnt inversion, we have worked with a fine discretization in imaginary time ($t\Delta\tau \leq 0.1$). For the lowest temperatures ($T < 0.4t$), we worked at fixed $\Delta\tau$ and β/L^2. We chose $\beta \propto L^2$ to control the finite-size effects because of the imaginary-time motion of the bosons is roughly diffusive, as discussed above. Our results are in fact not very sensitive to this choice, indicating that finite-size effects are small. For instance, the values of resistivity at $\beta t = 4$ and $n_b = 1/4$

for 4×4 and 6×6 are similar within statistical error.

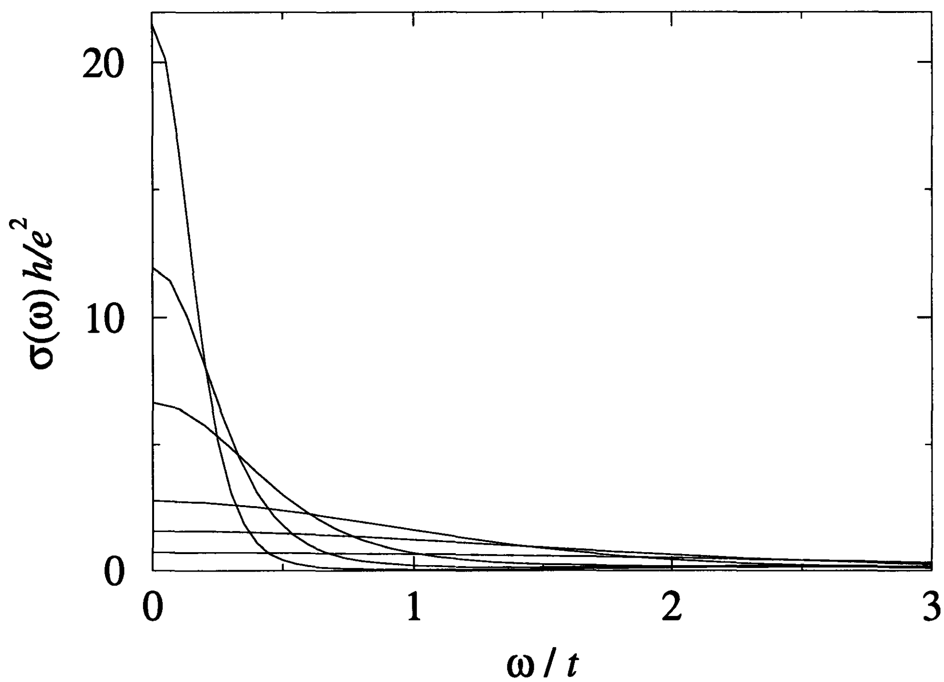

Figure 2-9: Optical conductivity for 6×6 lattice with 9 bosons at βt =9,6,4,2,1,0.5. $\langle \Phi^2 \rangle = 0.5\Phi_0^2$ and $U = 4t$.

We find that $\sigma(\omega)$ consists of a single Drude-like peak (Fig. 2-9). Since this peak exhausts the sum rule (2.31), its spectral weight is proportional to $-\langle K \rangle$. This spectral weight has a weak temperature dependence in this temperature range because, as already discussed, the kinetic energy approaches $-4t$ per particle as the temperature is lowered. This should be contrasted with the Brinkman-Rice result [83] for non-degenerate particles ($T \gg t$) where the weight under $\sigma(\omega)$ decreases as $\langle -K \rangle \sim T^{-1}$.

The width of $\sigma(\omega)$ gives a transport scattering rate consistent with: $1/\tau_{\rm tr} = \zeta k_{\rm B} T$ with $\zeta = 1.8 - 2.2$ (Fig. 2-10). This result has been obtained for two densities $n_b = 1/4$ and $1/6$ so that this scattering rate appears to be independent of density. Again this differs from the Brinkman-Rice result where $1/\tau_{\rm tr}$ is a constant of order t (as one begins to see at the highest T in Fig. 2-10). The resistivity ρ, given by the peak height, is consistent with a linear temperature dependence of $\rho e^2/h = (1/2\pi n_b)T/t$ for $T < 2t$ (Fig. 2-11). We estimate a statistical error of 5% for ρ by examining fluctuations due to statistical errors in the measurement of the current correlation function. There are also systematic errors due to the smoothing of structures.

There appears to be a systematic deviation from the linear-T behavior below $T = 0.3t$, in particular in the case of quarter filling. This deviation is stronger for ρ than for $1/\tau_{\rm tr}$. The difference can be attributed to the T-dependence of the Drude weight discussed above which should affect the resistivity but not the relaxation time. We speculate that the deviation from linearity at the lowest temperatures may indicate the approach to zero-temperature critical behavior. This is beyond the scope

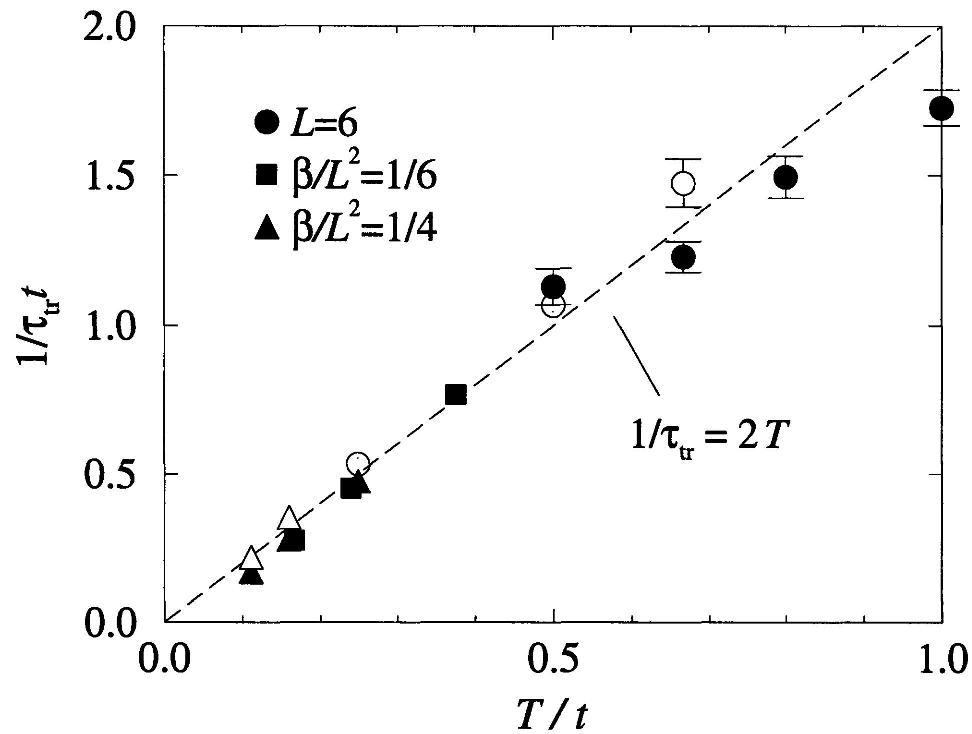

Figure 2-10: Scattering rate $1/\tau_{tr}$ as a function of temperature. Solid(hollow) symbols correspond to a boson density of n_b=1/4(1/6). $\langle\Phi^2\rangle = 0.5\Phi_0^2$ and $U = 4t$.

of this paper.

Our resistivity agrees, to within a factor of 2, with Jaklič and Prelovšek [98] who provided an approximate diagonalization of the t-J model on 4×4 lattices and found a Drude peak with width $2T$. They also found a broad background, and interpreted it with a frequency-dependent scattering rate $\tau(\omega)$. Indeed, some authors have interpreted the experimental optical conductivity as possessing a power-law tail and emphasized its importance[46]. This incoherent part of the conductivity is absent from our boson model, and may be due to inelastic scattering of the bosons with the gauge field or, more generally, with the fermionic degrees of freedom.

2.6 Magnetic Response

We now discuss the response of this degenerate Bose liquid to a weak external magnetic field perpendicular to the plane. In the absence of the random magnetic fields, a Bose liquid has a strong diamagnetic response as the temperature is lowered towards the transition to a superfluid when it develops a Meissner response. We argue here that the linear response of the system to a magnetic field is strongly suppressed by the gauge fluctuations. Qualitatively, this can be again understood by examining the world-line configurations. We have already demonstrated that the partition function

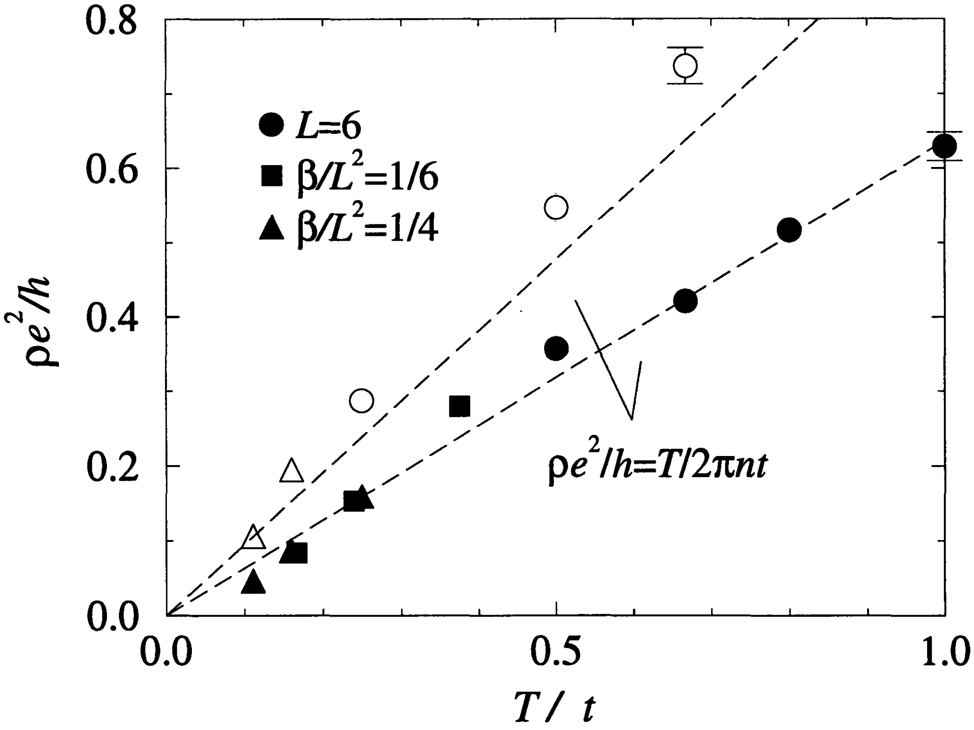

Figure 2-11: Resistivity as a function of temperature. Solid(hollow) symbols correspond to a boson density of n_b=1/4(1/6). $\langle\Phi^2\rangle = 0.5\Phi_0^2$ and $U = 4t$.

is dominated by world-line paths which are unaffected by the internal gauge fields $\sum_\alpha \int \mathbf{a} \cdot d\mathbf{x}_\alpha = 0$ for any $\mathbf{a}$. These configurations are therefore also unaffected by any external magnetic fields. Thus, we see that the system has a vanishing linear response to magnetic fields in the limit of strong gauge fluctuations. For the sake of completeness, we will now discuss more quantitatively the magnetic response of the system. Relevant physical quantities are the diamagnetic susceptibility $\chi_{\rm B}$, the Hall coefficient $R_{\rm H}$, and the magnetoresistance $\Delta\rho/\rho$.

Consider first the diamagnetic susceptibility. On the infinite plane, in the presence of a weak external field H, each world-line configuration picks up an extra factor of $\exp[-i\sum_\alpha \mathbf{A}_{\rm ext}\cdot d\mathbf{x}_\alpha] = \exp[-iH\mathcal{A}_0]$ where $\text{curl}\mathbf{A}_{\rm ext} = H$, and $\mathcal{A}_o = \sum_{\mathbf{r}} w_{\mathbf{r}}$ is the oriented area of the configuration. (In this section, we will use units where $\Phi_0 = 2\pi$.) Expanding this in a Taylor expansion, one can write the partition function $Z(H)$ as:

$$\begin{aligned} Z(H) &= \int \mathcal{D}\{\mathbf{x}\} \left(1 - iH\mathcal{A}_o - \frac{1}{2}H^2\mathcal{A}_o^2\right) e^{-S_{\rm eff}} \\ &= Z(0)\left(1 - \frac{1}{2}H^2\langle\mathcal{A}_o^2\rangle\right). \end{aligned} \quad (2.32)$$

where $\mathcal{A}_o$ is the oriented area of a world-line configuration and $\langle\ldots\rangle$ denotes an average for the system at $H = 0$. We have assumed here that the external magnetic field H has negligible effect on the spectrum of the gauge fluctuations. The diamagnetic susceptibility is given by:

$$\chi_{\rm B} = \frac{1}{\beta}\frac{\partial^2 \ln Z}{\partial H^2} = \frac{4\pi^2 T}{\Phi_0^2}\langle \mathcal{A}_o^2 \rangle. \tag{2.33}$$

Since $\mathcal{A}_a > \mathcal{A}_o$ by definition, we can see that, when the gauge fluctuations are strong so that configurations with zero Amperean area dominate, the system has no diamagnetic response, as suggested in Section IV B.

It should be noted that, with periodic boundary conditions, the total flux penetrating the torus is quantized in units of the flux quantum. One should use replace $\langle \mathcal{A}_o^2 \rangle$ by $4\langle \sin^2[H_0\mathcal{A}_o/2]\rangle/H_0^2$ where $H_0 = \Phi_0/L^2$ is the smallest uniform field allowed in a torus of size L. Moreover, as in the case for the Amperean area, a geometrical interpretation of the phase factor $\int \mathbf{A}_{\rm ext}\cdot d\mathbf{x}$ is not possible for paths which wrap around periodic boundaries. However, these wrapping configurations are strongly suppressed in the case of strong random flux and should give negligible contribution to the susceptibility.

We can also consider magnetotransport properties in terms of the conductivity tensor $\sigma^H_{\alpha\beta}$. For instance, the Hall number is given by $R_{\rm H} \approx \sigma^H_{xy}/(H\sigma^2_{xx})$. We need the current-current correlator at a small external field: $\langle j^\alpha(\tau) j^\beta(0)\rangle_H$. Expanding again in a Taylor series in H, one obtains the correlator:

$$\langle j^\alpha j^\beta \rangle_H = \frac{\langle j^\alpha j^\beta\rangle - iH\langle j^\alpha j^\beta \mathcal{A}_o\rangle - \frac{1}{2}H^2\langle j^\alpha j^\beta \mathcal{A}_o^2\rangle + \cdots}{1 - \frac{1}{2}H^2\langle \mathcal{A}_o^2\rangle + \cdots} \tag{2.34}$$

From (2.26), we get

$$\begin{aligned}
\frac{\sigma^H_{xy}(i\omega_n)}{H} &= \frac{2\pi i}{|\omega_n|\Phi_0}\int_0^\beta d\tau e^{i\omega_n\tau}\langle j^x_{\mathbf{q}=0}(\tau) j^y_{\mathbf{q}=0}(0)\mathcal{A}_o\rangle, \\
\frac{\Delta\sigma_{xx}(i\omega_n)}{H^2} &= \frac{2\pi^2}{|\omega_n|\Phi_0^2}\int_0^\beta d\tau e^{i\omega_n\tau}\times \\
&\quad [\langle j^x_{\mathbf{q}=0}(\tau) j^x_{\mathbf{q}=0}(0)\mathcal{A}_o^2\rangle - \langle j^x_{\mathbf{q}=0}(\tau) j^x_{\mathbf{q}=0}(0)\rangle\langle \mathcal{A}_o^2\rangle].
\end{aligned} \tag{2.35}$$

where $\Delta\sigma_{xx} = \sigma^H_{xx} - \sigma^{H=0}_{xx}$ is the magnetoconductivity. Since the oriented area $\mathcal{A}_o$ can be written as $\mathcal{A}_o = \frac{1}{2}\sum_{\mathbf{r}}\int_0^\beta \hat{z}\cdot(\mathbf{j}_{\mathbf{r}}(\tau)\times\mathbf{r})\,d\tau$, we see that we can relate this expression for the Hall conductivity σ^H_{xy} to the more familiar one involving the average of three currents[100].

Again, we see that the magnetotransport response is strongly suppressed by the gauge fluctuations because it is sensitive to the oriented area of the world-line configurations. In principle, the quantities $\operatorname{Im}\sigma^H_{xy}(\omega)$ (from which we can obtain $\operatorname{Re}\sigma^H_{xy}$ from a Kramers-Kronig relation) and $\Delta\sigma_{xx}(\omega)$ can be computed. However, these quantities are too small to measure in the regime of strong gauge fluctuations that we study.

Since we have argued that the gauge field fluctuations are indeed strong in the cuprates at temperatures above the superconducting transition, it appears that our simple boson model with a quasistatic gauge field cannot describe quantitatively the magnetotransport in these materials. This result is however qualitatively consistent with the experimental finding that these magnetotransport properties are generally suppressed from the classical values. To obtain a quantitative prediction for these properties, one may attempt to restore dynamics to the gauge fields. If the gauge field may relax in time, then the boson world lines no longer have to obey the condition of strictly retracing paths. This would allow the world lines to enclose a finite oriented area and hence a finite response to external magnetic fields. However, we emphasize that such an approach might not represent the physics completely. We believe that our model illustrates the general point that the influence of an external field on the system is strongly masked by the fluctuations of the internal magnetic field.

2.7 Density Correlation Function

2.7.1 phase separation

Non-interacting bosons are infinitely compressible. They would therefore collapse into a small region of the system in the presence of any quenched disorder which has a tail of localized states in the single-particle spectrum. An analogous collapse is also found in this problem with annealed random flux. Such an instability was discussed by Feigelman *et al.*[89] who have argued that it occurs also in the case of interacting bosons at low densities, leading to a hole-rich phase and a hole-absent phase. They further argued a long-range Coulomb repulsion would be necessary to stabilize the uniform phase.

Within the world-line picture, one can visualize the instability of the homogeneous phase in the limit of strong gauge fluctuations. The condition of retracing paths in this limit encourages the bosons to come close to each other so that their paths may retrace each other. This will allow individual boson paths to explore a larger area (in imaginary time), and hence lower the kinetic energy of the system compared to the case with each boson has to retrace its own path. In the absence of any repulsive interactions, this effect would dominate at low temperatures, making the homogeneous phase unstable to collapse.

We find that this instability towards the formation of dense aggregates indeed occurs in our model in the absence of boson repulsion, although the instability is prevented by on-site repulsion, at least for the moderate boson densities of interest here. We have studied the instability by examining the compressibility of the system: $\kappa = \lim_{q\to 0} \kappa(\mathbf{q})$ with

$$\kappa(\mathbf{q}) = \frac{1}{N n_b} \int_0^\beta d\tau \langle n_{\mathbf{q}}(\tau) n_{-\mathbf{q}}(0) \rangle, \tag{2.36}$$

where $n_{\mathbf{q}}(\tau)$ is the Fourier transform of the boson density at imaginary time τ. Alternatively, $\kappa = \lim_{q\to 0} \beta S(\mathbf{q})/n_b^2$ where $S(\mathbf{q})$ is the static structure factor:

$$S(\mathbf{q}) = \frac{1}{L^2} \langle n_{\mathbf{q}}(\tau) n_{-\mathbf{q}}(\tau) \rangle \tag{2.37}$$

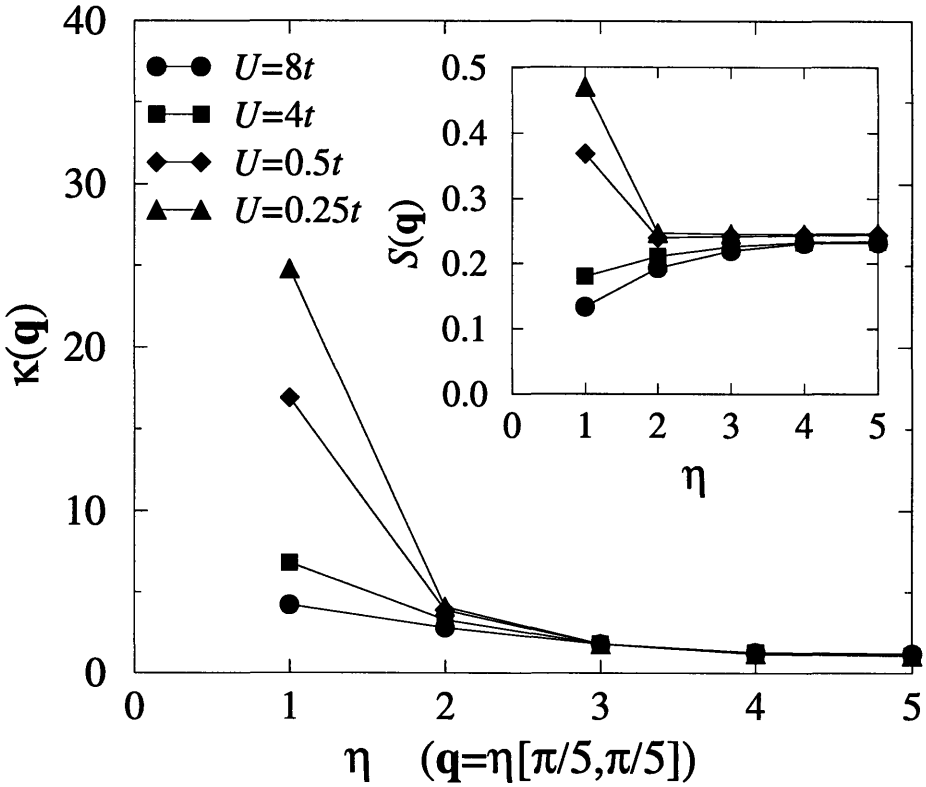

Figure 2-12: Static structure factor (inset) and the **q**-dependent compressibility as a function of q in the (π,π) direction for different values of U. $\beta t = 4$, $\langle \Phi^2 \rangle = 0.5\Phi_0^2$, $n_b = 0.25$.

In Fig. 2-12, we show the behavior of $S(\mathbf{q})$ and $\kappa(\mathbf{q})$ for different values of the on-site repulsion U for a 10×10 lattice with 25 bosons. The structure factor $S(\mathbf{q})$ as a function of $\mathbf{q}$ is qualitatively different for the cases of small U and large U (compared to t): $S(\mathbf{q})$ for $\mathbf{q} = (\frac{\pi}{5}, \frac{\pi}{5})$ is greater than the density n_b for when the on-site interaction is small. We can also look at the compressibility. Since we work with finite systems at fixed boson number, we will evaluate $\kappa(\mathbf{q})$ at the smallest wavevector of the system as an estimate of the $q = 0$ behavior. We see that, in the presence of random magnetic flux, the compressibility increases with decreasing U. This can be interpreted as a divergence as $q \to 0$ for small U, and hence an instability of the homogeneous phase. (This is also reflected in the magnitude of the fluctuations in our QMC results for $\kappa(\mathbf{q})$ which grows as $q \to 0$ for sufficiently small U.) However, for strong on-site repulsion, the density correlations show no sign of an instability at this density.

2.7.2 static structure factor

The density fluctuations in our boson model should be relevant to the charge fluctuations in the full t-J model. It has been pointed out that the density excitations of the t-J model does not resemble those of a conventional Fermi system. We will now compare our results with numerical results on the full t-J model in the literature.

The static structure factor (2.37) has been calculated by various means[67, 101]. Fig. 2-13 shows the static structure factor for our boson system together with that of the t-J model[67] at $T = 0.25t$. We see that our results are qualitatively similar to the t-J model, with improving quantitative agreement as one approaches the hard-core limit (see, for example, $U = 16t$). We should point out that this dependence on U should not be as strong for the transport properties of the system, because the particle currents are not directly affected by the repulsive density interactions.

It is also interesting to note that the magnitude of the gauge field fluctuations has a relative weak effect on $S(\mathbf{q})$ when the on-site repulsion U is strong. However, as we shall see in the next section, the *dynamics* of the density excitations is strongly modified by the interaction with the gauge fields.

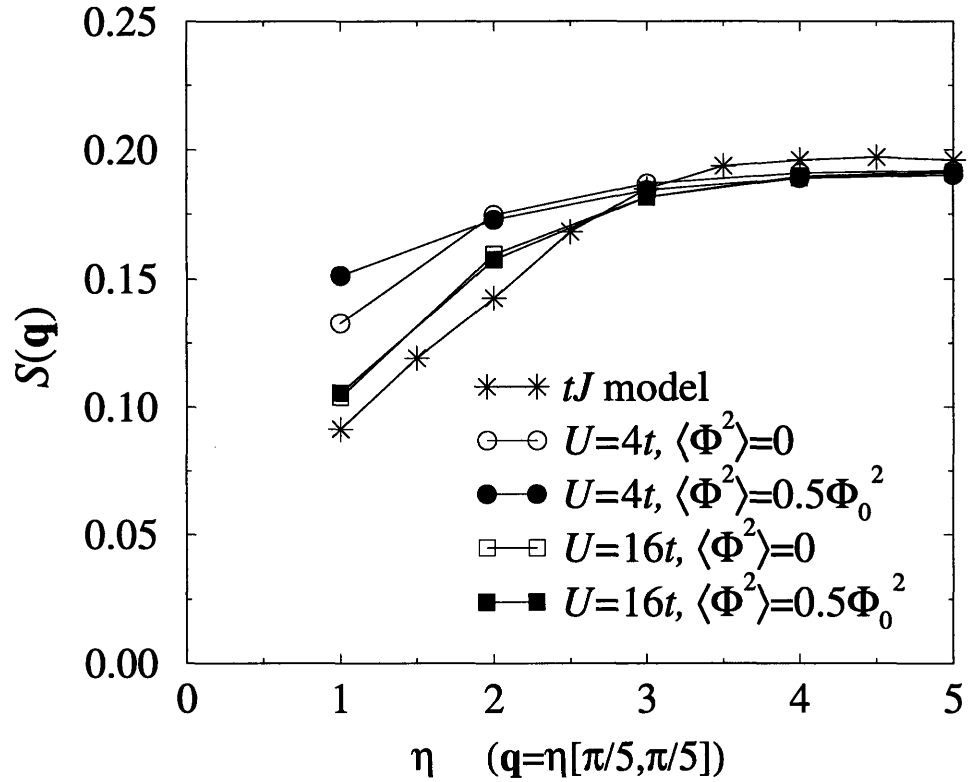

Figure 2-13: Static structure factor of the boson model at density $n_b = 0.2$ along the (π,π) direction at $T = 0.25t$. Asterisks: t-J model result[67] at electron density $n = 1 - n_b = 0.8$ and $t/J = 2$.

2.7.3 dynamic structure factor

We now look at the dynamic structure factor $S(\mathbf{q}, \omega)$:

$$S(\mathbf{q}, \omega) = \frac{1}{L^2} \int dt \, e^{i\omega t} \langle n_{\mathbf{q}}(t) n_{-\mathbf{q}}(0) \rangle \tag{2.38}$$

where $n_{\mathbf{q}}(t)$ is the Fourier transform of the density in real time. The dynamic structure factor is related to the imaginary-time density-density correlation function by

$$\frac{1}{L^2}\langle n_{\mathbf{q}}(\tau)n_{-\mathbf{q}}(0)\rangle = \int_0^\infty (e^{-\tau\omega}+e^{-(\beta-\tau)\omega})S(\mathbf{q},\omega)d\omega. \quad (2.39)$$

Again, we use MaxEnt to perform the inversion of this integral equation. Two sum rules can be used as a check of the MaxEnt procedure.

$$\begin{aligned} \int_0^\infty d\omega(1-e^{-\beta\omega})\omega S(\mathbf{q},\omega) &= -\frac{\langle K\rangle}{2L^2}(2-\cos q_x-\cos q_y)t \\ \int_0^\infty d\omega \frac{1-e^{-\beta\omega}}{\omega}S(\mathbf{q},\omega) &= \frac{1}{2}n_b^2\kappa(\mathbf{q}) \end{aligned} \quad (2.40)$$

These are lattice versions of the f-sum rule and the compressiblity sum rule. They are satisfied within 1% error in our results.

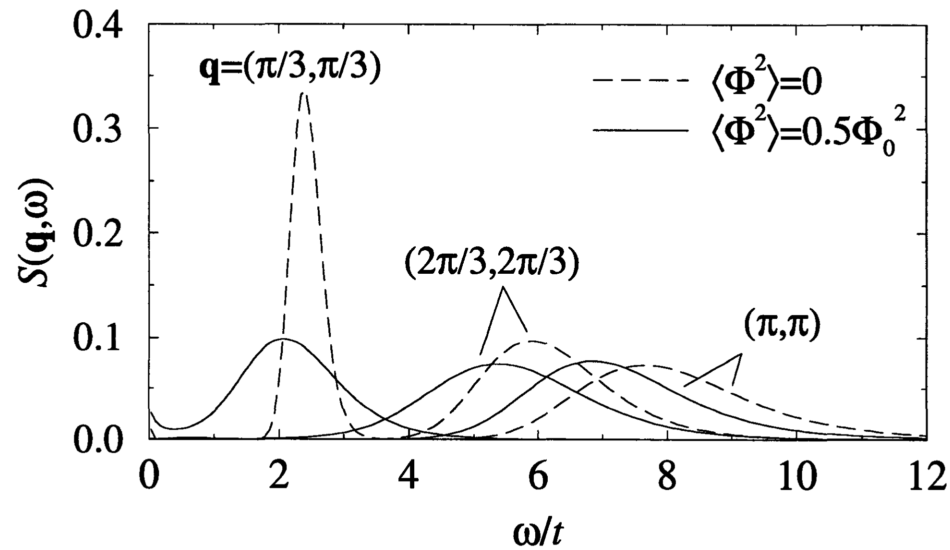

Figure 2-14: Dynamic structure factor of the superfluid phase ($\langle\Phi^2\rangle = 0$) and the normal phase ($\langle\Phi^2\rangle = 0.5\Phi_0^2$) in the (π,π) direction. $U = 4t$, $T = t/6$ at quarter-filling.

Fig. 2-14 shows $S(\mathbf{q},\omega)$ for our bosons with and without the random flux. The system in the absence of random flux should be a superfluid at the temperature and densities considered here, and therefore should possess well-defined phonon excitations. We see sharp phonon peaks in the density excitation spectrum, for instance, at wavevector $\mathbf{q} = (\frac{\pi}{3}, \frac{\pi}{3})$. These long-lived phonon excitations of the superfluid phase do not survive the coherence-breaking effect of the gauge-field interactions. We find only break peaks in $S(\mathbf{q}, \omega)$ in the presence of strong random flux.

Another effect of the presence of the gauge field is a reduction in the bandwidth of the density excitations. This might be expected because the gauge-field interaction tends to increase the compressibility of the system. Indeed, we see that the center of the (π, π) peak is pulled in from $7.6t$ to $6.8t$.

We also see that the dynamic structure factor has a simple scaling with the hole density (Fig. 2-15): $S(\mathbf{q}, \omega; n_h) = n_h S^0(\mathbf{q}, \omega)$ holds for $n_h = 0.1 \sim 0.3$. This is natural in a model of degenerate bosons where the boson density is equal to the hole density.

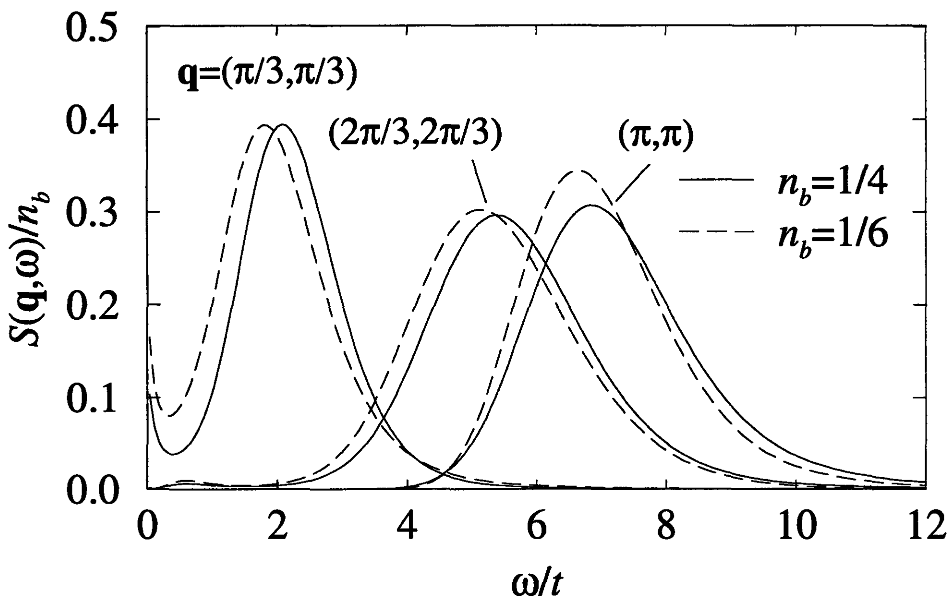

Figure 2-15: Scaling of $S(\mathbf{q}, \omega)$ with boson density in the (π, π) direction. The solid(dashed) lines are $S(\mathbf{q}, \omega; n_h)/n_h$ for 9(6) bosons on a 6×6 lattice. $\beta t = 6$, $U = 4t$, $\langle \Phi^2 \rangle = 0.5\Phi_0^2$.

We will now compare our results with numerical results on the full t-J model[68, 69]. It should be noted that, although we expect the electron density excitations of the t-J model to be dominated by its holon component, there is no quantitative equivalence between the structure factors of the t-J model and our boson-only model. Nevertheless, we argue that the dynamic structure factor of our model has qualitative similarities with that of the t-J model. For instance, the absence of sharp peaks in the dynamic structure factor is also found in the t-J model. An obvious similarity, built into our boson model *a priori*, is the lack of any structure indicating scattering across a Fermi surface at $q = 2k_F$. Another feature is the scaling of dynamic structure factor with the hole density [69].

We find that $S(\mathbf{q}, \omega)$ along the (π, π) direction agrees well with an exact diago-

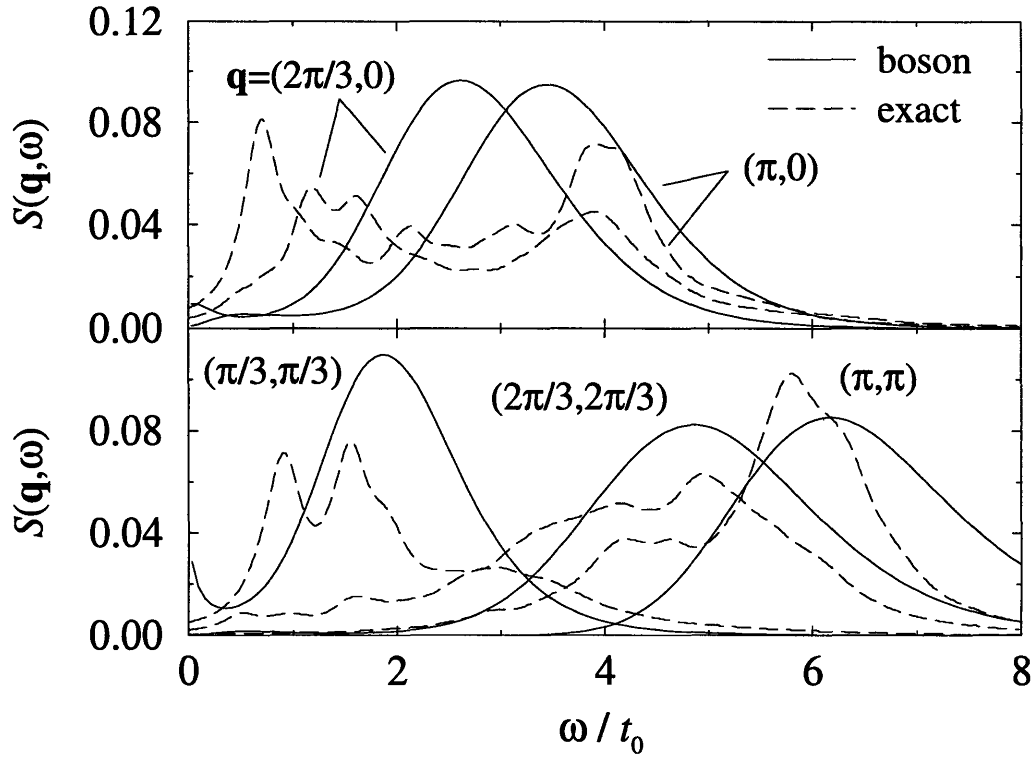

Figure 2-16: Dynamic structure factor. Solid lines denote our Monte Carlo results for 6×6 lattice with 9 bosons at $\beta t = 6$ with $t = 0.9t_0$. $\langle \Phi^2 \rangle = 0.5\Phi_0^2$ and $U = 4t$. Dashed lines denote exact diagonalization results[69] for 4 holes in an 18-site cluster with $t_0/J = 2.5$.

nalization study of the t-J model at a similar hole density, as shown in Fig. 2-16. (We have used a moderate rescaling of the hopping energy: $t = 0.9t_0$ where t_0 is the hopping energy in the t-J model.) The area under $\mathbf{q} = (\frac{\pi}{3}, \frac{\pi}{3})$ peak is larger in our model than in the t-J model. We believe that, as in the case of the static structure factor, this discrepancy can be improved with if we use a stronger on-site repulsion. However, the structure factor does not agree with the t-J model along the $(\pi, 0)$ direction. It might be that the spectrum of the holes at zero temperature is qualitatively different from the simple tight-binding spectrum that we have assumed here.

2.8 Conclusion

In summary, we have studied a degenerate Bose system which remains metallic below its degeneracy temperature due to elastic scattering with random and quasistatic gauge fields. In the path-integral picture, the bosons retrace their paths in the limit of strong gauge fluctuations in order to avoid the quantum frustration due to the fluctuating gauge field. We have demonstrated that many features of these "Brinkman-Rice bosons" indeed mimic the behavior of the full t-J model and the normal state of the cuprate superconductors. These features include the linear-T dependence of the longitudinal scattering rate and a charge excitation spectrum which consists of broad incoherent structures. This model itself has a strongly suppressed response to external magnetic fields, hinting that the behavior of the system as measured in Hall and magnetoresistance experiments have to understood in terms of a separate mechanism.

It would also be interesting to understand the behavior of the system in the zero-temperature limit. Although the limit of infinite gauge fluctuations (*i.e.*, a uniform flux distribution on a lattice) would strictly forbid any world lines to wrap around periodic boundaries, one may consider the case of weaker gauge fluctuations in the zero-temperature limit and ask whether there is a critical value of $\langle \Phi^2 \rangle$ below which the system is a superfluid at zero temperature. This will involve a study of the system at very low temperatures near a quantum critical point. This is beyond the scope of this paper.

This work was done in collaboration with D. K. K. Lee and P. A. Lee. We thank Wolfgang von der Linden for sending us his MaxEnt code and for many helpful correspondences. We also thank W. Putikka, R. Eder, and S. Maekawa for sending us their data for comparison. We acknowledge helpful conversations with J.T. Chalker, S.M. Girvin, D.H. Lee, E. Sorensen, X.-G. Wen, and S.C. Zhang.

Chapter 3

SPINONS AND MAGNETIC PROPERTIES

[THEORY OF SPIN EXCITATIONS IN UNDOPED AND UNDERDOPED CUPRATES]

Now sing my muse, for 'tis a weighty cause.
Explain the Magnet, why it strongly draws,..

Lucetius Carus, *De Rerum Natura*,
as quoted in D.C. Mattis, *The Theory of Magnetism I*

3.1 Introductory Remarks

The essence of the physics of high T_c cuprates boils down to the problem of how to treat the dual nature of the electrons that form local moments in the insulating compound, yet make up a Fermi surface when doped with $\sim 15\%$ holes. The problem conjures up an old ghost and invokes new ones. Many have struggled with the old ghost before, for example, in the localized–itinerant dichotomy of the f-electrons in the heavy fermion problem or in the familiar ferromagnetic metals like Ni and Fe. The present cuprate problem is simpler in the sense that a single electron in the unit cell (*i.e.* the Cu site) is believed to account for the transport and magnetic properties (superconductivity and antiferromagnetism), and there are no other electrons or orbitals of importance (see chapter 1), yet some new ghosts must be lurking underneath.

Central to our understanding of the problem is the physics of "underdoped" region which lies between the antiferromagnetic insulator and the optimally doped superconductor. How does the Fermi surface evolve from small hole pockets near $(\pm\pi/2, \pm\pi/2)$ in a slightly doped antiferromagnet to the full Fermi surface obeying Luttinger's theorem in the optimally doped materials? What are the magnetic properties in this intermediate doping region? Experimentalists have already answered a substantial part of these questions. In particular, the angle-resolved photoemission spectroscopy (ARPES) has shown the existence in the normal state of a gap with the same anisotropy as the d-wave gap of the superconducting state. Low-lying excitations are observed along a patch near $(\pm\pi/2, \pm\pi/2)$ ["Fermi surface segments"], but the Fermi surface, in the veritable sense of the word, does not exist. The ARPES results might have been accepted without much grudge simply as a plausible interpolation between the antiferromagnet and the optimally doped superconductor, had our understanding of metals not been so entrenched in the Fermi liquid theory; the notion of a metal without a Fermi surface is a serious embarrassment. At the same time, gaplike suppression of spin excitations are seen in NMR: the Knight shift and spin-lattice relaxation rates all decrease with decreasing temperature below certain temperatures.

Gaplike features in the underdoped cuprates might remind us of the spin liquids–the liquid of spin singlets. Yet the devil is in the details, and the underdoped cuprates deviate significantly from *generic spin liquids* like spin 1/2 ladders and integer spin chains. The latter, whose known examples are all nonmetals, are characterized by a clear gap (a scale below which there is no spectral weight for spin excitations) to the lowest (triplet) excitation that is inversely related to the correlation length; this gap can be seen in inelastic neutron scattering. The magnetic responses like uniform susceptibility and the NMR relaxation rate as a function of temperature have activated behaviors. Many of them can be satisfactorily described in terms of the "quantum disordered" phase of the nonlinear sigma model $\mathcal{L} = \frac{1}{g}(\partial_\mu \mathbf{n})^2, g > g_c$[102, 103] which can be also understood in terms of the CP^{N-1} model involving *bosonic* "spinons" (z fields) — these spin 1/2 excitations are said to be confined, as they do not appear in the basic physical spectra[104]. In other words, a description in terms of fluctuating spin 1 objects is most natural for them.

On the other hand, in the underdoped cuprates, inelastic neutron scattering does not show a clear gap like those of the generic spin liquids[105]. The decrease of the Knight shift with temperature looks more like a power-law. On the whole, the magnetic excitation spectrum of the cuprates seems to display a curious mixture of singlet and antiferromagnetic correlations. There are evidences for antiferromagnetic correlations from the $\mathbf{Q}$-space scan of neutron scattering cross section, and from the difference in the temperature dependence of the NMR relaxation rates : the Oxygen $1/T_1T$ (which has little contribution from spin excitations near wave vector $\mathbf{Q} = (\pi, \pi)$) monotonically decreases with decreasing temperature, while the Copper $1/T_1T$ (which weighs (π, π) spin excitation strongly) increases with decreasing temperature until around 150K and then falls[106, 107].

The antiferromaget-singlet debate has enormous ramifications for the theories of high T_c (for a succinct review, see Ref.[108]). Some theorists[109] have advocated the picture of Fermi liquid quasiparticles exchanging "antiparamagnons" for understanding the anomalous normal state properties and the superconductive pairing. Such a view (namely that the NMR is the isotope effect-equivalent for the cuprate superconductors) is not shared here. Instead of viewing the antiferromagnetic fluctuations as the cause of the superconductivity in a BCS-like scenario, in this chapter we would rather regard the antiferromagnetic correlations as a residual but important consequence of the local repulsive interactions that lead to superconductivity in the presence of doped holes, a part of Nature's conspiracy to find a compromise between a magnetic ground state and an itinerant metallic state. This line of thinking goes back to Anderson's seminal 1987 paper[29] on the resonating valence bond (RVB) theory, in which he reasoned that doped holes may propagate coherently in the liquid of spin singlets.

Theoretical attempts to realize Anderson's RVB picture are based on strong coupling models, such as the one band Hubbard model or the t-J model[47]. These models are considered to contain some essential physics of the cuprates at appropriate parameter values U/t or J/t. The t-J model, the simpler of the two, captures in a transparent way what is believed to be the basic physics, namely the competition between the magnetic exchange and the delocalization energy of holes. The no-double-occupancy constraint in the t-J model can be taken care of by writing the electron operator as a composite of a neutral fermion (spinon) and a spinless boson (holon) $[c_{i\sigma}^\dagger = f_{i\sigma}^\dagger b_i]$ and demanding each site be occupied by either a fermion or a boson $(\sum_\sigma f_{i\sigma}^\dagger f_{i\sigma} + b_i^\dagger b_i = 1)$. The theory then contains four-particle interactions which can be decoupled by introducing "mean fields" like $\chi_{ij} = \langle f_{i\sigma}^\dagger f_{j\sigma}\rangle$, $\Delta_{ij} = \langle f_{i\uparrow} f_{j\downarrow} - f_{i\downarrow} f_{i\uparrow}\rangle$, and $\eta_{ij} = \langle b_i^\dagger b_j\rangle$. Within the mean field approach, Kotliar and Liu[49] and Fukuyama and coworkers[48] have studied the phase diagram of the t-J model. At low doping (and below some temperature scale), it was found that the phases in which the fermions are paired into d-wave singlets ($\Delta_{i,i+x} = -\Delta_{i,i+y} \neq 0$)are favored. Depending on whether the bosons are condensed, they could be superconducting (SC phase) or normal ("d-wave RVB phase").

As noted by Rice[110] and others, the fermionic mean field theory captures some important features of the spin gap phenomena in underdoped cuprates which refuse clear-cut characterization in terms of a well-defined correlation length and relaxation times. The theory describes some kind of quantum spin liquid, but unlike the generic spin liquids, there is a particle-hole (spinon-antispinon) continuum, which would create some spectral weight for magnetic excitations at arbitrarily low energy. More specifically, the Dirac spectrum of the fermionic quasiparticles in the d-wave RVB phase gives the Knight shift $K \sim T$ and the Oxygen site NMR relaxation rate $1/T_1 \sim T^3$, in rough agreement with experiments in underdoped cuprates. Moreover, the absence of the gap in the charge response (for example, the in-plane optical conductivity) could be explained simply, since the spin and charge degrees of freedom are separated, *i.e.* the spin is carried by fermionic spinons while the charge is carried by bosonic holons.

A Dirac-type spectrum as in the d-wave RVB phase was also found by Affleck and Marston who considered the flux phase[111] as a possible spin liquid ground state of the cuprates. It turned out that at half filling the d-wave phase with $|\Delta_{ij}| = |\chi_{ij}|$ is equivalent to the flux phase, due to a local SU(2) symmetry[112]. Wen and Lee reasoned that this symmetry might still be a pretty good (and important) symmetry at small dopings, and came up with a slave boson theory that respects the SU(2) symmetry even away from half filling by introducing an SU(2) doublet of slave bosons, hoping to get a better description of underdoped cuprates[113]. In this theory, the mean field corresponding to the "spin gap" phase of the underdoped cuprates was identified as the "sFlux phase" which can be considered a combination of the d-wave RVB phase and the staggered Flux phase[114] of the U(1) theory. This phase also has fermions with a Dirac spectrum, but *in contrast to the d-wave RVB phase of the U(1) mean field theory*[115], *the fluctuations around this mean field include a massless gauge field which is expected to affect strongly the magnetic and transport properties of the system*[116]. With the inclusion of a residual attraction between bosons and fermions, the sF phase was shown to reproduce the gross features of the ARPES, such as the Fermi surface segments near $(\pm\pi/2, \pm\pi/2)$ and a large gap at $(\pi, 0)$.

Despite the successes, the mean field treatments (both the SU(2) theory and its predessesors) are unsatisfactory in several respects. For example, it is not clear how the spin gap phase is connected to the Néel ordered phase at zero doping. *The mean field ansatz loses a lot of antiferromagnetic correlation; within the mean field theory, the Copper site $1/T_1T$ has the same behavior as the Oxygen site $1/T_1T$, in disagreement with experiments.* Attempts to fix the problem by some kind of RPA scheme to enhance antiferromagnetic correlation do not seem to work naturally. Another serious question is the role of gauge fluctuations around the mean field solution which had not been studied carefully so far. In fact, a strong gauge fluctuation might destroy the mean field picture altogether, in which case we have to re-identify the elementary excitations of the theory[117].

In this chapter, we look into these questions. The basic point is that the gauge fluctuations ignored at the mean field level strongly enhance antiferromagnetic correlations. The gauge fluctuation could be so strong that the elementary excitations of the mean field theory disappear completely from the low energy spectrum. This is believed to be the case with the "flux phase" description of the undoped cuprates (antiferromagnet), which is a theory of massless Dirac fermions coupled to a U(1) gauge field (massless QED3). The QED3 can be treated in the $1/N$ perturbation theory. For physical $N(=2)$, a dynamic mass generation and spontaneous symmetry breaking corresponding to Néel ordering would occur[118], while for large enough N, the theory would still describe some kind of a spin liquid. The true low energy excitations of the symmetry-broken case are recognized as the Goldstone bosons–"mesons" which are a bound state of a particle and an antiparticle (spinon & antispinon). In the sFlux phase of the SU(2) theory for underdoped cuprates, again there are massless Dirac fermions coupled to a U(1) gauge field, but it is argued that *due to additional coupling of the gauge field to the holons (bosons), the gauge fluctuations will not destroy the validity of the mean field picture, but the picture of antiferromagnetic spin excitations will be much improved.* The coupling to the bosons would result in the screening of the time component of the gauge field which will prevent the Néel ordering. The gauge field will nevertheless mediate an attraction between spinons and antispinons and try to create a bound state with momentum $\sim (\pi, \pi)$, but due to the particle-hole continuum, this will appear only as a broad resonance. This can be viewed as a Goldstone boson precursor mode that comes down in energy as the transition is approached (as the boson density is reduced). The recent neutron scattering in underdoped cuprates which sees a broad peak in $\mathbf{Q} \approx (\pi, \pi)$ magnetic response whose energy scale is roughly proportional to doping might be consistent with this point of view[121]. We shall also discuss the issue of confinement, as there are lingering questions about the fate of "spinons" in the case of strong coupling gauge theories.

In order to illustrate some aspects of the foregoing ideas more concretely (*in particular the gauge-fluctuation restoration of antiferromagnetic correlation and the feasibility of the strong coupling gauge theories*), we'll first reexamine the well known spin half chain from the point of view of gauge theory — the Schwinger model.

3.2 Lessons from 1d Spin Chain

I do not know that I shall have anything particularly new in substance, but shall be contented if I can so choose my standpoint (as seems to me possible) as to get a simpler view of the subject.

Josiah Willard Gibbs, *Letter to John William Strutt (Lord Rayleigh), 1892*

At first, the prospect of describing the 1d quantum antiferromagnet in terms of massless quantum electrodynamics might seem dubious. After all, the original 1+1D massless QED of Schwinger is often discussed in particle physics as a model theory exhibiting quark confinement and chiral symmetry breaking[122] whose spin chain analogues are not obvious. Nonetheless, it has been known using conformal field theories (bosonization) that certain strong coupling gauge theories with massless Dirac fermions describe the 1d antiferromagnet[123, 124]. Here, we take a more pedestrian point of view. More specifically, we shall see that treating the effects of gauge fluctuations in $1/N$ perturbation theory systematically improves the mean field result, *i.e.* enhances the antiferromagnetic correlation. Although the 1d case is special in several important ways, some features of the perturbation theory may be viewed to persist, albeit in a less spectacular form, into the 2d case.

3.2.1 RVB theory of 1d spin chain

The Heisenberg model ($H = J\sum_{<ij>} \mathbf{S}_i \cdot \mathbf{S}_j$) in the fermion representation of the spin can be written

$$H = -\frac{J}{2} \sum_{<ij>} f^\dagger_{j\alpha} f_{i\alpha} f^\dagger_{i\beta} f_{j\beta} \tag{3.1}$$

with the constraint $\sum_\alpha f^\dagger_{i\alpha} f_{i\alpha} = 1$. The 4-fermion interactions and the constraint can be handled by the introduction of a Hubbard-Stratonovich field and a Lagrange multiplier, which gives

$$H = -J/2 \sum_{<ij>} (\chi_{ij} f^\dagger_{i\alpha} f_{j\alpha} + \text{h.c.}) + i \sum_i \lambda_i (f^\dagger_{i\alpha} f_{i\alpha} - 1). \tag{3.2}$$

Within the mean field theory, $\chi_{ij} = \chi, \lambda = 0$, hence the mean field hamitonian in the k-space is

$$H_{mf} = -\chi J \sum_k \cos(k) f^\dagger_{\alpha k} f_{\alpha k} \tag{3.3}$$
$$= -\chi J \Sigma'_k \cos(k)(f^\dagger_{\alpha k} f_{\alpha k} - f^\dagger_{\alpha,k-\pi} f_{\alpha,k-\pi}) \tag{3.4}$$
$$= -\chi J \Sigma'_k \cos(k)(f^\dagger_{\alpha e k} f_{\alpha o k} + f^\dagger_{\alpha o k} f_{\alpha e k}). \tag{3.5}$$

Here Σ'_k denotes sum over the magnetic BZ, say $0 < k < \pi$, and f_{ek}, f_{ok} are even and odd site operators ($f_{ek} = \frac{1}{\sqrt{2}}(f_k + f_{k-\pi}), f_{ok} = \frac{1}{\sqrt{2}}(f_k - f_{k-\pi})$). This hamiltonian has the same form as that of spin 1/2 XY model in 1d written in terms of *different* fermions (Jordan Wigner fermions).

Linearizing around $k = \pi/2 + k'$, we arrive at the continuum hamiltonian

$$H = -\int dk' \psi^\dagger_\alpha(k')\sigma_1 k' \psi_\alpha(k'), \tag{3.6}$$

where $\psi_\alpha = \begin{pmatrix} f_{\alpha e} \\ f_{\alpha o} \end{pmatrix}$ and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a Pauli matrix. This is just the hamiltonian of free Dirac fermions (we have set the velocity of the fermions =1). The corresponding (Euclidean space) lagrangian is

$$L = \bar{\psi}_\alpha \gamma_\mu \partial_\mu \psi_\alpha, \tag{3.7}$$

where $\bar{\psi} = \psi \gamma_0$, and $\mu = 0, 1$, and the γ matrices are

$$\gamma_0 = \sigma_3, \quad \gamma_1 = -\sigma_2. \tag{3.8}$$

In 1+1D, we define γ_5 matrix as $\gamma_5 = -i\gamma_0\gamma_1 = \sigma_1$, which has the property

$$\{\gamma_5, \gamma_\mu\} = 0, \quad \epsilon_{\mu\nu}\gamma_\nu = i\gamma_\mu\gamma_5. \tag{3.9}$$

Including the fluctuations around the mean field (the fluctuations of λ_i and the phase of χ_{ij}) amounts to coupling the fermions to a U(1) gauge field by the minimal prescription. Hence the continuum version of {the mean field + fluctuations} is the two flavor Schwinger model

$$L = \bar{\psi}_\alpha \gamma_\mu (\partial_\mu - i a_\mu) \psi_\alpha. \tag{3.10}$$

The apparent gauge coupling (bare coupling) is infinitely strong, as there is no kinetic term for the gauge field.

Above lagrangian is obviously invariant under the global SU(2) transform (spin rotation symmetry)

$$\psi_\alpha \to (\exp(i\phi^l \tau^l))_{\alpha\beta} \psi_\beta \tag{3.11}$$

where $\tau^l, l = 1, 2, 3$ are Pauli matrices (belonging to a space different from that of

σ_l). The lagrangian is also invariant under the "chiral transformation"

$$\psi_\alpha \to \exp(i\theta\gamma_5)\psi_\alpha. \tag{3.12}$$

This "chiral symmetry" is *explicitly* broken by higher derivative terms ignored in taking the continuum limit, like

$$L' = \bar{\psi}_\alpha \gamma_5 \gamma_\mu (\partial_\mu - i a_\mu)^2 \psi_\alpha. \tag{3.13}$$

We now consider spin correlation functions at the mean field level (*i.e.* ignoring gauge fields). The spin operators in the continuum has two contributions (uniform & staggered):

$$\begin{aligned} S^l(x_1) &\approx [f^\dagger_{\alpha e}(x_1)\frac{\tau^l_{\alpha\beta}}{2} f_{\beta e}(x_1) + f^\dagger_{\alpha o}(x_1)\frac{\tau^l_{\alpha\beta}}{2} f_{\beta o}(x_1)] \\ &+ (-1)^{x_1}[f^\dagger_{\alpha e}(x_1)\frac{\tau^l_{\alpha\beta}}{2} f_{\beta e}(x_1) - f^\dagger_{\alpha o}(x_1)\frac{\tau^l_{\alpha\beta}}{2} f_{\beta o}(x_1)] \\ &= \bar{\psi}_\alpha(x_1)\gamma_0 \frac{\tau^l_{\alpha\beta}}{2}\psi_\beta(x_1) + (-1)^{x_1}\bar{\psi}_\alpha(x_1)\frac{\tau^l_{\alpha\beta}}{2}\psi_\beta(x_1). \end{aligned} \tag{3.14}$$

To evaluate the spin correlation function

$$\begin{aligned} \langle S^+(x)S^-(0)\rangle &= \langle \bar{\psi}\gamma_0\tau^+\psi(x)\bar{\psi}\gamma_0\tau^-\psi(0)\rangle + (-1)^{x_1}\langle \bar{\psi}\tau^+\psi(x)\bar{\psi}\tau^-\psi(0)\rangle \\ &= \langle \bar{\psi}_1\gamma_0\psi_2(x)\bar{\psi}_2\gamma_0\psi_1(0)\rangle + (-1)^{x_1}\langle \bar{\psi}_1\psi_2(x)\bar{\psi}_2\psi_1(0)\rangle \end{aligned}$$

($\tau^\pm = (\tau^1 \pm i\tau^2)/2$), we need the fermion Green's function

$$\mathbf{G}_{\alpha\beta}(x) = \langle \psi_\alpha(x)\bar{\psi}_\beta(0)\rangle \equiv G(x)\delta_{\alpha\beta} \tag{3.15}$$

which can be obtained from the momentum space Green function $G(k) = -ik\gamma/k^2$ as

$$G(x) = \gamma_\mu \frac{\partial}{\partial x_\mu}\int \frac{d^2k}{(2\pi)^2}\frac{e^{-ik\cdot x}}{k^2} = \frac{-x\gamma}{2\pi x^2}. \tag{3.16}$$

Here and from now on, unless otherwise specified, we use the usual field theory notation: $k = (k_0, k_1), x = (x_0, x_1)$ [italics denote space time vectors]; $x\gamma \equiv x_\mu\gamma_\mu$; $x^2 = x_0^2 + x_1^2$, etc. Using Wick's theorem, we have

$$\begin{aligned} \langle S^+(x)S^-(0)\rangle = &= -\mathrm{tr}_{\gamma,\tau}[\mathbf{G}(x)\gamma_0\tau^+\mathbf{G}(-x)\gamma_0\tau^-] - (-1)^{x_1}\mathrm{tr}_{\gamma,\tau}[\mathbf{G}(x)\tau^+\mathbf{G}(-x)\tau^-] \\ &= \frac{1}{2\pi^2}\left[\frac{x_0^2 - x_1^2}{(x_0^2+x_1^2)^2} + (-1)^{x_1}\frac{1}{x_0^2+x_1^2}\right] \\ &= \frac{1}{4\pi^2}\left[\frac{1}{x_-^2} + \frac{1}{x_+^2} + (-1)^{x_1}\frac{2}{x_-x_+}\right], \end{aligned} \tag{3.17}$$

($x_\pm \equiv x_0 \pm ix_1$; $\mathrm{tr}_{\gamma,\tau}$ denotes trace over both the γ and τ spaces (spinor and spin spaces), which does (and should) equal the $\langle S_z(x)S_z(0)\rangle$ correlation function in the XY model[125, 126]. The equal time correlation function $\langle \mathbf{S}(x_1)\cdot\mathbf{S}(0)\rangle$ behaves as

$$\langle \mathbf{S}(x_1)\cdot\mathbf{S}(0)\rangle = \frac{3}{4\pi^2(x_1)^2}((-1)^{x_1}-1), \tag{3.18}$$

(the spins on the same sublattice are not correlated at all, while the correlation among spins on different sublattices are decaying algebraically as $1/x_1^2$). This peculiar behavior (that nevertheless agrees with Arovas and Auerbach's lattice version of the mean field theory[127], as well as Bulaevskii's Hartree-Fock treatment of the Jordon-Wigner fermionized Heisenberg model[128]) is viewed as a pathology of the mean field theory: *we have lost a substantial amount of antiferromagnetic correlation.*

3.2.2 Schwinger model

We now consider the effect of gauge fluctuations. It is natural to expect that the inclusion of gauge fluctuations will improve the mean field picture. The time component of the gauge field can be regarded to originate from the Lagrange multiplier field (for no-double-occupancy); this corresponds to Gutzwiller-projected (half filled tight binding) Fermi surface, which is known to be a pretty good description of 1d antiferromagnet[129, 130]. Haldane has implemented similar ideas by the bosonization method and shown that the correct correlation functions are easily reproduced[131].

As mentioned earlier, our theory with fluctuations is a Schwinger model[132]. For reasons that will become clear shortly, we consider a slightly more general case of N-flavors:

$$\begin{aligned} Z &= \int D\bar{\psi}D\psi Da_\mu \exp(-S), \\ S &= \int d^2x \sum_{\alpha=1}^{N} \bar{\psi}_\alpha\gamma_\mu(\partial_\mu - ia_\mu)\psi_\alpha + \frac{1}{4e^2}F_{\mu\nu}^2 \quad (e^2=\infty). \end{aligned} \tag{3.19}$$

The physical case is $N=2$; general (even) N corresponds to an SU(N) antiferromagnet.

Integrating out the fermions gives

$$Z = \int Da_\mu \exp(N\,\mathrm{Tr}\ln(1 - i\mathcal{G}\gamma_\mu a_\mu)) \tag{3.20}$$

where $\mathcal{G}(x,x') = (\gamma_\mu\partial_\mu)^{-1}\delta(x-x')$, and Tr denotes traces over the spinor space and the position space (Tr=tr$\int d^2x d^2x'...$). The logarithm can be expanded, giving

$$Z = \int Da_\mu \exp\left(-\frac{1}{2}\int d^2x d^2x' a_\mu(x)\Pi_{\mu\nu}(x-x')a_\nu(x')\right), \tag{3.21}$$

where $\Pi_{\mu\nu}(x) = -N\,\mathrm{tr}[G(x)\gamma_\mu G(-x)\gamma_\nu]$. Note that the beyond-Gaussian terms (like $\Gamma_{\mu\nu\rho\delta}a_\mu a_\nu a_\rho a_\delta$) are all zero; the proof can be found, for example, in Refs.[133, 134].

The polarization function $\Pi_{\mu\nu}(q)$ (in the momentum space) contains a divergence that has to be regulated using gauge invariant schemes, like the dimensional regularization or the Pauli-Villars regularization. Relegating the details to Appendix A.2, we have

$$\Pi_{\mu\nu}(k) = \frac{N}{\pi}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right), \tag{3.22}$$

which means that the gauge boson acquires an infinite mass ($= e\sqrt{N/\pi}$). The transversality of Eq.3.22 guarantees the conservation of the current $j_\mu = \bar{\psi}_\alpha \gamma_\mu \psi_\alpha$:

$$q_\mu j_\mu(q) = q_\mu(i\Pi_{\mu\nu}(q)a_\nu(q)) = 0 \ \rightarrow \partial_\mu j_\mu = 0. \tag{3.23}$$

On the other hand the current $j_{5\mu} (= i\bar{\psi}_\alpha\gamma_\mu\gamma_5\psi_\alpha = -\epsilon_{\mu\nu}j_\nu)$ associated with the chiral symmetry (Eq.3.12) is not conserved:

$$q_\mu j_{5\mu} = -q_\mu \epsilon_{\mu\nu} i\Pi_{\nu\rho}a_\rho = -\frac{iN}{\pi}\epsilon_{\mu\nu}q_\mu a_\nu \rightarrow \partial_\mu j_{5\mu} = -\frac{iN}{2\pi}\epsilon_{\mu\nu}F_{\mu\nu}. \tag{3.24}$$

This result, the so-called axial anomaly, can be regarded as either a consequence of or a condition for gauge invariance.

The exact spin correlation functions can be evaluated with the use of "chiral rotation"[135, 136, 137, 138]. In this approach, the gauge field is written as the sum of a div-free part and a curl-free part:

$$a_\mu = i\epsilon_{\mu\nu}\partial_\nu\phi_a + \partial_\mu\phi_b. \tag{3.25}$$

The transform

$$\psi_\alpha \rightarrow \psi'_\alpha = \exp(-i\gamma_5\phi_a - i\phi_b)\psi_\alpha \tag{3.26}$$

decouples the gauge field from the ψ' fermions, but the chiral part (ϕ_a) does not leave the Grassmann measure invariant. The jacobian $\mathcal{J}$ for the change of measure

$$D\psi D\bar{\psi} = \mathcal{J}^2 D\psi' D\bar{\psi}' \tag{3.27}$$

can be found straightforwardly by Fujikawa's technique[136]:

$$\mathcal{J} = \exp\left(-i\frac{N}{4\pi}\int d^2x\phi_a\epsilon_{\mu\nu}F_{\mu\nu}\right) = \exp\left(\frac{N}{4\pi}\int d^2x(\partial_\mu\phi_a)^2\right). \tag{3.28}$$

The same result can be obtained in a simple manner using the axial anomaly condition (Eq.3.24)[137] or by other methods[138]. In terms of the new fields, the functional integral is

$$Z = \int D\bar{\psi}' D\psi' D\phi_a D\phi_b \exp - \int d^2x\left(\bar{\psi}'_\alpha\gamma_\mu\partial_\mu\psi'_\alpha - \frac{N}{2\pi}(\partial_\mu\phi_a)^2\right). \tag{3.29}$$

These are "free" fields, but note that the ϕ_a-field has an indefinite (negative) metric! Now, the spin correlation function

$$\langle \mathbf{S}(x)\cdot\mathbf{S}(0)\rangle = \langle\bar{\psi}_1\gamma_0\psi_2(x)\bar{\psi}_2\gamma_0\psi_1(0)\rangle + (-1)^{x_1}\langle\bar{\psi}_1\psi_2(x)\bar{\psi}_2\psi_1(0)\rangle \tag{3.30}$$

can be evaluated easily:

$$\begin{aligned}
\langle\bar{\psi}_1\gamma_0\psi_2(x)\bar{\psi}_2\gamma_0\psi_1(0)\rangle &= \langle\bar{\psi}'_1\gamma_0\psi'_2(x)\bar{\psi}'_2\gamma_0\psi'_1(0)\rangle \\
&= \frac{1}{2\pi^2}\frac{x_0^2-x_1^2}{(x_0^2+x_1^2)^2}, \\
\langle\bar{\psi}_1\psi_2(x)\bar{\psi}_2\psi_1(0)\rangle &= \langle\bar{\psi}'_1 e^{-i2\gamma_5\phi_a}\psi'_2(x)\bar{\psi}'_2 e^{-i2\gamma_5\phi_a}\psi'_1(0)\rangle \\
&= \frac{1}{2\pi^2x^2}\langle e^{2i\phi_a(x)}e^{-2i\phi_a(0)}\rangle = \frac{1}{2\pi^2x^2}e^{-2\langle(\phi_a(x)-\phi_a(0))^2\rangle} \\
&= \frac{\mathcal{C}}{x^2}e^{\frac{1}{N}\ln(x^2)} = \frac{\mathcal{C}}{(x^2)^{1-1/N}},
\end{aligned} \tag{3.31}$$

where we have used

$$\langle(\phi_a(x)-\phi_a(0))^2\rangle = \frac{2\pi}{N}\int^{\Lambda}\frac{d^2k}{(2\pi)^2}\frac{1}{k^2}(e^{ik\cdot x}-1) = -\frac{1}{2N}\ln(x^2\Lambda^2), \tag{3.32}$$

(Λ is a UV cutoff originating from the lattice theory, and $\mathcal{C}$ is a nonuniversal constant that depends on high energy details Λ). In the physical case ($N=2$), we then have

$$\langle\mathbf{S}(x)\cdot\mathbf{S}(0)\rangle \sim (-1)^{x_1}\frac{1}{\sqrt{x_1^2+x_0^2}} \tag{3.33}$$

which agrees with the more accurate result[139] up to a $\ln^{1/2}(x^2)$ factor. The log factor, not captured by the Schwinger model, must be due to terms ignored in our derivation from the lattice theory, *e.g.*, the amplitude fluctuation of the RVB field; this is analogous to the bosonization theory of Heisenberg model, in which the Umklapp processes give rise to logarithms[140]. Sachdev[141] and others have used the correlation function of Eq.3.33, and shown that it captures the low energy (temperature) properties of the Heisenberg spin chains quite well.

Before moving on to the perturbative treatment of the same theory, we note that we have concentrated on correlation functions, rather than elementary excitations. It is well known that the basic elementary excitations (the true spinons) in the Heisenberg spin chain are solitonic objects with spin a half[142]. It is tempting to identify them with our ψ'_α fields: they carry spin 1/2, and the relation between ψ' and ψ fields ($\psi' = \exp(-i\gamma_5\phi_a - i\phi_b)\psi$) is reminiscent of the expression for spinons (Jordan Wigner fermions) of the XY model ($f_i = e^{i\phi_i}S_i^-$, $\phi_i = \pi\sum_{j=1}^{i-1}S_i^+S_i^-$). However, because ϕ_a is a field with a negative norm, such identification is rather premature, and it is perhaps necessary to bosonize the theory to pin down the real elementary excitations.

3.2.3 perturbation theory

We now examine the physics in terms of the perturbation theory in $1/N$. This can be regarded as an implementation of the program suggested by Arovas and Auerbach[127]. The key point is that the nature of the perturbative correction to the mean field results is very different for the uniform part and the antiferromagnetic

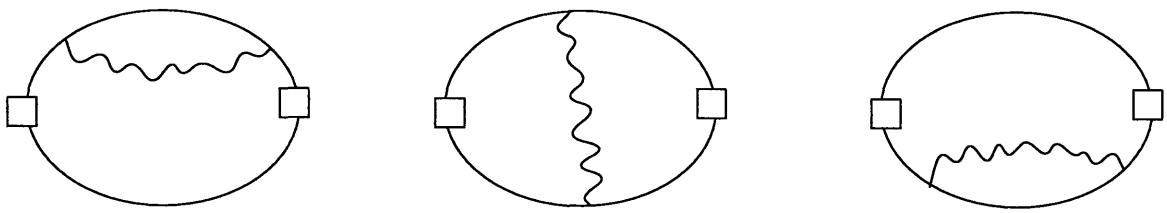

Figure 3-1: Leading $1/N$ correction to uniform spin correlation.

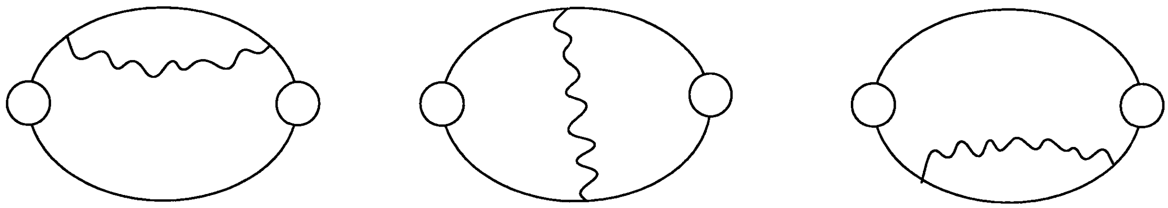

Figure 3-2: Leading $1/N$ correction to staggered spin correlation.

(staggered) part of the spin correlation: While $\mathbf{Q} = \pi$ response is strongly affected by perturbative correction, the $\mathbf{Q} = 0$ response receives no correction at all (*no* correction is a special feature of the 1d).

The leading $1/N$ correction to the spin correlation functions can be straightforwardly evaluated. They are represented by the Feynman diagrams in Fig.3-1 and Fig.3-2. Within the usual Faddeev-Popov scheme of gauge fixing (the introduction of the $\frac{1}{2\lambda}(\partial \cdot a)^2$ term to the lagrangian), the gauge propagator (represented by the wiggly line) is given by

$$D_{\mu\nu} = \langle a_\mu a_\nu \rangle = \frac{\pi}{N}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \lambda \frac{q_\mu q_\nu}{(q^2)^2}. \tag{3.34}$$

We choose the Landau gauge ($\lambda = 0$) which is a natural choice, since in this gauge no infrared divergence occurs in the perturbation theory[143]. The fermion propagator $G(p) = (ip\gamma)^{-1}$ is represented by a solid line. The fermion-gauge vertex is simply $i\gamma_\mu$; the external current vertex is γ_0 for the uniform part (represented by a square) and 1 for the staggered part (represented by a circle). Of course a trace is taken over the fermion loops.

In 1+1D, the transverse projector has a special property

$$\delta_{\mu\nu} - q_\mu q_\nu / q^2 = \epsilon_{\mu\rho}\epsilon_{\nu\delta} q_\rho q_\delta / q^2 \tag{3.35}$$

which, together with the "Ward identity"

$$G(p+q) q\gamma G(p) = i(G(p) - G(p+q)), \tag{3.36}$$

simplifies the algebra substantially (Note $\gamma_\mu \epsilon_{\mu\rho} p_\rho = -i\not{p}\gamma\gamma_5$). For example, the diagram 3.1b is

$$
\begin{aligned}
&\frac{\pi}{N}\int \frac{d^2p}{(2\pi)^2}\frac{d^2q'}{(2\pi)^2}\mathrm{tr}[G(p+q)\not{q}'\gamma\gamma_5 G(p+q+q')\gamma_0 G(p+q')\not{q}'\gamma\gamma_5 G(p)\gamma_0]/q'^2 \\
&= -\frac{\pi}{N}\int \frac{d^2p}{(2\pi)^2}\frac{d^2q'}{(2\pi)^2}\mathrm{tr}[(G(p+q)-G(p+q+q'))\gamma_0(G(p+q')-G(p))\gamma_0]/q'^2.
\end{aligned}
\tag{3.37}
$$

It's straightforward to show that sum of the diagrams 3.1a+3.1c is the same as Eq.3.37, except for a minus sign. Therefore in the uniform channel, the vertex correction and the self energy correction cancel. Similar cancellation is expected at all orders of perturbation theory; the nonrenormalization of uniform part of the spin correlation function is quite natural, since in our theory

$$
\langle \bar{\psi}_1 \gamma_0 \psi_2 \bar{\psi}_2 \gamma_0 \psi_1 \rangle \propto \langle j_0 j_0 \rangle = \Pi_{00}, \tag{3.38}
$$

and Eq.3.21 is an exact result.

On the other hand, the diagrams in the staggered channel do not cancel. The sum of the diagrams 3.2a and 3.2c are equal to 3.2b, which is given by

$$
\frac{\pi}{N}\int \frac{d^2p}{(2\pi)^2}\frac{d^2q'}{(2\pi)^2}\mathrm{tr}[(G(p+q)-G(p+q+q'))1(G(p+q')-G(p))1]/q'^2
$$

$$
= \frac{2\pi}{N}\int \frac{d^2p}{(2\pi)^2}\frac{d^2q'}{(2\pi)^2}\mathrm{tr}[G(p+q')G(p+q)-G(p+q)G(p))]/q'^2. \tag{3.39}
$$

(3.40)

Therefore, in the coordinate space, the $1/N$ correction is

$$
\mathrm{tr}[G(x)G(-x)]\frac{4\pi}{N}\int \frac{d^2q}{(2\pi)^2}\frac{1}{q^2}(e^{iq\cdot x}-1) = \frac{1}{2\pi^2 N x^2}\ln(x^2). \tag{3.41}
$$

Similar (but a lot more tedious) calculation would show that $1/N^2$ correction is given by $\frac{1}{4\pi^2 N^2 x^2}(\ln(x^2))$. In other words, the perturbation series exponentiates:

$$
\langle \bar{\psi}_1 \psi_2(x) \bar{\psi}_2 \psi_1(0) \rangle = \frac{1}{2\pi^2 x^2}(1 + (1/N)\ln(x^2) + \frac{1}{2}(1/N)^2 \ln^2(x^2) + ...) \propto \frac{1}{(x^2)^{1-1/N}}, \tag{3.42}
$$

giving the same result obtained from the chiral rotation approach.

3.3 2d Undoped Cuprates

The success of 1+1D gauge theory with Dirac fermions in describing the Heisenberg spin chain tempts us that a similar theory of massless Dirac fermions strongly coupled to a U(1) gauge field might describe a 2d quantum antiferromagnet. In fact, it is known from lattice gauge theories[144] that this is indeed so. SU(2) gauge theories with massless Dirac fermions may also describe the quantum antiferromagnet[112, 145], but we shall not consider this possibility because of the greater complexity of the nonabelian gauge theories.

3.3.1 Dirac fermons and the 2d Heisenberg antiferromagnet

A 2+1D theory of Dirac fermions

$$L = \bar{\psi}_\alpha \partial_\mu \gamma_\mu \psi_\alpha, \tag{3.43}$$

(μ=0,1,2) contains fermions whose density of states that behaves as $\sim |\epsilon|$ (in a general $D = d+1$ dimensions, the density of states will be $\sim |\epsilon|^{d-1}$). In condensed matter context, the paramagnetism of such fermions will result in the uniform susceptibility behaving as T^{d-1}. This seems to have little in common with the 2d antiferromagnet, but we shall see that a gauge field coupled to fermions can produce the correspondence with the physics of 2d antiferromagnet. Such a picture can be motivated from the flux phase mean field ansatz of the Heisenberg hamiltonian. The ansatz is so named since the phase of the product of the mean field parameter χ_{ij} around each plaquette ($\mathrm{Im}\ln(\chi_{12}\chi_{23}\chi_{34}\chi_{41})$) is π. Despite the "flux," the mean field does not break the parity and time reversal symmetry since the flux of π is equal to $-\pi$. This phase has a lower energy than the BZA phase (with a large Fermi surface); it also has the fermion spectrum

$$\epsilon(\mathbf{k}) = v\sqrt{\cos^2(k_x) + \cos^2(k_y)} \tag{3.44}$$

that roughly captures the high energy features of the undoped cuprates and the dispersion of a single hole in the antiferromagnet[146, 147]. The low energy fermionic excitations of the flux phase reside near two "Fermi points", $\mathbf{k}_{1,2} = (\pi/2, \pm\pi/2)$. Linearizing around these points gives the continuum theory[148]

$$L = \bar{\psi}_{\alpha a} \partial_\mu \gamma_\mu \psi_{\alpha a}, \tag{3.45}$$

where $\psi_{\alpha 1} = \begin{pmatrix} f_{\alpha 1e} \\ f_{\alpha 1o} \end{pmatrix}$, $\psi_{\alpha 2} = \begin{pmatrix} f_{\alpha 2o} \\ f_{\alpha 2e} \end{pmatrix}$, ($a = 1,2$ labels the two Fermi points; e,o denote even and odd sites.) Organizing the $\psi_{\alpha 1}, \psi_{\alpha 2}$ fields into a single spinor $\psi_\alpha \equiv \begin{pmatrix} \psi_{\alpha 1} \\ \psi_{\alpha 2} \end{pmatrix}$, we have a theory described by the lagrangian of Eq.3.43, with 4×4 γ-matrices:

$$\gamma_0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \; \gamma_1 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \; \gamma_2 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}. \tag{3.46}$$

Similarly to the 1d case, the uniform spin $S^l_{\mathbf{Q}=0}$ is given by $\bar{\psi}_\alpha \gamma_0 \tau^l_{\alpha\beta} \psi_\beta$, while the staggered spin $S^l_{\mathbf{Q}=(\pi,\pi)}$ is given by $\bar{\psi}_\alpha \tau^l_{\alpha\beta} \psi_\beta$. The spin correlation function

$$\langle S^+(x) \cdot S^-(0) \rangle = \langle \bar{\psi}_1 \gamma_0 \psi_2(x) \bar{\psi}_2 \gamma_0 \psi_1(0) \rangle + (-1)^{x_1+x_2} \langle \bar{\psi}_1 \psi_2(x) \bar{\psi}_2 \psi_1(0) \rangle \tag{3.47}$$

can be evaluated easily at the mean field level using the Green's function

$$\mathbf{G}_{\alpha\beta}(x) = \delta_{\alpha\beta}G(x) = \delta_{\alpha\beta}\gamma_\mu \frac{\partial_\mu}{\partial x_\mu}\int \frac{d^3q}{(2\pi)^3}\frac{e^{-iq\cdot x}}{q^2} = \delta_{\alpha\beta}\frac{x\gamma}{4\pi(x^2)^{3/2}} \tag{3.48}$$

(as in 1+1D, the italics like x, q denote space-time vectors, and $q^2 = q_0^2 + \mathbf{q}^2, x^2 = x_0^2 + \mathbf{x}^2$.) We have,

$$\begin{aligned}\langle S^+(x)S^-(0)\rangle &= -\mathrm{tr}[G(x)\gamma_0 G(-x)\gamma_0] - (-1)^{x_1+x_2}\mathrm{tr}[G(x)G(-x)] \\ &= \frac{1}{4\pi^2}\left[\frac{x_0^2 - \mathbf{x}^2}{x^6} + (-1)^{x_1+x_2}\frac{1}{x^4}\right]. \end{aligned} \tag{3.49}$$

The equal time correlation function is then

$$\langle S^+(0,\mathbf{x})S^-(0,0)\rangle = \frac{(-1)^{(x_1+x_2)} - 1}{4\pi^2\mathbf{x}^4}. \tag{3.50}$$

The mean field spin correlation function falls off as $1/\mathbf{x}^4$, and again as in 1d, the spins on the same sublattice are not correlated.

To go beyond the mean field level, gauge fluctuation is included by the minimal coupling scheme, leading to the following theory:

$$Z = \int D\bar{\psi}D\psi Da_\mu \exp\left(-\int d^3x \sum_{\alpha=1}^{N} \bar{\psi}_\alpha(\partial_\mu - ia_\mu)\gamma_\mu\psi_\alpha\right) \tag{3.51}$$

where N is a general number (in the physical case, $N = 2$). The absence of the $\frac{1}{4e^2}F_{\mu\nu}^2$ term means the bare coupling is infinite, but the theory is still sensible: the infrared behavior of the theory is well-behaved (within $1/N$ expansion, as we shall see), and the original lattice theory sets a ultraviolet cutoff. (Ultraviolet divergence can be also regulated by the the kinetic term $\frac{1}{4\tilde{e}^2}F_{\mu\nu}^2$ with $\tilde{e}^2 < \infty$.) As in 1+1D, the lagrangian contains more symmetries than the Heisenberg model. For example the theory is invariant under the transform $\psi_\alpha \to \exp(i\gamma_{4,5}\theta)\psi_\alpha$ (where $\gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma_5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$). Again, these symmetries are broken by higher order terms which have been ignored.

3.3.2 content of the gauge theory

We now explore the physical content of the theory with massless Dirac fermions. Integrating out the fermions generates the dynamics for the gauge field (see Appendix A.2 for details)

$$\begin{aligned} Z &= \int Da_\mu \exp\left(-\frac{1}{2}\int \frac{d^3q}{(2\pi)^3} a_\mu(q)\Pi_{\mu\nu}(q)a_\nu(q)\right), \\ \Pi_{\mu\nu} &= \frac{N}{8}\sqrt{q^2}(\delta_{\mu\nu} - q_\mu q_\nu/q^2). \end{aligned} \tag{3.52}$$

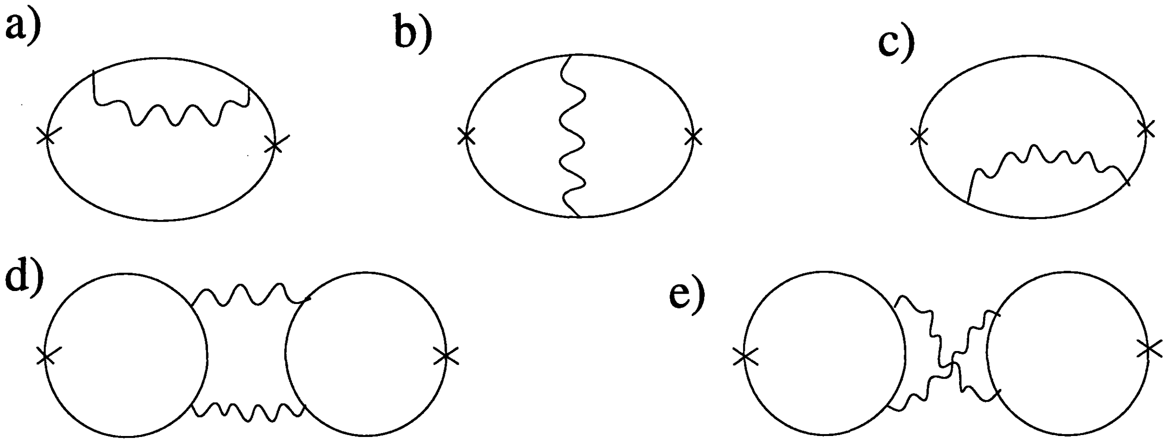

Figure 3-3: Leading $1/N$ correction to vacuum poloarization. The diagrams d) and e) are zero due to Furry's theorem.

The form of $\Pi_{\mu\nu}$ indicates that, unlike the 1+1D case, the gauge field is massless, though the infrared behavior is not as singular as the free gauge field (*i.e.*, $L = \frac{1}{4}F_{\mu\nu}^2$). The effect of the fermion-gauge field interaction can be analyzed perturbatively in $1/N$. The natural choice for the gauge fixing is the landau gauge

$$D_{\mu\nu}(q) = \frac{8}{N\sqrt{q^2}}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right). \tag{3.53}$$

The fermion self-energy at the leading order in $1/N$ is given by

$$\Sigma(k) = i\int \frac{d^3q}{(2\pi)^3}\gamma_\mu \frac{(k+q)\gamma}{(k+q)^2}\gamma_\nu D_{\mu\nu}(q) \sim ik\gamma \ln\left(\frac{\Lambda^2}{k^2}\right). \tag{3.54}$$

This log divergence does *not* occur in the $1/N$ correction to the polarization function represented by the diagrams in Fig.3-3. These diagrams have been calculated explicitly in a different context by Chen, Fisher, and Wu[149]. They found that the logarithmic divergences in the self energy correction (Figs.3-3 a+c) are cancelled by the vertex correction (Fig.3-3 b); the diagrams sum to

$$\Pi_{\mu\nu}^{1/N}(q) = \frac{3}{4\pi^2}\sqrt{q^2}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right), \tag{3.55}$$

which is of the same form as the zeroth order result (Eq.3.52) but down by some factor involving $1/N$. Thus, although we do not have a complete cancellation (like 1d), the gauge field is essentially unrenormalized, except for a some modification of the effective coupling constant ($1/N \to 1/N' \approx 1/N$). In other words, Z_3 (charge renormalization) ≈ 1. Weak correction to the gauge propagator (despite strong self energy correction) was seen in several other contexts, including the half-filled Landau level (fermion Chern-Simons theory)[150], bosonic Chern Simons theories[151, 152], and the uRVB gauge theory[52]. This robustness is due to the fact that $\Pi_{\mu\nu}$ is the correlation function of conserved currents[152], and conserved currents have no anomalous dimensions (as a consequence of the Ward identiy)[153, 154]. The foregoing argument provides some optimism for perturbation theory in $1/N$.

Because the uniform spin correlation $\Pi_u(x) = \langle \bar{\psi}_1\gamma_0\psi_2(x)\bar{\psi}_2\gamma_0\psi_1(0)\rangle \sim \Pi_{00}(x)$, it wouldn't be renormalized significantly. On the other hand, the staggered part $\Pi_s(x)$ of the spin correlation function does not involve conserved currents, and is expected to be strongly affected by gauge fluctuations. This difference can be more or less seen in perturbation theory. The diagramatic representation of Π_u is the same as in the 1+1D case (Fig.3-1). The external vertices in this case are a fermion-gauge field vertex (γ_0). Using the Ward identity

$$\frac{\partial}{\partial p_\mu}\Sigma(p) = i\Gamma_\mu(p,p), \tag{3.56}$$

we can see that the 2 overlapping divergences of Fig.3-1b are cancelled by self-energy bubbles of Fig.3-1a+c. For Π_s, the external vertices are 1 (unit matrix in the spinor space). In that case, the cancellation does not occur, and divergences develop in Π_s.

The question is, what would be ultimately the behavior of Π_s? Would $\Pi_s(x)$ be characterized by simple power law correlations like $\Pi_s(x) \sim 1/(x^2)^\alpha$, $(\alpha < 2)$ like the 1D case? This might be a realization of Anderson's 2d Luttinger liquid scenario[46], but we feel that *a priori* there is no reason to expect so; after all, 1+1D is rather special, with all sort of fascinating features like the conformal invariance[125, 126] and Coleman's theorem[155] which prohibits spontaneous symmetry breaking. More plausibly, we would expect a symmetry-broken phase (Néel order) or a symmetric phase with a more complicated magnetic correlations.

3.3.3 spontaneous symmetry breaking

The previous section has identified a possible antiferromagnetic instability, which corresponds to the staggered magnetization $\langle\bar{\psi}\tau^l\psi\rangle$ acquiring a definite orientation — an SU(2) symmetry breaking. We now examine this possibility more closely. In the symmetry-broken case, the fermions acquire a mass (dynamic mass generation). Without loss of generality, we assume that the rotation symmetry is broken in the z-direction. Then the fermion Green's function $\mathbf{G}(k)$ becomes $1/(ik\gamma + m(p)\tau^3)$. Note that $\mathbf{G}$ is a matrix Green's function in both the spinor space and the spin space. The "mass" $m(p)$ is related to the sublattice magnetization M by

$$M \sim \int \frac{d^3q}{(2\pi)^3}\mathrm{tr}_{\gamma,\tau}[\mathbf{G}(q)\tau_3]. \tag{3.57}$$

Self-consistent equation for $m(p)$ can be obtained from the Schwinger-Dyson equation. Expressed in terms of matrix (both in τ and γ space) Green's function and (matrix) self energy (pictorially represented by Fig.3-4a), the S-D equation is

$$\mathbf{\Sigma}(k) = -m(k)\tau^3 = -\int \frac{d^3q}{(2\pi)^3}\tilde{\Gamma}_\mu(k,q)\mathbf{G}(k-q)\gamma_\nu\tilde{D}_{\mu\nu}(q) \tag{3.58}$$

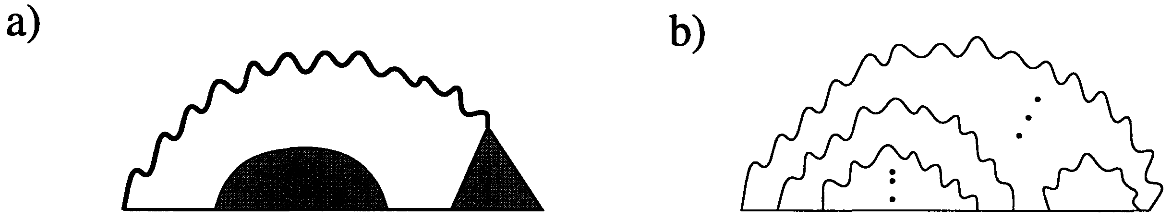

Figure 3-4: Schematic representation of the Schwinger-Dyson equation. In part a) the solid line with shaded blob is the self-consistent Green's function of the fermions **G**, the thick wiggly line is the dressed gauge field (which incorporates the changes in the vacuum polarization due to changes in fermion Green's function), and the shaded triangle is the dressed vertex. Part b) is a representation of the contribution to the self energy.

where $\tilde{\Gamma}_\mu, \tilde{D}_{\mu\nu}$ are fully renormalized vertex and gauge propagator. Within the approximation of replacing $\tilde{\Gamma}_\mu$ by γ_μ and $\tilde{D}_{\mu\nu}$ by $D_{\mu\nu}$, we have

$$m(p) = \int \frac{d^3k}{(2\pi)^3} \frac{\gamma_\mu m(k) \gamma_\nu}{k^2 + m^2(k)} D_{\mu\nu}(p-k). \tag{3.59}$$

Still this is a nonperturbative theory; it is easy to check that a finite order perturbation theory cannot generate a mass term in the fermion Green's function. Diagrams that contribute are shown in Fig.3-4b.

The context in which the self-consistent equation arises is similar to the SDW problem[156] and the superconductivity[157]. However, in our case, the mass $m(p)$ is dependent upon 3-momentum. Eq.3.59 has been already analyzed by Appelquist and coworkers in a different context ("chiral symmetry" breaking)[158]. Some of the steps are sketched here:

From the result

$$\gamma_\mu \gamma_\nu \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) = \gamma_\mu \gamma_\mu - \frac{q\gamma q \gamma}{q^2} = 3 - 1 = 2 \tag{3.60}$$

we have, after some angular integrals,

$$m(p) = \frac{4}{N\pi^2 p} \int_0^\Lambda dk \frac{k m(k)}{k^2 + m^2(k)} (k + p - |k-p|). \tag{3.61}$$

Note that the lattice origin of our theory sets the UV cutoff scale Λ, while in the theory of Appelquist *et al.* which retains the kinetic term $\sim \frac{1}{4e^2} F_{\mu\nu}^2$, the coupling constant e^2 sets the scale (QED3 is a superrenormalizable theory). The integral equation (3.61) is equivalent to the differential equation

$$\frac{d}{dp} \left(p^2 \frac{dm(p)}{dp} \right) = -\frac{8}{\pi^2 N} \frac{p^2 m(p)}{p^2 + m^2(p)} \tag{3.62}$$

with boundary conditions

$$\Lambda m'(\Lambda) + m(\Lambda) = 0 \tag{3.63}$$

and

$$0 \le m(0) < \infty. \tag{3.64}$$

It turns out that this nonlinear differential equation has a nontrivial solution only for $N < N_c = 32/\pi^2$. For the physical case of SU(2) antiferromagnet, N equals 2; therefore, the dynamical mass generation occurs, and the Néel-vector rotation symmetry is spontaneously broken. Thus, provided that we include the gauge fluctuations, we have a Néel order.

The foregoing argument, however, should be taken with a grain of salt. In principle, the gauge propagator that enters Schwinger Dyson equation must be a fully dressed one, and so should be the vertex. In the symmetry-broken phase, the gauge propagator is different from the symmetric phase, as the polarization function $\Pi_{\mu\nu}$ is different. At the crudest level, if we assume that the fermions acquire a constant mass of m, then the polarization function would be (See Appendix A.2).

$$\Pi_{\mu\nu}(q) \sim \frac{1}{m} q^2 \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right), \tag{3.65}$$

which means that a kinetic term $\sim \frac{1}{m} F_{\mu\nu}^2$ is generated. This results in the Coulomb potential for the fermions which in 2+1D is

$$\begin{aligned} V(\mathbf{x}) &= -m \int d^2\mathbf{q} \frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{\mathbf{q}^2} \\ &= m \ln(|\mathbf{x}|/R), \quad (R \sim 1/m). \end{aligned} \tag{3.66}$$

Since the potential increases at large distances, it is a confining potential (Sec. 3.5), and a different physical picture for the fermions might be expected in that case.

Arguments can be made that these considerations do not affect seriously the conclusion as far as the issue of determining *whether* dynamical mass generation occurs and (in the case of occurrence) the value of N_c [159]. (The point is that very near N_c, the polarization of the fermions in the symmetry-broken state must be pretty close to that of the massless fermions. On the other hand, above treatment is too crude to study the behaviors of quantities like $m(p)$. Lattice gauge theory simulation[160] does find that the symmetry breaking occurs, and $N_c \approx 3.5$, which is close to above analytical results.

Having "seen" the symmetry breaking, we can identify the elementary excitations — the Goldstone bosons. We all know that the Goldstone bosons in an ordered antiferromagnet are spin waves. In the fermion picture, the Goldstone bosons are a collective mode, and appear as a pole in the two-particle Green's function, the scattering amplitude in the appropriate channel, and in the related vertex[161]. Here we show this by considering the SU(2) vertex. The Bethe-Salpeter equation for the

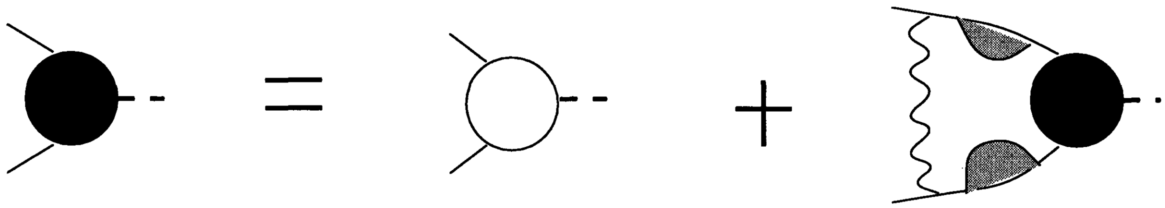

Figure 3-5: Bethe-Salpeter equation for the isovector vertex in the $\mathbf{Q} = (\pi, \pi)$ channel. The fermion lines willed a shaded blob represent the renormalized (self-consistent) Green's function.

vertex is given by

$$\Lambda^l(p; q) = 1 \cdot \tau^l - \int \frac{d^3k}{(2\pi)^3} \gamma_\mu \mathbf{G}(k) \Lambda^l(k; q) \mathbf{G}(k + q) \gamma_\nu D_{\mu\nu}(p - k), \tag{3.67}$$

which is represented by the ladder diagrams of Fig.3-5 (In Eq.3.67, the 1 in the first term on the RHS is the unit matrix in the spinor space.)

If there is a Goldstone boson, then $\Lambda^l(p; q)$ has a pole at $q^2 = 0$, in which case the homogeneous equation

$$\Lambda^l(p; 0) = - \int \frac{d^3k}{(2\pi)^3} \gamma_\mu \mathbf{G}(k) \Lambda^l(k; 0) \mathbf{G}(k) \gamma_\nu D_{\mu\nu}(p - k) \tag{3.68}$$

has a nontrivial solution. It is easy to see that

$$\Lambda^l(p; 0) = [m(p)\tau^3, \tau^l] \tag{3.69}$$

with $m(p)$ given by Eq.3.59 is the solution. Therefore, if there is a dynamical mass generation ($m(p) \neq 0$), then we have two Goldstone bosons (SU(2) symmetry breaking): $\tau^l = \tau^1, \tau^2$ (or τ^+, τ^-) in Eq.3.69 gives a nonvanishing commutator. The Lorentz invariance restricts the mesons to have a linear dispersion $q_0^2 = \mathbf{q}^2$ (Minkowski space), which is indeed the case with antiferromagnetic spin waves.

3.4 2d Underdoped Cuprates

3.4.1 antiferromagnetic correlations

The foregoing was one horrible way of looking at 2d quantum antiferromagnet, yet (hopefully) not without certain value. The qualitative idea that a strong attraction between spinons and antispinons via gauge field can result in the formation of a vector condensate with $\mathbf{Q}=(\pi, \pi)$ (the antiferromagnetic channel)[164] may shed some lights on the underdoped cuprates. In the underdoped cuprates, an effective theory based on the sFlux ansatz of the SU(2) mean field theory[165, 113, 116] consists of 2 flavors of massless Dirac fermions and a U(1) gauge field just like above, but now the gauge

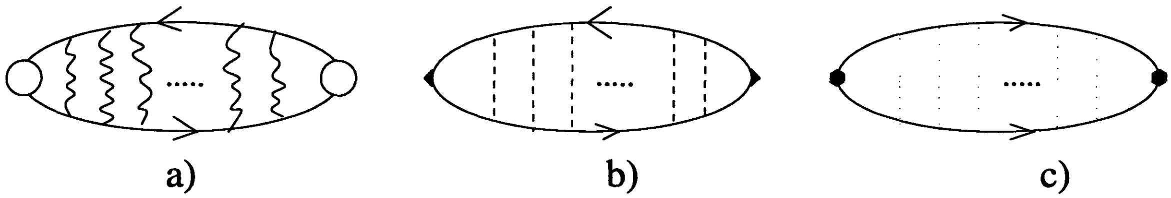

Figure 3-6: Ladder diagrams. a) Our case: staggered spin correlation. The wiggly lines are interactions mediated by the gauge field. Structurally similar examples: b) ferromagnetic spin correlation. The dotted lines are short-range repulsive interactions. c) superconducting correlation. The dotted lines are some kind of attractive interaction causing pairing.

field is also coupled to the bosons (holons). In other words, schematically,

$$L = \bar{\psi}_\alpha \gamma_\mu (\partial_\mu - i a_\mu) \psi_\alpha - i a_\mu J^B_\mu + L_B. \tag{3.70}$$

This additional coupling to the bosons will weaken the gauge field in the sense that it will screen the time component of the gauge field. In the simplest approximation, we ignore this massive part:

$$D_{\mu\nu}(q) = \frac{8}{N\sqrt{q^2}} \delta_{\mu i} \delta_{\mu j} \left(\delta_{ij} - \frac{q_i q_j}{\mathbf{q}^2} \right), \tag{3.71}$$

and examine the self-consistent equation for $m(p)$ (Eq.3.59). Since

$$\gamma_\mu \gamma_\nu D_{\mu\nu} = \gamma_i \gamma_i - \frac{(q_i \gamma_i)(q_i \gamma_i)}{\mathbf{q}^2} = 2 - 1 = 1 \tag{3.72}$$

(the gauge field in 2d has one transverse mode and one longitudinal mode the latter of which becomes massive), we have

$$m(p) = \frac{2}{N\pi^2 p} \int_0^\Lambda dk \frac{k m(k)}{k^2 + m^2(k)} (k + p - |k - p|). \tag{3.73}$$

This is identical to the Eq.3.59, except for a factor of 2 difference in the prefactor. Thus, we know immediately that there will be a symmetry breaking for $N < N_c' = N_c/2 = 16/\pi^2$. Now, for the physical case of $N = 2 > N_c'$, the spontaneous symmetry breaking would not occur!—This is what was hoped for our mean field theory.

The attraction in the $\mathbf{Q} = (\pi, \pi)$ channel mediated by the gauge field, although not strong enough to generate a condensate, will nevertheless have a strong effect on the spectrum of antiferromagnetic excitation. The fluctuation of the order parameter associated with the transition (staggered moment) can be examined by looking at the staggered-channel spin correlation function in the ladder approximation, similar to the more familiar problems like the superconducting fluctuations or the ferromagnetic spin fluctuations[166, 167] (See Fig.3-6).

In the problem of nearly ferromagnetic Fermi liquids, short range interaction between fermions are often modelled in terms of an on-site repulsion U ($UN(E_F)$ is the dimensionless coupling constant corresponding to our $1/N$). This problem is a lot simpler, as the ladder series sum immediately to the RPA form $\chi = \chi_0/(1-U\chi_0)$. The pole of the RPA propagator gives the diffusive mode ("paramagnons") associated with a conserved order parameter. This mode (more accurately the peak in $\chi''(\omega)$) comes down in energy and becomes sharper as U approaches U_c.

Unfortunately, in our problem the interaction is retarded and long-ranged, with the propagator taking the form

$$D_{\mu\nu} \approx \frac{1}{\sqrt{q^2}}\delta_{\mu i}\delta_{\nu j}\left(\delta_{ij} - \frac{q_i q_j}{\mathbf{q}^2}\right) + \frac{1}{xJ+\sqrt{q^2}}\delta_{\mu 0}\delta_{\nu 0}\frac{q^2}{\mathbf{q}^2} \tag{3.74}$$

(in the Coulomb gauge), hence the diagrams are not easy to evaluate. Nevertheless, on physical grounds, it is quite reasonable to expect that the same gauge field which caused the antiferromagnetic instability in the absence of holons will try to create a (massive) mode (particle-hole bound state in the $\mathbf{Q} = (\pi,\pi)$ channel) in this case. Because the symmetry is unbroken, there is a particle-hole continuum, as a result of which a sharp mode cannot exist, but a "broad resonance", *i.e.* a very unstable meson with a Minkowski-space dispersion

$$q^2 = \tilde{m}^2 - i\tilde{m}\Gamma, \tag{3.75}$$

is expected. The mass of the mode $\tilde{m}$ would come down as the transition is approached ("soft mode"). A physical consequence will be that the dynamic susceptibility $\chi''_{\mathbf{Q}=(\pi,\pi)}(\omega)$ (which can be probed by neutron scattering) has a broad peak, with a substantial rearrangement of the spectral weight compared to the mean field prediction. This heuristic picture is consistent with experiments of Keimer and collaborators[121] that find in the normal state (and in the superconducting state) of underdoped cuprates a magnetic scattering with a broad peak at some frequency scale that comes down in energy as the doping is reduced.

In principle, we should be able to study this by looking into the Bethe-Salpeter equation for spinon-antispinon scattering amplitude or the associated vertex. The analysis, however, entails a number of practical and technical difficulties. We can get around the problem of the complicated Lorentz-noninvariant gauge propagator for the doped case (Eq.3.74) by convincing ourselves that the we would obtain a similar physics by considering a Lorentz invariant one

$$D_{\mu\nu} = \frac{8}{\bar{N}\sqrt{q^2}}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \tag{3.76}$$

with $\bar{N} > N_c$ (Reducing the doping would correspond to reducing $\bar{N}$). Still, the fact that we are investigating a mode that is *massive and unstable* causes complications unseen in the B-S equation for the Goldstone bosons of the ordered antiferromagnet. Generally, the scattering amplitude and the vertex have many components ("invariant

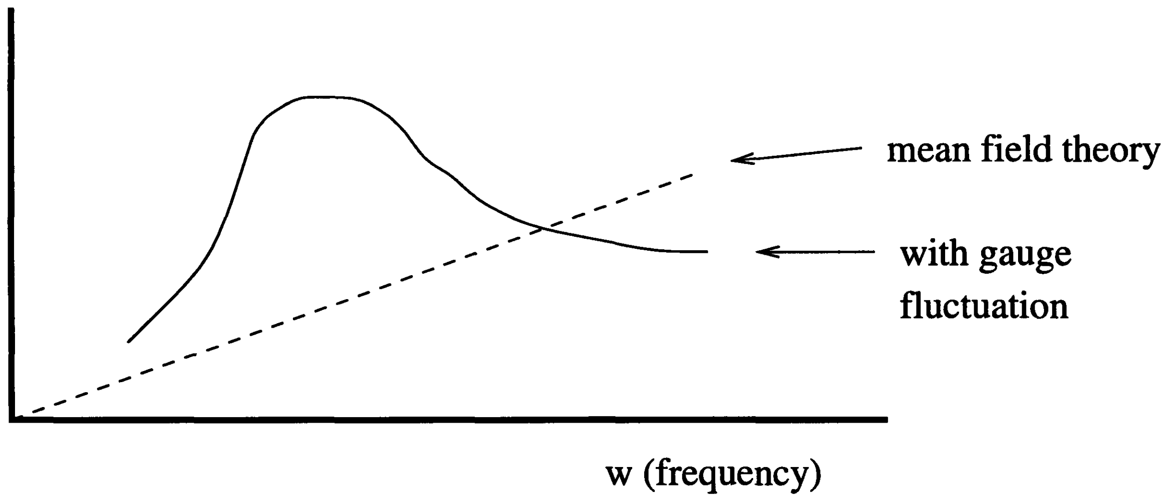

Figure 3-7: Rearrangement of spectral weight in $\chi''_{\mathbf{Q}=(\pi,\pi)}(\omega)$ due to gauge fluctuations.

amplitudes"), *e.g.*,

$$\Lambda^l(p;q) = f(p;q)\tau^l + g(p;q)p\gamma\tau^l + \cdots. \tag{3.77}$$

The Goldstone modes are massless, hence we can focus on $q = 0$ in which case $f(p;0)$ decouples from other amplitudes and has the same equation as that of the dynamical mass, as we have seen in Sec.3.3.3. The decoupling doesn't occur for $q \neq 0$. Appelquist *et al.*[168] have considered the scalar component of the Euclidean scattering amplitude in the ladder approximation, ignoring the coupling to other components; this corresponds to considering $f(p;q)$. They claim to find no light meson (soft mode) whose mass comes down to zero as the transition is approached, and hence the transition must be a novel one, *i.e.* neither a first order nor a second order transition. However, that conclusion seems questionable, as the analysis of Ref.[168] is not appropriate for finding an unstable meson which resides in the second Riemann sheet of the Minkowski space.

It would be difficult to calculate finite temperature properties within our gauge theory, but we believe that what has been discussed so far throws some light on the Copper and Oxygen NMR relaxation rates. Due to different effects of the gauge field on the $\mathbf{Q}$=0 and $\mathbf{Q}$=(π,π) response, there is no reason why the Copper site and the Oxygen site NMR relaxation rates should have the same temperature dependence. It is plausible that at high temperature side the Cu site $1/T_1T$ will be increasing with decreasing temperature, as the gauge fluctuation restores certain amount of antiferromagnetic correlations (note that above some energy scale, the screening effect of bosons won't be too important and the propagator would look like that of the undoped cuprates), while at low temperatures (below some scale) it will go down with decreasing temperature, since we still have the d-wave gap if the mean field picture is going to survive.

3.4.2 thermodynamic properties

In the underdoped cuprates the gauge field interaction also affects the uniform part of the spin response (the thermodynamic properties) though the effects are subtler. More specifically, the coupling of the gauge field to the nonrelativistic bosons results in the renormalization of the velocity of the fermions, which has the effect of enhancing the specific heat and the uniform susceptibility. This seems to be in accordance with experiments on underdoped cuprates. Although one might alternatively view the enhancement features in the normal state as having to do with the "Fermi surface segments", these are not really quasiparticle states in the strict sense (z=0) and there are likely to be theoretical complexity and possibility of overcounting. It seems a lot simpler to view that the fermions account for most of the entropy and spin response. A curious feature of our theory is that the Wilson ratio $W = C/T\chi_u$ has a value quite close to that of the "quantum critical" phase of the nonlinear sigma model[169, 170]: W in our theory is 0.128, while the $O(N)$ nonlinear sigma model gives $W = 0.124$ at zeroth order, and $W = 0.116$ with the inclusion of $1/N$ correction. Could this be a coincidence? The rest of this Section is an already published work (coauthored with P. A. Lee and X. -G. Wen) that discusses the effect of gauge fluctuations on the thermodynamic properties.

Recent experiments have indicated the existence in the normal state underdoped cuprate superconductor of a gap with the same anisotropy as the d-wave superconducting gap. One proposed explanation involves spin-charge separation: an electron in these highly correlated materials is a composite object made of a spin $\frac{1}{2}$ neutral fermion (spinon) and a spinless charged boson (holon). The suppression of normal state magnetic excitation seen in NMR and neutron scattering is thus viewed as a singlet pairing of neutral fermions in the absence of coherence among holons. As a possible realization of this idea, two of us have taken the t-J model (which is believed to capture essential physics of CuO_2 planes) and developed a slave boson mean field theory[113] that extends the local SU(2) symmetry at half-filling to the finite concentration of holes by introducing a SU(2) doublet of slave boson field. Among the mean field phases reported in Ref.[113] the so-called staggered flux (sF) phase (which is connected to d-wave pairing phase by a local SU(2) transformation) was argued to describe the pseudogap in underdoped cuprates. The low energy physics of this phase can be described by massless Dirac fermions, non-relativistic bosons, and a massless U(1) gauge field which together with two massive gauge fields forms SU(2) gauge fields that represent the fluctuations around the mean field.

The purpose of this paper is to address the low energy effective theory of the sF phase as a U(1) gauge theory problem. Although Dirac fermions coupled to a gauge field had been considered in several contexts in the past[120], we shall see that interesting new physics emerges when massless Dirac fermions are coupled to a gauge field that is also coupled to a compressible boson current. More specifically, the Lorentz symmetry breaking due to coupling to the bosons results in the renormalization of fermion velocity which have consequences on physical properties such as uniform susceptibility χ_u and electronic specific heat c_v^{el}. Experimentally, χ_u of underdoped cuprates begins to decrease with lowering of temperature far above the superconducting T_c, and decreases more rapidly below T_c[171, 106, 172]. Electronic specific heat experiments[173, 174] show that $\gamma(T)$ ($\equiv c_v^{el}(T)/T$) of the normal state behaves quite similar to χ_u. Although constant Wilson ratio (γ/χ_u) is a hallmark of Fermi liquid theory, the anomalous temperature dependence calls for a departure from the time-honored theory of most metals. We make a case that the puzzling normal state behavior of χ_u and γ may be viewed as *enhancement* over linear-in-T χ_u and γ of Dirac fermions due to logarithmic decrease of Dirac velocity caused by fermion-gauge field interaction.

We begin with the following continuum effective lagrangian for our problem

$$\begin{aligned}\mathcal{L} &= \bar{\Psi}_{\alpha s}(\partial_\mu\gamma^\mu + ia_\mu\gamma^\mu)\Psi_{\alpha s} + \\ &\quad b^*(\partial_0 - \mu_B + ia_0)b - \frac{1}{2m_B}b^*(\nabla + i\mathbf{a})^2 b. \end{aligned} \tag{3.78}$$

The Fermi field $\Psi_{\alpha s}$ is a 2×1 spinor: $\Psi_{1s}^\dagger = (f_{1se}^*, f_{1so}^*)$, $\Psi_{2s}^\dagger = (f_{2so}^*, f_{2se}^*)$, where $\alpha = 1,2$ labels the two Fermi points, $s = 1,..,N$ labels fermion species ($N = 2$ for physical case $s =\uparrow,\downarrow$), and e, o stands for even and odd sites, respectively. The γ^μ matrices are Pauli matrices $(\gamma^0,\gamma^1,\gamma^2) = (\sigma^3,\sigma^1,\sigma^2)$ and satisfy $\{\gamma^\mu,\gamma^\nu\} = 2\delta^{\mu\nu}$ $(\mu,\nu = 0,1,2)$. $\bar{\Psi}_{\alpha s} \equiv \Psi_{\alpha s}^\dagger\gamma^0$. In the sF phase of Ref.[113], the fermion dispersion near the fermi points is anisotropic, but we rescale it to an isotropic spectrum $E(\mathbf{k}) = v_D|\mathbf{k}|$ where $v_D = \sqrt{v_F v_2}$, the geometric mean of the two velocities (v_2 is proportional to the energy gap). We set $v_D = 1$, unless otherwise specified. The gauge field $a_\mu = (a_0, \mathbf{a})$ corresponds to the a_μ^3 part of the SU(2) gauge fields of Ref.[113]. The terms in Eq. (3.78) involving the Bose field b (representing charge degree of freedom) are believed to play several important roles, including the suppression of chiral symmetry breaking (Néel ordering[120]) and instanton effects[175]. Most importantly, the compressible boson current screens the a_0 field, making it massive. Unfortunately we do not have a detailed understanding of our boson subsystem. Therefore we shall draw upon only a few of qualitative features of the Bose sector while focusing mainly on the Fermi sector of the theory.

Eq. (3.78) carries certain similarity to the uniform resonating valence bond (uRVB) gauge theory[176, 52] proposed to describe optimally and slighly overdoped cuprates, and some of the theoretical framework can be carried over to our problem. As in the uRVB case, the internal gauge field a_μ does not have dynamics of its own, but it acquires dynamics from the polarization of fermions and bosons. Integrating out the matter fields generates the self energy term for the gauge field $\mathcal{L}_a = \frac{1}{2}a_\mu(\Pi_F^{\mu\nu} + \Pi_B^{\mu\nu})a_\nu$, up to quadratic order. The fermion polarization $\Pi_F^{\mu\nu}$ from the two Dirac points is given by

$$\Pi_F^{\mu\nu}(q) = -\frac{2N}{\beta}\sum_{k_0}\int\frac{d^2\mathbf{k}}{(2\pi)^2}\mathrm{tr}\left[G_F(k)\gamma^\mu G_F(k+q)\gamma^\nu\right], \tag{3.79}$$

where $G_F(k) = (ik_\mu\gamma^\mu)^{-1}$ is the fermion Green's function and k, q denote 3-momentum; for example, $k = (k_0 = (2n+1)\pi T, \mathbf{k})$. In the Coulomb gauge, the spatial part and the time part of the gauge field are decoupled, the propagators being $D^{00}(q) = (\Pi_F^{00}(q) + \Pi_B^{00}(q))^{-1}$ and $D^{ij}(q) = (\delta_{ij} - q_iq_j/\mathbf{q}^2)D^\perp(q)$ $(i,j = 1,2)$, with $D^\perp(q) =$

$(\Pi_F^{\perp}(q) + \Pi_B^{\perp}(q))^{-1}$. As mentioned earlier, the bosons should have a finite compressibility ($\Pi_B^{00}(q \to 0) \neq 0$) so the time component of the gauge field becomes massive (at finite temperature $\Pi_F^{00}(q \to 0)$ is also nonzero and contributes to the screening of a_0 field), but the spatial part of the gauge field, which is purely transverse, remains massless even at finite boson density and temperature, as long as the bosons are uncondensed (as in the spin gap phase). In the remainder of this paper, we will focus on the effect of this massless mode, ignoring the a_0 field.

In the absence of detailed understanding of the Bose sector, we assume that the transverse gauge propagator is dominated by the fermion part. In other words, $D^{\perp}(q) \approx D_F^{\perp}(q) \equiv 1/\Pi_F^{\perp}(q)$. This approximation, which is often used in the uRVB gauge theory, may not be fully justified in our case, but it allows us to organize the infrared behavior of our theory within $1/N$ expansion. The full expression for analytically continued transverse polarization function $\Pi_F^{\perp}(\omega, \mathbf{q})$ at finite temperature is rather complicated and therefore we shall not write it here, although it is used later in the evaluation of gauge fluctuation contribution to χ_u and c_v^{el}. In the limiting case of $T > |\mathbf{q}| > |\omega|$, we have

$$\Pi_F^{\perp}(\omega, \mathbf{q}) \approx -iC_1 \frac{\omega T}{|\mathbf{q}|} + C_2 \frac{\mathbf{q}^2}{T}, \tag{3.80}$$

while in the zero temperature limit,

$$\begin{aligned} \mathrm{Im}\Pi_F^{\perp}(\omega, \mathbf{q}) &= -N \mathrm{sign}(\omega)\theta(|\omega| - |\mathbf{q}|)\sqrt{\omega^2 - \mathbf{q}^2}/8 \\ \mathrm{Re}\Pi_F^{\perp}(\omega, \mathbf{q}) &= N\theta(|\mathbf{q}| - |\omega|)\sqrt{\mathbf{q}^2 - \omega^2}/8. \end{aligned} \tag{3.81}$$

To the leading order in $1/N$, fermion self energy due to transverse gauge fluctuations is

$$\Sigma(k) = -\frac{1}{\beta}\sum_{q_0}\int \frac{d^2\mathbf{q}}{(2\pi)^2}\gamma^i G_F(k+q)\gamma^j D_F^{ij}(q), \tag{3.82}$$

where $D_F^{ij}(q) = (\delta_{ij} - q_i q_j/\mathbf{q}^2)/\Pi_F^{\perp}$. At zero temperature, the self energy is [177]

$$\begin{aligned} \frac{N}{8}\Sigma(k) = i\gamma^0 \int \frac{d^3q}{(2\pi)^3} \frac{k_0 + q_0}{(k+q)^2\sqrt{q^2}} \\ -i\gamma^x \int \frac{d^3q}{(2\pi)^3} \frac{(k_x+q_x)(q_y^2 - q_x^2) - 2q_x q_y (k_y+q_y)}{\mathbf{q}^2 (k+q)^2\sqrt{q^2}} \\ -i\gamma^y \int \frac{d^3q}{(2\pi)^3} \frac{(k_y+q_y)(q_x^2 - q_y^2) - 2q_x q_y (k_x+q_x)}{\mathbf{q}^2 (k+q)^2\sqrt{q^2}}. \end{aligned} \tag{3.83}$$

We find, for $|\mathbf{k}| > |k_0|$,

$$\Sigma(k) = c\, ik_0\gamma^0 \mathcal{A}_0(k) - 2c\, i\mathbf{k}\cdot\vec{\gamma}\mathcal{A}_1(k) \tag{3.84}$$

with $c = 4/(3N\pi^2)$ and $\mathcal{A}_0(k) \approx \mathcal{A}_1(k) \approx \ln(\Lambda/|\mathbf{k}|)$, where Λ is a UV cutoff. Now the pole of the renormalized Green's function $G_F^R(k) = (G_F(k)^{-1} - \Sigma(k))^{-1}$ occurs at

$$E(\mathbf{k}) = |\mathbf{k}|(1 - 4/(N\pi^2)\ln(\Lambda/|\mathbf{k}|)). \tag{3.85}$$

Note that the presence of compressible bosons and the resulting breaking of Lorentz symmetry is crucial to have logarithmic velocity renormalization. Indeed, in the absence of bosons, the gauge propagator (gauge independent part) is given by $D^{\mu\nu}(q) = 8/N(\delta_{\mu\nu} - q_\mu q_\nu/q^2)/\sqrt{q^2}$ and the zero temperature fermion self energy takes the form $\Sigma = ik_\mu\gamma^\mu f(k^2)$; therefore the velocity is not renormalized.

Treating the quasiparticles described by Eq. (3.85) as "free", we calculate c_v and χ_u up to $\mathcal{O}(1/N^0)$:

$$\begin{aligned} c_v^{el} &= (9/\pi)\zeta(3)NT^2(1 + (8/N\pi^2)\ln(\Lambda/T) + ..) \\ \chi_u &= (2/\pi)\ln(2)NT(1 + (8/N\pi^2)\ln(\Lambda/T) + ..), \end{aligned} \tag{3.86}$$

($\zeta(3) = 1.202$). These results are believed to be valid for two reasons: 1) $\mathrm{Im}\mathcal{A}_{0,1}(\nu + i0^+, \mathbf{k}) = 0$ for $|\nu| < |\mathbf{k}|$, so the quasiparticles are well-defined. 2) Unlike the usual Fermi liquid theory, the free particle response function vanishes as $T \to 0$. To the extent the Landau parameters in Fermi liquid theory enter as in mean field theory, this means that Landau parameter correction vanish in $T \to 0$[179]. Indeed, it will be shown shortly that the calculation of χ_u and c_v^{el} from the free energy shift due to gauge fluctuation yields the same results.

The enhancement of c_v^{el} seen here finds its counterpart in the more familiar problems such as electron-phonon interaction in metals[180], uRVB gauge theory[52], and half-filled Landau level[181], where interactions induce mass enhancement which manifests iteself in the specific heat. In the nonrelativistic analogues, however, mass renormalization does not necessarily result in the enhancement of compressibility and uniform susceptibility[180, 182, 150], because the corrections are tied to the Fermi surface[180]. The crucial difference in our case is that there are only Fermi "points" instead of Fermi "surface". Thus in contrast to the nonrelativistic case, we find that the susceptibility is also renormalized such that the Wilson ratio $\gamma(T)/\chi_u(T)$ is constant. In fact, the Wilson ratio is the same as that of free Dirac fermions because quasiparticles are well-defined and Fermi-liquid type corrections are absent, as discussed earlier.

To check this conclusion, we calculate χ_u and c_v^{el} in a gauge invariant way, using the correction to the free energy due to gauge fluctuations. We consider only the leading correction in $1/N$, which is $\mathcal{O}(1/N^0)$:

$$\Delta F = \frac{1}{(2\pi)^3}\int d^2\mathbf{q}\int_{-\infty}^{\infty} d\omega n(\omega)\tan^{-1}\left(\frac{\mathrm{Im}\Pi_F^\perp(\omega,\mathbf{q})}{\mathrm{Re}\Pi_F^\perp(\omega,\mathbf{q})}\right). \tag{3.87}$$

The entropy shift ΔS $(= -\partial\Delta F/\partial T)$ due to gauge fluctuation has two contributions: ΔS_1 from the temperature dependence of the Bose function $n(\omega) = 1/(\exp(\omega/T) - 1)$ and ΔS_2 from the temperature dependence of fermion polarization. Numerically we find that the former gives a $\sim T^2$ contribution to entropy, while the latter which can be written as

$$\Delta S_2 = \frac{-1}{(2\pi)^3}\int^{|\mathbf{q}|<T_{UV}} d^2\mathbf{q} \int_{-\infty}^{\infty} d\omega n(\omega) \mathrm{Im}(D_F^{\perp} \frac{\partial}{\partial T} \Pi_F^{\perp}) \tag{3.88}$$

(T_{UV}=high energy cutoff) gives a singular contribution $\propto -T^2 \ln T$. The gauge fluctuation contribution to χ_u ($\Delta\chi_u$) is obtained by taking $-\partial^2/\partial H^2$ at $H = 0$ of $\Delta F(H)$. This approach corresponds to summing the bubble diagrams for the vertex correction and the self energy correction. It takes the form

$$\Delta\chi_u = \frac{-1}{(2\pi)^3}\int^{|\mathbf{q}|<T_{UV}} d^2\mathbf{q} \int_{-\infty}^{\infty} d\omega n(\omega) \mathrm{Im}(D_F^{\perp} \frac{\partial^2}{\partial \mu_F^2} \tilde{\Pi}_F^{\perp}), \tag{3.89}$$

where $\partial^2 \tilde{\Pi}_\perp / \partial \mu_F^2$ is a short-hand notation for $\partial^2 \Pi_\perp(\omega, \mathbf{q}; \mu_F)/\partial \mu_F^2|_{\mu_F=0}$ in which $\Pi_\perp(\omega, \mathbf{q}; \mu_F)$ is the transverse polarization function of Dirac fermions with finite chemical potential μ_F. This expression, which closely resembles that of ΔS_2, gives a singular contribution $\propto -T \ln T$. Note that the expressions for ΔS_2 and $\Delta\chi_u$ are also applicable to (nonrelativistic) uRVB gauge theory[52, 181], but they are usually ignored in that case because they give only subleading corrections while ΔS_1 generates a singular correction $\propto T^{2/3}$[181], unlike our case in which ΔS_2 dominate at low temperatures. Summarizing our numerical evaluation, we have

$$\Delta\chi_u = \frac{0.358}{v_D^2} T \ln \frac{T_{UV}}{2.4T}, \quad \Delta c_v^{el} = \frac{2.79}{v_D^2} T^2 \ln \frac{T_{UV}}{2.6T} \tag{3.90}$$

at low temperatures ($T <\sim T_{UV}/5$) in agreement with Eqs. (3.86).

We now discuss our results in light of the experiments. In Fig.3-8a we plot χ_u of $YBa_2Cu_3O_{6.63}$, a prototypical underdoped (bilayer) cuprate, from the Knight shift data of Takigawa *et al.*[106] We took the liberty of moving the zero of χ_u by 0.27 states/eV Cu(2), which is within the error bars corresponding to uncertainty in the orbital contributions K^{orb} ($\chi_u \propto K^{spin} = K - K^{orb}$). This change avoids the unphysical situation of Ref.[106] in which ${}^{63}K_{ab}^{spin}, {}^{17}K_{iso}^{spin}, {}^{17}K_c^{spin} < 0$ at $T = 0$. Further support for the adjustment of 0 comes from precision measurements of the Knight shifts in $YBa_2Cu_4O_8$ by Brinkmann and collaborators[172] who made substantial upward shift of K^{spin} from their previous values[183]. We find that the normal state data of Ref.[106] are well-fitted (solid line) by $\chi_u(T; v_D, T_{UV}) = \Delta\chi_u + \chi_u^0$. Here $\Delta\chi_u$ is the numerical evaluation of Eq. (3.89) whose low T behavior is given by Eq. (3.90), and χ_u^0 is the uniform susceptibility of bare Dirac fermions with the same upper cutoff T_{UV}: $\chi_u^0 = \frac{4}{v_D^2 \pi} T \mathcal{F}(T_{UV}/2T)$, $\mathcal{F}(x) = \int_0^x y/\cosh^2 y dy$. The two parameters in the fit are chosen to be $v_D = 0.76J$ and $T_{UV} = 0.63J$, where we set the antiferromagnetic exchange energy J=1500 K. We expect the gauge fluctuations to be suppressed in the superconducting state (due to Higgs mechanism) so that χ_u should cross-over to χ_u^0 (dashed line) at low temperatures. This is in qualitative agreement with the data below T_c. The inset of Fig.3-8a shows a similar fit for the spin Knight shifts of $YBa_2Cu_4O_8$[172], which is again very good. Thus our theory can account for the susceptibility in both the normal and superconducting states, without the need to adjust the energy scale of the gap parameter. *Using the same parameters* v_D and T_{UV}

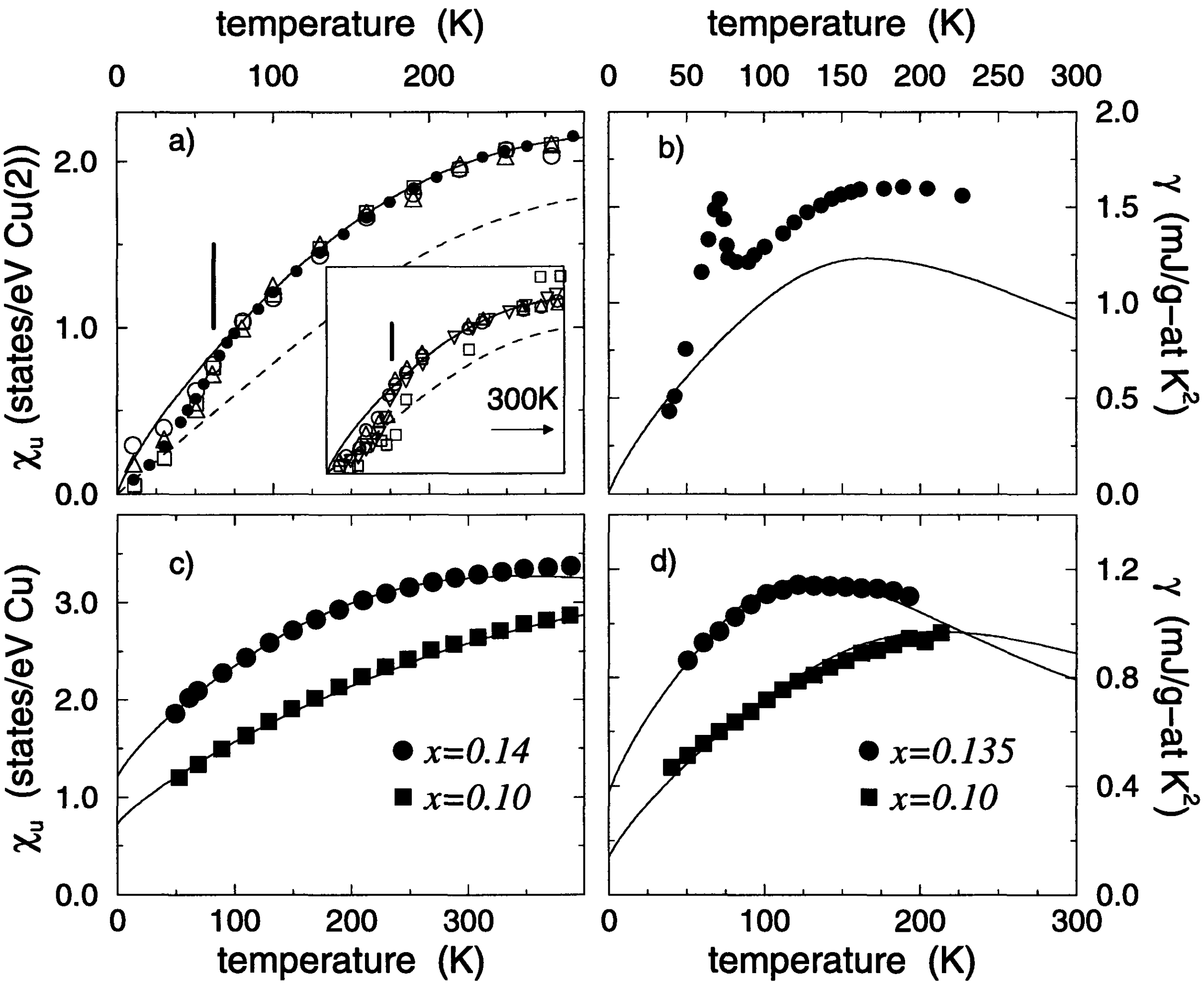

Figure 3-8: a)χ_u of $YBa_2Cu_3O_{6.63}$. The inset: spin Knight shifts of $YBa_2Cu_4O_8$. The vertical lines indicate T_c. Symbols are as in Ref.[106] and [172]. Dashed line is the susceptibility χ_u^0 of free Dirac fermions and solid line is the fit to our thoery, which includes gauge fluctuations. b) $\gamma(T)$ of $YBa_2Cu_3O_{6.67}$. c) χ_u of $La_{2-x}Sr_xCuO_4$ (see text). d) $\gamma(T)$ of $La_{2-x}Sr_xCuO_4$.

as in the fitting of $YBa_2Cu_3O_{6.63}$ Knight shift data, we plot $\gamma = \Delta c_v^{el}/T + \gamma^0$ (where $\gamma^0 = \frac{16}{v_D^2 \pi}\mathcal{G}(T_{UV}/2T), \mathcal{G}(x) = \int_0^x y^3/\cosh^2 y dy$) in Fig.3-8b. Also shown is the experimental data for $\gamma(T)$ of $YBa_2Cu_3O_{6.67}$[174]. Rough agreement of scales between the curves is quite encouraging.

In monolayer $La_{2-x}Sr_xCuO_4$, the uniform susceptibility is usually deduced from bulk susceptibility by subtracting the core diagmanetism χ_c and Van Vleck paramagnetism χ_{vv}. Fig.3-8c shows χ_u of $La_{2-x}Sr_xCuO_4$ obtained by subtracting the powder average value $\chi_{vv} + \chi_c = -0.5$ states/eV[184] (there's some uncertainty in the value of χ_{vv}) from the bulk susceptibility χ[171]. The data can be characterized by $\chi_u = \Delta\chi_u + \chi_u^0 + \chi_{const}$ with ($v_D = 0.99J, T_{UV} = 1.17J$) for $x = 0.10$ and ($v_D = 0.79J, T_{UV} = 0.65J$) for $x = 0.14$. Unlike the YBCO compounds, temperature independent part $\chi_{const} > 0$ is needed for a reasonable fit. Regarding the specific heat data of LSCO, cutoffs significantly smaller than the ones used for χ_u are needed to fit γ of the same compound in terms of $\gamma = \Delta c_v^{el}/T + \gamma^0 + \gamma_{const}$. In Fig.3-8d we have kept the same v_D as in Fig. 1c, but used smaller cutoffs ($T_{UV} = 0.8J$ for $x = 0.1$ and $T_{UV} = 0.49J$ for $x = 0.135$) to fit the γ-data[173]. This discrepancy and the origin of nonzero γ_{const} and χ_{const} are not well understood. The $\chi_{const} > 0$ feature in LSCO has been emphasized by some[185] to be an important evidence that the bilayer structure is important for spin gap behavior. Recent experiments on trilayer $HgBa_2Ca_2Cu_3O_{8+\delta}$[186] and monolayer $HgBa_2CuO_{4+\delta}$[187], however, find similar spin gap behaviors as in YBCO, suggesting that LSCO is a rather special case.

Despite reasonable agreement, we feel that above comparisons do not provide a conclusive test, because the T_c is too high to probe the normal state infrared behavior for a wide range of temperature. In fact, most of the bending feature seen in the $\gamma(T)$ data is presumably related to the high energy cutoff (the deviation from linear Dirac spectrum) which have been treated in a cavalier manner here by using a hard cutoff T_{UV}. The low-T curvature in χ_u data ($d^2\chi_u/dT^2 < 0$; faster decrease at lower temperature) seems to support the gauge fluctuation picture, but it may not be simple to separate this effect from the curvature due to high energy cutoff. Nevertheless, we view that the theory advocated here presents a simple and appealing picture of the spin gap phase. In this theory, no new energy scale is introduced to distinguish the spin gap phase and the superconducting phase; the Dirac velocity in both phases is taken to be the same. Rather, it is the gauge fluctuation that distinguishes the phases by causing the enhancement of χ_u and γ in the normal state.

Instead of conclusion, we recapitulate some issues that have been glossed over. We have ignored the a_0 field whose effect may not be totally innocuous[188]. In fact, we have checked that *in the absence of bosons* the contributions to χ_u and c_v^{el} derived from the free energy shift due to the a_0 fluctuation cancel the singular contributions from transverse gauge fluctuation, in agreement with non-renormalization of Dirac velocity in a Lorentz invariant situation. Secondly, we have not treated contributions from the Bose sector, especially in regard to the entropy. Lastly, we mention the issue of whether the renormalization of fermion propagator feeds back to the gauge propagator. In the non-relativistic gauge theory[52, 181], the density-density correlation function and the transverse gauge propagator receive only sub-leading corrections[150]. This might not hold any longer in our case. At present it is not clear to what extent the transverse propagator is modified by "feedback effect" and to what extent this affects the physical picture.

3.5 Concluding Remarks

We now conclude with a discussion of some difficult issues, hoping not so much as to resolve them but to content ourselves with placing them in perspective.

3.5.1 confinement, spinons, and all that

Confinement, like relativity and quantum mechanics, is not something that you can understand in your bones the way you understand Newtonian mechanics. You have to get used to it. — Howard Georgi

The term "spinon" has been used rather carelessly in this chapter. In the strict sense, it refers to well-defined neutral elementary excitations carrying spin 1/2. A well-known (and perhaps the only widely-accepted and physically-realized) example is solitonic (topological) objects in 1d spin chain that are usually understood within bosonization framework[131, 126].

The "spinons" in this chapter refer most of the times to spin 1/2 objects strongly coupled to fluctuating gauge fields. These spinons are far from being well-established. They were obtained by breaking a gauge symmetry (at the mean field level) which may be a questionable procedure. The pre-high T_c empiricism and experiences also provide resistance to such a notion. After all, in most metals (2 and 3d), the basic elementary excitation is quasiparticles that carry both spin 1/2 and charge e, and in most insulating antiferromagnets or in generic (gapped) spin liquids, the elementary excitations are spin 1 objects (spin waves or magnons). In other words, spinons and holons do not exist on their own, but are "confined" in spinon-antispinon composite objects (spin waves: $f^* f$) or in spinon-holon composite objects (quasiparticles: $f^* b$).

The sense in which this "confinement" is discussed is quite similar to the quark confinement in particle physics — the curious absence outside the nucleus of the quarks that make up hadrons (baryons and mesons). In fact, the strong interaction physics appears to share quite a few parallels with our problem[189]. Quantum chromodynamics (QCD), which is widely believed to be the correct theory of strong interactions, is a gauge theory whose low energy physics is as poorly understood as the high T_c cuprates. At a more substantial level, the phenomenology of strong interactions indicates that the (approximate) chiral symmetry is spontaneously broken, giving rise to mesons (such as pions) which are Goldstone bosons, in analogy with spin waves in a Néel ordered system. Just as the nonlinear sigma model gives an excellent description of the low energy spin dynamics of the undoped cuprates[12, 190], in particle physics it has been well known that similar effective lagrangians (sigma models, chiral perturbation theory[191, 192], etc.) give a very good description of the hadronic physics. However, attempts to "derive" the parameters of the effective theories, such as the pion decay constant, from first principles have not been entirely successful. The basic difficulty is that the same asymptotic freedom that led to the confirmation of the quark picture at the high energy side causes grave difficuties in analyzing the low energy physics. At present, no consensus exists as regards the "mechanism" of chiral symmetry breaking in QCD, but at least it seems clear that the problem is intimately connected to the general issue of confinement.

It is the basic underlying idea of this thesis that the confinement does not occur for the normal state of the superconducting cuprates, or more loosely, spinons and holons are "meaningful" objects. On the other hand, *some sort of* confinement seems necessary to describe the Néel ordered state in undoped cuprates and in the superconducting state of the doped cuprates: confinement *might* be necessary to explain why simple Ioffe-Larkin[176] rule does not work in the superconducting state[9] (*e.g.* in describing the temperature dependence of superfluid density); in an ordered antiferromagnet, we know that the low energy excitations are spin waves, not the spinons.

In considering confinement, we are (helped but also) burdened by previous studies in particle physics regarding the issue. The confinement motivated by the strong interaction phenomenology — no color nonsinglet particles exist in the physical spectrum — is a strong statement, and it is not clear to what extent the confinement in our case should match the quark confinement. The situation is not helped by the murky status of the quark confinement problem: a practical gauge invariant order parameter that can distinguish confined and deconfined phases is not known generally, and different definitions (of various degrees of rigor) are employed to identify confinement. Philosophies and semantics may often aggravate confusions.

Take the gauge theory of the spin chain (Sec. 3.2), for example. The original spinons (the fermions that couple to the gauge field) are regarded to be confined[122], since they don't appear in the physical spectrum which can be found exactly. This is in accordance with Green's function of the spinons, which (to all orders of perturbation theory) has the form $G(k) = -i\gamma k/k^{2-\eta}$; since it has a branchcut instead of a pole, these spinons do not have asymptotic states. However, the evaluation of the meaningfulness of these spinons involves preferences and opinions. From a positive point of view, they are *pretty good* spinons, since the mean field spinons give a reasonable description of the T-dependence of the specific heat and the uniform susceptibility[193], and the perturbative treatment of gauge fluctuation systematically improves the antiferromagnetic correlations. In fact, we may almost claim to "see" them in experiments. For example, the particle-hole excitation of these spinons might be one of the simplest explanation for the des-Cloizeaux-Pearson continuum seen by neutron scattering. A more spectacular example[194] is the changes in the continuum caused by a magnetic field (Zeeman splitting of up-spinons and down-spinons) and the associated incommensurate magnetic excitations which can be understood very naturally within our framework, although the Bethe ansatz can also produce the same result (in a far less transparent manner)[195].

In regard to the confinement of spinons in 2d undoped system, we can ask some definite questions that go beyond semantics. 1) On the low energy side, are the spin waves the only massless degrees of freedom, or could there be an additional mode hiding? 2) On the high energy side, do the spinon states exist, or have they disappeared completely? In other words, can we somehow "see" spinons at high energies?

We now consider the results of Sec. 3.3 in light of these questions. The answer to the first question is immediate: in addition to the spin waves, we have another massless mode — the gauge field. This is rather bothersome, in view of the common wisdom that the spin waves deplete the low energy excitations. Practically, it may be difficult to determine the absence or the existence of the gauge field, as it is an internal field. The gauge field may have $\sim T^2$ contribution to the specific heat, like the spin waves, but the modification due to gauge field might be considered a spin-wave renormalization effect. The second question (high energy side) is more difficult to answer. We have to determine whether the Minkowski-space Green's function of the spinons has a pole at some mass scale. If the dynamical symmetry breaking occurred in such a way that the spinons acquire a constant mass as in the NJL model[162], then the spinons should be found at high energies, and we might be able to "see" them[196] with the aforementioned method of Ref.[194]. If the symmetry is broken by (long-range, retarded) gauge field interactions as in our case, the problem is quite complicated; we need to continue the mass function $m(p)$ (Sec. 3.3.3) from the Euclidean space to the Minkowski space ($p^2 \to -p^2$), and examine whether $p^2 - m(-p^2) = 0$ has a solution near the real axis. The results from the literature[159] indicate that when the "feedback" effect of the spinon mass generation on the vacuum polarization is taken into account (in which case the gauge field dynamics looks like $F_{\mu\nu}^2$), no poles are found near the real axis. This may be connected to Witten's mechanism for confinement[198], namely that the strong dynamic gauge field produced by matter fields in turn confines the matter fields. In the 1+1D CP^{N-1} model ($L = |(\partial_\mu - ia_\mu)z|^2 + m^2 z^\dagger z$), integrating out the matter field (z-field) generates a kinetic term (self energy) for the gauge field $\frac{1}{m}F_{\mu\nu}^2$[199]. The Coulomb interaction in 1+1D is confining (i.e. $V(x_1) = -\int dq_1 \exp(iq_1 x_1)/q_1^2 \sim |x_1|$) while it is marginally confining in 2+1D ($V(\mathbf{x}) \sim \log|\mathbf{x}|$)[200].

We should like to point out that so far we have ignored one possibly important aspect of our gauge theory — the compactness. The models that we have examined in this chapter originate from the lattice, hence are compact gauge theories. The usual assumption is that the compact theory can be replaced by a more amenable noncompact theory in the continuum, but this is not always justified. A representative and scary example is the pure gauge theory in 2+1D. Polyakov[201] has shown that the compact pure gauge theory (2+1D photodynamics) differs from the noncompact one due to instantons — the topologically nontrivial, extended classical solutions of Euclidean gauge field equations, that can be viewed as tunneling events between topologically inequivalent vacua. Instantons cause the Wilson loop to follow the area law $\langle \exp(\oint dx_\mu a_\mu) \rangle \sim \exp(-Area)$, which means the presence of a linear potential $V(\mathbf{x}^a, \mathbf{x}^b) \sim |\mathbf{x}^a - \mathbf{x}^b|$ between static sources, a sign of (strong) confinement.

When matter fields are present as in our case and in QCD, the situation is not so simple. In QCD, for example, despite theoretical attempts[202] the relevance of instantons to quark confinement remains uncertain. Generally, the fluctuations of matter fields (especially the massless fields) are adverse to instantons. One specific scenario by which instantons are suppressed is fermion zero modes. For example, in the massless Schwinger model, the fermion determinant $\det[(\partial_\mu - ia_\mu)\gamma_\mu]$ in a gauge configuration a_μ with an instanton vanishes, so that only the topologically trivial sector contributes to the functional intergral (partition function). Similar mechanism turned out to be account for the famous U(1) problem[203] in QCD. In both cases (1+1D and 3+1D), the zero mode is connected to (axial) anomalies, whose analogue in odd space-time dimension (2+1D) is nonexistent.

In the context of our problem (compact 2+1D gauge theory with Dirac fermions), Marston[120] studied the possibility of fermion zero modes, and concluded zero modes do not exist, suggesting a possible relevance of instantons. Marston has also calculated the action of an instanton, and found that it is logarithmically divergent with a prefactor proportional to the number of flavors ($S \propto N \ln R$). Kosterlitz-Thouless type argument then indicates that below critical N_c which turns out to be 0.9, instantons may proliferate, while for $N > N_c$ which includes the physical case ($N = 2$), the instantons are suppressed. Ioffe & Larkin[176] have also calculated the action of an instanton, considering only the bilinear part of the gauge field action obtained by integrating out the fermions, and found logarithmic divergence, but N_c in this case turns out to be 24. It has been noted that this treatment may be too simple, and the question of instantons for physical N remains unclear[205]. We take the point of view that two massless fermions are enough to suppress the instantons.

The suppression of instantons may no longer be the case if the fermion masses are dynamically generated by spontaneous symmetry breaking, which is indeed our situation. We now have the possibility of symmetry-breaking-induced instantons. What are the consequences of instantons? Marston[204] has concluded that unlike the bosonic spinon theory[206] the instantons cannot induce the dimerization (valence bond solid order). The effect of instantons on the spinons is not clear. Does instantons lead to a very different picture of the spinon confinement? One thing reasonably clear is that the gauge field will be now massive[201], hence the spin waves will be the *only* low energy excitations of the ordered antiferromagnet. The difference between pictures with and without instantons would also affect issues like the calculation of Goldstone boson decay constant f_{GB} (related to the spin wave stiffness as $f_{GB} = \sqrt{\rho_s}$). In principle, it must be possible to work out f_{GB} microscopically, for example, in terms of the mass function $m(p)$ and other ingredients of the theory. In QCD, analogous attempts[207, 208] were made to calculate the pion decay constant f_π using the results of Jackiw and Johnson[209] or other methods. The result for f_{GB} might be quite different for the two pictures.

In the normal state of the underdoped cuprates, instantons are probably not relevant, as we have massless fermions and there are additional massless fluctuations due to the introduction of holes. In any case, the unusual phenomenology of high Tc cuprates points to that the spinons and holons are deconfined, and possible mechanisms of confinement can be ruled out on this ground. It is beyond the scope of this chapter to discuss the confinement in the superconducting state of the doped cuprate; we simply note that there the confinement might be understood from the equivalence of the confinement phase to the Higgs phase[210].

3.5.2 loose ends

In this chapter we have examined the magnetism of the undoped and underdoped cuprates from the point of view of neutral fermions with spin 1/2. Admittedly, the theory as it stands is far from rigorous. The philosophy has been to analyze possibly the simplest effective field theory of massless Dirac fermions, bosons and gauge fields, motivated by the sFlux phase that appears as a saddle point solution of the SU(2) mean field theory. In reality, the situation is a lot more complicated. The mean field fermion spectrum is anisotropic: $\epsilon(\mathbf{k}) = \sqrt{v_F^2 \tilde{k}_1^2 + v_2^2 \tilde{k}_2^2}$, and the velocities v_F, v_2 have some doping and temperature dependences. These dependences, however, do not account for the puzzling properties of the cuprates that we have discussed, and the photoemission does indicate that the quasiparticle gap remains large in the normal state of the underdoped cuprates (*i.e.* superconducting transition is not a gap-closing transition) and the gap is only weakly doping dependent. Therefore, the effective theory may be quite sensible for studying qualitative features not captured by mean field theories. We have made a number of approximations to treat the complicated dynamics of the gauge fields, but we hope it's not *too* optimistic to view that the qualitative conclusions are correct. In any case, the idea that the holon coupling to the gauge field prevents spontaneous symmetry breaking in the fermion system is an attractive one, and is very much in the spirit of the empirical fact that moving holes quickly destroy antiferromagetic order. Unfortunately, even with simplifications, the calculations quickly become rather intractable.

Evantually the "spin gap" has to close up as we go to the optimally doped regime, which in the mean field theory is modelled by the uRVB saddle point[52]. The details of this crossover is certainly beyond our hopes. This will severely limit us in considering some very interesting issues, like the relation between the neutron scattering peaks in the underdoped cuprates and the sharp 41meV peak in the superconducting phase of the optimally doped $YBCO_7$. In any case, the present theory does not say much about the spin excitations *in the superconducting state* of the underdoped cuprates, since the dynamics of gauge field in the superconducting state is different from that considered here. Another drawback is that in our framework it is difficult to consider possible incommensurate features in spin excitations. At present, it is not clear how universal and important the incommensurateness is. "Static stripes" are seen in the magic 1/8 filling in LSCO and its close relatives, while dynamic incommensurate features are well documented for LSCO at general dopings[211]. Some caution is necessary, however, since LSCO might not be a generic "clean cuprate material", as it does not display clear-cut spin gap behavior, and this class seems to suffer sample problems easily. At least for YBCO, the incommensurate aspects are weaker.

Granting these limitations, we still feel that the picture of spin excitations in underdoped cuprates in terms of deconfined fermionic spinons is reasonable and perhaps more natural than other descriptions like those based on fluctuating staggered moments. Features like the linear-in-T behavior of uniform susceptiblity and the specific heat coefficient in underdoped cuprates might be also explanable in terms of the "quantum critical" regime of the nonlinear sigma model, but it is not clear in that approach how to account for the strange behavior of the Copper $1/T_1T$ in the same temperature range. Attempts to explain the Copper $1/T_1T$ in terms of the "quantum disordered" regime then has to explain why activated behaviors are not seen in quantities like uniform susceptibility. Again, this seems to point to the difficulty of achieving a theoretical description of a system that involves a mysterious combination of gaplike (short range) correlations and critical correlations.

This work was done in collaboration with P. A. Lee, and X.-G. Wen (Sec. 3.4.2). Helpful conversations or correspondences with T. Imai, Y.B. Kim, C. Mudry, N. Nagaosa, K. Rajagopal, S. Sachdev, A. Shytov, and O. Syljuasen, are gratefully acknowledged.

Appendix A

A.1 Operator Averages

In this appendix, we discuss the evaluation of operator averages. In the path-integral representation of the partition function, world-line configurations are sampled according to a distribution proportional to $\exp(-S_B^0 - S_2)$. In our Monte Carlo scheme, we discretize the imaginary-time interval β into M segments of length $\Delta\tau$, and we define a world-line configuration by the boson coordinates at these discrete time points: $\{R_0, R_1, R_2, \ldots, R_M = P(R_0)\}$. ($R_m$ denotes the coordinates of the N bosons at m-th time slice: $R_m = (\mathbf{x}_1^{(m)}, \ldots, \mathbf{x}_N^{(m)})$, and R_M is a permutation of the coordinates of R_0.)

The configurations are sampled according to the probability:

$$\mathcal{P}(\{R\}) = \frac{1}{\mathcal{N}} e^{-S_2(\{R\})} \prod_{m=0}^{M-1} \rho_{\Delta\tau}(R_m, R_{m+1}), \tag{A.1}$$

where $\mathcal{N}$ is a normalization constant, and $\rho_{\Delta\tau}(R, R')$ is the short-time (high-temperature) density matrix in the absence of gauge fields. It is given by:

$$\begin{aligned}
\rho_{\Delta\tau}(R, R') &= \langle R| e^{-\Delta\tau(H_K^0 + H_U)} |R'\rangle \\
&\simeq \langle R| e^{-\frac{1}{2}\Delta\tau H_U} e^{-\Delta\tau H_K^0} e^{-\frac{1}{2}\Delta\tau H_U} |R'\rangle \\
&= \langle R| e^{-\Delta\tau H_K^0} |R'\rangle e^{-\frac{1}{2}\Delta\tau(\tilde{H}_U(R) + \tilde{H}_U(R'))}
\end{aligned} \tag{A.2}$$

with $\tilde{H}_U(R) = \langle R|H_U|R\rangle$, and

$$H_K^0 = -t \sum_{\langle ij \rangle} (b_i^\dagger b_j + \text{h.c.}) \tag{A.3}$$

$$H_U = \frac{U}{2} \sum_i n_i(n_i - 1). \tag{A.4}$$

The error involved in this approximation of the density matrix is $O(\Delta\tau^3)$.

Measurements which depend only on particle positions are simple to evaluate in this path-integral representation. The expectation value of such a measurement $\mathcal{O}$ is given by

$$\langle \mathcal{O} \rangle = \mathrm{Tr}\left[\tilde{\mathcal{O}}(R_0, R_1, R_2, \ldots) \mathcal{P}(R_0, R_1, R_2, \ldots)\right] \tag{A.5}$$

where $\tilde{O}(\{R\})$ is the measured value for the world-line configuration $\{R\}$, and the trace is taken over all such configurations.

Averages of operators which are non-local in position space are more cumbersome to evaluate. An example is the kinetic energy:

$$\langle H_K \rangle = -t \sum_{\langle ij \rangle} \langle (e^{i a_{ij}} b_i^\dagger b_j + \text{h.c.}) \rangle. \tag{A.6}$$

The gauge field a_{ij} is defined on the link between the neighboring sites i and j. The Peierls factor closes the gap in the imaginary-time loop caused by the action of kinetic energy operator. Inserting the operator H_K in the imaginary-time slice between the $m = 0$ and 1, it can be shown that the kinetic energy can be evaluated as:

$$\begin{aligned} \langle H_K \rangle &= \text{Tr}\left[\frac{\langle R_0 | H_K^0 \mathcal{U}_{\Delta\tau} | R_1 \rangle}{\langle R_0 | \mathcal{U}_{\Delta\tau} | R_1 \rangle} \mathcal{P}(\{R\}) \right] \\ &= \text{Tr}\left[\frac{\langle R_0 | H_K^0 e^{-\Delta\tau H_K^0} | R_1 \rangle}{\langle R_0 | e^{-\Delta\tau H_K^0} | R_1 \rangle} \times \right. \\ &\qquad \left. e^{\frac{1}{2}\Delta\tau(\tilde{H}_U(R_0) - \tilde{H}_U(K R_0))} \mathcal{P}(\{R\}) \right] \end{aligned} \tag{A.7}$$

where $\tilde{H}_U(KR)$ is defined by $H_U\{H_K^0|R\rangle\} \equiv \tilde{H}_U(KR)\{H_K^0|R\rangle\}$, and $\mathcal{U}_{\Delta\tau}$ is the short-time evolution operator:

$$\mathcal{U}_{\Delta\tau} = e^{-\frac{1}{2}\Delta\tau H_U} e^{-\Delta\tau H_K^0} e^{-\frac{1}{2}\Delta\tau H_U}. \tag{A.8}$$

A.2 Fermion Polarization

In this appendix, fermion vacuum-polarizations in 1+1D and 2+1D are worked out using "dimensional regularization" [197].

For massless Dirac fermions, the polarization function $\Pi_{\mu\nu}$ is given by

$$\begin{aligned}\Pi_{\mu\nu}(q) &= -N\int\frac{d^Dk}{(2\pi)^D}\mathrm{tr}[G(k)\gamma_\mu G(k+q)\gamma_\nu] \\ &= N\int\frac{d^Dk}{(2\pi)^D}\mathrm{tr}\left[\frac{k\gamma}{k^2}\gamma_\mu\frac{(k+q)\gamma}{(k+q)^2}\gamma_\nu\right] \\ &= N\int_0^1 dx\frac{d^Dk'}{(2\pi)^D}\frac{\mathrm{tr}[(k'+(1-x)q)\gamma\gamma_\mu(k'-xq)\gamma\gamma_\nu]}{(k'^2+q^2x(1-x))^2},\end{aligned} \tag{A.9}$$

where N is the number of flavors of fermions. Using the trace identity

$$\mathrm{tr}[\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu] = (\mathrm{tr}1)(\delta_{\rho\mu}\delta_{\sigma\nu}-\delta_{\rho\sigma}\delta_{\mu\nu}+\delta_{\rho\nu}\delta_{\mu\sigma}), \tag{A.10}$$

The numerator of the integrand is found as

$$2k'_\mu k'_\nu - 2x(1-x)(q_\mu q_\nu-\delta_{\mu\nu}q^2)-\delta_{\mu\nu}[k'^2+q^2x(1-x)]. \tag{A.11}$$

Substituting, we get

$$\begin{aligned}\Pi_{\mu\nu}(q) = N(\mathrm{tr}1)\int_0^1 dx\int\frac{d^Dk}{(2\pi)^D}&\left[\frac{2k_\mu k_\nu}{(k^2+q^2x(1-x))^2}\right. \\ &\left.-\frac{\delta_{\mu\nu}}{k^2+q^2x(1-x)}+\frac{2x(1-x)(\delta_{\mu\nu}q^2-q_\mu q_\nu)}{(k^2+q^2x(1-x))^2}\right].\end{aligned} \tag{A.12}$$

The first two terms cancel, and we have

$$\begin{aligned}\Pi_{\mu\nu}(q) &= 2N(\mathrm{tr}1)\frac{\Gamma(2-D/2)}{(4\pi)^{D/2}}(q^2\delta_{\mu\nu}-q_\mu q_\nu)\int_0^1 dx\, x(1-x)(q^2x(1-x))^{D/2-2} \\ &= \begin{cases}\frac{N(\mathrm{tr}1)}{32}\sqrt{q^2}\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right) & D=2+1 \\ \frac{N(\mathrm{tr}1)}{2\pi}\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right) & D=1+1.\end{cases}\end{aligned} \tag{A.13}$$

For the situations discussed in the main text, we have tr1=4 for the 2+1D (Sec.3.3) as we have 4 component fermions, while tr1=2 for the 1+1D (Sec.3.2).

If the fermions are massive, *i.e.*, $L=\bar\psi(\partial_\mu-ia_\mu)\gamma_\mu\psi+m\bar\psi\psi$, similar calculation shows that the polarization function is given by

$$\begin{aligned}\Pi_{\mu\nu}(q) &= 2N(\mathrm{tr}1)\frac{\Gamma(2-D/2)}{(4\pi)^{D/2}}(q^2\delta_{\mu\nu}-q_\mu q_\nu)\int_0^1 dx(1-x)x(m^2+q^2x(1-x))^{D/2-2} \\ &= \frac{(\mathrm{tr}1)N}{2\pi}(q^2\delta_{\mu\nu}-q_\mu q_\nu)\left(\frac{1}{q^2}-\frac{4m^2}{q^3\sqrt{4m^2+q^2}}\tanh^{-1}\left(\frac{q}{4m^2+q^2}\right)\right)\quad [D=1+1],\end{aligned}$$

$$= \frac{(\mathrm{tr}1)N}{4\pi}(q^2\delta_{\mu\nu} - q_\mu q_\nu)\left(\frac{m}{2q^2} + \frac{q^2-4m^2}{4q^3}\sin^{-1}\left(\frac{q}{\sqrt{4m^2+q^2}}\right)\right) \quad [D = 2+1].$$

For small q (i.e. $q^2 << m^2$)), we would have

$$\Pi_{\mu\nu}(q) \approx \frac{(\mathrm{tr}1)N}{4\pi}(q^2\delta_{\mu\nu} - q_\mu q_\nu)\left(\frac{1}{6m} - \frac{q^2}{60m^3} + ..\right) \quad D = 2+1$$

$$\Pi_{\mu\nu}(q) \approx \frac{(\mathrm{tr}1)N}{2\pi}(q^2\delta_{\mu\nu} - q_\mu q_\nu)\left(\frac{1}{6m^2} - \frac{q^2}{30m^4} + ..\right). \quad D = 1+1 \quad (A.14)$$

Note that the massive fermions give rise to a gauge field dynamics of the form $a\Pi a \sim (F_{\mu\nu})^2$ like the real electromagnetic field!

Appendix B

B.1 Remarks on Styles and Notations

The Euclidean space is used throughout, unless noted otherwise. In chapter 3, the convention for the Euclidean space lagrangian is chosen such that $\{\gamma_\mu, \gamma_\nu\} = +2\delta_{\mu\nu}$. The Greek indices μ, ν are space-time indices, while the Latin indices i, j are spatial indices. In most cases, spin indices are denoted by α, β or σ. Boldfaces are used for space vector, while italics are often used to denote space-time vector, for example, $x = (x_0, \mathbf{x}) = (\tau, \mathbf{x})$, $q = (q_0, \mathbf{q}) = (\omega_n, \mathbf{q})$, although now not always so. For example p could stand for either $(p_0, \mathbf{p})$ or $\sqrt{p^2}$. It should be pretty much clear from the contexts. Some abbreviations for integrals are: $\int \frac{d^3q}{(2\pi)^3} = \int \frac{dq_0 d^2\mathbf{q}}{(2\pi)^3}$ $(= \frac{1}{\beta}\sum_{\omega_n} \frac{d^2\mathbf{q}}{(2\pi)^2}$ at finite temperatures) and $\int d^3x = \int dx_0 d^2\mathbf{x}$.

Sec. 3.4.2 has been published as an independent article, and the notations and conventions can be a little bit different from the rest of Chap.3. For example, α in the fermion field denotes the two fermi points, while s is used for spin indices.

References

[1] J. G. Bednorz and K. A. Müller, Z. Phys. B **64**, 189 (1986).

[2] L. Gao *et al.*, Phys. Rev. B **50**, 4260 (1994).

[3] P.A. Lee, Phys. Rev. Lett. **71**, 1887 (1993).

[4] Z. -X. Shen *et al.*, Phys. Rev. Lett. **70**, 1553 (1993).

[5] J. R. Kirtley *et al.*, Nature **373**, 225 (1995); D. J. Van Harlingen, Rev. Mod. Phys. **67**, 515 (1995).

[6] K. Krishina, J. M. Harris, and N. P. Ong, Phys. Rev. Lett. **75**, 3529, (1995).

[7] H. Ding *et al.*, Nature **382**, 51 (1996).

[8] A. G. Loeser *et al.*, Science **273**, 325 (1996).

[9] X.-G. Wen and P.A. Lee, Phys. Rev. Lett. **78**, 4111 (1997).

[10] T. R. Chien, Z. Z. Wang, and N. P. Ong, Phys. Rev. Lett. **67**, 2088 (1991).

[11] D. S. Marshall *el al.*, Phys. Rev. Lett. **76**, 4841 (1996).

[12] S. Chakravarty, B. I. Halperin, and D. R. Nelson, Phys. Rev. B **39**, 1344 (1988).

[13] Z. Fisk, H. R. Ott, T. M. Rice, and J. L. Smith, Nature **320**, 124 (1986).

[14] F. Steglich, Phys. Rev. Lett. **43**, 1892 (1979).

[15] W. A. Little, Phys. Rev. **A134**, 1416 (1964).

[16] P. W. Anderson and R. Schrieffer, Physics Today, June 1991, p54.

[17] See, for example, R.H. McKenzie, Science **278**, 820 (1997).

[18] Y. Maeno *et al.*, Nature **372**, 532 (1994)

[19] T. M. Rice and M. Sigrist, J. Phys: Cond. Matt. **7**, L643, (1995).

[20] W. F. Brinkman and T. M. Rice, Phys. Rev. B **2**, 4302 (1970).

[21] See, for example, S. A. Carter *et al.*, Phys. Rev. B **48**, 16841 (1993).

[22] Y. Tokura *et al.*, Phys. Rev. Lett. **70**, 2126 (1993).

[23] M. Itoh *et al.*, J. Mag. Mag. Mat. **177**, 583 (1998); Y. Furukawa *et al.*, Physica C **282**, 1103 (1997).

[24] P.W. Anderson and W.F. Brinkman in *The Helium Liquids*, ed. J. G. M. Armitage and I. E. Farquhar (reproduced in P. W. Anderson, *Basic Notions of Condensed Matter Physics*, Addison-Wesley (1984)).

[25] P. W. Anderson, Phys. Rev. B **30**, 1549 (1984).

[26] D. Vollhardt, Rev. Mod. Phys. **56**, 99 (1984).

[27] J. Engelbrecht and K.S. Bedell, Phys. Rev. Lett. **74**, 4265 (1995).

[28] Y. Nagaoka, Phys. Rev. **147**, 392 (1966).

[29] P. W. Anderson, Science **235**, 1196 (1987).

[30] M. Takigawa *et al.*, Phys. Rev. Lett. **76**, 4612 (1996).

[31] D. A. Tennant *et al.*, Phys. Rev. Lett. **70**, 4003 (1993).

[32] P. W. Anderson, Mat. Res. Bull. **8**, 153 (1973).

[33] R. P. P. Singh and D. A. Huse, Phys. Rev. Lett. **68**, 1766 (1992).

[34] K. Ueda *et al.*, Phys. Rev. Lett. **76**, 1932 (1996).

[35] K. Kodama *et al.*, J. Phys. Soc. Jpn. **66**, 793 (1997).

[36] E. Dagotto and T. M. Rice, Science **271**, 618 (1996).

[37] S. Gopalan, T. M. Rice, and M. Sigrist, Phys. Rev. B **49**, 8901 (1994).

[38] Neutron scattering studies (R.S. Eccleston *et al.*, Phys. Rev. Lett. **73**, 2626 (1994)) on $(VO)_2P_2O_7$ (thought to be a spin ladder compound) showed a sharp triplet excitations and a large gap, but more recent investigations have revealed that the compound is a dimerized one. Some INS results are available for the ladder compound $(Sr_{0.8}Ca_{0.2})_{14}Cu_{24}O_{41}$ (R. S. Eccleston *et al.*, Phys. Rev. B53, R14721 (1996)).

[39] H. Mutka *et al.*, Phys. Rev. Lett. **67**, 497 (1991).

[40] L.P. Regnault *et al.*, Phys. Rev. B **50**, 9174 (1994).

[41] N. Nishi *et al.*, Phys. Rev. B **50**, 6508 (1994).

[42] B. S. Shastry and W. Sutherland, Phys. Rev. Lett. **47**, 974 (1981).

[43] C. K. Majumdar and D. K. Ghosh, J. Math. Phys. **10**, 1388 (1969).

[44] P. A. Lee, 1989 Los Alamos Symposium proceedings.

[45] P. W. Anderson, Adv. Phys. **46**, 3 (1997).

[46] P.W. Anderson, *The Theory of Superconductivity in the High T_c Cuprates*, Princeton Univ. Press (1997).

[47] F. C. Zhang and T. M. Rice, Phys. Rev. B **37**, 3759 (1988).

[48] Y. Suzumura, Y. Hasegawa, and H. Fukuyama, J. Phys. Soc. Jpn. **57**, 2768 (1988).

[49] G. Kotliar and J. Liu, Phys. Rev. B **38**, 5142 (1988).

[50] E.L. Pollock and D.M. Ceperley, Phys. Rev. B **36**, 8343 (1987).

[51] D.M. Ceperley and E.L. Pollock, Phys. Rev. B **39**, 2084 (1989).

[52] P.A. Lee and N. Nagaosa, Phys. Rev. B **46**, 5621 (1992); N. Nagaosa and P. A. Lee, Phys. Rev. Lett. **64**, 2450 (1990).

[53] H. Fukuyama and H. Kohno, J. Mag. Mag. Mat. **177-181**, 483 (1998); see also H. Fukuyama, J. Phys. Chem. Sol. **59**, 447 (1998).

[54] X. -G. Wen and P. A. Lee, Phys. Rev. Lett. **80**, 2193 (1998).

[55] P. Dai, H. A. Mook, and F. Dogan, Phys. Rev. Lett. **80**, 1738 (1998).

[56] V. J. Emery, S. A. Kivelson, and O. Zachar, Phys. Rev. B **56**, 6120 (1997).

[57] I am indebted with X. -G. Wen for discussions that lead to this paragraph.

[58] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).

[59] S. C. Zhang, H. Hansson, and S. Kivelson, Phys. Rev. Lett. **62**, 82 (1989).

[60] D. K. K. Lee, D. H. Kim, and P. A. Lee, Phys. Rev. Lett. **76**, 4801 (1996).

[61] B. Batlogg, H.Y. Hwang, H. Takagi, R.J. Cava, H.L. Kao, and J. Kwo, Physica C **235-240**, 130 (1994).

[62] For a review, see N.P. Ong, Y.F. Yan and J.M. Harris, in *Proceedings of the CCAST Symposium on High-T_c Superconductivity and the C_{60} Family, Beijing 1994* (Gordon and Breach, New York, 1995).

[63] J. Orenstein, G.A. Thomas, A.J. Millis, S.L. Cooper, D.H. Rapkine, T. Timusk, L.F. Schneemeyer, and J.V. Waszczak, Phys. Rev. B **42**, 6342 (1990).

[64] F. Gao, D.B. Romero, D.B. Tanner, J. Talvacchio, and M.G. Forrester, Phys. Rev. B **47**, 1036 (1993).

[65] D.B. Romero, C.D. Porter, D.B. Tanner, L. Forro, D. Mandrus, L. Mihaly, G.L. Carr, and G.P. Williams, Phys. Rev. Lett. **68**, 1590 (1992).

[66] J.M. Harris, Y.F. Yan, P. Matl, N.P. Ong, P.W. Anderson, T. Kimura, and K. Kitazawa, Phys. Rev. Lett. **75**, 1391 (1995).

[67] W.O. Putikka, R.R.P. Singh, R.L. Glenister, and H. Tsunetsugu, Phys. Rev. Lett. **73**, 170 (1994).

[68] T. Tohyama, P. Horsch, and S. Maekawa, Phys. Rev. Lett. **74**, 980 (1995).

[69] R. Eder, Y. Ohta, and S. Maekawa, Phys. Rev. Lett. **74**, 5124 (1995).

[70] L.B. Ioffe, and A.I. Larkin, Phys. Rev. B **39**, 8988 (1989).

[71] J.M. Wheatley and T.M. Hong, Phys. Rev. B **43**, 6288 (1991).

[72] J.M. Wheatley and A.J. Schofield, Int. J. Mod. Phys. B, **6**, 665 (1992).

[73] M. Ubbens, P.A. Lee, and N. Nagaosa, Phys. Rev. B **48**, 13762 (1993).

[74] M.U. Ubbens and P.A. Lee, Phys. Rev. B **49**, 13049 (1994).

[75] L.B. Ioffe and V. Kalmeyer, Phys. Rev. B **44**, 750 (1991).

[76] M. Yu. Reizer, Phys. Rev. B **40**, 11571 (1989); *ibid.* **44**, 5476 (1991).

[77] L.B. Ioffe, V. Kalmeyer, and P.B. Wiegmann, Phys. Rev. B **43**, 1219 (1991).

[78] R. Hlubina, W.O. Putikka, T.M. Rice, and D.V. Khveshchenko, Phys. Rev. B **46**, 11224 (1992).

[79] However, high temperature series studies of the gauge field model[78] have found that at least the thermodynamic properties of the t-J model are well described by the gauge field model within the Gaussian approximation.

[80] See, for example, R.P. Feynman, *Statistical Mechanics* (Addison Wesley, Reading, 1972), chapter 11.

[81] J.A. Hertz, Phys. Rev. B **18**, 197 (1978).

[82] N. Trivedi, in *Computer Simulations in Condensed Matter Physics V*, edited by D.P. Landau, K.K. Mon, and H.B. Schüttler, (Springer Verlag, Heidelberg, 1993).

[83] W.F. Brinkman and T.M. Rice, Phys. Rev. B **2**, 1324 (1970).

[84] R. Oppermann and F. Wegner, Z. Phys. B **34**, 327 (1979).

[85] C. Pryor and A. Zee, Phys. Rev. B **46**, 3116 (1992).

[86] A. Barelli, R. Fleckinger, and T.A.L. Ziman, Phys. Rev. B **49**, 3340 (1994).

[87] G. Gavazzi, J.M. Wheatley, and A.J. Schofield, Phys. Rev. B **47**, 15170 (1993).

[88] N. Nagaosa and P.A. Lee, Phys. Rev. B **45**, 966 (1992).

[89] M.V. Feigelman, V.B. Geshkenbein, L.B. Ioffe, and A.I. Larkin, Phys. Rev. B **48**, 16641 (1993).

[90] E. Fradkin, *Field Theories of Condensed Matter Systems* (Addison Wesley, Redwood City, 1991), section 10.9.

[91] J.M. Wheatley, Phys. Rev. B **41**, 7301 (1990).

[92] In Ref.[71], Wheatley and Hong have defined a "coherence temperature" $T_{\rm coh}$ in a similar way as our T_D. In their case of bosons coupled to dissipative bath, it was argued that $T_{\rm coh}$ is pushed down to zero.

[93] M. Wallin, E.S. Sorensen, S.M. Girvin, and A.P. Young, Phys. Rev. B **49**, 12115 (1994).

[94] G.G. Batrouni, B. Larson, R.T. Scalettar, J. Tobochnik, and J. Wang, Phys. Rev. B **48**, 9628 (1993).

[95] J. Skilling, in *Maximum Entropy and Baysian Methods* (Kluwer Academic, Dordrecht, 1989), p.45.

[96] J.E. Gubernatis, M. Jarrell, R.N. Silver, and D.S. Sivia, Phys. Rev. B **44**, 6011 (1991).

[97] W. von der Linden, Appl. Phys. A **60**, 155 (1995).

[98] J. Jaklic and P. Prelovsek, Phys. Rev. B **52**, 6903 (1995).

[99] P.W. Anderson, Phys. Rev. B **55**, 11785 (1997).

[100] H. Fukuyama, H. Ebisawa, and Y. Wada, Prog. Theor. Phys. **42**, 494 (1969).

[101] Y.C. Chen and T.K. Lee, Phys. Rev. B **51**, 6723 (1995).

[102] O. Syljuasen, S. Chakravarty, and M. Greven, Phys. Rev. Lett. **78**, 4115 (1997).

[103] T. Jolicouer and O. Golinelli, Phys. Rev. B **50**, 9265 (1994).

[104] A. V. Chubukov and O. A. Starykh, Phys. Rev. B **52**, 440 (1995).

[105] This point was also emphasized by S. Chakravarty, Science **278**, 1412 (1998).

[106] M. Takigawa *et al.*, Phys. Rev. B **43**, 247 (1991).

[107] See, for a review, C. P. Slichter, 1993 Los Alamos Symposium proceedings.

[108] N. Nagaosa, Science **275**, 1078 (1997).

[109] See, for example, D. Pines, Physica C, **282-287**, 273 (1997).

[110] T. M. Rice, in *Physics and Chemistry of Oxide Superconductors*, ed. Y. Iye and H. Yasuoka, Springer-Verlag (1992).

[111] I. Affleck and J. B. Marston, Phys. Rev. B **37**, 3774 (1988).

[112] I. Affleck, Z. Zou, T. Hsu, and P.W. Anderson, Phys. Rev. B **38**, 745 (1988); E. Dagotto, E. Fradkin, A. Moreo, Phys. Rev. B **38**, 2926 (1988).

[113] X. -G. Wen and P. A. Lee, Phys. Rev. Lett. **76**, 503 (1996).

[114] F. C. Zhang, Phys. Rev. Lett. **64**, 974 (1990).

[115] M. U. Ubbens and P. A. Lee, Phys. Rev. B **49**, 6853 (1994).

[116] P. A. Lee, N. Nagaosa, T. K. Ng, and X. -G. Wen, Phys. Rev. B **57**, 6003 (1998).

[117] J. Polchinski, Nuc. Phys. B **422**, 617 (1993).

[118] Such a scenario has been already mentioned in the literature. See, for example, Refs.[119, 120]

[119] R. B. Laughlin, and Z. Zou, Phys. Rev. B **41**, 664 (1990).

[120] J. B. Marston, Phys. Rev. Lett. **64**, 1166 (1990).

[121] H. F. Fong *et al.*, Phys. Rev. Lett. **78**, 713 (1997).

[122] S. Coleman, R. Jackiw, and L. Susskind, Ann. Phys. (New York) **93**, 267 (1975).

[123] Y. Hosotani, J. Phys. A **30**, L757, (1997)

[124] C. M. Mudry and E. Fradkin, Phys. Rev. B **50**, 11409 (1994).

[125] A. M. Tsvelik, *Quantum Field Theory in Condensed Matter Physics*, Cambridge Univ. Press (1995).

[126] I. Affleck, "Field Theory Methods and Quantum Critical Phenomena" in *Fields, Strings and Critical Phenomena*, Ed. E. Brezin and J. Zinn-Justin, 1989.

[127] D. P. Arovas and A. Auerbach, Phys. Rev. B **38**, 316 (1988).

[128] L. Bulaevskii, Soviet Physics.-JETP **16**, 685 (1963).

[129] T. A. Kaplan, P. Horsch, and P. Fulde, Phys. Rev. Lett. **49**, 889 (1982); F. Gebhard and D. Vollhardt, Phys. Rev. B **38**, 6911 (1988).

[130] The Gutzwiller-projected tight-binding band turns out to be the exact ground state of the generalized Heisenberg spin chain with $J(i-j) = 1/|i-j|^2$; See, F. D. M. Haldane, Phys. Rev. Lett. **60**, 635 (1988), and B. S. Shastry, Phys. Rev. Lett. **60**, 639 (1988).

[131] F. D. M. Haldane, in *Electron Correlation and Magnetism in Narrow-Band Systems*, ed. T. Moriya, Springer-Verlag (1981).

[132] J. Schwinger, Phys. Rev. **128**, 2425 (1962).

[133] Y. Frishman, in *Particles, Quantum Fields, and Statistical Mechanics*, Ed. M. Alexanian and A. Zepeda, Springer-Verlag (1975).

[134] G. T. Bodwin and E. V. Kovacs, Phys. Rev. D **35**, 3198 (1987).

[135] J. H. Lowenstein and J. A. Swieca, Ann. Phys. (New York), **68**, 172 (1971).

[136] K. Fujikawa, Phys. Rev. Lett. **42**, 1195 (1979); Phys. Rev. D **21**, 2848 (1980); pedagogical account is given in M. E. Peskin and D. V. Schroeder, *Introduction to Quantum Field Theory*, Addison-Wesley (1995).

[137] M. Stone ed. *Bosonization*, World-Scientific (1994).

[138] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Oxford Univ. Press (1997).

[139] See. for exampe, O. A. Starykh, R. R. P. Singh, and A. W. Sandvik, Phys. Rev. Lett. **78**, 539 (1997), and references therein.

[140] A. Luther and I. Peschel, Phys. Rev. B **12**, 3908 (1975).

[141] S. Sachdev, Phys. Rev. B **50**, 13006 (1994).

[142] L. D. Faddeev and L. A. Takhtajan, Phys. Lett. A **85A**, 375 (1981).

[143] Another convenient gauge choice in 1+1D is the light-cone gauge, used, for example, by t' Hooft (Nuc. Phys. B **75**, 461 (1974)).

[144] See, for example, M. C. Diamantini et al., Nuc. Phys. B **406**, 595 (1993).

[145] C. Mudry, E. Fradkin, Phys. Rev. B **49**, 5200 (1994).

[146] B. O. Wells *et al.*, Phys. Rev. Lett. **74**, 964 (1995).

[147] R. B. Laughlin, J. Phys. Chem. Sol. **56**, 1627 (1995); See also Wen and Lee (Ref.[113]).

[148] J. B. Marston and I. Affleck, Phys. Rev. B **39**, 11538 (1989).

[149] W. Chen, M. P. A. Fisher, and Y.-S. Wu, Phys. Rev. B **48**, 13749 (1993).

[150] Y. B. Kim, A. Furusaki, X. -G. Wen, and P. A. Lee, Phys. Rev. B **50**, 17917 (1994).

[151] G. W. Semenoff, P. Sodano, and Y. S. Wu, Phys. Rev. Lett. **62**, 715 (1989).

[152] X. -G. Wen and Y. -S. Wu, Phys. Rev. Lett. **70**, 1501 (1993).

[153] X.-G. Wen, Phys. Rev. B **46**, 2655 (1992).

[154] D. J. Gross, in *Methods in Field Theory*, North-Holland (1976).

[155] S. Coleman, Comm. Math. Phys. **31**, 259 (1973); similar "no go" theorems in condensed matter physics are due to N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966) and P. C. Hohenberg, Phys. Rev. **158**, 383 (1967).

[156] J. R. Schrieffer, X. -G. Wen, and S. C. Zhang, Phys. Rev. B **39**, 1663 (1989).

[157] Y. Nambu, Phys. Rev. **117**, 648 (1960).

[158] T. Appelquist, D. Nash, and L. C. R. Wijewardhana, Phys. Rev. Lett. **60**, 2575 (1988).

[159] P. Maris, Phys. Rev. D **52**, 6087 (1995).

[160] E. Dagotto, J. B. Kogut, and A Kocic, Phys. Rev. Lett. **62**, 1083 (1989).

[161] See, for example, V. G. Vaks, V. M. Galitskii, and A. I. Larkin, Soviet Phys-JETP **14**, 1177, (1962), and P. Nozieres, *Theory of Interacting Fermi Systems* Addison-Wesley (1964), as well as Refs.[162, 163].

[162] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).

[163] H. Pagels, Phys. Rev. D **7**, 3689 (1973).

[164] Similar idea was suggested for the large Fermi surface case by L. B. Ioffe, S. Kivelson, and A. I. Larkin, Phys. Rev. B **44**, 12537 (1991).

[165] D. H. Kim, P. A. Lee, and X. -G. Wen, Phys. Rev. Lett. **79**, 2109 (1997).

[166] J. A. Hertz, Phys. Rev. B **14**, 1165 (1976).

[167] S. Doniach and E. H. Sondheimer, *Green's Functions for Solid State Physicists*, Addison-Wesley (1974).

[168] T. Appelquist, J. Terning, and L. C. R. Wijewardhana, Phys. Rev. Lett. **75**, 2081 (1995).

[169] S. Sachdev, private communication.

[170] A. V. Chubukov, S. Sachdev, and J. Ye, Phys. Rev. B **49**, 11919 (1994); S. Sachdev and J. Ye, Phys. Rev. Lett. **69**, 2411 (1992).

[171] T. Nakano *et al.*, Phys. Rev. B **49**, 16000 (1994).

[172] M. Bankay *et al.*, Phys. Rev. B **50**, 6416 (1994).

[173] J.W. Loram *et al.*, *Proceedings of 10th Anniv. HTS Workshop,* Houston, March 1996.

[174] W.Y. Liang *et al.*, Physica C **263**, 277 (1996).

[175] N. Nagaosa, Phys. Rev. Lett. **71**, 4210 (1993).

[176] L.B. Ioffe and A.I. Larkin, Phys. Rev. B **39**, 8988 (1989).

[177] At finite temperature infrared divergence (analogous to those[52, 178] encountered in nonrelativistic gauge theory) occurs in $\Sigma(ik_0 \to \nu + i0^+, \mathbf{k})$ for $\nu \leq |\mathbf{k}|$ due to the soft overdamped mode that exists for $|\mathbf{q}| < T$ (Eq. (3.80)).

[178] P.A. Lee *et al.*, Phys. Rev. B **54**, 8782 (1996).

[179] A.J. Leggett, Rev. Mod. Phys. **47**, 331 (1975).

[180] C. Herring, in *Magnetism* IV, ed. G. Rado and H. Suhl.

[181] B.I. Halperin, P. A. Lee, and N. Read, Phys. Rev. B **47**, 7312 (1993).

[182] H. Fukuyama and K. Kuboki, J. Phys. Soc. Jpn. **59**, 2617 (1990).

[183] H. Zimmermann *et al.*, Physica C **185-189**, 1145 (1991).

[184] A.J. Millis, Phys. Rev. Lett. **71**, 3614 (1993).

[185] A.J. Millis *et al.*, J. Phys. Chem. Sol. **56**, 1641 (1995).

[186] M.-H. Julien *et al.*, Phys. Rev. Lett. **76**, 4238 (1996).

[187] Y. Itoh *et al.*, preprint; J. Bobroff *et al.*, preprint.

[188] Reduction of entropy at high temperatures due to a_0-fluctuation in the uRVB gauge theory is known[78].

[189] Laughlin has strongly pushed for this analogy. See, for example, R. B. Laughlin, *Proceedings of the Inauguration Conference of the Asia-Pacific Center for Theoretical Physics* ed. by Y. M. Cho, J. B. Hong, and C. N. Yang, World-Scientific (1998), and Phys. Rev. Lett. **79**, 1726 (1997); on a wilder note, the $\mu - T$ phase diagram of QCD (see, for example, K. Rajagopal, hep-ph/9808348) has tantalizing similarities to that of the high T_c cuprates.

[190] P. Hasenfrantz and F. Niedermayer, Z. Phys. B **92**, 91 (1993).

[191] J. Gasser and H. Leutwyler, Nuc. Phys. B **250**, 465 (1985).

[192] S. Weinberg, *Quantum Theory of Fields II*, Chap.19. Cambridge Univ. Press (1996).

[193] Admittedly the theory is a little bit crude, in view of the singular behavior of the uniform susceptibility $\chi_u(T)$ very close to $T = 0$ (S. Eggert, I. Affleck, and M. Takahashi, Phys. Rev. Lett. **73**, 332 (1994))

[194] D. C. Dender, Phys. Rev. Lett. **79**, 1750 (1997).

[195] Müller *et al.*, Phys. Rev. B **24**, 1429 (1981).

[196] We will need a big magnetic field; It is assumed that high magnetic field simply causes rigid shift of spinon bands.

[197] See, for example, P. Ramond, *Field Theory: a modern primer*, Addison-Wesley (1990).

[198] E. Witten, Nuc. Phys. B **149**, 285 (1979).

[199] A. D'Adda, M. Luscher. and P. Di Vecchia, Nuc. Phys. B **146**, 63 (1978).

[200] X. -G. Wen, Phys. Rev. B **39**, 7223 (1989).

[201] A. M. Polyakov, *Gauge Fields and Strings*, Harwood Academic Publishers (1987).

[202] See, for example, C. G. Callan, R. Dashen, and D. J. Gross, Phys. Rev. D **17**, 2717 (1978).

[203] G. 't Hooft, Phys. Rev. Lett. **37**, 8 (1976).

[204] J. B. Marston, Phys. Rev. B **42**, 10804 (1990).

[205] M. Yu. Khlebnikov, Phys. Rev. B **50**, 6954 (1994).

[206] N. Read and S. Sachdev, Phys. Rev. B **7B**, 4568 (1990).

[207] H. Pagels and S. Stoker, Phys. Rev. D **20**, 2947 (1979).

[208] P. Jain and H. J. Munczek, Phys. Rev. D **44**, 1873 (1991).

[209] R. Jackiw and K. Johnson, Phys. Rev. D **8**, 2386 (1973).

[210] E. Fradkin and S. H. Shenker, Phys. Rev. D **19**, 3682 (1979).

[211] M. A. Kastner *et al.*, Rev. Mod. Phys. **70**, 897 (1998).