

**REAL, COMPLEX, AND QUATERNIONIC
TORIC SPACES**

by

Richard A. Scott

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Submitted to the Department of Mathematics
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Signature of Author _____
Department of Mathematics
April 23, 1993

Certified by _____
Robert MacPherson
Professor of Mathematics
Thesis Supervisor

Accepted by _____
Professor Sigurdur Helgason
Director of Graduate Studies

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Abstract

In this thesis we give topological generalizations of complex toric varieties to the real numbers and quaternions. The resulting spaces, called, respectively, *real toric spaces* and *quaternionic toric spaces*, are characterized by a convex polytope P together with some algebraic data in the form of a *characteristic function* λ on the faces of P . In all cases we discuss conditions for nonsingularity and compute the cohomology ring for these nonsingular examples in terms of P and λ . The real part of a complex toric variety is the motivating example in the real case, and for these real varieties we give conditions on P for the existence of topological embeddings into real projective space. These conditions are shown to be weaker than those for projective embeddings of complex toric varieties.

In contrast to the real and complex cases, quaternionic toric spaces can be topologically nonsingular but fail to be smooth. Examples which arise easily are the Thom spaces of Milnor's exotic 7-spheres (given as 3-sphere bundles over the 4-sphere). Focusing on dimension 2 (8 real dimensions), we study a certain class of smooth examples. These 8-manifolds, being 3-connected, are classified by their intersection forms and first Pontrjagin classes. Formulas are given for these invariants, as well as other characteristic numbers, in terms of the characteristic function λ .

Thesis Supervisor: Dr. Robert MacPherson

Title: Professor of Mathematics

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0 Introduction

The theory of complex toric varieties was introduced formally in the early 1970's by Kempf, Knudsen, Mumford and Saint-Donat in [14]. The standard definition of a toric variety involves patching together various affine varieties according to the data of a rational cone complex Σ (or *fan*) in \mathbf{R}^d . It follows from this construction that a toric variety contains the algebraic torus $(\mathbf{C}^*)^d$ as a dense open subset and the natural action of the torus on itself extends to an action on the entire variety. Conversely, any variety with such a torus action (and dense orbit) is a toric variety (see, eg., [24]). This connection between the rational/combinatorial structure of cone complexes and torus actions on algebraic varieties is, historically, the motivation for the study of toric varieties, and leads to elegant descriptions of such objects as the cohomology ring and characteristic classes.

A classical problem in algebraic geometry is to determine when an algebraic variety is projective; in the case of toric varieties, the answer depends on the existence of certain “convex” functions on the cone complex. Such a function defines a rational polytope P (which is combinatorially dual to Σ), connecting the theory of toric varieties to that of convex polytopes. The use of geometric methods for toric varieties has led to progress in several areas relating to polytopes, two of the more notable applications being the counting of lattice points and the relations between face numbers (and flag numbers).

The entire subject was approached from a different angle by various people including Atiyah, Guillemin, Sternberg [1, 10] who studied (compact) torus actions on symplectic manifolds. Here the differential structure is enough to provide a description of the quotient space as the image of the *moment map*. In the event that a toric variety is nonsingular and projective, hence a symplectic manifold with a torus action, the moment map image coincides with the convex polytope mentioned above. A converse by Delzant [6] states that *every* symplectic $2d$ -manifold with an effective d -torus action (Hamiltonian) arises as such a variety.

More recently, Davis and Januszkiewicz [5] have generalized further to the case of a torus acting on an arbitrary manifold. They start with a manifold with a torus action, assuming only that the orbit space is a *simple* convex polytope and that the action is locally the standard representation of the torus on \mathbf{C}^n . They show that such objects are, in fact, more general than toric varieties and then proceed to compute their cohomology rings and characteristic classes.

In this thesis (starting with Section 2) we will make various additional generalizations arising from the description of a toric variety as a topological quotient of a product of the torus with a convex polytope. This description, due to MacPherson, starts with the space $P \times (S^1)^d$, where P is a d -dimensional convex polytope, and identifies points of the torus over the boundary of P . The first generalization (Sec-

tion 4) is to replace the torus with its *real* counterpart $(S^0)^d$ which we can think of additively as the group $(\mathbb{Z}/(2))^d$. We give necessary and sufficient conditions for the resulting *real toric space* to be nonsingular (Theorem 4.3.1), and in the event that it is, we compute the cohomology ring (Theorem 4.4.1). This computation of the cohomology ring is also given in [5]; we include it here both for the sake of completeness and because the author computed it without knowledge of this prior work. The real part of a toric variety (ie, a *real toric variety*) is perhaps the most interesting example of such a space, and Januszkiewicz has brought to my attention a paper [13] by Jurkiewicz where the cohomology ring in this case is also computed.

Another generalization (Section 5) is to replace the torus with the group $(S^3)^d$ and to specify appropriate identifications over the boundary of the polytope. Despite the fact that these spaces are not toric, we refer to them as *quaternionic toric spaces* to emphasize their heritage. Again, we give conditions for such a space to be nonsingular (Propositions 5.4.1 and 5.4.2) and, in this case, compute its cohomology ring (Theorem 5.5.1).

In all three cases (real, complex, and quaternionic), the collapsing of the space $(S^k)^d$ ($k = 0, 1, 3$) only occurs over the boundary of polytope, hence the corresponding space has a natural smooth structure over the interior of P . In contrast to the real and complex cases, a quaternionic toric space can be *topologically* nonsingular, yet have this smooth structure fail to extend over the entire polytope. In Section 6.1, we give the simplest 2-dimensional (8 *real* dimensional) examples of this phenomenon, these spaces being the Thom spaces of Milnor's exotic 7-spheres (thought of as 3-sphere bundles over the 4-sphere). The obstruction to extending the smooth structure is a certain element of the group of exotic 7-spheres, a complete invariant for which is given by Eells and Kuiper in [7] (see Proposition 6.1.1).

For the remainder of Section 6, we concentrate on a natural class of 2-dimensional examples for which the differential structure does extend. These smooth 8-manifolds are necessarily 3-connected hence, as in [31], are determined by their intersection form and first Pontrjagin class. The relationship between these two invariants is given by certain equations involving the Pontrjagin *numbers*, the \hat{A} -genus, and the rank and signature of the intersection form. For our class of smooth examples, the characteristic function determines a sequence of pairs of integers (Lemma 6.2.1); in Section 6.3, we give explicit formulas for all of the above invariants and formulate the relations among them as restrictions on this sequence of integer pairs.

The motivating examples of real toric spaces are real toric varieties. Section 1 discusses these real varieties (including the singular ones), setting up notation and definitions which will be useful in later sections. As a culmination of this section, we prove a result for real toric varieties which is related to the complex projective embedding criterion mentioned above. In particular, an integral convex polytope P gives a natural map from a (uniquely determined) toric variety X to projective space. This map is an embedding if the polytope is large enough (ie., if the pullback of the

universal line bundle has enough global sections). The image of the restriction of this map to the real toric variety $X_{\mathbf{R}}$ is a real projective space, and Theorem 1.4.1 gives conditions on the size of P for this restricted map to be a topological embedding. In Section 1.5 we show that these conditions are weaker than those for complex projective embeddings.

0.1 Notation

$\mathbf{R}, \mathbf{C}, \mathbf{H}$	the real numbers, complex numbers, and quaternions (resp.)
$\mathbf{Z}, \mathbf{Z}/(n), \mathbf{Q}$	the integers, integers modulo n , and rational numbers (resp.)
$\mathbf{RP}^n, \mathbf{P}^n, \mathbf{HP}^n$	the real, complex and quaternionic projective n -spaces
N	a free \mathbf{Z} -module of rank d
M	the dual \mathbf{Z} -module $\text{Hom}(N, \mathbf{Z})$
$N_{\mathbf{R}}, M_{\mathbf{R}}$	the real vector spaces $N \otimes_{\mathbf{Z}} \mathbf{R}, M \otimes_{\mathbf{Z}} \mathbf{R}$
$\langle *, * \rangle$	the natural bilinear pairing $M_{\mathbf{R}} \times N_{\mathbf{R}} \rightarrow \mathbf{R}$
Σ	a rational cone complex (fan) in $N_{\mathbf{R}}$
P, \mathcal{P}	a d -dimensional polytope in $M_{\mathbf{R}}$, and its poset of faces
η	the number of facets of P
$X(\Sigma), X_{\mathbf{R}}(\Sigma)$	the complex and real (resp.) toric varieties associated to Σ
T_k	the unit k -sphere in \mathbf{R} ($k = 0$), \mathbf{C} ($k = 1$), and \mathbf{H} ($k = 3$)
λ_k	real ($k = 0$), complex ($k = 1$), and quaternionic ($k = 3$) characteristic functions
$\mathcal{T}_{\mathbf{R}}\lambda_0, \mathcal{T}_{\mathbf{C}}\lambda_1, \mathcal{T}_{\mathbf{H}}\lambda_3$	real, complex, and quaternionic toric spaces for the given characteristic function
μ	the natural projection from $\mathcal{T}_{\mathbf{R}}\lambda_0, \mathcal{T}_{\mathbf{C}}\lambda_1$, or $\mathcal{T}_{\mathbf{H}}\lambda_3$ to the polytope P
$\langle E \rangle$	free group on the set E
$\langle E R \rangle, \langle E \rangle // \langle R \rangle$	the group given by generators E and relations R
$\Gamma \langle E R \rangle$	the function space $\text{Hom}_{gp}(\langle E R \rangle, \Gamma)$ where Γ is a topological group (usually S^3)

In addition, we mention the following conventions. A **cone** will be denoted by a lower case bold letter as in \mathbf{c} , with dual cone $\check{\mathbf{c}}$. The smallest \mathbf{R} -subspace (say in $N_{\mathbf{R}}$) spanned by \mathbf{c} will be denoted by \mathbf{Rc} and the induced sublattice $N \cap \mathbf{Rc}$ by \mathbf{Zc} . The **faces** of a polytope (in $M_{\mathbf{R}}$) will be denoted by lower case Greek letters, usually σ or τ . $\mathbf{R}\sigma$ will denote the affine \mathbf{R} -span of σ , translated to the origin.

1 Real Projective Embeddings

A toric variety $X(\Sigma)$ is uniquely determined by a complex Σ of rational cones in \mathbf{R}^d . It will have a projective embedding $X(\Sigma) \rightarrow \mathbf{P}^n$ if and only if Σ has a dual cell complex with the same face lattice as an integral convex polytope P (cf. Appendix [25]). Having chosen such a P , there is a natural map ϵ_P from $X(\Sigma)$ to a certain projective space, and in the event that P is large enough, this map is an embedding. In particular, for some large k , ϵ_{kP} is an embedding. Restricting the natural map ϵ_P to the real part $X_{\mathbf{R}}(\Sigma)$ of a complex toric variety gives a map from $X_{\mathbf{R}}(\Sigma)$ to a real projective space, and again, if P is large enough, this map will be an embedding. The main result of this section is to show that P does not have to be as large in the real case to ensure a topological embedding.

In Sections 1.1 and 1.2 we present the necessary definitions for complex toric varieties. In Section 1.3, we give the explicit conditions on the size of P to ensure a *complex* projective embedding. In Section 1.4 we state and prove the main result: conditions for real embeddings. And finally, in Section 1.5 we show that these conditions are weaker than those for complex projective embeddings.

1.1 Cones and Affine Toric Varieties

Let N be a free \mathbf{Z} -module of rank d and let M be the dual module $\text{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$. Denote by $N_{\mathbf{R}}$ and $M_{\mathbf{R}}$ (respectively, $N_{\mathbf{Q}}$ and $M_{\mathbf{Q}}$) the vector spaces $N \otimes_{\mathbf{Z}} \mathbf{R}$ and $M \otimes_{\mathbf{Z}} \mathbf{R}$ (resp., $N \otimes \mathbf{Q}$ and $M \otimes \mathbf{Q}$). Geometrically, we just think of M and N as standard lattices in their respective Euclidean spaces. The natural pairing $\langle \cdot, \cdot \rangle : M_{\mathbf{R}} \times N_{\mathbf{R}} \rightarrow \mathbf{R}$ clearly restricts to $M \times N$ and to $M_{\mathbf{Q}} \times N_{\mathbf{Q}}$.

A *rational cone* \mathbf{c} in $N_{\mathbf{R}}$ is the convex hull of a finite set of rays passing through nonzero points of N . For an arbitrary set $S \subset N$, we will often use the notation $\mathbf{R}_{\geq 0}S$ for the convex hull of the rays passing through points of S . For a rational cone \mathbf{c} in $N_{\mathbf{R}}$ there is, in fact, a unique minimal set $\{n_1, n_2, \dots, n_s\}$ of *primitive* lattice points in N , called the *extreme set* for \mathbf{c} and written $\text{ext } \mathbf{c}$, such that

$$\mathbf{c} = \mathbf{R}_{\geq 0}n_1 + \mathbf{R}_{\geq 0}n_2 + \cdots + \mathbf{R}_{\geq 0}n_s = \mathbf{R}_{\geq 0}\text{ext } \mathbf{c}.$$

Dually, \mathbf{c} can be written as the intersection of a finite number of rational half spaces:

$$\mathbf{c} = \bigcap_{i=1}^r \{q \in N_{\mathbf{R}} \mid \langle m_i, q \rangle \geq 0\}$$

where the m_i are unique, primitive lattice points of M . The smallest \mathbf{R} -subspace of $N_{\mathbf{R}}$ containing \mathbf{c} will be denoted by $\mathbf{R}\mathbf{c}$, and the *dimension* of the cone \mathbf{c} is the dimension of the vector space $\mathbf{R}\mathbf{c}$. If the cardinality of $\text{ext } \mathbf{c}$ is equal to the dimension of \mathbf{c} , then \mathbf{c} is called *simplicial*. Any hyperplane which does not intersect the interior of \mathbf{c} is

called a *supporting hyperplane*, and a *face* \mathbf{b} of \mathbf{c} , written $\mathbf{b} \prec \mathbf{c}$, is the intersection of \mathbf{c} with a supporting hyperplane. A cone \mathbf{c} is *strictly convex* if it contains no nonzero subspace of $N_{\mathbf{R}}$ and is *maximal* if $\mathbf{R}\mathbf{c} = N_{\mathbf{R}}$. A maximal, simplicial cone for which the d spanning rays are generated by a basis n_1, n_2, \dots, n_d for N will be called *basic*. Throughout this work all cones will be assumed to be rational.

If \mathbf{c} is a cone in $N_{\mathbf{R}}$, we denote by \mathbf{c}^\perp the vector subspace

$$\{p \in M_{\mathbf{R}} \mid \langle p, q \rangle = 0 \text{ for all } q \in \mathbf{c}\}$$

of $M_{\mathbf{R}}$. And we define the *dual cone* $\check{\mathbf{c}}$ to be the rational cone in $M_{\mathbf{R}}$ defined by

$$\check{\mathbf{c}} = \{p \in M_{\mathbf{R}} \mid \langle p, q \rangle \geq 0 \text{ for all } q \in \mathbf{c}\}.$$

Notice that this duality has the following properties ([25]):

- i. $(\check{\check{\mathbf{c}}})^\vee = \mathbf{c}$.
- ii. If \mathbf{b} is a face of \mathbf{c} , $\check{\mathbf{c}}$ is contained $\check{\mathbf{b}}$.
- iii. If \mathbf{c} is strictly convex, $\check{\mathbf{c}}$ is maximal.
- iv. \mathbf{c} is simplicial and maximal if and only if $\check{\mathbf{c}}$ is simplicial and maximal.
- v. \mathbf{c} is strictly convex and maximal if and only if $\check{\mathbf{c}}$ is strictly convex and maximal.
- vi. \mathbf{c} is basic if and only if $\check{\mathbf{c}}$ is basic.

Property (ii) has a much stronger formulation, the proof of which can also be found in [25]. Namely, there is an inclusion reversing bijection between k -faces of \mathbf{c} and codimension k faces of $\check{\mathbf{c}}$ given by

$$\mathbf{b} \longmapsto \mathbf{b}^\perp \cap \check{\mathbf{c}}.$$

If \mathbf{c} is a strictly convex cone in $N_{\mathbf{R}}$, then $\check{\mathbf{c}} \cap M$ has the structure of an additive semigroup with unit. $\mathbf{C}[\check{\mathbf{c}} \cap M]$ denotes the semigroup \mathbf{C} -algebra generated by $\check{\mathbf{c}} \cap M$, and the *affine toric variety* associated to \mathbf{c} is the variety

$$U(\mathbf{c}) = \text{Spec } \mathbf{C}[\check{\mathbf{c}} \cap M] = \text{Hom}_{\text{s.gp.}}(\check{\mathbf{c}} \cap M, \mathbf{C})$$

where Hom denotes unitary semigroup homomorphisms (\mathbf{C} being a multiplicative semigroup with unit 1). We will concentrate primarily on the second of these presentations for $U(\mathbf{c})$, regarding a *point* of $U(\mathbf{c})$ as a semigroup homomorphism from $\check{\mathbf{c}} \cap M$ to \mathbf{C} .

REMARK. The reader is encouraged to explore the correspondence between the two definitions of $U(\mathbf{c})$. Choosing, say n , generators for the semigroup $\check{\mathbf{c}} \cap M$ and finding all additive relations among these generators gives a presentation of the algebra $\mathbb{C}[\check{\mathbf{c}} \cap M]$ as a quotient of the polynomial ring with n indeterminates by some ideal I . We can then think of $U(\mathbf{c})$ as the zeros in \mathbb{C}^n of a set of polynomials generating I . On the other hand, each such point in \mathbb{C}^n determines a unique element of $\text{Hom}_{\text{s.gp.}}(\check{\mathbf{c}} \cap M, \mathbb{C})$ by sending the i th generator of $\check{\mathbf{c}} \cap M$ to the i th coordinate of the point. As a special case, notice that $U(\mathbf{c})$ is isomorphic to \mathbb{C}^d if and only if \mathbf{c} is basic ([25, page 15]).

1.2 Cone Complexes and Toric Varieties

The following fact motivates the construction of a general toric variety: if \mathbf{b} is a face of \mathbf{c} , then the affine variety $U(\mathbf{b})$ is a dense open subset of $U(\mathbf{c})$. Therefore, if two maximal cones in $N_{\mathbf{R}}$ share a face, there is a natural way to glue the associated affine varieties together along an open subset of each variety. To obtain this natural inclusion, notice that if $\mathbf{b} \prec \mathbf{c}$, $\check{\mathbf{b}} \cap M$ is a subsemigroup of $\check{\mathbf{c}} \cap M$, and that this induces a map

$$U(\mathbf{b}) = \text{Hom}_{\text{s.gp.}}(\check{\mathbf{b}} \cap M, \mathbb{C}) \rightarrow U(\mathbf{c}) = \text{Hom}_{\text{s.gp.}}(\check{\mathbf{c}} \cap M, \mathbb{C}).$$

$x \in U(\mathbf{c})$ is in the image of this map if and only if $x(p) \neq 0$ for all $p \in (\mathbf{b}^\perp \cap \check{\mathbf{c}}) \cap M$. Because any $p \in \check{\mathbf{b}} \cap M$ can be written $p = p_1 - p_2$ where $p_1 \in \check{\mathbf{c}} \cap M$ and $p_2 \in (\mathbf{b}^\perp \cap \check{\mathbf{c}}) \cap M$, any $x \in U(\mathbf{c})$ in the image of this map is the image of the *unique* semigroup homomorphism $x' \in U(\mathbf{b})$ defined by

$$x'(p) = x(p_1)/x(p_2).$$

It follows that $U(\mathbf{b})$ is a dense open subset of $U(\mathbf{c})$.

Notice, in particular, that the variety $U(\mathbf{o})$ corresponding to the zero cone is the *algebraic* torus

$$\text{Hom}_{\text{s.gp.}}(M, \mathbb{C}) = N \otimes_{\mathbf{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^d.$$

This torus sits inside *every* affine toric variety as a Zarisky open subset. (In fact, the subgroup $T^d = \text{Hom}_{\text{s.gp.}}(M, S^1)$ is the compact torus of Section 1.)

A *rational cone complex* Σ in $N_{\mathbf{R}}$ is a collection of rational cones satisfying the two conditions:

- i. If $\mathbf{c} \in \Sigma$ and $\mathbf{b} \prec \mathbf{c}$, then $\mathbf{b} \in \Sigma$.
- ii. If $\mathbf{c}_1 \in \Sigma$ and $\mathbf{c}_2 \in \Sigma$, then $\mathbf{c}_1 \cap \mathbf{c}_2 \in \Sigma$.

Because of the combinatorial structure of Σ and the natural inclusions mentioned above, all of the varieties $\{U(\mathbf{c})|\mathbf{c} \in \Sigma\}$ glue together to form the *complex toric variety* $X(\Sigma)$. In the category of spaces, this construction is simply the topological union or *pushout* of the directed system $\{U(\mathbf{c})|\mathbf{c} \in \Sigma\}$. For important properties of $X(\Sigma)$ (for example, that it is a Hausdorff complex analytic space), the reader is referred to [25, Theorem 1.4].

1.3 Complex Projective Varieties

Let P be an integral convex polytope in $M_{\mathbf{R}}$ (ie, P is the convex hull of a finite subset of M). Assume that P is d -dimensional. We can always translate P so that 0 is in the interior and the vertices of P are in $M_{\mathbf{Q}}$. Hence, we can write this translate of P as the intersection of *affine* halfspaces (one for each facet):

$$P = \bigcap_{i=1}^{\eta} \{p \in M_{\mathbf{R}} | \langle n_i, p \rangle \leq 1\}$$

where each $n_i \in N_{\mathbf{Q}}$ is uniquely determined by the chosen translation of P . We now let Σ be the collection of cones in $N_{\mathbf{R}}$ of the form

$$\mathbf{c} = \mathbf{R}_{\geq 0} S$$

where $S \subset \{n_1, \dots, n_{\eta}\}$ is such that

$$\bigcap_{n \in S} \{p \in M_{\mathbf{R}} | \langle n, p \rangle = 1\}$$

is an affine subspace of $M_{\mathbf{R}}$ containing a face of P . Details of the following fact can be found in [25].

Proposition 1.3.1 Σ is a complete rational cone complex and does not depend on the choice of 0 (ie, the translation).

Σ is called the cone complex *dual* to P and the variety $X(\Sigma)$ is called the toric variety associated to P .

If $\{m_0, m_2, \dots, m_r\} = P \cap M$ is the set of lattice points in P , then there is a natural algebraic map ϵ from the toric variety associated to P to the projective space \mathbf{P}^r . A maximal cone $\mathbf{c} \in \Sigma$ corresponds to a vertex, say m_0 . If $P - m_0$ is the translation of P which moves m_0 to the origin, then the $\mathbf{R}_{\geq 0}$ -span of $P - m_0$ is the dual cone $\check{\mathbf{c}}$. In particular, $m - m_0 \in \check{\mathbf{c}} \cap M$ for all $m \in R$. ϵ is then defined on $U(\mathbf{c})$ as follows:

$$x \in U(\mathbf{c}) \longmapsto [1, x(m_1 - m_0), x(m_2 - m_0), \dots, x(m_r - m_0)] \in \mathbf{P}^r.$$

ϵ is defined similarly on $U(\mathbf{c})$ for each maximal cone $\mathbf{c} \in \Sigma$, and one can check that these maps agree on the overlaps (see eg, [25]).

We are now ready to state the main theorem of complex projective embeddings. For the proof, the reader is referred to [25].

Theorem 1.3.1 *The map $\epsilon : X(\Sigma) \rightarrow \mathbf{P}^r$ is an embedding if and only if for each vertex $m_0 \in P$ the set $\{m - m_0 \in M \mid m \in P \cap M\}$ generates the semigroup $\check{c} \cap M$ where \mathbf{c} is the cone dual to m_0 .*

1.4 Real Projective Embeddings

By a real toric variety, we mean the real part of a complex toric variety. To be precise, the *real part* of an affine toric variety $U(\mathbf{c})$ is the subspace

$$U_0(\mathbf{c}) = \text{Hom}_{\text{s.gp.}}(\check{c} \cap M, \mathbf{R}) \subset U(\mathbf{c})$$

and, as in the complex case, for a fixed cone complex Σ , the $U_0(\mathbf{c})$, as \mathbf{c} runs through Σ , glue together along the natural inclusions. The resulting space is again a pushout in the category of spaces and is called the *real toric variety* $X_{\mathbf{R}}(\Sigma)$.

If P is an integral convex polytope in $M_{\mathbf{R}}$ and Σ is the dual cone complex defined in Section 1.3, then ϵ_0 will denote the restriction of the map $\epsilon : X(\Sigma) \rightarrow \mathbf{P}^r$ to the subspace $X_{\mathbf{R}}(\Sigma)$. Clearly, ϵ_0 has image in the *real* projective space $\mathbf{RP}^d \subset \mathbf{P}^d$. In addition, for each l -face σ of P , we denote by $\mathbf{R}\sigma$ the unique l dimensional subspace of $M_{\mathbf{R}}$ which is parallel to σ (ie, the \mathbf{R} -linear extension of $\sigma - p$ for some $p \in \sigma$). We let $\mathbf{Z}\sigma$, then, be the rank l , unimodular sublattice $\mathbf{R}\sigma \cap M$ of M . We now state the main result of this section.

Theorem 1.4.1 *The map $\epsilon_0 : X_{\mathbf{R}}(\Sigma) \rightarrow \mathbf{RP}^d$ is a topological embedding if and only if for every pair (v, σ) , where σ is a face of P and v is a vertex of σ , the image in $M \otimes \mathbf{Z}/(2)$ of the set $\{m \in M \mid m + v \in \sigma \cap M\}$ generates $\mathbf{Z}\sigma \otimes \mathbf{Z}/(2)$.*

Just as we defined the sublattice $\mathbf{Z}\sigma$ for a face σ of P , for any convex l dimensional cone $\mathbf{c} \in M$, we let $\mathbf{Z}\mathbf{c}$ denote the rank l unimodular sublattice $\mathbf{R}\mathbf{c} \cap M$. In particular, $\mathbf{Z}\mathbf{c} = M$ for every maximal cone \mathbf{c} . For any finite subset S of M , we denote by $\mathbf{Z}_{\geq 0}S$ the subsemigroup of M generated by S (including 0), and for

$$x \in \text{Hom}_{\text{s.gp.}}(\mathbf{Z}_{\geq 0}S, \mathbf{R})$$

we define the *support of x in S* , written $\text{supp } x$, to be the set $\{m \in S \mid x(m) \neq 0\}$. Theorem 1.4.1 is a straightforward consequence of the following *affine* version.

Theorem 1.4.2 *Let c be a maximal, strictly convex cone in $N_{\mathbf{R}}$ with dual cone (as in Section 1.1)*

$$\check{c} = \mathbf{R}_{\geq 0}m_1 + \mathbf{R}_{\geq 0}m_2 + \cdots + \mathbf{R}_{\geq 0}m_s = \mathbf{R}_{\geq 0}\text{ext } \check{c}.$$

Let S be a subset of $\check{c} \cap M$ which contains $\text{ext } \check{c}$. Then the natural map

$$\phi : U_0(c) = \text{Hom}_{\text{s.gp.}}(\check{c} \cap M, \mathbf{R}) \rightarrow \text{Hom}_{\text{s.gp.}}(\mathbf{Z}_{\geq 0}S, \mathbf{R})$$

is a homeomorphism if and only if for every face $b \prec \check{c}$, the image of $S \cap b$ generates $\mathbf{Z}b \otimes \mathbf{Z}/(2)$ as a $\mathbf{Z}/(2)$ -vector space.

Lemma 1.4.1 *Let c be a strictly convex, l -dimensional cone in $M_{\mathbf{R}}$, let p be a point in $\text{relint } c \cap M$, and let m_0 be an element of $\text{ext } c$. Then there is a set $\Gamma \subset \text{ext } c$ containing m_0 and a map $a : \Gamma \rightarrow \mathbf{Q}_{\geq 0}$ such that $a(m_0) > 0$ and*

$$p = \sum_{m \in \Gamma} a(m) \cdot m.$$

Proof. The proof is by induction on $l = \dim c$. If $l = 0$, p is a positive integral multiple of m_0 ; it suffices, then, to let $\Gamma = \{m_0\}$ and to let $a(m_0)$ be this integral multiple. More generally, the ray starting at m_0 and passing through p will intersect $\text{relint } b$ for a unique proper face b of c . Let p' be this intersection point and notice that there are positive rational constants α and β such that

$$p = \alpha m_0 + \beta p'.$$

Next choose $m'_0 \in \text{ext } b$. By induction, there is a set $\Gamma' \subset \text{ext } b \subset \text{ext } c$ containing m'_0 and a map $a' : \Gamma' \rightarrow \mathbf{Q}_{\geq 0}$ such that $a'(m'_0) > 0$ and

$$p' = \sum_{m \in \Gamma'} a'(m) \cdot m.$$

Finally, let $\Gamma = \Gamma' \cup \{m_0\}$ and define $a(m_0) = \alpha$ and $a(m) = \beta \cdot a'(m)$ for $m \in \Gamma'$. ■

Lemma 1.4.2 *Let \check{c} be a maximal, strictly convex cone in $M_{\mathbf{R}}$. If S is any finite subset of $\check{c} \cap M$ containing $\text{ext } \check{c}$ and $x \in \text{Hom}_{\text{s.gp.}}(\mathbf{Z}_{\geq 0}S, \mathbf{R})$, then $\text{supp } x = S \cap b$ for some face b of \check{c} .*

Proof. We first show that $\text{supp } x$ is of the form

$$\bigcup_{b \in \mathfrak{B}} S \cap b$$

for some collection \mathfrak{B} of faces of \check{c} . Then we show that $\cup \mathfrak{B}$ is itself a face of \check{c} .

Each $p \in S$ is in the relative interior of exactly one face of \check{c} . We will show that if $p \in \text{supp } x$ and $p \in \text{relint } \mathbf{b}$ then $\mathbf{b} \cap S$ is contained in $\text{supp } x$. We then take \mathfrak{B} to be the collection of all faces containing a point $p \in \text{supp } x$. Suppose, then, that $p \in \text{relint } \mathbf{b}$ for some l -face \mathbf{b} and $p \in \text{supp } x$. For each $m_0 \in \text{ext } \mathbf{b}$, we can write

$$p = \sum_{m \in \Gamma} a(m) \cdot m$$

with a and Γ defined as in Lemma 1.4.1. But $x(p) \neq 0$ implies, then, that $x(m_0) \neq 0$ since all coefficients $a(m)$ are non-negative and $a(m_0)$ is strictly positive. It follows that $\text{ext } \mathbf{b}$ is contained in the support of x , and since any other point in $\mathbf{b} \cap S$ is a non-negative rational combination of these extreme points, it must also be an element of $\text{supp } x$.

Next we will show that if $\mathbf{b}_1 \cap S$ and $\mathbf{b}_2 \cap S$ are both in $\text{supp } x$, then so is $\mathbf{b}_3 \cap S$ where \mathbf{b}_3 is the smallest face of \check{c} containing \mathbf{b}_1 and \mathbf{b}_2 . If

$$p_1 = \sum_{m \in \text{ext } \mathbf{b}_1} m \quad \text{and} \quad p_2 = \sum_{m \in \text{ext } \mathbf{b}_2} m \quad (1)$$

then $p_1 \in \text{relint } \mathbf{b}_1$, $p_2 \in \text{relint } \mathbf{b}_2$, and $p_1 + p_2 \in \text{relint } \mathbf{b}_3$. For any $m_0 \in \text{ext } \mathbf{b}_3$, we can find (by Lemma 1.4.1) $\Gamma \subset \text{ext } \mathbf{b}_3$ and $a : \Gamma \rightarrow \mathbf{Q}_{\geq 0}$ with $a(m_0) > 0$ such that

$$p_1 + p_2 = \sum_{m \in \Gamma} a(m) \cdot m.$$

Substituting the expressions (1) for p_1 and p_2 , we have

$$\sum_{m \in \text{ext } \mathbf{b}_1} m + \sum_{m \in \text{ext } \mathbf{b}_2} m = \sum_{m \in \Gamma} a(m) \cdot m. \quad (2)$$

Multiplying both sides by a suitable positive integer D to clear denominators (so that all terms of equation (2) are in the semigroup $\mathbf{Z}_{\geq 0} S$), and applying x to the result gives the equation

$$\prod_{m \in \text{ext } \mathbf{b}_1} x(m)^D \prod_{m \in \text{ext } \mathbf{b}_2} x(m)^D = x(m_0)^{a(m_0)D} \prod_{m \in \Gamma \setminus \{m_0\}} x(m)^{a(m)D}.$$

Since the left hand side of this equation is nonzero, $x(m_0)$ is nonzero. Repeating the argument for all extreme points of \mathbf{b}_3 gives $\text{ext } \mathbf{b}_3 \subset \text{supp } x$, and using the argument of the previous paragraph, we have $\mathbf{b}_3 \cap S \subset \text{supp } x$. Finally, since \mathfrak{B} is finite, we can continue this process until we find a maximal face $\mathbf{b} \in \mathfrak{B}$ with

$$\mathbf{b} = \cup \mathfrak{B}.$$

■

Proof of Theorem 1.4.2. Assume first that for every face $\mathbf{b} \prec \check{\mathbf{c}}$, the image of $S \cap \mathbf{b}$ generates $\mathbf{Z}\mathbf{b} \otimes \mathbf{Z}/(2)$ as a $\mathbf{Z}/(2)$ -vector space. We need only to produce an inverse for ϕ ; ϕ , being polynomial is certainly continuous, and the continuity of ϕ^{-1} will follow from its explicit formulation. Let $x \in \text{Hom}_{\text{s.gp.}}(\mathbf{Z}_{\geq 0}S, \mathbf{R})$ and find (by Lemma 1.4.2) an l -face \mathbf{b} of $\check{\mathbf{c}}$ such that $\text{supp } x = S \cap \mathbf{b}$. We need to find an element $y \in U_0(\mathbf{c})$ such that $\phi(y) = x$. If $m \notin \mathbf{b} \cap M$ define $y(m) = 0$. Otherwise, let m_1, m_2, \dots, m_l be a set of elements of $S \cap \mathbf{b}$ whose images $\overline{m}_1, \overline{m}_2, \dots, \overline{m}_l$ form a basis for $\mathbf{Z}\mathbf{b} \otimes \mathbf{Z}/(2)$. Then, in fact, the m_i 's form a basis for $\mathbf{Z}\mathbf{b} \otimes \mathbf{Q}$, because for any linear dependence $\sum a_i m_i = 0$ with $a_i \in \mathbf{Q}$, we can clear the denominators and reduce mod 2 to get a nontrivial dependence in $\mathbf{Z}\mathbf{b} \otimes \mathbf{Z}/(2)$. For any primitive element $m \in \mathbf{b} \cap M$, then, we can write

$$bm = a_1 m_1 + a_2 m_2 + \dots + a_l m_l \quad (3)$$

for $b, a_1, a_2, \dots, a_l \in \mathbf{Z}$ relatively prime. We can assume further that b is odd: because m is primitive, $\overline{m} \neq 0$, and reducing equation (3) mod 2 would produce a non-trivial dependence among the \overline{m}_i 's. Finally, $m_i \in \text{supp } x$ means $x(m_i) \neq 0$ for $i = 1, 2, \dots, l$, so the expression

$$y(m) = \left(\prod_{i=1}^l x(m_i)^{a_i} \right)^{1/b}$$

is unique and well-defined over \mathbf{R} . If m is not primitive, we can find a primitive m' with $m = am'$ ($a \in \mathbf{Z}_{\geq 0}$), then define $y(m) = y(m')^a$. It is clear from the construction that y is a semigroup homomorphism from $\check{\mathbf{c}} \cap M$ to \mathbf{R} and that $\phi(y) = x$.

Conversely, assume \mathbf{b} is an l -face of $\check{\mathbf{c}}$ and the mod 2 reduction of $S \cap \mathbf{b}$ does not generate $\mathbf{Z}\mathbf{b} \otimes \mathbf{Z}/(2)$. Our conditions on S guarantee that there are l elements $m_1, m_2, \dots, m_l \in S \cap \mathbf{b}$ which form a \mathbf{Q} -basis for $\mathbf{Z}\mathbf{b} \otimes \mathbf{Q}$. Choose a primitive $m_0 \in \mathbf{b} \cap M$ such that $\overline{m}_0 \in \mathbf{Z}\mathbf{b} \otimes \mathbf{Z}/(2)$ cannot be expressed as a $\mathbf{Z}/(2)$ -combination of elements in $S \cap \mathbf{b}$. Because we can write m_0 as a rational combination of m_1, m_2, \dots, m_l , some multiple bm_0 is an *integral* combination

$$bm_0 = a_1 m_1 + a_2 m_2 + \dots + a_l m_l$$

But b must be *even* since otherwise the mod 2 reduction of this equation would contradict our choice of m_0 .

We can now show that ϕ is at least two-to-one. Consider the semigroup homomorphism $x : \mathbf{Z}_{\geq 0}S \rightarrow \mathbf{R}$ defined by

$$x(m) = \begin{cases} 1 & \text{if } m \in \mathbf{b} \cap \mathbf{Z}_{\geq 0}S \\ 0 & \text{otherwise} \end{cases}$$

Now define $y^+, y^- \in U_0(\mathbf{c})$ by $y^\pm(m) = x(m)$ for $m \in \mathbf{Z}_{\geq 0}S$, $y^\pm(m_0) = \pm 1$, and choose any consistent extensions (possibly not unique) to the rest of $\check{\mathbf{c}} \cap M$. Then $\phi(y^\pm) = x$. ■

Proof of Theorem 1.4.1. We first prove the “if” direction. Let \mathbf{c} be a maximal cone of Σ and let $\epsilon_{\mathbf{c}}$ be the restriction of ϵ_0 to $U_0(\mathbf{c})$. As we said above, $\check{\mathbf{c}}$ is the $\mathbf{R}_{\geq 0}$ -span of the translated polytope $P - v$ where v is the vertex of P dual to the cone \mathbf{c} . Let S be the set $(P - v) \cap M$. For each face \mathbf{b} of $\check{\mathbf{c}}$, the set $\mathbf{b} \cap S$ is precisely the set $\{m \in M \mid m + v \in \sigma \cap M\}$ where σ is the face of P corresponding to \mathbf{b} . Moreover, the map $\epsilon_{\mathbf{c}}$ factors through the natural map ϕ of Theorem 1.4.2, giving the following commutative diagram:

$$\begin{array}{ccc} U_0(\mathbf{c}) & \xrightarrow{\epsilon_{\mathbf{c}}} & \mathbf{RP}^r \\ & \searrow \phi & \nearrow \\ & \text{Hom s.gp.}(\mathbf{Z}_{\geq 0}S, \mathbf{R}) & \end{array}$$

Since $\mathbf{Z}\sigma = \mathbf{Z}\mathbf{b}$, the hypotheses of our theorem guarantee that $\mathbf{b} \cap S$ generates the $\mathbf{Z}/(2)$ vector space $\mathbf{Z}\mathbf{b} \otimes \mathbf{Z}/(2)$; hence, by Theorem 1.4.2, the map ϕ is a homeomorphism. But it is clear that the natural map

$$\text{Hom s.gp.}(\mathbf{Z}_{\geq 0}S, \mathbf{R}) \rightarrow \mathbf{RP}^r$$

is injective, because each element of S corresponds to a distinct projective coordinate. It follows that $\epsilon_{\mathbf{c}}$ is injective (being a polynomial map, it is obviously continuous).

It remains to show that if x_1 and x_2 are two points of $X_{\mathbf{R}}(\Sigma)$ with $x_i \in U_0(\mathbf{c}_i)$ for maximal cones \mathbf{c}_1 and \mathbf{c}_2 of Σ such that $\epsilon_0(x_1) = \epsilon_0(x_2)$, then $x_1 = x_2$. We will show that if x_2 has the same image as x_1 , then $x_1 \in \text{Im} \{U_0(\mathbf{c}_1 \cap \mathbf{c}_2) \rightarrow U_0(\mathbf{c}_1)\}$; in other words, $x_1 \in U_0(\mathbf{c}_1) \cap U_0(\mathbf{c}_2)$. But then x_1 and x_2 are both in $U_0(\mathbf{c}_2)$ and by the previous paragraph must coincide.

Let \mathbf{b} be the common face $\mathbf{c}_1 \cap \mathbf{c}_2$ and recall from the beginning of Section 1.2 that x_1 is in the image of the inclusion

$$U_0(\mathbf{b}) \rightarrow U_0(\mathbf{c}_1)$$

if and only if the support of x_1 (in $\check{\mathbf{c}}_1 \cap M$) is precisely the set $(\mathbf{b}^{\perp} \cap \check{\mathbf{c}}_1) \cap M$. Let v_1 and v_2 be the vertices of P dual to the cones \mathbf{c}_1 and \mathbf{c}_2 . Then $x(v_2 - v_1) \neq 0$, since $x_1(v_2 - v_1)$ and $x_2(v_2 - v_2) = 1$ are the same projective coordinate for the point $\epsilon_0(x_1) = \epsilon_0(x_2) \in \mathbf{RP}^r$. Since $v_2 - v_1 \in \text{relint } \mathbf{b}^{\perp} \cap \check{\mathbf{c}}_1$, we can invoke Lemma 1.4.2 (or its proof, rather) to show that $\text{supp } x_1 = (\mathbf{b}^{\perp} \cap \check{\mathbf{c}}_1) \cap M$ as desired.

For the “only if” direction, assume we can find (v, σ) such that $\{m \mid m + v \in \sigma \cap M\}$ does *not* generate $\mathbf{Z}\sigma_v \otimes \mathbf{Z}/(2)$. Let \mathbf{c} be the maximal cone dual to v and let \mathbf{b} be the face of $\check{\mathbf{c}}$ corresponding to the face σ . With $S = \{m \mid m + v \in P \cap M\}$, we have

$S \cap \mathfrak{b} = \{m \mid m + v \in \sigma \cap M\}$, and by Theorem 1.4.2, the restriction of ϵ_0 to $U_0(\mathfrak{c})$ is at least two-to-one. Hence ϵ_0 is not injective. ■

1.5 Real Embeddings Are More Often

In this section we show that the conditions imposed on P in Theorem 1.4.1 are strictly weaker than those of Theorem 1.3.1. Obviously, if ϵ is a topological embedding, its restriction, ϵ_0 , to $X_{\mathbf{R}}(\Sigma)$ will also be an embedding, and one can easily check that the conditions in the first theorem on P imply the conditions in the second theorem. Our goal, then, is to find a P (with dual cone complex Σ) such that the natural map $\epsilon_0 : X_{\mathbf{R}}(\Sigma) \rightarrow \mathbf{RP}^r$ is an embedding while $\epsilon : X(\Sigma) \rightarrow \mathbf{P}^r$ is not.

Consider the rank-3 lattice L in \mathbf{R}^3 whose elements are all integral lattice points (p, q, r) with $p \equiv q \equiv r \pmod{3}$. A basis for this lattice consists of the points $\{(3, 0, 0), (0, 3, 0), (1, 1, 1)\}$. The convex hull of the four points $p_0 = (3, 3, 0), p_1 = (3, 0, 0), p_2 = (0, 3, 0)$, and $p_3 = (3, 3, 3)$ is a tetrahedron containing no other lattice points of L . Clearly, $\{p_1 - p_0, p_2 - p_0, p_3 - p_0\}$ fails to generate L over \mathbf{Z} , since the third coordinate is always a multiple of 3. On the other hand, for every i and every subset I of $\{0, 1, 2, 3\}$ containing i , the set $\{p_j - p_i \mid j \in I\}$ is independent in $L \otimes \mathbf{Z}/(2)$. Choosing a linear isomorphism $\alpha : \mathbf{R}^3 \rightarrow M_{\mathbf{R}}$ which carries L to the standard lattice M , and taking the convex hull of the images $\alpha(p_i)$ for $i = 0, 1, 2, 3$ then gives the desired polytope P in $M_{\mathbf{R}}$.

2 Stratified Toric Spaces

In Sections 3, 4, and 5, we present generalizations of toric varieties as abstract stratified spaces in the sense of Thom and Mather [18]. As mentioned in the introduction, the underlying spaces of these three generalizations are topological quotients of $P \times T_k^d$, where P is a d -dimensional convex polytope and T_k is the k -sphere for $k = 0, 1$, and 3 . The resulting stratified objects will be called *real*, *complex*, or *quaternionic toric spaces* for $k = 0, 1$, or 3 , respectively.

As topological spaces, these three classes have several similarities. For example, they all have a map to a convex polytope (mentioned above), they all have a perfect cell decomposition, and they all have the same cohomology ring (when appropriately graded and reduced mod 2 in the real case). The additional structure of a stratified space, however, reveals some significant differences. Most notably, in the quaternionic case, some of the spaces are topologically nonsingular, but admit no differential structure.

2.1 Abstract Stratified Spaces

The primary ingredients of a stratified space X are the following:

- i. A collection \mathcal{S} of disjoint sets whose union is all of X . $S \in \mathcal{S}$ is called a *stratum* of X and is assumed to have a fixed smooth structure.
- ii. For each stratum S , an open subset $\text{Tub}(S)$ of X containing S (ie, a *tubular neighborhood*).
- iii. For each S , a *tubular distance function* $\rho_S : \text{Tub}(S) \rightarrow [0, \infty)$ and a projection $\pi_S : \text{Tub}(S) \rightarrow S$.

The additional structure provided by $\text{Tub}(S)$, ρ_S , and π_S is often called the *control data* for X . This data is subject to various compatibility and smoothness conditions listed in [18]. The *dimension* of a stratum is its dimension as a manifold; the dimension d of X is the maximum dimension of all strata.

Following [9] (with different notation), we define $\text{Tub}_\epsilon(S)$ to be the level set $\{p \in \text{Tub}(S) | \rho_S(p) = \epsilon\}$ and $\text{Tub}_{<\epsilon}(S)$ to be the neighborhood $\{p \in \text{Tub}(S) | \rho_S(p) < \epsilon\}$. For ϵ suitably small, $\pi_S|_{\text{Tub}_\epsilon(S)}$ is a fiber bundle with fiber $L_\epsilon(S)$. More generally, if R is in the closure of S , then $\pi_{R,S} \equiv \pi_R|_{\text{Tub}_\epsilon(R) \cap S}$ is a fiber bundle ([18, Corollary 10.6]) and we denote the fiber by $L_\epsilon^S(R)$. It is well known that for small δ and ϵ , $L_\epsilon^S(R)$ is isotopic to $L_\delta^S(R)$. With this in mind, for each R , we fix ϵ and a point $p \in R$, and define the *link of R in S* , written $L^S(R)$, to be the fiber $\pi_{R,S}^{-1}(p)$. For simplicity we write $L(R)$ instead of $L^X(R)$. Although the link is defined with respect to a fixed point $p \in S$ and a fixed ϵ , its homeomorphism class is independent of this point.

Proposition 2.1.1 *If for every stratum S , the link of S (in X) is a $(d-l-1)$ -sphere where $l = \dim S$, then X is a topological manifold.*

Proof. Let p be a point in S . Then p has a neighborhood homeomorphic to the topological join of the link of S with a small disk about p in S . If the link is a sphere, this neighborhood is a ball. ■

Proposition 2.1.2 *If for some stratum S , the link of S is not a homology sphere, X is topologically singular (ie, is not a topological manifold).*

Proof. Let l be the dimension of S . For $p \in S$, the link $L(p)$ is the join of S^{l-1} with $L(S)$. Recall that the join of A and B , written $A * B$ is defined as $I \times A \times B / \sim$ where $(0, a, b) \sim (0, a, b')$ and $(1, a, b) \sim (1, a', b)$. Removing the subset $0 \times A \times B / \sim$ leaves a space which retracts to B , and removing the subset $1 \times A \times B / \sim$ leaves a space which retracts to A . The intersection of these two open subsets retracts to $A \times B$ yielding a Mayer-Vietoris sequence for joins. In our case, we obtain

$$\cdots \rightarrow H_*(S^{l-1}) \otimes H_*(L(S)) \rightarrow H_*(S_{l-1}) \oplus H_*(L(S)) \rightarrow H_*(L(p)) \rightarrow \cdots$$

From this exact sequence, one sees that if $L(S)$ is not a homology $(d-l-1)$ -sphere, $L(p)$ is not a homology $(d-1)$ -sphere, hence is not *homeomorphic* to a sphere. ■

2.2 Convex Polytopes as Stratified Spaces

As a simple example of a stratified space, consider a convex polytope P in \mathbf{R}^d (assume 0 is in the interior of P). Let \mathcal{P} be the corresponding partially ordered set (*poset*) of faces, and let \mathcal{P}_l denote the subset of l -faces for $l = 0, \dots, d$. If $\text{relint } \sigma$ denotes the relative interior of a face σ , then $\{\text{relint } \sigma \mid \sigma \in \mathcal{P}\}$ is a disjoint collection of (smooth) manifolds whose union is P . If V_σ is the affine linear subspace spanned by σ , we define the tubular distance function ρ_σ to be the usual Euclidean distance to V_σ , and the projection π_σ to be orthogonal projection onto V_σ . Let $f : \sigma \rightarrow \mathbf{R}$ be any smooth function which is positive on $\text{relint } \sigma$ and zero on the boundary of σ , and define the tubular neighborhood $\text{tub}(\sigma)$ to be the set

$$\{x \in P \mid \rho_\sigma(x) < f(x)\}.$$

It is clear that this control data makes P into a Thom-Mather stratified space.

The stratified link of an l -face $\sigma \in \mathcal{P}$ (as defined above) is itself stratified by intersecting with subfaces of σ , and the combinatorial data of this stratification (ie,

the poset of closed strata ordered by inclusion) is equivalent to that of a certain convex polytope, called the *combinatorial link* L_σ of σ in P . Explicitly, given a vertex v of P , let H be a hyperplane separating v from the other vertices of P . Then L_v is the convex polytope $H \cap P$, and has the collection $\{H \cap \sigma \mid \sigma \in \mathcal{P}\}$ as its faces. More generally, for any l -face σ , we can choose a transverse $d - l$ plane E which meets σ in exactly one point p . Then $E \cap P$ is a convex polytope and the link of σ in P is defined to be the link of p in $E \cap P$. Notice that the combinatorial type of the link does not depend on the choices of H or E .

When referring to the link of a face as a (stratified) topological space, we will always mean the stratified link with respect to a fixed point of the face, a fixed ϵ , and the control data for P defined above. If we are only interested in the combinatorial information of this space, however, we will often use the convex polytope link of the previous paragraph.

3 The Complex Case $k = 1$

3.1 Toric Spaces

The key ingredient in the construction of a toric space is the collapsing data, the idea being that as one approaches a k -face of the polytope from the interior, the d -torus *collapses* to a k -torus. In order to retain nice topological properties (eg, Hausdorff) and to provide a tubular neighborhood as in Section 2, we assume that the collapsing is a fibration $(S^1)^d \rightarrow (S^1)^k$. That is, points of the d -torus are identified when they are in the same fiber of this map. Following the algebraic notation [24], we fix the d -torus $T_1^d = N \otimes_{\mathbf{Z}} S^1$ where N is a free \mathbf{Z} -module of rank d . A sublattice of N is *unimodular* if it is a direct summand. A rank $d - l$ unimodular sublattice Λ of N defines a $d - l$ dimensional subtorus $T_\Lambda = \Lambda \otimes S^1$; the projection $f(\Lambda) : T_1^d \rightarrow T_1^d/T_\Lambda$ onto the l -dimensional quotient is a (trivial) fibration. With this model in mind, we let $G_l(N)$ denote the Grassmanian of unimodular rank l sublattices of N and make the following definition (as in [5]).

Definition 3.1.1 *A (complex) characteristic function is a map*

$$\lambda_1 : \mathcal{P}_l \rightarrow G_{d-l}(N)$$

for $0 \leq l \leq d$ satisfying the condition: if τ is a face of σ , then $\lambda_1(\sigma)$ is a sublattice of $\lambda_1(\tau)$.

By the *characteristic value* of λ_1 at σ we mean the sublattice $\lambda_1(\sigma)$.

Definition 3.1.2 *The toric space $\mathcal{T}_{\mathbf{C}}\lambda_1$ associated to the polytope P and characteristic function λ_1 is the topological space $P \times T_1^d / \sim$ where two points (x, s) and (x, t) are equivalent if $x \in \text{relint } \sigma$ and s and t have the same image in the quotient torus $T_1^d/T_{\lambda_1(\sigma)}$.*

Proposition 3.1.1 *Let $\mu : \mathcal{T}_{\mathbf{C}}\lambda_1 \rightarrow P$ be the natural projection. Then $\mathcal{T}_{\mathbf{C}}\lambda_1$ admits the structure of an abstract stratified space with strata $S_\sigma \equiv \mu^{-1}(\text{relint } \sigma)$ as σ ranges through all faces of P .*

Proof. Let P be stratified as in Section 2.2. We “lift” the stratification from P to $\mathcal{T}_{\mathbf{C}}\lambda_1$. Let $S = S_\sigma$ and define a tubular neighborhood $\text{Tub}(S)$ by $\mu^{-1}(\text{tub}(\sigma))$, a distance function by $\rho_S = \rho_\sigma \circ \mu$, and a projection map $\pi_S = (\pi_\sigma, f(\lambda_1(\sigma)))$.

■

The action of the torus T_1^d on itself defines an action on the quotient torus T_1^d/T_Λ which is equivariant with respect to the projection. As a result, $\mathcal{T}_{\mathbf{C}}\lambda_1$ has a T_1^d action with P as the orbit space.

REMARK. Definition 3.1.1 works in a slightly more general setting. Let P be a regular, finite CW decomposition of the d -ball with one maximal d -cell. Let \mathcal{P} be collection of closed cells of P partially ordered by inclusion, and generalize the notion of l -faces to include l -cells of P . It is clear that a convex polytope is just a restricted case of such a cell decomposition and that the definitions of λ_1 and $\mathcal{T}_{\mathbf{C}}\lambda_1$ extend to the more general complex. In order to describe all toric *varieties*, it is necessary to use this more general setting, but for most of our results we will focus on the case where P is a convex polytope. (As we remark in the next section, all *projective varieties* fall into this class.)

3.2 Toric Varieties

To see that the above construction adapts to the algebraic theory of toric varieties, recall that a toric variety is given by gluing together various *affine* toric varieties according to the data of a rational cone complex Σ (Section 1.2). Recall also that if $X(\Sigma)$ is projective, there exists a convex polytope whose associated cell complex is dual to Σ in the following sense. For every l -dimensional cone in Σ there is a unique codimension l face of the polytope and this correspondence is inclusion reversing. Oda gives a detailed account of how this dual polytope is obtained from Σ [25]. We now take P to be this dual complex and define a characteristic function λ_1 by mapping each l -face to the unimodular sublattice spanned by the intersection of the dual cone with the standard lattice $N \subset \mathbf{R}^d$. Notice that the definition of a cone complex is precisely the information necessary to guarantee that λ_1 be a characteristic function. The toric space $\mathcal{T}_{\mathbf{C}}\lambda_1$ is homeomorphic to the underlying topological space (in the Euclidean topology) for the projective variety.

Even if $X(\Sigma)$ is not projective, Σ has a dual cell complex P which is a regular, finite cell decomposition of the d -ball with one maximal cell. In light of the remark of Section 3.1, we can still form $\mathcal{T}_{\mathbf{C}}\lambda_1$ and, again, it is homeomorphic to $X(\Sigma)$.

A priori, two adjacent facets of a toric space could have the same characteristic values, in which case the value of the characteristic function on the intersection would not be uniquely determined. In the case of toric varieties, however, the characteristic function is completely determined by its values on the facets of P (that is, the 1-dimensional sublattices spanned by the rays of Σ). The geometry of the cone complex restricts the characteristic values on the facets in such a way that a unique $(n - l)$ -dimensional unimodular sublattice of N is determined for each l -face. In fact, a topologically concise presentation of the toric variety is as a cell complex P with an

integral d -tuple (a generator for the 1-dimensional sublattice) labeling each facet.

REMARK. Another approach to toric varieties, and perhaps more historically accurate, is to consider a d -dimensional subtorus of the standard algebraic torus $(\mathbf{C}^*)^n$ acting on \mathbf{P}^{n+1} . The closure of the orbit of a point under this action is, after normalization, a toric variety. The algebraic action is given by $n + 1$ Laurent monomials in the d coordinates of the subtorus. The exponents of each monomial give a d -tuple of integers corresponding to points in the lattice of characters of the d -torus. The results of Section 2 are more in the spirit of this approach.

3.3 Nonsingularity and Cohomology of Toric Spaces

The conditions for a toric variety to be a nonsingular variety are well known. The following theorem in the algebro-geometric setting can be found in [25].

Theorem 3.3.1 *A toric variety $X(\Sigma)$ is nonsingular if and only if for every $c \in \Sigma$, c is basic.*

If P is a convex polytope dual to Σ and λ_1 is defined as above, this *basic* condition in the theorem is equivalent to saying that every vertex of P is contained in exactly d facets and that the sum of the d characteristic values on these facets is precisely M . We call such a characteristic function *basic* as well. In [5] one direction of the analogous statement for *toric spaces* is proved:

Proposition 3.3.1 *$\mathcal{T}_{\mathbf{C}}\lambda_1$ is topologically (and differentiably) nonsingular if P is simple and λ_1 is basic.*

Nonsingular toric spaces are precisely the objects studied in [5]; we restate one of the main results, the cohomology calculation, and refer the reader to this paper for a topological proof. One of the key ingredients in this proof is the geometric interpretation of the *face ring* of P as the equivariant cohomology of the toric space. (The algebro-geometric version of this theorem, the computation of the Chow ring, can be found in [4].)

We first introduce the following notation. The f -vector of a d -dimensional simple polytope P is the vector (f_0, f_1, \dots, f_d) where f_l is the number of l -faces. We will denote by $f(t)$ the degree d polynomial whose l th coefficient is f_l . The h -vector of P is the vector (h_0, h_1, \dots, h_d) where h_l is the l th coefficient of the polynomial $f(t - 1)$. There has been much research focusing on these two combinatorial invariants, the most notable being McMullen's conjectures concerning which vectors can arise as f -vectors of a simplicial polytope [19, 28] (note that in most of these papers, the definitions of f and h are dual to the ones given here).

Just as in the projective variety case mentioned previously, in the nonsingular case the characteristic function λ_1 is determined by its values on the facets. In fact, if $\sigma_1, \sigma_2, \dots, \sigma_\eta$ are the facets of P , it is enough to specify primitive elements $\alpha_i \in \lambda_1(\sigma_i)$ for $i = 1, 2, \dots, \eta$. Assume without loss of generality that the first d facets in this list all share a vertex, so that $\alpha_1, \alpha_2, \dots, \alpha_d$ form a basis for N . Let $\delta_1, \delta_2, \dots, \delta_d$ be the dual basis.

Consider the polynomial ring $A = \mathbb{Z}[X_1, X_2, \dots, X_\eta]$ where η is the number of facets of P . Let I be the ideal generated by all monomials of the form $X_{i_1} X_{i_2} \cdots X_{i_s}$, where $\sigma_{i_1} \cap \sigma_{i_2} \cap \cdots \cap \sigma_{i_s}$ is not a face of P . The ring A/I is called the *face ring* of P . Finally, let J be the ideal generated by the linear relations

$$r_j = \sum_{i=1}^{\eta} \delta_j(\alpha_i) \cdot X_i \tag{1}$$

one for each j .

Theorem 3.3.2 (*Davis and Januszkiewicz*) *The cohomology ring of a nonsingular toric space $\mathcal{T}_{\mathbb{C}}\lambda_1$ is isomorphic to $A/(I + J)$ where the generators X_i are degree 2. The betti numbers are given as $\dim H^{2l}(\mathcal{T}_{\mathbb{C}}\lambda_1) = h_l$ in even degrees and zero in odd degree.*

4 The Real Case $k = 0$

4.1 Real Toric Spaces

Again we start with a convex polytope P in \mathbf{R}^d with corresponding cell complex \mathcal{P} . Sitting inside the torus T_1^d is the discrete subgroup $T_0^d = N \otimes S^0$ where $S^0 = \{\pm 1\} \subset S^1$. Thinking of S^0 as the additive group $\mathbf{Z}/(2)$, we see that T_0^d is naturally isomorphic to $N \otimes \mathbf{Z}/(2) \cong (\mathbf{Z}/(2))^d$. The product $P \times T_0^d$ is, therefore, 2^d copies of the polytope P , and because it is a subset of $P \times T^d$, we can restrict the equivalence relation defined in Section 3.1. We call the resulting space the *real part* of $\mathcal{T}_{\mathbf{C}}\lambda_1$.

Of course, we can define such a space without any reference to the complex case. Let $G_l(N \otimes \mathbf{Z}/(2))$ denote the Grassmanian of l dimensional $\mathbf{Z}/(2)$ -subspaces of $N \otimes \mathbf{Z}/(2)$.

Definition 4.1.1 *A real characteristic function is a map*

$$\lambda_0 : \mathcal{P}_l \rightarrow G_{d-l}(N \otimes \mathbf{Z}/(2))$$

which satisfies the condition: if τ is a face of σ then $\lambda_0(\sigma)$ is a subspace of $\lambda_0(\tau)$.

If Λ_0 is a codimension l , $\mathbf{Z}/(2)$ -subspace of T_0^d , denote by $f(\Lambda_0)$ the projection $T_0^d \rightarrow T_0^d/\Lambda_0$.

Definition 4.1.2 *The real toric space associated to λ_0 is*

$$\mathcal{T}_{\mathbf{R}}\lambda_0 = P \times T_0^d / \sim$$

where $(x, s) \sim (x, t)$ if $x \in \text{relint } \sigma$ and $s \equiv t \pmod{\lambda_0(\sigma)}$. That is $(x, s) \sim (x, t)$ if $f(\lambda_0(\sigma))(s) = f(\lambda_0(\sigma))(t)$.

As in the complex case, the control data of P lifts to $\mathcal{T}_{\mathbf{R}}\lambda_0$, showing that

Proposition 4.1.1 *$\mathcal{T}_{\mathbf{R}}\lambda_0$ admits the structure of a stratified space.*

The strata obtained in this way consist of 2^l connected components, each homeomorphic to an open l -cell, over each l -face σ . These components are indexed in a natural way by the elements of $T_0^d/\lambda_0(\sigma)$. As in the complex case we let S_σ be the union of these components, so that

$$S_\sigma = \mu^{-1}(\text{relint } \sigma)$$

(where μ is the projection $\mathcal{T}_{\mathbf{R}}\lambda_0 \rightarrow P$). In fact, μ is the quotient map of the natural action of T_0^d on $\mathcal{T}_{\mathbf{R}}\lambda_0$.

The image $\Lambda_0 = \Lambda \otimes \mathbf{Z}/(2)$ in T_0^d of a unimodular sublattice $\Lambda \subset N$ of rank $d - l$ is a $\mathbf{Z}/(2)$ -subspace with codimension l . Consequently, for a given λ_1 , we can define the *real reduction* λ_0 by $\lambda_0(\sigma) = \lambda(\sigma) \otimes \mathbf{Z}/(2)$; λ_0 is obviously a real characteristic function, and the associated real toric space is the real part of $\mathcal{T}_{\mathbf{C}}\lambda_1$.

4.2 Real Toric Varieties

By a *real toric variety* we mean the real points of a complex toric variety. Because we are only concerned with the Euclidean rather than Zariski topology of such a space, we will not get into the pathologies of real algebraic geometry. Regarding this difference, there is one clarification that needs to be made. We call a real variety nonsingular if it is a topological manifold, whereas real algebraic geometers (eg, [3]) impose the more stringent condition that the differentials of the regular functions at each point span a fixed-dimensional vector space (the cotangent space at the point). Our definition of nonsingular, then, allows for the possibility of a cusp.

As an immediate consequence of this understanding of real toric varieties, we see that all real toric varieties are real toric spaces. Indeed, just take the real part of the complex toric space corresponding to the complexification of the real toric variety.

4.3 Nonsingularity

We now state and prove the following conditions for nonsingularity of $\mathcal{T}_{\mathbf{R}}\lambda_0$. As in the complex case, it is the necessity of these conditions that requires work; the sufficiency is an easy exercise and can be found in [5]. Because $\mathcal{T}_{\mathbf{R}}\lambda_0$ will be nonsingular only if the links of the strata are all homology spheres, we will need the following concise presentation of these links.

Lemma 4.3.1 *Suppose $\mathcal{T}_{\mathbf{R}}\lambda_0$ is a real toric space and σ is an l -face of P with link L_σ . The topological link $L(S)$ of the stratum $S = S_\sigma$ is the stratified space*

$$L(S) = L_\sigma \times \lambda_0(\sigma) / \sim$$

where $(x, s) \sim (x, t)$ if $x \in \text{relint}(\tau \cap L_\sigma)$ and $s \equiv t \pmod{\lambda_0(\tau)}$.

As in section 3.3, we call the real characteristic function λ_0 $\mathbf{Z}/(2)$ -basic if its values on the facets about any vertex of P form a $\mathbf{Z}/(2)$ -basis for T_0^d . In particular, P must be simple.

Theorem 4.3.1 *$\mathcal{T}_{\mathbf{R}}\lambda_0$ is nonsingular if and only if λ_0 is $\mathbf{Z}/(2)$ -basic.*

Proof. We prove the necessity. If λ_0 is not $\mathbf{Z}/(2)$ -basic, choose an l -face $\sigma \in \mathcal{P}$ whose adjacent facets have linearly dependent characteristic values and such that l is maximal. Consider the stratum $S = S_\sigma$. We will show that the link $L = L(S)$ of this stratum is not a homology $(d - l - 1)$ -sphere; hence, by Proposition 2.1.2, $\mathcal{T}_{\mathbf{R}}\lambda_0$ is singular.

Consider the presentation of L given in the lemma. Choosing l maximal implies that L_σ is simple and for a fixed vertex $v \in L_\sigma$, the surrounding facets have independent characteristic values (otherwise, the $l + 1$ face of P corresponding to v would contradict the maximality of σ). Form a CW-filtration for L as follows: let L_i be the preimage in L of all i -faces of L_σ not containing v , for $i = 0, 1, \dots, d - l - 2$. Notice that L is obtained from L_{d-l-2} by adjoining two $(d - l - 1)$ -disks so that with $\mathbf{Z}/(2)$ coefficients

$$H_{d-l-1}(L, L_{d-l-2}) = \mathbf{Z}/(2) \oplus \mathbf{Z}/(2).$$

The chain complex $C_* = H_*(L_*, L_{*-1})$ computes the $\mathbf{Z}/(2)$ homology of L , so we need only show that it is different from the homology of a $(d - l - 1)$ -sphere. The preimage in L of each facet of L_σ is a $\mathbf{Z}/(2)$ -cycle in C_{d-l-2} and these are clearly independent. Because the two $(d - l - 1)$ -cells of the complex were obtained by gluing along $d - l - 1$ of these facets of L_σ , the kernel of the differential $C_{d-l-2} \rightarrow C_{d-l-3}$ has dimension at least 1. If it is greater than 1, the $(d - l - 2)$ Betti number of L is positive, in which case L is not a homology $(d - l - 1)$ -sphere. If the kernel dimension is equal to 1, L_σ is a simplex and because λ_0 fails to be basic at σ , the facet of L_σ opposite v has a characteristic value which is dependent on those of the other facets. In terms of the chain complex, this means the boundary map $C_{d-l-1} \rightarrow C_{d-l-2}$ is zero; hence, $H_{d-l-1} = \mathbf{Z}/(2) \oplus \mathbf{Z}/(2) \neq H_{d-l-1}(S^{d-l-1})$. In this case, $\mathcal{T}_{\mathbf{R}}\lambda_0$ fails even to be normal, let alone nonsingular. ■

REMARK. The requirement that P be a convex polytope is not necessary for the above proof. We could use, instead, any simple cell decomposition of the $(d - 1)$ -sphere together with a $\mathbf{Z}/(2)$ -basic λ_0 . The existence of a Morse function in the computation of the homology in the next section, however, does need the convexity of P .

4.4 The Cohomology of Real Toric Spaces

In this section we show that the $\mathbf{Z}/(2)$ -cohomology ring of a nonsingular, real toric space has the same presentation as the complex case (section 3.3). Throughout this section $\mathbf{Z}/(2)$ coefficients will be assumed, and A will denote the polynomial ring $\mathbf{Z}/(2)[X_1, X_2, \dots, X_\eta]$ where η is the number of facets of P . Again we label the facets $\sigma_1, \sigma_2, \dots, \sigma_\eta$ and assume that the first d of these facets all share a vertex. Let I be the ideal of A generated by monomials not supported on a face of P (ie, as in section 3.3). We choose the non-zero element $\alpha_i \in \lambda_0(\sigma_i)$ for $i = 1, 2, \dots, \eta$. If $\mathcal{T}_{\mathbf{R}}\lambda_0$ is nonsingular, we know from the previous section that λ_0 is basic, hence $\alpha_1, \alpha_2, \dots, \alpha_d$ form a basis for T_0^d . Let $\delta_1, \delta_2, \dots, \delta_d$ be the $\mathbf{Z}/(2)$ -dual basis and let J be the ideal generated by the linear relations (1) of Section 3.3, now understood mod 2. The main point of the following theorem is that the argument of [4] works as well in the *real*

case assuming only topological nonsingularity.

Theorem 4.4.1 (*Mostly Jurkiewicz*) *If $\mathcal{T}_{\mathbf{R}}\lambda_0$ is nonsingular, then*

$$H^*(\mathcal{T}_{\mathbf{R}}\lambda_0; \mathbf{Z}/(2)) = A/(I + J).$$

Lemma 4.4.1 *The mod2 Poincaré polynomial is the h -vector*

$$h(t) = h_0 + h_1t + \cdots + h_d t^d.$$

Proof. By duality, it is enough to compute homology. Choose a vector u in \mathbf{R}^d such that any affine hyperplane perpendicular to u contains at most one vertex of P . Define a height function $\mu : P \rightarrow \mathbf{R}$ by the formula $\mu(x) = \langle x, u \rangle$ where \langle, \rangle denotes the standard inner product on \mathbf{R}^d . Order the vertices $v_1 < v_2 < \cdots < v_m$ of P such that $\mu(v_1) < \mu(v_2) < \cdots < \mu(v_m)$, and consider the filtration

$$P_1 \subset P_2 \subset \cdots \subset P_m = P$$

where P_s is the union of all faces σ with $\mu(\sigma) \subset (-\infty, \mu(v_s)]$. Let

$$Q_1 \subset Q_2 \subset \cdots \subset Q_m$$

denote the induced filtration in $\mathcal{T}_{\mathbf{R}}\lambda_0$. Because P is simple, P_s is obtained from P_{s-1} by adjoining exactly one face of dimension, say, ι_s and Q_s from Q_{s-1} by adjoining exactly one ι_s -cell.

Finally, observe that the spectral sequence $E_{p,q}^1 = H_{p+q}(Q_q, Q_{q-1})$ degenerates at E^2 since all of the attaching maps are two to one. In other words, the cell decomposition for $\mathcal{T}_{\mathbf{R}}\lambda_0$ is *perfect*, and a simple counting argument completes the proof. ■

REMARK. Notice that by Poincaré duality, the h -vector is symmetric: $h(t) = h(1/t)t^d$. Writing this equality in terms of the f -vector for the simplicial polytope dual to P and comparing coefficients, reveals the Dehn-Somerville equations for this dual polytope.

For $i = 1, 2, \dots, \eta$, let D_i be the preimage in $\mathcal{T}_{\mathbf{R}}\lambda_0$ of the facet σ_i . D_i is a codimension 1 cycle (a *divisor* in the case of a toric variety), and we denote by D_i^* the Poincaré dual cohomology class in $H^1(\mathcal{T}_{\mathbf{R}}\lambda_0)$. Define a graded ring homomorphism $\Phi : A \rightarrow H^*(\mathcal{T}_{\mathbf{R}}\lambda_0)$ by sending the degree-one generator X_i to the class D_i^* for $i = 1, 2, \dots, \eta$. It turns out that this map is surjective; that is, the cohomology ring

is generated in degree one. Indeed, the preimage in $\mathcal{T}_{\mathbf{R}}\lambda_0$ of any l -face of P is the transverse intersection of $d - l$ distinct D_i 's, and the fact that these l -cycles generate $H_k(\mathcal{T}_{\mathbf{R}}\lambda_0)$ follows from the explicit filtration given in the previous lemma. Transverse intersections are dual to the cup product in that the following diagram commutes:

$$\begin{array}{ccc} H^{d-p}(\mathcal{T}_{\mathbf{R}}\lambda_0) \otimes H^{d-q}(\mathcal{T}_{\mathbf{R}}\lambda_0) & \xrightarrow{\cup} & H^{2d-(p+q)}(\mathcal{T}_{\mathbf{R}}\lambda_0) \\ \downarrow & & \downarrow \\ H_p(\mathcal{T}_{\mathbf{R}}\lambda_0) \otimes H_q(\mathcal{T}_{\mathbf{R}}\lambda_0) & \xrightarrow{\cap} & H_{p+q-d}(\mathcal{T}_{\mathbf{R}}\lambda_0) \end{array}$$

where the vertical maps are Poincaré duality. Thus, the intersections

$$D_{i_1} \cap D_{i_2} \cap \cdots \cap D_{i_{d-l}}$$

of distinct codimension-one cycles correspond to cohomology classes

$$D_{i_1}^* \cup \cdots \cup D_{i_{d-l}}^*,$$

and these classes generate $H^{d-l}(\mathcal{T}_{\mathbf{R}}\lambda_0)$. But these classes are simply the images of the monomials in the polynomial ring; hence, Φ is surjective.

Lemma 4.4.2 Φ factors through $A/(I + J)$.

Proof. Glue the 2^d polytopes together along the facets $\sigma_1, \sigma_2, \dots, \sigma_d$; the result is a single cell which, near the common vertex, looks like \mathbf{R}^d near the origin. The preimages of the facets lift to coordinate hyperplanes (the divisors have *normal crossings*), and the linear relations, r_1, \dots, r_d , are simply the boundaries of the half-spaces containing the cell relint $P \times 0 / \sim$. Notice that this argument does not depend on the vertex: a different choice of vertex gives a different set of generators for J . We have shown then that $J \subset \ker \Phi$.

To see that $I \subset \ker \Phi$, just notice that if distinct $\sigma_{i_1}, \dots, \sigma_{i_s}$ have empty intersection, we can regroup any monomial

$$\Xi = X_{i_1}^{p_1} X_{i_2}^{p_2} \cdots X_{i_s}^{p_s}$$

as

$$(X_{i_1} X_{i_2} \cdots X_{i_s}) \cdot (X_{i_1}^{p_1-1} X_{i_2}^{p_2-1} \cdots X_{i_s}^{p_s-1}).$$

But $\Phi(X_{i_1} \cdots X_{i_s}) = 0$ since the corresponding cycles D_{i_1}, \dots, D_{i_s} are in general position and have empty intersection. Hence $\Phi(\Xi) = 0$ and $I \subset \ker \Phi$. ■

Proof of theorem 4.4.1. Finally, we invoke some algebra to show that the dimension of $A/(I+J)$ in degree l is precisely h_l ; hence, by Lemma 4.4.1, $A/(I+J) \rightarrow H^*(\mathcal{T}_{\mathbf{R}}\lambda_0)$ is an isomorphism.

The face ring A/I has Poincaré series (see, eg., [27])

$$\sum_{l=0}^{\infty} H(l)t^l = \sum_{l=0}^{\infty} \frac{h_l t^l}{(1-t)^d}, \quad (2)$$

where $H(l)$ is the dimension of A/I in degree l .

Furthermore, A/I is Cohen Macaulay; hence, our sequence r_1, \dots, r_d is *regular* (ie., in the ring

$$\frac{A}{I + (r_1, \dots, r_{i-1})},$$

r_i is not a zero divisor) if and only if $A/(I+J)$ is finite dimensional. But if r_1, \dots, r_d is regular, then passing from

$$\frac{A}{I + (r_1, \dots, r_{i-1})}$$

to

$$\frac{A}{I + (r_1, \dots, r_i)}$$

alters the Poincaré series by a factor of $(1-t)$. Applying this factor d times to equation (2) gives the Poincaré polynomial for $H^*(\mathcal{T}_{\mathbf{R}}\lambda_0)$. It remains to show, then, that $A/(I+J)$ is finite dimensional.

In fact, $A/(I+J)$ is generated by monomials $X_{i_1} X_{i_2} \cdots X_{i_s}$, where $\sigma_{i_1}, \dots, \sigma_{i_s}$ are distinct facets whose intersection is a face of P . To see this, let

$$\Xi = X_{i_1}^{p_1} X_{i_2}^{p_2} \cdots X_{i_s}^{p_s}$$

where $\sigma_{i_1}, \dots, \sigma_{i_s}$ are distinct and intersect in a face. In particular, $\alpha_{i_1}, \dots, \alpha_{i_s}$ are linearly independent. If we can write Ξ as a sum of monomials with strictly fewer repetitions of factors, then applying induction to the number of these repetitions would complete the proof. Toward this end, suppose without loss of generality, that $p_1 > 1$. Choose $\delta \in (N \otimes \mathbf{Z}/(2))^*$ such that

$$\delta(\alpha_{i_j}) = \begin{cases} 1 & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

Substituting

$$X_{i_1} = \sum_{i \neq i_1} \delta(\alpha_i) \cdot X_i$$

for just one of the X_i 's, produces a polynomial whose monomial summands all have fewer repetitions than Ξ . ■

5 The Quaternionic Case $k = 3$

The fundamental obstacle to defining toric spaces over the quaternions \mathbf{H} is that S^3 , the unit quaternions, is not an abelian group. In particular, it is no longer the case that any monomial $x_1^{e_1} x_2^{e_2} \cdots x_d^{e_d}$ defines an $(S^3)^d$ -equivariant projection from $(S^3)^d$ to S^3 . In order to obtain an interesting class of spaces, then, we drop this condition, maintaining instead just enough structure to produce a stratified space. As a result, quaternionic toric spaces will not have a natural T_3^d action. The group S^3 , however, does have a non-trivial group of inner automorphisms (unlike the real and complex cases), namely $SO(3)$. It turns out that the spaces we define *do* have a natural $SO(3)$ -action.

Recall that for real and complex toric spaces, the collapsing data over the faces of P is determined by the characteristic function λ_k . The range of the characteristic function simply indexes a suitable class of fibrations $(S^k)^d \rightarrow (S^k)^l$ (where l is the dimension of the face). In the quaternionic case, the range of λ_3 will be certain subgroups of the free group on d letters, and again these will index an interesting class of fibrations. These fibrations will be given, in general, by l -tuples of words in the d coordinates (quaternionic) of $(S^3)^d$. If $SO(3)$ acts on an arbitrary product of S^3 's by acting on each factor simultaneously (by inner automorphisms), then all of the fibrations $(S^k)^d \rightarrow (S^k)^l$ are $SO(3)$ -equivariant. This fact allows us to define an $SO(3)$ action on quaternionic toric spaces.

5.1 Free Groups and Notation

Let E be a finite set with d elements. We denote by $\langle E \rangle$ the free group on the set E , and for any subset $W \subset \langle E \rangle$, $\langle W \rangle_E$ denotes the subgroup of $\langle E \rangle$ generated by W . If H is *any* subgroup of $\langle E \rangle$, then it is itself a free group on some subset $W \subset \langle E \rangle$ ([26]). The *rank* of a subgroup H is defined to be the cardinality of a minimal generating set. We use the notation $\langle E \rangle // H$ or $\langle E | W \rangle$ (depending on the context) for the quotient group $\langle E \rangle / N$ where N is the smallest normal subgroup containing H .

A subset $W \subset \langle E \rangle$ of l words will be called *unimodular of rank l* if there exists another subset W' of $d - l$ elements such that $\langle W \cup W' \rangle_E = \langle E \rangle$. Likewise, a subgroup H of $\langle E \rangle$ will be called *unimodular of rank l* if there exists a unimodular, rank l generating set for H (it is clear that this is the same rank defined in the previous paragraph); in this case, we will call W a *basis* for H . Notice that W' is also unimodular (of rank $d - l$), and any such unimodular pair $\{W, W'\}$ will be called *complementary*. Because the subgroup generated by a unimodular set W is itself a free group on W we will often write $\langle W \rangle$ instead of $\langle W \rangle_E$.

An obvious, but useful fact about a unimodular, rank l subgroup $\langle W \rangle$ is that the quotient group $\langle E | W \rangle$ is free of rank $d - l$. Explicitly, if W' is any complement for

W , then the composition

$$\langle W' \rangle_E \hookrightarrow \langle E \rangle \rightarrow \langle E|W \rangle$$

is an isomorphism.

Definition 5.1.1 For any topological group Γ , define $\Gamma\langle E|W \rangle$ to be the function space

$$\text{Hom}_{gp}(\langle E|W \rangle, \Gamma).$$

$\Gamma\langle E|W \rangle$ is naturally pointed with base point $*$: $w \mapsto 1$ for all $w \in \langle E|W \rangle$.

REMARK. Although in the nonabelian case $\Gamma\langle E|W \rangle$ does not have a natural group structure, it always has the structure of an $\text{Aut } \Gamma$ -space where $\text{Aut } \Gamma$ is the group of inner automorphisms of Γ . For $g \in \Gamma$, let $\rho_g \in \text{Aut } \Gamma$ be the automorphism $h \mapsto ghg^{-1}$. If $t \in \Gamma\langle E|W \rangle$, then $w \mapsto gt(w)g^{-1}$ is a well-defined homomorphism from $\langle E|W \rangle$ to Γ , hence $t \mapsto gtg^{-1}$ defines a natural $\text{Aut } \Gamma$ action on $\Gamma\langle E|W \rangle$.

There are several straightforward consequences of the definition, which we list as follows.

- i. $\Gamma\langle E \rangle$ ($= \Gamma\langle E|\emptyset \rangle$) is homeomorphic to the pointed space $(\Gamma^d, 1)$.
- ii. If W is unimodular of rank l , then $\langle W \rangle \hookrightarrow \langle E \rangle$ induces an surjective $\text{Aut } \Gamma$ -equivariant pointed map $\Gamma\langle E \rangle \rightarrow \Gamma\langle W \rangle$ and $\Gamma\langle W \rangle$ is homeomorphic to Γ^l .
- iii. If W is unimodular of rank l , then the surjection $\langle E \rangle \rightarrow \langle E|W \rangle$ induces an injective (equivariant) pointed map $\Gamma\langle E|W \rangle \rightarrow \Gamma\langle E \rangle$. $\Gamma\langle E|W \rangle$ is homeomorphic to Γ^{d-l} .

Proposition 5.1.1 In fact, if $W \subset \langle E \rangle$ is unimodular of rank l , then $\Gamma\langle E \rangle \rightarrow \Gamma\langle W \rangle$ is a trivial fiber bundle with fiber $\Gamma\langle E|W \rangle \cong \Gamma^{d-l}$.

Proof. Let $W = \{w_1, \dots, w_l\}$. Then we can find a $W' = \{w'_1, \dots, w'_{d-l}\}$ such that W and W' are complementary. A point $t \in \Gamma\langle W \rangle$ is uniquely determined by its coordinates $t_1 = t(w_1), \dots, t_l = t(w_l), t_{l+1} = t(w'_1), \dots, t_d = t(w'_{d-l})$. This determines a homeomorphism $\Gamma\langle W \rangle \rightarrow \Gamma^d$ by sending t to the d -tuple (t_1, \dots, t_d) . Likewise, a homeomorphism $\Gamma\langle W \rangle \rightarrow \Gamma^l$ is given by $t \mapsto (t(w_1), \dots, t(w_l))$. With respect to these homeomorphisms, the map $\Gamma\langle E \rangle \rightarrow \Gamma\langle W \rangle$ corresponds to the projection $\Gamma^d \rightarrow \Gamma^l$ onto the first l factors.

It remains to show that the inclusion of the fiber over $*$ $\in \Gamma\langle W \rangle$ is induced by the map $\langle E \rangle \rightarrow \langle E|W \rangle$. But $\langle E|W \rangle$ is freely generated by the image of W' ,

say $\{\bar{w}'_1, \dots, \bar{w}'_{d-l}\}$. As before we have a homeomorphism $\langle E|W \rangle \rightarrow \Gamma^{d-l}$ given by $t \mapsto (t(\bar{w}'_1), \dots, t(\bar{w}'_{d-l}))$, and the map $\Gamma\langle E|W \rangle \rightarrow \Gamma\langle E \rangle$ corresponds to the inclusion $(t_1, \dots, t_{d-l}) \mapsto (1, 1, \dots, 1, t_1, \dots, t_{d-l})$ of the last $d-l$ factors. ■

5.2 Quaternionic Toric Spaces

In this section we will define quaternionic spaces. It is helpful to use a coordinate free definition of the *quaternionic torus* T_3^d . Fix a set $E = \{e_1, \dots, e_d\}$ and define $T_3^d = S^3\langle E \rangle$. (This definition is dual to that of T_1^d in that $T_1^d = N \otimes S^1$ is naturally isomorphic to $\text{Hom}(M, S^1)$ where $M = \text{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$. We will elaborate more on this connection in Section 5.5.) Let $G_l(\langle E \rangle)$ denote the “Grassmannian” of all unimodular, rank l subgroups of $\langle E \rangle$, and for $\langle W \rangle \in G_l(\langle E \rangle)$ let $f(\langle W \rangle)$ be the induced map $T_3^d \rightarrow S^3\langle W \rangle$.

Definition 5.2.1 A quaternionic characteristic function is a map

$$\lambda_3 : \mathcal{P}_l \rightarrow G_l(\langle E \rangle)$$

(for $l = 0, \dots, d$) satisfying the condition: if τ is a j -face of σ , $\lambda_3(\tau)$ is a unimodular, rank j subgroup of $\lambda_3(\sigma)$.

Definition 5.2.2 The quaternionic toric space associated to λ_3 is the topological quotient

$$\mathcal{T}_{\mathbf{H}}\lambda_3 = P \times T_3^d / \sim$$

where $(x, s) \sim (x, t)$ if $x \in \text{relint } \sigma$ and $f(\lambda_3(\sigma))(s) = f(\lambda_3(\sigma))(t)$.

Proposition 5.2.1 If μ is the projection $\mathcal{T}_{\mathbf{H}}\lambda_3 \rightarrow P$, then $\mathcal{T}_{\mathbf{H}}\lambda_3$ admits the structure of a stratified space with strata $\{S_\sigma | \sigma \in \mathcal{P}\}$ where $S_\sigma = \mu^{-1}(\text{relint } \sigma)$.

Proof. Lift the control data on P (Section 2.2) as follows. For $S = S_\sigma$, let $\text{Tub}(S) = \mu^{-1}(\text{tub}(\sigma))$, $\rho_S = \rho_\sigma \circ \mu$, and $\pi_S = (\pi_\sigma, f(\lambda_3(\sigma)))$. The compatibility condition in the definition of λ_3 together with the fact that f is a trivial fibration guarantee that this data makes $\mathcal{T}_{\mathbf{H}}\lambda_3$ into a stratified space. ■

As mentioned in Section 5.1, the spaces $T_3^d = S^3\langle E \rangle$ and $S^3\langle \lambda_3(\sigma) \rangle$ all have natural actions by the inner automorphism group of S^3 , and all of the maps $f(\lambda_3(\sigma))$ are equivariant with respect to this action. The inner automorphism group itself is isomorphic to the abstract group $SO(3)$ (conjugation by $g \in S^3$ fixes every element if and only if $g = \pm 1$, and $S^3/\{\pm 1\} = SO(3)$). Combining these facts gives the following:

Proposition 5.2.2 The natural action of $SO(3)$ on T_3^d extends to an action on $\mathcal{T}_{\mathbf{H}}\lambda_3$.

5.3 Generalized Quaternionic Spaces and Links

Just as the links of strata in Section 4 had nice topological presentations, so do the links of quaternionic spaces. In fact, a simple generalization of the characteristic function λ_3 produces a class of spaces which *includes* the links of all strata. (A close examination of the presentation of links in Section 4 will reveal a similar situation). The generalization is to allow the number of 3-sphere factors in the generic fiber over the polytope to be greater than the dimension of the polytope.

Definition 5.3.1 *Let P be a c -dimensional polytope (with $c < d$) and let \mathcal{P}_l be the set of l -faces of Q . A generalized characteristic function λ^g on P is a map*

$$\lambda^g : \mathcal{P}_l \rightarrow G_{l+(d-c)}(\langle E \rangle)$$

satisfying the condition: if τ is a j -face of σ , then $\lambda^g(\tau)$ is a unimodular, rank j subgroup of $\lambda^g(\sigma)$.

Definition 5.3.2 *The generalized quaternionic space $\mathcal{T}_{\mathbf{H}}\lambda^g$ is the quotient*

$$\mathcal{T}_{\mathbf{H}}\lambda^g = P \times T_3^d / \sim$$

where $(x, s) \sim (x, t)$ if $x \in \text{relint } \sigma$ and $f(\lambda^g(\sigma))(s) = f(\lambda^g(\sigma))(t)$.

The important example of a generalized quaternionic space is the link of a stratum in a quaternionic space. Let $\mathcal{T}_{\mathbf{H}}\lambda_3$ be a quaternionic space over the d -dimensional polytope P . Let σ be a codimension $c+1$ face of P . Then the link L_σ of σ (see Section 2.2) is a c -dimensional convex polytope inside P . Let $\mathcal{L}_l = \{L_\sigma \cap \tau \mid \tau \in \mathcal{P}_{l+(d-c)}\}$ be the collection of l -faces of L_σ .

Lemma 5.3.1 *The quotient group $\langle E \rangle // \lambda_3(\sigma)$ is free of rank $c+1$ and for any $L_\sigma \cap \tau \in \mathcal{L}_l$, the image of the natural inclusion $\lambda_3(\tau) // \lambda_3(\sigma) \rightarrow \langle E \rangle // \lambda_3(\sigma)$ is unimodular of rank $l+1$.*

Proof. Choose a basis W_1 for $\lambda_3(\sigma)$. Since $\lambda_3(\sigma)$ is unimodular in $\lambda_3(\tau)$, we can find a complement W_2 in $\lambda_3(\tau)$. Since $\lambda_3(\tau)$ is unimodular (with basis $W_1 \cup W_2$) in $\langle E \rangle$ we can choose a further complement W_3 in $\langle E \rangle$. The respective ranks of the unimodular sets W_1 , W_2 , and W_3 are $d-c-1$, $l+1$, and $c-l$. The composite

$$\langle W_2 \rangle \hookrightarrow \langle \lambda_3(\tau) \rangle \rightarrow \lambda_3(\tau) // \lambda_3(\sigma)$$

is an isomorphism, so $\lambda_3(\tau) // \lambda_3(\sigma)$ is free of rank $l+1$. Likewise, the composite

$$\langle W_2 \cup W_3 \rangle \hookrightarrow \langle E \rangle \rightarrow \langle E \rangle // \lambda_3(\sigma)$$

is an isomorphism, so $\langle E \rangle // \lambda_3(\sigma)$ is free of rank $c + 1$. The image of the inclusion

$$\lambda_3(\tau) // \lambda_3(\sigma) \hookrightarrow \langle E \rangle // \lambda_3(\sigma)$$

is generated by the image of W_2 , and the image of W_3 in $\langle E \rangle // \lambda_3(\sigma)$ is the desired complement. ■

We can now define a generalized characteristic function on the link L_σ

$$\lambda^g : \mathcal{L}_l \rightarrow G_{l+1}(\langle E \rangle // \lambda_3(\sigma))$$

by $\lambda^g(L_\sigma \cap \tau) = \lambda_3(\tau) // \lambda_3(\sigma)$.

Theorem 5.3.1 *The link of S_σ in $\mathcal{T}_{\mathbf{H}}\lambda_3$ is homeomorphic to the generalized quaternionic space $\mathcal{T}_{\mathbf{H}}\lambda^g$.*

Proof. Let $S = S_\sigma$ and recall that the link L_σ in P is a level set of the tubular distance function ρ_σ restricted to a fiber of the projection π_σ . The map $\pi_\sigma|_{L_\sigma} : L_\sigma \rightarrow \{p\}$ lifts to a fiber bundle

$$\mu^{-1}(L_\sigma) \rightarrow \mu^{-1}(p). \quad (1)$$

It is clear that $\mu^{-1}(L_\sigma)$ is the space

$$L_\sigma \times T_3^d / \sim$$

where the equivalence relation is the same as the one defined by λ_3 : that is, $(x, s) \sim (x, t)$ if $x \in \text{relint } \tau$ (notice $\sigma \prec \tau$) and $f(\lambda_3(\tau))(s) = f(\lambda_3(\tau))(t)$. π_S maps this space to $\mu^{-1}(p) = S^3 \langle \lambda_3(\sigma) \rangle$ and the fiber over $*$ is clearly the space $\mathcal{T}_{\mathbf{H}}\lambda^g$ defined above.

Finally, recall that the link $L(S)$ is defined as a generic fiber of the bundle $\text{Tub}_\epsilon(S) \rightarrow S$. Using basic facts about stratified spaces, one can show that this bundle is equivalent to the bundle

$$\mathbf{R}^{d-c-1} \times \mu^{-1}(L_\sigma) \rightarrow \mathbf{R}^{d-c-1} \times \mu^{-1}(p).$$

Hence the fiber is homeomorphic to the fiber of (1), completing the proof. ■

5.4 Nonsingularity

Having an explicit presentation of the links of strata makes it possible to determine (at least in part) when $\mathcal{T}_{\mathbf{H}}\lambda_3$ is topologically nonsingular. A necessary condition for nonsingularity is that the polytope P be simple. A sufficient condition is that the

characteristic function satisfy a certain set of conditions relating to automorphisms of the free group $\langle E \rangle$. As we will see later (in Section 6.1), these conditions are not the best possible, but for now, they are the easiest to state and will suffice.

If σ is a face of a convex polytope P , the *open star* of σ is the subset

$$\text{st}^\circ(\sigma) = \bigcup_{\tau \succeq \sigma} \text{relint } \tau.$$

The collection $\{\text{st}^\circ(v) | v \in \mathcal{P}_0\}$ is an open cover of P , hence lifts by μ^{-1} to an open cover of $\mathcal{T}_{\mathbf{H}}\lambda_3$. When P is a simple polytope, $\text{st}^\circ(v)$ is homeomorphic (and combinatorially equivalent) to $(\mathbf{R}_{\geq 0})^d$. This motivates the following local construction.

The Standard Local Model

Let $B = (\mathbf{R}_{\geq 0})^d$ with coordinate axes labeled $1, 2, \dots, d$. Let \mathcal{B} be the associated complex of faces, indexed by subsets of $\{1, 2, \dots, d\}$. If $\langle E \rangle$ is the free group on E and $W = \{w_1, \dots, w_d\}$ is an ordered basis for $\langle E \rangle$, for each $I \subset \{1, 2, \dots, d\}$, let W_I be the unimodular subset $\{w_i | i \in I\}$, and define

$$\Lambda_W : \mathcal{B}_I \rightarrow G_I(\langle E \rangle)$$

by $\Lambda(I) = \langle W_I \rangle$. It is clear that the resulting space

$$\mathcal{T}_{\mathbf{H}}\Lambda_W = B \times T_3^d / \sim$$

is homeomorphic to \mathbf{H}^d .

Proposition 5.4.1 *Let $\mathcal{T}_{\mathbf{H}}\lambda_3$ be defined over the simple polytope P . For each vertex v , index the edges containing v with the numbers $1, \dots, d$, so that the remaining faces containing v are indexed by subsets of $\{1, 2, \dots, d\}$. If for each vertex of P , there is some basis W such that $\lambda_3(\sigma_I) = \Lambda_W(I)$ for all subsets $I \subset \{1, 2, \dots, d\}$, then $\mathcal{T}_{\mathbf{H}}\lambda_3$ is topologically nonsingular.*

Proof. $\mathcal{T}_{\mathbf{H}}\lambda_3$ is covered by open sets of the form $\mu^{-1}(\text{st}^\circ(v))$, and each of these is homeomorphic to $\mathcal{T}_{\mathbf{H}}(\Lambda_W)$. ■

Proposition 5.4.2 *If P is not simple, $\mathcal{T}_{\mathbf{H}}\lambda_3$ is singular.*

Proof. If P is not simple we can find an l -face σ which is the intersection of $N > d - l$ facets. Then L_σ is a $c = d - l - 1$ dimensional polytope having N codimension 1

faces. By Proposition 2.1.2, it will be enough to show that the link of S_σ is not a homology $(4c + 3)$ -sphere.

Let λ^g be as in Section 5.3 so that the link of S_σ is homeomorphic to the generalized quaternionic space $\mathcal{T}_{\mathbf{H}}\lambda^g$. Let L_i be the union of all i -faces of L_σ ; that is,

$$L_0 \subset L_1 \subset \cdots \subset L_c = L_\sigma$$

is the usual skeletal filtration of the polytope L_σ . Setting $X_i = \mu^{-1}(L_i)$ gives the induced filtration

$$X_0 \subset X_1 \subset \cdots \subset X_c = \mathcal{T}_{\mathbf{H}}\lambda^g$$

whose associated rational homology spectral sequence $E_{i,j}^1 = H_{i+j}(X_i, X_{i-1}; \mathbf{Q})$ abuts to $H_*(\mathcal{T}_{\mathbf{H}}\lambda^g)$.

If \mathcal{L}_i is the collection of all i -faces of L_σ , then the pair (X_i, X_{i-1}) is homeomorphic to

$$\bigvee_{\tau \in \mathcal{L}_i} (D^i \times (S^3)^{i+1}, S^{i-1} \times (S^3)^{i+1})$$

where D^i denotes the i -disk; hence,

$$H_*(X_i, X_{i-1}) \cong \bigoplus_{\tau \in \mathcal{L}_i} \overline{H}_*(S^i) \otimes H_*((S^3)^{i+1}) \cong \bigoplus_{\tau \in \mathcal{L}_i} H_{*+i}((S^3)^{i+1}).$$

If f_i is the number of i -faces of L_σ , then the dimension of the spectral sequence entry $E_{i,3j} = H_{i+3j}(X_i, X_{i-1})$ is

$$f_i \cdot \binom{i+1}{j}, \quad (2)$$

and zero for other entries. Consider the block of the spectral sequence

$$\begin{array}{cccc}
 & 0 & 0 & E_{c,3(c+1)} & 0 \\
 i+j=4c-1 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 \\
 & 0 & E_{c-1,3c} & \xleftarrow{d} E_{c,3c} & 0 \\
 & 0 & 0 & 0 & 0
 \end{array}$$

Because d is the only nonzero differential hitting $E_{c-1,3c} = H_{4c-1}(X_{c-1}, X_{c-2})$, the dimension of the cokernel of d is the same as the dimension of $H_{4c-1}(\mathcal{T}_{\mathbf{H}}\lambda^g)$. But

according to (2), $H_{4c}(X_c, X_{c-1})$ has dimension $c + 1$ while $H_{4c-1}(X_{c-1}, X_{c-2})$ has dimension $f_{c-1} = N$. Since $N > c + 1$, the cokernel has positive dimension, and $\mathcal{T}_{\mathbf{H}}\lambda^g$ is, therefore, not a homology $(4c + 3)$ -sphere. ■

5.5 Cohomology of Nonsingular Quaternionic Spaces

Throughout this section we will assume that $\mathcal{T}_{\mathbf{H}}\lambda_3$ satisfied the conditions of Proposition 5.4.1. Namely, P will be a simple polytope and the characteristic function will be locally equivalent to a standard model. We will show that under these conditions, the cohomology ring of $\mathcal{T}_{\mathbf{H}}\lambda_3$ is exactly what one would expect given the real and complex analogues.

Let M be the abelianization of the free group $\langle E \rangle$, and for any word $w \in \langle E \rangle$, let w^{ab} be the image of w under the natural map $\langle E \rangle \rightarrow M$. Then M , isomorphic to \mathbf{Z}^d , is naturally generated by the images $e_1^{ab}, e_2^{ab}, \dots, e_d^{ab}$. The image of any unimodular subgroup $\langle W \rangle \subset \langle E \rangle$ is a unimodular sublattice of M . Our choice of the letter M is not coincidental. Recall that the characteristic function in the quaternionic case is *dual* to that of the real and complex case. If N is the dual \mathbf{Z} -module $\text{Hom}(M, \mathbf{Z})$, the following table lists the possible presentations for the “tori” as well as the range of the characteristic function.

	Real	Complex	Quaternionic
$k =$	0	1	3
$T_k^d =$	$N \otimes \mathbf{Z}/(2)$	$N \otimes S^1$	
	$\text{Hom}_{\mathbf{Z}}(M, \mathbf{Z}/(2))$	$\text{Hom}_{\mathbf{Z}}(M, S^1)$	$\text{Hom}_{\text{gp}}(\langle E \rangle, S^3)$
$\lambda_k(\sigma) \subset$	$G_{d-l}(N \otimes \mathbf{Z}/(2))$	$G_{d-l}(N)$	$G_l(\langle E \rangle)$

As in the real and complex cases, we let $\sigma_1, \sigma_2, \dots, \sigma_\eta$ be the facets of P and assume the first d of them share a vertex. Then there is some basis W for $\langle E \rangle$ such that for $j = 1, 2, \dots, d$, $\lambda_3(\sigma_j) = \Lambda_W(I)$ where $I = \{1, 2, \dots, \hat{j}, \dots, d\}$. By replacing $\langle E \rangle$ with an appropriate automorphic image, we might as well assume that this basis W is the standard basis $\{e_1, e_2, \dots, e_d\}$. For each facet σ_i , the image $\lambda_3(\sigma_i)^{ab}$ is a rank $d - 1$ unimodular sublattice of M , whose complement (ie, nullspace) in

$N = \text{Hom}(M, \mathbf{Z})$ is a unimodular rank 1 sublattice. Let $\alpha_i \in N$ be a primitive element of the complementary sublattice for $\lambda_3(\sigma_i)^{ab}$ for $i = 1, 2, \dots, \eta$. According to the assumptions of Proposition 5.4.1, $\alpha_1, \dots, \alpha_d$ is a basis for N . In fact, by design, it is dual to the basis $\{e_1^{ab}, e_2^{ab}, \dots, e_d^{ab}\}$ for M .

Let A be the polynomial ring $\mathbf{Z}[X_1, X_2, \dots, X_\eta]$. Let I be the ideal generated by monomials of the form $X_{i_1} X_{i_2} \cdots X_{i_l}$ where $\sigma_{i_1} \cap \sigma_{i_2} \cap \cdots \cap \sigma_{i_l}$ is not a face of P , so the ring A/I is the face ring of P . Let J be the ideal generated by the linear relations

$$r_j = \sum_{i=1}^{\eta} \alpha_i(e_j^{ab}) \cdot X_i \quad (3)$$

for $j = 1, 2, \dots, d$. In this section we will prove:

Theorem 5.5.1 *The integral cohomology ring of a quaternionic space $\mathcal{T}_{\mathbf{H}}\lambda_3$ satisfying the conditions of Proposition 5.4.1 is isomorphic to $A/(I + J)$ where the generators X_i are in degree 4.*

As in the real and complex cases, we first determine the Betti numbers of $\mathcal{T}_{\mathbf{H}}\lambda_3$, and then compare these to the dimensions in each degree of a certain quotient of the face ring of the polytope P .

Lemma 5.5.1 *The Poincaré polynomial of $\mathcal{T}_{\mathbf{H}}\lambda_3$ is the h -vector*

$$h(t^4) = h_0 + h_1 t^4 + h_2 t^8 + \dots + h_d t^{4d}$$

Proof. The proof is the same as the real and complex case. We construct a relative cell decomposition such that the dimension of every cell is a multiple of 4, and then we count the number of cells using the combinatorics of the polytope. Choose a vector u in \mathbf{R}^d such that any affine hyperplane perpendicular to u contains at most one vertex of P , and define a height function $\mathfrak{h} : P \rightarrow \mathbf{R}$ by the formula $\mathfrak{h}(x) = \langle x, u \rangle$ (the standard inner product on \mathbf{R}^d). Order the vertices $v_1 < v_2 < \cdots < v_m$ of P such that $\mathfrak{h}(v_1) < \mathfrak{h}(v_2) < \cdots < \mathfrak{h}(v_m)$, and consider the filtration

$$P_1 \subset P_2 \subset \cdots \subset P_m = P$$

where P_i is the union of all faces σ with $\mathfrak{h}(\sigma) \subset (-\infty, \mathfrak{h}(v_k)]$. Let

$$Q_1 \subset Q_2 \subset \cdots \subset Q_m = \mathcal{T}_{\mathbf{H}}\lambda_3$$

be the induced filtration of $\mathcal{T}_{\mathbf{H}}\lambda_3$. Because P is simple, P_i is obtained from P_{i-1} by adjoining exactly one face of dimension, say, l . The local conditions on the characteristic function λ_3 guarantee that Q_i is obtained from Q_{i-1} by attaching exactly one cell of dimension $4l$.

Since nonzero entries of the associated spectral sequence $E_{p,q}^1 = H_{p+q}(Q_q, Q_{q-1})$ occur only in every fourth diagonal, the spectral sequence degenerates at E^2 . The cell decomposition is, therefore, perfect and a simple counting argument completes the proof. ■

We now give a class of geometric chains (certain *oriented* subsets) for $\mathcal{T}_{\mathbf{H}}\lambda_3$ which generate the homology and whose intersections will determine, via Poincaré duality, the ring structure of $H^*(\mathcal{T}_{\mathbf{H}}\lambda_3)$. There are various ways to make the concept of “geometric chain” rigorous. For example, one could find a triangulation of $\mathcal{T}_{\mathbf{H}}\lambda_3$ for which these geometric subsets are genuine simplicial chains (for an exposition of the subject with further references, see [17, Appendix 2]).

Let $\mu : \mathcal{T}_{\mathbf{H}}\lambda_3 \rightarrow P$ be the natural projection, and for each l -face σ of P , consider the subset $\mu^{-1}(\sigma) \subset \mathcal{T}_{\mathbf{H}}\lambda_3$ (so $\mu^{-1}(\sigma)$ is the closure of the stratum S_σ from Proposition 5.2.1). Because $\mu^{-1}(\sigma) = \sigma \times T_3^d / \sim$ has relative interior homeomorphic to $\text{relint } \sigma \times S^3\langle\lambda_3(\sigma)\rangle$, an orientation for $\mu^{-1}(\sigma)$ is determined by an orientation for σ together with an orientation for $S^3\langle\lambda_3(\sigma)\rangle \cong (S^3)^l$. We fix, once and for all, an orientation for each face σ and fiber $S^3\langle\lambda_3(\sigma)\rangle$ and denote by D_σ the *oriented* geometric chain $\mu^{-1}(\sigma)$. In the case of *facets* we choose orientations more carefully, to simplify later calculations. Having fixed an orientation for P , the geometric boundary is given by

$$\partial P = \sum_{i=1}^{\eta} \pm \sigma_i$$

where $\sigma_1, \dots, \sigma_\eta$ are the facets of P . We choose the orientations for σ_i which make all the signs positive.

In fact, the D_σ are all geometric *cycles*, since the boundaries are always supported on a union of chains with dimension 4 less than that of D_σ . And because the collection $\{D_\sigma \mid \sigma \in \mathcal{P}\}$ refines the perfect cell decomposition of the previous lemma (ie, each cell of the CW decomposition is a union of interiors of D_σ 's), these cycles generate the homology of $\mathcal{T}_{\mathbf{H}}\lambda_3$. However, there are relations among this set of generators which arise from boundaries of certain chains which we have not yet included. We describe these chains next.

For $j = 1, 2, \dots, d$, let E_j be the singleton set $\{e_j\}$ which generates the unimodular rank 1 subgroup $\langle E_j \rangle$. Then the quotient map $E \rightarrow \langle E | E_j \rangle$ induces an inclusion

$$S^3\langle E | E_j \rangle \hookrightarrow S^3\langle E \rangle = T_3^d.$$

Since T_3^d is the d -fold product of 3-spheres, this map is simply the inclusion of all but the j th factor. Fix an orientation for the standard product $(S^3)^{d-1}$ and choose an orientation for $S^3\langle E | E_j \rangle$ so that the homeomorphism $S^3\langle E | E_j \rangle \rightarrow (S^3)^{d-1}$ given by

$$t \mapsto (t(e_1), \dots, \widehat{t(e_j)}, \dots, t(e_d))$$

is orientation preserving/reversing depending on whether j is even/odd. Having already fixed an orientation for P , this determines an orientation for the subset

$$P \times S^3\langle E|E_j \rangle / \sim$$

of $\mathcal{T}_{\mathbf{H}}\lambda_3$. Let R_j be the geometric chain given by this subset and its fixed orientation.

Lemma 5.5.2 *Let D_1, D_2, \dots, D_η be the geometric cycles corresponding to the facets $\sigma_1, \sigma_2, \dots, \sigma_\eta$. Then for $j = 1, \dots, d$,*

$$\partial R_j = \sum_{i=1}^{\eta} \alpha_i(e_j^{ab}) \cdot D_i$$

where $\alpha_i \in \text{Hom}(M, \mathbf{Z})$ is one of the two primitive elements in $[\lambda_3(\sigma_i)^{ab}]^\perp$.

REMARK. The correct choice of α_i obviously depends on the (arbitrarily) fixed orientation for D_i .

Proof. The support of ∂R_j , $\text{supp } \partial R_j$, is contained in

$$\bigcup_{i=1}^{\eta} \text{supp } D_i.$$

Hence if $\langle \partial R_j : D_i \rangle$ denotes the incidence of D_i in ∂R_j , we need only show that

$$e_j^{ab} \mapsto \langle \partial R_j : D_i \rangle \quad (j = 1, 2, \dots, d)$$

defines a primitive element of $[\lambda_3(\sigma_i)^{ab}]^\perp$.

Having fixed all of the orientations, it is not hard to see that $\langle \partial R_j : D_i \rangle$ is nothing more than the degree of the map

$$S^3\langle E|E_j \rangle \rightarrow S^3\langle \lambda_3(\sigma_i) \rangle$$

obtained by restricting the natural projection $S^3\langle E \rangle \rightarrow S^3\langle \lambda_3(\sigma_i) \rangle$ to $S^3\langle E|E_j \rangle$. To understand this map, we fix a unimodular set of words w_1, w_2, \dots, w_{d-1} which generate $\lambda_3(\sigma_i)$ and, hence, a homeomorphism $S^3\langle \lambda_3(\sigma_i) \rangle \xrightarrow{\cong} (S^3)^{d-1}$ given by $t \mapsto (t(w_1), t(w_2), \dots, t(w_{d-1}))$. We can assume further that this homeomorphism is orientation preserving (otherwise replace w_1 with w_1^{-1}). Using our fixed orientation for $S^3\langle E|E_j \rangle$ as well as the fixed homeomorphism from $S^3\langle E|E_j \rangle$ to $(S^3)^{d-1}$ (orientation preserving/reversing depending on whether j is even/odd), it follows that $\langle \partial R_j : D_i \rangle$ is $(-1)^j$ times the degree of the self-map $\pi_j : (S^3)^{d-1} \rightarrow (S^3)^{d-1}$ given by:

$$(t_1, \dots, t_{d-1}) \mapsto (w_1(t_1, \dots, t_{j-1}, 1, t_j, \dots, t_{d-1}), \dots, w_{d-1}(t_1, \dots, t_{j-1}, 1, t_j, \dots, t_{d-1}))$$

where $w(a_1, a_2, \dots, a_d)$ is the element of S^3 obtained by replacing the letters e_i in the word w with the coordinates a_i in S^3 and evaluating. Determining the degree of π_j , then, reduces to the following linear algebra computation.

By assumption, the elements $w_1^{ab}, w_2^{ab}, \dots, w_{d-1}^{ab}$ span the unimodular sublattice $\lambda_3(\sigma_i)^{ab}$ of M . Relative to the natural basis for M , we can then form the $(d-1) \times d$ matrix $[W_i]$ whose u th row is w_u^{ab} . Unimodularity implies that we can complete this matrix to a $d \times d$ matrix $[\widetilde{W}_i]$ with determinant ± 1 . The last column of the inverse matrix can be naturally identified with an element $W_i^\perp \in N = \text{Hom}(M, \mathbf{Z})$ which is clearly a generator of $[\lambda_3(\sigma_i)^{ab}]^\perp$. Using the cofactor algorithm for inverting matrices, we see that $W_i^\perp(e_j^{ab})$ is $\pm(-1)^j$ times the determinant of the $(d-1) \times (d-1)$ matrix obtained by deleting the j th column from $[W_i]$ (the \pm depending on the sign of $\det[\widetilde{W}_i]$).

Now let θ_u be the homology class in $H_3((S^3)^{d-1}; \mathbf{Z})$ corresponding to the u th factor of $(S^3)^{d-1}$, and let $[\pi_j]$ be the $(d-1) \times (d-1)$ matrix for the map $(\pi_j)_*$ in degree 3 where we identify $\{\theta_u\}$ with the u th elementary column vector. Using, for example, the dual map in cohomology (the transpose of $[\pi_j]$) and the cup product, one can show that the degree of π_j is the determinant $\det[\pi_j]$. Hence $\langle \partial R_j : D_i \rangle$ is $(-1)^j$ times the determinant of $[\pi_j]$. The proof is completed, finally, by observing that the matrix $[\pi_j]$ is precisely the cofactor matrix given in the preceding paragraph (obtained by deleting the j th column from $[W_i]$). Thus the linear map defined by

$$e_j \mapsto \langle \partial R_j : D_i \rangle \quad (j = 1, \dots, d)$$

is, up to sign, precisely the primitive element $W_i^\perp \in N$, and we take this to be our covector α_i . ■

Proof of Theorem 5.5.1. We will compute the ring structure of $H^*(\mathcal{T}_{\mathbf{H}}\lambda_3)$ using Poincaré duality and transverse intersections.

For each facet σ_i , let D_i be the corresponding geometric cycle. Let D_i^* be the Poincaré dual class in $H^*(\mathcal{T}_{\mathbf{H}}\lambda_3)$, and define a graded ring homomorphism

$$\Phi : A \rightarrow H^*(\mathcal{T}_{\mathbf{H}}\lambda_3)$$

by sending the degree 4 generator X_i to the cohomology class D_i^* .

Any l -face σ can be written as the intersection of exactly $d-l$ facets of P , say

$$\sigma = \sigma_{i_1} \cap \sigma_{i_2} \cap \dots \cap \sigma_{i_{d-l}},$$

and the corresponding cycle D_σ is the transverse intersection

$$D_{i_1} \cap D_{i_2} \cap \dots \cap D_{i_{d-l}}.$$

As we have already noted, these cycles generate the homology. By Poincaré duality, it follows that the cohomology classes

$$D_{i_1}^* \cup D_{i_2}^* \cup \cdots \cup D_{i_{d-l}}^*$$

generate $H^*(\mathcal{T}_H \lambda_3)$. But these classes are simply images of monomials in the polynomial ring A , hence Φ is surjective.

The ring $A/(I + J)$ is well understood; we refer the reader to any of a number of sources (eg, [4]) for the proof that the degree $4i$ summand is free of rank h_i (an outline of this argument in the $\mathbf{Z}/(2)$ case was given in the proof of Theorem 4.4.1). With this in mind, it will be enough to show that the ideal $I + J$ is contained in the kernel of the map Φ , since we would then have a surjective map

$$A/(I + J) \rightarrow H^*(\mathcal{T}_H \lambda_3)$$

which preserves rank in each degree (recall Lemma 5.5.1). Such a map must be an isomorphism.

To this end, let Ξ be any monomial generator of I . Then there are disjoint, distinct facets $\sigma_{i_1}, \sigma_{i_2}, \dots, \sigma_{i_l}$ such that Ξ is of the form

$$\Xi = X_{i_1}^{p_1} X_{i_2}^{p_2} \cdots X_{i_l}^{p_l}$$

where $p_i > 0$ for all i . Because the corresponding cycles D_{i_1}, \dots, D_{i_l} have empty intersection, the image $\Phi(X_{i_1} X_{i_2} \cdots X_{i_l})$ must be zero. But this degree l monomial $X_{i_1} X_{i_2} \cdots X_{i_l}$ divides Ξ ; hence, $\Phi(\Xi) = 0$ and $I \subset \ker \Phi$.

It remains to show that $J \subset \ker \Phi$. This is just Lemma 5.5.2. J is generated by r_1, \dots, r_d given in (3), and $\Phi(r_j) = \sum_i \alpha_i (e_j^{ab}) D_i^*$. By the lemma, the Poincaré dual class of this image is zero, hence $\Phi(r_j) = 0$. This completes the proof. ■

6 2 Dimensional Quaternionic Spaces

In this section we will focus on the case $d = 2$, so the resulting spaces $\mathcal{T}_{\mathbb{H}}\lambda_3$ will have 8 *real* dimensions. We first show that $\mathcal{T}_{\mathbb{H}}\lambda_3$ can be topologically nonsingular, but admit no differential structure. In [21], Milnor describes certain exotic 7-spheres as principal 3-sphere bundles over the 4-sphere. Our examples of nonsmoothable manifolds are precisely the Thom spaces of these bundles.

For the remainder of the section, we will concentrate on smooth (oriented) examples, giving explicit transition functions for the tangent bundle. As a result of our explicit cell decomposition into $4l$ -dimensional cells, it follows that all quaternionic spaces are 3-connected, hence our smooth examples are 3-connected 8-manifolds. Two invariants which characterize such a manifold are the intersection form and the first Pontrjagin class. In Section 6.3 we give formulas for these as well as other characteristic numbers in terms of a certain sequence of pairs of integers arising from the characteristic function λ .

6.1 Milnor's Thom Spaces

Let P be a triangle in \mathbb{R}^2 with vertices m_0, m_1, m_2 and edges $\sigma_0, \sigma_1, \sigma_2$ (ordered as in Figure 6.1.1). A characteristic function on P is completely determined by its values on the 3 edges, which we give as

$$\lambda_3(\sigma_0) = \langle e_1 \rangle \quad \lambda_3(\sigma_1) = \langle e_2 \rangle \quad \lambda_3(\sigma_2) = \langle e_2^a e_1 e_2^b \rangle.$$

Let $M_{a,b}$ be the quaternionic space $\mathcal{T}_{\mathbb{H}}\lambda_3$.

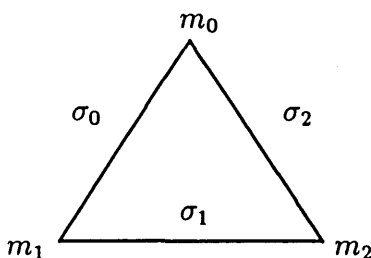


FIGURE 6.1.1

Lemma 6.1.1 $M_{a,b}$ is a topological manifold if and only if $a + b = \pm 1$.

Proof. By Proposition 5.4.1, $M_{a,b}$ is nonsingular, except possibly at the point $p = \mu^{-1}(m_0)$, and by Theorem 5.3.1, the link of this point $L(p)$ is homeomorphic to

the space

$$I \times S^3 \times S^3 / \sim$$

where the identifications at the ends of the interval I are given by

$$(0, t_1, t_2) \sim (0, s_1, s_2) \text{ where } t_2 = s_2$$

and

$$(1, t_1, t_2) \sim (1, s_1, s_2) \text{ where } t_1^a t_2^b = s_1^a s_2^b.$$

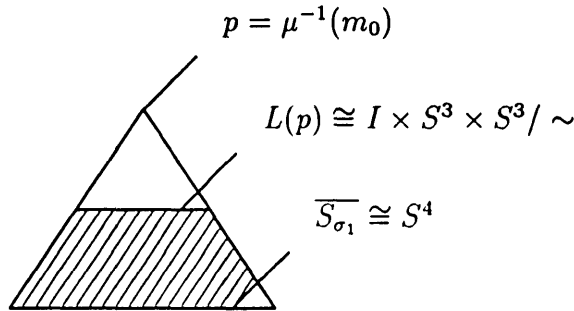


FIGURE 6.1.2

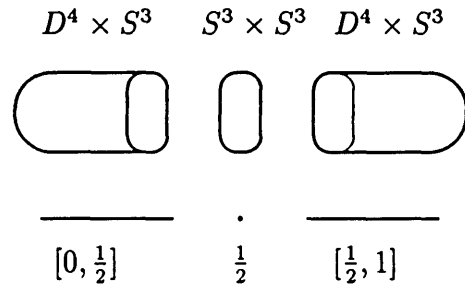


FIGURE 6.1.3

The subsets $[0, \frac{1}{2}] \times S^3 \times S^3 / \sim$ and $[\frac{1}{2}, 1] \times S^3 \times S^3 / \sim$ are mapping cylinders of the two fibrations

$$S^3 \times S^3 \rightrightarrows S^3$$

given by

$$(t_1, t_2) \mapsto t_2 \text{ and } (t_1, t_2) \mapsto t_1^a t_2^b.$$

Each of these mapping cylinders is homeomorphic to $D^4 \times S^3$. If (t_1, t_2) and (s_1, s_2) are the respective coordinates of these two pieces, then on the intersection $S^3 \times S^3$, a point with coordinates (t_1, t_2) in the first piece has coordinates $(s_1, s_2) = (t_1^{-1}, t_1^a t_2^b)$ in the second piece. The link of the point p is, therefore, homeomorphic to two copies of $D^4 \times S^3$ glued along the above diffeomorphism of their boundaries (see Figure 6.1.3). Such a link is certainly simply connected, and by a Mayer-Vietoris argument, is a homology 7-sphere if and only if $a + b = \pm 1$. By the Poincaré conjecture in dimension 7, it follows that the link is homeomorphic to the 7-sphere, and (by Proposition 2.1.1) that $M_{a,b}$ is topologically nonsingular. ■

Because $\{e_2, e_1^a e_2 e_1^b\}$ does not always generate the free group, we can see that the sufficient conditions of Proposition 5.4.1 to guarantee nonsingularity are not necessary.

We are now interested in putting a smooth structure on $M_{a,b}$. If we remove the point p , the resulting space is homeomorphic to the open manifold obtained by gluing

together 2 copies of \mathbf{H}^2 along the open subset $\mathbf{H}^* \times \mathbf{H}$ where the identification is given by the map

$$(t_1, t_2) \mapsto (t_1^{-1}, t_1^a t_2 t_1^b).$$

Notice that the second component of this map is linear in t_2 while the first is simply the transition function for the 2 contractible charts on the 4-sphere. In other words, this open manifold is a 4-dimensional real vector bundle over the 4-sphere, and the clutching function $S^3 \rightarrow SO(4, \mathbf{R})$ is given by

$$t_1 \mapsto (t_2 \mapsto t_1^a t_2 t_1^b).$$

The associated sphere bundle is the link of the missing point p , hence $M_{a,b}$ is homeomorphic to the Thom space of this bundle. The preimage of the shaded region in Figure 6.1.2 is the disk bundle over the 4-sphere $\overline{S_{\sigma_1}}$. If $a + b = \pm 1$, then the link is a manifold homeomorphic to S^7 and represents an element $[L(p)]$ of the group of exotic structures (known to be cyclic of order 28). $[L(p)]$ is the obstruction to extending the differential structure over the point p .

Proposition 6.1.1 *If $a + b = \pm 1$ then $M_{a,b}$ is smoothable if and only if 28 divides the integer*

$$\hat{a} = \pm \frac{(a - b)^2 - 1}{8}.$$

The invariant \hat{a} is essentially the μ -invariant given by Eells and Kuiper in [7] ($\hat{a} = 28\mu$).

Proof. It is enough to show that the smooth structure on the link $L(p)$ is the standard smooth structure on the 7-sphere. A complete invariant for the exotic 7-spheres is obtained by considering a smooth, spin manifold X which is bounded by $L(p)$. Let $p_1^2[X]$ and $\tau(X)$ denote the first Pontrjagin number of X (to be well-defined we need to assume $H^4(X, L(p)) \rightarrow H^4(X)$ is an isomorphism) and the signature of X , respectively. Then the number

$$\hat{A}(X) = \frac{1}{896}(p_1^2[X] - 4\tau)$$

is an invariant mod \mathbf{Z} of the homotopy sphere $L(p)$. This follows from the additivity of the Pontrjagin class and signature under connected sum, as well as the fact that the \hat{A} -genus is an integer for closed spin manifolds (Index Theorem). It can be shown that the number $p_1^2[X] - 4\tau$ is always a multiple of 32, hence

$$\hat{a} = 28\hat{A} = \frac{1}{32}(p_1^2[X] - 4\tau)$$

is an integer invariant mod 28 of $[L(p)]$.

To complete the proof, we just take X to be the 4-disk bundle mentioned above. According to [21], the signature is ∓ 1 and the Pontrjagin number is $\pm 4(a - b)^2$. ■

A few comments will be helpful in the subsequent sections. The homology of $M_{a,b}$ is generated in degree 4 by the *embedded* 4-sphere $\overline{S_{\sigma_1}} = \mu^{-1}(\sigma_1)$, and the normal bundle of this embedding is precisely the 4-dimensional real vector bundle described above. It is not hard to show (see [21]) that the first Pontrjagin class of this bundle is $\pm 2(a - b)$ times a generator of $H^4(S^4)$. The presentation of the cohomology ring for $M_{a,b}$ ($a + b = \pm 1$) goes through as in Section 5.5. As a consequence of this ring structure, it is easy to see that the Euler class of the normal bundle of this embedded 4-sphere is $(a + b) = \pm 1$ times a generator of $H^4(S^4)$ (the Euler class corresponds to the self intersection of the embedded 4-sphere).

REMARK. As a special case, notice that the manifold $M_{1,0}$ is \mathbf{HP}^2 . The other manifolds $M_{a,b}$ are, in some sense, “twisted” quaternionic projective planes.

6.2 Smooth Examples

In order to obtain a class of smooth examples we will exploit the rigidity of the cone complex (cf. Sections 1.1, 1.2, and 3.5) and consider “liftings” of 2-dimensional complex toric *varieties*.

Let M be the abelianization of the free group $\langle E \rangle = \langle \{e_1, e_2\} \rangle$, and let N be the dual \mathbf{Z} -module $\text{Hom}(M, \mathbf{Z})$. As in Section 1, we let $M_{\mathbf{R}}$ and $N_{\mathbf{R}}$ be the vector spaces obtained by tensoring with \mathbf{R} , and denote by \langle, \rangle the natural pairing $M_{\mathbf{R}} \times N_{\mathbf{R}} \rightarrow \mathbf{R}$. We think of e_1^{ab}, e_2^{ab} as the standard basis for M , with dual basis ϵ_1, ϵ_2 for N .

Let Σ be a complete rational cone complex in $N_{\mathbf{R}}$ with 1 dimensional cones

$$c_0, c_1, \dots, c_{\eta-1},$$

numbered counterclockwise. Let n_i ($i \in \mathbf{Z}/(\eta)$) be primitive vectors in the direction of the ray c_i such that the polytope

$$P = \bigcap_i \{x \in M_{\mathbf{R}} \mid \langle x, r_i \rangle \leq 1\}$$

is dual to Σ . (The existence of such elements $n_i \in N_{\mathbf{R}}$ is the condition for a complex projective embedding; in 2 dimensions this condition is always met.) Let m_i be the vertex of P dual to the 2 dimensional cone

$$\mathbf{R}_{\geq 0}r_{i-1} + \mathbf{R}_{\geq 0}r_i$$

and let σ_i be the *oriented edge* $[m_i, m_{i+1}]$ dual to c_i (see Figures 6.2.1 and 6.2.2).

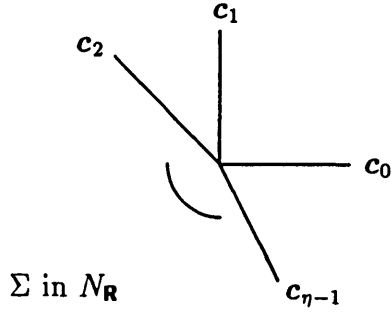


FIGURE 6.2.1

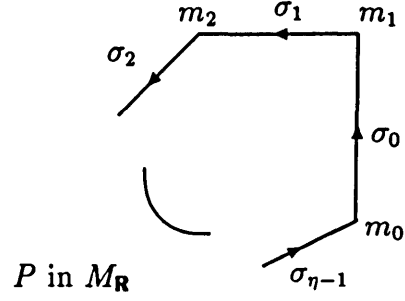


FIGURE 6.2.2

Recall from Sections 1 and 3.2 that Σ determines a complex characteristic function λ_1 by mapping σ_i to the rank 1 sublattice $\mathbf{R} \cdot \mathbf{c}_i \cap N$.

Definition 6.2.1 A quaternionic characteristic function $\lambda_3 : \mathcal{P}_l \rightarrow G_l(\langle E \rangle)$ is a lift of λ_1 if for all $i \in \mathbf{Z}/(\eta)$ and all $w \in \lambda_3(\sigma_i)$, $r_i(w^{ab}) = 0$. The associated space $\mathcal{T}_{\mathbf{H}}\lambda_3$ will be called a lift of the toric variety $X(\Sigma)$.

For the remainder of this section we will consider only those quaternionic spaces which are lifts of 2-dimensional toric varieties, and which satisfy the nonsingularity conditions of Proposition 5.4.1. In this case, for each edge σ_i of P , $\lambda_3(\sigma_i)$ has a unique generator w_i such that w_i^{ab} lies on the ray $\mathbf{R}_{\geq 0}(m_{i+1} - m_i)$; the other generator $(w_i^{-1})^{ab}$ will lie on the opposite ray $\mathbf{R}_{\geq 0}(m_i - m_{i+1})$. The conditions of Proposition 5.4.1 guarantee that $\{w_i, w_{i+1}\}$ is a basis for $\langle E \rangle$. We can assume (by some automorphism of the free group) that $w_0 = e_2^{-1}$ and $w_1 = e_1$.

Lemma 6.2.1 Let $\mathcal{T}_{\mathbf{H}}\lambda_3$ be a lift of a 2-dimensional complex toric variety, and let $\{w_i\}$ be chosen as above. Then for each i , there are unique integers a_i and b_i such that

$$w_{i+1} = w_i^{a_i} w_{i-1}^{-1} w_i^{b_i}.$$

Proof. Because $\{w_{i-1}, w_i\}$ is a basis for $\langle E \rangle$ we can write w_{i+1} as a word in these two letters. But since $\{w_i, w_{i+1}\}$ is itself a basis, the only possibilities for this expression are of the form

$$w_i^{a_i} w_{i-1}^{\pm 1} w_i^{b_i}.$$

Our choice of generators $\{w_j\}$ ensures a negative power of w_{i-1} . ■

We will construct a smooth manifold X by gluing together η copies of $\mathbf{H} \times \mathbf{H}$ along dense open submanifolds. The construction is entirely analogous to the complex structure on a nonsingular toric variety except the transition functions are *noncommutative* monomials in the coordinates.

For each $i \in \mathbf{Z}/(\eta)$, let $U_i = \mathbf{H} \times \mathbf{H}$ with quaternionic coordinates (x_i, y_i) , and let (a_i, b_i) be as in Lemma 6.2.1. We set $X_1 = U_1$ and define X_{i+1} inductively as follows: identify the point $(x_i, y_i) \in U_i \subset X_i$ ($y_i \neq 0$) with $(x_{i+1}, y_{i+1}) \in U_{i+1}$ whenever $x_{i+1} = y_i^{-1}$ and $y_{i+1} = y_i^{a_i} x_i y_i^{b_i}$. U_{i+1} is then an open subset of X_{i+1} and we can continue the process by gluing on U_{i+2} . The space X is obtained from the space X_η by the final identification of U_η ($= U_0$) with U_1 . That is, $(x_0, y_0) \in U_0$ is identified with $(x_1, y_1) \in U_1$ whenever $x_1 = y_0^{-1}$ and $y_1 = y_0^{a_1} x_0 y_0^{b_1}$. One checks easily that this last identification is compatible with all previous identifications, hence X is well-defined. Furthermore, since the quotient map to X is injective on each U_i , we may think of the U_i 's as subsets of X with coordinate maps $\phi_i : U_i \cap X \rightarrow \mathbf{H} \times \mathbf{H}$ given by $p \mapsto (x_i(p), y_i(p))$.

Theorem 6.2.1 *The subsets U_i are coordinate charts, making X into a smooth, compact, oriented manifold, homeomorphic to $\mathcal{T}_{\mathbf{H}}\lambda_3$.*

Proof. That X is compact will follow from the homeomorphism with $\mathcal{T}_{\mathbf{H}}\lambda_3$. Orientability and smoothness will follow from explicit formulas for the transition functions

$$\phi_i \cdot \phi_j^{-1} : \phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j).$$

By symmetry (and a possible automorphism of the free group $\langle E \rangle$), it suffices to consider the case $j = 1$. For $i = 2$, we have

$$\phi_2 \cdot \phi_1^{-1}(x, y) = (y^{-1}, y^{a_2} x y^{b_2})$$

which defines a diffeomorphism

$$\phi_1(U_2 \cap U_1) = \mathbf{H} \times \mathbf{H}^* \rightarrow \phi_2(U_2 \cap U_1) = \mathbf{H}^* \times \mathbf{H}.$$

Similarly, for $i = 0$, $\phi_0 \cdot \phi_1^{-1}$ is the inverse of the diffeomorphism $\mathbf{H} \times \mathbf{H}^* \rightarrow \mathbf{H}^* \times \mathbf{H}$ given by $(x, y) \mapsto (y^{-1}, y^{a_1} x y^{a_2})$. For all other values of i , $\phi_i(U_i \cap U_1) = \phi_1(U_i \cap U_1) = \mathbf{H}^* \times \mathbf{H}^*$. If $w_{i-1}(x, y)$ and $w_i(x, y)$ denote the elements of \mathbf{H}^* obtained by evaluating the words on the coordinates $(x, y) \in \mathbf{H}^* \times \mathbf{H}^*$, then

$$\phi_i \cdot \phi_1^{-1}(x, y) = (w_{i-1}(x, y)^{-1}, w_i(x, y)).$$

This is clearly a diffeomorphism.

To see that the transition maps are orientation preserving, it is sufficient to check on those of the form $\phi_{i+1} \cdot \phi_i^{-1}$ since any other is a composite of these. But all of

these are of the form $(x, y) \mapsto (y^{-1}, y^a x y^b)$, and the determinant of the corresponding derivative at $x = y = 1$ is $+1$.

We define the homeomorphism $\psi : \mathcal{T}_{\mathbf{H}\lambda_3} \rightarrow X$ by first constructing a surjective map $\Psi : P \times S^3 \times S^3 \rightarrow X$, and showing that the induced equivalence relation on $P \times S^3 \times S^3$ is precisely the defining relation for $\mathcal{T}_{\mathbf{H}\lambda_3}$. To define Ψ we make use of the underlying cone complex Σ .

Lemma 6.2.2 *X is the union of the retracted subsets*

$$\widetilde{U}_i = \{p \in X \mid \|x_i(p)\| \leq 1, \|y_i(p)\| \leq 1\}.$$

Proof. By symmetry it is enough to show that every point of U_1 is contained in at least one of the \widetilde{U}_i 's. Fix $p \in U_1$ and let $(x_i, y_i) = \phi_i(p)$ whenever this makes sense. If $x_1 = 0$, then using the formula for the transition function $\phi_2 \cdot \phi_1^{-1}$ (given above) we see that either $\|y_1\| \leq 1$ in which case $p \in \widetilde{U}_1$ or $\|x_2\| \leq 1$ and $y_2 = 0$ in which case $p \in \widetilde{U}_2$. A similar argument shows that in case $y_1 = 0$, p must be contained in either \widetilde{U}_1 or $\widetilde{U}_{\eta-1}$.

We can now assume $\|x_1\| > 0$ and $\|y_1\| > 0$. Having fixed the standard basis ϵ_1, ϵ_2 for $N_{\mathbf{R}}$, we consider the covector $\mathbf{n} = -(\log \|x_1\|)\epsilon_1 - (\log \|y_1\|)\epsilon_2$ in $N_{\mathbf{R}}$. Let \mathbf{c}_i be a cone of Σ which contains the covector \mathbf{n} . Recall from the definitions of Section 1.1 and Definition 6.2.1 that the dual cone $\check{\mathbf{c}}_i$ is spanned by the vectors $-w_{i-1}^{ab}$ and w_i^{ab} , which we write as $r_1 e_1^{ab} + r_2 e_2^{ab}$ and $s_1 e_1^{ab} + s_2 e_2^{ab}$. From the definition of the dual cone it follows that the covector \mathbf{n} must be *nonnegative* when evaluated on these spanning vectors, giving the inequalities

$$r_1 \log \|x_1\| + r_2 \log \|y_1\| \leq 0$$

and

$$s_1 \log \|x_1\| + s_2 \log \|y_1\| \leq 0.$$

But exponentiating these inequalities and using the formula for (x_i, y_i) in terms of (x_1, y_1) given by the appropriate transition function, we have

$$\|x_i\| = \|w_{i-1}^{-1}(x_1, y_1)\| = \|x_1\|^{r_1} \|y_1\|^{r_2} \leq 1$$

and

$$\|y_i\| = \|w_i(x_1, y_1)\| = \|x_1\|^{s_1} \|y_1\|^{s_2} \leq 1.$$

It follows that p is in \widetilde{U}_i . ■

Subdivide P into η quadrilaterals by joining a fixed interior point to the midpoint of each edge. Let \square_i be the quadrilateral containing the vertex m_i , as in Figure 6.2.3.

Define homeomorphisms $\psi_i : \square_i \xrightarrow{\cong} [0, 1] \times [0, 1]$ which preserve faces, map m_i to $(0, 0)$, and satisfy the overlap compatibility condition

$$\psi_{i+1}\psi_i^{-1}(x, 1) = (1, x)$$

for all $i \in \mathbb{Z}/(\eta)$ and $x \in [0, 1]$. In other words, P is homeomorphic to the space obtained by gluing η copies of $[0, 1] \times [0, 1]$ together by the maps $\psi_j\psi_i^{-1}$ along the faces $\{1\} \times [0, 1]$ and $[0, 1] \times \{1\}$.

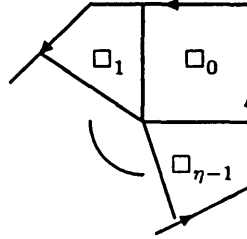


FIGURE 6.2.3

Let $\pi_1, \pi_2 : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be the respective coordinate projections. For each i and $(q, x, y) \in \square_i \times S^3 \times S^3$, define

$$\Psi(q, x, y) = \phi_i^{-1}(\pi_1\psi_i(q)w_{i-1}^{-1}(x, y), \pi_2\psi_i(q)w_i(x, y)).$$

Ψ maps $\square_i \times S^3 \times S^3$ onto the subset \widetilde{U}_i and for $q \in \square_i \cap \square_j$, the corresponding images of Ψ coincide. Hence, these maps patch together, giving a continuous surjection $\Psi : P \times S^3 \times S^3 \rightarrow X$.

Finally, one checks that the equivalence relation given by identifying the fibers of Ψ is the same as the defining relation (Definition 5.2.2) for $\mathcal{T}_{\mathbb{H}}\lambda_3$, giving a continuous bijection $\psi : \mathcal{T}_{\mathbb{H}}\lambda_3 \rightarrow X$. It follows from Lemma 6.2.2 and the explicit transition functions that X is Hausdorff. Since $\mathcal{T}_{\mathbb{H}}\lambda_3$ is compact, ψ is a homeomorphism. ■

6.3 Characteristic Numbers

In this section, we work toward a topological classification of the smooth examples defined in Section 6.2. The combinatorial structure of these manifolds allows one to give straightforward expressions for various characteristic classes and other topological invariants. The two fundamental invariants we work with are the first Pontrjagin class and the intersection form. According to Wall [31], the topological type of a

3-connected 8-manifold is determined by these two invariants. As a result of the cohomology calculation Theorem 5.5.1, we know that our smooth examples are 3-connected, so following Wall's program, we compute these invariants explicitly.

As in the previous section, we let λ_3 be a characteristic function on the η -gon P satisfying the nonsingularity conditions of Proposition 5.4.1 and let $X = \mathcal{T}_{\mathbf{H}}\lambda_3$ be the associated quaternionic space. In addition, we assume X is a lift of a toric variety, hence has the smooth structure defined in Theorem 6.2.1. According to Lemma 6.2.1, λ_3 is completely determined by a sequence of integral pairs $(a_0, b_0), \dots, (a_{\eta-1}, b_{\eta-1})$, and it is clear from the the last section that the smooth structure depends solely on this sequence. We will call such a sequence of pairs *admissible* if it arises (via Lemma 6.2.1) from a characteristic function for *some* smooth example X .

Let $D_i = \overline{S_{\sigma_i}}$ for $i \in \mathbf{Z}/(\eta)$ so that $D_0^*, D_1^*, \dots, D_{\eta-1}^*$ (the Poincaré dual cohomology classes of Section 5.5) generate the ring $H^*(X)$. It is clear from the last section that the cycle D_i is an embedded 4-sphere. Let ν_i be the normal bundle of the embedding with first Pontrjagin number $\xi_i = p_1(\nu_i)[D_i]$.

Lemma 6.3.1 *The cochain given by $\xi : D_i \mapsto \xi_i$ is a cocycle and is a representative for the Pontrjagin class $p_1(X)$.*

Proof. The normal bundle of D_i is represented by an element $\overline{\nu_i}$ of $\pi_3(SO_4)$, and according to [31, p. 165] the rule $D_i \mapsto \overline{\nu_i}$ induces a map $H_4(X) \rightarrow \pi_3(SO_4)$ (which Wall denotes by α). Composing with the stable map $S : \pi_3(SO_4) \rightarrow \pi_3(SO) \cong \mathbf{Z}$ thus defines a cohomology class $S\alpha$ in $H^4(X)$ which is $1/2$ the first Pontrjagin class ([31, p. 179]). Because the image of $\overline{\nu_i}$ in $\pi_3(SO_4)$ is precisely $\frac{1}{2}\xi_i$ (see, eg., [30]), we have

$$\frac{1}{2}\xi = S\alpha = \frac{1}{2}p_1(X)$$

as desired. ■

To determine $p_1(X)$ explicitly, then, we need to determine the numbers ξ_i . According to Section 6.2, a smooth neighborhood of the embedded 4-sphere D_i is given by identifying the subset $\mathbf{H} \times \mathbf{H}^*$ of U_i with $\mathbf{H}^* \times \mathbf{H}$ of U_{i+1} by the diffeomorphism

$$(x, y) \mapsto (y^{-1}, y^{a_i}xy^{b_i}).$$

As in Section 6.1, this defines a 4-dimensional real vector bundle over D_i which is precisely the normal bundle of D_i in X . The Pontrjagin class can be computed as in [21] by noting that the associated sphere bundle has clutching function $y \mapsto (x \mapsto y^{a_i}xy^{b_i})$, defining an element of $\pi_3(SO_4) \cong \mathbf{Z} \oplus \mathbf{Z}$. The Pontrjagin class is known to be ± 2 times the image of this element in the stable group $\pi_3(SO) \cong \mathbf{Z}$. Using, for example, [30, pages 115-117] this number can then be determined explicitly

$$\xi_i = 2(a_i - b_i).$$

The intersection form $H_4(X) \otimes H_4(X) \xrightarrow{\cap} H_0(X) = \mathbf{Z}$ can be computed by using the explicit description of the intersection ring in Theorem 5.5.1. If D_i and D_j are consecutive, they intersect in a point. By choosing appropriate orientations, we have $D_i \cap D_j = 1$. If D_i and D_j ($i \neq j$) are not consecutive, they do not intersect, hence $D_i \cap D_j = 0$. To compute the self intersection of D_i , the linear relations among the cycles can be used to push D_i away from itself. This can be done in such a way that the coefficients of D_{i+1} , D_i , and D_{i-1} in the resulting cycle are $-(a_i + b_i)$, 0, and 0, respectively. Intersecting with D_i then gives $-(a_i + b_i)$. In short,

$$D_i \cap D_j = \begin{cases} 1 & \text{if } i - j = \pm 1 \pmod{\eta} \\ \chi_i & \text{if } i = j \pmod{\eta} \\ 0 & \text{otherwise} \end{cases}$$

where $\chi_i = -(a_i + b_i)$. Note that χ_i is the Euler number obtained by evaluating the Euler class of ν_i on the generator $[D_i] \in H_4(D_i)$.

We will now compute some of the relations among the integers a_i and b_j for an admissible sequence $(a_0, b_0), (a_1, b_1), \dots, (a_{\eta-1}, b_{\eta-1})$. We focus on those relations arising from well known geometric formulas involving certain characteristic numbers for smooth manifolds. The numbers are the signature, the first and second Pontrjagin numbers, and the \hat{A} -genus, which for a given manifold X , we denote by $\tau(X)$, $p_1^2[X]$ and $p_2[X]$, and $\hat{A}(X)$, respectively. The first formula we recall is the signature theorem of Hirzebruch:

$$\tau(X) = \frac{1}{45}(7p_2[X] - p_1^2[X]), \quad (1)$$

and the second is the definition of $\hat{A}(X)$:

$$\hat{A}(X) = \frac{1}{5760}(7p_1^2[X] - 4p_2[X]). \quad (2)$$

Since in our case, X is 3-connected, it admits a unique spin structure (see [22], eg.); hence $\hat{A}(X)$, as the index of a certain elliptic operator, is an integer (see, eg., [12]). In fact, for our examples we can say more.

Lemma 6.3.2 *Let X be one of the smooth 2-dimensional quaternionic spaces of Section 6.2. Then $\hat{A}(X) = 0$.*

Proof. A smooth manifold admits a spin structure provided the second Stiefel-Whitney class vanishes, hence X is spin. A well-known fact about spin manifolds (see, eg., [16]) is that the \hat{A} genus vanishes if the manifold admits an S^1 -action. One checks easily that the $SO(3)$ -action on X (defined in Section 5.2) is smooth. Since the subgroup $SO(2) = S^1$ also acts smoothly, $\hat{A}(X) = 0$. ■

According to (2), then, we have

$$p_2[X] = \frac{7}{4}p_1^2[X],$$

and substituting into (1) gives

$$\tau(X) = \frac{1}{4}p_1^2[X]. \quad (3)$$

All of the above numbers, then, can be written in terms of the single number $p_1^2[X]$, which we now compute explicitly in terms of the ξ_i and χ_i defined above. According to Lemma 6.3.1, the Pontrjagin cohomology class $\xi = p_1$ is given by $D_i \mapsto \xi_i$ for $i \in \mathbf{Z}/(\eta)$. To square this class, we use the nondegenerate intersection form to obtain the Poincaré dual class x in $H_4(X)$. Then $\xi(x) = x \cap x \in \mathbf{Z}$ is precisely the number $p_1^2[X]$. For any representative cycle

$$\sum_{i=0}^{\eta-1} \gamma_i \cdot D_i$$

for the homology class x , intersecting with D_j must give the same result as evaluating ξ on D_j . Thus for any $j \in \mathbf{Z}/(\eta)$, we have

$$\xi_j = \gamma_{j-1} + \chi_j \gamma_j + \gamma_{j+1}. \quad (4)$$

Similarly, evaluating ξ on this representative cycle gives

$$p_1^2[X] = \xi(x) = \sum_i \gamma_i \xi_i \quad (5)$$

Since any $\eta - 1$ consecutive D_i 's form a basis for $H_4(X)$, we can take $\gamma_0 = \gamma_1 = 0$. The remaining coefficients are then determined inductively by (4)

$$\begin{aligned} \gamma_2 &= \xi_1 - \chi_1 \gamma_1 - \gamma_0 = \xi_1 \\ \gamma_3 &= \xi_2 - \chi_2 \gamma_2 - \gamma_1 = \xi_2 - \chi_2 \xi_1 \\ &\vdots \\ \gamma_{i+1} &= \xi_i - \chi_i \gamma_i - \gamma_{i-1} \\ &\vdots \\ \gamma_{\eta-1} &= \xi_{\eta-2} - \chi_{\eta-2} \gamma_{\eta-2} - \gamma_{\eta-3} \end{aligned}$$

Substituting into (5) then gives the desired formula.

On the other hand, since X is the lift of a toric variety associated to some fan Σ , it has the same cohomology ring (up to a shift in degree) as the toric variety $X(\Sigma)$. Hence, X has the same intersection form (and signature) as the complex toric variety associated with the η -gon P . (Incidentally, this toric variety embeds as a submanifold

of X by restricting each the coordinate charts of Section 6.2 to $\mathbf{C} \times \mathbf{C} \subset \mathbf{H} \times \mathbf{H}$.) Moreover, the numbers $\chi_i = -(a_i + b_i)$ are the self intersection numbers of the generating 2-cycles (Theorem 3.3.2) for the toric variety. P. Melvin, in [20, Theorem 1], gives a formula for the signature in terms of this sequence $\chi_1, \chi_2, \dots, \chi_\eta$ of Euler numbers. Because of the cyclic symmetry, one would hope for a symmetric expression, and indeed

$$\tau(X) = \frac{1}{3} \sum_i \chi_i. \quad (6)$$

This formula holds for the signature of *any* 2-dimensional nonsingular complex toric space; in the case of a toric *variety*, the rigidity of the cone complex gives, as in [8], the much simpler formula

$$\tau(X) = 4 - \eta. \quad (7)$$

Theorem 6.3.1 *Let (a_i, b_i) , $i \in \mathbf{Z}/(\eta)$ be an admissible sequence of integral pairs, and let $\xi_i = 2(a_i - b_i)$ and $\chi_i = -(a_i + b_i)$. Then*

$$4 - \eta = \frac{1}{3} \sum_i \chi_i.$$

In addition, if γ_i , $i \in \mathbf{Z}/(\eta)$, is any sequence satisfying (4), then

$$4 - \eta = \frac{1}{4} \sum_i \gamma_i \xi_i.$$

Proof. The first equation follows from (6) and (7), the second from (3) and (7).

■

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