Published by Institute of Physics Publishing for SISSA

Received: *April 27, 2006*
Accepted: *May 9, 2006*
Published: *May 24, 2006*

Matrix factorizations, minimal models and Massey products

Johanna Knapp and Harun Omer

Department of Physics, Theory Division, CERN
CH-1211 Geneva 23, Switzerland
E-mail: johanna.knapp@cern.ch, harun.omer@cern.ch

Abstract: We present a method to compute the full non-linear deformations of matrix factorizations for ADE minimal models. This method is based on the calculation of higher products in the cohomology, called Massey products. The algorithm yields a polynomial ring whose vanishing relations encode the obstructions of the deformations of the D-branes characterized by these matrix factorizations. This coincides with the critical locus of the effective superpotential which can be computed by integrating these relations. Our results for the effective superpotential are in agreement with those obtained from solving the A-infinity relations. We point out a relation to the superpotentials of Kazama-Suzuki models. We will illustrate our findings by various examples, putting emphasis on the E_6 minimal model.

Keywords: D-branes, Topological Strings.

JHEP05(2006)064

Contents

JHEP05(2006)064

1. Introduction and summary

It is by now well-known that topological D-branes for the B-model are characterized by matrix factorizations of the Landau-Ginzburg superpotential [1 – 4]. In the mathematics literature the problem is approached from the category theoretic point of view [5 – 8]. The purpose of this paper is to discuss methods to compute the effective superpotential for B-type Landau-Ginzburg models with boundary.

The effective superpotential $\mathcal{W}_{\text{eff}}$ can be understood in various ways. In $\mathcal{N} = 1$ string compactifications with D-branes it can be interpreted as the four dimensional space-time superpotential. This description is valid for critical string theories. Furthermore, the effective superpotential $\mathcal{W}_{\text{eff}}$ represents the generating functional for open string disk amplitudes. Once all the amplitudes are known, they can be integrated to give the effective superpotential. The values of the amplitudes are constrained by worldsheet consistency constraints they have to satisfy. In the case without D-branes the constraints are the WDVV equations [9]. Generalizing to worldsheets with boundary, these constraints have to be extended. This was done in [10] for the case without integrated insertions while the general case with insertions of integrated operators was derived in [11], where it was found that amplitudes have to satisfy the A_∞-relations. Correlators with bulk insertions have to satisfy the bulk-boundary crossing constraint in addition. The CFT Cardy constraint of [11], however, does not necessarily hold due to possible anomalies. We will demonstrate that it indeed breaks down for all but the simplest problems. A further sewing constraint for the amplitudes are the quantum A_∞-relations [12, 13] but we will not use them in this work. The focus of this paper is the derivation of $\mathcal{W}_{\text{eff}}$ by means of deformation theory.

An interesting aspect of the effective superpotential is the fact that it encodes the obstructions of the deformations of D-branes. Turning on generic deformations usually leads away from the critical point and the problem cannot be approached within a CFT context. It is thus not surprising that the deformation problem has not yet been considered in a systematic way in the physics literature. First steps towards understanding this problem were made in [14 – 16]. In [17] some interesting results were found for the quintic. See also [18] for a recent account of the deformation problem in the context of $K3$ surfaces. In the field of mathematics, however, deformation theory of matrix factorization is an active area of research. It is the aim of this work to present methods known to mathematicians in the context of deformation theory, extend them and use them to calculate $\mathcal{W}_{\text{eff}}$ for a number of examples. It seems ironic that mathematics provides efficient and elegant tools to calculate deformations of matrix factorizations, whereas the existence of an effective superpotential seems to be unknown in the mathematics literature in that context.

In our discussion we will mostly deal with the ADE minimal models. In these models all the possible deformations are massive and there are no marginal deformations. Consequently, it is possible to work in a purely algebraic setup which simplifies matters significantly. So far, various aspects of the A– and D– series minimal models have been discussed in the literature [2, 4, 11, 19]. The classification of the matrix factorizations for the minimals model was solved some time ago [20]. The complete spectra for all the ADE minimal models were given in [21], where also the relation to the Dynkin diagrams of the

JHEP05(2006)064

simply laced Lie algebras was discussed. See also [8]. The mathematical framework for the deformation theory of matrix factorizations was established in [22, 23]. An explicit method to calculate deformations of matrix factorizations was presented in [24, 25]. In the mathematics literature the deformation problem is sometimes referred to as "the method of computing formal moduli of modules". The idea of the method is to find the most general deformation of a matrix factorization such that the factorization condition still holds. This can be done by an iterative procedure which amounts to calculating higher cohomology elements of the complex defined by the BRST operator $Q = \left(\begin{smallmatrix} 0 & E \\ J & 0 \end{smallmatrix}\right)$, where $W = E \cdot J = J \cdot E$ is the matrix factorization. The central object in these calculations is the Massey product which is the generalization of the cup product to higher cohomologies. We will give a detailed description of the algorithm in section 2. The spectrum of Q is graded, $H(Q) = H^e \oplus H^o$, where the superscripts denote even and odd states. The odd states give the deformations, the even states the obstructions. Associating to every odd state in H^o a deformation parameter u the deformed Q-operator has the following form:

$$Q_{\text{def}} = Q + \sum_{\vec{m}} \alpha_{\vec{m}} u^{\vec{m}}, \tag{1.1}$$

where $u^{\vec{m}} = u_1^{m_1} \dots u_r^{m_r}$ with $r = \dim H^o$ and $\alpha_{\vec{m}}$ are matrices which can be calculated with the Massey product algorithm. Whenever a Massey product is non-zero it contributes to Q_{def} through $\alpha_{\vec{m}}$ unless it lies in H^e. In that latter case it contributes to the obstructions which are represented by a polynomial ring: $k[[u_1, \dots, u_r]]/(f_1, \dots, f_r)$, where the vanishing relations f_i are polynomials in the deformation parameters u_i, which give precisely the critical locus of $\mathcal{W}_{\text{eff}}$: $f_i \sim \frac{\partial \mathcal{W}_{\text{eff}}}{\partial u_i}$. That these relations can be integrated to a superpotential seems a priori not obvious, but turns out to be the case for all examples we investigated. The deformed Q-operator then squares to the Landau-Ginzburg superpotential up to these vanishing relations. Further interesting mathematics references on this subject are [26 – 30].

Without bulk insertions the effective superpotential agrees with the result obtained from the A_∞-relations. We will show that it is possible to incorporate bulk deformations into the algorithm. In this case, however, the results disagree with those obtained from the consistency constraints. We suspect that the reason for this discrepancy is that the deformation algorithm only sees the A_∞-structure whereas the bulk-boundary crossing constraint gives additional conditions that may be imposed consistently. This is in agreement with the structure of open-closed homotopy algebras (OCHAs) proposed in [31].

In addition we will point out a close connection between *effective* superpotentials associated to rank two matrix factorizations and the LG superpotentials of Kazama-Suzuki type coset models [32 – 35]. For all tested cases our results agree with these superpotentials up to field redefinitions and we will present a general reason why this is so.

The paper is organized as follows: The next section is devoted to explaining the algorithm to compute deformations of matrix factorizations and the effective superpotential. Some explicit examples will be given. Using the example of the E_6 minimal model, we will demonstrate in section 3 how the Massey product formalism can be extended to incorporate bulk deformations. Section 4 is devoted to the relation between the boundary superpotential and the superpotentials obtained from coset models. In section 5 we give

a summary of the techniques necessary to calculate the the effective superpotential using consistency constraints for the open string amplitudes. We will then merge both of the discussed methods in order to calculate the effective superpotential for the E_6-example and compare the results. In section 6 we give further examples for superpotentials for minimal models of type E. Section 7 is devoted to conclusions and open questions. Finally, in the appendix we summarize the data for the E_6 model and give some new results for the A minimal models, which is the only case where the Cardy constraint works.

1.1 Matrix factorizations

For a given ADE superpotential W one must first find its matrix factorizations $W\,\mathbf{1}_{N\times N} = E\cdot J = J\cdot E$. Two matrix factorizations (E,J) and (E',J') are called equivalent if they are related by a similarity transformation:

$$E' = U_1 E U_2^{-1} \qquad J' = U_2 J U_1^{-1}, \tag{1.2}$$

where $U_1, U_2 \in GL(N,\mathbb{R})$ are invertible matrices with polynomial entries. We consider only reduced factorizations, i.e. $E(0) = J(0) = 0$. The matrix factorizations for the ADE singularities were already completely classified by mathematicians in [20]. Now, having a set of factorizations which we label by capital letters, one constructs a BRST operator,

$$Q^A = \begin{pmatrix} 0 & E^A \\ J^A & 0 \end{pmatrix}, \tag{1.3}$$

These operators define a graded differential d by

$$d\psi = Q^A \psi^{AB} - (-1)^{|\psi|}\psi^{AB} Q^B. \tag{1.4}$$

The physical states lie in the cohomology $H(d) = \frac{\mathrm{Ker}(d)}{\mathrm{Im}(d)}$ of the differential d. In the category theoretic description, d is the differential of a differential graded category and the fermionic states correspond to $\mathrm{Hom}^1(A,B) \cong \mathrm{Ext}^1(A,B)$, the bosonic states are in $\mathrm{Hom}^2(A,B) \cong \mathrm{Ext}^2(A,B)$.

A method for computing the R-charges $q_{\psi^{AB}}$ of a open string $\psi^{AB} \in H(d)$ stretching between the branes A and B was presented in [36]; they are determined by the equation

$$E\psi^{AB} + R^A\psi^{AB} - \psi^{AB}R^B = q_{\psi^{AB}}\psi^{AB}, \tag{1.5}$$

where

$$E = \sum_i q_i x_i \frac{\partial}{\partial x_i} \quad \text{and} \quad W(e^{i\lambda q_i}x_i) = e^{2i\lambda}W(x_i)\ \forall\lambda\in\mathbb{R}. \tag{1.6}$$

The defining equation for the matrix R^A is

$$EQ^A + [R^A, Q^A] = Q^A. \tag{1.7}$$

2. Massey products

We will demonstrate that an effective superpotential can be derived by calculating Massey products. That is the way a mathematician would approach this problem and we will introduce it in this paper in a form digestible to physicists.

In particular, an easy example will be calculated, showing that computing deformations by this method is not too hard in practice although the mathematical framework needed for a rigorous treatment is quite extensive. The method has the advantage, that a general, straightforward algorithm exists, at least for the effective superpotentials of simple singularities. The calculation not only gives the superpotential, but as a by-product also the full deformed matrices of the factorization. This allows to double-check that the result is indeed a valid deformation, minimizing the risk of an error during the computation. The entire algorithm is already implemented in the SINGULAR–package [37, 38] to which the tedious work can be outsourced. The whole formalism is, however, restricted to compute only fermionic deformations.

The inclusion of bulk deformations into the algorithm has not yet been done by mathematicians and we will extend the algorithm in this paper in order to include them.

2.1 Mathematics

We will review deformations only very briefly and start in the context of differential graded Lie algebras (DGLAs). For a rigorous treatment of this and the slightly involved definitions of Massey products in this context we refer to the work of mathematicians [39 – 41].

A DGLA is a $\mathbb{Z}_2$- or $\mathbb{Z}$-graded vector space V over a field $\mathbb{K}$ with a commutator of degree zero and a differential $d : V \to V$ of degree 1, satisfying the conditions

$$\begin{aligned} &[\alpha,\beta] = -(-1)^{|\alpha||\beta|}[\beta,\alpha], \\ &d[\alpha,\beta] = [d\alpha,\beta] + (-1)^{|\alpha|}[\alpha,d\beta], \\ &[[\alpha,\beta],\gamma] + (-1)^{|\alpha|(|\beta|+|\gamma|)}[[\beta,\gamma],\alpha] + (-1)^{|\gamma|(|\alpha|+|\beta|)}[\gamma,\alpha,\beta] = 0. \end{aligned} \tag{2.1}$$

A *formal deformation* of V is defined as the power series

$$[g,h]_u = [g,h] + \sum_{i=1}^{\infty} u^i(\alpha_i(g,h) + \theta\beta_i(g,h)), \tag{2.2}$$

where θ is an odd parameter. Requiring the bracket to remain bilinear and antisymmetric and to fulfill the Jacobi identity places constraints on the α_i and β_i. Deformations exist, if certain conditions of Massey products are fulfilled and the deformations can be calculated explicitly from the Massey products.

It is not necessary to restrict to just a single parameter u, i.e. L_u as a deformation of an algebra over $k[u]$; we can also generalize this to $k[u_1, u_2, \ldots, u_r]$. Such deformations are called versal deformations and they are defined as follows.

Definition. *A deformation L_R of L parametrized by $R \in \mathbb{C}$ is a versal deformation, if for any L_A parametrized by $A \in \mathbb{C}$ there is a morphism $f : R \to A$ such that*

(i) $L_R \otimes_R A \cong A$,

(ii) the map $m_R/m_R^2 \to m_A/m_A^2$ *induced by* f *is unique.*

The whole procedure can be extended to A_∞ algebras by a simple redefinition of the commutator (see [41]):

$$\{\phi, \psi\} = (-1)^{(k-1)}\psi[\phi, \psi]. \tag{2.3}$$

The A_∞ structure is defined by an odd element Q satisfying $\{Q, Q\} = 0$ and the action of the differential d by $d\phi = \{Q, \phi\}$. The formal deformation of such an A_∞ algebra V is still given by

$$Q_u = Q + \sum_{i=1}^{\infty} u^i(\alpha_i + \theta\beta_i), \tag{2.4}$$

with the condition $\{Q_u, Q_u\} = 0$.

Dealing with matrix factorizations involves modules, which can be treated in a similar manner. There, the object of interest is the infinitesimal deformation functor of a module M over a k-algebra A, defined as

$$\mathrm{Def}_M(S) = \{(M, \theta) | M \text{ an } A \otimes_R S - \text{module, flat over } S, M \otimes_S R \cong^{\theta} M\}/\cong .$$

For finite-dimensional $\mathrm{Ext}^1(M, M)$, Schlessingers theorem [42] ensures the existence of a hull $\hat{H}_M$ of the deformation functor, which has also been called the *formal moduli* of M, as well as the existence of a formal versal family. This has been studied extensively in [24, 25] and we refer to these references for proofs, details and the explicit algorithm. We restrict ourselves to give a less rigorous down-to-earth reasoning in the next section to make the algorithm plausible. Afterwards an explicit example will be calculated for illustration.

2.2 The idea

Suppose we have a matrix factorization $Q^2 = W\mathbb{1}$ of a superpotential W. A deformation Q_{def} of the original module Q can always be written in the form

$$Q_{\mathrm{def}} = Q + \sum_{\vec{m}} \alpha_{\vec{m}} u^{\vec{m}}. \tag{2.5}$$

Here, we consider only fermionic deformations and leave the bosonic ones for future work. Just like Q, the $\alpha_{\vec{m}}$ are modules and the deformation is parametrized by $u_1, \ldots, u_r$ where we used the notation

$$u^{\vec{m}} = u_1^{m_1} u_2^{m_2} \ldots u_d^{m_d}, \qquad \vec{m} \in \mathbb{N}^d \tag{2.6}$$

for convenience. Squaring Q_{def}, and comparing with W, we find

$$\sum_{\vec{m}} (Q\alpha_{\vec{m}} + \alpha_{\vec{m}} Q) u^{\vec{m}} + \sum_{\substack{\vec{m}_1, \vec{m}_2 \\ \vec{m_1} + \vec{m_2} = \vec{m}}} \alpha_{\vec{m}_1} \alpha_{\vec{m}_2} u^{\vec{m}_1 + \vec{m}_2} = 0, \tag{2.7}$$

which must be valid at all orders of u. This constraint can be solved iteratively. At order $|\vec{m}| = \sum_{i=1}^N m_i = 1$ the condition $Q\alpha_{\vec{m}} + \alpha_{\vec{m}} Q = 0$ holds, which means that the basis for the odd $\alpha_{\vec{m}}$ with $|\vec{m}| = 1$ is precisely the basis for the odd cohomology.

JHEP05(2006)064

The $\alpha_{\vec{m}}$ for some order $n-1$ now determine those of order $|\vec{m}| = n$ as follows. We define

$$y(\vec{m}) := \sum_{\vec{m}_1+\vec{m}_2=\vec{m}} \alpha_{\vec{m}_1}\alpha_{\vec{m}_2}, \tag{2.8}$$

neglecting a subtlety for a moment. The $y(\vec{m})$ are called Massey Products. For later convenience we define $\beta_{\vec{m}} = y(\vec{m})$.

All we would need in order to satisfy eq. (2.7) at order $|\vec{m}| = n$ is an arbitrary $\alpha_{\vec{m}}$ such that

$$d\alpha_{\vec{m}} = Q\alpha_{\vec{m}} + \alpha_{\vec{m}}Q = -\beta_{\vec{m}}. \tag{2.9}$$

Since we started at lowest order with α_{e_i} spanning a basis of $\mathrm{Ext}^1(M,M)$, all $\beta_{\vec{m}}$ will lie either in $\mathrm{Ext}^2(M,M)$ or be a polynomial multiple of an element lying in $\mathrm{Ext}^2(M,M)$. In the latter case, a matrix $\alpha_{\vec{m}}$ satisfying eq. (2.9) can always be found and this matrix will again be an element of $\mathrm{Ext}^1(M,M)$. In the former case, where $\beta_{\vec{m}}$ lies in the even cohomology, a counterterm cancelling the Massey product does not exist, therefore eq. (2.9) holds only mod $\mathrm{Ext}^2(M,M)$.

In each iteration step, such non-cancelling terms add up so that for a matrix factorization with $\dim_k \mathrm{Ext}^2(M,M) = r$, there are r polynomials $f_i \in k[u_1,\ldots,u_d]$, each associated with a basis element $\phi_i \in \mathrm{Ext}^2(M,M)$, so that $Q^2_{\mathrm{def}} = W + \sum_i f_i\phi_i$.

For the deformed factorization to be exactly equal to W, the f_i must vanish and the hull $\hat{H}_M$ of the deformation is therefore,

$$\hat{H}_M \simeq k[[u_1,\ldots,u_d]]/(f_1,\ldots f_r). \tag{2.10}$$

The building up of the f_i must be kept track of during the iteration. At each order n, we add the new terms to the f_i,

$$f_i^1 = 0, \qquad f_i^n = \sum y_i^*(\langle x^*;\vec{m}\rangle)u^{\vec{m}} + f_i^{n-1}, \tag{2.11}$$

where the expression $y_i^*(\langle x^*;\vec{m}\rangle)$ denotes the proportionality constant between $y(\vec{m})$ and the appropriate ϕ_i.

Up to now, we have neglected one important subtlety. Namely, the ring relations $k[u_1,\ldots,u_d]/f_i^n$ must also be applied to the $y(\vec{m})$ and the $\beta_{\vec{m}}$, so the above equations have to be modified. As an ingredient to be able to do this we need to define appropriate bases B_i at each order. We start with

$$\bar{B}_1 = \{\vec{n} \in \mathbb{N}^d || n| \leq 1\}, \qquad B_1 = \{\vec{n} \in \mathbb{N}^d || n| = 1\}. \tag{2.12}$$

at lowest order; in general, the B_i denote the bases for

$$\vec{m}^i/\vec{m}^{i+1} + \vec{m}^i \bigcap \vec{m}(f_1^{i-1},\ldots,f_r^{i-1}). \tag{2.13}$$

At each order, we take

$$\beta_{\vec{k},\vec{l}} \in k[\vec{u}]/(\vec{m}^{n+1} + (f_1^n,\ldots,f_r^n)), \tag{2.14}$$

where these $\beta_{\vec{k},\vec{l}}$ are defined by the unique relation

$$u^{\vec{n}} = \sum_{\vec{m}\in\bigcup B_i} \beta_{\vec{n},\vec{m}} u^{\vec{m}}. \tag{2.15}$$

with the sum running from $i = 1$ up to the the order n of the iteration step. The new defining equation for $\beta_{\vec{m}}$ is given by

$$\beta_{\vec{m}} = \sum_{\vec{m}\in\bigcup B_i} \beta_{\vec{n},\vec{m}} y(\vec{n}), \tag{2.16}$$

which, of course, reduces to the previous $\beta_{\vec{m}} = y(\vec{m})$ if $\beta_{\vec{n},\vec{m}} = \delta_{\vec{n},\vec{m}}$. That is only the case when there is no relation for the appropriate $\vec{n}$, i.e. $u^{\vec{n}}$ or a term proportional to it does not appear in one of the f_i. A similar correction must be made for eq. (2.8):

$$y(\vec{n}) := \sum_{\vec{m}_1+\vec{m}_2=\vec{m}} \gamma_{\vec{m},\vec{n}} \alpha_{\vec{m}_1} \alpha_{\vec{m}_2}, \tag{2.17}$$

where γ is defined by

$$u^{\vec{k}} = \textstyle\sum_{\vec{l}\in\bigcup B_i} \gamma_{\vec{k},\vec{l}} u^{\vec{l}} + \sum_j \gamma_{\vec{k},j} f_j^n. \tag{2.18}$$

We will now clarify the algorithm by calculating an explicit example.

2.3 A simple example

Here, the A-series superpotential $W = -\frac{1}{5}x^5$ shall be calculated step by step. The factorization Q and the four basis elements of the cohomology are given by

$$\begin{aligned}
Q &= \begin{pmatrix} 0 & x^2 \\ -1/5x^3 & 0 \end{pmatrix}, \\
\alpha_{(1,0)} &= \psi_1 = \begin{pmatrix} 0 & -x \\ -1/5x^2 & 0 \end{pmatrix}, \\
\alpha_{(0,1)} &= \psi_2 = \begin{pmatrix} 0 & -1 \\ -1/5x & 0 \end{pmatrix}, \\
\phi_1 &= \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \\
\phi_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned} \tag{2.19}$$

The second order Massey products are

$$\begin{aligned}
y(2,0) &= \alpha_{(1,0)}\alpha_{(1,0)} = \frac{1}{5}x^2\phi_1, \\
y(0,2) &= \alpha_{(0,1)}\alpha_{(0,1)} = \frac{1}{5}\phi_1, \\
y(1,1) &= \alpha_{(1,0)}\alpha_{(0,1)} + \alpha_{(0,1)}\alpha_{(1,0)} = \frac{2}{5}x^2\phi_1.
\end{aligned} \tag{2.20}$$

$y(0,2)$ is the only Massey product that lies in the cohomology, therefore $f_1^2 = \frac{1}{5}\vec{u}^{(0,2)} = \frac{1}{5}u_2^2$ while f_2^2 remains zero. The new basis for $\vec{m}^2/(\vec{m}^3 + (u_1^2, 0))$ is

$$B_2 = \{\vec{m} \in \mathbb{N}^2 : |\vec{m}| = 2\} - \{(0,2)\}. \tag{2.21}$$

For the two elements not in the cohomology, we take

$$\alpha_{(2,0)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \alpha_{(1,1)} = \begin{pmatrix} 0 & 0 \\ -2/5 & 0 \end{pmatrix}, \tag{2.22}$$

to satisfy $d\alpha_{(2,0)} = -y(2,0)$ and $d\alpha_{(1,1)} = -y(1,1)$. A different choice is possible and corresponds to a field redefinition of the effective potential.
As a basis for the third order, we choose

$$B_3 = \{|\vec{m} \in \mathbb{N}^3 : |\vec{m}| = 3\} - \{(0,2) + \vec{m}\} = \{(3,0),(2,1)\}, \tag{2.23}$$

and we can always find a basis of the same structure for higher orders:

$$B_n = \{\vec{m} \in \mathbb{N}^n : |\vec{m}| = n\} - \{(0,2) + \vec{m}\}. \tag{2.24}$$

We get

$$\begin{aligned} y(3,0) &= \alpha_{(2,0)}\alpha_{(1,0)} + \alpha_{(1,0)}\alpha_{(2,0)} = -\frac{1}{5}x^2\phi_2, \\ y(2,1) &= \alpha_{(2,0)}\alpha_{(0,1)} + \alpha_{(0,1)}\alpha_{(2,0)} + \alpha_{(1,1)}\alpha_{(1,0)} + \alpha_{(1,0)}\alpha_{(1,1)} = \frac{1}{5}\phi_1. \end{aligned} \tag{2.25}$$

All Massey products not listed are zero. f_1 at third order becomes $f_1^3 = \frac{1}{5}u_2^2 + \frac{1}{5}u_1^2u_2$ while f_2^3 is still zero. The $\beta_{\vec{m}}$ are

$$\begin{aligned} \beta_{(3,0)} &= y(3,0), \\ \beta_{(2,1)} &= y(2,1) - y(0,2) = 0, \end{aligned} \tag{2.26}$$

according to their definition in eq. (2.16). The relation for the $\beta_{\vec{n},\vec{m}}$ derives here from $f_1^3 = 0$, according to which $\frac{1}{5}u^{(2,1)} = -\frac{1}{5}u^{(0,2)}$, fixing therewith $\beta_{(0,2),(2,1)} = -1$ from eq. (2.15). The choice

$$\alpha_{(3,0)} = \begin{pmatrix} 0 & 0 \\ 1/5 & 0 \end{pmatrix}, \tag{2.27}$$

satisfies $d\alpha_{(3,0)} = -\beta_{(3,0)}$. The fourth order result is

$$\begin{aligned} y(4,0) &= \alpha_{(3,0)}\alpha_{(1,0)} + \alpha_{(1,0)}\alpha_{(3,0)} + \alpha_{(2,0)}\alpha_{(2,0)} = -\frac{1}{5}\phi_1 \\ y(3,1) &= \alpha_{(3,0)}\alpha_{(0,1)} + \alpha_{(0,1)}\alpha_{(3,0)} + \alpha_{(2,0)}\alpha_{(1,1)} + \alpha_{(1,1)}\alpha_{(2,0)} \\ &\quad -(\alpha_{(1,1)}\alpha_{(0,1)} + \alpha_{(0,1)}\alpha_{(1,1)}) = -\phi_2 \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} f_1^4 &= \frac{1}{5}(u_2^2 + u_1^2u_2 - u_1^4), \\ f_2^4 &= -u_1^3u_2. \end{aligned} \tag{2.29}$$

At fifth order - the last non-vanishing order - we find

$$\begin{aligned} y(5,0) &= \alpha_{(3,0)}\alpha_{(2,0)} + \alpha_{(2,0)}\alpha_{(3,0)} + \alpha_{(1,1)}\alpha_{(0,1)} + \alpha_{(0,1)}\alpha_{(1,1)} = \frac{3}{5}\phi_2, \\ f_1 := f_1^5 &= \frac{1}{5}(u_2^2 + u_1^2 u_2 - u_1^4), \\ f_2 := f_2^5 &= \frac{3}{5}u_1^5 - u_1^3 u_2. \end{aligned} \tag{2.30}$$

Now we have all necessary data and can assemble Eq (2.5),

$$\begin{aligned} E_{\text{def}} &= x^2 - u_1 x - u_2 + u_1^2, \\ J_{\text{def}} &= \frac{1}{5}(-x^3 - u_1 x^2 - u_2 x - 2u_1 u_2 + u_1^3). \end{aligned} \tag{2.31}$$

Squaring Q_{def} gives

$$Q_{\text{def}}^2 = \mathbb{1}\frac{1}{5}(-x^5 + (u_2^2 + u_1^2 u_2 - u_1^4)x + (2u_1 u_2^2 - 3u_1^3 u_2 + u_1^5)). \tag{2.32}$$

Using the ring relations

$$\begin{aligned} f_1 &= \frac{1}{5}(-u_1^4 + u_1^2 u_2 + u_2^2) = 0, \\ f_2 &= -u_1^3 u_2 + \frac{3}{5}u_1^5 = 0, \end{aligned} \tag{2.33}$$

we see that Q_{def}^2 reduces to the undeformed $W = -\frac{1}{5}x^5$, confirming that the result is indeed a valid deformation.

The conditions $f_i = 0$ are also called critical locus. In terms of the effective superpotential $\mathcal{W}_{\text{eff}}$, the critical locus is given by

$$\mathcal{Z}_{\text{crit}} = \{u \in \mathbb{C}^d | \partial_u \mathcal{W}_{\text{eff}}(u) = 0\}. \tag{2.34}$$

With no bulk insertions it is of course trivial and identical to the origin. We will now derive the effective superpotential by using the commutativity of the partial derivatives and charge reasoning. Using the R-charges (denoted by brackets),

$$[u_i] = i/5 \qquad [\mathcal{W}_{\text{eff}}] = 6/5 \qquad [f_1] = 4/5 \qquad [f_2] = 5/5 \tag{2.35}$$

we find that $[\partial_{u_1}\partial u_2 \mathcal{W}_{\text{eff}}(u,t)] = 3/5$ and therefore

$$\begin{aligned} \partial u_1 \mathcal{W}_{\text{eff}}(u; t=0) &= \mathbb{R} f_2 + \mathbb{R} u_1 f_1 \quad \text{and} \\ \partial u_2 \mathcal{W}_{\text{eff}}(u; t=0) &= \mathbb{R} f_1. \end{aligned} \tag{2.36}$$

The coefficients can be determined by using the commutativity of the second order partial derivatives. The effective superpotential is now

$$\mathcal{W}_{\text{eff}}(u; t=0) = \frac{u_1^6}{15} - \frac{u_1^4 u_2}{5} + \frac{u_1^2 u_2^2}{10} + \frac{u_2^3}{15}. \tag{2.37}$$

The result is equivalent to that in [11] and related to it by the field redefinition $u_2 \to u_2 + u_1^2$. This can be understood by remembering that during the iteration, we chose an arbitrary element $\alpha_{\vec{m}}$ to satisfy eq. (2.9). Since the operator d is nilpotent, each $\alpha_{\vec{m}}$ is only fixed up to the addition of an exact matrix. With a specific choice of the $\alpha_{\vec{m}}$, the cited results can be reproduced directly without need for a field redefinition.

2.4 The case E_6

The E_6 superpotential $W = x^3 + y^4 - z^2$ has a matrix factorization

$$Q = \begin{pmatrix} 0 & 0 & -y^2 - z & x \\ 0 & 0 & x^2 & y^2 - z \\ -y^2 + z & x & 0 & 0 \\ x^2 & y^2 + z & 0 & 0 \end{pmatrix}. \tag{2.38}$$

A basis for the cohomology is

$$\begin{aligned} \alpha_{(1,0)} &= \psi_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -x & 0 \\ 0 & 1 & 0 & 0 \\ -x & 0 & 0 & 0 \end{pmatrix}, \\ \alpha_{(0,1)} &= \psi_2 = \begin{pmatrix} 0 & 0 & 0 & y \\ 0 & 0 & -xy & 0 \\ 0 & y & 0 & 0 \\ -xy & 0 & 0 & 0 \end{pmatrix}, \\ \phi_1 &= \mathbb{1}, \\ \phi_2 &= y\mathbb{1}. \end{aligned} \tag{2.39}$$

The full deformation is found to be

$$\hat{H}_M \simeq k[[u_1, u_4]] / \left(u_4^3 - \frac{3}{4} u_4 u_1^8 - \frac{5}{64} u_1^{12},\ \ 3u_4^2 u_1 + \frac{3}{2} u_4 u_1^5 + \frac{1}{8} u_1^9 \right), \tag{2.40}$$

and explicitly as

$$\begin{aligned} E_{\text{def}} &= \begin{pmatrix} -y^2 + z + \frac{u_1^3}{2} y + \frac{u_1^6}{8} + \frac{3}{2} u_4 u_1^2 & x + u_1 y + u_4 \\ x^2 - u_1 x y + u_1^2 y^2 + 2u_4 u_1 y - u_4 x + u_4^2 & y^2 + z - \frac{u_1^3}{2} y - \frac{u_1^6}{8} - \frac{3}{2} u_4 u_1^2 \end{pmatrix}, \\ J_{\text{def}} &= \begin{pmatrix} -y^2 - z + \frac{u_1^3}{2} y + \frac{u_1^6}{8} + \frac{3}{2} u_4 u_1^2 & x + u_1 y + u_4 \\ x^2 - u_1 x y + u_1^2 y^2 + 2u_4 u_1 y - u_4 x + u_4^2 & y^2 - z - \frac{u_1^3}{2} y - \frac{u_1^6}{8} - \frac{3}{2} u_4 u_1^2 \end{pmatrix}. \end{aligned} \tag{2.41}$$

Note that we have slightly changed notation and labelled the deformation parameters by their R-charge. Squaring Q_{def} gives

$$J_{\text{def}} E_{\text{def}} = E_{\text{def}} J_{\text{def}} = W(x, y, z) + f_2 y + \left(f_1 + \frac{1}{2} u_1^3 f_2 \right), \tag{2.42}$$

proving that it is indeed a deformation. The corresponding effective superpotential is

$$\mathcal{W}_{\text{eff}}(u) = u_4^3 u_1 + \frac{3}{4} u_4^2 u_1^5 + \frac{1}{8} u_4 u_1^9 + \frac{5}{832} u_1^{13}. \tag{2.43}$$

We can check this result by using of the computer algebra package SINGULAR which can also perform this calculation. The output gives the full deformed matrices as well as the critical locus. All that remains to do is to integrate a combination of the latter to a superpotential.

3. Bulk deformations

Instead of restricting ourselves to deformations of matrices whose product is again the superpotential W, we now allow for perturbations of W as well.

3.1 Polynomial division

Polynomial division was used in [16] to derive the bulk deformations of the A model. The polynomial division gives

$$J_{\rm def}(x,u,t) = \frac{W(x,t)}{E_{\rm def}(x,u)} = \frac{1}{5}(x^3+u_1x^2+(u_1^2+u_2-5t_2)x+2u_1u_2-5t_3-5t_2u_1+u_1^3)+r, \quad (3.1)$$

where $W(x,t)$ is the deformed bulk superpotential. Vanishing of the remainder r,

$$\begin{aligned} r &= r_1(u,t)x + r_2(u,t), \\ r_1(u,t) &= t_2t_3 - t_5 - t_3u_2 - t_2u_1u_2 + \frac{u_1^3u_2}{5} + \frac{2u_1u_2^2}{5}, \\ r_2(u,t) &= t_2^2 - t_4 - t_3u_1 - t_2u_1^2 + \frac{1}{5}u_1^4 - t_2u_2 + \frac{3}{5}u_1^2u_2 + \frac{1}{5}u_2^2, \end{aligned} \quad (3.2)$$

gives the critical locus for this case. It defines the effective superpotential by

$$\begin{aligned} \partial_{u_1}\mathcal{W}_{\rm eff}(u;t) &= r_1(u,t) + r_2(u,t)u_1, \\ \partial_{u_2}\mathcal{W}_{\rm eff}(u;t) &= r_2(u,t), \end{aligned} \quad (3.3)$$

Of course, the A model is special, in that it is the only factorization with 1×1 matrices and for higher dimensional matrix factorizations we would have to take the determinant of the equation. The graver problem is that polynomial division does not necessarily have a solution. The E_6 model with the superpotential derived earlier is an example for that. This is not too surprising since the polynomial division implies that only either J *or* E have bulk deformations whereas we have chosen the boundary deformation to be symmetrical in E and J in the E_6 computation. Therefore, polynomial division generally requires at least non-trivial guesswork in starting with a suitable boundary potential, and normally a solution does not even exist.

3.2 Adapting the Massey product method

The method of computing formal moduli can be adapted to incorporate bulk insertions. It shall be demonstrated again at hand of the E_6 example. The most general deformation for this singularity is obtained by deforming with all elements in the chiral ring,

$$W_{\rm def}(x,s) = W(x) - s_2xy^2 - s_5xy - s_6y^2 - s_8x - s_9y - s_{12}. \quad (3.4)$$

In the physical theory the s_i are functions of bulk parameters t_i as given in [43]. Parallel to the reasoning in computing the formal moduli, we are looking for a deformation

$$\tilde{Q} = Q_{\rm def} + \sum_{\vec{m}} \tilde{\alpha}_{\vec{m}} s^{\vec{m}}, \quad (3.5)$$

which squares to $W(x,s)$. The analogue to eq. (2.7) becomes

$$\sum_{\vec{m}}(Q_{\rm def}\tilde{\alpha}_{\vec{m}}+\tilde{\alpha}_{\vec{m}}Q_{\rm def})s^{\vec{m}}+\sum_{\vec{m}_1,\vec{m}_2}\tilde{\alpha}_{\vec{m}_1}\tilde{\alpha}_{\vec{m}_2}s^{\vec{m}_1+\vec{m}_2}=W_{\rm def}(x,s)-W(x). \tag{3.6}$$

The s_{12}- and s_8-terms lie in $\mathrm{Ext}^2(E,E)$ and are therefore of no concern. They will simply be subtracted from the f_i' in $Q_{\rm def}^2=W+f_1'y+f_2'$. For the other four deformations we chose odd $\tilde{\alpha}_{e_i}$ such that

$$\begin{aligned}
[Q_{\rm def},\tilde{\alpha}_{e_1}] &= -x,\\
[Q_{\rm def},\tilde{\alpha}_{e_2}] &= -y^2,\\
[Q_{\rm def},\tilde{\alpha}_{e_3}] &= -xy,\\
[Q_{\rm def},\tilde{\alpha}_{e_4}] &= -xy^2.
\end{aligned} \tag{3.7}$$

Possible choices are

$$\begin{aligned}
\tilde{\alpha}_{(1,0,0,0)} &= -1\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\1&0&0&0\end{pmatrix},\\
\tilde{\alpha}_{(0,1,0,0)} &= \frac{1}{2}\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix},\\
\tilde{\alpha}_{(0,0,1,0)} &= y\tilde{\alpha}_{(1,0,0,0)}-u_1\tilde{\alpha}_{(0,1,0,0)},\\
\tilde{\alpha}_{(0,0,0,1)} &= \left(\frac{1}{8}u_1^4+\frac{1}{2}x\right)\tilde{\alpha}_{(0,1,0,0)}\\
&\quad+\begin{pmatrix}0&0&0&\frac{1}{4}u_1^2\\0&0&-(u_2u_4^2+\frac{u_1^6}{8})-\frac{1}{4}u_1^2x&0\\0&\frac{1}{4}u_1^2&0&0\\-(u_1^2u_4+\frac{u_1^6}{8})-\frac{1}{4}u_1^2x&0&0&0\end{pmatrix}.
\end{aligned}$$

At second order we find,

$$\begin{aligned}
y(0,0,0,2) &= \left(\frac{1}{4}x^2+\frac{1}{16}x\right)\mathbb{1} \quad \mathrm{mod}\ \mathrm{Ext}^2(E,E),\\
y(0,1,0,1) &= \frac{1}{2}x\mathbb{1} \qquad\qquad\quad \mathrm{mod}\ \mathrm{Ext}^2(E,E),\\
y(0,0,1,1) &= -\frac{1}{2}u_1x\mathbb{1} \qquad\quad \mathrm{mod}\ \mathrm{Ext}^2(E,E),
\end{aligned} \tag{3.8}$$

all others are zero (mod $\mathrm{Ext}^2(E,E)$). The associated second order $\tilde{\alpha}$'s are,

$$\begin{aligned}
\tilde{\alpha}(0,0,0,2) &= \left(\frac{1}{16}u_1^4-\frac{1}{4}u_2+\frac{1}{4}x\right)\tilde{\alpha}(1,0,0,0)-\frac{1}{4}u_1\tilde{\alpha}(0,0,1,0),\\
\tilde{\alpha}(0,1,0,1) &= \frac{1}{2}\tilde{\alpha}(1,0,0,0),\\
\tilde{\alpha}(0,0,1,1) &= \frac{1}{2}u_1\tilde{\alpha}(1,0,0,0).
\end{aligned} \tag{3.9}$$

At third (and last order) only one term remains,

$$\begin{aligned} y(0,0,0,3) &= \frac{1}{16}u_1^2 x\mathbb{1}, \\ \tilde{\alpha}(0,0,0,3) &= \frac{1}{16}u_1^2\tilde{\alpha}(1,0,0,0). \end{aligned} \tag{3.10}$$

Adding these terms up to get the complete deformed factorization $\tilde{Q}$ and squaring, the result is $W_{\rm def}(x,t) + \tilde{f}_1 y + \tilde{f}_2$ with some $\tilde{f}_1$ and $\tilde{f}_2$. The relations

$$\begin{aligned} \partial_{u_1}\mathcal{W}_{\rm eff}(u;s) &:= \tilde{f}_1, \\ \partial_{u_2}\mathcal{W}_{\rm eff}(u;s) &:= \left(\frac{1}{2}u_1^3\tilde{f}_2 + \tilde{f}_1\right), \end{aligned} \tag{3.11}$$

allow to integrate to the full superpotential,

$$\begin{aligned} \mathcal{W}_{\rm Massey}(u;s) = &\frac{5}{64\cdot 13}u_1^{13} + \frac{1}{8}u_4u_1^9 + \frac{3}{4}u_4^2u_1^5 + u_4^3u_1 + \frac{1}{32\cdot 11}s_2u_1^{11} \\ &+\frac{3}{64\cdot 9}s_2^2u_1^9 - \frac{3}{8\cdot 8}s_5u_1^8 + \frac{3}{8\cdot 7}s_6u_1^7 + \frac{3}{64\cdot 7}s_2^3u_1^7 \\ &+\frac{1}{16}s_2^2u_4u_1^5 - \frac{1}{10}\left(s_8 + \frac{1}{4}s_6s_2\right)u_1^5 - \frac{1}{2}s_5u_4u_1^4 \\ &+\frac{1}{2\cdot 4}s_9u_1^4 - \frac{1}{4}s_2u_4^2u_1^3 + \frac{1}{2}s_6u_4u_1^3 - \frac{1}{12}\left(s_8s_2 - s_5^2\right)u_1^3 \\ &+\frac{1}{4}s_5s_2u_4u_1^2 - \frac{1}{4}s_6s_5u_1^2 + \frac{1}{4}s_2^2u_4^2u_1 - \frac{1}{2}s_2s_6u_4u_1 \\ &-s_8u_4u_1 + (s_{12} + \frac{1}{4}s_6^2)u_1 - \frac{1}{2}s_5u_4^2 + s_9u_4 + \text{const.} \end{aligned} \tag{3.12}$$

The integration constant is an arbitrary function of the s_i but, of course, independent from the u_i. In the bulk theory, usually the coordinate transformation $s_i \to t_i(s_i)$ is used for the sake of adherence to the constant metric coordinate system [43],

$$\begin{aligned} s_2 &= t_2, \\ s_5 &= t_5, \\ s_6 &= \left(t_6 - \frac{t_2^3}{2}\right), \\ s_8 &= \left(t_8 - t_6t_2 + \frac{t_2^4}{12}\right), \\ s_9 &= \left(t_9 - t_5t_2^2\right), \\ s_{12} &= \left(t_0 - \frac{t_6^2}{2} - \frac{t_5^2t_2}{2} - \frac{t_8t_2^2}{2} + \frac{t_6t_2^3}{6}\right). \end{aligned} \tag{3.13}$$

4. Relation to the hermitian symmetric space coset models

In this section we show that the *effective* superpotentials associated with rank two matrix factorizations can be related to the *Landau-Ginzburg* potentials of simply-laced, level one,

hermitian symmetric space (SLOHSS) models [32].[1] These Kazama-Suzuki type models are represented by cosets G/H, where the group G is divided by its maximal subgroup H. In the E_6 case this is

$$\frac{E_6}{\mathrm{SO}(10)\times \mathrm{U}(1)}.$$

Landau-Ginzburg potentials for the deformed SLOHSS models were derived in [35]. They are obtained by first expressing the Casimirs V_i of the group G, in terms of the Casimirs of H, which are called x_i. Next, we set $V(x_i) = v_i$, identifying the Casimirs with deformation parameters v_i of the superpotential. This yields a system of equations, where the x_i that appear linearly can be eliminated. The remaining equations can then be integrated to a superpotential. The explicit form of it was given as [35]:

$$\begin{aligned} W(x,z,w) = {} & x^{13} - \frac{25}{169}x\,z^3 + \frac{5}{26}x^2\,w_2 \\ & + z\left(x^9 + x^7\,w_1 + \frac{1}{3}x^5\,w_1^2 - x^4\,w_2 - \frac{1}{3}x^2\,w_1\,w_2 + \frac{1}{12}x^3\,w_3 - \frac{1}{6}x\,w_4 + \frac{1}{3}w_5\right) \\ & + \frac{247}{165}x^{11}\,w_1 + \frac{13}{15}x^9\,w_1^2 - \frac{39}{20}x^8\,w_2 + \frac{169}{945}x^7\,w_1^3 + \frac{13}{105}x^7\,w_3 - \frac{26}{15}x^6\,w_1\,w_2 \\ & + \frac{13}{225}x^5\,w_1\,w_3 - \frac{13}{50}x^5\,w_4 - \frac{91}{180}x^4\,w_1^2\,w_2 + \frac{13}{30}x^4\,w_5 + \frac{13}{15}x^3w_2^2 - \frac{13}{90}x^3\,w_1\,w_4 \\ & - \frac{13}{120}x^2\,w_2\,w_3 + \frac{13}{90}x^2\,w_1\,w_5 - \frac{13}{270}x\,w_6 - \frac{13}{360}w_1^4\,w_2 + \frac{13}{90}w_1^2\,w_5. \end{aligned} \tag{4.1}$$

This superpotential is, up to a (quite complicated) field redefinition and the choice of an integration constant precisely the effective superpotential associated to the 2×2-factorization of the minimal model in our results. The ansatz for such a field redefinition looks as follows:

$$u_1 \to \alpha_1 x \qquad u_4 \to \alpha_2 z + \alpha_3 x^4 + \alpha_4 w_1^2 + \alpha_5 w_1 x^2 \tag{4.2}$$

$$\begin{aligned} s_{12} &\to \beta_1 w_6 + \beta_2 w_4 w_1^2 + \beta_3 w_3^2 + \beta_4 w_3 w_1^3 + \beta_5 w_2^2 w_1 + \beta_6 w_1^6 \\ s_9 &\to \beta_7 w_5 + \beta_8 w_2 w_1^2 \qquad s_8 \to \beta_9 w_4 + \beta_{10} w_3 w_1 + \beta_{11} w_1^4 \\ s_6 &\to \beta_{12} w_3 + \beta_{13} w_1^3 \quad s_5 \to \beta_{14} w_2 \quad s_2 \to \beta_{15} w_1. \end{aligned} \tag{4.3}$$

Plugging this into eq. (3.12) and comparing with eq. (4.1) all the parameters α_i, β_i can be fixed. It turns our that the integration constant in eq. (3.12) is a crucial degree of freedom to get the potentials in agreement. Without it, agreement can only be achieved by setting one deformation parameter to zero.

Note that for the E_8 there is no SLOHSS model. However, there is also no rank 2 matrix factorization of the E_8 Landau-Ginzburg potential. With the formalism of matrix factorizations we can also derive superpotentials for matrix factorizations of higher rank.

It thus seems that only the effective superpotentials associated to 2×2 matrix factorizations have a direct connection to the superpotentials coming from SLOHSS models. It is interesting to ask why and how the matrix factorizations are encoded in these coset

[1] Quite recently an interesting relation between Kazama Suzuki superpotentials, matrix factorizations and knot homology was discovered [44 – 46].

models. We now give a qualitative explanation of how this happens.[2] One way to calculate such superpotentials (at least in principle) is to eliminate as many variables as possible [35]. Take for instance the E_6-example:

$$W = x^3 + y^4 - z^2 \ (+ \text{ bulk deformations}). \tag{4.4}$$

The equation $W = 0$ describes an ALE space, a two complex dimensional surface in $\mathbb{C}^3$. The idea is to start with the simplest algebraic objects and find lines and quadrics on this surface. Using the ansatz

$$x = \lambda y + \alpha(\lambda), \tag{4.5}$$

where λ is a variable of weight one and α is a homogeneous polynomial of weight four, the variable z can be eliminated from W:

$$z = \sqrt{x^3 + y^4 \ (+ \text{ bulk deformations})} \tag{4.6}$$

In this relation the ansatz for x can be inserted. The equation on z describes a quadric if the expression under the square root is a perfect square. In this case one gets

$$z = y^2 + \gamma_1(\lambda)y + \gamma_2(\lambda). \tag{4.7}$$

Lines and quadrics on the surface $W = 0$ are then given by

$$\begin{aligned} A_1 &= \lambda y + \alpha(\lambda) \ = \ 0 \\ A_2 &= y^2 + \gamma_1(\lambda)y + \gamma_2(\lambda) \ = \ 0. \end{aligned} \tag{4.8}$$

The Nullstellensatz tells us that this is consistent with $W = 0$ if W has the form

$$W = A_1 B_1 + A_2 B_2, \tag{4.9}$$

for some B_1, B_2. This is precisely a 2×2 matrix factorization of the superpotential! Our results for the ADE minimal models thus suggest that there is a direct relation between the coset model LG superpotentials and $\mathcal{W}_{\text{eff}}$ for 2×2 matrix factorizations of the ADE minimal models. The results are therefore consistent with the argument presented above. Note that, at least in the presently discussed form, such a computation cannot be generalized to higher dimensional matrix factorizations.

5. Using the method of CFT consistency conditions

As we have demonstrated, the results of the different approaches are in agreement, confirming the correctness of the method we introduced and establishing a link between these approaches. However, there is no reason why these general solutions should be consistent with all physical constraints. Therefore, we will now use the CFT consistency conditions to rederive the superpotential. This approach does not give the deformed matrices J_{def} and E_{def}, but on the other hand one gets all the correlators.

[2]We thank Nicholas Warner for helpful discussions.

The idea is to determine the values of the allowed correlators of the model by imposing consistency constraints. These constraints can be viewed as generalizations of the WDVV equations for the bulk [9]. Adding boundaries on the worldsheet, one gets additional sewing constraints which were derived in [10] for the case without integrated insertions. For the case where insertions of integrated boundary states are allowed, the consistency conditions were derived in [11]. In the following, we will give a short summary of the necessary steps to compute the superpotential. The input data are the matrix factorizations, the spectrum and the R-charges as defined in [36].

5.1 Two- and three-point functions

We begin by determining the values of the boundary three-point functions and the bulk-boundary two-point functions [4, 47]:

$$\langle \phi_i \psi_a^{AA} \rangle = \frac{1}{(2\pi i)^n} \oint \mathrm{d}^n x \frac{\phi_i \mathrm{STr}\left(\left((\partial Q^A)^{\wedge n} \right) \psi_a^{AA} \right)}{\partial_1 W \dots \partial_n W} \tag{5.1}$$

Here n is the number of variables in Q, ϕ_i are the elements of the bulk chiral ring $\frac{\mathbb{C}[x_i]}{\partial_i W}$, STr is the supertrace and the wedge product is

$$\left(\partial Q^A\right)^{\wedge n} = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{|\sigma|} \partial_{\sigma(1)} Q^A \dots \partial_{\sigma(n)} Q^A. \tag{5.2}$$

The boundary three-point function is:

$$\langle \psi_a^{AB} \psi_b^{BC} \psi_c^{CA} \rangle = \frac{1}{(2\pi i)^n} \oint \mathrm{d}^n x \frac{\mathrm{STr}\left((\partial Q^A)^{\wedge n} \psi_a^{AB} \psi_b^{BC} \psi_c^{CA} \right)}{\partial_1 W \dots \partial_n W} \tag{5.3}$$

5.2 Correlators and selection rules

In the second step we use a set of selection rules to find out which correlators are possibly non-vanishing. First define correlators with arbitrary numbers of bulk and boundary insertions:[3]

$$\begin{aligned} B_{a_0 \dots a_m; i_1 \dots i_m} &:= (-1)^{\tilde{a}_1 + \dots \tilde{a}_{m-1}} \left\langle \psi_{a_0} \psi_{a_1} P \int \psi_{a_2}^{(1)} \dots \psi_{a_{m-1}}^{(1)} \psi_{a_m} \int \phi_{i_1}^{(2)} \dots \phi_{i_n}^{(2)} \right\rangle \\ &= -\left\langle \phi_{i_1} \psi_{a_0} P \int \psi_{a_1}^{(1)} \dots \int \psi_{a_m}^{(1)} \int \phi_{i_2}^{(2)} \dots \int \phi_{i_n}^{(2)} \right\rangle. \end{aligned} \tag{5.4}$$

The integrated insertions of bulk and boundary fields are defined in [9, 11]. The second equality can be derived from Ward identities. Here we introduced the suspended grade of the boundary field ψ^a:

$$\tilde{a} := |\psi_a| + 1. \tag{5.5}$$

Furthermore we can introduce a metric on the boundary ring:

$$\omega_{ab} = \langle \psi_a \psi_b \rangle = (-1)^{\tilde{a}} B_{0ab} = (-1)^{\tilde{\omega}} (-1)^{\tilde{a}\tilde{b}} \omega_{ba}, \tag{5.6}$$

[3] From now on we will suppress the indices labelling the brane.

where the "0" stands for the insertion of the unit operator. It can be used to raise and lower indices:

$$B^{a}{}_{a_1\ldots a_m} := \omega^{ab} B_{ba_1\ldots a_m}. \tag{5.7}$$

The correlators (5.4) are cyclic in the boundary insertions:

$$B_{a_0\ldots a_m;i_1\ldots i_m} = (-1)^{\tilde{a}_m(\tilde{a}_0+\cdots+\tilde{a}_{m-1})} B_{a_m a_0\ldots a_{m-1};i_1\ldots i_m}. \tag{5.8}$$

Furthermore they are symmetric under permutations of the bulk indices.

It turns out to be convenient to define

$$B_{a_0 a_1} = B_{a_0} = B_i = 0. \tag{5.9}$$

A correlator (5.4) satisfies the following selection rules:

- Charge selection rule.

  The R-charge of the correlator must be equal to the background charge. For the minimal models the background charge is given by:

  $$q_b = \frac{k-2}{k}, \tag{5.10}$$

  where k is the dual Coxeter number.

  The R-charges of the integrated insertions are

  $$q^I_\psi = q_\psi - 1, \tag{5.11}$$
  $$q^I_\phi = q_\phi - 2. \tag{5.12}$$

  For minimal models these charges are always negative.

- The correlators must have the same suspended degree as the boundary metric:

  $$B_{a_0 a_1\ldots a_m;i_1\ldots i_n} = 0 \qquad \text{unless } \tilde{a}_0 + \cdots + \tilde{a}_m = \tilde{\omega}. \tag{5.13}$$

- Insertions of the unit operator are only allowed if there are no integrated insertions.

  $$B_{0a_1\ldots a_m;i_1\ldots i_n} = 0 \qquad \text{for } m \geq 3 \text{ or } n \geq 1. \tag{5.14}$$

5.3 Consistency conditions

Next, we determine the correlators by imposing the generalized WDVV-constraints. For this we introduce generating functions for the bulk perturbations, given set of boundary insertions, which satisfy the property

$$B_{a_0\ldots a_m;i_1\ldots i_n} = \partial_{i_1}\ldots\partial_{i_n}\mathcal{F}_{a_0\ldots a_m}(t)|_{t=0}. \tag{5.15}$$

For $m \geq 2$ the generating functions are given by

$$\mathcal{F}_{a_0 \dots a_m} = (-1)^{\tilde{a}_1 + \dots + \tilde{a}_{m-1}} \left\langle \psi_{a_0} \psi_{a_1} P \int \psi_{a_2} \dots \int \psi_{a_{m-1}} \psi_{a_m} e^{\sum_p t_p \int \phi_p} \right\rangle \tag{5.16}$$

$$= (-1)^{\tilde{a}_1 + \dots + \tilde{a}_{m-1}} \sum_{N_0 \dots N_{h_c-1}=0}^{\infty} \prod_{p=0}^{h_c-1} \frac{t_p^{N_p}}{N_p!} \left\langle \psi_{a_0} \psi_{a_1} P \int \psi_{a_2} \dots \int \psi_{a_{m-1}} \psi_{a_m} \left[\int \phi_p \right]^{N_p} \right\rangle,$$

where h_c is the dimension of the bulk chiral ring. For $m = 0$ and $m = 1$ we define $\mathcal{F}_a(t)$ and $\mathcal{F}_{ab}(t)$ by

$$\partial_i \mathcal{F}_a(t) = -\langle \phi_i \psi_a e^{\sum_p t_p \int \phi_p} \rangle, \tag{5.17}$$

$$\partial_i \mathcal{F}_{ab}(t) = -\langle \phi_i \psi_a P \int \psi_b e^{\sum_p t_p \int \phi_p} \rangle. \tag{5.18}$$

Now we impose the consistency constraints on the correlators. We start with the A_∞-relations:

$$\sum_{\substack{k,j=0 \\ k \leq j}}^{m} (-1)^{\tilde{a}_1 + \dots \tilde{a}_k} \mathcal{F}^{a_0}{}_{a_1 \dots a_k c a_{j+1} \dots a_m}(t) \mathcal{F}^{c}{}_{a_{k+1} \dots a_j}(t) = 0. \tag{5.19}$$

The second consistency condition is the bulk-boundary crossing constraint:

$$\partial_i \partial_j \partial_k \mathcal{F}(t) \eta^{kl} \partial_l \mathcal{F}_{a_0 \dots a_m}(t) = \sum_{0 \leq m_1 \leq \dots m_4 \leq m} (-1)^{\tilde{a}_{m+1} + \dots + \tilde{a}_{m_3}} \mathcal{F}_{a_0 \dots a_{m_1} b a_{m_2+1} \dots a_{m_3} c a_{m_4+1} \dots a_m}$$
$$\times \partial_i \mathcal{F}^{b}{}_{a_{m_1+1} \dots a_{m_2}} \partial_j \mathcal{F}^{c}{}_{a_{m_3+1} \dots a_{m_4}}. \tag{5.20}$$

Here, $\mathcal{F}(t)$ is the bulk WDVV potential and η^{kl} is the inverse of the topological metric $\eta_{kl} = \langle \phi_0 \phi_k \phi_l \rangle$.

These two constraints alone do not determine the values of all the amplitudes. For the A-minimal models one can use the Cardy constraint to fix all the correlators. The Cardy constraint was derived in [11] and takes the form

$$\partial_i \mathcal{F}_{a_0 \dots a_n} \eta^{ij} \partial_j \mathcal{F}_{b_0 \dots b_n} = \sum_{\substack{0 \leq n_1 \leq n_2 \leq n \\ 0 \leq m_1 \leq m_2 \leq m}} (-1)^{s + \tilde{c}_1 + \tilde{c}_2} \omega^{c_1 d_1} \omega^{c_2 d_2} \mathcal{F}_{a_0 \dots a_{n_1} d_1 b_{m_1+1} \dots b_{m_2} c_2 a_{n_2+1} \dots a_n}$$
$$\times \mathcal{F}_{b_0 \dots b_{m_1} c_1 a_{n_1+1} \dots a_{n_2} d_2 b_{m_2+1} \dots b_m} \tag{5.21}$$

It turns out that this sewing constraint is only valid in the case of the A-models. (We give some new results for these models in appendix C.) For other minimal models the Cardy-condition turns out to be in contradiction with (5.19) and (5.20). This does not come as a surprise since in the derivation of the above formula it was assumed that the annulus amplitude is metric independent, which is not necessarily true due to the possible existence of anomalies in the Q-symmetry. Still, it had been hoped that the Cardy constraint eq. (5.21) could nevertheless be imposed to get the topological part of the amplitude, but our results show that this is not the case.

JHEP05(2006)064

5.4 Calculation of the effective superpotential

With these preparations we are now able to compute the superpotential. When the values of all the correlators have been fixed, the effective superpotential is given by the following expression:

$$\mathcal{W}_{\text{eff}}(s,t) = \sum_{m\geq 1} \frac{1}{m!} s_{a_m} \dots s_{a_1} \mathcal{A}_{a_1 \dots a_m}, \tag{5.22}$$

where

$$\mathcal{A}_{a_1 \dots a_m} := (m-1)! \mathcal{F}_{(a_1 \dots a_m)} := \frac{1}{m} \sum_{\sigma \in S_m} \eta(\sigma; a_1, \dots a_m) \mathcal{F}_{a_{\sigma(1)} \dots a_{\sigma(m)}}(t) \tag{5.23}$$

Note that the parameters s_i are super-commuting since those associated to even boundary states are anticommuting. The sign factor η comes from permuting the boundary operators.

5.5 Deformations and the bulk-boundary crossing constraint

As noted before, we can not rely on eq. (5.21) for models other than the A-minimal models. Therefore some of the amplitudes will not be fixed by the A_∞– and the bulk-boundary crossing relations. We now give a prescription how to determine the values of all the correlators with bulk insertions by merging the generalized WDVV equations with the methods from section 2. The procedure can be cast into the following recipe:

- Without bulk insertions, the bulk-boundary crossing constraint does not contain any extra information. This is why we may assume that in this case the superpotential coming from the A_∞-relations and the one coming from the versal deformation of the Q-operator will agree. We thus start by computing the superpotentials for the bulk parameters set to zero with either method.

- The superpotential obtained from solving the A_∞-relations will contain as undetermined parameters non-linear functions of the unknown correlators. Comparing with the result of the Massey product algorithm one obtains an overdetermined system of non-linear equations for the correlators which has, at least for the examples we checked, a unique solution.

- Next we set up the WDVV-constraints with the bulk parameters turned on and use the boundary correlators whose values we found by comparison of the two superpotentials as input for solving the equations. It turns out that this is enough to uniquely determine all the values of the remaining correlators and the complete superpotential is fixed up to reparameterizations.

5.6 Comparing results

For the E_6 case, we find

$$\begin{aligned}\overline{\mathcal{W}}_{\rm CFT}(u;t) = {} & u_4^3u_1 + \frac{3}{4}u_4^2u_1^5 + \frac{1}{8}u_4u_1^9 + \frac{5}{832}u_1^{13} \\ & + \frac{1}{2}u_1\left(t_{12} - \frac{1}{2}t_6^2 - \frac{1}{2}t_5^2t_2 - \frac{1}{2}t_8t_2^2 + \frac{1}{6}t_6t_2^3 + \frac{1}{8}\left(t_6^2 - t_6t_2^3 + \frac{1}{4}t_2^6\right)\right) \\ & + \frac{1}{2}\left(u_4 + \frac{1}{8}u_1^4\right)\left(t_9 - t_5t_2^2\right) + \frac{1}{4}\left(u_4u_1^3 + \frac{3}{28}u_1^7\right)\left(t_6 - \frac{1}{2}t_2^3\right), \end{aligned} \tag{5.24}$$

From the point of view of the consistency constraints it is natural to use the "flat" parameters t_i since the bulk-boundary crossing constraint contains the bulk prepotential $\mathcal{F}(t)$ [9]. We can now make a field redefinition of the open string parameters in order to get maximum agreement with (3.12). This yields

$$\begin{aligned}\mathcal{W}_{\rm CFT}(u;s) = {} & \frac{5}{64\cdot 13}u_1^{13} + \frac{1}{8}u_4u_1^9 + \frac{3}{4}u_4^2u_1^5 + u_4^3u_1 + u_1\left(s_{12} + \frac{1}{4}s_6^2\right) \\ & + s_9u_4 + \frac{1}{8}s_9u_1^4 + \frac{1}{2}s_6u_4u_1^3 + \frac{3}{56}s_6u_1^7. \end{aligned} \tag{5.25}$$

where the s_i are given in eq. (3.13). Here only the A_∞- and Crossing constraint were used, not the Cardy equation. This solution corresponds to eq. (3.12) with some deformation parameters set to zero,

$$\mathcal{W}_{\rm CFT}(u;s) = \mathcal{W}_{\rm Massey}(u; s_2 = 0, s_5 = 0, s_6, s_8 = 0, s_9, s_{12}).$$

Thus, incorporating the bulk-boundary crossing constraint leads to a reduced version of the effective superpotential. This may be explained as follows: If one deforms the matrix factorization one only captures the A_∞-structure. The bulk-boundary crossing equations impose an additional constraint on the superpotential.[4] This additional interaction between bulk and boundary is incorporated in a mathematical structure termed Open Closed Homotopy Algebra (OCHA) which has recently been introduced in the literature [31].

6. Further results

6.1 The exceptional singularities

In this section we list some more results for the superpotentials of the minimal models of types E_6, E_7, E_8. For simplicity, all bulk parameters are set to 0.

First, we consider the the self-dual matrix factorization M_3 for E_6 given in (A.3) where we choose $\varepsilon = -1$ in order to have only real entries in the matrix. The fermionic spectrum can be read off from table (A.8). There are four fermionic states and their deformation parameters u_i have charges $\{1, 3, 4, 6\}$ and the four corresponding polynomials f_i of the ring $k[u_i]/(f_i)$ have charges $\{12, 10, 9, 7\}$.

$$f_1 = -u_6^2 + u_4^3 - u_3^4 - 18u_4u_3^2u_1^2 + 7u_6u_3u_1^3 - 6u_4^2u_1^4 - 26u_3^2u_1^6 - 9u_4u_1^8 - 3u_1^{12}$$

[4] In technical terms, this means that the bulk-boundary crossing constraint sets certain correlators to zero which are left undetermined when only imposing the A_∞ relations.

$$
\begin{aligned}
f_2 &= 4u_3^3u_1 + 12u_4u_3u_1^3 + 8u_3u_1^7 \\
f_3 &= 3u_4^2u_1 - 6u_3^2u_1^3 - u_1^9 \\
f_4 &= -2u_6u_1 + 4u_3u_1^4
\end{aligned} \tag{6.1}
$$

These polynomials can be integrated to the effective superpotential

$$
\begin{aligned}
\mathcal{W}_{\text{eff}}(u) = &-u_6^2u_1 + u_4^3u_1 - u_3^4u_1 - 6u_4u_3^2u_1^3 + 4u_6u_3u_1^4 \\
&-8u_3^2u_1^7 - u_4u_1^9 - \frac{5}{13}u_1^{13}.
\end{aligned} \tag{6.2}
$$

Our last example for the E_6 model is the factorization M_4 given in eq. (A.4), where, once again, we set $\varepsilon = -1$. We find the following expression for the superpotential:

$$
\begin{aligned}
\mathcal{W}_{\text{eff}}(u) = &\frac{5}{1664}u_1^{13} - \frac{1}{4}u_1^8u_5 + \frac{3}{4}u_1^6u_2u_5 - 2u_2^4u_5 + 2u_2u_3^2u_5 - u_4^2u_5 \\
&-2u_3u_5^2 + u_1^3\left(-u_2u_3u_5 - \frac{1}{2}u_5^2\right) + u_1^9\left(-\frac{1}{16}u_2^2 + \frac{1}{16}u_4 + \frac{1}{16}v_4\right) \\
&-u_4u_5v_4 + u_5v_4^2 + u_1^4\left(-\frac{3}{2}u_2^2u_5 + u_4u_5 + \frac{1}{2}u_5v_4\right) \\
&+u_1^5\Big(-\frac{3}{8}u_2^4 - \frac{3}{8}u_4^2 - \frac{1}{2}u_3u_5 + u_2^2\left(\frac{3}{4}u_4 + \frac{3}{4}v_4\right) - \frac{3}{4}u_4v_4 - \frac{3}{8}v_4^2\Big) \\
&+u_1^2\left(2u_2^3u_5 - u_3^2u_5 + u_2\left(-2u_4u_5 - u_5v_4\right)\right) + u_2^2\left(3u_4u_5 + 3v_4u_5\right) \\
&+u_1\Big(-\frac{1}{2}u_2^6 + \frac{1}{2}u_4^3 + u_2u_5^2 + \frac{3}{2}u_4^2v_4 + \frac{3}{2}u_4v_4^2 + \frac{1}{2}v_4^3 \\
&+u_2^4\left(\frac{3}{2}u_4 + \frac{3}{2}v_4\right) + u_3\left(u_4u_5 + 2u_5v_4\right) \\
&+u_2^2\left(-\frac{3}{2}u_4^2 + u_3u_5 - 3u_4v_4 - \frac{3}{2}v_4^2\right)\Big).
\end{aligned} \tag{6.3}
$$

With (3.12), (6.2) and (6.3) we have actually given the superpotentials of five of the six branes since a brane yields the same superpotential as its anti-brane.

For the E_7 singularity with $W = x^3 + xy^3 + z^2$ we consider its simplest factorization, a self-dual one,

$$
E = J = \begin{pmatrix} z & x \\ x^2 + y^3 & -z \end{pmatrix}. \tag{6.4}
$$

The odd spectrum consists of three states with charges $\{0, 8, 16\}$ to which we associate parameters u_i with charges $\{1, 5, 9\}$. The deformed matrix is given by:

$$
E_{\text{def}} = \begin{pmatrix} z + u_1y^2 + u_5y + u_9 \\ y^3 + x^2 - u_1^2xy - 2u_1u_5x + u_1^4y^2 + 4u_1^3u_5y - 8u_1^3u_9 + u_1^6x - 2u_1^8y + 20u_1^7u_5 - 11u_1^{12} \\ x + u_1^2y + 2u_1u_5 - u_1^6 \\ -z + u_1y^2 + u_5y + u_9 \end{pmatrix} \tag{6.5}
$$

The polynomials defining the deformation ring are:

$$
f_1 = -u_9^2 - 16u_1^4u_5u_9 + 40u_1^8u_5^2 + 8u_1^9u_9 - 42u_1^{13}u_5 + 11u_1^{18}
$$

$$
\begin{aligned}
f_2 &= -2u_5u_9 + 8u_1^4u_5^2 - 8u_1^5u_9 + 12u_1^9u_5 - 9u_1^{14} \\
f_3 &= -u_5^2 - 2u_1u_9 + 6u_1^5u_5 - 3u_1^{10}
\end{aligned} \tag{6.6}
$$

These polynomials of degrees $\{18, 14, 10\}$ are easily integrated to the corresponding effective superpotential of degree 19,

$$
\mathcal{W}_{\text{eff}}(u) = -\frac{55}{19}u_1^{19} + 12u_1^{14}u_5 - 15u_1^9u_5^2 + 5u_1^4u_5^3 - 3u_1^{10}u_9 + 6u_1^5u_5u_9 - u_5^2u_9 - u_1u_9^2. \tag{6.7}
$$

This can be related as follows to the coset model

$$
\frac{E_7}{E_6 \times \mathrm{U}(1)}.
$$

From [35] we take the expression for the coset Landau-Ginzburg potential:

$$
\begin{aligned}
W(x,y,z) = &\frac{1016644}{817887699}x^{19} + \frac{33326}{177147}x^{14}y + \frac{266}{6561}x^{10}z + \frac{16850}{2187}x^9y^2 \\
&+\frac{80}{27}x^5yz + \frac{124}{9}x^4y^3 + xz^2 + \frac{3}{2}y^2z
\end{aligned} \tag{6.8}
$$

And indeed, this expression can be obtained from (6.7) by a field redefinition.

Finally we also give an example for the E_8 model. The Landau-Ginzburg potential for this model is $W = x^3 + y^5 + z^2$ and its dual Coxeter number is $k = 30$. The simplest matrix factorization has rank 4:

$$
E = J = \begin{pmatrix} z & 0 & x & y \\ 0 & z & y^4 & -x^2 \\ x^2 & y & -z & 0 \\ y^4 & -x & 0 & -z \end{pmatrix} \tag{6.9}
$$

The corresponding superpotential is

$$
\begin{aligned}
\mathcal{W}_{\text{eff}} = &-\frac{11}{31}u_1^{31} + u_1^{25}u_6 - 10u_1^{19}u_6^2 + 45u_1^{13}u_6^3 - 55u_1^7u_6^4 - u_1u_6^5 + 3u_1^{21}u_{10} \\
&-15u_1^{15}u_6u_{10} + 15u_1^9u_6^2u_{10} + 10u_1^3u_6^3u_{10} - u_1u_{10}^3 - 3u_1^{16}u_{15} \\
&+10u_1^{10}u_6u_{15} - 10u_1^4u_6^2u_{15} - u_1u_{15}^2 + 3u_1^{21}u_{10} - u_1u_{10}^3 - 3u_1^{16}u_{15}.
\end{aligned} \tag{6.10}
$$

We cannot relate this result to a coset model, since such models do not exist for E_8. This supports the conjecture that matrix factorizations of rank greater than 2 can not be related to such coset models.

7. Conclusions and open questions

In this paper we discussed various methods to calculate the effective superpotential for ADE minimal models with D-branes. The method of computing formal moduli provides a very efficient and elegant method to calculate the effective superpotential. It circumvents the problem that the Cardy condition as given in [11] does not hold in the general case, which had been uncertain before. We were able to calculate various examples of effective superpotentials, most of which would have required an exceedingly high amount of

JHEP05(2006)064

computing time if tackled by implementing the consistency constraints. Superpotentials of matrix factorizations whose spectrum contained more than 4 (even or odd) states were not calculated since computing time quickly increases as more states are added. The deformation calculus using the Massey products as implemented in SINGULAR allowed us to go further than that and compute for example the potentials for all E_6 branes except the most complicated one, for which the calculation exceeded the powers of an ordinary PC.

As always, a number of open questions remain. One issue is the deformation of the matrix factorization by bosonic states. In the context of obstruction theory the fermionic states provide the deformations and the bosonic states give the obstructions. In this formalism the deformations with even states can not be computed. From the point of view of the CFT constraints, even and odd states are treated on equal footing, the only difference being that the even states are associated to anticommuting deformation parameters whereas the parameters for the odd states are commuting. We will leave a possible generalization of the Massey product formalism to incorporate the even deformations as a future task.

An even more interesting question would be the generalization to Calabi-Yau manifolds which is of course the ultimate goal. In principle the computation should proceed just as in the ADE-case but then we also have marginal deformations. The results of [17] may be useful for a better understanding of this problem.

In this paper we dealt only with the boundary preserving sector but it would be interesting to consider systems with various D-branes. Some rather naïve experimentation with the A-minimal models leads to the conclusion that a generalization of the algorithm to the case of multiple D-branes is straightforward although the technical complexity grows quickly. It seems likely that this is a powerful framework to describe phenomena like tachyon condensation and bound state formation of D-branes.

Another interesting issue is the relation to coset models. We have been very brief about the relations between the superpotentials and we only gave an idea of how the matrix factorizations could be recovered from these models. It may be useful to investigate this relation further. In particular, one should try to find out whether also matrix factorizations of rank higher than two have connections to coset models.

Last but not least it remains to be understood why the formulation of the Cardy constraint from [11] works perfectly for the A-minimal models and fails for other cases. Although it generically leads to contradictions, the information contained in the Cardy condition does not seem to be entirely wrong (see for example appendix A in [13]). It may be worthwhile to look for possible modifications of this constraint.

Acknowledgments

We are deeply grateful to our advisor Wolfgang Lerche for introducing us to the topic of this paper and for many useful explanations and enlightening discussions.

We would also like to thank Gerhard Pfister, Johannes Walcher and Nicholas Warner for helpful comments.

A. E_6 — three-variable case

A.1 Matrix factorizations

These results were already given in [21]. The superpotential is:

$$W = x^3 + y^4 + \varepsilon\, z^2, \tag{A.1}$$

where $\varepsilon = \pm 1$. Introducing this parameter is just for calculational convenience since we can always choose the matrix factorization to be real. The choice of sign has no influence on the dimensions and charges of the spectrum or the form of the superpotential. The following matrices satisfy $W = E_i \cdot J_i$:

$$E_1 = J_2 = \begin{pmatrix} -y^2 + \sqrt{-\varepsilon}z & x \\ x & y^2 + \sqrt{-\varepsilon}z \end{pmatrix}$$

$$E_2 = J_1 = \begin{pmatrix} -y^2 - \sqrt{-\varepsilon}z & x \\ x & y^2 - \sqrt{-\varepsilon}z \end{pmatrix} \tag{A.2}$$

$$E_3 = J_3 = \begin{pmatrix} -\sqrt{\varepsilon}z & 0 & x^2 & y^3 \\ 0 & -\sqrt{\varepsilon}z & y & -x \\ x & y^3 & \sqrt{\varepsilon}z & 0 \\ y & -x^2 & 0 & -\sqrt{\varepsilon}z \end{pmatrix} \tag{A.3}$$

$$E_4 = J_5 = \begin{pmatrix} -y^2 + \sqrt{-\varepsilon} & 0 & xy & x \\ -xy & y^2 + \sqrt{-\varepsilon} & x^2 & 0 \\ 0 & x & \sqrt{-\varepsilon}z & y \\ x^2 & -xy & y^3 & \sqrt{-\varepsilon}z \end{pmatrix} \tag{A.4}$$

$$E_5 = J_4 = \begin{pmatrix} -y^2 - \sqrt{-\varepsilon}z & 0 & xy & x \\ -xy & y^2 - \sqrt{-\varepsilon}z & x^2 & 0 \\ 0 & x & -\sqrt{-\varepsilon}z & y \\ x^2 & -xy & y^3 & -\sqrt{-\varepsilon}z \end{pmatrix} \tag{A.5}$$

$$E_6 = \begin{pmatrix} -\sqrt{-\varepsilon}z & -y^2 & xy & 0 & x^2 & 0 \\ -y^2 & -\sqrt{-\varepsilon}z & 0 & 0 & 0 & x \\ 0 & 0 & -\sqrt{-\varepsilon}z & -x & 0 & y \\ 0 & xy & -x^2 & -\sqrt{-\varepsilon}z & y^3 & 0 \\ x & 0 & 0 & y & -\sqrt{-\varepsilon}z & 0 \\ 0 & x^2 & y^3 & 0 & xy^2 & -\sqrt{-\varepsilon}z \end{pmatrix} \tag{A.6}$$

$$J_6 = \begin{pmatrix} \sqrt{-\varepsilon}z & -y^2 & xy & 0 & x^2 & 0 \\ -y^2 & \sqrt{-\varepsilon}z & 0 & 0 & 0 & x \\ 0 & 0 & \sqrt{-\varepsilon}z & -x & 0 & y \\ 0 & xy & -x^2 & \sqrt{-\varepsilon}z & y^3 & 0 \\ x & 0 & 0 & y & \sqrt{-\varepsilon}z & 0 \\ 0 & x^2 & y^3 & 0 & xy^2 & \sqrt{-\varepsilon}z \end{pmatrix} \tag{A.7}$$

A.2 Spectrum

The spectrum for this model has already been discussed in [21]. There are six matrix factorizations, one for each node in the Dynkin diagram. We summarize the data of the boundary preserving spectrum in the following table:

Factorization	Rank	Spectrum bosonic	Spectrum fermionic
M_1	2	$0\,6$	$4\,10$
M_2	2	$0\,6$	$4\,10$
M_3	4	$0\,4\,6\,10$	$0\,4\,6\,10$
M_4	4	$0\,2\,4\,6^2\,8$	$2\,4^2\,6\,8\,10$
M_5	4	$0\,2\,4\,6^2\,8$	$2\,4^2\,6\,8\,10$
M_6	6	$0\,2^2\,4^3\,6^3\,8^2\,10$	$0\,2^2\,4^3\,6^3\,8^2\,10$

(A.8)

We labelled the matrix factorizations by M_i, the second column indicates the ranks of the matrices. The last two columns give the even and the odd spectrum. The numbers correspond to the R-charges multiplied by the number 12 — the Coxeter number of E_6 — and the exponents give the multiplicities. Note that there are six possible values of the charges, $q_\psi \in \{0, 2, 4, 6, 8, 10\}$. To fermionic states with these charges we associate fermionic deformation parameters u_i with charges $q_{u_i} = \frac{1}{2}(12 - q_{\psi_i}) \in \{6, 5, 4, 3, 2, 1\}$. We observe that, concerning the spectrum, there are two types of matrix factorizations. The factorizations M_1, M_2 and M_4, M_5 have the same spectra, respectively. These branes are the antibranes of each other. M_3 and M_6 belong to a different class of D-branes. The even spectrum is identical to the odd spectrum, these branes are "self-dual" — the brane is its own antibrane [4].

We observe that the highest charge, which is equal to the background charge, is always in the fermionic sector, whereas the charge 0 state is always in the bosonic sector. In fact, it is possible to determine the degree of the effective superpotential just by charge considerations. Remember the selection rules for the correlators given in section 5.2. The allowed correlators must have an R-charge which is equal to the background charge q_b. We can now determine the correlator with the maximal number of insertions. This correlator will have three unintegrated insertions of the field with the highest charge q_b, we will call this field ψ_b, and a certain number of integrated operators, which have negative charge. To get the maximum number of insertions one must use only insertions of $\int \psi_b$, which has charge $b - 1$, which is the least negative. From the charge selection rule we can now calculate the the number x of integrated insertions of the top element:

$$3 \cdot b + x \cdot (b - 1) \stackrel{!}{=} b \tag{A.9}$$

This yields $x = \frac{2b}{1-b}$. Now take into account that for the minimal models the background charge is related to the Coxeter number k via $b = \frac{k-2}{k}$. Inserting this, we find that the number of integrated insertions of the top-element is $k - 2$. Adding the three unintegrated operators, one finds that the top-correlator is a $k + 1$-point function. Looking more closely,

one also finds that the selection rule for the $\mathbb{Z}_2$-charge is satisfied and that this correlator will not vanish and contribute to the superpotential. The deformation parameter u associated to ψ_b has charge one and we will get a term u_1^{k+1} in the superpotential. Since the effective superpotential is a homogeneous polynomial, we conclude:

The effective superpotentials for the ADE minimal models always have degree $k+1$, where k is the Coxeter number.

B. E_6 — two-variable case

There is also a two-variable description for the E_6 minimal model:

$$W = x^3 + y^4 \tag{B.1}$$

For completeness we list all the matrix factorizations and give the complete spectrum. From the point of view of conformal field theory these two incarnations of the E_6-model correspond to two different GSO projections [4, 19, 48].

B.1 Matrix factorizations

We find the following matrix factorizations:

$$E_1 = J_2 = \begin{pmatrix} x & y \\ y^3 & -x^2 \end{pmatrix} \qquad E_2 = J_1 = \begin{pmatrix} x^2 & y \\ y^3 & -x \end{pmatrix} \tag{B.2}$$

$$E_3 = \begin{pmatrix} x & y^2 \\ y^2 & -x^2 \end{pmatrix} \qquad J_3 = \begin{pmatrix} x^2 & y^2 \\ y^2 & -x \end{pmatrix} \tag{B.3}$$

$$E_4 = J_5 = \begin{pmatrix} x & y & 0 \\ 0 & x & y \\ y^2 & 0 & x \end{pmatrix} \qquad E_5 = J_4 = \begin{pmatrix} x^2 & -xy & y^2 \\ y^3 & x^2 & -xy \\ -xy^2 & y^3 & x^2 \end{pmatrix} \tag{B.4}$$

$$E_6 = \begin{pmatrix} x & y^2 & 0 & 0 \\ y^2 & -x^2 & 0 & 0 \\ 0 & -xy & x^2 & y^2 \\ y & 0 & y^2 & -x \end{pmatrix} \quad J_6 = \begin{pmatrix} x^2 & y^2 & 0 & 0 \\ y^2 & -x & 0 & 0 \\ 0 & -y & x & y^2 \\ xy & 0 & y^2 & -x^2 \end{pmatrix} \tag{B.5}$$

B.2 Spectrum

For the even spectrum we find:

	1	2	3	4	5	6
1	0 10	4 6	3 7	1 3 5	5 7 9	2 4 6 8
2	4 6	0 10	3 7	5 7 9	1 3 5	2 4 6 8
3	3 7	3 7	0 4 6 10	2 4 6 8	2 4 6 8	$1\,3\,5^2\,7\,9$
4	5 7 9	1 3 5	2 4 6 8	0 2 4 6 8 10	$2\,4^2\,6^2\,8$	$1\,3^2\,5^2\,7^2\,9$
5	1 3 5	5 7 9	2 4 6 8	$2\,4^2\,6^2\,8$	0 2 4 6 8 10	$1\,3^2\,5^2\,7^2\,9$
6	2 4 6 8	2 4 6 8	$1\,3\,5^2\,7\,9$	$1\,3^2\,5^2\,7^2\,9$	$1\,3^2\,5^2\,7^2\,9$	$0\,2^2\,4^3\,6^3\,8^2\,10$

(B.6)

The odd spectrum is summarized in the following table:

	1	2	3	4	5	6
1	4 6	0 10	3 7	5 7 9	1 3 5	2 4 6 8
2	0 10	4 6	3 7	1 3 5	5 7 9	2 5 6 8
3	3 7	3 7	0 4 6 10	2 4 6 8	2 4 6 8	$1\,3\,5^2\,7\,9$
4	1 3 5	5 7 9	2 4 6 8	$2\,4^2\,6^2\,8$	0 2 4 6 8 10	$1\,3^5\,5^2\,7^2\,9$
5	5 7 9	1 3 5	2 4 6 8	0 2 4 6 8 10	$2\,4^2\,6^2\,8$	$1\,3^2\,5^2\,7^2\,9$
6	2 4 6 8	2 4 6 8	$1\,3\,5^2\,7\,9$	$1\,3^2\,5^2\,7^2\,9$	$1\,3^2\,5^2\,7^2\,9$	$0\,2^2\,4^3\,6^3\,8^2\,10$

(B.7)

C. Some results for the A-series

The minimal models of the A-Series satisfy an additional constraint, the Cardy Condition [11]:

$$\partial_i \mathcal{F}_{a_0\ldots a_n}\eta^{ij}\partial_j\mathcal{F}_{b_0\ldots b_m} = \sum_{\substack{0\le n_1\le n_2\le n\\ 0\le m_1\le m_2\le m}} (-1)^{(\tilde{c}_1+\tilde{a}_0)(\tilde{c}_2+\tilde{b}_0)+\tilde{c_1}+\tilde{c_2}}\omega^{c_1d_1}\omega^{c_2d_2} \times\mathcal{F}_{a_0\ldots a_{n_1}d_1b_{m_1+1}\ldots m_2c_2a_{n_2+1}\ldots a_n}\mathcal{F}_{b_0\ldots b_{m_1}c_1a_{n_1+1}\ldots a_{n_2}d_2b_{m_2+1}\ldots b_m} \tag{C.1}$$

Setting $t=0$ and $m=0, n=0$ one recovers the CFT Cardy constraint, which is satisfied for all the minimal models. The full constraint in the form given above is only satisfied for the A-Series. For the generic case it has to be replaced by the Quantum A_∞ Structure [12, 13]. The Cardy constraint fixes the reparameterization freedom of the superpotential. The superpotential for the A_k-model is:

$$W^{(k+2)}(x) = \frac{x^{k+2}}{k+2}, \tag{C.2}$$

where the exponents in brackets give the degrees of the polynomials. The matrix factorizations are:

$$W^{k+2}(x) = E^{\kappa+1}(x)J^{k+1-\kappa}(x), \qquad \kappa = 0,\ldots,[k/2] \tag{C.3}$$

We denote by h the greatest common denominator of E and J, i.e $E^{\kappa+1}(x) = p(x)h^{\ell+1}(x)$ and $J^{k+1-\kappa}(x) = q(x)h^{\ell+1}(x)$. The pair (k,ℓ) then uniquely labels the D-brane we consider. For the A_3-model with $W=\frac{x^5}{5}$ and $(k,\ell)=(3,1)$ only one incarnation of the superpotential satisfies the Cardy constraint because this condition fixes the reparameterization freedom of the effective superpotential [11]:

$$\begin{aligned}\mathcal{W}_{\text{eff}}(t,u) = &-\frac{1}{5}\left(\frac{u_1^6}{6}+u_1^4u_2+\frac{3}{2}u_1^2u_2^2+\frac{u_2^3}{3}\right)-t_2\left(-\frac{u_1^4}{4}-u_1^2u_2-\frac{u_2^2}{2}\right)+t_3\left(\frac{u_1^3}{3}+u_2u_2\right)\\ &-\left(t_4-t_2^2\right)\left(-\frac{u_1^2}{2}-u_2\right)+\left(t_5-t_2t_3\right)u_1\end{aligned} \tag{C.4}$$

It turns out that the Cardy constraint is only valid in the boundary preserving sector when bulk perturbations are turned on. But this information is enough to obtain superpotentials

for the boundary changing sector. We now state some new results for superpotentials in the boundary changing sector. For the model $(3,1)\oplus(3,0)$, we find:

$$\begin{aligned}\mathcal{W}_{\text{eff}} = &-\frac{1}{5}\Big(\frac{u_1^6}{6}+\frac{w_1^6}{6}+u_1^4u_2+\frac{3}{2}u_2^2u_1^2+\frac{u_2^3}{3}+2u_2u_1v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}+u_1^3v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}+\frac{1}{2}v_{\frac{3}{2}}^2\tilde{v}_{\frac{3}{2}}^2\\ &+u_2v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}w_1+u_1^2v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}w_1+u_1v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}w_1^2+v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}w_1^3\Big)\\ &-t_2\left(-\frac{u_1^4}{4}-\frac{w_1^4}{4}-u_1^2u_2-\frac{u_2^2}{2}-u_1v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}-v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}w_1\right)\\ &+t_3\left(\frac{u_1^3}{3}+\frac{w_1^3}{3}+u_1u_2+v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}\right)-\left(t_4-t_2^2\right)\left(-\frac{u_1^2}{2}-\frac{w_1^2}{2}-u_2\right)\\ &+(t_5-t_2t_3)(-u_1-w_1)\end{aligned}\tag{C.5}$$

Here we chose the convention that the indices correspond to the charges, the parameters u and w are related to odd states on the branes labeled with $\ell=1$ and $\ell=0$, respectively, whereas the parameters v and $\tilde{v}$ are related to the open string states stretching between the two branes.

For the model $(4,1)\oplus(4,0)$ we find:

$$\begin{aligned}\mathcal{W}_{\text{eff}} = &-\frac{1}{6}\Big(\frac{u_1^7}{7}+\frac{w_1^7}{7}+2u_1^3u_2^2+u_1^5u_2+u_1u_2^3+u_1^4v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}+u_1^3v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}w_1+3u_1^2u_2v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}+u_1^2v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}w_1\\ &+u_1v_{\frac{3}{2}}^2\tilde{v}_{\frac{3}{2}}^2+u_1v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}w_1^3+u_2^2v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}+2u_2u_1v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}w_1+u_2v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}w_1^2+v_{\frac{3}{2}}^2\tilde{v}_{\frac{3}{2}}w_1+v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}w_1^4\Big)\\ &+t_2\left(\frac{u_1^5}{5}+\frac{w_1^5}{5}+u_1^3u_2+u_1^2v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}+u_1u_2^2+u_1v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}w_1+u_2v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}+v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}w_1^2\right)\\ &+t_3\left(\frac{u_1^4}{4}+\frac{w_1^4}{4}+u_1^2u_2+\frac{u_2^2}{2}+u_1v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}+v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}w_1\right)\\ &+\left(t_4-\frac{3}{2}t_2^2\right)\left(\frac{u_1^3}{3}+\frac{w_1^3}{3}+u_1u_2+v_{\frac{3}{2}}\tilde{v}_{\frac{3}{2}}\right)\\ &+(t_5-2t_2t_3)\left(\frac{u_1^2}{2}+\frac{w_1^2}{2}+u_2\right)+\left(t_6-\frac{1}{2}t_3^2-t_2t_4+\frac{1}{3}t_2^3\right)(u_1+w_1)\end{aligned}\tag{C.6}$$

Finally, we give the result for $(4,0)\oplus(4,0)$

$$\begin{aligned}\mathcal{W}_{\text{eff}} = &-\frac{1}{6}\Big(\frac{u_1^7}{7}+\frac{w_1^7}{7}+u_1^5v_1\tilde{v}_8+u_1^4v_1\tilde{v}_8w_1+2u_1^3v_1^2\tilde{v}_8^2+u_1^3v_1\tilde{v}_8w_1^2+u_1^2v_1\tilde{v}_8w_1^3+3u_1^2v_1^2\tilde{v}_8^2w_1\\ &+u_1v_1^3\tilde{v}_8^3+3u_1v_1^2\tilde{v}_8w_1^2+u_1v_1\tilde{v}_8w_1^4+v_1\tilde{v}_8w_1^5+2v_1^2\tilde{v}_8^2w_1^3+v_1^3\tilde{v}_8^3w_1\Big)\\ &+t_2\left(\frac{u_1^5}{5}+\frac{w_1^5}{5}+u_1^3v_1\tilde{v}_8+u_1^2v_1\tilde{v}_8w_1+u_1v_1^2\tilde{v}_8^2+u_1v_1\tilde{v}_8w_1^2+v_1^2\tilde{v}_8^2w_1+v_1\tilde{v}_8w_1^3\right)\\ &+t_3\left(\frac{u_1^4}{4}+\frac{w_1^4}{4}+u_1^2v_1\tilde{v}_8+u_1v_1\tilde{v}_8w_1+\frac{1}{2}v_1^2\tilde{v}_8^2+v_1\tilde{v}_8w_1^2\right)\\ &+\left(t_4-\frac{3}{2}t_2^2\right)\left(\frac{u_1^3}{3}+\frac{w_1^3}{3}+u_1v_1\tilde{v}_8+v_1\tilde{v}_8w_1\right)+(t_5-2t_2t_3)\left(\frac{u_1^2}{2}+\frac{w_1^2}{2}+v_1\tilde{v}_8\right)\end{aligned}$$

$$+\left(t_6-\frac{1}{2}t_3^2-t_2t_4+\frac{1}{3}t_2^3\right)(u_1+w_1) \tag{C.7}$$

All of these results can be obtained from a residue formula [16]:

$$\mathcal{W}_{\text{eff}}(t;u)=-\oint\frac{\mathrm{d}x}{2\pi i}\log(\det J(x;u))W(x;t), \tag{C.8}$$

where $J(x;u)$ is the matrix factorization of the system with linear odd deformations turned on and $W(x;t)$ is the bulk superpotential [9]. Note that this equation cannot be generalized to the multi-variable case in a straight forward manner. The fact that only linear deformations in the boundary parameters appear in $J(x;u)$ is a peculiarity of the A_k-models and does not hold in general. A possible hint for the generalization of this formula may be found in [35] where it was shown that in the context of coset models it is possible to obtain the superpotentials as period integrals over Calabi-Yau 4-folds which are fibrations of ALE singularities. In may prove worthwhile to investigate these results from the point of view of D-branes and matrix factorizations.

References

[1] A. Kapustin and Y. Li, *D-branes in Landau-Ginzburg models and algebraic geometry*, *JHEP* **12** (2003) 005 [hep-th/0210296].

[2] I. Brunner, M. Herbst, W. Lerche and B. Scheuner, *Landau-Ginzburg realization of open string TFT*, hep-th/0305133.

[3] A. Kapustin and Y. Li, *Topological correlators in Landau-Ginzburg models with boundaries*, *Adv. Theor. Math. Phys.* **7** (2004) 727 [hep-th/0305136].

[4] A. Kapustin and Y. Li, *D-branes in topological minimal models: the Landau-Ginzburg approach*, *JHEP* **07** (2004) 045 [hep-th/0306001].

[5] D. Eisenbud, *Homological algebra on a complete intersection with an application to group representations*, *Trans. Amer. Math. Soc* **260** (1980) 35.

[6] M. Kontsevich, unpublished.

[7] D. Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, math.AG/0302304.

[8] Y. Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*. Cambridge University Press, Cambridge 1990.

[9] R. Dijkgraaf, H.L. Verlinde and E.P. Verlinde, *Topological strings in $D<1$*, *Nucl. Phys.* **B 352** (1991) 59.

[10] C.I. Lazaroiu, *On the structure of open-closed topological field theory in two dimensions*, *Nucl. Phys.* **B 603** (2001) 497 [hep-th/0010269].

[11] M. Herbst, C.-I. Lazaroiu and W. Lerche, *Superpotentials, A_∞ relations and WDVV equations for open topological strings*, *JHEP* **02** (2005) 071 [hep-th/0402110].

[12] M. Herbst, *Quantum A_∞ structures for open-closed topological strings*, hep-th/0602018.

[13] M. Herbst, W. Lerche and D. Nemeschansky, *Instanton geometry and quantum A_∞ structure on the elliptic curve*, hep-th/0603085.

[14] K. Hori and J. Walcher, *D-branes from matrix factorizations*, *Comptes Rendus Physique* **5** (2004) 1061 [hep-th/0409204].

[15] K. Hori and J. Walcher, *F-term equations near Gepner points*, *JHEP* **01** (2005) 008 [hep-th/0404196].

[16] M. Herbst, C.-I. Lazaroiu and W. Lerche, *D-brane effective action and tachyon condensation in topological minimal models*, *JHEP* **03** (2005) 078 [hep-th/0405138].

[17] S.K. Ashok, E. Dell'Aquila, D.-E. Diaconescu and B. Florea, *Obstructed D-branes in Landau-Ginzburg orbifolds*, *Adv. Theor. Math. Phys.* **8** (2004) 427 [hep-th/0404167].

[18] I. Brunner, M.R. Gaberdiel and C.A. Keller, *Matrix factorisations and D-branes on K3*, hep-th/0603196.

[19] I. Brunner and M.R. Gaberdiel, *The matrix factorisations of the D-model*, *J. Phys.* **A 38** (2005) 7901 [hep-th/0506208].

[20] G.M. Greuel and K. Knörrer, *Einfache Kurvensingularitäten und Torsionsfreie Moduln*, *Math. Ann.* **270** (1985) 417.

[21] H. Kajiura, K. Saito and A. Takahashi, *Matrix factorizations and representations of quivers II: type ADE case*, math.AG/0511155.

[22] O.A. Laudal, *Formal moduli of algebraic structures*, *Lecture Notes in Mathematics* **754** (1978).

[23] O.A. Laudal, *Matric Massey products and formal moduli I*, *Lecture Notes in Mathematics* **1183** (1986) 218.

[24] A. Siqveland, *Matric Massey products and formal moduli local and global*, Ph.d. thesis, University of Oslo, Dept. of Mathematics, 1995.

[25] A. Siqveland, *The method of computing formal moduli*, *J. Alg.* **241** (2001) 292.

[26] A. Siqveland, *Global matric Massey products and the compactified jacobian of the E_6-singularity*, *J. Alg.* **241** (2001) 259.

[27] A. Siqveland, *Generalized matric Massey products for graded modules*, math.AG/0603425.

[28] E. Eriksen, *Connections and monodromy on modules*, Technical Report, University of Oslo, 1995.

[29] E. Eriksen and T.S. Gustavsen, *Computing connections on modules*, math.AG/0602616.

[30] E. Eriksen, *Connections on modules over simple curve singularities*, math.AG/0603259.

[31] H. Kajiura and J. Stasheff, *Open-closed homotopy algebra in mathematical physics*, *J. Math. Phys.* **47** (2006) 023506 [hep-th/0510118].

[32] W. Lerche and N.P. Warner, *Polytopes and solitons in integrable, $N = 2$ supersymmetric Landau-Ginzburg theories*, *Nucl. Phys.* **B 358** (1991) 571.

[33] T. Eguchi and S.-K. Yang, *A new description of the E_6 singularity*, *Phys. Lett.* **B 394** (1997) 315 [hep-th/9612086].

JHEP05(2006)064

[34] W. Lerche and N.P. Warner, *Exceptional sw geometry from ALE fibrations*, *Phys. Lett.* **B 423** (1998) 79 [hep-th/9608183].

[35] T. Eguchi, N.P. Warner and S.-K. Yang, *ADE singularities and coset models*, *Nucl. Phys.* **B 607** (2001) 3 [hep-th/0105194].

[36] J. Walcher, *Stability of Landau-Ginzburg branes*, *J. Math. Phys.* **46** (2005) 082305 [hep-th/0412274].

[37] G.-M. Greuel, G. Pfister and H. Schönemann, SINGULAR *3.0.1 — A computer algebra system for polynomial computations*, Center for Computer Algebra, University of Kaiserslautern, 2006. http://www.singular.uni-kl.de.

[38] G.-M. Greuel and G. Pfister, deform.lib. *A* SINGULAR *3.0.1 Library for computing deformations*, 2006.

[39] D. Fuchs and L. Lang, *Massey products and deformations*, q-alg/9602024.

[40] A. Fialowski, *Deformations of some infinite-dimensional Lie algebras*, *J. Math. Phys.* **31** (1990).

[41] M. Penkava and A.S. Schwarz, A_∞ *algebras and the cohomology of moduli spaces*, hep-th/9408064.

[42] M. Schlessinger, *Functors of artin rings*, *Trans. Am. Math. Soc.* **130** (1968) 208.

[43] A. Klemm, S. Theisen and M.G. Schmidt, *Correlation functions for topological Landau-Ginzburg models with* $c \leq 3$, *Int. J. Mod. Phys.* **A 7** (1992) 6215.

[44] M. Khovanov and L. Rozansky, *Matrix factorizations and link homology*, math.QA/0401268.

[45] M. Khovanov and L. Rozansky, *Matrix factorizations and link homology II*, math.QA/0505056.

[46] S. Gukov and J. Walcher, *Matrix factorizations and Kauffman homology*, hep-th/0512298.

[47] M. Herbst and C.-I. Lazaroiu, *Localization and traces in open-closed topological Landau-Ginzburg models*, *JHEP* **05** (2005) 044 [hep-th/0404184].

[48] I. Brunner and M.R. Gaberdiel, *Matrix factorisations and permutation branes*, *JHEP* **07** (2005) 012 [hep-th/0503207].

JHEP05(2006)064