Coordination and Competition in Resource-Constrained Channels

by

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Abstract

This thesis deals with five important ideas pertaining to supply chains and supply contracts: *coordination, flexibility in allocating profit, the push-pull boundary, the valuation of capacity,* and *cooperation versus competition and its effects on profit and prices.* Throughout the thesis, we focus on capacity-constrained supply channels, motivated by the fact that most real-world supply chains have physical or monetary constraints.

In the first part of this thesis, we show that when a supply channel is capacity-constrained and the constraint is tight, there is a set of linear wholesale price contracts that coordinates the channel while allowing the supplier to make a profit. We prove this for the one-supplier/one-newsvendor supply channel as well as the many-supplier/one-newsvendor channel configuration (with each supplier selling a unique product). We analyze how this set of wholesale prices changes as we change the channel's capacity constraint. We also explore conditions under which these channel-efficient linear wholesale price contracts result from the equilibrium behavior of a newsvendor procurement game. Our newsvendor procurement game generalizes the Stackelberg game introduced in Lariviere and Porteus (2001) to allow for multiple suppliers as well as a capacity constraint at the newsvendor. In order to convey the worst-case channel performance when these channel-efficient contracts are not used in equilibrium, we quantify the worst-case efficiency loss for the supply channel using a distribution-free method. We also identify the set of Pareto-dominated contracts in a negotiation setting. Furthermore, unlike the unconstrained setting, we show that in the constrained setting wholesale price contracts can be flexible in allocating the channel profit without necessarily sacrificing coordination. Finally, we find the set of risk-sharing contracts (such as buy-back and revenue-sharing contracts) that coordinate a constrained supply channel and contrast that set with the set of risk-sharing contracts that coordinate an unconstrained channel. We show that in a capacity constrained channel, even risk-sharing contracts gain extra flexibility because for any given level of risk, there is now a range of possible allocations of the system optimal profit between the supplier and retailer. (Without a capacity constraint, for any given level of risk, there is only one allowable allocation of channel optimal profit between the supplier and retailer.) In other words, in a capacity-constrained environment, using risk-sharing contracts, for any given level of risk, we show there is flexibility in allocating the channel optimal profit.

In the second part of this thesis, we consider a supply channel with a capacity constraint in which the retailer makes an order quantity decision that depends only on realized demand rather than a forecast, and instead the supplier is the newsvendor for the channel making a stocking decision based on a forecast. In other words, the retailer now 'pulls' inventory from the supplier as demand is realized which differs from the model in the first part of this thesis wherein the supplier 'pushes' inventory onto the retailer before the sales season begins. We find that for the new supply channel similar results hold. Namely, when the retailer is operating in 'pull-mode', there is a set of 'pull' wholesale price contracts that coordinates the channel while allowing the retailer to make a profit. We analyze how this set of wholesale prices changes as we change the channel's capacity constraint. We also explore conditions under which these channel-efficient 'pull' wholesale price contracts result from the equilibrium behavior of a newsvendor procurement game. Our newsvendor procurement game generalizes the Stackelberg game introduced in Cachon and Lariviere (2001) to allow for multiple retailers as well as a capacity constraint at the newsvendor. We assess the worst-case channel performance in equilibrium when these channel-efficient contracts are not selected. Furthermore, we identify the set of Pareto-dominated 'pull' contracts in a negotiation setting. Finally, we identify the wholesale price contracts that coordinate regardless of the supply chain's mode of operation.

In the third part of this thesis, we consider a supply channel, operating in 'push'-mode, with multiple suppliers selling differentiated products to one newsvendor with limited capacity, using wholesale price contracts. We show that both in a negotiation setting as well as in an equilibrium setting, with the suppliers selecting wholesale prices followed by the newsvendor choosing order quantities, each supplier incurs an endogenous price for their share of the newsvendor's capacity. We intrepret this price as the value of the newsvendor's capacity and analyze the capacity's price in both a negotiation and an equilibrium setting. Furthermore, we show that our capacity valuation technique can be applied to different supply chain settings by analyzing the capacity price for a different supply chain, operating in 'pull'-mode, with one supplier with limited capacity selling differentiated products to multiple retailers. Finally, we analyze the effects of collusion on prices and profits in both these settings.

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Chapter 1

Introduction and Contributions

This thesis deals with five important ideas pertaining to supply chains and supply contracts: *coordination*, *flexibility in allocating profit*, *the push-pull boundary*, *the valuation of capacity*, and *cooperation versus competition and its effects on profit and prices.* Throughout the thesis, we focus on capacity-constrained supply channels, motivated by the fact that most real-world supply chains have physical or monetary constraints.

■ 1.1 Coordinating a constrained channel

There is a wealth of supply contracts available that coordinate a newsvendor's decision for unconstrained supplier-retailer channels: buy-back contracts, revenue-sharing contracts, etc. (Cachon 2003) A contract *coordinates* the actions of a newsvendor *for* a supply channel if the contract causes the newsvendor to take actions when solving his *own* decision problem that are also optimal for the *channel.*[1] Our thesis shows that simpler contracts, namely linear wholesale price contracts (which are thought to be unable to coordinate a newsvendor's decision for unconstrained channels) can, in

[1]Sometimes we also say a contract *channel-coordinates* a newsvendor's decision. Therefore, *achieving coordination* for the channel equates to attaining channel optimality (and thus efficiency) when the newsvendor is allowed to decide for himself.

fact, coordinate a newsvendor's procurement decision for resource-constrained channels. This is relevant for supply channels in which capacity of some resource is limited. For example, shelf space at retail stores, seats on airlines, warehouse space, procurement budgets, time available for manufacturing, raw materials, etc. (Corsten 2006)

In this thesis, we also show how risk-sharing contracts such as buy-back and revenue-sharing coordinate the procurement decision of a resource-constrained newsvendor thereby generalizing the treatment of these contracts. But the primary insight we show in the first part of this thesis, is that if newsvendor capacity is a binding constraint, then a *set* of linear wholesale-price contracts can coordinate the procurement decision of a capacity-constrained newsvendor (when the supply chain operates in 'pull-mode').[2] Furthermore, this set includes wholesale prices that allow both the supplier and the newsvendor to profit.

■ 1.2 Additional flexibility in allocating profit without sacrificing coordination

In addition to coordination capability, another important feature of any supply contract is its flexibility in allocating profit while maintaining coordination (Cachon 2003). Buyback contracts and revenue sharing contracts in unconstrained channels are well known to have this advantage. But wholesale price contracts in unconstrained settings lack the flexibility in allocating channel profit while maintaining coordination.

1.2.1 Wholesale price contracts

Our thesis shows that when the channel is constrained, wholesale price contracts gain some flexibility (in allocating channel profit) while maintaining coordination.

[2] In addition to capacity being a binding constraint, the relative power of the parties and their competitive environments are also important for the wholesale-price contract to coordinate the actions of the newsvendor in practice. For example, even if the set of wholesale prices $\mathcal{W}(k)$ that coordinate the retailer's actions is enlarged beyond the supplier's marginal cost (due to the retailer's capacity constraint k), the retailer and supplier still need to agree upon some wholesale price in that set. Their outside-alternatives and the power in the supply channel could determine if some wholesale price in the set $\mathcal{W}(k)$ is acceptable for the parties involved.

1.2.2 Risk-sharing contracts

However, this extra gain in flexibility is not limited to wholesale price contracts. We also show that when the channel is constrained, buyback contracts also gain some flexibility. In particular, we show that buyback contracts gain a feature that they do not have in the unconstrained setting: the flexibility in allocating channel optimal profit, for any fixed level of risk.

■ 1.3 Push versus Pull

In the supply chain literature, the *'push-pull boundary'* in a supply chain refers to the point in the supply chain at which the supply chain's mode of operation switches from 'building to forecast' to 'reacting to realized demand' (Chopra and Lariviere 2005). This is also called The 'Fulcrum Point' by Martin Christopher; the BTF/BTO boundary (build to forecast/build to order).

In this thesis, we also show that our results on the coordination capability of wholesale price contracts are independent of where we place the 'push-pull boundary', i.e., the supply chain's mode of operation. We go on to highlight the wholesale price contracts that coordinate the supply chain regardless of mode of operation.

■ 1.4 A valuation technique for capacity

When considering multiple suppliers selling to a capacity constrained newsvendor (i.e., push-mode) or multiple retailers buying from a capacity constrained newsvendor (i.e., pull-mode), we analyze the capacity constraint's shadow price in *equilibrium*, motivated by the fact that the shadow price is important in 'valuing' the newsvendor's capacity.

■ 1.5 Cooperation versus Competition

The theme of cooperation versus competition runs throughout this thesis.

1.5.1 Equilibrium setting: 1 supplier/1 retailer

In addition to understanding the wholesale prices that coordinate a channel, we also analyze equilibrium settings (push and pull modes) to answer the question: when do wholesale price contracts coordinate in a competitive setting? Or equivalently, when is the equilibrium outcome equivalent to the outcome of the integrated firm. We find that when capacity is small enough, wholesale price contracts induce a channel profit that is as large as any cooperative or integrated outcome. In other words, for small enough channel capacities there are no gains to be had from integration or cooperation. We show this for a single supplier/single retailer supply chain and we show that this feature holds regardless of the supply chain's mode of operation (push or pull).

1.5.2 Competition versus Collusion within an echelon

We also consider a supply chain operating in push-mode with multiple suppliers selling to a retailer. We show that when there is supplier collusion, *every* supplier can be better off in terms of profit.

Furthermore, we consider a supply chain operating in pull-mode with multiple retailers buying from a single supplier. We show that when there is retailer collusion, *every* retailer can be better off in terms of profit.

■ 1.6 Organization of this Thesis

In Section 1.7, we provide an overview of the supply contracts literature, emphasizing the point that the literature has underestimated the coordination capability of wholesale price contracts for a constrained supply channel.

Chapter 2 focuses on single supplier/single retailer supply chain (with a capacity constraint) operating in push-mode (i.e., we have a make-to-stock retailer). We formally define coordination and analyze the set of coordinating wholesale price contracts. Then we consider an equilibrium setting, proving a unique equilibrium exists, and providing necessary and sufficient conditions for the equilibrium wholesale price contract to coordinate the supply chain. Furthermore, we analyze the set of Pareto-dominated contracts (contracts that should be avoided in both a negotiation and equilibrium setting). Finally, recognizing that the outcome can be inefficient in an equilibrium setting, we provide a worst-case efficiency bound for the equilibrium setting using a distribution-independent technique.

Chapter 3 uses the model presented in Chapter 2 in order to characterize the fractions of revenue and profit that can be allocated to a supplier and retailer when they use a coordinating contract. In particular, we show that wholesale price contracts have some flexibility in allocating the channel-optimal profit between the supplier and retailer (a flexibility that does not exist in the unconstrained setting). We conduct some comparative statics and analyze how this flexibility changes as a function of capacity and market demand. Then we move on and consider risk-sharing contracts for the same supply chain model. We show that they still coordinate a capacity-constrained channel and, furthermore, there is even more flexibility in the choice of risk-sharing contracts (for coordinating the channel). In particular, for any given level of risk (represented by the buyback parameter of a buyback contract), there is now flexibility in allocating the channel profit (without sacrificing coordination), a flexibility that is not present in the unconstrained setting.

Chapter 4 extends the push-mode supply chain model used in Chapter 2 and Chapter 3 by having multiple suppliers (each offering one differentiated good) instead of a single supplier. The chapter's focus is on coordination in this expanded setting. We provide conditions for wholesale price contracts to coordinate the channel. Then we consider an equilibrium setting and find conditions guaranteeing that the equilibrium wholesale price is a coordinating contract.

Chapter 5 considers the supply chain model in Chapter 2 but changes the mode of operation to pull and places the capacity constraint at the supplier. In other words, we have a make-to-order 'lean' retailer. In this chapter, we analyze the set of coordinating 'pull' wholesale price contracts, showing that when capacity is 'small enough', coordination becomes possible. Then we consider an equilibrium setting, proving that a unique equilibrium exists and providing necessary and sufficient conditions for the equilibrium 'pull' wholesale price contract to be a coordinating contract. We then analyze the set of Pareto-dominated 'pull' wholesale price contracts which are to be avoided in a negotiation setting (and will be avoided in an equilibrium setting). Recognizing that in an equilibrium setting we may not achieve the channel-optimal outcome, we analyze the worst-case efficiency loss using a distribution-independent technique. Finally, combining our results from Chapter 2 and Chapter 5, we describe the wholesale price contracts that coordinate the supply chain regardless of its mode of operation (push or pull).

Chapter 6 considers a more general supply chain model with multiple suppliers selling multiple goods to one capacity constrained retailer, extending the model in Chapter 2. The supply chain operates in push-mode (i.e., the retailer is a make-to-stock retailer). We analyze the retailer's order decision and derive an endogenous price for the retailer's capacity, i.e., the retailer's shadow price. Focusing on an equilibrium setting, we provide conditions for the existence and uniqueness of an equilibrium shadow price. We conduct comparative statics. Finally, we consider the effect of supplier collusion on the retailer's (shadow) price for capacity and on supplier profit.

Chapter 7 considers an extension of the 'pull' supply chain model presented in Chapter 5 by having multiple retailers pulling multiple goods from a single supplier. We focus on an equilibrium setting, conduct comparative statics and provide conditions for the existence and uniqueness of an equilibrium. Finally, we consider the effect of retailer collusion on the supplier's (shadow) price for capacity and on retailer profit.

Finally, we summarize our findings and provide insights in Chapter 8.

1.7 A survey of the literature

The supply contracts literature has been based on the observation, pointed out, for example, by Lariviere and Porteus (2001), that wholesale price contracts are simple but *do not* coordinate the retailer's order quantity decision for a supplier-retailer supply chain in a newsvendor setting. This observation has led to the study of an assortment of alternative contracts. For example, buy back contracts (Pasternack 1985), quantity flexibility contracts (Tsay 1999), and many others. Cachon (2003) provides an excellent survey of the many contracts and models that have been studied in the supply contracts literature. The mindset surrounding wholesale price contract's inability to channel-coordinate is true under appropriate assumptions— which the supply contracts literature has been implicitly assuming: that there are no capacity constraints (e.g., shelf space, budget, etc.).

As mentioned before, we also consider the case of multiple suppliers serving a single retailer. This exploration is motivated, in part, by Cachon (2003) and Cachon and Lariviere (2005), who emphasize that coordination for channel configurations with multiple suppliers has yet to be explored. The relevant literature on multi-product newsvendors with side constraints (which has developed independently from the coordination literature) includes Lau and Lau (1995), Abdel-Malek and Montanari (2005a,b).

Considering capacity constraints in a supply channel is not new to the supply contracts literature. However, most other papers in the literature consider choosing capacity as one stage of a game (before downstream demand is realized) that also involves a production decision after demand is finally realized (Cachon and Lariviere 2001, Gerchak and Wang 2004, Wang and Gerchak 2003, Tomlin 2003). Our paper, although complementary to this stream of literature, does not involve an endogenous capacity choice for any party but rather analyzes how an exogenous capacity constraint determines the set of wholesale prices that can coordinate the retailer's decision for the channel. Pasternack (2001) considers an exogenous budget constraint, but not for the purposes of studying coordination. Rather, he analyzes a retailer's optimal procurement decision when the retailer has two available strategies: buying on consignment and outright purchase.

Also our work is not the first to reconsider wholesale price contracts and their benefits beyond simplicity. Cachon (2004) looks at how inventory risk is allocated according to wholesale price contracts and the resulting impact on supply chain efficiency. As far as we are aware, our paper is the first to consider the coordination-capability of linear wholesale price contracts under a simple capacity-constrained production/procurement newsvendor model.

CHAPTER 2

Coordinating a constrained channel

Wholesale price contracts are commonplace since they are straightforward and easy to implement. While risk-sharing contracts such as revenue-sharing agreements can coordinate a retailer's decision in a newsvendor setting, Cachon and Lariviere (2005) note that these alternative contracts impose a heavier administrative burden. For example, these alternative contracts may require an investment in information technology or a higher level of trust between the trading partners due to the additional processes involved. Our stylized capacity-constrained newsvendor setting provides a laboratory for understanding the set of wholesale price contracts that lead the retailer to take coordinating actions under various channel configurations: one-supplier/one-retailer (this chapter's focus) and multiple-suppliers/one-retailer (Chapter 4's focus).

In this chapter, we are concerned with the coordination capability of wholesale price contracts for a supply channel in both a negotiation setting and an equilibrium setting. In our negotiation setting, we are concerned with the entire set of coordinating wholesale-price contracts. The wholesale prices in this set are Pareto-optimal, a useful property for getting 'win/win' results in negotiation settings. This is in contrast to an equilibrium setting, where choosing the wholesale price(s) is an initial stage of a game for the supplier(s). In the equilibrium setting we explore conditions for the game's equilibrium wholesale-price vectors to coordinate the newsvendor's procurement decision for the channel (i.e., necessary and sufficient conditions so that the game's equilibria are included in the set of coordinating wholesale price contracts), and characterize the extent of the efficiency loss when these conditions are violated.

Chapter Outline

In Section 2.1, we provide a stylized 1-supplier/1-retailer model and formally define what it means for a wholesale price contract to coordinate the retailer's ordering decision for a supply channel. Then, in Section 2.2, we describe the set of coordinating wholesale price contracts for this model and analyze the size of this set in Section 2.2.1. In Section 2.3, we consider a 1-supplier/1-retailer equilibrium model and prove that a unique equilibrium exists. Then, we provide necessary and sufficient conditions for the equilibrium wholesale price contract to coordinate the retailer's ordering decision (i.e., the equilibrium wholesale price contract is included in the set of coordinating contracts). In Section 2.4, we analyze the set of wholesale price contracts that are Pareto-dominated (i.e., a different contract exists that enables one firm to better off without making the other firms worse off). The Pareto-dominated contracts are important because they should be avoided in both a negotiation setting as well as an equilibrium setting. Recognizing that in an equilibrium setting the equilibrium wholesale price contract need not be a coordinating contract (due to the conditions we state in Section 2.3), in Section 2.5, we characterize the worst case efficiency loss in an equilibrium setting. In order to maintain the flow of presentation, the proofs for all our results in this chapter are contained in Section 2.6.

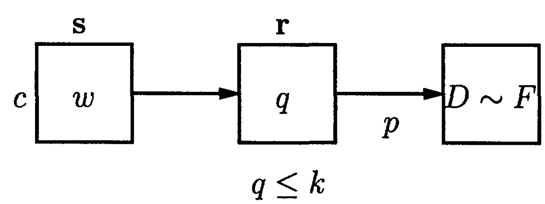
■ 2.1 Model

A risk-neutral retailer r faces a newsvendor problem in ordering from a risk-neutral supplier for a single good: there is a single sales season, the retailer decides on an order quantity q and orders well in advance of the season, the entire order arrives before the start of the season, and finally demand is realized, resulting in sales for the retailer (without an opportunity for replenishment). Without loss of generality, we assume that units remaining at the end of the season have no salvage value and that there is no cost for stocking out.

The model's parameters are summarized in Figure 2-1 with the arrows denoting the direction of product flow. In particular, the supplier has a fixed marginal cost of c per unit supplied and charges the retailer a wholesale price $w \geq c$ per unit ordered. The retailer's price p per unit to the market is fixed, and we assume that $p > w$. For that price, the demand D is random with probability density function (p.d.f.) f and cumulative distribution function (c.d.f.) F. We also define $\bar{F}(x) \stackrel{\text{def}}{=} 1 - F(x) = P(D > x)$. We say that a c.d.f. F has the *IGFR property* (increasing generalized failure rate), if $g(x) \stackrel{\text{def}}{=} \frac{x \cdot f(x)}{\bar{F}(x)}$ is weakly increasing on the set of all x for which $\bar{F}(x) > 0$ (Lariviere and Porteus 2001). Most distributions used in practice (such as the Normal, the Uniform, the Gamma, and the Weibull distribution) have the IGFR property.

We assume that the retailer's capacity is constrained by some $k > 0$; for example, the retailer can only hold k units of inventory, or accept a shipment not larger than k. For a different interpretation, k could represent a constraint on the capacity of the channel or a budget constraint.

Figure 2-1 "single supplier & single capacity constrained retailer" model.



Note. Supplier s with marginal cost c (per unit) offers a product at wholesale price w (per unit) to a capacity-constrained retailer r that faces uncertain demand D downstream, when the price for the product is fixed at p (per unit). The retailer must decide on a quantity q to order from the supplier.

ASSUMPTION 2.1. *The probability density function (p.d.f.) f for the demand D has support $[0, l]$, with $l > k$, on which it is positive and continuous.*

As a consequence, $\bar{F}(0) = 1$ and $\bar{F}$ is continuously differentiable, strictly decreasing, and invertible on $(0, l)$. There is no additional restriction on the value of l. This is not a restrictive assumption and is made for technical reasons as shown in our proofs.

2.1.1 Retailer's problem

Faced with uncertain sales $S(q) \stackrel{\text{def}}{=} \min\{q, D\}$ (when ordering q units) and a wholesale price w (from the supplier), the retailer decides on a quantity to order from the supplier in order to maximize expected profit $\pi_r(q) \stackrel{\text{def}}{=} E[pS(q)] - wq$ while satisfying the capacity constraint k. Namely, it solves the following convex program with linear constraints in the decision variable, q:

RETAILER(k,w)

$$\begin{aligned} \text{maximize} \quad & pE[S(q)] - wq \\ \text{subject to} \quad & k - q \geq 0 \\ & q \geq 0. \end{aligned} \tag{2.1}$$

Because of our assumptions on the c.d.f. F, it can be shown that *RETAILER(k,w)* has a unique solution which we denote by $q^r(w)$.

2.1.2 Channel's problem

Denote the channel's expected profit by $\pi_s(q) \stackrel{\text{def}}{=} E[pS(q) - cq]$. Under capacity constraint k, the optimal order quantity q^s for the system/channel is the solution to convex program (2.2), *CHANNEL(k)*. Note that *CHANNEL(k)* has identical linear constraints but a slightly altered objective function when compared to *RETAILER(k,w)*:

CHANNEL(k)

$$\begin{aligned} \text{maximize} \quad & pE[S(q)] - cq \\ \text{subject to} \quad & k - q \geq 0 \\ & q \geq 0. \end{aligned} \tag{2.2}$$

Again because of our assumptions on the c.d.f. F it can be shown that *CHANNEL(k)* also has a unique solution which we denote by q^s. We denote the unique solution, $\arg\max_{0 \leq q < \infty} \pi_s(q)$, for the unconstrained channel problem by q^*. It is well known that $q^* = \bar{F}^{-1}(c/p)$ (e.g., Cachon and Terwiesch (2006)). Because of convexity, it is also easily seen that $q^s = \min\{q^*, k\}$.

2.1.3 Definition: Coordinating the retailer's action

A wholesale price contract w *coordinates* the retailer's ordering decision *for* the supply channel when it causes the retailer to order the channel-optimal amount, i.e., $q^r(w) = q^s$. In Section 2.2 we are interested in the following questions: For a fixed capacity k, what is the set of wholesale prices $\mathcal{W}(k)$ for which $q^r(w) = q^s$? What does this set $\mathcal{W}(k)$ resemble geometrically?

If there is no capacity constraint (or equivalently if k is very large), 'double marginalization' results in the retailer not ordering enough (i.e., $q^r(w) < q^s$) under any wholesale price contract, $w > c$. In the next section, we will show that when the capacity constraint k is small relative to demand, there exist a set of wholesale price contracts $w > c$ that can coordinate the retailer's order quantity, i.e., $q^r(w) = q^s$.

■ 2.2 Set of coordinating wholesale prices

Our first result describes the set of coordinating wholesale prices under a capacity constraint.

THEOREM 2.1. *In a 1-supplier/1-retailer configuration where the retailer faces a newsvendor problem and has a capacity constraint k, any wholesale price*

$$w \in \mathcal{W}(k) \stackrel{def}{=} [c, p\bar{F}(\min\{q^*, k\})]$$

will coordinate the retailer's ordering decision for the supply channel, i.e., $q^r(w) = q^s$. Furthermore, if $q^r(w) = q^s$ and $c \leq w \leq p$, then $w \in \mathcal{W}(k)$.

Proof. See Section 2.6.1.

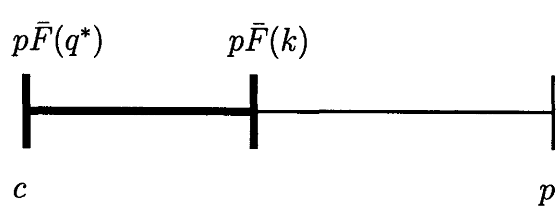
Notice that if the capacity constraint k is larger than or equal to the unconstrained channel's optimal order quantity, q^*, then $p\bar{F}(\min\{q^*, k\}) = p\bar{F}(q^*) = c$, reducing to the 'classic' result in the supply contracts literature. However, this is true only when *the capacity constraint is not binding for the channel* (i.e., $q^* \leq k$). When *the capacity constraint* k *is binding for the channel* (i.e., $q^* > k$), then any wholesale price $w \in [c, p\bar{F}(k)]$ will coordinate the retailer's action *and* only wholesale prices in the range $[c, p\bar{F}(k)]$ can coordinate the retailer's action.

Many factors such as 'power in the channel', 'outside alternatives', 'inventory risk exposure', and 'competitive environment' ultimately influence the actual wholesale price (selected from the set $[c, p]$) charged by the supplier. In the unconstrained setting, regardless of these factors, coordination is not possible with a linear wholesale price contract (because the supplier presumably would not agree to price at cost). However, when the capacity constraint is binding for the channel, coordination becomes *possible* (because the set of coordinating wholesale price contracts becomes $[c, p\bar{F}(k)]$ (rather than $\{c\}$) and ultimately depends on these other factors. Theorem 2.2 in Section 2.3 considers a equilibrium setting where the retailer takes on all the inventory risk (akin to the 'Stackelberg game' in Lariviere and Porteus (2001) and 'push mode' in Cachon (2004)), and provides additional conditions that must be met so that the 'equilibrium' wholesale price contract is a member of the set of coordinating wholesale price contracts, $[c, p\bar{F}(k)]$.

2.2.1 Size of $\mathcal{W}(k)$.

The geometry of the set of wholesale prices $\mathcal{W}(k)$ that coordinate the retailer's decision for the supply channel is depicted in Figure 2-2.

Figure 2-2 **The set of wholesale prices that coordinates the actions of a single retailer when procuring from a single supplier.**



Note. Note that $p\bar{F}(q^*) = c$ and $\mathcal{W}(k) = [c, p\bar{F}(k)]$ (the interval denoted in bold) when $k \leq q^*$.

Note that the size of $\mathcal{W}(k)$ is increasing as k decreases. Corollary 2.1 formalizes this notion and follows directly from Theorem 2.1 because $\bar{F}(k)$ is decreasing in k.

COROLLARY 2.1. *If* $0 \leq k_1 \leq k_2$, *then* $\mathcal{W}(k_2) \subseteq \mathcal{W}(k_1) \subseteq [c, p]$.

Thus, the more constrained the channel is with respect to the channel optimal order quantity, q^*, the larger the set of coordinating wholesale price contracts $\mathcal{W}(k)$.

Consider two supply channels selling the same good with the same retail price p and supplier cost c. Assume that the probability of excess demand in the first channel is larger, in the sense $\bar{F}_1(k) \geq \bar{F}_2(k)$. Let $\mathcal{W}_i(k)$ denote the set of coordinating wholesale price contracts for channel i when the channel is constrained by k units. The channel with the higher probability of excess demand has a larger set of coordinating wholesale prices. Corollary 2.2 to Theorem 2.1 makes this precise.

COROLLARY 2.2. *Given two demand distributions* F_1 *and* F_2, *if* $\bar{F}_1(k) \geq \bar{F}_2(k) > 0$, *then*

$$\mathcal{W}_2(k) \subseteq \mathcal{W}_1(k) \subseteq [c, p].$$

Proof. See Section 2.6.2.

2.3 Equilibrium setting.

The equilibrium setting we analyze is a two-stage (Stackelberg) game. In the *first* stage, the supplier (the *'leader'*) sets a wholesale price w. In the *second* stage, the retailer (the *'follower'*) chooses an optimal response q, given the wholesale price w. The supplier produces and delivers q units before the sales season starts and offers no replenishments. Both the supplier and retailer aim to maximize their own profit. The supplier's payoff function is $\pi_s(w; q) = (w - c)q$ and the retailer's payoff function is $\pi_r(q; w) = E[pS(q) - wq]$. Lariviere and Porteus (2001) analyze this Stackelberg game, for an unconstrained channel with one supplier and one retailer. They find that when F has the IGFR property, the game results in a unique outcome (q^e, w^e) defined implicitly in terms of the equations

$$p\bar{F}(q^e)\,(1 - g(q^e)) - c = 0, \tag{2.3}$$

$$p\bar{F}(q^e) - w^e = 0, \tag{2.4}$$

where g is the generalized failure rate function $g(y) \stackrel{\text{def}}{=} yf(y)/\bar{F}(y)$. Furthermore, they show that the outcome is not channel optimal. In this section, and in Section 2.5, we explore the efficiency of the outcome when the channel has a capacity constraint (i.e., $q \leq k$).

Theorem 2.2 provides necessary and sufficient conditions on the channel's capacity constraint k for the Stackelberg game to result in a channel-optimal equilibrium.

THEOREM 2.2. *Assume F has the IGFR property. Consider the above described game, when the channel capacity is k units. This game has a unique equilibrium, given by $q^{eq}(k) = \min\{k, q^e\}$ and $w^{eq}(k) = \max\{p\bar{F}(k), w^e\}$, where q^e and w^e are defined by equations* (2.3) *and* (2.4), *respectively. This equilibrium is channel optimal if and only if*

$$k \leq q^e. \tag{2.5}$$

Under this condition, we have $q^{eq} = k$ and $w^{eq} = p\bar{F}(k)$.

Proof. See Section 2.6.3.

The function $p\bar{F}(y)\,(1 - g(y)) - c$ represents the supplier's marginal profit on the yth unit, when $y < k$. When F has the IGFR property, the supplier's marginal profit is decreasing in y, while the marginal profit is nonnegative. This fact and equation (2.3) imply that inequality (2.5) is equivalent to the inequality $p\bar{F}(k)\,(1 - g(k)) - c \geq 0$, which can be interpreted as a statement that the supplier's marginal profit (when relaxing the capacity constraint) on the kth unit is greater than zero. Therefore, inequality (2.5) suggests that when the capacity constraint is binding for the supplier's problem (the 'leader' in the Stackelberg game), then the outcome of the game is channel optimal and vice-versa.

If the channel capacity k is 'large enough', so that inequality (2.5) is not satisfied, how inefficient is the channel? In Section 2.5, we provide a distribution-free 'measuring stick' for the efficiency loss in channels with a capacity constraint.

■ 2.4 When can both parties be better off?

The set of coordinating wholesale price contracts $\mathcal{W}(k)$ introduced in Theorem 2.1 has many merits in a negotiation setting. For example, such contracts are Pareto optimal. In contrast, Theorem 2.3 examines the set of wholesale price contracts $\mathcal{D}(k)$ that have little merit in that they are Pareto-dominated by some other wholesale price contract in $[c, p]$. A contract is *Pareto-dominated* if there exists an alternative linear wholesale price contract that makes one party better off without making any other party worse off. Having a complete picture of the contracts that are channel-optimal and the contracts that are Pareto-dominated is helpful in a negotiation setting.

THEOREM 2.3. *Assume F has the IGFR property and that the quantity q^e and wholesale price w^e are defined implicitly in terms of equations* (2.3) *and* (2.4). *If $k \leq q^*$, then the set of Pareto-dominated wholesale price contracts $\mathcal{D}(k)$ is*

$$\mathcal{D}(k) \stackrel{def}{=} (\max\{w^e, p\bar{F}(k)\}, p] = (p\bar{F}(\min\{q^e, k\}), p].$$

Proof. See Section 2.6.4.

Note that $\mathcal{W}(k)$ and $\mathcal{D}(k)$ are disjoint. Corollary 2.3 to Theorem 2.3 formalizes the idea that when k is 'small enough', $\mathcal{W}(k)$ and $\mathcal{D}(k)$ partition the set $[c, p]$. Figure 2-3 illustrates these ideas when demand has a Gamma distribution.

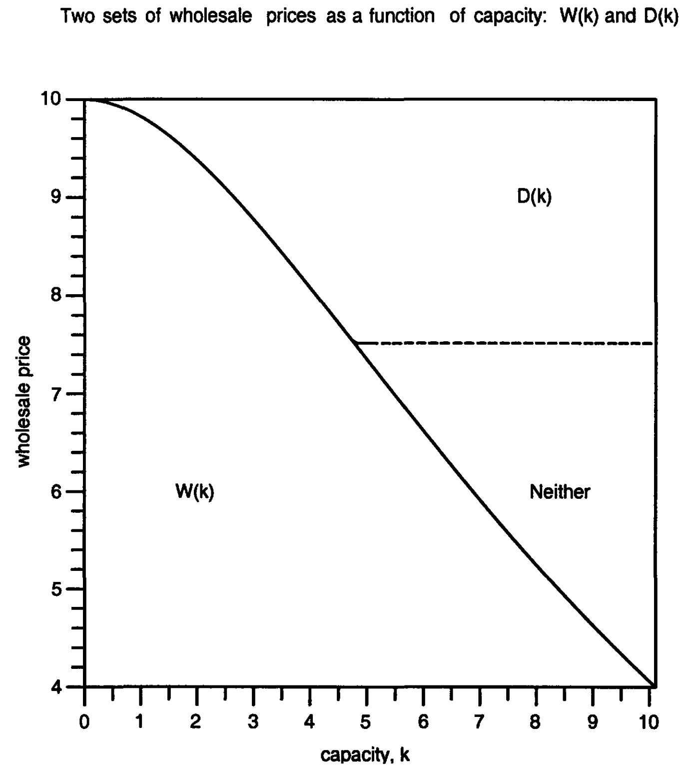
COROLLARY 2.3. *Assume F has the IGFR property. If $k \leq q^e$, then*

$$\mathcal{W}(k) \cup \mathcal{D}(k) = [c, p], \tag{2.6}$$

$$\mathcal{W}(k) \cap \mathcal{D}(k) = \emptyset. \tag{2.7}$$

Corollary 2.3 is especially interesting: it asserts that when capacity is small enough there are only two types of contracts: 'good contracts', $\mathcal{W}(k)$, and 'bad contracts', $\mathcal{D}(k)$. Furthermore, both parties will always have a reason to avoid the 'bad contracts' because they are Pareto-dominated by some channel-optimal contract in the set $\mathcal{W}(k)$.

Figure 2-3 **An example illustrating $\mathcal{W}(k)$ and $\mathcal{D}(k)$.**



Note. We use the same parameters as in Figure 3-1, resulting in $q^* \approx 10.112$, $q^e \approx 4.784$, and $w^e \approx 7.516$. The set of coordinating wholesale price contracts $\mathcal{W}(k)$ lies under the solid curve. The set of Pareto-dominated wholesale price contracts $\mathcal{D}(k)$ lies above both the solid and dashed curves. The set of contracts that lie *between* the solid and dashed curves are neither in $\mathcal{W}(k)$ nor in $\mathcal{D}(k)$. Such contracts do not coordinate the channel, but nevertheless, are not Pareto dominated by coordinating wholesale contracts.

■ 2.5 Efficiency Loss.

When the outcome of the Stackelberg game we described in Section 2.3 results in a wholesale price contract that is not channel optimal, how much does the channel 'lose' as a result? What is the 'price' paid for the 'gaming' between the supplier and retailer? To quantify the answer we analyze the worst-case efficiency. Our definition of efficiency is related to the concept of *Price of Anarchy*, "PoA", as used by Koutsoupias and Papadimitriou (1999), and Papadimitriou (2001). PoA has been used as a 'measuring stick' in an assortment of gaming contexts: facility location (Vetta 2002), traffic networks (Schulz and Moses 2003), resource allocation (Johari and Tsitsiklis 2004). More recently Perakis and Roels (2006) analyze the PoA for an assortment of supply channel configurations with the IGFR restriction, but not for resource constrained channels. Theorem 2.4 complements their results, by providing an efficiency result for the Stackelberg game of Section 2.3, in the presence of a capacity constraint k.

For a channel with a capacity constraint k and probability $\bar{F}(k)$ of excess demand, we define the parameter $\beta \stackrel{\text{def}}{=} \frac{\max\{\bar{F}(k), c/p\}}{c/p}$. The parameter β depends on the probability $\bar{F}(k)$ of excess demand and takes values from the set $[1, p/c]$. It quantifies how constrained the channel is with respect to the channel optimal order quantity q^*, because $\beta \stackrel{\text{def}}{=} \frac{\max\{\bar{F}(k), c/p\}}{c/p} = \frac{\max\{\bar{F}(k), \bar{F}(q^*)\}}{\bar{F}(q^*)}$. In the Stackelberg game with a capacity constraint k and parameter β, the efficiency, Eff(k, β), is defined according to equation (2.8) below.

$$\text{Eff}(k, \beta) = \inf_{F \in \mathcal{F}(k,\beta)} \frac{\text{Channel profit under 'gaming'}}{\text{Optimal channel profit}} = \inf_{F \in \mathcal{F}(k,\beta)} \frac{E[pS(q^{eq}(k)) - cq^{eq}(k)]}{E[pS(q^s(k)) - cq^s(k)]} \tag{2.8}$$

The set $\mathcal{F}(k, \beta)$ represents the set of probability distributions that satisfy Assumption 2.1, have the IGFR property, and such that the probability $\bar{F}(k)$ of excess demand satisfies $\frac{\max\{\bar{F}(k), c/p\}}{c/p} = \beta$. Note that Eff$(k, \beta)$ is a distribution-free method of quantifying the *worst-case* efficiency. When Eff(k, β) is low (much smaller than one), there is significant efficiency loss due to 'gaming'.

THEOREM 2.4. *Define $m \stackrel{def}{=} (p - c)/p$ (the channel's gross profit margin). For the Stackelberg game described in Section 2.3, we have*

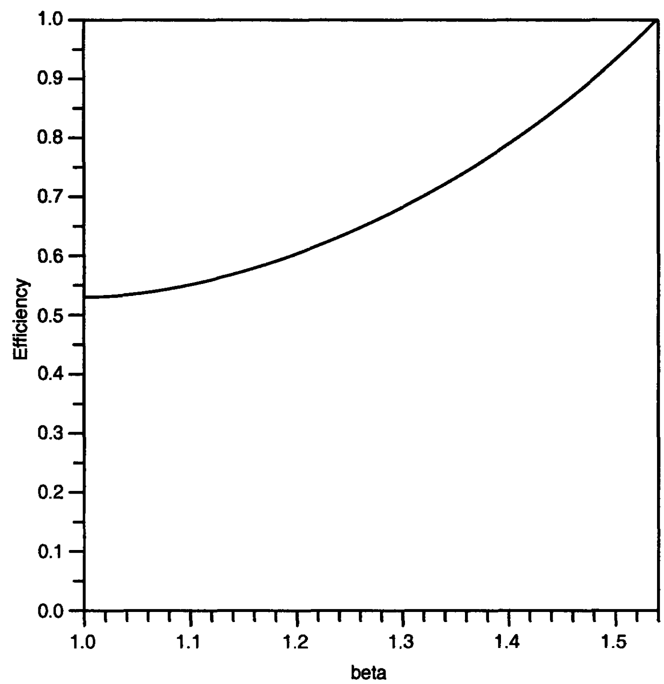
$$Eff(k, \beta) = \left(\left(\frac{\beta - 1 + m}{m} \right) \left(\frac{1}{\beta} \cdot \frac{1}{1-m} \right)^{1/m} - \left(\frac{1}{1-m} \right) \right)^{-1}. \tag{2.9}$$

Proof. See Section 2.6.5.

Note that Eff(k, β) is decreasing in the channel's gross profit margin m and increasing in β. When $\beta = 1$, the channel is not constrained and Eff(k, β) equals $\left(\left(\frac{1}{1-m} \right)^{1/m} - \frac{1}{1-m} \right)^{-1}$ which, after some algebraic manipulation, matches the result in Perakis and Roels (2006). On the other hand, when the channel is most constrained (i.e., $k \approx 0$, $\bar{F}(k) \approx 1$, and $\beta \approx p/c$), then Eff(k, β) simplifies to 1. In other words there is no efficiency loss because the equilibrium outcome involves the retailer ordering exactly k. Our result is thus a more general version of the 'two-stage push-mode

PoA' result in Perakis and Roels (2006) in that we account for a capacity constraint. Also our proof technique differs from and complements Perakis and Roels (2006), in that we indirectly optimize over the space of probability distributions by optimizing over the space of generalized failure rates.

Figure 2-4 An example illustrating Eff(k, β) when $m = 0.35$.



Note. We fix the margin $(p - c)/p = 0.35$ and see how Eff(k, β) changes as a function of β.

Figure 2-4 provides an example of the Eff(k, β) when the channel's gross profit margin is 35 percent. Figure 2-4 illustrates that for channels with smaller capacity (i.e., higher β), the worst-case efficiency (as measured by Eff(k, β)) is larger.

■ 2.6 Proofs

In order to not disrupt the flow of presentation, the proofs for our results in this chapter are contained here.

2.6.1 Proof: 1-supplier/1-retailer, Set of wholesale prices $\mathcal{W}(k)$

Proof of Theorem 2.1. We start by proving that if $w \in \mathcal{W}(k)$, then $q^r(w) = q^s$. Suppose first that $q^* \leq k$. We then have $p\bar{F}(\min\{q^*, k\}) = p\bar{F}(q^*) = c$. Therefore, $\mathcal{W}(k) = \{c\}$. Thus, for any $w \in \mathcal{W}(k)$, the problems *RETAILER(k,w)* and *CHANNEL(k)* are the same and $q^r(w) = q^s$.

Suppose now that $q^* > k$. We then have $q^s = k$ and, furthermore, $p\bar{F}(\min\{q^*, k\}) = p\bar{F}(k) > p\bar{F}(q^*) = c$. (The strict inequality is obtained because $\bar{F}$ is strictly decreasing.) Therefore, $\mathcal{W}(k) = [c, p\bar{F}(k)]$. Solving $\frac{\partial}{\partial x}\left(E[pS(x)] - p\bar{F}(k)x\right) = 0$ for $x \in [0, l]$ and noting $\frac{\partial S(x)}{\partial x} = \bar{F}(x)$, we obtain $q^r(p\bar{F}(k)) = k$. Since $q^r(w)$ is nondecreasing as we decrease w, we see that for all $w \in \mathcal{W}(k)$, $q^r(w) = k = q^s$.

Suppose now that $q^r(w) = q^s$ and $c \leq w \leq p$. We have shown that

$$\mathcal{W}(k) = \begin{cases} \{c\}, & \text{if } q^* \leq k; \\ [c, p\bar{F}(k)], & \text{if } q^* > k. \end{cases}$$

When $q^* \leq k$, the first order conditions imply that $p\bar{F}(q^r(w)) - w = 0 = p\bar{F}(q^s) - c$ for any $w \geq c$, which implies w must equal c. When $q^* > k$, we know that $q^s = k$. Assume $w > p\bar{F}(k)$ when $q^r(w) = q^s$. Due to invertibility around k, $q^r(w) < k$. This is a contradiction because $q^s = q^r(w) < k$. □

2.6.2 Proof: Impact of size of Market on size of $\mathcal{W}(k)$

Proof of Corollary 2.2. Let $q_i^* = \bar{F}_i^{-1}(c/p)$ be the order quantity (for an unconstrained channel) under the demand distribution F_i.

If $k \leq q_2^*$, then $c/p \leq \bar{F}_2(k) \leq \bar{F}_1(k)$, which implies that $k \leq q_1^*$. Thus, $\mathcal{W}_i(k) = [c, p\bar{F}_i(k)]$ for $i \in 1, 2$. Since $\bar{F}_2(k) \leq \bar{F}_1(k)$, we can conclude that $\mathcal{W}_2(k) \subseteq \mathcal{W}_1(k) \subseteq [c, p]$.

Similarly, if $q_2^* < k$, then $\mathcal{W}_2(k) = \{c\}$. Thus, $\mathcal{W}_2(k) \subseteq \mathcal{W}_1(k)$. □

2.6.3 Proof: When is the equilibrium of the Stackelberg game channel optimal?

Proof of Theorem 2.2. The retailer's profit function $\pi_r(q; w)$ under a wholesale price contract w is defined as $\pi_r(q; w) \stackrel{\text{def}}{=} E[pS(q) - wq]$. Since $\pi_r(q; w)$ is concave, in q, we can use the first order conditions and conclude that for a wholesale price $w \in [c, p]$, the constrained retailer's order quantity $q^r(w)$ is given by

$$q^r(w) = \min\{k, \bar{F}^{-1}(w/p)\}. \tag{2.10}$$

The supplier's profit function $\pi_s(w; q)$ under a wholesale price contract w is defined as $\pi_s(w; q) \stackrel{\text{def}}{=} (w - c)q$. Since $q^r(w)$ is the retailer's best response in the second stage to a wholesale price w by the supplier in the first stage, equation (2.10) allows us to express the supplier's objective function as follows:

$$\pi_s(w) = \begin{cases} (w - c)\, k, & \text{if } c \le w \le \max\{c, p\bar{F}(k)\}; \\ \left(p\bar{F}(q^r(w)) - c\right) q^r(w), & \text{if } \max\{c, p\bar{F}(k)\} < w \le p. \end{cases} \tag{2.11}$$

For $w > \max\{c, p\bar{F}(k)\}$, note that $\frac{\partial \pi_s(w)}{\partial w} = \left(p\bar{F}(q^r(w))\,(1 - g(q^r(w))) - c\right) \cdot \frac{\partial q^r(w)}{\partial w}$. Since the function $p\bar{F}(y)\,(1 - g(y)) - c$ is strictly decreasing in y when it is nonnegative and equals zero at q^e (see equation (2.3)), we can deduce that $\left(p\bar{F}(q^r(w))\,(1 - g(q^r(w))) - c\right) > 0$ for $w > w^e$ (because $q^r(w) < q^e$). Furthermore, $\frac{\partial q^r(w)}{\partial w} < 0$ for $w > p\bar{F}(k)$. Therefore, we can conclude that $\frac{\partial \pi_s(w)}{\partial w} < 0$ for $w > \max\{w^e, p\bar{F}(k)\}$.

Either the inequality $p\bar{F}(k) < w^e$ holds or the inequality $w^e \le p\bar{F}(k)$ holds. First assume the inequality $p\bar{F}(k) < w^e$ holds. Equation (2.11) implies that $\pi_s(w)$ is increasing linearly between c and $\max\{c, p\bar{F}(k)\}$. Furthermore, since

$$\left(p\bar{F}(q^r(w))\,(1 - g(q^r(w))) - c\right) < 0$$

for $w < w^e$ (because $q^r(w) > q^e$), we can deduce that

$$\frac{\partial \pi_s(w)}{\partial w} = \left(p\bar{F}(q^r(w))\left(1 - g(q^r(w))\right) - c\right) \cdot \frac{\partial q^r(w)}{\partial w} > 0$$

for $w \in (\max\{c, p\bar{F}(k)\}, w^e)$. And we know

$$\frac{\partial \pi_s(w)}{\partial w} < 0$$

for $w > \max\{w^e, p\bar{F}(k)\} = w^e$. Therefore, $w^{eq}(k) = w^e$ and equations (2.10) and (2.4) imply $q^{eq}(k) = q^e$. The inequality $p\bar{F}(k) < w^e$ is equivalent to the inequality $q^e < k$ (see equation (2.4)). Therefore, when $q^e < k$ holds, the inequality $w^{eq}(k) = w^e > \max\{c, p\bar{F}(k)\} = p\bar{F}(\min\{q^*, k\})$ holds and we can deduce that $w^{eq}(k) \notin \mathcal{W}(k)$ (using Theorem 2.1).

Next assume $w^e \leq p\bar{F}(k)$ holds. Since $\frac{\partial \pi_s(w)}{\partial w} < 0$ for $w > \max\{w^e, p\bar{F}(k)\} = \max\{c, p\bar{F}(k)\}$, equation (2.11) implies $w^{eq}(k) = p\bar{F}(k)$ and equation (2.10) implies $q^{eq}(k) = k$. The inequality $w^e \leq p\bar{F}(k)$ is equivalent to the inequality $k \leq q^e$ (see equation (2.4)). Therefore, when $k \leq q^e$ holds, the equality $w^{eq}(k) = p\bar{F}(k) = \max\{c, p\bar{F}(k)\} = p\bar{F}(\min\{q^*, k\})$ holds and we can deduce that $w^{eq}(k) \in \mathcal{W}(k)$ (again using Theorem 2.1). □

2.6.4 Proof: The set of Pareto-dominated contracts $\mathcal{D}(k)$ as a function of capacity

Proof of Theorem 2.3. Equation (2.10) allows us to express the retailer's objective function as follows:

$$\pi_r(w) = \begin{cases} pE[S(k)] - wk, & \text{if } c \leq w \leq p\bar{F}(k); \\ pE[S(q^r(w))] - p\bar{F}(q^r(w))q^r(w), & \text{if } p\bar{F}(k) < w \leq p. \end{cases} \tag{2.12}$$

Note that $\pi_r(w)$ is strictly decreasing in w, when $w \in (c, p\bar{F}(k))$. Furthermore, when $w \in (p\bar{F}(k), p)$, note that $\pi_r(w)$ is strictly decreasing in w because $\frac{\partial \pi_r(w)}{\partial w} =$

$p\bar{F}(q^r(w))g(q^r(w)) \cdot \frac{\partial q^r(w)}{\partial w} < 0$. From the proof of Theorem 2.2, we know that the supplier's profit $\pi_s(w)$ is also strictly decreasing for $w > \max\{w^e, p\bar{F}(k)\}$. Therefore, any wholesale price contract in the set $(\max\{w^e, p\bar{F}(k)\}, p]$ is Pareto-dominated by $\max\{w^e, p\bar{F}(k)\}$.

Since the supplier's profit is decreasing as the wholesale price w decreases from $\max\{w^e, p\bar{F}(k)\}$ (see the proof of Theorem 2.2) but the retailer's profit is increasing as the wholesale price decreases, we can conclude that any wholesale price contract in the set $[c, \max\{w^e, p\bar{F}(k)\}]$ is not Pareto-dominated. Thus, the set of Pareto-dominated wholesale price contracts in $[c, p]$ is exactly $\mathcal{D}(k) = (w^e, p] = (\max\{w^e, p\bar{F}(k)\}, p]$. □

2.6.5 Proof: Efficiency loss for a two-stage push channel with capacity constraint

LEMMA 2.1. *Assume F has the IGFR property and that the quantity q^e is defined implicitly in terms of equation (2.3). If $q^e \leq k \leq q^s$, then*

$$\frac{p\left(\int_0^k \bar{F}(x)\,dx\right) - ck}{p\left(\int_0^{q^e} \bar{F}(x)\,dx\right) - cq^e} \leq \left(\left(\frac{k}{q^e}\right)\left(m - 1 + \frac{1}{1-m}\left(\frac{k}{q^e}\right)^{-m}\right) - \frac{1}{1-m} + 1\right)/m. \tag{2.13}$$

Proof of Lemma 2.1. Recall the generalized failure rate function $g(y)$ for c.d.f. F is defined as $g(y) \stackrel{\text{def}}{=} -y\frac{\partial \bar{F}(y)}{\partial y}/\bar{F}(y)$. Since $\bar{F}(y) = e^{-\int_0^y f(t)/\bar{F}(t)\,dt} = e^{-\int_0^y g(t)/t\,dt}$, we have

$$\begin{aligned}\frac{p\left(\int_0^k \bar{F}(x)\,dx\right) - ck}{p\left(\int_0^{q^e} \bar{F}(x)\,dx\right) - cq^e} &= \frac{p\left(\int_0^k e^{-\int_0^x g(t)/t\,dt}\,dx\right) - ck}{p\left(\int_0^{q^e} e^{-\int_0^x g(t)/t\,dt}\,dx\right) - cq^e} \\ &= 1 + \frac{p\left(\int_{q^e}^k e^{-\int_0^x g(t)/t\,dt}\,dx\right) - c(k - q^e)}{p\left(\int_0^{q^e} e^{-\int_0^x g(t)/t\,dt}\,dx\right) - cq^e}.\end{aligned} \tag{2.14}$$

For any $y \in [q^e, k]$, define the *profit-gain factor* $a(y)$ by

$$a(y) \stackrel{\text{def}}{=} \left(p\left(\int_{q^e}^{y} e^{-\int_0^x g(t)/t\,dt}\,dx\right) - c(y-q^e)\right) \Big/ \left(p\left(\int_0^{q^e} e^{-\int_0^x g(t)/t\,dt}\,dx\right) - cq^e\right). \tag{2.15}$$

The derivative $\frac{\partial a(y)}{\partial y}$ is expressed via equation (2.16) below, when $y \in [q^e, k]$, leading to the following nonnegative upper bound:

$$\frac{\partial a(y)}{\partial y} = \left(pe^{-\int_0^y g(t)/t\,dt} - c\right) \Big/ \left(p\left(\int_0^{q^e} e^{-\int_0^x g(t)/t\,dt}\,dx\right) - cq^e\right) \tag{2.16}$$

$$\leq \left(pe^{-\int_0^{q^e} g(t)/t\,dt - \int_{q^e}^y g(q^e)/t\,dt} - c\right) \Big/ \left(p\left(\int_0^{q^e} e^{-\int_0^x g(t)/t\,dt}\,dx\right) - cq^e\right) \tag{2.17}$$

$$= \left(p\left(\frac{y}{q^e}\right)^{-g(q^e)} e^{-\int_0^{q^e} g(t)/t\,dt} - c\right) \Big/ \left(p\left(\int_0^{q^e} e^{-\int_0^x g(t)/t\,dt}\,dx\right) - cq^e\right)$$

$$\leq \left(p\left(\frac{y}{q^e}\right)^{-g(q^e)} e^{-\int_0^{q^e} g(t)/t\,dt} - c\right) \Big/ \left(p\left(q^e e^{-\int_0^{q^e} g(t)/t\,dt}\right) - cq^e\right) \tag{2.18}$$

$$= \left(p\left(\frac{y}{q^e}\right)^{-g(q^e)} \bar{F}(q^e) - c\right) \Big/ \left(p\bar{F}(q^e) - c\right) q^e \tag{2.19}$$

$$\leq \left(p\left(\frac{y}{q^e}\right)^{-(p-c)/p} - c\right) \Big/ (p-c)\, q^e \tag{2.20}$$

$$= \left(\left(\frac{y}{q^e}\right)^{-m} + (m-1)\right) \Big/ (mq^e). \tag{2.21}$$

Therefore,

$$\frac{p\left(\int_0^k \bar{F}(x)\,dx\right) - ck}{p\left(\int_0^{q^e} \bar{F}(x)\,dx\right) - cq^e} = 1 + \int_{q^e}^{k} \frac{\partial a(y)}{\partial y}\,dy$$

$$\leq 1 + \int_{q^e}^{k} \left(\left(\frac{y}{q^e}\right)^{-m} + (m-1)\right) / (mq^e)\,dy$$

$$= 1 + \left(\frac{k}{1-m}\left(\frac{k}{q^e}\right)^{-m} - \frac{q^e}{1-m}\left(\frac{q^e}{q^e}\right)^{-m} + (m-1)(k-q^e)\right) / (mq^e)$$

$$= 1 + \left(\frac{1}{1-m}\left(\frac{k}{q^e}\right)^{1-m} - \frac{1}{1-m} + (m-1)(\frac{k}{q^e} - 1)\right) / m$$

$$= \left(\left(\frac{k}{q^e}\right)\left(m - 1 + \frac{1}{1-m}\left(\frac{k}{q^e}\right)^{-m}\right) - \frac{1}{1-m} + 1\right) / m. \quad \square$$

LEMMA 2.2. *Under the same assumptions as in Lemma 2.1, when $\bar{F}(k) = \delta$ we have $k \cdot \delta^{1/m} \leq q^e$.*

Proof of Lemma 2.2. Assume $q^e < k \cdot \delta^{1/m}$. This leads to a contradiction (inequality (2.23)):

$$\delta = \bar{F}(k) = e^{-\int_0^k g(t)/t\,dt} = e^{-\int_0^{q^e} g(t)/t\,dt} \cdot e^{-\int_{q^e}^k g(t)/t\,dt} \leq 1 \cdot e^{-\int_{q^e}^k g(q^e)/t\,dt} = (k/q^e)^{-g(q^e)}$$

$$\leq (k/q^e)^{-m} \tag{2.22}$$

$$< (k/(k \cdot \delta^{1/m}))^{-m} = \delta. \tag{2.23}$$

Inequality (2.22) holds because $p\bar{F}(q^e)(1 - g(q^e)) \leq c$, implying $g(q^e) \geq m$. Inequality (2.23) follows from our assumption, $q^e < k \cdot \delta^{1/m}$. $\square$

Proof of Theorem 2.4. The case where $\beta = 1$ is equivalent to the unconstrained problem which is addressed in Perakis and Roels (2006). Therefore, fix channel capacity k and assume $\beta > 1$, so that $q^s = k$. When $\beta > 1$, the probability of excess demand, which we will denote by δ, is fixed and satisfies $\beta = \delta p/c$.

Fix a c.d.f. $F \in \mathcal{F}(k, \beta)$. The efficiency Eff$(F)$ of F satisfies the following lower bound:

$$\begin{aligned}
\text{Eff}(F) &\stackrel{\text{def}}{=} E[pS(q^{eq}) - cq^{eq}]/E[pS(k) - ck] \\
&\geq E[pS(q^{e}) - cq^{e}]/E[pS(k) - ck] \qquad (2.24) \\
&= \left(p\left(\int_0^{q^e} \bar{F}(x)\,dx\right) - cq^e\right) \Big/ \left(p\left(\int_0^{k} \bar{F}(x)\,dx\right) - ck\right) \\
&\geq m \Big/ \left(\left(\frac{k}{q^e}\right)\left(m - 1 + \frac{1}{1-m}\left(\frac{k}{q^e}\right)^{-m}\right) - \frac{1}{1-m} + 1\right) \qquad (2.25) \\
&\geq \left(\left(\frac{1}{\beta}\cdot\frac{1}{1-m}\right)^{1/m}\left(\frac{\beta - 1 + m}{m}\right) - \left(\frac{1}{1-m}\right)\right)^{-1}. \qquad (2.26)
\end{aligned}$$

In particular, inequality (2.24) follows because $q^e \leq q^{eq} \leq q^s$. Inequality (2.25) follows from Lemma 2.1. The function on the right-hand side of inequality (2.25) is decreasing as q^e decreases and from Lemma 2.2 we know that the equilibrium order quantity q^e satisfies $q^e \geq k \cdot \delta^{1/m}$. Therefore, inequality (2.26) follows when we substitute in $q^e = k \cdot \delta^{1/m} = k\,(\beta(1-m))^{1/m}$.

It can be verified that the lower bound in inequality (2.26) is attained when the c.d.f. F is taken equal to H, where the c.d.f. H satisfies

1. $\bar{H}(x) = 1$ for $x \in [0, k \cdot \delta^{1/m}]$,

2. $\bar{H}(x) = (k/x)^m \cdot \delta$ for $x \in [k \cdot \delta^{1/m}, \infty)$.

(To verify this claim confirm that $q^e = k \cdot \delta^{1/m}$, using eq. (2.3), implying that we can convert the inequalities in eqs. (2.24) and (2.26) into equalities. Furthermore, since the c.d.f. F is taken equal to H, we can convert the inequalities in eqs. (2.17), (2.18), and (2.20) into equalities. Therefore, inequality (2.25) becomes an equality.) The c.d.f. H does not satisfy Assumption 2.1, because the corresponding density is zero for $x \leq k \cdot \delta^{1/m}$. However, it can be approximated arbitrarily closely by c.d.f.s in the class $\mathcal{F}(k, \beta)$ (in particular, that satisfy Assumption 2.1), with an arbitrarily small change in the resulting efficiency. □

Chapter 3

Flexibility in allocating profit

Wholesale price contracts in unconstrained channels lack flexibility in allocating the channel-optimal profit; the only coordinating wholesale price contract gives the entire channel profit to the retailer.

In this chapter, we analyze the flexiblity of wholesale price contracts in allocating the channel-optimal profit in a constrained setting. That is, we consider whether or not wholesale price contracts possess some flexibility without sacrificing coordination. Then, we reconsider risk-sharing contracts which are known to possess flexibility in allocating channel profit (without sacrificing coordination) in an unconstrained setting and investigate how this flexibility changes in a constrained channel.

Chapter Outline

We continue analyzing the model described in Chapter 2. In Section 3.1, we characterize the fractions of revenue allocated to a supplier and retailer when they use a wholesale price from the set $\mathcal{W}(k)$ of coordinating contracts and we analyze how the fractions change as the market changes. Then, in Section 3.2, we show that wholesale price contracts have some flexibility in allocating the channel-optimal profit, a feature that has motivated the study of risk-sharing contracts. We analyze how this flexibility changes as a function of capacity and the market. In Section 3.3 we focus on risk-sharing contracts. We show in Section 3.3.1, that risk-sharing contracts still coordinate a capacity-constrained channel. Furthermore, in Section 3.3.2, we show that there is even more flexibility in the choice of risk-sharing contract that can be used for coordination. In particular, the set of coordinating risk-sharing contracts is larger in the presence of constraints. That is, the set of revenue-sharing and buyback contracts that coordinates a newsvendor's decision for a constrained channel is a superset of the set of coordinating contracts in the unconstrained setting. And for any given level of risk, there is now flexibility in allocating the channel-optimal profit. We make this last point precise in Section 3.3.3. In order to maintain the flow of presentation, the proofs for all our results in this chapter are contained in Section 3.4.

■ 3.1 Revenue requirement implicit in $\mathcal{W}(k)$.

By agreeing to focus on the set $\mathcal{W}(k)$ in negotiating over a wholesale price for coordination purposes, the supplier and retailer are implicitly agreeing to a '*minimum* share of expected revenue' requirement for the retailer and thus a '*maximum* share of expected revenue' restriction for the supplier. This notion is formalized in Theorem 3.1.

THEOREM 3.1. *If the capacity constraint k is binding for the channel (i.e., $q^* > k$), then any coordinating linear wholesale price contract $w \in \mathcal{W}(k)$ guarantees that the retailer receive at least a fraction $\frac{\int_0^k \bar{F}(x)\,dx - k\cdot\bar{F}(k)}{\int_0^k \bar{F}(x)\,dx}$ of the channel's expected revenue, and that the supplier receive at most a fraction $\frac{k\cdot\bar{F}(k)}{\int_0^k \bar{F}(x)\,dx}$ of the channel's expected revenue. Furthermore, if F has the IGFR property, then the supplier's maximum revenue share is weakly decreasing as k increases.*

Proof. See Section 3.4.1.

An important distinction regarding the supplier and retailer 'share of expected revenue' guarantees formalized in Theorem 3.1 is that the supplier's share results in a guaranteed income (i.e., no uncertainty) whereas the retailer's share results in an uncertain income. For example, from Theorem 3.1 there exists some wholesale price $w \in \mathcal{W}(k)$, where the supplier receives a fraction $\frac{k\cdot\bar{F}(k)}{\int_0^k \bar{F}(x)\,dx}$ of the expected channel revenue, $pE[S(k)]$. But the supplier's income is certain, wk, whereas the retailer's income is an uncertain amount, $pS(k) - wk$.

As a numerical example, if $\frac{k \cdot \bar{F}(k)}{\int_0^k \bar{F}(x)\,dx} = 1/2$, the supplier can receive *up to* fifty percent of the expected channel revenue and still keep the channel coordinated, whereas we require that the retailer receive *at least* fifty percent of the revenue in order for the wholesale price to coordinate the actions of the retailer.

Recall that the set of coordinating wholesale price contracts $\mathcal{W}(k)$ increases with the probability $\bar{F}(k)$ of excess demand, when k is held fixed (Corollary 2.2). Theorem 3.2 formalizes a related idea: the larger the expected excess demand, the greater the maximum possible share of revenue at the supplier without sacrificing channel-coordination.

THEOREM 3.2. *Consider two different demands D_1 and D_2, with each D_i associated with a c.d.f. F_i, that have the same mean and such that $\bar{F}_1(k) \geq \bar{F}_2(k)$. Suppose that (a) the capacity constraint k is binding for the channel under both distributions (i.e., $\min\{q_1^*, q_2^*\} > k$), and (b) $E[(D_1 - k)^+] \geq E[(D_2 - k)^+]$ (i.e., the expected excess demand under D_1 is higher than that under D_2). Then,*

$$\frac{k \cdot \bar{F}_1(k)}{\int_0^k \bar{F}_1(x)\,dx} \geq \frac{k \cdot \bar{F}_2(k)}{\int_0^k \bar{F}_2(x)\,dx}.$$

Proof. See Section 3.4.2.

■ 3.2 Wholesale price contracts and flexibility in allocating channel-optimal profit

The benefits of risk sharing contracts in the unconstrained setting include the ability to channel-coordinate the retailer's decision *as well as* flexibility (due to the extra contract parameters) that allows for any allocation of the optimal channel profit between the supplier and retailer. Cachon (2003) provides excellent examples of the 'channel-profit allocation flexibility' inherent in these more complex contracts.

Theorem 3.3 demonstrates that in a resource constrained setting, wholesale price contracts also have flexibility in allocating the channel-optimal profit. Namely, these simpler contracts allow for a range of divisions of the optimal channel profit among the firms. The divisions allowed (without losing coordination) depend on the channel's capacity, k. Similar to our observations in Section 3.1 for the implicit revenue requirements, the supplier's share results in a guaranteed income (i.e., no uncertainty) whereas the retailer's share results in an uncertain income.

THEOREM 3.3. *If the capacity constraint is binding for the channel (i.e., $q^* > k$), there exists a wholesale price contract $w \in \mathcal{W}(k)$ that can allocate a fraction t_s of the channel-optimal profit to the supplier and a fraction $1 - t_s$ to the retailer, if and only if $t_s \in [0, t_s^{max}(k; \bar{F})]$, where*

$$t_s^{max}(k; \bar{F}) \stackrel{def}{=} \frac{k \cdot \left(\bar{F}(k) - c/p\right)}{\int_0^k \left(\bar{F}(x) - c/p\right)\, dx}.$$

Furthermore, if F has the IGFR property, then $t_s^{max}(k; \bar{F})$ is weakly decreasing as k increases in the range $[0, q^*)$.

Proof. See Section 3.4.3.

Let us interpret Theorem 3.3 at two extremes values for the capacity k. As k approaches q^*, $t_s^{\max}(k; \bar{F})$ approaches zero. Thus the supplier can not get any fraction of the channel-optimal profit with any wholesale price contract from $\mathcal{W}(k)$ (this was to be expected because $\mathcal{W}(k) = \{c\}$ when $k \geq q^*$). At the other extreme, as k tends to zero, $t_s^{\max}(k; \bar{F})$ tends to one. Thus any allocation of the channel-optimal profit becomes possible with some wholesale price contract from $\mathcal{W}(k)$ (this is natural, because as k tends to zero, the interval $\mathcal{W}(k)$ becomes $[c, p]$). See Figure 3-1.

Theorem 3.4 parallels Theorem 3.2 and makes precise the idea that when we serve a larger market the 'flexibility' in allocating the channel-optimal profit 'increases'.

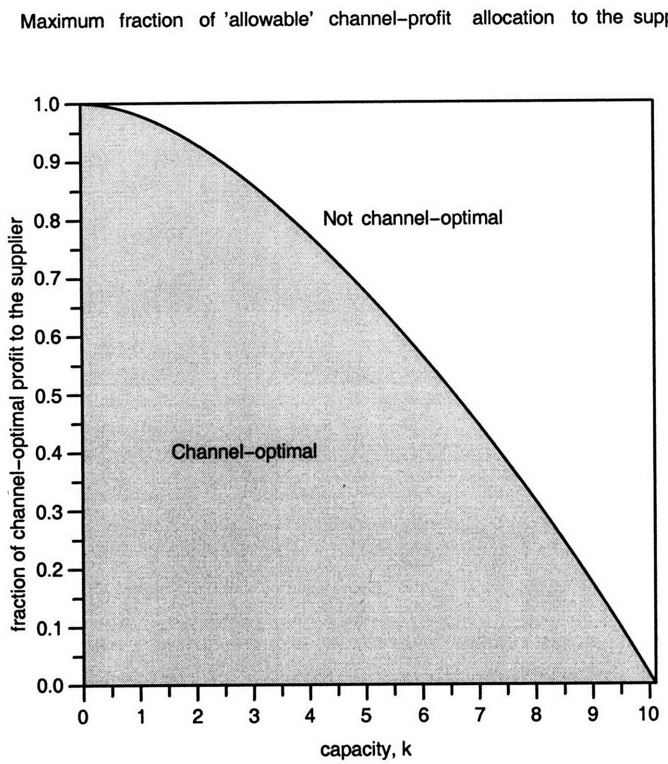
THEOREM 3.4. *Under the same assumptions as in Theorem 3.2, we have*

$$t_s^{max}(k; \bar{F}_1) \geq t_s^{max}(k; \bar{F}_2).$$

Proof. See Section 3.4.4.

Theorem 3.4 suggests that a supplier (and retailer) can find flexibility in profit allocation by joining a supply channel that serves a larger market.

Figure 3-1 **Flexibility in allocating channel-optimal profit as a function of the capacity constraint.**



Note. Demand is distributed according to a Gamma distribution with mean 10 and coefficient of variation $2^{-1/2} \approx .707$. The retail price is $p = 10$, and the cost is $c = 4$. (these are similar to parameters used in Cachon (2004)). Thus, $q^* \approx 10.112$. The shaded region denotes the fractions of profit to the supplier consistent with a channel-optimal outcome (i.e., the set $[0, t_s^{\max}(k; \bar{F})]$). Or in other words, the shaded region represents the fractional allocations of channel-optimal profit to the supplier that are achievable with some wholesale price contract $w \in \mathcal{W}(k)$.

3.3 Risk sharing contracts

We have provided necessary and sufficient conditions so that linear wholesale price contracts coordinate a newsvendor's procurement decision and allow both the supplier(s) and the newsvendor to profit. A natural related question is whether more complicated contracts such as buy-back contracts and revenue-sharing contracts also coordinate a newsvendor's procurement decision when the newsvendor is capacity-constrained.

In this section, we prove that revenue-sharing contracts and buy-back contracts continue to coordinate a newsvendor's ordering decision even when the newsvendor has a constrained resource. Furthermore, we examine the advantages of these more complex contracts over a linear wholesale price contract for a constrained newsvendor.

3.3.1 Buyback and revenue-sharing contracts for unconstrained newsvendor's still coordinate

In Theorem 3.5, we show that buyback contracts, which are known to coordinate an *unconstrained* newsvendor's procurement decision, continue to coordinate a constrained newsvendor's procurement decision.

THEOREM 3.5. *Consider a 1-supplier/1-retailer configuration in the presence of a capacity constraint* $k \geq 0$. *Buyback and revenue sharing contracts coordinate the retailer's ordering decision for the channel, and allow for any profit allocation. In particular, the buyback and revenue sharing contracts that coordinate an unconstrained retailer (in the corresponding unconstrained channel) continue to coordinate the constrained retailer's order decision and allow for any profit allocation.*

Proof. See Section 3.4.5.

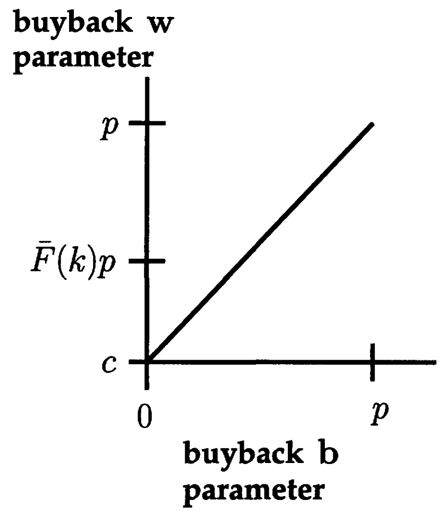
Figure 3-2 illustrates the set of buyback contracts (w, b) that channel-coordinate a capacity-constrained newsvendor (as well as unconstrained retailer) as described in Theorem 3.5. The buyback contracts in Figure 3-2 are the *only* buyback contracts that can coordinate an unconstrained newsvendor. However, the buyback contracts in Figure 3-2 are *not* the only buyback contracts that can coordinate a constrained newsvendor. There are more. In Subsection 3.3.2 we find necessary and sufficient conditions for a buyback contract (w, b) to coordinate a capacity-constrained newsvendor.

3.3.2 Necessary and sufficient conditions for coordination

In Theorem 3.6, we show that the set of buyback contracts that coordinate an constrained newsvendor's procurement decision is a superset of the set of buyback contracts that coordinate an unconstrained newsvendor's procurement decision.

THEOREM 3.6. *Consider a 1-supplier/1-retailer configuration in the presence of a capacity constraint* $k \geq 0$*, and assume that* $\bar{F}(k) > c/p$. *A buyback contract* $(w, b) \in \{(u, v) \mid c \leq u \leq p,\ v \leq u\}$ *coordinates a newsvendor's procurement decision*

Figure 3-2 **Some buyback contracts (w, b) that channel-coordinate a constrained newsvendor.**



Note. The buyback contracts (w, b) that channel-coordinate an unconstrained newsvendor's ordering decision (the ones graphed in this figure) still coordinate a capacity-constrained newsvendor. $\bar{F}(k)p$ is labelled on the y-axis purely for comparison with Figure 3-3.

for the channel if and only if

$$(w, b) \in \mathcal{B}(k) \stackrel{def}{=} \{(u, v) \mid u = (1 - \lambda)v + \lambda p,\ \lambda \in [c/p, \bar{F}(k)]\}.$$

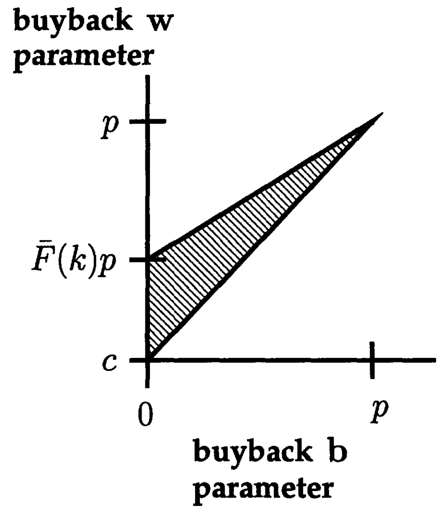
Proof. See Section 3.4.6.

Notice that if capacity becomes large enough (so that $k \geq q^*$), then the set of coordinating buyback contracts implied by Theorem 3.6 and Figure 3-3 simplifies to the 'classical' set of coordinating buyback contracts implied by Theorem 3.5 and Figure 3-2.

3.3.3 Flexibility in allocating channel optimal profit, for a given level of risk

For the constrained newsvendor, notice from Figure 3-3, that for any given buyback parameter b, there is a set of wholesale price parameters such that the resulting buyback contract coordinates the retailer's ordering decision for the channel. However for the unconstrained newsvendor, from Figure 3-2, we see that for any fixed buyback parameter b, there is only one wholesale price parameter that coordinates the channel. In other words, in the unconstrained setting, for any given level of inventory risk that

Figure 3-3 **Necessary and sufficient conditions for a buyback contract (w, b) to channel-coordinate a constrained newsvendor.**



Note. The shaded area represents $\mathcal{B}(k)$, all the buyback contracts (w, b) that channel-coordinate a capacity-constrained newsvendor when $k \le q^c$. Compare with Figure 3-2.

the supplier takes on (represented by the buyback parameter b), there is no flexibility in allocating the channel profit while maintaining coordination. However, in the constrained setting, for any level of inventory risk that the supplier accepts, there is still flexibility in allocating the channel profit. For revenue-sharing contracts, a similar flexibility exists in the constrained setting that is absent in the unconstrained, when the revenue share parameter is held fixed.

Theorem 3.7 formalizes the idea that in a resource constrained setting, buyback contracts have flexibility in allocating the channel-optimal profit when the inventory risk (of loss) is held fixed for the supplier (i.e., the buyback parameter is held fixed). These contracts allow for a range of divisions of the optimal channel profit among the firms. The divisions allowed (without losing coordination) depend on the channel's capacity, k. Unlike our observations for wholesale price contracts in Section 3.1 and Section 3.2 for the implicit revenue requirements, the supplier's share results in an uncertain income similar to the retailer, whose share also results in an uncertain income.

THEOREM 3.7. *Consider a buyback parameter $b \le p$. If the capacity constraint is binding for the channel (i.e., $q^* > k$), there exists a buyback contract $(w, b) \in \mathcal{B}(k)$ that can allocate a fraction t_s of the channel-optimal profit to the supplier and a* *fraction* $1 - t_s$ *to the retailer, if and only if* $t_s \in [t_s^{min}(k; \bar{F}, b), t_s^{max}(k; \bar{F}, b)]$*, where* $t_s^{min}(k; \bar{F}, b) \stackrel{def}{=} b/p$ *and*

$$t_s^{max}(k; \bar{F}, b) \stackrel{def}{=} (1 - b/p) \cdot \frac{(\bar{F}(k) - c/p) \cdot k}{\int_0^k \left(\bar{F}(x) - c/p\right) dx} + b/p.$$

Furthermore, if F *has the IGFR property, then* $t_s^{max}(k; \bar{F}, b)$ *is weakly decreasing as* k *increases in the range* $[0, q^*)$.

Proof. See Section 3.4.7.

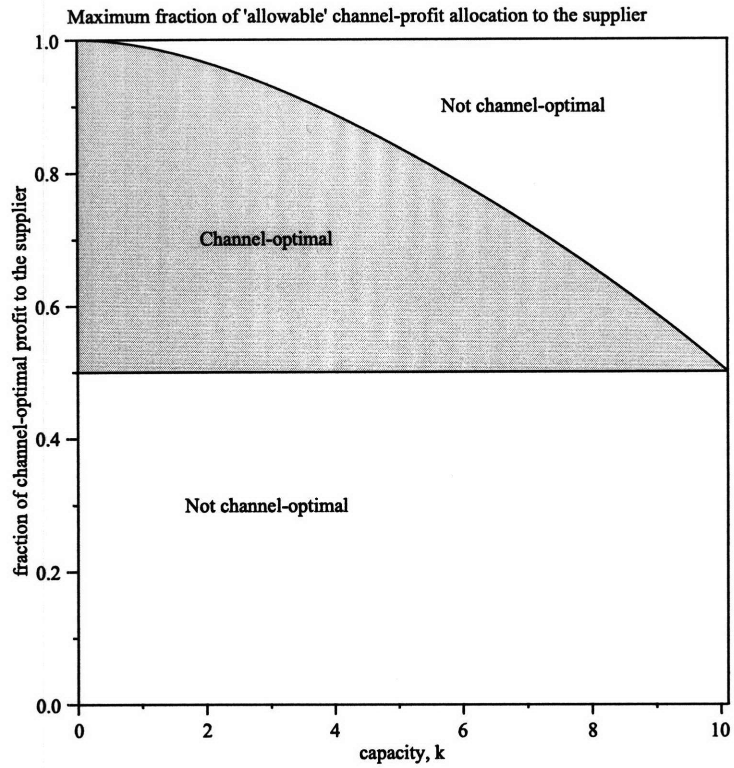
Let us interpret Theorem 3.7 at two extremes values for the capacity k. As k approaches q^*, $t_s^{\max}(k; \bar{F}, b)$ approaches $b/p = t_s^{\min}(k; \bar{F}, b)$. Thus the supplier can only obtain one particular fraction of the channel-optimal profit with any wholesale price contract from the set of coordinating buyback contracts that has a fixed level of inventory risk b (this was to be expected because $\bar{F}(k)p = c$ when $k = q^*$ so that Figure 3-2 and Figure 3-3 are identical and for any b there is only w). At the other extreme, as k tends to zero, $t_s^{\max}(k; \bar{F}, b)$ tends to one. Thus, for a buyback parameter b, *any* allocation of the channel-optimal profit that allocates *at least* b/p of the channel-optimal profit to the supplier becomes possible with some buyback contract from the set of coordinating contracts (this is natural, because as k tends to zero, the set of coordinating contracts becomes the entire region above the rectangle's diagonal in Figure 3-2). See Figure 3-4 for an example illustrating feasible allocations of the channel-optimal profit at intermediate capacity values.

Corollary 3.1 points out that as the supplier takes larger inventory risk (by increasing the buyback parameter), the fraction of optimal channel profit that the supplier can obtain while keeping the channel coordinated increases. This corollary follows directly from Theorem 3.7.

COROLLARY 3.1. *Both* $t_s^{min}(k; \bar{F}_1, b)$ *and* $t_s^{max}(k; \bar{F}_1, b)$ *are strictly increasing and continuous in* b *when* $b \in [0, p)$.

Theorem 3.8 parallels Theorem 3.4 and formalizes the idea that when we serve a larger market the 'flexibility' in allocating the channel-optimal profit 'increases'.

Figure 3-4 Flexibility in allocating channel-optimal profit as a function of the capacity constraint.



Note. We use the same parameters as in Figure 3-1, resulting in $q^* \approx 10.112$. Furthermore, the buyback parameter is $b = p/2$, representing the level of inventory risk the supplier accepts. The shaded region denotes the fractions of profit to the supplier consistent with a channel-optimal outcome (i.e., the set $[b/p, t_s^{\max}(k; \bar{F}, b)]$). Or in other words, the shaded region represents the fractional allocations of channel-optimal profit to the supplier that are achievable with some buyback contract that has a buyback parameter $p/2$ and comes from the set of coordinating contracts defined in Theorem 3.6.

THEOREM 3.8. *Consider a buyback parameter $b \leq p$. Under the same assumptions as in Theorem 3.2, we have*

$$t_s^{max}(k; \bar{F}_1, b) \geq t_s^{max}(k; \bar{F}_2, b).$$

Proof. See Section 3.4.8.

Theorem 3.8 suggests that a supplier (and retailer) can find flexibility in profit allocation by joining a supply channel that serves a larger market.

3.4 Proofs

In order to not disrupt the flow of presentation, the proofs for our results in this chapter are contained here.

3.4.1 Proof: Revenue requirement implicit in $\mathcal{W}(k)$

Proof of Theorem 3.1. If the capacity constraint k is binding for the channel (i.e., $q^* > k$), then $\mathcal{W}(k) = \left[c, p\bar{F}(k)\right]$. For any wholesale price, the supplier's fraction of expected revenue is $r_s(w) \stackrel{\text{def}}{=} wq(w)/E[pS(q(w))]$ where $q(w)$ is the retailer's order quantity for a wholesale price w. Thus for any coordinating linear wholesale price contract $w \in \mathcal{W}(k)$,

$$r_s(w) = \frac{wk}{E[pS(k)]} = \frac{wk}{p\int_0^k \bar{F}(x)\,dx}.$$

The maximum possible value for $r_s(w)$, when $w \in \mathcal{W}(k)$, is

$$r_s^{\max}(k;\bar{F}) = \frac{\left(p\bar{F}(k)\right)k}{pE[S(k)]} = \frac{k\cdot\bar{F}(k)}{\int_0^k \bar{F}(x)\,dx}.$$

Accordingly, the expected revenue that the retailer receives with any linear wholesale price contract $w \in \mathcal{W}(k)$ is at least a fraction

$$1 - \frac{k\cdot\bar{F}(k)}{\int_0^k \bar{F}(x)\,dx} = \frac{\int_0^k \bar{F}(x)\,dx - k\cdot\bar{F}(k)}{\int_0^k \bar{F}(x)\,dx}$$

of the total.

Next we show that if F has the IGFR property, then $r_s^{\max}(k;\bar{F})$ is weakly decreasing as k increases. We first note that

$$\frac{\partial r_s^{\max}(k;\bar{F})}{\partial k} = \frac{\bar{F}(k)}{\int_0^k \bar{F}(x)\,dx}\cdot\left(1 - g(k) - r_s^{\max}(k;\bar{F})\right), \tag{3.1}$$

where $g(x) \stackrel{\text{def}}{=} \frac{xf(x)}{\bar{F}(x)}$ is the generalized failure rate function. From L'Hôpital's rule, we also have $\lim_{k\to 0} r_s^{\max}(k;\bar{F}) = 1$. Furthermore, the function $r_s^{\max}(k;\bar{F})$ is bounded above by 1 and goes to zero as $k \to \infty$. If this function is not weakly decreasing, there must exist some value t such that the derivative of $r_s^{\max}(k;\bar{F})$ at t is zero, and positive for values slightly larger than t. We then have

$$r_s^{\max}(t;\bar{F}) = 1 - g(t) \tag{3.2}$$

since the derivative of $r_s^{\max}(k; \bar{F})$ at t is zero. For k slightly larger than t, the function $r_s^{\max}(k; \bar{F})$ increases, and $g(k)$ is nondecreasing, by the IGFR assumption. But then, equation (3.1) implies that the derivative of $r_s^{\max}(k; \bar{F})$ is negative, which is a contradiction. □

3.4.2 Proof: Revenue requirement as we 'vary' F

Proof of Theorem 3.2. Note that $\int_0^k \bar{F}_i(x)\,dx = (\int_0^\infty \bar{F}_i(x)\,dx) - (\int_k^\infty \bar{F}_i(x)\,dx) = E[D_i] - E[(D_i - k) \cdot 1_{\{D_i > k\}}]$.

Thus,

$$\begin{aligned}\int_0^k \bar{F}_1(x)\,dx &= E[D_1] - E[(D_1 - k) \cdot 1_{\{D_1 > k\}}] \\ &= E[D_2] - E[(D_1 - k) \cdot 1_{\{D_1 > k\}}] \\ &\leq E[D_2] - E[(D_2 - k) \cdot 1_{\{D_2 > k\}}] \\ &= \int_0^k \bar{F}_2(x)\,dx. \end{aligned} \tag{3.3}$$

The inequalities (3.3) and $\bar{F}_1(k) \geq \bar{F}_2(k)$ imply that $\frac{\bar{F}_1(k)}{\int_0^k \bar{F}_1(x)\,dx} \geq \frac{\bar{F}_2(k)}{\int_0^k \bar{F}_2(x)\,dx}$. □

3.4.3 Proof: $\mathcal{W}(k)$'s flexibility in allocating the channel-optimal profit

Proof of Theorem 3.3. We first recall that given our assumption $k < q^*$, the set of coordinating wholesale price contracts is $\mathcal{W}(k) = \left[c, p\bar{F}(k)\right]$.

First we prove that $t_s \in [0, t_s^{\max}(k; \bar{F})]$, if and only if there exists a wholesale price contract $w \in \mathcal{W}(k)$ such that w allocates a fraction t_s of the channel-optimal profit to the supplier (and thus a fraction $1 - t_s$ to the retailer).

For any wholesale price w, the supplier's fraction of the channel's expected profit is $t_s(w) \stackrel{\text{def}}{=} \frac{(w-c)q(w)}{E[pS(q(w))-cq(w)]}$ where $q(w)$ is the retailer's order quantity for a wholesale price w. For any coordinating linear wholesale price contract $w \in \mathcal{W}(k)$, the retailer orders k units; thus we can simplify $t_s(w)$:

$$t_s(w) = \frac{(w-c)k}{E[pS(k)] - ck} = \frac{k(w/p - c/p)}{\int_0^k \left(\bar{F}(x) - c/p\right)\, dx}. \tag{3.4}$$

Observe that $t_s(c) = 0$, $t_s(p\bar{F}(k)) = t_s^{\max}(k;\bar{F})$, and $t_s(w)$ is strictly increasing and continuous in w for $w \in \left[c, p\bar{F}(k)\right]$. Thus, $t_s(w)$ is a one-to-one and onto map from the domain $\left[c, p\bar{F}(k)\right]$ to the range $[0, t_s^{\max}(k;\bar{F})]$.

Next we show that if F has the IGFR property, then $t_s^{\max}(k;\bar{F}) \stackrel{\text{def}}{=} \frac{k\cdot\left(\bar{F}(k)-c/p\right)}{\int_0^k \left(\bar{F}(x)-c/p\right) dx}$ is weakly decreasing as k increases. Define $\bar{H}(x) = \frac{\bar{F}(x)-c/p}{1-c/p}$. Since $\bar{F}(q^*) = c/p$, $\bar{H}(x)$ restricted to the domain $[0, q^*)$ is equal to $1 - H(x)$, where H is a c.d.f. with support $[0, q^*)$.

The generalized failure rate function $g_H(x)$ for H, defined in equation (3.5) below, can be rewritten in terms of the generalized failure rate function $g_F(x)$ for F, as follows:

$$\begin{aligned} g_H(x) &\stackrel{\text{def}}{=} -\frac{x\frac{\partial \bar{H}(x)}{\partial x}}{\bar{H}(x)} && (3.5)\\ &= \frac{xf(x)}{\bar{F}(x) - c/p} \\ &= \frac{\bar{F}(x)}{\bar{F}(x) - c/p} \cdot \frac{xf(x)}{\bar{F}(x)} \\ &= \frac{\bar{F}(x)}{\bar{F}(x) - c/p} \cdot g_F(x). && (3.6) \end{aligned}$$

Furthermore,

$$\frac{\partial}{\partial x}\left(\frac{\bar{F}(x)}{\bar{F}(x) - c/p}\right) = \frac{f(x)\cdot c/p}{(\bar{F}(x) - c/p)^2} \geq 0, \tag{3.7}$$

which implies that $\frac{\bar{F}(x)}{\bar{F}(x)-c/p}$ is weakly increasing (over the domain $[0, q^*)$).

Since $\frac{\bar{F}(x)}{\bar{F}(x)-c/p}$ is positive and weakly increasing and F has the IGFR property, we can deduce that H also has the IGFR property when restricted to the domain $[0, q^*)$ (because of equation (3.6)).

Then, Theorem 3.1 (applied to $\bar{H}$) implies that $\frac{k\cdot\bar{H}(k)}{\int_0^k \bar{H}(x)\, dx}$ is weakly decreasing as

k increases (while k is restricted to the domain $[0, q^*)$). But $t_s^{\max}(k; \bar{F}) = \frac{k \cdot \bar{H}(k)}{\int_0^k \bar{H}(x)\,dx}$, which proves that $t_s^{\max}(k; \bar{F})$ is weakly decreasing as k increases (and $k < q^*$). □

3.4.4 Proof: Flexibility in allocating the channel-optimal profit as we 'vary' F

Proof of Theorem 3.4. Given the definition of $t_s^{\max}(k; \bar{F})$ (cf. Theorem 3.3), we need to prove that

$$\frac{\bar{F}_1(k) - c/p}{\int_0^k \left(\bar{F}_1(x) - c/p\right)\,dx} \geq \frac{\bar{F}_2(k) - c/p}{\int_0^k \left(\bar{F}_2(x) - c/p\right)\,dx}. \tag{3.8}$$

We know that $\bar{F}_1(k) \geq \bar{F}_2(k)$ and that the capacity constraint is binding for the channel's problem under both distributions. Thus,

$$\bar{F}_1(k) - c/p \geq \bar{F}_2(k) - c/p > 0. \tag{3.9}$$

From inequality (3.3) in the proof of Theorem 3.2, we also know that $\int_0^k \bar{F}_1(x)\,dx \leq \int_0^k \bar{F}_2(x)\,dx$. Thus, we can deduce that

$$0 < \int_0^k \left(\bar{F}_1(x) - c/p\right)\,dx \leq \int_0^k \left(\bar{F}_2(x) - c/p\right)\,dx. \tag{3.10}$$

Inequalities (3.9) and (3.10) imply that inequality (3.8) holds. □

3.4.5 Proof: Buyback and revenue-sharing contracts continue to coordinate

Proof of Theorem 3.5. Our proof follows the proof technique given in Cachon (2003) for the 1-supplier, 1-retailer channel in the absence of a capacity constraint.

Our proof has two parts. The first part shows that buyback contracts coordinate a capacity-constrained newsvendor, allocating any fraction of the channel optimal profit among the parties. The second part shows that buyback contracts are equivalent to revenue sharing contracts in a constrained setting.

Under a buyback contract (w, b) the newsvendor pays w per unit to the supplier for each unit ordered and is compensated b per unit for any unit unsold at the end of the sales season. We show that if

$$w = b + c(p - b)/p, \quad b \in [0, p], \tag{3.11}$$

then the buyback contract (w, b) coordinates the capacity-constrained newsvendor's ordering decision, giving the newsvendor $(p - b)/p$ fraction of the channel-optimal profit and the supplier b/p fraction of the channel-optimal profit.

We show that under the above buyback contract, (w, b), the channel-optimal order quantity, q^c, equals the retailer-optimal order quantity, q^r, as well as the supplier-optimal order quantity (i.e., the retailer's order quantity that is optimal from the supplier's point of view), q^s. Indeed,

$$\begin{aligned} q^c &\stackrel{\text{def}}{=} \arg\max_{0 \le q \le k} pS(q) - cq \\ &= \arg\max_{0 \le q \le k} \big((p - b)/p\big)\big(pS(q) - cq\big) \qquad (3.12) \\ &= \arg\max_{0 \le q \le k} (p - b)S(q) - (w - b)q \qquad \text{(Using buyback contract (3.11))} \\ &= \arg\max_{0 \le q \le k} pS(q) - wq + b(q - S(q)) \\ &\stackrel{\text{def}}{=} q^r \end{aligned}$$

and

$$\begin{aligned}
q^c &\stackrel{\text{def}}{=} \arg\max_{0\le q\le k} pS(q) - cq \\
&= \arg\max_{0\le q\le k} \big(b/p\big)\big(pS(q) - cq\big) && (3.13)\\
&= \arg\max_{0\le q\le k} bS(q) - (c - w + b)q && \text{(Using buyback contract (3.11))}\\
&= \arg\max_{0\le q\le k} wq - cq - b(q - S(q))\\
&\stackrel{\text{def}}{=} q^s
\end{aligned}$$

Equations (3.12) and (3.13) prove that the newsvendor and supplier receive $\big((p-b)/p\big)$ and $\big(b/p\big)$ fractions, respectively, of the channel-optimal profit.

Next, we remind the reader that buyback contracts and revenue sharing contracts are equivalent (regardless of the channel's capacity constraint). Under a revenue sharing contract the newsvendor purchases each unit from a supplier at a price of w_r per unit, keeps a fraction f of the revenue, and shares a fraction $(1-f)$ of the revenue with the supplier. A given buyback contract, (w, b), is a revenue sharing contract where the newsvendor purchases at $w - b$ per unit from the supplier and in return shares a fraction b/p of the revenue with the supplier. Similiarly, a given revenue sharing contract, (w_r, f), is a buyback contract where the newsvendor purchases at $w_r + (1 - f)p$ per unit and is compensated $(1 - f)p$ per unit by the supplier for any unsold items at the end of the sales season. Since there is a one-to-one mapping from buyback contracts to revenue sharing contracts and because buyback contracts coordinate a constrained newsvendor's ordering decision, we conclude that revenue sharing contracts also coordinate a constrained newsvendor's ordering decision. □

3.4.6 Proof: Necessary and suff. conditions for risk-sharing contracts to coordinate

Proof of Theorem 3.6. Let

$$B \stackrel{\text{def}}{=} \{(u,v) \mid u = (1-\lambda)v + \lambda p,\ \lambda \in [c/p, F(k)]\}$$

and

$$A \stackrel{\text{def}}{=} \{(u,v) \mid c \leq u \leq p, v \leq u\}.$$

The proof has two parts. First we show every buyback contract $(w,b) \in B \subseteq A$ channel-coordinates the newsvendor's decision. Then, we show that there are no other buyback contracts in the set A that can channel-coordinate the newsvendor's decision. Before we proceed note that the optimal order quantity for the constrained channel is k (because $\bar{F}(k) > c/p$). Thus, the capacity constraint is tight.

First we show that every buyback contract $(w,b) \in B$ channel-coordinates. If $(w,b) \in B$, then $w - b = \lambda(p-b)$ for some $\lambda \in [c/p, \bar{F}(k)]$. The newsvendor orders $\min\{k, \bar{F}^{-1}(\frac{w-b}{p-b})\}$. But $\frac{w-b}{p-b} \in [c/p, \bar{F}(k)]$, therefore $\bar{F}^{-1}(\frac{w-b}{p-b}) \geq k$ and $\min\{k, \bar{F}^{-1}(\frac{w-b}{p-b})\} = k$. The newsvendor thus orders the channel-optimal order quantity for this capacity-constrained channel.

Next we show that there is no buyback contract (w,b) outside of B but in set A that channel-coordinates the newsvendor's action. Assume the contrary. Namely, assume a buyback contract $(w,b) \in A \setminus B$ channel-coordinates the newsvendor's action. Under buyback contract (w,b), the constrained newsvendor orders $\min\{k, \bar{F}^{-1}(\frac{w-b}{p-b})\}$. But since (w,b) channel-coordinates the newsvendor's decision, we have $\min\{k, \bar{F}^{-1}(\frac{w-b}{p-b})\} = k$, since the newsvendor's constraint is tight. Therefore, $\bar{F}^{-1}(\frac{w-b}{p-b}) \geq k$, implying $\frac{w-b}{p-b} \leq \bar{F}(k)$. Furthermore, $\min_{(w,b)\in A} \frac{w-b}{p-b} = \frac{c}{p}$, implying $\frac{w-b}{p-b} \geq \frac{c}{p}$. Thus, $(w,b) \in B$, because $w - b = \lambda(p-b)$ for some $\lambda \in [c/p, \bar{F}(k)]$. But this is a contradiction. □

3.4.7 Proof: Buyback flexibility in allocating the channel-optimal profit

Proof of Theorem 3.7. We first recall that given our assumption $k < q^*$, the set of coordinating buyback contracts is $\mathcal{B}(k) \overset{\text{def}}{=} \{(u,v) \mid u = (1-\lambda)v + \lambda p,\ \lambda \in [c/p, \bar{F}(k)]\}$.

First we prove that $t_s \in [t_s^{\min}(k;\bar{F},b), t_s^{\max}(k;\bar{F},b)]$, if and only if there exists a buyback contract $(w,b) \in \mathcal{B}(k)$ such that (w,b) allocates a fraction t_s of the channel-optimal profit to the supplier (and thus a fraction $1-t_s$ to the retailer).

For any buyback contract (w,b), the supplier's fraction of the channel's expected profit is $t_s(w;b) \overset{\text{def}}{=} \frac{(w-c)q(w,b)-b(q-S(q(w,b)))}{E[pS(q(w,b))-cq(w,b)]}$ where $q(w,b)$ is the retailer's order quantity for a buyback contract (w,b). For any coordinating buyback contract $(w,b) \in \mathcal{B}(k)$, the retailer orders k units; thus we can simplify $t_s(w;b)$:

$$t_s(w;b) = \frac{(w-c)k - b(k-S(k))}{E[pS(k)] - ck} = \frac{1}{p}\frac{(w-c)k - b(\int_0^k \left(1-\bar{F}(x)\right)\,dx)}{\int_0^k \left(\bar{F}(x)-c/p\right)\,dx}. \tag{3.14}$$

Therefore, for any $\lambda \in [c/p, \bar{F}(k)]$, we have

$$\begin{aligned}
t_s\left((1-\lambda)b+\lambda p; b\right) &= \frac{1}{p}\cdot\frac{((1-\lambda)b+\lambda p - c)\cdot k + b\cdot\int_0^k \left(\bar{F}(x)-1\right)\,dx}{\int_0^k \left(\bar{F}(x)-c/p\right)\,dx} \\
&= \frac{1}{p}\cdot\frac{(-\lambda b+\lambda p - c)\cdot k + b\cdot\int_0^k \bar{F}(x)\,dx}{\int_0^k \left(\bar{F}(x)-c/p\right)\,dx} \\
&= \frac{(1-b/p)\lambda k - (c/p)k + (b/p)\cdot\int_0^k \bar{F}(x)\,dx}{\int_0^k \left(\bar{F}(x)-c/p\right)\,dx} \\
&= \frac{(1-b/p)\cdot(\lambda-c/p)\cdot k + (b/p)\cdot\int_0^k \left(\bar{F}(x)-c/p\right)\,dx}{\int_0^k \left(\bar{F}(x)-c/p\right)\,dx} \\
&= \frac{(1-b/p)\cdot(\lambda-c/p)\cdot k}{\int_0^k \left(\bar{F}(x)-c/p\right)\,dx} + b/p.
\end{aligned} \tag{3.15}$$

From equation (3.15), observe that $t_s((1-c/p)b + (c/p)p; b) = b/p$ and $t_s((1-\bar{F}(k))b + \bar{F}(k)p; b) = t_s^{\max}(k;\bar{F},b)$. Furthermore, from equation (3.14), we have that

$t_s(w;b)$ is strictly increasing and continuous in w when w is in the set

$$\left[(1-c/p)b+(c/p)p, (1-\bar{F}(k))b+\bar{F}(k)p\right].$$

Thus, $t_s(w;b)$ is a one-to-one and onto map from the domain $\{(1-\lambda)b+\lambda p \mid \lambda \in [c/p, \bar{F}(k)]\}$ to the range $[t_s^{\min}(k;\bar{F},b), t_s^{\max}(k;\bar{F},b)]$.

From Theorem 3.3, we have that if F has the IGFR property, then $t_s^{\max}(k;\bar{F},b) \stackrel{\text{def}}{=} (1-b/p)\cdot\frac{(\bar{F}(k)-c/p)\cdot k}{\int_0^k\left(\bar{F}(x)-c/p\right)dx}+b/p$ is weakly decreasing as k increases in the range $[0,q^*]$. □

3.4.8 Proof: Buyback flexibility in allocating the channel-optimal profit as we 'vary' F

Proof of Theorem 3.8. From Theorem 3.4, we have that

$$\frac{(\bar{F}_1(k)-c/p)\cdot k}{\int_0^k\left(\bar{F}_1(x)-c/p\right)\,dx} \geq \frac{(\bar{F}_2(k)-c/p)\cdot k}{\int_0^k\left(\bar{F}_2(x)-c/p\right)\,dx}. \tag{3.16}$$

Therefore, we have that

$$(1-b/p)\cdot\frac{(\bar{F}_1(k)-c/p)\cdot k}{\int_0^k\left(\bar{F}_1(x)-c/p\right)\,dx}+b/p \geq (1-b/p)\cdot\frac{(\bar{F}_2(k)-c/p)\cdot k}{\int_0^k\left(\bar{F}_2(x)-c/p\right)\,dx}+b/p. \tag{3.17}$$

□

Chapter 4

An Extension: Coordinating a constrained channel with multiple suppliers

We consider a retailer who orders from multiple suppliers (where each supplier offers one differentiated product), subject to a constraint on the total amount of inventory that can be stocked. The market price for each product is fixed. The retailer faces a random demand for each one of the products (product substitution is not allowed), which is independent of the quantities stocked. In this context, the retailer must make a portfolio decision: which suppliers to order from, and how much to order from each.

For this model, we explore questions similiar to those studied for the single-product case. Do there exist nontrivial wholesale contracts (with the wholesale price different from the unit cost) that coordinate the retailer's portfolio decision, resulting in an order quantity vector which is optimal from the channel's point of view? How does the set of coordinating wholesale price vectors change as we change the retailer's capacity constraint? Is everyone better off or no worse off by picking a wholesale price vector in this set? We will show that our main findings for the 1-supplier/1-retailer case (Theorems 2.1 and 2.2) extend to this more general setting with many suppliers.

Chapter Outline

In Section 4.1, we extend the 1-supplier/1-retailer model (presented in Section 2.1) to include more suppliers and describe the set of coordinating wholesale price contracts for this extended model in Section 4.2. In Section 4.3, we consider an equilibrium model and find conditions that guarantee that the equilibrium wholesale price contract is a coordinating contract. In order to maintain the flow of presentation, the proofs for all our results in this chapter are contained in Section 4.4.

■ 4.1 Many-suppliers/1-retailer model

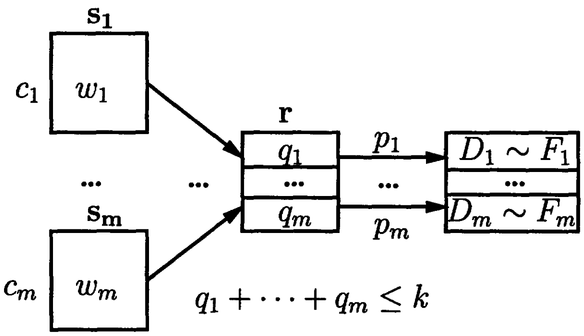
A risk-neutral retailer r orders from $m \geq 2$ risk-neutral suppliers, for m different goods, differentiated by supplier. There is a single sales season, the retailer decides on an order quantity vector/portfolio $(q_1, q_2, \ldots, q_m)$ and orders well in advance of the season, the entire order arrives before the start of the season, and finally demand is realized, resulting in sales for the retailer (without an opportunity for replenishment). Without loss of generality, units remaining at the end of the season are assumed to have no salvage value, and there is no cost for stocking out.

Supplier i has a fixed marginal cost of c_i per unit supplied and charges the retailer a wholesale price $w_i \geq c_i$ per unit ordered. The retailer's price p_i per unit to the market for good i is fixed and, at that price, the demand for good i, D_i, is random with p.d.f. f_i and c.d.f. F_i. We assume that the distribution for demands D_i does not depend on the ordered quantities $(q_1, q_2, \ldots, q_m)$.

The retailer's total capacity is again constrained by some $k > 0$. We assume that the capacity as well as the quantities of the different products are measured with a common set of units (e.g., shelf-space), so that the capacity constraint can be expressed in the form $q_1 + \cdots + q_m \leq k$. The models parameters are summarized in Figure 4-1, with the arrows denoting the direction of product flow.

As before, we assume that the p.d.f. f_i for the demand D_i has support $[0, l_i]$, with $l_i > k$, on which it is positive and continuous. As a consequence, $\bar{F}_i(0) = 1$ and $\bar{F}_i$ is continuously differentiable, strictly decreasing, and invertible on $(0, l_i)$.

Figure 4-1 **"m suppliers & 1 capacity constrained retailer" model with independent downstream demands.**



Note. There are m suppliers. Supplier s_i with marginal cost c_i (per unit) offers good i at wholesale price w_i (per unit) to a capacity-constrained retailer r who faces uncertain demand D_i downstream with c.d.f. F_i (for good i) when the price for the good is fixed at p_i (per unit). The retailer decides on a portfolio of goods to order from the suppliers.

4.1.1 Retailer's problem

For product $i \in \{1, \ldots, m\}$, let $S_i(q_i) \stackrel{\text{def}}{=} \min\{q_i, D_i\}$ denote the (uncertain) amount of sales for product i given that the retailer orders q_i units of product i. The retailer decides on a quantity vector $q^r(w) = (q_1^r, q_2^r, \ldots, q_m^r)$ to order (for a given wholesale price vector w) that maximizes the expected profit $\pi_r(q) \stackrel{\text{def}}{=} E[\sum_{i=1}^m p_i \min\{q_i, D_i\} - w_i q_i]$, subject to the capacity constraint k. In particular, it solves the following convex program with linear constraints in the decision vector, q:

RETAILER(k,w)

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{m} \left(p_i E[S_i(q_i)] - w_i q_i\right) \qquad (4.1) \\ \text{subject to} \quad & k - \sum_{i=1}^{m} q_i \geq 0 \\ & q_i \geq 0, \quad i = 1, \ldots, m. \end{aligned}$$

Because of our assumptions on the distribution of the demand D_i for each product, it can be shown that *RETAILER(k,w)* has a unique solution (vector), which we denote by $q^r(w)$.

4.1.2 Channel's problem

Given the channel's expected profit $\pi_s(q) \stackrel{\text{def}}{=} E[\sum_{i=1}^{m} p_i \min\{q_i, D_i\} - c_i q_i]$ and capacity constraint k, the optimal order quantity vector q^s for the system/channel is the solution to the following convex program, *CHANNEL(k)*, with the same linear constraints on the decision vector, q, but a slightly altered objective function:
CHANNEL(k)

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{m} \left(p_i E[S_i(q_i)] - c_i q_i\right) \qquad (4.2) \\ \text{subject to} \quad & k - \sum_{i=1}^{m} q_i \geq 0 \\ & q_i \geq 0, \quad i = 1, \ldots, m. \end{aligned}$$

Again, because of our assumptions on the demand distributions, it can be shown that *CHANNEL(k)* also has a unique solution (vector) which we denote by q^s. Finally, we denote the unique solution for the unconstrained channel problem, $\max_{\{q \in \mathbb{R}_+^m\}} \pi_s(q)$, by q^*.

■ 4.2 The set $\mathcal{W}(k)$.

In this subsection, (cf. Theorem 4.1 below), we derive conditions under which the vector $w = (w_1, ..., w_m)$ belongs to the set $\mathcal{W}(k)$ of wholesale price vectors that coordinate the retailer's order quantity vector, i.e., $q^r(w) = q^s$.

Throughout this subsection, we assume that the capacity constraint is binding for the channel, that is, $\sum_{i=1}^{m} q_i^* \geq k$ or equivalently

$$\sum_{i=1}^{m} \bar{F}_i^{-1}\left(\frac{c_i}{p_i}\right) \geq k.$$

Otherwise, the problem degenerates into m standard 1-supplier/1-retailer problems in which the only way to coordinate the retailer's action for the supply channel is with a wholesale price contract $w = (c_1, .., c_m)$.

THEOREM 4.1. *Let $Z = \{i \mid q_i^s = 0\} \subset M = \{1, \ldots, m\}$ be the set of products that are not ordered in the channel's portfolio decision problem, and define λ_{m+1} implicitly by the equation:*

$$\sum_{j \in M \setminus Z} F_j^{-1}\left(\frac{p_j - c_j - \lambda_{m+1}}{p_j}\right) = k.$$

For any wholesale price vector $w = (w_1, w_2, ..., w_m)$, the following two conditions are equivalent.

(a) The vector w coordinates the retailer's portfolio decision, i.e., $q^r(w) = q^s$.

(b) There exists some α that satisfies

$$\alpha \in [0, \lambda_{m+1}], \tag{4.3}$$

$$w_j = c_j + \alpha, \qquad \forall\, j \in M \setminus Z, \tag{4.4}$$

$$w_j \geq p_j - \lambda_{m+1} + \alpha, \qquad \forall\, j \in Z. \tag{4.5}$$

Proof. See Section 4.4.1.

Let $\mathcal{W}(k)$ be the set of all w for which $q^r(w) = q^s$. If $Z = \emptyset$ (so that every product is in the channel's optimal portfolio), $\mathcal{W}(k)$ can be represented geometrically by a line segment that starts at the point $(c_1, c_2, \ldots, c_m)$, has unit partial derivatives, and ends at the intersection of the line with the set of vectors w that satisfy $\sum_{i=1}^m \bar{F}_i^{-1}(\frac{w_i}{p_i}) = k$. More generally, if $Z \neq \emptyset$, then $\mathcal{W}(k)$ is the set described by the conditions (4.3) through (4.5).

■ 4.3 The Stackelberg game with multiple suppliers.

We now consider a generalization of the Stackelberg game analyzed in Section 2.3. In the first stage, all the suppliers (the 'leaders') simultaneously choose their wholesale prices w_i. In the second stage, the retailer (the 'follower') chooses an order quantity vector q. When does an equilibrium wholesale price vector of this game belong to the set $\mathcal{W}(k)$? A full exploration of this game is beyond the scope of this chapter and is considered in Chapter 6. We only provide here one result that connects to and generalizes Theorem 2.2.

THEOREM 4.2. *Assume the game is symmetric for the suppliers, that is,* $c_i = c$, $p_i = p$, *and* $F_i = F$, *for every supplier* i. *Furthermore, assume that* F *has the IGFR property and that the retailer is service constrained in the sense that* f *is decreasing. Recall the definition of* q^e *given in equation* (2.3). *If* $k \leq m \cdot q^e$, *then there exists a symmetric equilibrium that belongs to* $\mathcal{W}(k)$.

Proof. See Section 4.4.2.

■ 4.4 Proofs

In order to not disrupt the flow of presentation, the proofs for our results in this chapter are contained here.

4.4.1 Proof: m-suppliers/1-retailer, Set of wholesale prices $\mathcal{W}(k)$

Proof of Theorem 4.1. First, we write the Lagrangian $\mathcal{L}_s(q,\lambda)$ for *CHANNEL(k)* and the Lagrangian $\mathcal{L}_r(q,\gamma)$ for *RETAILER(k,w)*:

$$\mathcal{L}_s(q,\lambda) = \sum_{i=1}^{m} \big(p_i E[\min(q_i, D_i)] - c_i q_i\big) + \sum_{i=1}^{m} \lambda_i q_i + \lambda_{m+1}\big(k - \sum_{i=1}^{m} q_i\big),$$

$$\mathcal{L}_r(q,\gamma) = \sum_{i=1}^{m} \big(p_i E[\min(q_i, D_i)] - w_i q_i\big) + \sum_{i=1}^{m} \gamma_i q_i + \gamma_{m+1}\big(k - \sum_{i=1}^{m} q_i\big).$$

Note that $\pi_s(q)$ and $\pi_r(q)$ are strictly concave for $q \in [0, l_1) \times \cdots [0, l_m)$ because each c.d.f. F_i is strictly increasing over $[0, l_i)$. Because the feasible sets are convex and compact, *CHANNEL(k)* and *RETAILER(k,w)* have unique solutions. Furthermore, because of the concavity of the objective function and the fact that the Slater condition is satisfied, any critical point of the respective Lagrangian (that satisfies the Karush-Kuhn-Tucker conditions) is the unique maximizer in the respective constrained decision problem. Conversely, the optimal solution in the respective constrained decision problem must correspond to a unique critical point of the respective Lagrangian (Sundaram 1996, chap. 7).

The Karush-Kuhn-Tucker conditions for the channel's decision problem, *CHANNEL(k)*, are:

$$p_j \bar{F}_j(q_j) - c_j + \lambda_j - \lambda_{m+1} = 0, \quad j = 1, \ldots, m;$$

$$q_i \geq 0, \quad i = 1, \ldots, m;$$

$$k - \sum_{i=1}^{m} q_i \geq 0;$$

$$\lambda_i q_i = 0, \quad i = 1, ..m;$$

$$\lambda_{m+1}(k - \sum_{i=1}^{m} q_i) = 0;$$

$$\lambda_i \geq 0, \quad i = 1, \ldots, m+1.$$

Let (q^s, λ) denote the unique vector that satisfies these conditions.

The Karush-Kuhn-Tucker conditions for the retailer's decision problem, *RETAILER(k,w)*, are:

$$p_j \bar{F}_j(q_j) - w_j + \gamma_j - \gamma_{m+1} = 0, \quad j = 1, \ldots, m;$$

$$q_i \geq 0, \quad i = 1, \ldots, m;$$

$$k - \sum_{i=1}^{m} q_i \geq 0;$$

$$\gamma_i q_i = 0, \quad i = 1, \ldots, m;$$

$$\gamma_{m+1}(k - \sum_{i=1}^{m} q_i) = 0;$$

$$\gamma_i \geq 0, \quad i = 1, \ldots, m+1.$$

Let $(q^r(w), \gamma)$ denote the unique vector that satisfies these conditions.

Let $M = \{1, \ldots, m\}$ and $Z \stackrel{\text{def}}{=} \{i \in M \mid q_i^s = 0\}$. Therefore, $M \setminus Z$ is the set of items ordered by the system when solving its decision problem. Similarly, let $Z_r(w) \stackrel{\text{def}}{=} \{i \in M \mid q_i^r(w) = 0\}$, so that $M \setminus Z_r(w)$ is the set of items ordered by the retailer when solving its decision problem. Because of the uniqueness of the channel optimal solution, a wholesale price vector $(w_1, \ldots, w_m)$ will coordinate the retailer's portfolio decision (i.e., $q^r(w) = q^s$) if and only if $Z_r(w) = Z$ and $q_i^r(w) = q_i^s$ for every $i \notin Z$.

We claim that $q^r(w) = q^s$ if and only if conditions (4.6)–(4.8) hold:

$$\alpha \in [0, \lambda_{m+1}], \tag{4.6}$$

$$w_j - c_j = w_i - c_i \stackrel{\text{def}}{=} \alpha, \quad \forall i, j \notin Z, \tag{4.7}$$

$$w_t \geq p_t - \lambda_{m+1} + \alpha, \quad \forall t \in Z. \tag{4.8}$$

Suppose $q_i^r(w) = q_i^s$, for all i. Eq. (4.6) follows because

$$0 \leq \gamma_{m+1} \leq \lambda_{m+1}$$

which implies that there exists an $\alpha \in [0, \lambda_{m+1}]$ such that

$$0 \leq \lambda_{m+1} - \alpha = \gamma_{m+1}.$$

Necessity for condition (4.7) follows because $-c_j + \lambda_j - \lambda_{m+1} = -w_j + \gamma_j - \gamma_{m+1}$ and $\gamma_j = \lambda_j = 0$, when $j \notin Z$, implying

$$c_j + \lambda_{m+1} = w_j + \gamma_{m+1} \quad \forall j \notin Z.$$

Necessity for condition (4.8) follows because, when $t \in Z$, $p_t - w_t + \gamma_t - \gamma_{m+1} = 0$ and $\gamma_t \geq 0$ hold, implying

$$\lambda_{m+1} - \alpha = \gamma_{m+1} \geq p_t - w_t \quad \forall t \in Z.$$

Now we show sufficiency by showing that conditions (4.6),(4.7),(4.8) imply $Z_r(w) = Z$ and $q_i^r(w) = q_i^s$ for every $i \notin Z_r(w)$. Using conditions (4.7) and (4.8) we rewrite the KKT conditions for the retailer's decision problem, *RETAILER(k,w)*:

$$p_j\bar{F}_j(q_j) - c_j + \gamma_j - (\gamma_{m+1} + \alpha) = 0, \quad \forall j \notin Z; \tag{4.9}$$

$$p_t\bar{F}_t(q_t) - (p_t - \lambda_{m+1} + \alpha + \delta_t) + \gamma_t - \gamma_{m+1} = 0, \quad \forall t \in Z; \tag{4.10}$$

$$\delta_t = w_t - (p_t - \lambda_{m+1} + \alpha) \geq 0, \quad \forall t \in Z;$$

$$q_i \geq 0, \quad i = 1, \ldots, m;$$

$$k - \sum_{i=1}^{m} q_i \geq 0;$$

$$\gamma_i q_i = 0, \quad i = 1, \ldots, m;$$

$$\gamma_{m+1}(k - \sum_{i=1}^{m} q_i) = 0;$$

$$\gamma_i \geq 0, \quad i = 1, \ldots, m+1.$$

When $\gamma_{m+1} = \lambda_{m+1} - \alpha$, $\gamma_i = 0$ for all $i \notin Z$, and $\gamma_i = \delta_i$ for all $i \in Z$, we have that (q^s, γ) satisfies the KKT conditions for *RETAILER(k,w)*. Note that (q^s, γ) satisfies (4.9) because (q^s, λ) satisfies the KKT conditions for *CHANNEL(k)* and (4.10) is satisfied because $q_t = 0$. Therefore, $q_i^r(w) = q_i^s$ for every $i \in M$. □

4.4.2 Proof: m-suppliers/1-retailer, equilibrium setting

Proof of Theorem 4.2. It can be shown that each supplier's payoff function is continuous and quasi-concave with respect to their own wholesale price; continuity and quasi-concavity follow from our results in Chapter 6 (Lemma 6.1 and the proof of Theorem 6.7, respectively). Furthermore, the game is symmetric and the strategy space (the hypercube of possible wholesale price vectors) is compact and convex. Therefore, by Theorem 2 in Cachon and Netessine (2004), there exists at least one symmetric pure strategy Nash equilibrium (i.e., wholesale price vector), in which all the suppliers charge the same wholesale price w.

Due to the symmetry in the problem, $Z = \emptyset$ (i.e., all the products are included in the channel's optimal portfolio). Furthermore, the capacity constraint is tight for the channel, thus the channel's optimal order vector is $(k/m, \dots, k/m)$. The symmetric equilibrium (identical wholesale prices across products) results in the retailer order vector $(k/m, \dots, k/m)$ because the retailer's capacity constraint is also tight under the condition $k \leq m \cdot q^e$. Thus the wholesale price vector $(w, \dots, w)$ is in the set $\mathcal{W}(k)$ by definition. $\square$

CHAPTER 5

Coordinating a constrained channel: 'make-to-order' retailer

In the supply chain literature, the *'push-pull boundary'* in a supply chain refers to the point in the supply chain at which the supply chain's mode of operation switches from 'building to forecast' to 'reacting to realized demand' (Chopra and Lariviere 2005). This is also called The 'Fulcrum Point' by Martin Christopher and the 'BTF/BTO boundary' (build to forecast/build to order). In Chapter 2, we considered a 1-supplier/1-retailer model and analyzed the set of wholesale price contracts that coordinate that channel in both a negotiation and equilibrium setting when the retailer has a capacity constraint. The 'push-pull boundary' for that channel is between the retailer and the retailer's customers because the retailer makes the order quantity decision based, in part, on the cumulative distribution function for demand (i.e., the 'forecast'). And, therefore, as pointed out in Cachon (2004), the retailer takes on the inventory risk for the channel (i.e., the retailer 'makes to stock/forecast') under a wholesale price contract.

In this chapter, we consider a 1-supplier/1-retailer model similar to the model in Section 2.1 except that we move the 'push-pull boundary' in between the supplier and retailer, so that the retailer makes an order quantity decision that depends only on realized demand and not on the 'forecast', and rather the supplier becomes a 'newsvendor', making a decision based, in part, on the cumulative distribution function for demand (i.e., the 'forecast'). This means that the retailer is running a 'lean' supply chain with no safety stock. Furthermore, we remove the capacity constraint at the retailer and instead place it at the supplier. Cachon (2004) considers a similar model but without a capacity constraint in order to analyze the allocation of inventory risk using different contracts. He notes that the supplier takes on the inventory risk for the channel under a wholesale price contract (because the supplier now 'makes to stock/forecast', for example as a 'drop-shipper', while the retailer 'makes to order'). Cachon and Lariviere (2001) also considers a similar model without a capacity constraint for inducing credible forecast sharing. Our purpose in considering this model differs. In particular, with a capacity constraint at the newsvendor (i.e., the supplier), we analyze the set of wholesale price contracts that coordinate that channel in both a negotiation and equilibrium setting, in the spirit of Chapter 2, comparing these coordinating contracts with the contracts that coordinate the channel described in Section 2.1.

Chapter Outline

In Section 5.1, we provide a stylized 1-supplier/1-retailer model (with the push-pull boundary in between the supplier and retailer) and formally define what it means for a wholesale price contract to coordinate the supplier's decision for a supply channel. Then, in Section 5.2, we describe the set of coordinating 'pull' wholesale price contracts for this model and analyze the size of this set in Section 5.2.1. In Section 5.3, we consider a 1-supplier/1-retailer equilibrium model and prove that a unique equilibrium exists. Then, we provide necessary and sufficient conditions for the equilibrium wholesale price contract to coordinate the supplier's decision (i.e., for the equilibrium wholesale price contract to be included in the set of coordinating contracts). In Section 5.4, we analyze the set of wholesale price contracts that are Pareto-dominated (i.e., for which a different contract exists that enables one firm to better off without making the other firms worse off). The Pareto-dominated contracts are important because they should be avoided in both a negotiation setting as well as an equilibrium setting. Recognizing that in an equilibrium setting the equilibrium wholesale price contract need not be a coordinating contract (due to the conditions we state in Section 5.3), in Section 5.5, we characterize the worst case efficiency loss in an equilibrium setting. Then, in Section 5.6, relating our results from Sections 2.2 and 5.2, we describe the wholesale price contracts that coordinate a supply channel regardless of whether it is operating in push-mode or pull-mode. In order to maintain the flow of presentation, the proofs for all our results in this chapter are contained in Section 5.7.

5.1 Model

A risk-neutral supplier v faces a newsvendor problem when deciding on how much inventory of a single good to prepare and make readily available upon request from a retailer: there is a single sales season, a wholesale price w that a retailer is willing to pay for 'at-once' orders[1], the supplier decides on a quantity q to prepare well in advance of the season, the entire amount q is ready before the start of the season, and finally demand is realized, resulting in sales for the retailer that are immediately satisfied by the supplier (e.g., by drop-shipping) if the supplier has enough inventory in stock. The supplier has no opportunity to prepare more goods during the sales season. Without loss of generality, we assume that units remaining at the end of the season have no salvage value and that there is no cost for stocking out.

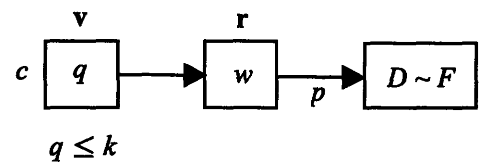
The model's parameters are summarized in Figure 5-1 with the arrows denoting the direction of product flow. Similar to our model in Section 2.1, the supplier has a fixed marginal cost of c per unit supplied and the retailer pays the supplier a wholesale price $w \geq c$ per unit ordered. The retailer's price p per unit to the market is fixed, and we assume that $p > w$. For that price, the demand D is random with probability density function (p.d.f.) f and cumulative distribution function (c.d.f.) F.

We assume that the supplier's capacity is constrained by some $k > 0$; for example, the supplier can only hold k units of inventory, or accept a shipment not larger than

[1]Following the convention in Cachon (2004), we sometimes refer to wholesale price contracts for 'at-once' orders (the main contracts we consider in this chapter) as *pull contracts* and wholesale price contracts for 'prebook' orders (i.e., the contracts in Chapter 2) as *push contracts*.

k. For a different interpretation, k could represent a constraint on the capacity of the channel or a budget constraint.

Figure 5-1 "single capacity constrained supplier & single build-to-order retailer" model.



Note. A capacity-constrained supplier v offers a product, with marginal cost c (per unit), to a retailer r. The retailer faces uncertain demand D downstream with c.d.f. F when the retail price for the product is fixed at p (per unit). However, the retailer orders (or 'pulls') from the supplier only after demand is realized and pays the supplier the wholesale price w per unit ordered. The supplier must decide on a quantity q to prepare in anticipation of the retailer's order.

We assume Assumption 2.1 (from Section 2.1) holds, so that, as a consequence, $\bar{F}(0) = 1$ and $\bar{F}$ is continuously differentiable, strictly decreasing, and invertible on $(0, l)$, where $l > k$.

5.1.1 Supplier's problem

Faced with the retailer's uncertain sales $S(q) \stackrel{\text{def}}{=} \min\{q, D\}$ (and hence the retailer's uncertain order), when preparing q units, and and an agreed upon wholesale price w, the supplier decides on a quantity to prepare for the retailer's future order $S(q)$ so as to maximize expected profit $\pi_v(q) \stackrel{\text{def}}{=} wE[S(q)] - cq$ while satisfying the capacity constraint k. Namely, it solves the following convex program with linear constraints in the decision variable, q:

SUPPLIER(k,w)

$$\begin{aligned} \text{maximize} \quad & wE[S(q)] - cq \qquad (5.1) \\ \text{subject to} \quad & k - q \geq 0 \\ & q \geq 0. \end{aligned}$$

Because of our assumptions on the c.d.f. F, it can be shown that *SUPPLIER(k,w)* has a unique solution, which we denote by $q^v(w)$.

5.1.2 Channel's problem

As in Section 2.1.2, we denote the channel's expected profit by $\pi_s(q) \stackrel{\text{def}}{=} E[pS(q) - cq]$ and observe that under capacity constraint k, the optimal order quantity q^s for the system/channel is the (unique) solution to convex program (2.2), *CHANNEL(k)*, described in Section 2.1.2. Continuing with our convention from that section, we denote the unique solution, $\arg\max_{0 \le q < \infty} \pi_s(q)$, for the unconstrained channel problem by q^*.

5.1.3 Definition: Coordinating the supplier's action

A wholesale price contract w *coordinates* the supplier's quantity preparation decision *for* the supply channel when it causes the supplier to prepare the channel-optimal amount, i.e., $q^v(w) = q^s$. In Section 5.2 we are interested in the following questions: For a fixed capacity k, what is the set of 'pull' wholesale prices $\mathcal{W}_{\text{pull}}(k)$ for which $q^v(w) = q^s$? What does this set $\mathcal{W}_{\text{pull}}(k)$ resemble geometrically?

If there is no capacity constraint (or equivalently if k is very large), 'double marginalization' results in the supplier not preparing enough (i.e., $q^v(w) < q^s$) under any wholesale price contract, $w < p$. In the next section, we will show that when the capacity constraint k is small relative to demand, there exist a set of wholesale price contracts $w < p$ that can coordinate the supplier's preparation quantity, i.e., $q^v(w) = q^s$.

■ 5.2 Set of coordinating wholesale prices

Our first result describes the set of coordinating 'pull' wholesale price contracts under a capacity constraint.

THEOREM 5.1. *In the 1-supplier/1-retailer configuration described in Section 5.1 where the supplier faces a newsvendor problem and has a capacity constraint k, any wholesale price*

$$w \in \mathcal{W}_{pull}(k) \stackrel{def}{=} [c/\bar{F}(\min\{q^*, k\}), p]$$

will coordinate the supplier's decision for the supply channel, i.e., $q^v(w) = q^s$. *Furthermore, if* $q^v(w) = q^s$ *and* $c \leq w \leq p$, *then* $w \in \mathcal{W}_{pull}(k)$.

Proof. See Section 5.7.1.

Notice that if the capacity constraint k is larger than or equal to the unconstrained channel's optimal order quantity, q^*, then $c/\bar{F}(\min\{q^*, k\}) = c/\bar{F}(q^*) = p$, reducing to the 'classic' result in the supply contracts literature stating that wholesale price contracts can not coordinate a channel. However, this is true only when *the capacity constraint is not binding for the channel* (i.e., $q^* \leq k$). When *the capacity constraint* k *is binding for the channel* (i.e., $q^* > k$), then any wholesale price $w \in [c/\bar{F}(k), p]$ will coordinate the retailer's action *and* only wholesale prices in the range $[c/\bar{F}(k), p]$ can coordinate the retailer's action.

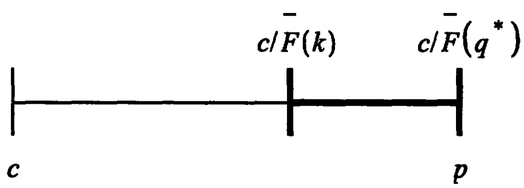
Again, many factors ultimately influence the actual wholesale price (selected from the set $[c, p]$) that the retailer pays the supplier. In the unconstrained setting, regardless of these factors, coordination is not possible with a linear 'pull' wholesale price contract (because the retailer presumably would not agree to pay the retail price to the supplier, making zero profit). However, when the capacity constraint is binding for the channel, coordination becomes *possible* (because the set of coordinating wholesale price contracts becomes $[c/\bar{F}(k), p]$ (rather than $\{p\}$) and ultimately depends on these other factors. Theorem 5.2 in Section 5.3 considers a equilibrium setting where the supplier takes on all the inventory risk (akin to the 'Stackelberg game' in Cachon and Lariviere (2001) and 'pull mode' in Cachon (2004)), and provides additional conditions that must be met so that the 'equilibrium' wholesale price contract is a member of the set of coordinating 'pull' wholesale price contracts, $[c/\bar{F}(k), p]$.

5.2.1 Size of $\mathcal{W}_{\text{pull}}(k)$.

The geometry of the set of wholesale prices $\mathcal{W}_{\text{pull}}(k)$ that coordinate the retailer's decision for the supply channel is depicted in Figure 5-2.

Note that the size of $\mathcal{W}_{\text{pull}}(k)$ is increasing as k decreases. Corollary 5.1 formalizes this notion and follows directly from Theorem 5.1 because $\bar{F}(k)$ is decreasing in k.

Figure 5-2 **The set of wholesale prices that coordinates the actions of a single supplier when building to stock for a single retailer that 'pulls' from that supplier.**



Note. Note that $c/\bar{F}(q^*) = p$ and $\mathcal{W}_{\text{pull}}(k) = [c/\bar{F}(k), p]$ (the interval denoted in bold) when $k \leq q^*$.

COROLLARY 5.1. *If* $0 \leq k_1 \leq k_2$, *then* $\mathcal{W}_{pull}(k_2) \subseteq \mathcal{W}_{pull}(k_1) \subseteq [c, p]$.

Thus, the more constrained the channel is with respect to the channel optimal order quantity, q^*, the larger the set of coordinating 'pull' wholesale price contracts $\mathcal{W}_{\text{pull}}(k)$.

Consider two supply channels selling the same good with the same retail price p and supplier cost c. Assume that the probability of excess demand in the first channel is larger, in the sense $\bar{F}_1(k) \geq \bar{F}_2(k)$. Let $\mathcal{W}^i_{\text{pull}}(k)$ denote the set of coordinating 'pull' wholesale price contracts for channel i when the channel is constrained by k units. The channel with the higher probability of excess demand has a larger set of coordinating wholesale prices. Corollary 5.2 to Theorem 5.1 makes this precise.

COROLLARY 5.2. *Given two demand distributions* F_1 *and* F_2, *if* $\bar{F}_1(k) \geq \bar{F}_2(k) > 0$, *then*

$$\mathcal{W}^2_{pull}(k) \subseteq \mathcal{W}^1_{pull}(k) \subseteq [c, p].$$

Proof. See Section 5.7.2.

■ 5.3 Equilibrium setting.

The equilibrium setting we analyze is a two-stage (Stackelberg) game. In the *first* stage, the retailer (the *'leader'*) offers a wholesale price w to the supplier. In the *second* stage, the supplier (the *'follower'*) chooses an optimal response q, given the wholesale price w. The supplier produces q units before the sales season starts and has no replenishment opportunity. Demand occurs and then the supplier delivers the units to the retailer or the end customer (e.g., drop-shipping) and collects payment for those units from the retailer. Both the supplier and retailer aim to maximize their own profit. The retailer's payoff function is $\pi_r(w;q) = E[(p-w)S(q)]$ and the supplier's payoff function is $\pi_v(q;w) = E[wS(q)-cq]$. Cachon and Lariviere (2001) and Cachon (2004) analyze this Stackelberg game, for an unconstrained channel with one supplier and one retailer. Cachon (2004) finds that when F has the IGFR property, the game results in a unique outcome (q^e, w^e) defined implicitly in terms of the equations

$$p\bar{F}(q^e) - c\left(1 + \left(E[S(q^e)]/\bar{F}(q^e)\right) \cdot \left(f(q^e)/\bar{F}(q^e)\right)\right) = 0, \tag{5.2}$$

$$w^e\bar{F}(q^e) - c = 0. \tag{5.3}$$

Furthermore, they show that the outcome is not channel optimal. In this section, and in Section 5.5, we explore the efficiency of the outcome when the channel has a capacity constraint (i.e., $q \leq k$).

Theorem 5.2 provides necessary and sufficient conditions on the channel's capacity constraint k for the Stackelberg game to result in a channel-optimal equilibrium.

THEOREM 5.2. *Assume F has the IGFR property. Consider the above described game, when the channel capacity is k units. This game has a unique equilibrium, given by $q^{eq}(k) = \min\{k, q^e\}$ and $w^{eq}(k) = \min\{c/\bar{F}(k), w^e\}$, where q^e and w^e are defined by equations* (5.2) *and* (5.3)*, respectively. This equilibrium is channel optimal if and only if*

$$k \leq q^e. \tag{5.4}$$

Under this condition, we have $q^{eq} = k$ and $w^{eq} = c/\bar{F}(k)$.

Proof. See Section 5.7.3.

The function $p\bar{F}(y) - c\left(1 + \left(E[S(y)]/\bar{F}(y)\right) \cdot \left(f(y)/\bar{F}(y)\right)\right)$ represents the retailer's marginal profit on the yth unit, when $y < k$. When F has the IGFR property, the retailer's marginal profit is decreasing in y, while the marginal profit is nonnegative. This fact and equation (5.2) imply that inequality (5.4) is equivalent to the inequality $p\bar{F}(k) - c\left(1 + \left(E[S(k)]/\bar{F}(k)\right) \cdot \left(f(k)/\bar{F}(k)\right)\right) \geq 0$, which can be interpreted as a statement that the retailer's marginal profit (when relaxing the capacity constraint) on the kth unit is greater than zero. Therefore, inequality (5.4) suggests that when the capacity constraint is binding for the retailer's problem (the 'leader' in the Stackelberg game), then the outcome of the game is channel optimal and vice-versa.

If the channel capacity k is 'large enough', so that inequality (5.4) is not satisfied, how inefficient is the channel? In Section 5.5, we provide a distribution-free 'measuring stick' for the efficiency loss in channels with a capacity constraint.

■ 5.4 When can both parties be better off?

The set of coordinating 'pull' wholesale price contracts $\mathcal{W}_{\text{pull}}(k)$ introduced in Theorem 5.1 has many merits in a negotiation setting. For example, such contracts are Pareto optimal. In contrast, Theorem 5.3 examines the set of wholesale price contracts $\mathcal{D}_{\text{pull}}(k)$ that have little merit in that they are Pareto-dominated by some other wholesale price contract in $[c, p]$. A contract is *Pareto-dominated* if there exists an alternative linear wholesale price contract that makes one party better off without making any other party worse off. Having a complete picture of the contracts that are channel-optimal and the contracts that are Pareto-dominated is helpful in a negotiation setting.

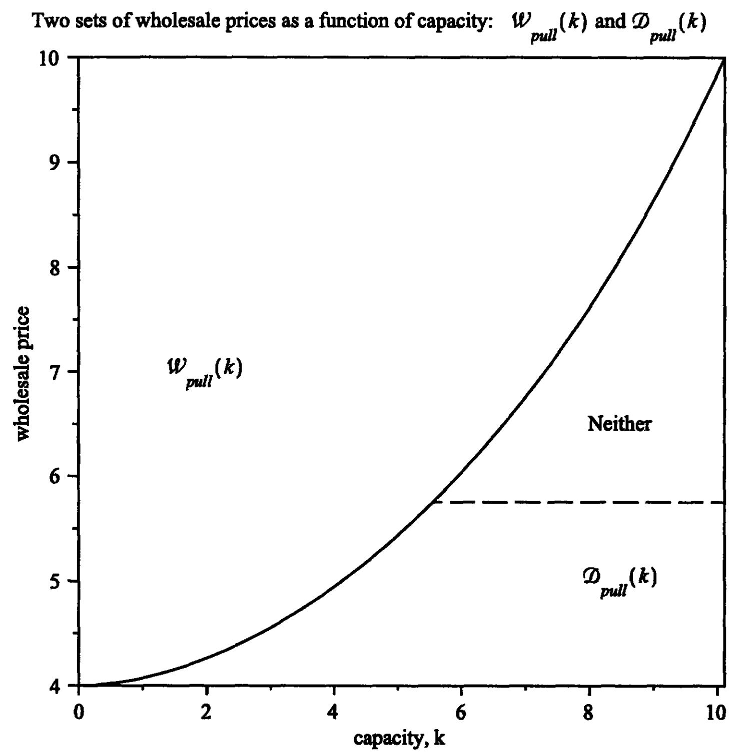
THEOREM 5.3. *Assume F has the IGFR property and that the quantity q^e and wholesale price w^e are defined implicitly in terms of equations* (5.2) *and* (5.3). *If $k \le q^*$, then the set of Pareto-dominated 'pull' wholesale price contracts $\mathcal{D}_{pull}(k)$ is*

$$\mathcal{D}_{pull}(k) \stackrel{def}{=} \left[c, \min\{w^e, c/\bar{F}(k)\}\right) = \left[c, c/\bar{F}(\min\{q^e, k\})\right).$$

Proof. See Section 5.7.4.

Note that $\mathcal{W}_{\text{pull}}(k)$ and $\mathcal{D}_{\text{pull}}(k)$ are disjoint. Corollary 5.3 to Theorem 5.3 formalizes the idea that when k is 'small enough', $\mathcal{W}_{\text{pull}}(k)$ and $\mathcal{D}_{\text{pull}}(k)$ partition the set $[c, p]$. Figure 5-3 illustrates these ideas when demand has a Gamma distribution.

Figure 5-3 **An example illustrating $\mathcal{W}_{\text{pull}}(k)$ and $\mathcal{D}_{\text{pull}}(k)$.**



Note. We use the same parameters as in Figure 3-1, resulting in $q^* \approx 10.112$, the difference being that the push-pull boundary is now between the supplier and retailer so that $q^e \approx 5.552$, and $w^e \approx 5.753$. The set of coordinating wholesale price contracts $\mathcal{W}_{\text{pull}}(k)$ lies above the solid curve. The set of Pareto-dominated wholesale price contracts $\mathcal{D}_{\text{pull}}(k)$ lies under both the solid and dashed curves. The set of contracts that lie *between* the solid and dashed curves are neither in $\mathcal{W}_{\text{pull}}(k)$ nor in $\mathcal{D}_{\text{pull}}(k)$. Such contracts do not coordinate the channel, but nevertheless, are not Pareto dominated by coordinating wholesale contracts. See Figure 2-3 to compare with the set of 'push' contracts $\mathcal{W}(k)$ and $\mathcal{D}(k)$ for the same problem parameters.

COROLLARY 5.3. *Assume F has the IGFR property. If $k \leq q^e$, then*

$$\mathcal{W}_{pull}(k) \cup \mathcal{D}_{pull}(k) = [c, p], \tag{5.5}$$

$$\mathcal{W}_{pull}(k) \cap \mathcal{D}_{pull}(k) = \emptyset. \tag{5.6}$$

Corollary 5.3 is especially interesting: it asserts that when capacity is small enough there are only two types of contracts: 'good contracts', $\mathcal{W}_{\text{pull}}(k)$, and 'bad contracts', $\mathcal{D}_{\text{pull}}(k)$. Furthermore, both parties will always have a reason to avoid the 'bad contracts' because they are Pareto-dominated by some channel-optimal contract in the set $\mathcal{W}_{\text{pull}}(k)$.

5.5 Efficiency Loss.

When the outcome of the Stackelberg game we described in Section 5.3 results in a 'pull' wholesale price contract that is not channel optimal, how much does the channel 'lose' as a result? What is the 'price' paid for the 'gaming' between the supplier and retailer? To quantify the answer we analyze the worst-case efficiency using the same definition of efficiency introduced in Section 2.5.

In particular, for a channel with a capacity constraint k and probability $\bar{F}(k)$ of excess demand, we define the parameter $\beta \stackrel{\text{def}}{=} \frac{\max\{\bar{F}(k), c/p\}}{c/p}$. The parameter β depends on the probability $\bar{F}(k)$ of excess demand and takes values from the set $[1, p/c]$. It quantifies how constrained the channel is with respect to the channel optimal order quantity q^*, because $\beta \stackrel{\text{def}}{=} \frac{\max\{\bar{F}(k), c/p\}}{c/p} = \frac{\max\{\bar{F}(k), \bar{F}(q^*)\}}{\bar{F}(q^*)}$. In the Stackelberg game with a capacity constraint k and parameter β, the efficiency, $\text{Eff}(k, \beta)$, is defined according to equation (5.7) below.

$$\text{Eff}(k, \beta) = \inf_{F \in \mathcal{F}(k,\beta)} \frac{\text{Channel profit under 'gaming'}}{\text{Optimal channel profit}} = \inf_{F \in \mathcal{F}(k,\beta)} \frac{E[pS(q^{eq}(k)) - cq^{eq}(k)]}{E[pS(q^s(k)) - cq^s(k)]} \tag{5.7}$$

The set $\mathcal{F}(k, \beta)$ represents the set of probability distributions that satisfy Assumption 2.1, have the IGFR property, and such that the probability $\bar{F}(k)$ of excess demand satisfies $\frac{\max\{\bar{F}(k), c/p\}}{c/p} = \beta$. Note that $\text{Eff}(k, \beta)$ is a distribution-free method of quantifying the *worst-case* efficiency. When $\text{Eff}(k, \beta)$ is low (much smaller than one), there is significant efficiency loss due to 'gaming'.

THEOREM 5.4. *Define* $m \stackrel{def}{=} (p - c)/p$ *(the channel's gross profit margin). Also when* $\beta \in [1, 1/(1 - m)]$*, define the function*

$$x(m, \beta) \stackrel{def}{=} \underset{(\beta-1) \le x \le m/(1-m)}{\arg\max} \frac{1}{x} \cdot \left(\frac{1 + x}{\beta}\right)^{1/x} \cdot \left(\frac{\beta}{1 - x} - 1\right) - \frac{2x}{1 - x} - 1. \tag{5.8}$$

For the Stackelberg game described in Section 5.3, we have

$$\mathit{Eff}(k,\beta) = \left(\frac{1}{x(m,\beta)} \cdot \left(\frac{1 + x(m,\beta)}{\beta} \right)^{1/x(m,\beta)} \cdot \left(\frac{\beta}{1 - x(m,\beta)} - 1 \right) - \frac{2x(m,\beta)}{1 - x(m,\beta)} - 1 \right)^{-1}. \tag{5.9}$$

Proof. See Section 5.7.5.

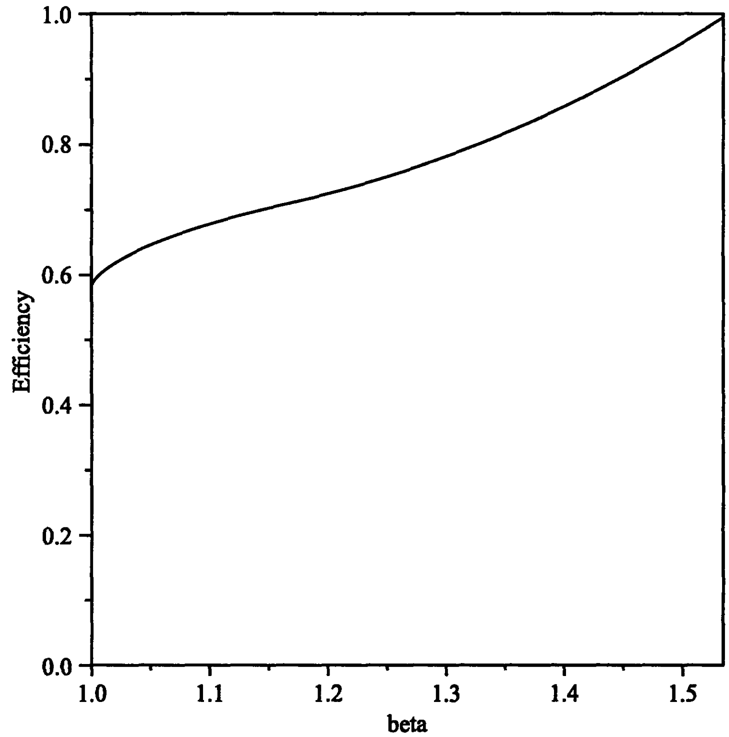
Note that $\text{Eff}(k,\beta)$ is decreasing in the channel's gross profit margin m and increasing in β. When $\beta = 1$, the channel is not constrained and $\text{Eff}(k,\beta)$ equals $(e-1)^{-1}$ which matches the result in Perakis and Roels (2006). On the other hand, when the channel is most constrained (i.e., $k \approx 0$, $\bar{F}(k) \approx 1$, and $\beta \approx p/c$), then $\text{Eff}(k,\beta)$ simplifies to 1. In other words there is no efficiency loss because the equilibrium outcome involves the retailer ordering exactly k. Our result is thus a more general version of the 'two-stage pull-mode PoA' result in Perakis and Roels (2006) in that we account for a capacity constraint. Also our proof technique differs from and complements Perakis and Roels (2006), in that we indirectly optimize over the space of probability distributions by optimizing over the space of generalized failure rates.

Figure 5-4 provides an example of the $\text{Eff}(k,\beta)$ when the channel's gross profit margin is 35 percent. Figure 5-4 illustrates that for channels with smaller capacity (i.e., higher β), the worst-case efficiency (as measured by $\text{Eff}(k,\beta)$) is larger. Comparing the (push-mode) supply chain example in Figure 2-4 with the (pull-mode) supply chain example in Figure 5-4, we see that the supply chain's worst case efficiency in pull-mode is better than in push-mode.

■ 5.6 Coordinating wholesale prices for both push-mode and pull-mode

Consider both the 1-supplier/1-retailer configuration described in Section 2.1 (i.e., push-mode) where the retailer faces a newsvendor problem and has a capacity constraint k_r and the 1-supplier/1-retailer configuration described in Section 5.1 (i.e.,

Figure 5-4 An example illustrating Eff(k, β) when $m = 0.35$.



Note. We fix the margin $(p - c)/p = 0.35$ and see how Eff(k, β) changes as a function of β.

pull-mode) where the supplier faces a newsvendor problem and has a capacity constraint k_v. From observing Figure 2-2 and Figure 5-2, we see that when the capacities k_r, k_v are 'small enough' (relative to demand) there exist wholesale price contracts that coordinate a supply chain regardless of whether it is operating in push-mode or pull-mode.

Theorem 5.5, below, formalizes how small the capacities must be relative to demand so that there exist such wholesale price contracts. Furthermore, our theorem provides the interval of wholesale price contracts that coordinate a supply chain regardless of whether it is operating in push-mode or pull-mode. In the statement of the theorem, we use the parameters $\beta_{\text{push}} \stackrel{\text{def}}{=} \frac{\max\{\bar{F}(k_r), \bar{F}(q^*)\}}{\bar{F}(q^*)}$ and $\beta_{\text{pull}} \stackrel{\text{def}}{=} \frac{\max\{\bar{F}(k_v), \bar{F}(q^*)\}}{\bar{F}(q^*)}$. As pointed out in Sections 2.5 and 5.5, respectively, those parameters express how constrained the channel is with respect to the channel optimal order quantity q^* when it's operating in push-mode and pull-mode, respectively.

THEOREM 5.5. *Any wholesale price*

$$w \in \mathcal{W}_{both}(k_r, k_v) \stackrel{def}{=} \{w \mid c/\bar{F}(\min\{q^*, k_v\}) \leq w \leq p\bar{F}(\min\{q^*, k_r\})\}$$

will coordinate the supply chain regardless of whether the supply chain is operating in push-mode or pull-mode. Furthermore, if a wholesale price w *coordinates the supply chain in both push-mode and pull-mode and* $c \leq w \leq p$*, then* $w \in \mathcal{W}_{both}(k_r, k_v)$*. The set* $\mathcal{W}_{both}(k_r, k_v)$ *is not empty if and only if* $\beta_{push} \cdot \beta_{pull} \geq p/c$*.*

Proof. See Section 5.7.6.

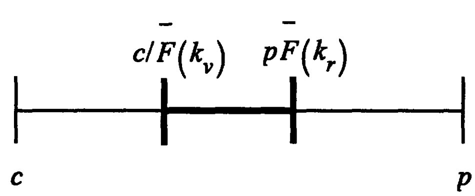
Notice that if the capacities k_r, k_v are larger than or equal to the unconstrained channel's optimal order quantity, q^*, then $c/\bar{F}(\min\{q^*, k_v\}) = c/\bar{F}(q^*) = p$ and $p\bar{F}(\min\{q^*, k_r\}) = p\bar{F}(q^*) = c$, so that no wholesale price coordinates both the channel operating in pull-mode and the channel operating in push-mode. However, even when *the capacity constraints* k_r, k_v *are binding for the channel* (i.e., $q^* > k_r$, k_v or, equivalently, β_{push}, $\beta_{\text{pull}} > 1$), we do not have a guarantee that there exist wholesale prices that coordinate both the channel operating in push-mode and the channel operating pull-mode. Rather, the channel needs to be constrained 'enough' (i.e., $\beta_{\text{push}} \cdot \beta_{\text{pull}} \geq p/c$) for a wholesale price $w \in \mathcal{W}_{\text{both}}(k_r, k_v)$ to coordinate the supply chain regardless of it's mode of operation *and* only wholesale prices in the set $\mathcal{W}_{\text{both}}(k_r, k_v)$ can coordinate the supply chain regardless of it's mode of operation.

As pointed out already, many factors ultimately influence the actual wholesale price (selected from the set $[c, p]$) that the retailer pays the supplier. In the unconstrained setting, regardless of these factors, coordinating both a supply chain operating in pull-mode and a supply chain operating in push-mode is not possible with any wholesale price contract. However, when the capacity constraint is small enough for the channel, coordination becomes *possible* and ultimately depends on these other factors.

5.6.1 Size of $\mathcal{W}_{\text{both}}(k_r, k_v)$.

The geometry of the set of wholesale prices $\mathcal{W}_{\text{both}}(k_r, k_v)$ that coordinate the retailer's decision for the supply channel (operating in push-mode) and the supplier's decision for the supply channel (operating in pull-mode) is depicted in Figure 5-5.

Figure 5-5 The set of wholesale prices that coordinates a 1-supplier/1-retailer configuration regardless of whether it is operating in push-mode or pull-mode.



Note. Note that $\mathcal{W}_{\text{both}}(k) = [c/\bar{F}(k_v), p\bar{F}(k_r)]$ (the interval denoted in bold) when $\beta_{\text{push}} \cdot \beta_{\text{pull}} \geq p/c$ and $k_v,\ k_r \leq q^*$.

Note that the size of $\mathcal{W}_{\text{both}}(k_r, k_v)$ is increasing as k_r or k_v decreases. Corollary 5.4 formalizes this notion and follows directly from Theorem 5.5 because $\bar{F}(k)$ is decreasing in k.

COROLLARY 5.4. *If* $0 \leq k_r^1 \leq k_r^2$, *and* $0 \leq k_v^1 \leq k_v^2$ *then* $\mathcal{W}_{both}(k_r^2, k_v^2) \subseteq \mathcal{W}_{both}(k_r^1, k_v^1) \subseteq [c, p]$.

Thus, the more constrained the channel is with respect to the channel optimal order quantity, q^*, the larger the set of coordinating wholesale price contracts $\mathcal{W}_{\text{both}}(k_r, k_v)$ that coordinate a channel regardless of its mode of operation (i.e., push or pull).

In fact, we can state a stronger result, i.e., Corollary 5.5, below. Consider two channels, 1 and 2, (each with one supplier and one retailer) that sell a single good. Channel i sells its good at retail price p_i per unit, facing uncertain demand with c.d.f. F_i, with the supplier facing a cost c_i per unit, so that $q_i^* = \bar{F}_i^{-1}(c_i/p_i)$ is the optimal quantity for the channel to prepare before the sales season begins. When channel i operates in push-mode, suppose the retailer faces a capacity constraint k_r^i, and when channel i operates in pull-mode, suppose the supplier faces a capacity constraint k_v^i. Recall that we can measure how constrained channel i is when operating in push-mode by considering the parameter $\beta_{\text{push}}^i \stackrel{\text{def}}{=} \frac{\max\{\bar{F}_i(k_r^i), \bar{F}_i(q_i^*)\}}{\bar{F}_i(q_i^*)}$ and how constrained channel i

is when operating in pull-mode by considering the parameter $\beta^i_{\text{pull}} \stackrel{\text{def}}{=} \frac{\max\{\bar{F}_i(k^i_v),\bar{F}_i(q^*_i)\}}{\bar{F}_i(q^*_i)}$. Let $\mathcal{W}^i_{\text{both}}$ denote the set of coordinating wholesale price contracts for channel i, regardless of its mode of operation (i.e., $\mathcal{W}^i_{\text{both}} = \mathcal{W}_{\text{both}}(k^i_r, k^i_v)$).

COROLLARY 5.5. *The inequalities $c_2 \cdot \beta^2_{push} \leq c_1 \cdot \beta^1_{push}$ and $\beta^2_{pull}/p_2 \leq \beta^1_{pull}/p_1$ hold, if and only if, $\mathcal{W}^2_{both} \subseteq \mathcal{W}^1_{both} \subseteq [c,p]$.*

Proof. See Section 5.7.7.

We illustrate Corollary 5.5, by applying it to answer the question: how does the set $\mathcal{W}_{\text{both}}$ change when 'demand increases'? More formally, consider two supply channels selling the same good with the same retail price p and supplier cost c. Assume that the probability of excess demand in the first channel is larger, in the sense $\bar{F}_1(k) \geq \bar{F}_2(k)$ when $k = k_r, k_v$. Let $\mathcal{W}^i_{\text{both}}(k_r, k_v)$ denote the set of coordinating wholesale price contracts for channel i (regardless of the mode of operation). Applying Corollary 5.5, we have that the channel with the higher probability of excess demand has a larger set of coordinating wholesale prices. Corollary 5.6 makes this precise.

COROLLARY 5.6. *Given two demand distributions F_1 and F_2, if $\bar{F}_1(k) \geq \bar{F}_2(k) > 0$ when $k = k_r,\ k_v$, then*

$$\mathcal{W}^2_{both}(k_r, k_v) \subseteq \mathcal{W}^1_{both}(k_r, k_v) \subseteq [c,p].$$

Observe that Corollary 5.6 follows from Corollary 5.5 because the inequality $\bar{F}_1(k) \geq \bar{F}_2(k) > 0$ when $k = k_r, k_v$ implies that the inequalities $\beta^2_{\text{push}} \leq \beta^1_{\text{push}}$ and $\beta^2_{\text{pull}} \leq \beta^1_{\text{pull}}$ hold.

■ 5.7 Proofs

In order to not disrupt the flow of presentation, the proofs for our results in this chapter are contained here.

5.7.1 Proof: 1-supplier/1-retailer, Set of wholesale prices $\mathcal{W}_{\textbf{pull}}(k)$

Proof of Theorem 5.1. Observe that the objective function for the channel's problem (see Section 5.1.2) is the same as in our model in Chapter 2 (see Section 2.1.2). Also, we can express the supplier's objective function as

$$wE[S(q)] - cq = \frac{p}{w} \cdot \left(pE[S(q)] - \frac{pc}{w}q\right).$$

Therefore, from Theorem 2.1 in Chapter 2, we have that q^s is a solution to the supplier's problem when offered wholesale price $w \in [c,p]$ (i.e., $q^v(w) = q^s$) if and only if $pc/w \in \mathcal{W}(k) \overset{\text{def}}{=} \left[c, p\bar{F}(\min\{q^*,k\})\right]$. But $pc/w \in \left[c, p\bar{F}(\min\{q^*,k\})\right]$ is equivalent to $w \in \left[c/\bar{F}(\min\{q^*,k\}), p\right]$. □

5.7.2 Proof: Impact of size of Market on size of $\mathcal{W}_{\textbf{pull}}(k)$

Proof of Corollary 5.2. Let $q_i^* = \bar{F}_i^{-1}(c/p)$ be the order quantity (for an unconstrained channel) under the demand distribution F_i.

If $k \leq q_2^*$, then $c/p \leq \bar{F}_2(k) \leq \bar{F}_1(k)$, which implies that $k \leq q_1^*$. Thus, $\mathcal{W}_{\text{pull}}^i(k) = [c/\bar{F}_i(k), p]$ for $i \in 1,2$. Since $\bar{F}_2(k) \leq \bar{F}_1(k)$, we can conclude that $\mathcal{W}_{\text{pull}}^2(k) \subseteq \mathcal{W}_{\text{pull}}^1(k) \subseteq [c,p]$.

Similarly, if $q_2^* < k$, then $\mathcal{W}_{\text{pull}}^2(k) = \{c\}$. Thus, $\mathcal{W}_{\text{pull}}^2(k) \subseteq \mathcal{W}_{\text{pull}}^1(k)$. □

5.7.3 Proof: When is the equilibrium of the Stackelberg game channel optimal?

Proof of Theorem 5.2. The supplier's profit function $\pi_v(q;w)$ under a wholesale price contract w is defined as $\pi_v(q;w) \overset{\text{def}}{=} E[wS(q) - cq]$. Since $\pi_v(q;w)$ is concave, in q, we can use the first order conditions and conclude that for a wholesale price $w \in [c,p]$, the constrained supplier's order quantity $q^v(w)$ is given by

$$q^v(w) = \min\{k, \bar{F}^{-1}(c/w)\}. \tag{5.10}$$

The retailer's profit function $\pi_r(w;q)$ under a wholesale price contract w is defined as $\pi_r(w;q) \stackrel{\text{def}}{=} E[(p-w)S(q)]$. Since $q^v(w)$ is the supplier's best response in the second stage to a wholesale price w by the retailer in the first stage, equation (5.10) allows us to express the retailer's objective function as follows:

$$\pi_r(w) = \begin{cases} E[(p - c/\bar{F}(q^v(w)))\, S(q^v(w))], & \text{if } c \le w \le \min\{c/\bar{F}(k), p\}; \\ E[(p-w)\, S(k)], & \text{if } \min\{c/\bar{F}(k), p\} < w \le p. \end{cases} \tag{5.11}$$

For $w < \min\{c/\bar{F}(k), p\}$, note that

$$\frac{\partial \pi_r(w)}{\partial w} = \left(p\bar{F}(q^v(w)) - c\left(1 + \frac{E[S(q^v(w))]}{\bar{F}(q^v(w))} \cdot \frac{f(q^v(w))}{\bar{F}(q^v(w))}\right)\right) \cdot \frac{\partial q^v(w)}{\partial w}. \tag{5.12}$$

Cachon (2004) (Lemma 1) shows that when F has the IGFR property, the function $\left(E[S(y)]/\bar{F}(y)\right) \cdot \left(f(y)/\bar{F}(y)\right)$ is increasing in y, for $y > 0$. Therefore, the function $p\bar{F}(y) - c\left(1 + \left(E[S(y)]/\bar{F}(y)\right) \cdot \left(f(y)/\bar{F}(y)\right)\right)$ is strictly decreasing in y when it is nonnegative and equals zero at q^e (see equation (5.2)). We can deduce that

$$p\bar{F}(q^v(w)) - c\left(1 + \left(E[S(q^v(w))]/\bar{F}(q^v(w))\right) \cdot \left(f(q^v(w))/\bar{F}(q^v(w))\right)\right) > 0 \tag{5.13}$$

for $w < w^e$ (because $q^v(w) < q^e$). Furthermore, $\frac{\partial q^v(w)}{\partial w} > 0$ for $w < c/\bar{F}(k)$. Therefore, we can conclude that $\frac{\partial \pi_r(w)}{\partial w} > 0$ for $w < \min\{c/\bar{F}(k), w^e\}$.

Either the inequality $c/\bar{F}(k) \le w^e$ holds or the inequality $w^e < c/\bar{F}(k)$ holds. First assume the inequality $w^e < c/\bar{F}(k)$ holds. We know $\frac{\partial \pi_r(w)}{\partial w} > 0$ for $w < \min\{c/\bar{F}(k), w^e\} = w^e$. Furthermore, the function

$$p\bar{F}(y) - c\left(1 + \left(E[S(y)]/\bar{F}(y)\right) \cdot \left(f(y)/\bar{F}(y)\right)\right)$$

is negative when $y > q^e$, so that, we have $\frac{\partial \pi_r(w)}{\partial w} < 0$ for $w^e < w < \min\{c/\bar{F}(k), p\}$. Equation (5.11) implies that $\pi_r(w)$ is decreasing linearly between $\min\{c/\bar{F}(k), p\}$ and p. Since $\pi_r(w)$ is continuous over $[c,p]$, we have $w^{eq}(k) = w^e$ and equations (5.10) and (5.3) imply $q^{eq}(k) = q^e$. The inequality $w^e < c/\bar{F}(k)$ is equivalent to the inequality

$q^e < k$ (see equation (5.3)). Therefore, when $q^e < k$ holds, the inequality $w^{eq}(k) = w^e < \min\{c/\bar{F}(k), p\} = c/\bar{F}(\min\{q^*, k\})$ holds and we can deduce that $w^{eq}(k) \notin \mathcal{W}_{\text{pull}}(k)$ (using Theorem 5.1).

Next assume $c/\bar{F}(k) \leq w^e$ holds. We know $\frac{\partial \pi_r(w)}{\partial w} > 0$ for $w < \min\{c/\bar{F}(k), w^e\} = c/\bar{F}(k)$. And since $\pi_r(w)$ is decreasing linearly between $\min\{c/\bar{F}(k), p\}$ and p, equation (5.11) implies $w^{eq}(k) = c/\bar{F}(k)$ and equation (5.10) implies $q^{eq}(k) = k$. The inequality $c/\bar{F}(k) \leq w^e$ is equivalent to the inequality $k \leq q^e$ (see equation (5.3)). Therefore, when $k \leq q^e$ holds, the equality $w^{eq}(k) = c/\bar{F}(k) = \min\{c/\bar{F}(k), p\} = c/\bar{F}(\min\{q^*, k\})$ holds and we can deduce that $w^{eq}(k) \in \mathcal{W}_{\text{pull}}(k)$ (again using Theorem 5.1). □

5.7.4 Proof: The set of Pareto-dominated contracts $\mathcal{D}_{\text{pull}}(k)$ as a function of capacity

Proof of Theorem 5.3. Equation (5.10) allows us to express the supplier's objective function as follows:

$$\pi_v(w) = \begin{cases} (c/\bar{F}(q^v(w)))E[S(q^v(w))] - cq^v(w), & \text{if } c \leq w < c/\bar{F}(k); \\ wE[S(k)] - ck, & \text{if } c/\bar{F}(k) \leq w \leq p. \end{cases} \tag{5.14}$$

Note that $\pi_v(w)$ is strictly increasing in w, when $w \in (c, c/\bar{F}(k))$ because

$$\frac{\partial \pi_v(w)}{\partial w} = c\left(E[S(q^v(w))]/\bar{F}(q^v(w))\right) \cdot \left(f(q^v(w))/\bar{F}(q^v(w))\right) \cdot \frac{\partial q^v(w)}{\partial w} > 0.$$

Furthermore, when $w \in (c/\bar{F}(k), p)$, note that $\pi_v(w)$ is strictly increasing in w. From the proof of Theorem 5.2, we know that the retailer's profit $\pi_r(w)$ is also strictly increasing for $w < \min\{w^e, c/\bar{F}(k)\}$. Therefore, any wholesale price contract in the set $[c, \min\{w^e, c/\bar{F}(k)\})$ is Pareto-dominated by $\min\{w^e, c/\bar{F}(k)\}$.

Since the retailer's profit is decreasing as the wholesale price w increases from $\min\{w^e, c/\bar{F}(k)\}$ (see the proof of Theorem 5.2) but the supplier's profit is increasing as the wholesale price increases, we can conclude that any wholesale price contract in the set $[\min\{w^e, c/\bar{F}(k)\}, p]$ is not Pareto-dominated. Thus, the set of Pareto-dominated wholesale price contracts in $[c, p]$ is exactly $\mathcal{D}_{\text{pull}}(k) = [c, \min\{w^e, c/\bar{F}(k)\})$. □

5.7.5 Proof: Efficiency loss for a two-stage pull channel with capacity constraint

LEMMA 5.1. *Assume F has the IGFR property and that the quantity q^e is defined implicitly in terms of equation* (5.2)*. If $q^e \le k \le q^s$, then*

$$\frac{p\left(\int_0^k \bar{F}(x)\,dx\right) - ck}{p\left(\int_0^{q^e} \bar{F}(x)\,dx\right) - cq^e} \le \frac{1+g(q^e)}{g(q^e)(1-g(q^e))}\left(\frac{k}{q^e}\right)^{1-g(q^e)} - \frac{2g(q^e)}{1-g(q^e)} - \frac{1}{g(q^e)}\frac{k}{q^e} - 1. \tag{5.15}$$

Proof of Lemma 5.1. Recall the generalized failure rate function $g(y)$ for c.d.f. F is defined as $g(y) \stackrel{\text{def}}{=} -y\frac{\partial \bar{F}(y)}{\partial y}/\bar{F}(y)$. Since $\bar{F}(y) = e^{-\int_0^y f(t)/\bar{F}(t)\,dt} = e^{-\int_0^y g(t)/t\,dt}$, we have

$$\begin{aligned}\frac{p\left(\int_0^k \bar{F}(x)\,dx\right) - ck}{p\left(\int_0^{q^e} \bar{F}(x)\,dx\right) - cq^e} &= \frac{p\left(\int_0^k e^{-\int_0^x g(t)/t\,dt}\,dx\right) - ck}{p\left(\int_0^{q^e} e^{-\int_0^x g(t)/t\,dt}\,dx\right) - cq^e}\\ &= 1 + \frac{p\left(\int_{q^e}^k e^{-\int_0^x g(t)/t\,dt}\,dx\right) - c(k-q^e)}{p\left(\int_0^{q^e} e^{-\int_0^x g(t)/t\,dt}\,dx\right) - cq^e}.\end{aligned} \tag{5.16}$$

For any $y \in [q^e, k]$, define the *profit-gain factor* $a(y)$ by

$$a(y) \stackrel{\text{def}}{=} \left(p\left(\int_{q^e}^y e^{-\int_0^x g(t)/t\,dt}\,dx\right) - c(y-q^e)\right) \Big/ \left(p\left(\int_0^{q^e} e^{-\int_0^x g(t)/t\,dt}\,dx\right) - cq^e\right). \tag{5.17}$$

The derivative $\frac{\partial a(y)}{\partial y}$ is expressed via equation (5.18) below, when $y \in [q^e, k]$, leading to the following nonnegative upper bound:

$$\frac{\partial a(y)}{\partial y} = \left(pe^{-\int_0^y g(t)/t\,dt} - c\right) / \left(p\left(\int_0^{q^e} e^{-\int_0^x g(t)/t\,dt}\,dx\right) - cq^e\right) \tag{5.18}$$

$$\leq \left(pe^{-\int_0^{q^e} g(t)/t\,dt - \int_{q^e}^y g(q^e)/t\,dt} - c\right) / \left(p\left(\int_0^{q^e} e^{-\int_0^x g(t)/t\,dt}\,dx\right) - cq^e\right) \tag{5.19}$$

$$= \left(p\left(\frac{y}{q^e}\right)^{-g(q^e)} e^{-\int_0^{q^e} g(t)/t\,dt} - c\right) / \left(p\left(\int_0^{q^e} e^{-\int_0^x g(t)/t\,dt}\,dx\right) - cq^e\right)$$

$$\leq \left(p\left(\frac{y}{q^e}\right)^{-g(q^e)} e^{-\int_0^{q^e} g(t)/t\,dt} - c\right) / \left(p\left(q^e e^{-\int_0^{q^e} g(t)/t\,dt}\right) - cq^e\right) \tag{5.20}$$

$$= \left(p\left(\frac{y}{q^e}\right)^{-g(q^e)} \bar{F}(q^e) - c\right) / \left(p\bar{F}(q^e) - c\right) q^e \tag{5.21}$$

$$\leq \left(p\left(\frac{y}{q^e}\right)^{-g(q^e)} (1-m)(1+g(q^e)) - c\right) / \left(p(1-m)(1+g(q^e)) - c\right) q^e \tag{5.22}$$

$$= \left(\left(\frac{y}{q^e}\right)^{-g(q^e)} (1+g(q^e)) - 1\right) / \left((1+g(q^e)) - 1\right) q^e. \tag{5.23}$$

Therefore,

$$
\begin{aligned}
\frac{p\left(\int_0^k \bar{F}(x)\,dx\right) - ck}{p\left(\int_0^{q^e} \bar{F}(x)\,dx\right) - cq^e} &= 1 + \int_{q^e}^{k} \frac{\partial a(y)}{\partial y}\,dy \\
&\leq 1 + \int_{q^e}^{k} \left(\left(\frac{y}{q^e}\right)^{-g(q^e)} (1+g(q^e)) - 1 \right) / \left(g(q^e)q^e\right)\,dy \\
&= 1 + \left(\frac{k(1+g(q^e))}{1-g(q^e)} \left(\frac{k}{q^e}\right)^{-g(q^e)} - \frac{q^e(1+g(q^e))}{1-g(q^e)} - (k-q^e) \right) / \left(g(q^e)q^e\right) \\
&= 1 + \left(\frac{1+g(q^e)}{1-g(q^e)} \left(\frac{k}{q^e}\right)^{1-g(q^e)} - \frac{1+g(q^e)}{1-g(q^e)} - \left(\frac{k}{q^e} - 1\right) \right) / g(q^e) \\
&= 1 + \left(\left(\frac{k}{q^e}\right) \left(-1 + \frac{1+g(q^e)}{1-g(q^e)} \left(\frac{k}{q^e}\right)^{-g(q^e)}\right) - \frac{1+g(q^e)}{1-g(q^e)} + 1 \right) / g(q^e). \\
&= \frac{1+g(q^e)}{g(q^e)(1-g(q^e))} \left(\frac{k}{q^e}\right)^{1-g(q^e)} - \frac{1}{g(q^e)}\frac{k}{q^e} - \frac{1+g(q^e)}{g(q^e)(1-g(q^e))} + \frac{1}{g(q^e)} + 1 \\
&= \frac{1+g(q^e)}{g(q^e)(1-g(q^e))} \left(\frac{k}{q^e}\right)^{1-g(q^e)} - \frac{1}{g(q^e)}\frac{k}{q^e} \\
&\qquad + \frac{-1-g(q^e)+1-g(q^e)+g(q^e)-g(q^e)^2}{g(q^e)(1-g(q^e))} + \frac{1}{g(q^e)} + 1 \\
&= \frac{1+g(q^e)}{g(q^e)(1-g(q^e))} \left(\frac{k}{q^e}\right)^{1-g(q^e)} - \frac{1}{g(q^e)}\frac{k}{q^e} + \frac{-g(q^e)^2-g(q^e)}{g(q^e)(1-g(q^e))} \\
&= \frac{1+g(q^e)}{g(q^e)(1-g(q^e))} \left(\frac{k}{q^e}\right)^{1-g(q^e)} - \frac{1}{g(q^e)}\frac{k}{q^e} - \frac{1+g(q^e)}{1-g(q^e)} \\
&= \frac{1+g(q^e)}{g(q^e)(1-g(q^e))} \left(\frac{k}{q^e}\right)^{1-g(q^e)} - \frac{1}{g(q^e)}\frac{k}{q^e} - \frac{2g(q^e)+1-g(q^e)}{1-g(q^e)} \\
&= \frac{1+g(q^e)}{g(q^e)(1-g(q^e))} \left(\frac{k}{q^e}\right)^{1-g(q^e)} - \frac{1}{g(q^e)}\frac{k}{q^e} - \frac{2g(q^e)}{1-g(q^e)} - 1 \quad \square
\end{aligned}
$$

LEMMA 5.2. *Under the same assumptions as in Lemma 2.1, when* $\bar{F}(k) = \delta$ *and* $\bar{F}(q^e) = (1-m)(1+g(q^e))$ *we have* $k \cdot ((1-m)(1+g(q^e))/\delta)^{-1/g(q^e)} \leq q^e$.

Proof of Lemma 5.2. Assume $q^e < k \cdot ((1-m)(1+g(q^e))/\delta)^{-1/g(q^e)}$. This leads to a contradiction (inequality (5.24)):

$$\begin{aligned}
\delta = \bar{F}(k) &= e^{-\int_0^k g(t)/t\,dt} = e^{-\int_0^{q^e} g(t)/t\,dt} \cdot e^{-\int_{q^e}^k g(t)/t\,dt} \le \bar{F}(q^e) \cdot e^{-\int_{q^e}^k g(q^e)/t\,dt} \\
&= (1-m)(1+g(q^e)) \cdot (k/q^e)^{-g(q^e)} \\
&< (1-m)(1+g(q^e)) \cdot \left(k/\left(k \cdot ((1-m)(1+g(q^e))/\delta)^{-1/g(q^e)}\right)^{-g(q^e)} = \delta. \quad (5.24)
\end{aligned}$$

Inequality (5.24) follows from our assumption, $q^e < k \cdot (\delta/((1-m)(1+g(q^e))))^{1/g(q^e)}$. □

Proof of Theorem 5.4. The case where $\beta = 1$ is equivalent to the unconstrained problem which is addressed in Perakis and Roels (2006). Therefore, fix channel capacity k and assume $\beta > 1$, so that $q^s = k$. When $\beta > 1$, the probability of excess demand, which we will denote by δ, is fixed and satisfies $\beta = \delta p/c$. Fix a c.d.f. $F \in \mathcal{F}(k, \beta)$. The efficiency Eff($F$) of F satisfies the following lower bound:

$$\begin{aligned}
\text{Eff}(F) &\stackrel{\text{def}}{=} E[pS(q^{eq}) - cq^{eq}]/E[pS(k) - ck] \\
&\ge E[pS(q^e) - cq^e]/E[pS(k) - ck] \quad (5.25) \\
&= \left(p\left(\int_0^{q^e} \bar{F}(x)\,dx\right) - cq^e\right) \Big/ \left(p\left(\int_0^{k} \bar{F}(x)\,dx\right) - ck\right) \\
&\ge \left(\frac{1+g(q^e)}{g(q^e)(1-g(q^e))}\left(\frac{k}{q^e}\right)^{1-g(q^e)} - \frac{2g(q^e)}{1-g(q^e)} - \frac{1}{g(q^e)}\frac{k}{q^e} - 1\right)^{-1} \quad (5.26) \\
&\ge \left(\frac{1+g(q^e)}{g(q^e)(1-g(q^e))}\left(\frac{1+g(q^e)}{\beta}\right)^{-1+1/g(q^e)} - \frac{2g(q^e)}{1-g(q^e)} - \frac{1}{g(q^e)}\left(\frac{1+g(q^e)}{\beta}\right)^{1/g(q^e)} - 1\right)^{-1} \quad (5.27) \\
&= \left(\frac{1}{g(q^e)} \cdot \left(\frac{1+g(q^e)}{\beta}\right)^{1/g(q^e)} \cdot \left(\frac{1+g(q^e)}{1-g(q^e)} \cdot \frac{\beta}{1+g(q^e)} - 1\right) - \frac{2g(q^e)}{1-g(q^e)} - 1\right)^{-1} \\
&= \left(\frac{1}{g(q^e)} \cdot \left(\frac{1+g(q^e)}{\beta}\right)^{1/g(q^e)} \cdot \left(\frac{\beta}{1-g(q^e)} - 1\right) - \frac{2g(q^e)}{1-g(q^e)} - 1\right)^{-1} \\
&\ge \left(\max_{(\beta-1)\le x \le m/(1-m)} \frac{1}{x} \cdot \left(\frac{1+x}{\beta}\right)^{1/x} \cdot \left(\frac{\beta}{1-x} - 1\right) - \frac{2x}{1-x} - 1\right)^{-1}.
\end{aligned}$$

In particular, inequality (5.25) follows because $q^e \le q^{eq} \le q^s$. Inequality (5.26) follows from Lemma 5.1. The function on the right-hand side of inequality (5.26) is decreasing as q^e decreases and from Lemma 5.2 we know that the equilibrium order quantity q^e satisfies $q^e \geq k \cdot ((1+g(q^e))/\beta)^{-1/g(q^e)}$. Therefore, inequality (5.27) follows when we substitute in $q^e = k \cdot ((1+g(q^e))/\beta)^{-1/g(q^e)}$.

Define the function

$$x(m,\beta) \stackrel{\text{def}}{=} \underset{(\beta-1)\leq x\leq m/(1-m)}{\arg\max} \ \frac{1}{x} \cdot \left(\frac{1+x}{\beta}\right)^{1/x} \cdot \left(\frac{\beta}{1-x} - 1\right) - \frac{2x}{1-x} - 1. \tag{5.28}$$

It can be verified that the lower bound in inequality (5.27) is attained when the c.d.f. F is taken equal to H, where the c.d.f. H satisfies

1. $\bar{H}(t) = 1$ for $t \in \left[0, k \cdot \left(\frac{\beta}{1+x(m,\beta)}\right)^{1/x(m,\beta)}\right]$,

2. $\bar{H}(t) = (k/t)^{x(m,\beta)} \cdot \delta$ for $t \in \left[k \cdot \left(\frac{\beta}{1+x(m,\beta)}\right)^{1/x(m,\beta)}, \infty\right)$.

(To verify this claim confirm that $q^e = k \cdot \left(\frac{\beta}{1+x(m,\beta)}\right)^{1/x(m,\beta)}$, using eq. (5.2), implying that we can convert the inequalities in eqs. (5.25) and (5.27) into equalities. Furthermore, since the c.d.f. F is taken equal to H, we can convert the inequalities in eqs. (5.19),(5.20), and (5.22) into equalities. Therefore, inequality (5.26) becomes an equality.) The c.d.f. H does not satisfy Assumption 2.1, because the corresponding density is zero for $t \leq k \cdot \left(\frac{\beta}{1+x(m,\beta)}\right)^{1/x(m,\beta)}$. However, it can be approximated arbitrarily closely by c.d.f.s in the class $\mathcal{F}(k,\beta)$ (in particular, that satisfy Assumption 2.1), with an arbitrarily small change in the resulting efficiency. □

5.7.6 Proof: 1-supplier/1-retailer, Set of wholesale prices $\mathcal{W}_{\text{both}}(k_r, k_v)$

Proof of Theorem 5.5. From the definition $\mathcal{W}_{\text{both}}(k_r,k_v) \stackrel{\text{def}}{=} \{w \mid c/\bar{F}(\min\{q^*,k_v\}) \leq w \leq p\bar{F}(\min\{q^*,k_r\})\}$, the set $\mathcal{W}_{\text{both}}(k_r,k_v)$ is non-empty if and only if the inequality

$$c/\bar{F}(\min\{q^*,k_v\}) \leq p\bar{F}(\min\{q^*,k_r\}) \tag{5.29}$$

holds. Inequality (5.29) can be rewritten as $c/p \leq \bar{F}(\min\{q^*, k_r\}) \cdot \bar{F}(\min\{q^*, k_v\})$, or

$$p/c \leq \frac{\max\{\bar{F}(k_r), \bar{F}(q^*)\}}{c/p} \cdot \frac{\max\{\bar{F}(k_v), \bar{F}(q^*)\}}{c/p} = \beta_{\text{push}} \cdot \beta_{\text{pull}} \tag{5.30}$$

because $\max\{\bar{F}(k), \bar{F}(q^*)\} = \bar{F}(\min\{q^*, k\})$ for any k. Therefore, inequality (5.29) is equivalent to the condition $\beta_{\text{push}} \cdot \beta_{\text{pull}} \geq p/c$. And, we have the set $\mathcal{W}_{\text{both}}(k_r, k_v)$ is non-empty if and only if $\beta_{\text{push}} \cdot \beta_{\text{pull}} \geq p/c$.

In Theorem 2.1 we defined the set $\mathcal{W}(k) \stackrel{\text{def}}{=} \left[c, p\bar{F}(\min\{q^*, k\})\right]$ and in Theorem 5.1 we defined the set $\mathcal{W}_{\text{pull}}(k) \stackrel{\text{def}}{=} \left[c/\bar{F}(\min\{q^*, k\}), p\right]$. Observe that $\mathcal{W}_{\text{both}}(k_r, k_v) = \mathcal{W}(k_r) \cap \mathcal{W}_{\text{pull}}(k_v)$. Therefore, any wholesale price $w \in \mathcal{W}_{\text{both}}(k_r, k_v)$ is also a member of $\mathcal{W}(k_r)$ so that from Theorem 2.1 we have that w will coordinate the retailer's ordering decision for the supply channel when operating in push-mode. Furthermore, any wholesale price $w \in \mathcal{W}_{\text{both}}(k_r, k_v)$ is also a member of $\mathcal{W}_{\text{pull}}(k_v)$. From Theorem 5.1, we have that w will coordinate the supplier's decision for the supply channel when operating in pull-mode.

Suppose now that there is a wholesale price w that coordinates the retailer's ordering decision for the supply channel (in push-mode) and that coordinates the supplier's decision for the supply channel (in pull-mode) and that $c \leq w \leq p$ holds. From Theorem 2.1 we have that $w \in \mathcal{W}(k_r)$. From Theorem 5.1, we have that $w \in \mathcal{W}_{\text{pull}}(k_v)$. Therefore, $w \in \mathcal{W}(k_r) \cap \mathcal{W}_{\text{pull}}(k_v) = \mathcal{W}_{\text{both}}(k_r, k_v)$. □

5.7.7 Proof: Impact of changing β_{push}, β_{pull}

Proof of Corollary 5.5. Since $\bar{F}_i(\min\{q_i^*, k\}) = \max\{\bar{F}_i(k), \bar{F}_i(q_i^*)\}$ for any k, we have that

$$c_i/\bar{F}_i(\min\{q_i^*, k_v^i\}) = p_i \cdot (c_i/p_i) \cdot 1/\max\{\bar{F}_i(k_v^i), \bar{F}_i(q_i^*)\} = p_i \cdot (1/\beta_{\text{pull}}^i)$$

and

$$p_i\bar{F}_i(\min\{q_i^*, k_r^i\}) = c_i \cdot \max\{\bar{F}_i(k_r^i), \bar{F}_i(q_i^*)\}/(c_i/p_i) = c_i \cdot \beta_{\text{push}}^i.$$

Therefore, the inequalities $c_1/\bar{F}_1(\min\{q_1^*, k_v^1\}) \le c_2/\bar{F}_2(\min\{q_2^*, k_v^2\})$ and $p_2\bar{F}_2(\min\{q_2^*, k_r^2\}) \le p_1\bar{F}_1(\min\{q_1^*, k_r^1\})$ hold, if and only if, inequalities $\beta^2_{\text{pull}}/p_2 \le \beta^1_{\text{pull}}/p_1$ and $c_2 \cdot \beta^2_{\text{push}} \le c_1 \cdot \beta^1_{\text{push}}$ hold. $\square$

CHAPTER 6

Multiple suppliers selling to a newsvendor

Capacity is *not* free. Rather, distributors and retailers provide a valued service to their immediate upstream suppliers: sales capacity (i.e., access to downstream demand). So that in many supply channels, suppliers pay their distributor or retailer (e.g., with 'slotting' fees, 'pay-to-stay' fees, or favorable contractual terms) in order to obtain that capacity. For example, two large pharmaceutical manufacturers, Pfizer and Roche Diagnostics, pay distributors, such as Cardinal Health, in order to distribute their medical supplies to pharmacies, hospitals, and clinics. And, as another example, two large producers of household goods, Procter & Gamble and Unilever, pay retailers (e.g., Wal-Mart) for the shelf-space that ultimately delivers their goods to consumers. But how much value does (sales) capacity actually have?

This chapter studies supplier competition (with wholesale price contracts) for a single retailer's downstream capacity and competition's influence on the price and, therefore, value of that capacity, when the (downstream) buyer is a newsvendor. Our analysis can be interpreted as a capacity valuation technique that applies in both a negotiation and equilibrium setting for this particular supply chain configuration.

Furthermore, using a multiple supplier/single retailer model, we show that when suppliers collude they decrease the value of the retailer's capacity. Our multiple supplier/single retailer model differs from the model in Chapter 4 in that each supplier can sell more than one good. Also the focus of this chapter is on competition and the value of capacity, whereas, the focus of Chapter 4 is on coordination.

Chapter Outline

In Section 6.1, we explain the supply chain setting considered. In Section 6.2, we analyze the newsvendor's capacity allocation decision and derive the (endogenous) price for the newsvendor's capacity. We conduct comparative statics in Section 6.3. Then, in Section 6.4, we analyze the equilibrium setting, by providing conditions for the existence of an equilibrium in Section 6.4.2 and for uniqueness in Section 6.4.4. Finally, in Section 6.5, we consider supplier collusion/integration and show that the retailer's shadow price for capacity decreases and that every supplier can achieve more profit.

■ 6.1 Model Framework

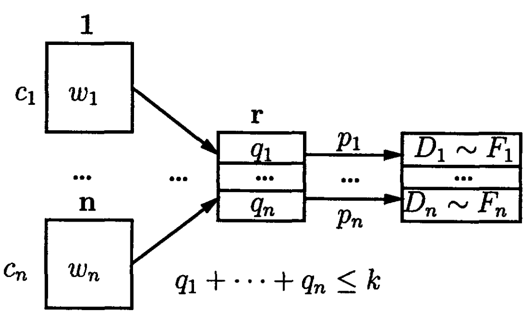
There are multiple suppliers, offering multiple goods, to a risk-neutral retailer. There is some initial exogenous negotiation process, whereby the retailer decides on order quantities (e.g., to guarantee certain service levels), followed by an equilibrium process (that we model) involving the suppliers (and their goods) who received orders (in the negotiation process) in competition for the remainder of the retailer's capacity. The equilibrium environment for the suppliers that go on to 'sell to a newsvendor' is described below.

We consider s risk-neutral suppliers, offering n different goods in aggregate (where $n \geq s$ and each good is offered by exactly one supplier) to a risk-neutral retailer r, facing a newsvendor problem: there is a single sales season, the retailer decides on an order quantity vector/portfolio $(q_1, q_2, \ldots, q_n)$ and orders well in advance of the season, the entire order arrives before the start of the season, and finally demand is realized, resulting in sales for the retailer (without an opportunity for replenishment). Without loss of generality, units remaining at the end of the season are assumed to have no salvage value, and there is no (additional) cost for stocking out.

The model's parameters are summarized in Figure 6-1 with the arrows denoting the direction of product flow. In particular, the supplier for good i has a fixed marginal cost of c_i per unit supplied and charges the retailer a wholesale price $w_i \geq c_i$ per unit ordered. The retailer's price $p_i \geq w_i$ per unit to the market for good i is fixed and, at that price, the demand for good i, D_i, is random with probability density function (p.d.f.) f_i and cumulative distribution function (c.d.f.) F_i. We assume that the distribution for demands D_i does not depend on the ordered quantities $(q_1, q_2, \ldots, q_n)$. We also define $\bar{F}_i(x) \stackrel{\text{def}}{=} 1 - F_i(x) = Pr\{D_i > x\}$.

We assume that the retailer's total capacity is constrained by some $k > 0$; for example, the retailer can only hold up to k units of inventory, or accept a shipment not larger than k. For a different interpretation, k could represent a constraint on the capacity of the channel or a budget constraint. We assume that the capacity as well as the quantities of the different products are measured with a common set of units (e.g., shelf-space), so that the capacity constraint can be expressed in the form $q_1 + \cdots + q_n \leq k$.

Figure 6-1 **"n goods & 1 capacity constrained retailer" model.**



Note. There are n goods, each offered by exactly one supplier. The suppliers are not depicted here. The supplier for good i faces marginal cost c_i (per unit) and offers wholesale price w_i (per unit) to a capacity-constrained retailer r who faces uncertain demand D_i downstream with c.d.f. F_i (for good i) when the price for the good is fixed at p_i (per unit). The retailer must decide on a portfolio q of goods to order from the suppliers.

The p.d.f. f_i for the demand D_i has support $[0, l_i]$, with $l_i > k$, on which it is positive and continuous. As a consequence, $\bar{F}_i(0) = 1$ and $\bar{F}_i$ is continuously differentiable, strictly decreasing, and invertible on $(0, l_i)$.

We say that the retailer is *(mode) service constrained* for good $i \in N$, if f_i is nonincreasing for good i. The name follows from one scenario in which the p.d.f. f_i

could be nonincreasing in practice. In particular, recall that there is an initial negotiation process (which we do not model) where the retailer orders from some suppliers followed by an equilibrium process (which we model) to determine which suppliers will obtain the remainder of the retailer's capacity. Suppose that initially, before the negotiation process, the demand for good i has a unimodal distribution. And suppose that the retailer receives a demand update for good i (before commencing the negotiation process), namely, that x_i units of good i have been pre-ordered and therefore are 'guaranteed' sales. If x_i is larger than the mode of the demand distribution for good i, and the retailer orders x_i units in the negotiation process, then the updated demand distribution f_i for the equilibrium process will have the property that it is nonincreasing.

6.1.1 Equilibrium setting

The equilibrium setting we analyze is a two-stage (Stackelberg) game. In the *first* stage, the suppliers (the *'leaders'*) simultaneously set the wholesale prices for their goods. In the *second* stage, the retailer (the *'follower'*) chooses an optimal response q, given the wholesale prices w. The suppliers produce and deliver $\sum_{i=1}^{n} q_i$ units before the sales season starts and offer no replenishments. The suppliers and the retailer aim to maximize their own profit.

6.1.2 Retailer's problem in the second stage

Faced with uncertain sales $S_i(x) \stackrel{\text{def}}{=} \min\{x, D_i\}$ for product $i \in \{1, \ldots, n\}$ (when ordering x units) and a wholesale price vector w (from the suppliers), the retailer decides on a vector of quantities to order from the suppliers in order to maximize expected profit $\pi_r(q) \stackrel{\text{def}}{=} E[\sum_{i=1}^{n} p_i S_i(q_i) - w_i q_i]$ while keeping in mind the capacity constraint k. Namely, the retailer solves the following convex program with linear constraints in the decision vector, q:

RETAILER-PRIMAL(k,w):

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} (p_i E[S_i(q_i)] - w_i q_i) \\ \text{subject to} \quad & q_i \geq 0, \quad i = 1, \ldots, n \\ & k - \sum_{i=1}^{n} q_i \geq 0. \end{aligned} \tag{6.1}$$

Because of our assumptions on the distribution of the demand D_i for each product, it can be shown that *RETAILER-PRIMAL(k,w)* has a unique solution (vector), which we denote by $q^r(w)$. We denote the unique solution, $\arg\max_{q \in \mathbb{R}_+^n} \pi_r(q)$, for the unconstrained retailer's problem by $q^*(w)$. Note that the unconstrained retailer's problem can be decomposed into n independent newsvendor problems, each of which decides on an order quantity for a single good. Therefore, $q_i^*(w)$ equals the optimal order quantity for a newsvendor ordering good i only, which is well known to be $\bar{F}_i^{-1}(w_i/p_i)$ units (e.g., Cachon and Terwiesch (2006)).

The dual problem in the decision variables $\gamma_1, \gamma_2, \ldots, \gamma_n$ (the shadow prices for the nonnegativity constraints) and λ (the shadow price for the capacity constraint) is:

RETAILER-DUAL(k,w):

$$\text{minimize} \quad \max_{\{q \in \mathbb{R}_+^n | k - \sum_{t=1}^{n} q_t \geq 0\}} \sum_{i=1}^{n} (p_i E[S_i(q_i)] - w_i q_i) + \sum_{i=1}^{n} \gamma_i q_i + \lambda \left(k - \sum_{i=1}^{n} q_i \right) \tag{6.2}$$

$$\begin{aligned} \text{subject to} \quad & \gamma_i \geq 0, \quad i = 1, \ldots, n \\ & \lambda \geq 0. \end{aligned}$$

Also, *RETAILER-DUAL(k,w)* has a unique solution which we denote by $(\gamma_1^r(w), \ldots, \gamma_n^r(w), \lambda^r(w))$.

6.1.3 Supplier's problem in the first stage

When the suppliers charge wholesale price vector w and the retailer, in response, orders $q^r(w)$, a supplier, offering the set $Y \subseteq N \stackrel{\text{def}}{=} \{1, \ldots, n\}$ of goods, obtains profit $\pi_Y(w) \stackrel{\text{def}}{=} \sum_{i \in Y}(w_i - c_i)q_i^r(w)$. If there exist other good(s) $\overline{Y} \stackrel{\text{def}}{=} N \setminus Y$, then supplier Y's profit depends on the wholesale prices of the other supplier(s) (due to the terms $\{q_i^r(w)\}_{i \in Y}$).[1] And, therefore, supplier Y competes in a simultaneous-move *game* in the first-stage against the other supplier(s).

If there exist other good(s) $\overline{Y}$ and the corresponding wholesale price vector $w_{\overline{Y}}$ is held fixed, a supplier, offering the good(s) Y, determines Y's wholesale price(s) by solving the following program with linear constraints in the decision vector, w_Y:
Y-SUPPLIER($w_{\overline{Y}}$):

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in Y} (w_i - c_i) \cdot q_i^r(w) \\ \text{subject to} \quad & p_i - w_i \geq 0, \quad i \in Y, \\ & w_i - c_i \geq 0, \quad i \in Y. \end{aligned} \tag{6.3}$$

In Section 6.4.1 (cf. Theorem 6.8, Theorem 6.7, and Equation (6.14)), we characterize the solution set $\mathcal{W}_Y^{\text{br}}(w_{\overline{Y}})$ for *Y-SUPPLIER($w_{\overline{Y}}$)*, when the retailer is service constrained for good(s) $\overline{Y}$ and the c.d.f. for each good $y \in Y$ has the IGFR property.[2] Following the convention in game theory, we refer to the set-valued mapping $\mathcal{W}_Y^{\text{br}}$ as supplier Y's *best response* to the wholesale price(s) of the other supplier(s).

Furthermore, we denote the vector of best response mappings by $\mathcal{W}^{\text{br}} \stackrel{\text{def}}{=} (\mathcal{W}_{Y_1}^{\text{br}}, \ldots, \mathcal{W}_{Y_s}^{\text{br}})$ (where Y_i represents all the goods offered by supplier i) and refer to it, by convention, as the *best response correspondence.* Note that any (pure-strategy) equilibrium in the simultaneous-move game (and, thus, in the overall Stackelberg game) corresponds to some fixed point of the correspondence $\mathcal{W}^{\text{br}}$, i.e., a vector w^{eq} of wholesale prices

[1] Supplier Y denotes the supplier that offers *only* the good(s) Y.

[2] Section 6.4.1 culminates with Theorem 6.8 (equation (6.21)), showing that supplier Y's best response $\mathcal{W}_Y^{\text{br}}(w_{\overline{Y}})$ equals the intersection of the set $\mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$ (which is also described in Section 6.4.1) with the hypercube of feasible wholesale price vectors.

for all n goods, such that $w^{\mathrm{eq}} \in \mathcal{W}^{\mathrm{br}}(w^{\mathrm{eq}})$.

6.1.4 Equilibrium with an unconstrained newsvendor

Lariviere and Porteus (2001) analyze this Stackelberg game, for an unconstrained channel with one supplier, one good, and one newsvendor. However, their equilibrium results are applicable in a setting with multiple suppliers supplying multiple goods to one *unconstrained* newsvendor. In particular, since for any good $i \in N$, the newsvendor's order $q_i^r(w)$ equals $q_i^*(w) = \bar{F}_i^{-1}(w_i/p_i)$ when the newsvendor is unconstrained, we have that good i's profit, $(w_i - c_i) \cdot q_i^r(w)$, is not dependent on the wholesale price of any other good. Therefore, in the first stage, any supplier offering only one good faces a 'selling to the newsvendor' problem and any supplier Y offering more than one good can decompose its problem into $|Y|$ independent 'selling to the newsvendor' problems.

Thus, applying Lariviere and Porteus (2001) to our setting: when F_t has the IGFR property for every good $t \in N$ and the newsvendor is unconstrained (i.e., k is sufficiently large), the game results in a unique outcome (q^e, w^e) defined implicitly in terms of the equations

$$p_t \bar{F}_t(q_t^e)\,(1 - g_t(q_t^e)) - c_t = 0, \quad t = 1, \dots, n; \tag{6.4}$$

$$p_t \bar{F}_t(q_t^e) - w_t^e = 0, \quad t = 1, \dots, n \tag{6.5}$$

where g_t is the generalized failure rate function $g_t(y) \stackrel{\mathrm{def}}{=} y f_t(y)/\bar{F}_t(y)$.

6.1.5 Definition: Valuation for capacity

In Section 6.2, we show that when suppliers charge wholesale price vector w in the first round of the Stackelberg game, they induce an *endogenous* valuation, $\lambda^r(w) \cdot k$, for the retailer's capacity.[3] In this chapter, we are interested in understanding the valuations that are feasible in our equilibrium setting. In particular, if we denote the

[3] So that any supplier that obtains x units of the newsvendor's capacity, in effect, pays the newsvendor $x \cdot \lambda^r(w)$.

set of equilibrium wholesale price vectors for the Stackelberg game (when the retailer has a capacity of k units) by $\mathcal{W}^{\text{eq}}(k) \stackrel{\text{def}}{=} \{w \mid w \in \mathcal{W}^{\text{br}}(w)\}$, we are interested in determining properties of the set of *equilibrium* valuations (per unit capacity), i.e., the set $\Lambda^{\text{eq}}(k) \stackrel{\text{def}}{=} \{\lambda \mid \lambda = \lambda^{\text{r}}(w), w \in \mathcal{W}^{\text{eq}}(k)\}$.[4]

Although the analysis in Section 6.3 is specific to a newsvendor setting and a wholesale price contract, our analysis can be generalized for other equilibrium settings under different supply contracts.

6.2 The newsvendor's problem and an endogenous price for capacity k

Our first result describes properties of the retailer's shadow price $\lambda^{\text{r}}(w)$ for capacity k and formalizes the retailer's usage of $\lambda^{\text{r}}(w)$, when deciding how much of each good to order, in a 'threshold rule' (on the marginal expected profit curve of each good).

THEOREM 6.1. *Let $Z(w) \stackrel{def}{=} \{i \mid q_i^r(w) = 0\} \subset N$ be the set of products that are not ordered in the retailer's decision problem when faced with wholesale price vector $w = (w_1, w_2, ..., w_n)$. For any wholesale price vector w, there exists some $\lambda^{\text{r}}(w)$ such that the following conditions hold:*

$$\lambda^{\text{r}}(w) = p_j \bar{F}_j \left(q_j^r(w)\right) - w_j, \qquad \forall\, j \in N \setminus Z(w), \tag{6.6}$$

$$\lambda^{\text{r}}(w) \geq p_i - w_i, \qquad \forall\, i \in Z(w). \tag{6.7}$$

Furthermore, $\lambda^{\text{r}}(w) = 0$ if and only if $\sum_{t=1}^{n} q_t^*(w) \leq k$.

Proof. See Section 6.6.1.

Notice that when the capacity constraint k is larger than or equal to the unconstrained retailer's total optimal order quantity, $\sum_{t=1}^{n} q_t^*(w)$, we have $\lambda^{\text{r}}(w) = 0$, so that equation (6.6) reduces to the 'classic' optimal order quantity result for a newsven-

[4]Theorem 6.9 states conditions under which the set $\mathcal{W}^{\text{eq}}(k)$ is non-empty, and, therefore, conditions under which the set $\Lambda^{\text{eq}}(k)$ is, also, non-empty.

dor: $\bar{F}_j\left(q_j^r(w)\right) = w_j/p_j$ for any ordered good j. And, furthermore, equation (6.7) implies every good is included in the retailer's portfolio, if $w < p$.

However, when the capacity constraint is binding for the retailer (i.e., $\sum_{t=1}^n q_t^*(w) > k$), the retailer's shadow price $\lambda^r(w)$ for the capacity constraint is strictly positive. And, therefore, equation (6.6), which can be reexpressed as $w_j = p_j\bar{F}_j\left(q_j^r(w)\right) - \lambda^r(w)$, implies that the supplier for good j, included in the retailer's portfolio, *charges* the retailer $p_j\bar{F}_j\left(q_j^r(w)\right)$ per unit of good j, a known result for unconstrained channels (Lariviere and Porteus 2001), but, in addition, *pays* the retailer $\lambda^r(w)$ per unit of capacity allocated to good j. Thus, the retailer obtains an income, $k \cdot \lambda^r(w)$, from 'selling' capacity k, in addition to its uncertain income, $\sum_{i=1}^n p_i S_i(q_i^r(w))$.

In other words, the portfolio $q^r(w)$ of goods that the retailer orders, would have cost the retailer extra, i.e., $\sum_{i=1}^n q_i^r(w) \cdot \lambda^r(w) = k \cdot \lambda^r(w)$, if the retailer was unconstrained.

Figure 6-2 illustrates the 'threshold rule' when the capacity constraint is binding. Corollary 6.1 suggests a simple algorithm for calculating the shadow price $\lambda^r(w)$ when given a single plot displaying the retailer's marginal expected profit curve for each available good (e.g., Figure 6-2): start with initial threshold $\lambda = 0$ and increase λ until the sum of implied order quantities equals $\min\left\{\sum_{t=1}^n q_t^*(w), k\right\}$.

COROLLARY 6.1. *For any wholesale price vector w, the retailer's shadow price for capacity k is*

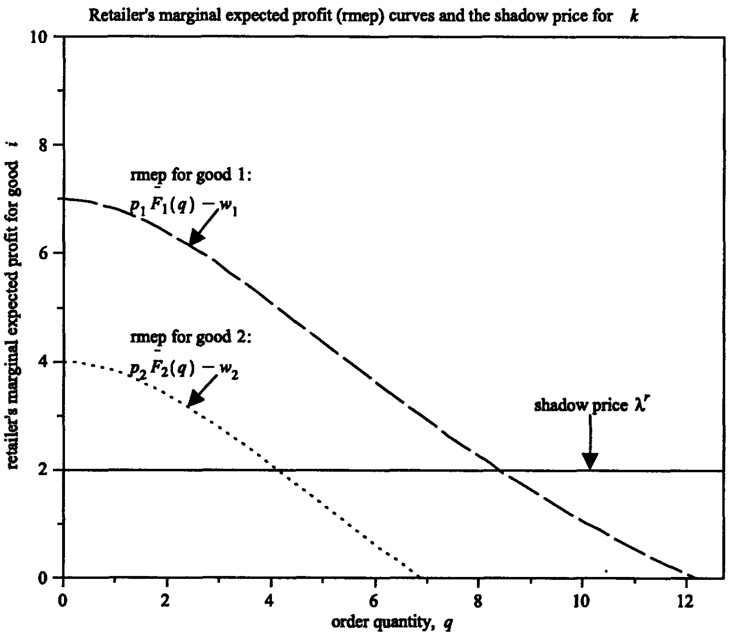
$$\lambda^r(w) = \min\left\{\lambda \mid p_t\bar{F}_t(q_t) - w_t \leq \lambda \ \ \forall t \in N, \ \sum_{t=1}^n q_t = \min\left\{\sum_{t=1}^n q_t^*(w), k\right\}, \ q \in \mathbb{R}_+^n\right\}. \tag{6.8}$$

Proof. See Section 6.6.2.

Regardless of whether or not the capacity constraint is binding, the newsvendor's optimal order quantity for any good can be expressed, more generally, as a function that depends on the shadow price $\lambda^r(w)$ as shown in Corollary 6.2. This result follows directly from equations (6.6) and (6.7).

COROLLARY 6.2. *For any wholesale price vector w, the retailer orders $q_t^r(w) =$*

Figure 6-2 **The shadow price $\lambda^r(w)$ as a 'threshold rule' on a retailer's marginal expected profit (rmep) curve.**



Note. Retailer's capacity is $k \approx 12.7$ units. The retailer considers two goods (dash & dot) when ordering. Demand for each good is identically distributed according to a Gamma distribution with mean 10 units and coefficient of variation $2^{-1/2} \approx .707$. Retail price (per unit) for each good is $p = 10$, but the wholesale prices are: $w_1 = 3$ (dash), $w_2 = 6$ (dot).

$\bar{F}_t^{-1}\left(\frac{\max\{\lambda^r(w)+w_t,p_t\}}{p_t}\right)$ *units of good* $t \in N$.

As a result, under the 'threshold order/allocation rule' the ratio of the service levels (i.e., fill rates) for any two goods (that the retailer orders) equals the corresponding ratio of the retailer's gross profit margins of those goods, for the uncertain income[5], as formalized in Corollary 6.3.

COROLLARY 6.3. *For good* $t \in N$ *and wholesale price vector* w, *define* $u_t(w) \stackrel{def}{=} (p_t - w_t - \lambda^r(w))/p_t$ *(the retailer's gross profit margin for the uncertain income). For any two goods* $a, b \in N \backslash Z(w)$ *that the retailer orders, we have* $F_a(q_a^r(w))/F_b(q_b^r(w)) = u_a(w)/u_b(w)$.

[5]For each unit of good $t \in N$ that the retailer orders when facing wholesale price vector w, the uncertain profit margin $p_t - w_t - \lambda^r(w)$ for good t accounts for the certain income $\lambda^r(w)$ received from 'selling' a unit of capacity.

6.3 Comparative statics, the game's geometry, and reformulation.

We show how changes in supplier wholesale prices effect the newsvendor's charge for capacity in Section 6.3.1. In Section 6.3.2, we derive a useful property that simplifies our analysis when considering suppliers offering more than one good. Then, in Section 6.3.3 we partition the set of supplier wholesale prices into equivalence classes based on the newsvendor capacity price they induce or the newsvendor order vector they induce. Then, in Section 6.4.1, we can recast supplier's problem *Y-SUPPLIER*($w_{\overline{Y}}$) into a (simpler) problem of choosing an aggregate order quantity to induce (see Decision Problem (6.18)). Finally, in Section 6.4.4, we provide conditions for the existence and uniqueness of an equilibrium (endogenous) capacity price and conclude with a section analyzing a special case of the Stackelberg game, i.e., when the suppliers collude on pricing.

6.3.1 The newsvendor's shadow price for capacity when a wholesale price drops

In Theorem 6.2, we show that the shadow price for capacity is nondecreasing when one good's wholesale price drops (and provide conditions on when the shadow price is strictly increasing). In addition, we provide a simple upper bound on the increase in the shadow price.

THEOREM 6.2. *Consider two different wholesale price vectors w and w'. Suppose that w' differs from w on exactly one good $d \in N$ so that $w'_d < w_d$ and $w'_{-d} = w_{-d}$. Then, $\lambda^r(w) \leq \lambda^r(w')$. And, $\lambda^r(w) < \lambda^r(w')$ if and only if good d is included in the retailer's order under w' (i.e., $d \in N \setminus Z(w')$) and the capacity constraint is binding for the retailer under w' (i.e., $\sum_{t=1}^{n} q_t^*(w') > k$). Furthermore,*

$$\lambda^r(w') - \lambda^r(w) \leq w_d - w'_d. \tag{6.9}$$

And $\lambda^r(w') - \lambda^r(w) = w_d - w'_d$, *if and only if, the retailer orders* k *units of good* d *under* w *and* w'.

Proof. See Section 6.6.3.

Therefore, when the retailer's capacity constraint is binding (and thus the retailer 'charges' for capacity), a supplier that competes with other suppliers on price (by lowering its wholesale price(s)) creates two effects: the price-lowering supplier increases *every* supplier's cost λ^r in obtaining a unit of the retailer's capacity, and the price-lowering supplier increases its share of the retailer's capacity when the good is in the retailer's portfolio at the lower price (cf. Corollary 6.4).

COROLLARY 6.4. *Under the same assumptions as in Theorem 6.2, we have* $q^r_d(w) \leq q^r_d(w')$ *and* $q^r_o(w') \leq q^r_o(w)$ *for any other good* $o \neq d$. *Furthermore, the following two conditions are equivalent.*

(a) The retailer orders more of good d *under* w', *i.e.,* $q^r_d(w) < q^r_d(w')$, *if* $q^r_d(w) < k$. *And the retailer orders less of any other good* $o \neq d$ *under* w', *i.e.,* $q^r_o(w') < q^r_o(w)$, *if* $o \in N \setminus Z(w)$.

(b) The retailer orders good d *under* w', *i.e.,* $d \in N \setminus Z(w')$, *and the capacity constraint is binding, i.e.,* $\sum_{t=1}^{n} q^*_t(w') > k$.

Proof. See Section 6.6.4.

6.3.2 An invariance property on the retailer's shadow price for capacity

As shown in Theorem 6.2 and Corollary 6.4, any supplier Y can induce a change in the retailer's shadow price λ^r for capacity by dropping the wholesale price(s) for good(s) Y, or, equivalently, taking away retailer capacity from competing goods $\overline{Y}$.[6] In particular, from Theorem 6.3, it follows that when supplier Y takes away $x < k$ units of capacity from competing suppliers (when the wholesale prices $w_{\overline{Y}}$ for competing

[6]Sometimes, in order to affect a retailer's shadow price for capacity, a supplier Y may be required to drop the wholesale price(s) for good(s) Y to below cost, which would not occur in our formulation.

goods $\overline{Y}$ are fixed), supplier Y induces the retailer to have shadow price $\lambda^r(x; w_{\overline{Y}})$, as defined in equation (6.10) below, for capacity k.

THEOREM 6.3. *Consider a supplier $Y \subset N$ competing with good(s) $\overline{Y}$ for a retailer's capacity k. Suppose $w \stackrel{def}{=} (w_Y, w_{\overline{Y}})$ and the wholesale price vector $w_{\overline{Y}}$ is held fixed. If supplier Y's wholesale price vector w_Y induces the retailer to allocate $x < k$ units of capacity to supplier Y (i.e., $\sum_{t \in Y} q_t^r(w) = x$), then the retailer's shadow price $\lambda^r(w)$ equals*

$$\lambda^r(x; w_{\overline{Y}}) \stackrel{def}{=} \min \left\{ \lambda \mid p_t \bar{F}_t(q_t) - w_t \leq \lambda \;\; \forall t \in \overline{Y}, \; \sum_{t \in \overline{Y}} q_t = \min \left\{ \sum_{t \in \overline{Y}} q_t^*(w), k - x \right\}, \; q \in \mathbb{R}_+^{|\overline{Y}|} \right\}. \tag{6.10}$$

Furthermore, if $\lambda^r(w) = \lambda^r(x; w_{\overline{Y}}) > 0$, then $\sum_{t \in Y} q_t^r(w) = x$ holds.

Proof. See Section 6.6.5.

In other words, when $w_{\overline{Y}}$ is held fixed and $\sum_{t=1}^n q_t^r(w) = k$, the retailer's shadow price for capacity $\lambda^r(w)$ is invariant to changes in the wholesale price vector w_Y as long as the aggregate order quantity, $\sum_{t \in Y} q_t^r(w)$, remains the same. Furthermore, the retailer's shadow price $\lambda^r(x; w_{\overline{Y}})$ is a nondecreasing function of the aggregate order quantity x as formalized in Corollary 6.5.

COROLLARY 6.5. *Under the same assumptions as in Theorem 6.3, $\lambda^r(x; w_{\overline{Y}})$ is continuous. When x satisfies $0 \leq x \leq k - \sum_{t \in \overline{Y}} q_t^*(w)$, we have $\lambda^r(x; w_{\overline{Y}}) = 0$, and, when x satisfies $\max\left\{k - \sum_{t \in \overline{Y}} q_t^*(w), 0\right\} \leq x < k$, we have $\lambda^r(x; w_{\overline{Y}})$ is strictly increasing.*

Proof. See Section 6.6.6.

Therefore, when $\lambda^r(x; w_{\overline{Y}})$ is strictly positive and $x \in [0, k)$, the function $\lambda^r(x; w_{\overline{Y}})$ is strictly increasing. Furthermore, since the average capacity cost supplier Y incurs for inducing the retail to order an aggregate of x units equals $x \cdot \lambda^r(x; w_{\overline{Y}})/x$, the supplier average capacity cost is increasing in the induced aggregate order x (from Corollary 6.5). And, from Corollary 6.6, below, we have that the marginal capacity cost (i.e., $x \cdot \frac{\partial \lambda^r(x; w_{\overline{Y}})}{\partial x^+} + \lambda^r(x; w_{\overline{Y}})$) is also increasing (in the induced aggregate order x) when the retailer is service constrained for good(s) $\overline{Y}$. Therefore, supplier Y does not benefit from any economies of scale, when obtaining more than x units of newsvendor capacity and the capacity charge $\lambda^r(x; w_{\overline{Y}})$ is positive.

COROLLARY 6.6. *Under the same assumptions as in Theorem 6.3,* $\lambda^r(x; w_{\overline{Y}})$ *is differentiable (i.e.,* $\frac{\partial \lambda^r(x;w_{\overline{Y}})}{\partial x^+} = \frac{\partial \lambda^r(x;w_{\overline{Y}})}{\partial x^-}$*) and* $\frac{\partial \lambda^r(x;w_{\overline{Y}})}{\partial x}$ *is continuous at* $x > 0$ *when* $\lambda^r(x; w_{\overline{Y}}) > 0$ *and* $\lambda^r(x; w_{\overline{Y}}) \neq p_i - w_i$ *for any* $i \in \overline{Y}$*. If* x *satisfies the equation* $\max\left\{k - \sum_{t \in \overline{Y}} q_t^*(w), 0\right\} \leq x < k$ *and the retailer is service constrained for good(s)* $\overline{Y}$*, then* $\frac{\partial \lambda^r(x;w_{\overline{Y}})}{\partial x^+}$ *and* $\frac{\partial \lambda^r(x;w_{\overline{Y}})}{\partial x^-}$ *are strictly increasing.*

Proof. See Section 6.6.7.

6.3.3 Set of wholesale prices for a particular capacity price λ or capacity allocation q

Theorem 6.4, below, provides the set $\mathcal{W}(\lambda)$ of wholesale prices for good(s) N that induce the retailer to have shadow price λ. Therefore, from Theorem 6.4, we have that $\mathcal{W}(\lambda^r(x; w_{\overline{Y}})) \cap \{w' \mid w'_{\overline{Y}} = w_{\overline{Y}}\}$ is the set of all wholesale price vectors for good(s) Y that induce the retailer to have shadow price, $\lambda^r(x; w_{\overline{Y}})$, for allocated retail capacity x, when the wholesale price(s) for competing good(s) $\overline{Y}$ is held fixed at $w_{\overline{Y}}$.

THEOREM 6.4. *When* $\lambda \geq 0$*, any wholesale price vector in the set*

$$\mathcal{W}(\lambda) \stackrel{def}{=} \left\{ w \;\middle|\; \begin{array}{l} w_t = p_t \bar{F}_t(q_t) - \lambda + \gamma_t \cdot 1_{\{q_t = 0\}} \quad \forall t \in N, \;\; q, \gamma \in \mathbb{R}_+^{|N|}, \\ \sum_{t \in N} q_t = \min\left\{ \left(\sum_{t \in N} q_t^*(w) \right) \cdot 1_{\{\lambda = 0\}} + k \cdot 1_{\{\lambda > 0\}}, \;\; k \right\} \end{array} \right\} \tag{6.11}$$

induces the retailer to have shadow price λ *for the capacity constraint* k*. Furthermore, if a wholesale price vector* w *induces retailer shadow price* λ *for capacity* k*, then* $w \in \mathcal{W}(\lambda)$*.*

Proof. See Section 6.6.8.

When $\lambda^r > 0$, Theorem 6.1 implies that $\min\left\{\sum_{t \in N} q_t^*(w), k\right\} = k$, and, thus, Theorem 6.4 suggests that the set $\mathcal{W}(\lambda^r)$ can be indexed by the simplex $\{q \mid \sum_{t \in N} q_t = k, \; q \in \mathbb{R}_+^{|N|}\}$ of order quantities. Furthermore, when $\lambda^r(x; w_{\overline{Y}}) > 0$, the set $\mathcal{W}(\lambda^r(x; w_{\overline{Y}})) \cap \{w' \mid w'_{\overline{Y}} = w_{\overline{Y}}\}$ can be indexed by the (lower dimensional) simplex $\{q \mid \sum_{t \in Y} q_t =$

$x,\ q \in \mathbb{R}_+^{|Y|}\}$ of order quantities. Also, when $\lambda^r(x; w_{\overline{Y}}) > 0$, we have that $\lambda^r(x; w_{\overline{Y}})$ is invertible (from Corollary 6.5), so that for every $w \in \mathcal{W}(\lambda^r(x; w_{\overline{Y}})) \cap \{w' \mid w'_{\overline{Y}} = w_{\overline{Y}}\}$, we have $\sum_{t \in Y} q_t^r(w) = x$ (from Theorem 6.3).

Only wholesale prices in the set $\mathcal{W}(\lambda^r(x; w_{\overline{Y}})) \cap \{w' \mid w'_{\overline{Y}} = w_{\overline{Y}}\}$ induce the retailer to order x units from supplier Y's goods (in aggregate), when the wholesale price(s) for competing good(s) $\overline{Y}$ is held fixed at $w_{\overline{Y}}$. This set may be large, but Section 6.4.1 shows that there is a unique division of induced (aggregate) order x among supplier Y's goods that is optimal for supplier Y when the demand for every good $t \in Y$ has the IGFR property, so that the subset of wholesale price vectors of interest to supplier Y is much smaller. In particular, Theorem 6.1 and Theorem 6.3 imply that the optimal wholesale price vectors (for supplier Y) from the set $\mathcal{W}(\lambda^r(x; w_{\overline{Y}})) \cap \{w' \mid w'_{\overline{Y}} = w_{\overline{Y}}\}$ are identical in every good (component) $j \in Y$ included in the newsvendor's order. Therefore, if for the unique division of induced order x we have $x_t > 0$ for every good $t \in Y$, then there is a unique maximizing wholesale price vector in the set $\mathcal{W}(\lambda^r(x; w_{\overline{Y}})) \cap \{w' \mid w'_{\overline{Y}} = w_{\overline{Y}}\}$ for supplier Y.

The set of wholesale price vectors $\{w' \mid \min\{\sum_{t \in N} q_t^*(w), k\} = k\}$ can, also, be partitioned according to the retailer's allocation (vector) q of capacity k (where $\sum_{t \in N} q_t = k$), as shown in Theorem 6.5.

THEOREM 6.5. *Suppose $q \in \mathbb{R}_+^{|N|}$ and $\sum_{t \in N} q_t = k$. Any wholesale price vector in the set*

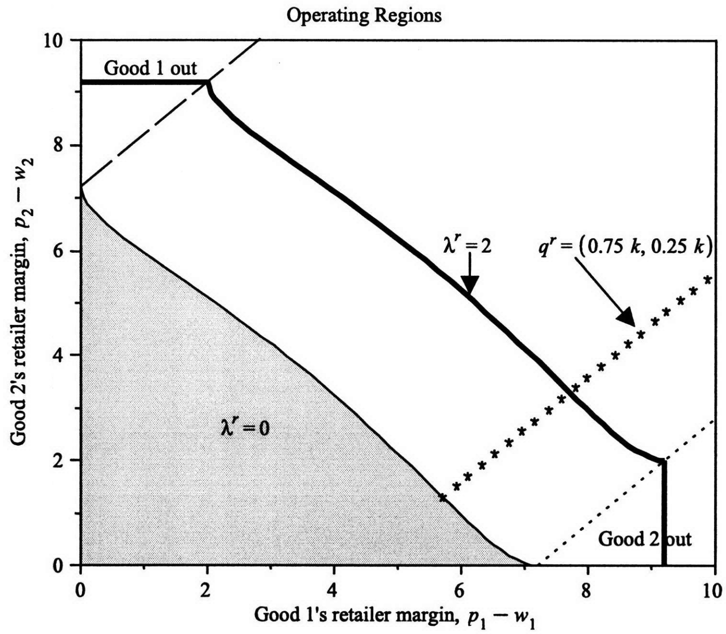
$$\mathcal{A}(q) \stackrel{def}{=} \left\{ w \mid w_t = p_t \bar{F}_t(q_t) - \lambda + \gamma_t \cdot 1_{\{q_t = 0\}}\ \ \forall t \in N,\ \lambda \in \mathbb{R}_+,\ \gamma \in \mathbb{R}_+^{|N|} \right\} \tag{6.12}$$

induces the retailer to order the vector q. Furthermore, if a wholesale price vector w induces the retailer to order the vector q, then $w \in \mathcal{A}(q)$.

Proof. See Section 6.6.9.

Figure 6-3 illustrates Theorem 6.4 and Theorem 6.5 for the example depicted in Figure 6-2. Notice, in Figure 6-3, that if the suppliers choose wholesale prices farther along the ray of asterisks, their allocation stays the same, but they end up being charged more for their allocated capacity.

Figure 6-3 **Wholesale price vectors that induce a particular capacity charge or capacity allocation.**



Note. The retailer considers two goods (dash & dot), as in Figure 6-2, when ordering and faces the same capacity, demand distributions, and retail prices. The shaded region denotes the wholesale prices that induce the retailer to have a zero shadow price for capacity. Whereas, the thickest line denotes the set of wholesale prices that induce the retailer to have a shadow price of two units for capacity. Also the region above the dashed ray represents the wholesale prices that induce the retailer to oust the good 'dash' from the portfolio (and order only 'dot'), whereas, the region below the dotted ray denotes the wholesale prices that induce the retailer to oust 'dot' from the portfolio. The ray denoted by asterisks represents the wholesale prices that induce the retailer to order $(.75 \cdot k, .25 \cdot k)$.

■ 6.4 Analysis for the two-stage game.

In this section we analyze the equilibria for the two stage game. We start by reformulating the supplier's best response problem.

6.4.1 Recasting a supplier's problem & its shadow price for allocated retail capacity

Consider a supplier $Y \subset N$ faced with the problem Y*-SUPPLIER*$(w_{\overline{Y}})$ in the decision vector w_Y when competing with good(s) $\overline{Y}$ (whose wholesale price vector $w_{\overline{Y}}$ is held fixed) for a retailer's capacity k. From the proof of Theorem 6.1, we have that every wholesale price vector w_Y is associated with some shadow price, $\lambda^r(w)$, for a retailer's capacity (where $w = (w_Y, w_{\overline{Y}})$) so that the set of wholesale prices can be partitioned into equivalence classes (i.e., $\{\mathcal{W}(\lambda)\}_{\lambda \geq 0}$) indexed by shadow price λ for a newsvendor's capacity (cf. Theorem 6.4). And from Theorem 6.3 and Corollary 6.5, we have that every positive retailer shadow price for capacity is associated with a unique aggregate capacity allocation/induced order (i.e., $\sum_{t \in Y} q_t$) by the retailer for supplier Y. Therefore, supplier Y has a simple algorithm for solving *Y-SUPPLIER($w_{\overline{Y}}$)* in order to maximize profit: 1) start with an initial aggregate number of units $x = 0$ to induce the retailer to order, 2) if $\lambda^r(x; w_{\overline{Y}}) > 0$, find the wholesale price vector in the set $\mathcal{W}(\lambda^r(x; w_{\overline{Y}})) \cap \{w' \mid w'_{\overline{Y}} = w_{\overline{Y}}\}$ that maximizes profit (which, thereby, determines the optimal number of units q_t of each good $t \in Y$ the retailer is induced to order, such that $\sum_{t \in Y} q_t = x$), otherwise, if $\lambda^r(x; w_{\overline{Y}}) = 0$, find the wholesale price vector in the set $\mathcal{W}(0) \cap \{w' \mid w'_{\overline{Y}} = w_{\overline{Y}}, \sum_{t \in Y} q_t^r(w') = x\}$ that maximizes profit, 3) keep track of the maximum attainable profit, thus far, and the associated capacity allocation x and optimal wholesale price vector, 4) increase x and go to step two, if $x < \bar{x}$ where $\bar{x}$ is an upper bound on the aggregate quantity of goods that supplier Y would induce the retailer to order. The upper bound $\bar{x}$ is formally defined later in this section (i.e., Corollary 6.7).

Supplier Y's optimal wholesale price(s) when inducing (aggregate) order x.

Suppose $x \in [0, \bar{x}]$. When the c.d.f. F_t has the IGFR property for every good $t \in Y$, step two of this algorithm can be described by a convex program with linear constraints in the decision vector q_Y to induce the retailer to order. In particular, from Theorem 6.3 and Theorem 6.4, we have that maximizing the objective function $\sum_{i \in Y}(w_i - c_i)q_i^r(w)$ of the program *Y-SUPPLIER($w_{\overline{Y}}$)* (i.e., equation (6.3)) over the set of wholesale prices $\mathcal{W}(\lambda^r(x; w_{\overline{Y}})) \cap \{w' \mid w'_{\overline{Y}} = w_{\overline{Y}}, \sum_{t \in Y} q_t^r(w') = x\}$ can be re-expressed as maximizing $\sum_{i \in Y}(p_i \bar{F}_i(q_i) - \lambda^r(\sum_{t \in Y} q_t; w_{\overline{Y}}) - c_i)q_i$ over the set of induced order vectors $\{q \mid q_t \geq 0 \ \forall t \in Y, \sum_{t \in Y} q_t = x\}$. Therefore, the convex program with linear constraints in the decision vector q_Y that solves step two of the algorithm is:

Y-SUPPLIER-PRICING-PRIMAL(x,$w_{\overline{Y}}$):

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in Y} \left(p_i \bar{F}_i(q_i) - \lambda^r\left(x; w_{\overline{Y}}\right) - c_i \right) \cdot q_i \\ \text{subject to} \quad & q_i \geq 0, \quad i \in Y \\ & x - \sum_{i \in Y} q_i = 0. \end{aligned} \tag{6.13}$$

When the c.d.f. F_t has the IGFR property for each good $t \in Y$, it can be shown that *Y-SUPPLIER-PRICING-PRIMAL(x,$w_{\overline{Y}}$)* has a unique solution (vector), which we denote by $q^Y(x; w_{\overline{Y}})$. So that the set of wholesale prices $\mathcal{W}^Y(x; w_{\overline{Y}})$ that maximize supplier Y's profit when the supplier induces the retailer to order x units in aggregate and when the other suppliers charge $w_{\overline{Y}}$ is

$$\mathcal{W}^Y(x; w_{\overline{Y}}) \stackrel{\text{def}}{=} \left\{ w' \,\middle|\, \begin{array}{l} w'_t = p_t \bar{F}_t(q_t^Y(x; w_{\overline{Y}})) - \lambda^r(x; w_{\overline{Y}}) + \gamma_t \cdot 1_{\{q_t^Y(x; w_{\overline{Y}})=0\}} \quad \forall t \in Y, \\ w'_{\overline{Y}} = w_{\overline{Y}}, \ \gamma \in \mathbb{R}_+^{|Y|} \end{array} \right\}. \tag{6.14}$$

Note that every good $t \in Y$ that is included in the retailer's portfolio has a unique wholesale price.

The dual problem in the decision variables $\gamma_1, \gamma_2, \ldots, \gamma_{|Y|}$ (the shadow prices for the nonnegativity constraints) and λ (the shadow price for the aggregate induced order) is:

Y-SUPPLIER-PRICING-DUAL(x,$w_{\overline{Y}}$):

$$\begin{aligned} \text{minimize} \quad & \max_{\{q \in \mathbb{R}_+^{|Y|} \mid x - \sum_{i \in Y} q_i = 0\}} \sum_{i \in Y} \left(p_i \bar{F}_i(q_i) - \lambda^r\left(x; w_{\overline{Y}}\right) - c_i \right) \cdot q_i \\ & \qquad + \sum_{i \in Y} \gamma_i q_i + \lambda \left(x - \sum_{i \in Y} q_i \right) \\ \text{subject to} \quad & \gamma_i \geq 0, \quad i \in Y. \end{aligned} \tag{6.15}$$

Also, *Y-SUPPLIER-PRICING-DUAL(x,$w_{\overline{Y}}$)* has a unique solution which we denote by $(\gamma_1^Y(x; w_{\overline{Y}}), \ldots, \gamma_{|Y|}^Y(x; w_{\overline{Y}}), \lambda^Y(x; w_{\overline{Y}}))$.

Theorem 6.6 formalizes the idea that supplier Y's shadow price $\lambda^Y(x; w_{\overline{Y}})$ describes a threshold for the marginal profit of an additional unit of any good in the set Y (when inducing an aggregate order quantity x and facing a fixed retailer shadow price $\lambda^r(x; w_{\overline{Y}})$ for capacity).

THEOREM 6.6. *Suppose that for every good $t \in Y$, the c.d.f. F_t has the IGFR property. Let $Z^Y(x; w_{\overline{Y}}) \stackrel{def}{=} \{i \in Y \mid q_i^Y(x; w_{\overline{Y}}) = 0\}$ be the set of products that are not ordered in supplier Y's decision problem when faced with wholesale price vector $w_{\overline{Y}}$ and inducing retailer aggregate order x. For any wholesale price vector $w_{\overline{Y}}$ and induced aggregate order $x \in \left(0, \min\left\{\sum_{t\in Y} q_t^e, k\right\}\right]$, the following conditions hold:*

$$\lambda^Y(x; w_{\overline{Y}}) = p_j \bar{F}_j\left(q_j^Y(x; w_{\overline{Y}})\right) \cdot \left(1 - g_j\left(q_j^Y(x; w_{\overline{Y}})\right)\right) - c_j - \lambda^r(x; w_{\overline{Y}}), \quad \forall\, j \in Y \setminus Z^Y(x; w_{\overline{Y}}), \tag{6.16}$$

$$\lambda^Y(x; w_{\overline{Y}}) \geq p_i - c_i - \lambda^r(x; w_{\overline{Y}}), \qquad \forall\, i \in Z^Y(x; w_{\overline{Y}}). \tag{6.17}$$

Furthermore, $\lambda^Y(x; w_{\overline{Y}}) + \lambda^r(x; w_{\overline{Y}}) = 0$ if and only if $x = \sum_{t\in Y} q_t^e \leq k$. And, the function $\lambda^Y(x; w_{\overline{Y}}) + \lambda^r(x; w_{\overline{Y}})$ is strictly decreasing as $x \in \left(0, \min\left\{\sum_{t\in Y} q_t^e, k\right\}\right]$ increases.

Proof. See Section 6.6.10.

From Equation (6.16), we have that supplier Y's shadow price $\lambda^Y(x; w_{\overline{Y}})$ represents an upper bound for the supplier's marginal profit on the x^{th} unit that the retailer orders (when supplier Y chooses the optimal number of units of each good $y \in Y$ to induce the retailer to order, so that the retailer orders x units in aggregate) and accounts for the marginal cost of the good as well as the marginal cost for the retailer's capacity, $\lambda^r(x; w_{\overline{Y}})$. From Theorem 6.6, we have that the function $\lambda^Y(x; w_{\overline{Y}})$ is strictly decreasing in x, because the function $\lambda^Y(x; w_{\overline{Y}}) + \lambda^r(x; w_{\overline{Y}})$ is strictly decreasing and from Corollary 6.5 we know that $\lambda^r(x; w_{\overline{Y}})$ is nondecreasing. Therefore, supplier Y only considers inducing the retailer to order up to some $\bar{x}$ units (in aggregate) where $\bar{x}$ is defined in Corollary 6.7.

COROLLARY 6.7. *Under the same assumptions as in Theorem 6.6, supplier Y would never induce the retailer to order more than $\bar{x}$ units of good(s) Y in aggregate* *where $\bar{x}$ is defined according to the following conditions. If the conditions*

$$0 < \max\left\{p_i - c_i - \lambda^r\left(0; w_{\overline{Y}}\right) \mid i \in Y\right\} \text{ and } \lambda^Y\left(\min\left\{\sum_{t\in Y} q_t^e, k\right\}; w_{\overline{Y}}\right) \le 0,$$

hold, then $\bar{x}$ is the positive value that satisfies the equation $\lambda^Y(\bar{x}; w_{\overline{Y}}) = 0$. But, if the conditions

$$0 < \max\left\{p_i - c_i - \lambda^r\left(0; w_{\overline{Y}}\right) \mid i \in Y\right\} \text{ and } 0 < \lambda^Y\left(\min\left\{\sum_{t\in Y} q_t^e, k\right\}; w_{\overline{Y}}\right),$$

hold, then $\bar{x}$ equals $\min\left\{\sum_{t\in Y} q_t^e, k\right\} = k$. Finally, if the condition

$$\max\left\{p_i - c_i - \lambda^r\left(0; w_{\overline{Y}}\right) \mid i \in Y\right\} \le 0,$$

holds, then $\bar{x}$ equals zero. Under any of these conditions, $\bar{x} \le \sum_{t\in Y} q_t^e$.

Proof. See Section 6.6.11.

Recall that via equation (6.14), the supplier can map any induced aggregate order x to the set $\mathcal{W}^Y(x; w_{\overline{Y}})$ of wholesale prices that should be charged to achieve that aggregate order. Next, we analyze the optimal aggregate order that the supplier should induce (and hence the wholesale prices the supplier should charge) when faced with wholesale price vector $w_{\overline{Y}}$ from competing good(s) $\overline{Y}$.

Supplier Y's optimal induced order x and best response to $w_{\overline{Y}}$.

Consider a supplier $Y \subset N$ competing with good(s) $\overline{Y}$ for a retailer's capacity k. Suppose $w = (w_Y, w_{\overline{Y}})$ and the wholesale price vector $w_{\overline{Y}}$ is held fixed. From Theorem 6.3 and Theorem 6.4, we have that the objective function $\sum_{i\in Y}(w_i - c_i)q_i^r(w)$ of the program *Y-SUPPLIER($w_{\overline{Y}}$)* (i.e., equation (6.3)) can be re-expressed as $\sum_{i\in Y}(p_i\bar{F}_i(q_i) - \lambda^r(\sum_{t\in Y} q_t; w_{\overline{Y}}) - c_i)q_i$. Therefore, as suggested in the beginning of this section, supplier Y's problem of maximizing profit and deciding the optimal wholesale price vector $w_Y^{\text{br}}(w_{\overline{Y}})$ when solving *Y-SUPPLIER($w_{\overline{Y}}$)* can be recast as the equivalent problem of deciding upon an aggregate quantity x to induce the retailer to order and then deciding how to split the aggregate order x among the goods Y. Formally, the program with linear constraints in the decision quantity x and decision vector q_Y that solves *Y-SUPPLIER($w_{\overline{Y}}$)* is:

Y-SUPPLIER-INDUCING-AGGREGATE-ORDER(k, $w_{\overline{Y}}$):

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in Y} \left(p_i \bar{F}_i(q_i) - \lambda^r \left(\sum_{t \in Y} q_t; w_{\overline{Y}} \right) - c_i \right) \cdot q_i \qquad (6.18) \\ \text{subject to} \quad & q_i \geq 0, \quad i \in Y \\ & x \geq 0, \\ & x - \sum_{i \in Y} q_i = 0, \\ & k - x \geq 0. \end{aligned}$$

Theorem 6.7, below, provides sufficient conditions for *Y-SUPPLIER-INDUCING-AGGREGATE-ORDER($w_{\overline{Y}}$)* to have a unique solution and, under those conditions, we denote the optimal aggregate order quantity by $x^Y(w_{\overline{Y}})$ and optimal induced order vector by $q^Y(w_{\overline{Y}})$. Therefore, from the proof of Theorem 6.6, we have that the optimal induced order quantity vector $q^Y(w_{\overline{Y}})$ must equal $q^Y(x^Y; w_{\overline{Y}})$. And, from equation (6.14), supplier Y's best response to competing wholesale prices $w_{\overline{Y}}$ is the set of wholesale prices $\mathcal{W}^Y(x^Y; w_{\overline{Y}})$. Furthermore, when supplier Y is faced with competing wholesale price vector $w_{\overline{Y}}$ and when it is optimal for supplier Y to induce the retailer to order every good $y \in Y$ (i.e., $Z^Y(x^Y; w_{\overline{Y}}) = \emptyset$), from equation (6.14), we have that supplier Y's best response is unique (i.e., the set $\mathcal{W}^Y(x^Y; w_{\overline{Y}})$ has only one wholesale price vector).

THEOREM 6.7. *Y-SUPPLIER-INDUCING-AGGREGATE-ORDER($w_{\overline{Y}}$) has a unique solution $\left(x^Y(w_{\overline{Y}}), q^Y(w_{\overline{Y}})\right)$ defined implicitly by the conditions*

$$x^Y(w_{\overline{Y}}) = \sup \left\{ x \in [0, \bar{x}] \mid \lambda^Y(x; w_{\overline{Y}}) - x \cdot \frac{\partial \lambda^r(x; w_{\overline{Y}})}{\partial x^-} \geq 0 \right\}, \qquad (6.19)$$

$$q^Y(w_{\overline{Y}}) = q^Y\left(x^Y(w_{\overline{Y}}); w_{\overline{Y}}\right), \qquad (6.20)$$

when $\bar{x} \neq 0$, the retailer is service constrained for good(s) $\overline{Y}$, and the cumulative distribution function for demand of each good $y \in Y$ has the IGFR property.

Proof. See Section 6.6.12.

Now we can state the main result of this section.

THEOREM 6.8. *Consider a supplier $Y \subset N$ that faces the problem Y-SUPPLIER($w_{\overline{Y}}$) (when the wholesale price vector $w_{\overline{Y}}$ for its competitors is held fixed) and, therefore, decides upon an optimal set $\mathcal{W}_Y^{br}(w_{\overline{Y}})$ of wholesale price vectors from the hypercube $\prod_{t \in Y}[c_t, p_t]$. Suppose the retailer is service constrained for good(s) $\overline{Y}$, and the cumulative distribution function for demand of each good $y \in Y$ has the IGFR property. Then, the solution set $\mathcal{W}_Y^{br}(w_{\overline{Y}})$ is non-empty, convex, and satisfies*

$$\mathcal{W}_Y^{br}(w_{\overline{Y}}) = \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}}) \bigcap \prod_{t \in Y}[c_t, p_t]. \tag{6.21}$$

Proof. See Section 6.6.13.

6.4.2 Existence of equilibrium

In Theorem 6.9, we provide sufficient conditions so that the two-stage game described in Section 6.1.1 has at least one equilibrium (supplier) wholesale price vector, and, therefore, resulting retailer order vector and shadow price for capacity.

THEOREM 6.9. *With more than one supplier (i.e., when $s \geq 2$), an equilibrium wholesale price vector exists when the retailer is service constrained for goods N and the demand for each good $t \in N$ has the IGFR property.*

Proof. See Section 6.6.14.

Denote the set of equilibrium wholesale price vectors for the Stackelberg game (when the retailer has a capacity of k units) by $\mathcal{W}^{\text{eq}}(k) \stackrel{\text{def}}{=} \{w \mid \text{ for every supplier } Y, w_Y \in \mathcal{W}_Y^{\text{br}}(w_{\overline{Y}})\}$. Furthermore, denote the set of (resulting) equilibrium capacity prices by $\Lambda^{\text{eq}}(k) \stackrel{\text{def}}{=} \{\lambda \mid \lambda = \lambda^r(w), w \in \mathcal{W}^{\text{eq}}(k)\}$. From Theorem 6.9, we know that the set $\mathcal{W}^{\text{eq}}(k)$ is non-empty, so that the set $\Lambda^{\text{eq}}(k)$ is, also, non-empty. Therefore, two values that allow us to bound the valuation for the retailer's capacity are

$\lambda^{\min}(k) \stackrel{\text{def}}{=} \inf \Lambda^{\text{eq}}(k)$ and $\lambda^{\max}(k) \stackrel{\text{def}}{=} \sup \Lambda^{\text{eq}}(k)$. But, often times, we can do better, and give an exact valuation for the retailer's capacity. In the next section, we describe an economic assumption that guarantees a unique equilibrium capacity price, so that the set $\Lambda^{\text{eq}}(k)$ has exactly one element.

6.4.3 An economic assumption

When suppliers have larger allocations of the newsvendor's capacity, the newsvendor's capacity price is more sensitive to attempts to increase that allocation. Formally, consider s suppliers and two different (aggregate) capacity allocations to those suppliers (represented by the vectors $a = (a_1, \ldots, a_s)$ and $b = (b_1, \ldots, b_s)$) induced by two optimal wholesale price vectors w^a and w^b (respectively) for those aggregate allocations (see Section 6.4.1) that cause the newsvendor to allocate his entire capacity k (i.e., $\sum_{t=1}^{s} a_t = k$ and $\sum_{t=1}^{s} b_t = k$). From Theorem 6.3, we have that supplier competition induces some virtual 'charge' for capacity (λ_a^r and λ_b^r, respectively), paid to the newsvendor. From Theorem 6.6, we have that the i^{th} supplier (i.e., supplier Y_i) has thresholds $\lambda_a^{Y_i}$ and $\lambda_b^{Y_i}$ for its marginal profit when faced with competing wholesale price vectors $w^a_{\overline{Y_i}}$ and $w^b_{\overline{Y_i}}$, respectively. Denote the subset of suppliers that have a larger share of the newsvendor's capacity under allocation a when compared to allocation b by $L(a, b)$ (i.e., formally, $L(a, b) \stackrel{\text{def}}{=} \{i \in \{1, \ldots, s\} \mid a_i > b_i\}$).[7]

ASSUMPTION 6.1. *Consider the suppliers $L(a,b)$ that have a higher allocation under allocation a when compared to allocation b. The marginal increase in the price of capacity, $\lambda^r(w^a)$, for a percent increase in the induced order (by each supplier in $L(a,b)$) is larger than the marginal increase in the price of capacity, $\lambda^r(w^b)$, for a percent increase in the induced order (by each supplier in $L(a,b)$), i.e.,*

$$\sum_{i \in L(a,b)} a_i \cdot \frac{\partial \lambda^r(a_i; w^a_{\overline{Y_i}})}{\partial a_i^-} \geq \sum_{i \in L(a,b)} b_i \cdot \frac{\partial \lambda^r(b_i; w^b_{\overline{Y_i}})}{\partial b_i^+}. \tag{6.22}$$

[7] From the 'pigeon-hole principle', we have that the subset, $L(a, b)$, is not empty because the allocation vectors are not equal.

Inequality (6.22) can be interpreted as stating the 'average' over the capacity price elasticities of the (aggregate) induced orders is decreasing as the induced orders increase. In particular, inequality (6.22) can be reexpressed as a statement about the two scaled harmonic means over the capacity price elasticities of the induced orders (that are larger in allocation a):

$$\frac{1}{\lambda_a^r}\cdot\frac{L(a,b)}{\sum_{i\in L(a,b)}\left((\lambda_a^r/a_i)\cdot\left(\frac{\partial\lambda^r(a_i;w_{\overline{Y_i}}^a)}{\partial a_i^-}\right)^{-1}\right)^{-1}} \leq \frac{1}{\lambda_b^r}\cdot\frac{L(a,b)}{\sum_{i\in L(a,b)}\left((\lambda_b^r/b_i)\cdot\left(\frac{\partial\lambda^r(b_i;w_{\overline{Y_i}}^b)}{\partial b_i^+}\right)^{-1}\right)^{-1}}. \tag{6.23}$$

Furthermore, note that if Inequality (6.24) between the harmonic means

$$\frac{L(a,b)}{\sum_{i\in L(a,b)}\left((\lambda_a^r/a_i)\cdot\left(\frac{\partial\lambda^r(a_i;w_{\overline{Y_i}}^a)}{\partial a_i^-}\right)^{-1}\right)^{-1}} \leq \frac{L(a,b)}{\sum_{i\in L(a,b)}\left((\lambda_b^r/b_i)\cdot\left(\frac{\partial\lambda^r(b_i;w_{\overline{Y_i}}^b)}{\partial b_i^+}\right)^{-1}\right)^{-1}} \tag{6.24}$$

holds and we also know that $\lambda_b^r \leq \lambda_a^r$, then Inequality (6.23) and, therefore, Assumption 6.1 follow.[8]

In the case of two suppliers with two goods, if the newsvendor is serviced constrained for both goods, we can show that Inequality (6.22) is a necessary condition.

6.4.4 Uniqueness of equilibrium shadow price

In Theorem 6.10, we provide sufficient conditions so that the two-stage game described in Section 6.1.1 has a unique equilibrium shadow price for capacity.

THEOREM 6.10. *With one supplier (i.e., when $s = 1$), any equilibrium wholesale price vector results in the retailer having a shadow price for capacity of zero units. Furthermore, with more than one supplier (i.e., when $s \geq 2$), if the retailer is service constrained for goods N, the demand for each good $t \in N$ has the IGFR property, and Assumption 6.1 holds, then for any two equilibrium wholesale price vectors $\widehat{w}$ and w' that induce different orders (i.e., $q^r(\widehat{w}) \neq q^r(w')$) the induced shadow prices*

[8]So that Assumption 6.1 is weaker than Inequality (6.24) when $\lambda_b^r \leq \lambda_a^r$ holds.

for capacity ($\lambda^r(\widehat{w})$ and $\lambda^r(w')$) are the same (i.e., there is a unique equilibrium shadow price for the retailer's capacity).

Proof. See Section 6.6.15.

When there is more than one supplier (i.e., $s \geq 2$), this theorem implies that if there are two equilibrium wholesale price vectors inducing different allocations of the retailer's capacity, then there is a unique equilibrium shadow price which we denote by λ^{eq}. And so, geometrically, the equilibrium wholesale price vectors are a subset of the set $\mathcal{W}(\lambda^{\text{eq}})$ as defined in Theorem 6.4 and depicted in Figure 6-3. In Theorem 6.11, we consider the scenario when there are two equilibrium wholesale price vectors that induce the retailer to order the same vector of goods.

THEOREM 6.11. *Consider the two-stage game described in Section 6.1.1 with more than one supplier (i.e., $s \geq 2$). Suppose the retailer is service constrained for goods N, the demand for each good $t \in N$ has the IGFR property, and that there are two equilibrium wholesale price vectors $\widehat{w}$ and w' that induce the same retailer order (i.e., $q^r(\widehat{w}) = q^r(w')$) but induce shadow prices for capacity satisfying $\lambda^r(w') \leq \lambda^r(\widehat{w})$. Denote supplier Y_j's marginal profit for inducing the retailer to order an extra unit (when the retailer faces wholesale price vector w') by the function*

$$m_{Y_j}(w') \stackrel{def}{=} \lambda^{Y_j}\left(\sum_{i \in Y_j} q_i^r(w'); w'_{\overline{Y_j}}\right) - \left(\sum_{i \in Y_j} q_i^r(w')\right) \cdot \left.\frac{\partial \lambda^r(x; w'_{\overline{Y_j}})}{\partial x^-}\right|_{x=\sum_{i\in Y_j} q_i^r(w')}.$$

Then, we have the following upper bound on the shadow price $\lambda^r(\widehat{w})$ when the retailer orders from two or more suppliers when facing wholesale price vector w':

$$\lambda^r(\widehat{w}) \leq \lambda^r(w') + \min\left\{m_{Y_j}(w') \mid j \in \{1, \ldots, s\},\ Y_j \cap (N \setminus Z(w')) \neq \emptyset\right\}. \quad (6.25)$$

And, we have the following upper bound on the shadow price $\lambda^r(\widehat{w})$ when the retailer orders from exactly one supplier when facing wholesale price vector w':

$$\lambda^r(\widehat{w}) \leq \lambda^r(w'). \quad (6.26)$$

Proof. See Section 6.6.16.

Therefore, under the same assumptions as in Theorem 6.10, Theorem 6.11 (in conjunction with Theorem 6.10) implies there are three possible scenarios in an equilibrium setting: either there is a unique equilibrium shadow price (with multiple equilibrium orders), or there is a unique equilibrium order (with multiple equilibrium shadow prices), or there is a unique equilibrium order and shadow price. These two theorems rule out the possibility of having two different equilibrium wholesale price vectors, $\widehat{w}$ and w', that simultaneously induce different retailer orders and different retailer shadow prices for capacity (i.e., such that both $q^r(\widehat{w}) \neq q^r(w')$ and $\lambda^r(\widehat{w}) \neq \lambda^r(w')$ hold).

■ 6.5 Supplier collusion.

Theorem 6.12 formalizes the idea that if there are more than two suppliers and they collude by making pricing decisions as if they were one firm, then they'd make more profit in aggregate than they would from any equilibrium that induces a positive equilibrium retailer shadow price for capacity. Consequently, there exists a division of the collusion profit such that every supplier would receive more profit than they would from the equilibrium wholesale price that induces a positive shadow price.

THEOREM 6.12. *Consider the two-stage game described in Section 6.1.1 with more than one supplier (i.e., $s \geq 2$). Suppose there is an equilibrium wholesale price vector w' that induces a positive shadow price $\lambda^r(w') > 0$. If the suppliers collude on pricing by setting prices as if they were one firm, then the aggregate supplier profit would be larger than the aggregate supplier profit from price vector w'.*

Proof. See Section 6.6.17.

Many interesting questions remain. For example, does the supply chain's profits increase or decrease when the suppliers collude? We know that the retailer will have a shadow price for capacity of zero units when the suppliers collude, but will the retailer also see higher wholesale prices for every good? We leave these and other questions for future work.

6.6 Proofs

In order to not disrupt the flow of presentation, the proofs for our results in this chapter are contained here.

6.6.1 Proof: The shadow price for capacity and the goods ordered.

Proof of Theorem 6.1. First, we write the Lagrangian $\mathcal{L}(q, \gamma_1, \ldots, \gamma_n, \lambda)$ for *RETAILER-PRIMAL(k,w)*:

$$\mathcal{L}(q, \gamma_1, \ldots, \gamma_n, \lambda) = \sum_{i=1}^{n} (p_i E[\min(q_i, D_i)] - w_i q_i) + \sum_{i=1}^{n} \gamma_i q_i + \lambda \left(k - \sum_{i=1}^{n} q_i \right).$$

Note that $\pi_r(q)$ is strictly concave for $q \in [0, l_1) \times \cdots [0, l_n)$ because each c.d.f. F_i is strictly increasing over $[0, l_i)$. Because the feasible set is convex and compact, *RETAILER-PRIMAL(k,w)* has a unique solution.

The Karush-Kuhn-Tucker conditions for the retailer's decision problem, *RETAILER-PRIMAL(k,w)*, are:

$$p_t \bar{F}_t(q_t) - w_t + \gamma_t - \lambda = 0, \quad t = 1, \ldots, n; \tag{6.27}$$

$$q_t \geq 0, \quad t = 1, \ldots, n;$$

$$k - \sum_{t=1}^{n} q_t \geq 0;$$

$$\gamma_t q_t = 0, \quad t = 1, \ldots, n; \tag{6.28}$$

$$\lambda \left(k - \sum_{t=1}^{n} q_t \right) = 0; \tag{6.29}$$

$$\lambda \geq 0; \quad \gamma_t \geq 0, \quad t = 1, \ldots, n.$$

Because of the concavity of the objective function and the fact that the Slater condition is satisfied, the Karush-Kuhn-Tucker conditions are both necessary and sufficient conditions for any primal optimal vector q and dual optimal vector (γ, λ). As a result, since the primal problem has a unique solution, it can be shown that the dual problem also has a unique solution, using these conditions. Let $(q^r(w), \gamma^r(w), \lambda^r(w))$ denote the unique vector that satisfies the Karush-Kuhn-Tucker conditions.

When $j \in N \setminus Z(w)$, from equation (6.28) we have $\gamma_j^r(w) = 0$. Therefore, from equation (6.27) we have $\lambda^r(w) = p_j \bar{F}_j\left(q_j^r(w)\right) - w_j$. When $i \in Z(w)$, from equation (6.27), we have $\lambda^r(w) = p_i - w_i + \gamma_i^r(w) \geq p_i - w_i$. Thus, the conditions in equations (6.6) and (6.7) hold.

Furthermore, if $\sum_{t=1}^n q_t^*(w) \leq k$, we have $q^r(w) = q^*(w)$. Therefore, when $j \in N \setminus Z(w)$, from equation (6.27) we have $\lambda^r(w) = p_j \bar{F}_j\left(q_j^*(w)\right) - w_j = 0$.

On the other hand, assume $\lambda^r(w) = 0$. When $j \in N \setminus Z(w)$, we have $p_j \bar{F}_j\left(q_j^r(w)\right) - w_j = 0$ from equation (6.27). Therefore, $q_j^r(w) = q_j^*(w)$. When $i \in Z(w)$, from equation (6.27), we have $p_i - w_i \leq p_i - w_i + \gamma_i^r(w) = 0$. Thus, $q_i^*(w) = 0 = q_i^r(w)$. And so we have $\sum_{t=1}^n q_t^*(w) = \sum_{t=1}^n q_t^r(w) \leq k$. □

6.6.2 Proof: The shadow price for capacity as the minimum of some set.

Proof of Corollary 6.1. Let $\Lambda \overset{\text{def}}{=} \{\lambda \mid p_t \bar{F}_t(q_t) - w_t \leq \lambda \;\; \forall t \in N, \; \sum_{t=1}^n q_t = \min\left\{\sum_{t=1}^n q_t^*(w), k\right\}, \; q \in \mathbb{R}_+^n\}$.

The vector $q^r(w) \in \mathbb{R}_+^n$ satisfies $\sum_{t=1}^n q_t^r(w) = \min\left\{\sum_{t=1}^n q_t^*(w), k\right\}$. Furthermore, from equation (6.27), we have $\lambda^r(w) = p_t \bar{F}_t(q_t^r(w)) - w_t + \gamma_t^r(w) \geq p_t \bar{F}_t(q_t^r(w)) - w_t$ when $t = 1, \ldots, n$. Therefore, we have $\lambda^r(w) \in \Lambda$.

Assume there exists a $\lambda' < \lambda^r(w)$ such that $\lambda' \in \Lambda$. Then there must exist a vector $q' \in \mathbb{R}_+^n$ such that $\sum_{t=1}^n q_t' = \min\left\{\sum_{t=1}^n q_t^*(w), k\right\}$ and $p_t \bar{F}_t(q_t') - w_t \leq \lambda'$ when $t = 1, \ldots, n$. When $j \in N \setminus Z(w)$, from equation (6.27) we have $\lambda^r(w) = p_j \bar{F}_j\left(q_j^r(w)\right) - w_j$. Since $\lambda' < \lambda^r(w)$, when $j \in N \setminus Z(w)$, we have $p_j \bar{F}_j(q_j') - w_j < p_j \bar{F}_j\left(q_j^r(w)\right) - w_j$, implying $\bar{F}_j(q_j') < \bar{F}_j\left(q_j^r(w)\right)$ and, thus, $q_j' > q_j^r(w)$. Therefore, $\sum_{j \in N \setminus Z(w)} q'_j > \sum_{j \in N \setminus Z(w)} q^r_j(w) = \sum_{t \in N} q^r_t(w) = \min\{\sum_{t=1}^n q^*_t(w), k\}$, implying $\sum_{t=1}^n q'_t > \min\{\sum_{t=1}^n q^*_t(w), k\}$. But this is a contradiction. Thus, $\lambda^r(w) = \min_{\lambda \in \Lambda} \lambda$. □

6.6.3 Proof: $\lambda^r(w)$ is nondecreasing as w_t decreases, and the increase is bounded.

Proof of Theorem 6.2. Let $\Lambda(w) \stackrel{\text{def}}{=} \{\lambda \mid p_t \bar{F}_t(q_t) - w_t \leq \lambda \;\; \forall t \in N, \; \sum_{t=1}^n q_t = \min\{\sum_{t=1}^n q^*_t(w), k\}, \; q \in \mathbb{R}^n_+\}$. Since $w'_d < w_d$ and $w'_{-d} = w_{-d}$, we have $q^*_d(w) < q^*_d(w')$ and $q^*_o(w) = q^*_o(w')$ for any other good $o \neq d$. Therefore, $\min\{\sum_{t=1}^n q^*_t(w), k\} \leq \min\{\sum_{t=1}^n q^*_t(w'), k\}$. If $\min\{\sum_{t=1}^n q^*_t(w), k\} = k$, then $\Lambda(w') \subseteq \Lambda(w)$ so that we have $\min \Lambda(w) \leq \min \Lambda(w')$. And, from Corollary 6.1, we have $\lambda^r(w) \leq \lambda^r(w')$. Otherwise, if $\min\{\sum_{t=1}^n q^*_t(w), k\} < k$, then from Theorem 6.1 we have $\lambda^r(w) = 0$ so that $\lambda^r(w) \leq \lambda^r(w')$ because the shadow prices are nonnegative.

Next, we show that $\lambda^r(w) < \lambda^r(w')$ holds if and only if the conditions $d \in N \setminus Z(w')$ and $\sum_{t=1}^n q^*_t(w') > k$ hold by proving the statement: $\lambda^r(w) = \lambda^r(w')$ holds if and only if $d \in Z(w')$ or $\sum_{t=1}^n q^*_t(w') \leq k$ holds.

First, we prove the direction: if $d \in Z(w')$ or $\sum_{t=1}^n q^*_t(w') \leq k$ holds, then $\lambda^r(w) = \lambda^r(w')$ holds. If $\sum_{t=1}^n q^*_t(w') \leq k$ holds, then from Theorem 6.1 we have $\lambda^r(w') = 0$. Since $\lambda^r(w) \leq \lambda^r(w')$ and the shadow prices are nonnegative, we have $\lambda^r(w) = \lambda^r(w')$. When $d \in Z(w')$ and $\sum_{t=1}^n q^*_t(w') > k$ hold, assume $\lambda^r(w) < \lambda^r(w')$ holds, instead. Then, for any $j \in N \setminus Z(w)$, we have $q^r_j(w') < q^r_j(w)$ when either $j \in N \setminus Z(w')$ (from equation (6.6)) or $j \in Z(w')$. Since $\lambda^r(w) < \lambda^r(w')$, from equation (6.7), we have $Z(w) \subseteq Z(w')$ so that $N \setminus Z(w') \subseteq N \setminus Z(w)$. Therefore, $\sum_{j \in N \setminus Z(w')} q^r_j(w') < \sum_{j \in N \setminus Z(w)} q^r_j(w) \leq k$. From equation (6.29), we have $\lambda^r(w') = 0$, implying $\sum_{t=1}^n q^*_t(w') \leq k$ (by Theorem 6.1). But this is a contradiction since $\sum_{t=1}^n q^*_t(w') > k$ holds. Thus, $\lambda^r(w) = \lambda^r(w')$.

Next, we show that $\lambda^r(w) = \lambda^r(w')$ implies $d \in Z(w')$ or $\sum_{t=1}^n q^*_t(w') \leq k$ holds. Assume $d \in N \setminus Z(w')$ and $\sum_{t=1}^n q^*_t(w') > k$ hold, instead. Therefore, we have $\lambda^r(w) = \lambda^r(w') > 0$ from Theorem 6.1. And equation (6.29) implies $\sum_{j \in N \setminus Z(w)} q^r_j(w) = k$. If

$d \in N \backslash Z(w)$, then $q_d^r(w') > q_d^r(w)$ from equation (6.6) since $w_d' < w_d$. If $d \in Z(w)$, we also have $q_d^r(w') > q_d^r(w) = 0$. Furthermore, equation (6.7) implies that for any good $i \neq d$: $i \in Z(w')$ if and only if $i \in Z(w)$ (because $w_i = w_i'$ and $\lambda^r(w) = \lambda^r(w')$). Thus, the sets $Z(w)$ and $Z(w')$ are identical unless $d \in Z(w)$. Therefore, $N \setminus Z(w') = (N \setminus Z(w)) \cup \{d\}$. And for any good $j \neq d$ such that $j \in N \setminus Z(w')$ we have $q_j^r(w') = q_j^r(w)$ (from equation (6.6)). Therefore, $\sum_{j \in N \setminus Z(w')} q_j^r(w') > \sum_{j \in N \setminus Z(w)} q_j^r(w) = k$. But this is a contradiction. Thus, $d \in Z(w')$ or $\sum_{t=1}^n q_t^*(w') \leq k$ holds.

Next, we prove inequality (6.9) holds. If $\lambda^r(w) = \lambda^r(w')$, inequality (6.9) follows. If $\lambda^r(w) < \lambda^r(w')$, then $d \in N \setminus Z(w')$ and $\sum_{t=1}^n q_t^*(w') > k$ hold, as proven. Therefore, we have $\lambda^r(w') > 0$ from Theorem 6.1 and equation (6.29) implies $\sum_{j \in N \setminus Z(w')} q_j^r(w') = k$. Assume the inequality $\lambda^r(w') - \lambda^r(w) > w_d - w_d'$ holds, instead. Therefore, rearranging terms, $\lambda^r(w') + w_d' > \lambda^r(w) + w_d$ holds. Corollary 6.2, then, implies $q_d^r(w') < q_d^r(w)$ and that for any $j \in N \setminus Z(w)$ such $j \neq d$, we have $q_j^r(w') < q_j^r(w)$ (because $\lambda^r(w') + w_j' > \lambda^r(w) + w_j$ holds). The inequalities $q_d^r(w') < q_d^r(w)$ and $\lambda^r(w) < \lambda^r(w')$ imply $Z(w) \subseteq Z(w')$ (from equation (6.7)). Therefore, $N \setminus Z(w') \subseteq N \setminus Z(w)$. And we have $k = \sum_{j \in N \setminus Z(w')} q_j^r(w') < \sum_{j \in N \setminus Z(w)} q_j^r(w)$. But this is a contradiction. Thus, inequality (6.9) follows.

If the retailer orders k units of good d under w and w', then, from equation (6.6), we have that $\lambda^r(w') + w_d' = \lambda^r(w) + w_d$ holds. If $\lambda^r(w') + w_d' = \lambda^r(w) + w_d$ holds, then, from Corollary 6.2, we have that $q_d^r(w') = q_d^r(w)$. Furthermore, since $w_d' < w_d$, we have $\lambda^r(w) < \lambda^r(w')$. Therefore, as proven, we have that good d is included in the retailer's order under w' (and thus w) and that the capacity constraint is binding for the retailer under w'. From Theorem 6.1, we have $\lambda^r(w') > 0$ and equation (6.29) implies $\sum_{j \in N \setminus Z(w')} q_j^r(w') = k$. Assume the inequality $q_d^r(w') < k$ holds, instead of the equality $q_d^r(w') = k$. Then, there exists at least one other good $o \in N \setminus Z(w')$, where $o \neq d$. Good d is included in the retailer's order under both w and w', and $\lambda^r(w) < \lambda^r(w')$, therefore, we have $Z(w) \subseteq Z(w')$ (from equation (6.7)), implying $N \setminus Z(w') \subseteq N \setminus Z(w)$. And for any good $j \neq d \in N \setminus Z(w)$, from Corollary 6.2, we have $q_j^r(w') < q_j^r(w)$ because $\lambda^r(w) + w_j < \lambda^r(w') + w_j'$ holds. Therefore, we have $k = \sum_{j \in N \setminus Z(w')} q_j^r(w') < \sum_{j \in N \setminus Z(w)} q_j^r(w)$. But this is a contradiction. Thus, the equality $q_d^r(w') = k$ follows.

□

6.6.4 Proof: The effect of a price drop on the retailer's order.

Proof of Corollary 6.4. From Theorem 6.2, we have $\lambda^r(w) \leq \lambda^r(w')$. Since $w'_o = w_o$, we have $\lambda^r(w) + w_o \leq \lambda^r(w') + w'_o$. The inequality $q_o^r(w') \leq q_o^r(w)$, then, follows from Corollary 6.2. Furthermore, from inequality (6.9), we have $\lambda^r(w') + w'_d \leq \lambda^r(w) + w_d$. Therefore, from Corollary 6.2, we have $q_d^r(w) \leq q_d^r(w')$.

First, we show condition (b) implies condition (a). If $q_d^r(w) < k$, then, from Theorem 6.2, we have the strict inequality $\lambda^r(w') + w'_d < \lambda^r(w) + w_d$. Since the retailer orders good d under w', from Corollary 6.2, we have $\max\{\lambda^r(w') + w'_d, p_d\} < p_d$. Therefore, $\max\{\lambda^r(w') + w'_d, p_d\} < \max\{\lambda^r(w) + w_d, p_d\}$ holds. And, since the c.d.f. F_d is strictly increasing over $[0, l_d)$, we have $q_d^r(w) < q_d^r(w')$ (from Corollary 6.2). Furthermore, condition (b) and Theorem 6.2 imply $\lambda^r(w) < \lambda^r(w')$. Since $w'_o = w_o$, we have $\lambda^r(w) + w_o < \lambda^r(w') + w'_o$. If $o \in N \setminus Z(w)$, then, from Corollary 6.2, we have $\max\{\lambda^r(w) + w_o, p_o\} < p_o$. Therefore, $\max\{\lambda^r(w) + w_o, p_o\} < \max\{\lambda^r(w') + w'_o, p_o\}$. And, from Corollary 6.2, we have $q_o^r(w') < q_o^r(w)$ because the c.d.f. F_o is strictly increasing over $[0, l_o)$.

Next, we show condition (a) implies condition (b). Assume $d \in Z(w')$ or $\sum_{t=1}^n q_t^*(w') \leq k$ holds, instead. From Theorem 6.2, we have $\lambda^r(w) = \lambda^r(w')$. Therefore, for any good $o \neq d$, we have $\lambda^r(w) + w_o = \lambda^r(w') + w'_o$ because $w_o = w'_o$. From Corollary 6.2, then, we have $q_o^r(w) = q_o^r(w')$. But this contradicts condition (a) when $o \in N \setminus Z(w)$. It can be shown that the set $N \setminus Z(w)$ includes some good $o \neq d$ when $\sum_{t=1}^n q_t^*(w') \leq k$ holds. When $\sum_{t=1}^n q_t^*(w') > k$ and $d \in Z(w')$ hold, we have $q_d^r(w) = q_d^r(w') = 0$ because $q_d^r(w) \leq q_d^r(w')$ holds and $q_d^r(w)$ must be nonnegative. But this contradicts condition (a) since $q_d^r(w) < k$ holds, yet $q_d^r(w) = q_d^r(w')$. Thus, condition (b) holds.

□

6.6.5 Proof: A supplier effects the price for capacity via its induced allocation.

Proof of Theorem 6.3. Since the wholesale price vector w induces the retailer to order x units of goods from supplier Y, we have that when the retailer solves the convex program *RETAILER-PRIMAL(k,w)* in the decision vector q, the optimal order quantity vector $q^r(w)$ is also the unique solution to the following convex program in the decision vector q:

RETAILER-WITH-Y-GUARANTEE(k,x,w):

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} (p_i E[S_i(q_i)] - w_i q_i) \qquad (6.30) \\ \text{subject to} \quad & q_i \geq 0, \quad i = 1, \dots, n \\ & \sum_{i \in Y} q_i = x \\ & k - \sum_{i=1}^{n} q_i \geq 0. \end{aligned}$$

Therefore, since the objective function in (6.30) is separable into the sum of two independent expressions,

$$\sum_{i=1}^{n} (p_i E[S_i(q_i)] - w_i q_i) = \sum_{i \in Y} (p_i E[S_i(q_i)] - w_i q_i) + \sum_{i \in \overline{Y}} (p_i E[S_i(q_i)] - w_i q_i), \qquad (6.31)$$

the order quantity vector $q^r_{\overline{Y}}(w)$ is the solution to the following convex program in the decision vector q:

RETAILER-RESTRICTED-TO-$\overline{Y}$-PRIMAL(k,x,$w_{\overline{Y}}$):

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in \overline{Y}} (p_i E[S_i(q_i)] - w_i q_i) \qquad (6.32) \\ \text{subject to} \quad & q_i \geq 0, \quad i \in \overline{Y} \\ & (k - x) - \sum_{i \in \overline{Y}} q_i \geq 0. \end{aligned}$$

The dual problem in the decision variables $\gamma_{\overline{Y}}$ (the shadow price vector for the nonnegativity constraints) and λ (the shadow price for the capacity constraint) is:
RETAILER-RESTRICTED-TO-$\overline{Y}$-DUAL(k,x,$w_{\overline{Y}}$):

$$\text{minimize} \quad \max_{\{q \in \mathbb{R}_+^{|\overline{Y}|} \mid k - \sum_{t \in \overline{Y}} q_t \geq 0\}} \sum_{i \in \overline{Y}} (p_i E[S_i(q_i)] - w_i q_i) + \sum_{i \in \overline{Y}} \gamma_i q_i + \lambda \left((k-x) - \sum_{i \in \overline{Y}} q_i \right) \tag{6.33}$$

$$\text{subject to} \quad \gamma_i \geq 0, \quad i \in \overline{Y}$$

$$\lambda \geq 0.$$

Note that *RETAILER-RESTRICTED-TO-$\overline{Y}$-DUAL(k,x,$w_{\overline{Y}}$)* is identical to *RETAILER-DUAL(k−x,$w_{\overline{Y}}$)*, when $N = \overline{Y}$. Therefore, we have from Theorem 6.1 that *RETAILER-RESTRICTED-TO-$\overline{Y}$-DUAL(k,x,$w_{\overline{Y}}$)* has a unique solution which we denote by $(\gamma^r_{|\overline{Y}|}(x; w_{\overline{Y}}), \lambda^r(x; w_{\overline{Y}}))$. Furthermore, from Corollary 6.1 we have that

$$\lambda^r(x; w_{\overline{Y}}) = \min \left\{ \lambda \mid p_t \bar{F}_t(q_t) - w_t \leq \lambda \ \ \forall t \in \overline{Y}, \ \sum_{t \in \overline{Y}} q_t = \min \left\{ \sum_{t \in \overline{Y}} q_t^*(w), k - x \right\}, \ q \in \mathbb{R}_+^{|\overline{Y}|} \right\}. \tag{6.34}$$

Since $x < k$, there exists at least one good $j \in \overline{Y}$ such that $q_j^r(w) > 0$. Since the vector $q^r_{\overline{Y}}(w)$ is the solution to *RETAILER-RESTRICTED-TO-$\overline{Y}$-PRIMAL(k,x,$w_{\overline{Y}}$)*, from Equation (6.6) in Theorem 6.1, we have that

$$\lambda^r(x; w_{\overline{Y}}) = p_j \bar{F}_j \left(q_j^r(w) \right) - w_j. \tag{6.35}$$

Since the vector $q^r(w)$ is the solution to *RETAILER-PRIMAL(k,w)*, from Equation (6.6) in Theorem 6.1, we also have that

$$\lambda^r(w) = p_j \bar{F}_j \left(q_j^r(w) \right) - w_j. \tag{6.36}$$

Therefore, from Equations (6.35) and (6.36), we have that $\lambda^r(w) = \lambda^r(x; w_{\overline{Y}})$.

Next, we prove the partial converse. Denote the solution to *RETAILER-RESTRICTED-*

TO-$\overline{Y}$-PRIMAL(k,x,$w_{\overline{Y}}$) by the vector $q^r_{\overline{Y}}(x; w_{\overline{Y}})$. From Equation (6.6) in Theorem 6.1, we have that for every good $j \in \overline{Y}$ such that $q^r_j(x; w_{\overline{Y}}) > 0$,

$$\lambda^r(x; w_{\overline{Y}}) = p_j \bar{F}_j\left(q^r_j(x; w_{\overline{Y}})\right) - w_j. \tag{6.37}$$

Furthermore, from Equation (6.6), we also have that for every good $j \in N$ such that $q^r_j(w) > 0$,

$$\lambda^r(w) = p_j \bar{F}_j\left(q^r_j(w)\right) - w_j. \tag{6.38}$$

Therefore, since $\lambda^r(w) = \lambda^r(x; w_{\overline{Y}})$, from Equations (6.37) and (6.38), we have that $q^r_j(w) = q^r_j(x; w_{\overline{Y}})$ for every $j \in \overline{Y}$, implying that

$$\sum_{j\in\overline{Y}} q^r_j(w) = \sum_{j\in\overline{Y}} q^r_j(x; w_{\overline{Y}}) \tag{6.39}$$

holds. Since $\lambda^r(x; w_{\overline{Y}}) > 0$, from Theorem 6.1 it follows that $\sum_{j\in\overline{Y}} q^r_j(x; w_{\overline{Y}}) = k - x$. So that from Equation (6.39), we can conclude that $\sum_{j\in\overline{Y}} q^r_j(w) = k - x$. Since $\lambda^r(w) > 0$, from Theorem 6.1 we also have that $\sum_{j\in N} q^r_j(w) = k$. Therefore, since $\sum_{j\in N} q^r_j(w) = \sum_{t\in\overline{Y}} q^r_t(w) + \sum_{t\in Y} q^r_t(w)$, we have $\sum_{j\in Y} q^r_j(w) = x$. □

6.6.6 Proof: A newsvendor's price for capacity is continuous and increasing.

Proof of Corollary 6.5. Suppose $0 \le k - \sum_{t\in\overline{Y}} q^*_t(w)$. Then, there exists an x that satisfies $0 \le x \le k - \sum_{t\in\overline{Y}} q^*_t(w)$. For any such x, we have $\sum_{t\in\overline{Y}} q^*_t(w) \le k - x$. Therefore, since *RETAILER-RESTRICTED-TO-$\overline{Y}$-DUAL(k,x,$w_{\overline{Y}}$)* is identical to *RETAILER-DUAL(k − x,$w_{\overline{Y}}$)*, when $N = \overline{Y}$, we have from Theorem 6.1 that $\lambda^r(x; w_{\overline{Y}}) = 0$ when x satisfies $0 \le x \le k - \sum_{t\in\overline{Y}} q^*_t(w)$.

Suppose that x_1 and x_2 satisfy $\max\left\{k - \sum_{t\in\overline{Y}} q^*_t(w), 0\right\} \le x_1 < x_2 < k$. We show that $\lambda^r(x_1; w_{\overline{Y}}) < \lambda^r(x_2; w_{\overline{Y}})$. Assume that $\lambda^r(x_2; w_{\overline{Y}}) \le \lambda^r(x_1; w_{\overline{Y}})$ holds, instead. Denote the solution to *RETAILER-RESTRICTED-TO-$\overline{Y}$-PRIMAL(k,x_i,$w_{\overline{Y}}$)* by the vector $q^r_{\overline{Y}}(x_i; w_{\overline{Y}})$ for $i = 1, 2$. From Equation (6.6) in Theorem 6.1, we have that for every good $j \in \overline{Y}$ such that $q_j^r(x_i; w_{\overline{Y}}) > 0$,

$$\lambda^r(x_i; w_{\overline{Y}}) = p_j \bar{F}_j \left(q_j^r(x_i; w_{\overline{Y}})\right) - w_j, \tag{6.40}$$

for $i = 1, 2$. Because of our assumption on the cumulative distribution functions and $\lambda^r(x_2; w_{\overline{Y}}) \leq \lambda^r(x_1; w_{\overline{Y}})$, from equation (6.40), we have

$$\sum_{j \in \overline{Y}} q_j^r(x_1; w_{\overline{Y}}) \leq \sum_{j \in \overline{Y}} q_j^r(x_2; w_{\overline{Y}}). \tag{6.41}$$

Since $k - x_2 < \sum_{t \in \overline{Y}} q_t^*(w)$, from Theorem 6.1, we have $0 < \lambda^r(x_2; w_{\overline{Y}})$, implying that $0 < \lambda^r(x_1; w_{\overline{Y}})$ holds too. Therefore both $\sum_{j \in \overline{Y}} q_j^r(x_2; w_{\overline{Y}}) = k - x_2$ and $\sum_{j \in \overline{Y}} q_j^r(x_1; w_{\overline{Y}}) = k - x_1$ hold. But then equation (6.41) implies that $k - x_1 \leq k - x_2$ so that $x_2 \leq x_1$ holds. But this is a contradiction. Thus, the inequality $\lambda^r(x_1; w_{\overline{Y}}) < \lambda^r(x_2; w_{\overline{Y}})$ follows.

Next, we prove $\lambda^r(x; w_{\overline{Y}})$ is continuous when $x \in [0, k)$. Since $\lambda^r(x; w_{\overline{Y}}) = 0$ when x satisfies $0 \leq x \leq k - \sum_{t \in \overline{Y}} q_t^*(w)$, we need to show that $\lambda^r(x; w_{\overline{Y}})$ is continuous when x satisfies $\max\left\{k - \sum_{t \in \overline{Y}} q_t^*(w), 0\right\} \leq x < k$. Suppose that x satisfies $\max\left\{k - \sum_{t \in \overline{Y}} q_t^*(w), 0\right\} \leq x < k$. Denote the inverse of $\lambda^r(x; w_{\overline{Y}})$ by $\lambda^{-1} : [0, \max_{t \in \overline{Y}} w_t - c_t) \to [\max\left\{k - \sum_{t \in \overline{Y}} q_t^*(w), 0\right\}, k)$. (Note this exists since λ^r is strictly increasing and onto the set $[0, \max_{t \in \overline{Y}} w_t - c_t)$). Pick any number $\epsilon > 0$. Consider the neighborhoods around x defined by the radiuses $\delta_1 \stackrel{\text{def}}{=} x - \lambda^{-1}(\max\{\lambda^r(x; w_{\overline{Y}}) - \epsilon, 0\}) \geq 0$ and $\delta_2 \stackrel{\text{def}}{=} \lambda^{-1}(\min\{\lambda^r(x; w_{\overline{Y}}) + \epsilon, \max_{t \in \overline{Y}} w_t - c_t\}) - x \geq 0$. It can be shown that both δ_1 and δ_2 can not be zero. If either δ_1 or δ_2 is zero, consider the radius $\delta \stackrel{\text{def}}{=} \max\{\delta_1, \delta_2\} > 0$, otherwise we set $\delta \stackrel{\text{def}}{=} \min\{\delta_1, \delta_2\} > 0$. Denote the δ neighborhood around a number z by $N_\delta(z)$. It can be shown that if $x' \in N_\delta(x) \cap [\max\left\{k - \sum_{t \in \overline{Y}} q_t^*(w), 0\right\}, k)$, then $\lambda^r(x'; w_{\overline{Y}}) \in N_\epsilon(\lambda^r(x; w_{\overline{Y}}))$. □

6.6.7 Proof: The marginal price for capacity is increasing.

Proof of Corollary 6.6. From Corollary 6.5, when x satisfies $0 \leq x < k - \sum_{t \in \overline{Y}} q_t^*(w)$ we have $\frac{\partial \lambda^r(x; w_{\overline{Y}})}{\partial x^+} = 0$ and when x satisfies $0 < x \leq k - \sum_{t \in \overline{Y}} q_t^*(w)$ we have

$\frac{\partial \lambda^r(x;w_{\overline{Y}})}{\partial x^-} = 0.$

Denote the solution to *RETAILER-RESTRICTED-TO-$\overline{Y}$-PRIMAL(k,x,$w_{\overline{Y}}$)* by the vector $q^r_{\overline{Y}}(x;w_{\overline{Y}})$. Let $Z(k,x,w_{\overline{Y}}) \stackrel{\text{def}}{=} \{i \mid q^r_i(x;w_{\overline{Y}}) = 0\} \subset \overline{Y}$ be the set of products that are not ordered in the retailer's decision problem. Suppose x satisfies $\max\left\{k - \sum_{t\in\overline{Y}} q^*_t(w), 0\right\} \leq x < k$. Observe that the equation $\sum_{t\in\overline{Y}} q^r_t(x;w_{\overline{Y}}) = k - x$ holds. Therefore, we can express x via the equation $x = k - \sum_{t\in\overline{Y}} q^r_t(x;w_{\overline{Y}})$. Denote the inverse of $\lambda^r(x;w_{\overline{Y}})$ by $x^r(\lambda;w_{\overline{Y}})$. Since $\lambda^r(x;w_{\overline{Y}})$ is strictly increasing and continuous, when $\frac{\partial x^r(\lambda;w_{\overline{Y}})}{\partial \lambda^+}$ exists it must be the case that $\frac{\partial \lambda^r(x;w_{\overline{Y}})}{\partial x^+} = \left(\frac{\partial x^r(\lambda;w_{\overline{Y}})}{\partial \lambda^+}\right)^{-1}$ where the equations $\lambda^r(x;w_{\overline{Y}}) = \lambda$ and $x^r(\lambda;w_{\overline{Y}}) = x$ hold. Furthermore,

$$\begin{aligned}
\frac{\partial x^r(\lambda;w_{\overline{Y}})}{\partial \lambda^+} &= \frac{\partial}{\partial \lambda^+}\left(k - \sum_{t\in\overline{Y}} q^r_t(x;w_{\overline{Y}})\right) \\
&= \frac{\partial k}{\partial \lambda^+} - \sum_{t\in\overline{Y}} \frac{\partial q^r_t(x;w_{\overline{Y}})}{\partial \lambda^+} \\
&= -\sum_{t\in\overline{Y}\setminus Z(k,x,w_{\overline{Y}})} -\frac{1}{p_t \cdot f_t\left(q^r_t(x;w_{\overline{Y}})\right)} && (6.42) \\
&= \sum_{t\in\overline{Y}\setminus Z(k,x,w_{\overline{Y}})} \frac{1}{p_t \cdot f_t\left(q^r_t(x;w_{\overline{Y}})\right)}. && (6.43)
\end{aligned}$$

(Equation 6.42 follows from Theorem 6.1 because from Equation (6.6) we have that for every good $j \in \overline{Y}$ such that $q^r_j(x_i;w_{\overline{Y}}) > 0$, the equation

$$\lambda^r(x;w_{\overline{Y}}) = p_j\bar{F}_j\left(q^r_j(x;w_{\overline{Y}})\right) - w_j,$$

holds.) Therefore, we have that $\frac{\partial \lambda^r(x;w_{\overline{Y}})}{\partial x^+} = \left(\sum_{t\in\overline{Y}\setminus Z(k,x,w_{\overline{Y}})} \frac{1}{p_t\cdot f_t\left(q^r_t(x;w_{\overline{Y}})\right)}\right)^{-1}$ when x satisfies $\max\left\{k - \sum_{t\in\overline{Y}} q^*_t(w), 0\right\} \leq x < k$.

Suppose x satisfies $\max\left\{k - \sum_{t\in\overline{Y}} q^*_t(w), 0\right\} < x < k$. Consider the retailer's problem *RETAILER-RESTRICTED-TO-$\overline{Y}$-PRIMAL(k,x,$w_{\overline{Y}}$)*. Let $A(k,x,w_{\overline{Y}}) \stackrel{\text{def}}{=} \{i \in Z(k,x,w_{\overline{Y}}) \mid \lambda^r(x;w_{\overline{Y}}) = p_i - w_i\} \subset \overline{Y}$ be the set of products that were almost ordered by retailer. Then, we have

$$\begin{aligned}
\frac{\partial x^r(\lambda; w_{\overline{Y}})}{\partial \lambda^-} &= \frac{\partial}{\partial \lambda^-}\left(k - \sum_{t\in\overline{Y}} q_t^r(x; w_{\overline{Y}})\right) \\
&= \frac{\partial k}{\partial \lambda^-} - \sum_{t\in\overline{Y}} \frac{\partial q_t^r(x; w_{\overline{Y}})}{\partial \lambda^-} \\
&= -\sum_{t\in A(k,x,w_{\overline{Y}})\cup\left(\overline{Y}\setminus Z(k,x,w_{\overline{Y}})\right)} -\frac{1}{p_t\cdot f_t\left(q_t^r(x; w_{\overline{Y}})\right)} \qquad (6.44)\\
&= \sum_{t\in A(k,x,w_{\overline{Y}})\cup\left(\overline{Y}\setminus Z(k,x,w_{\overline{Y}})\right)} \frac{1}{p_t\cdot f_t\left(q_t^r(x; w_{\overline{Y}})\right)} \\
&= \sum_{t\in \overline{Y}\setminus Z(k,x,w_{\overline{Y}})} \frac{1}{p_t\cdot f_t\left(q_t^r(x; w_{\overline{Y}})\right)} + \sum_{t\in A(k,x,w_{\overline{Y}})} \frac{1}{p_t\cdot f_t(0)}. \qquad (6.45)
\end{aligned}$$

(Equation 6.44 follows from Theorem 6.1 because from Equation (6.6) we have that for every good $j \in A(k, x, w_{\overline{Y}}) \cup \left(\overline{Y} \setminus Z(k, x, w_{\overline{Y}})\right)$, the equation

$$\lambda^r(x; w_{\overline{Y}}) = p_j\bar{F}_j\left(q_j^r(x; w_{\overline{Y}})\right) - w_j, \qquad (6.46)$$

holds.) Therefore, we have that

$$\frac{\partial \lambda^r(x; w_{\overline{Y}})}{\partial x^-} = \left(\sum_{t\in A(k,x,w_{\overline{Y}})\cup\left(\overline{Y}\setminus Z(k,x,w_{\overline{Y}})\right)} \frac{1}{p_t\cdot f_t\left(q_t^r(x; w_{\overline{Y}})\right)}\right)^{-1} \qquad (6.47)$$

when x satisfies $\max\left\{k - \sum_{t\in\overline{Y}} q_t^*(w), 0\right\} < x < k$.

Suppose $\lambda^r(x; w_{\overline{Y}}) > 0$ and $\lambda^r(x; w_{\overline{Y}}) \neq p_i - w_i$ for any $i \in \overline{Y}$. From Corollary 6.5, we have that x satisfies $\max\left\{k - \sum_{t\in\overline{Y}} q_t^*(w), 0\right\} < x < k$. And from the definition of $A(k, x, w_{\overline{Y}})$ we have that $A(k, x, w_{\overline{Y}}) = \emptyset$. Therefore, from Equations (6.43) and (6.45), we have that $\frac{\partial x^r(\lambda; w_{\overline{Y}})}{\partial \lambda^+} = \frac{\partial x^r(\lambda; w_{\overline{Y}})}{\partial \lambda^-}$ which implies that $\frac{\partial \lambda^r(x; w_{\overline{Y}})}{\partial x^+} = \frac{\partial \lambda^r(x; w_{\overline{Y}})}{\partial x^-}$. From Corollary 6.5, we have $\lambda^r(x; w_{\overline{Y}})$ is continuous at x, implying that $x^r(\lambda; w_{\overline{Y}})$ is continuous at λ (where the equations $\lambda^r(x; w_{\overline{Y}}) = \lambda$ and $x^r(\lambda; w_{\overline{Y}}) = x$ hold). Therefore, from equation (6.46), we have that $q_t^r\left(x^r(\lambda; w_{\overline{Y}}); w_{\overline{Y}}\right)$ is continuous at λ. Since $\frac{\partial x^r(\lambda; w_{\overline{Y}})}{\partial \lambda} = \sum_{t\in\overline{Y}\setminus Z(k,x,w_{\overline{Y}})} \frac{1}{p_t\cdot f_t\left(q_t^r(x; w_{\overline{Y}})\right)}$ and the p.d.f. f_t is continuous for every good $t \in Y$, we have that $\frac{\partial x^r(\lambda;w_{\overline{Y}})}{\partial \lambda}$ is continuous at λ since the set of goods $\overline{Y}\backslash Z(k,x,w_{\overline{Y}})$ do not change for small changes in λ. Therefore, $\frac{\partial \lambda^r(x;w_{\overline{Y}})}{\partial x}$ is continuous at x.

Suppose x satisfies the equation $\max\left\{k - \sum_{t\in\overline{Y}} q_t^*(w), 0\right\} \leq x < k$ and the retailer is service constrained for good(s) $\overline{Y}$. From equation (6.46), we have that for every good $t \in \overline{Y}\setminus Z(k,x,w_{\overline{Y}})$, the order quantity $q_t^r\left(x^r(\lambda;w_{\overline{Y}});w_{\overline{Y}}\right)$ is strictly decreasing in λ. Since the retailer is service constrained for good t, we have $f_t\left(q_t^r\left(x^r(\lambda;w_{\overline{Y}});w_{\overline{Y}}\right)\right)$ is strictly increasing in λ. Therefore, $\frac{\partial x^r(\lambda;w_{\overline{Y}})}{\partial \lambda^+}$ and $\frac{\partial x^r(\lambda;w_{\overline{Y}})}{\partial \lambda^-}$ are strictly decreasing in λ. So that we have $\frac{\partial \lambda^r(x;w_{\overline{Y}})}{\partial x^+}$ and $\frac{\partial \lambda^r(x;w_{\overline{Y}})}{\partial x^-}$ are strictly increasing in x. $\square$

6.6.8 Proof: Partitioning the set of wholesale prices by 'capacity charge'.

Proof of Theorem 6.4. Suppose $\lambda \geq 0$. Consider a wholesale price vector w from the set

$$\mathcal{W}(\lambda) \stackrel{\text{def}}{=} \left\{ w \;\middle|\; \begin{array}{l} w_t = p_t\bar{F}_t(q_t) - \lambda + \gamma_t \cdot 1_{\{q_t=0\}} \;\; \forall t \in N,\; q,\gamma \in \mathbb{R}_+^{|N|}, \\ \sum_{t\in N} q_t = \min\left\{\left(\sum_{t\in N} q_t^*(w)\right)\cdot 1_{\{\lambda=0\}} + k\cdot 1_{\{\lambda>0\}},\; k\right\} \end{array} \right\}. \quad (6.48)$$

For this vector w there exist vectors q and γ that satisfy the conditions in (6.48) which guarantee w's membership in the set $\mathcal{W}(\lambda)$. From the proof of Theorem 6.1, we have that The Karush-Kuhn-Tucker conditions for the retailer's decision problem, *RETAILER-PRIMAL(k,w)*, are:

$$p_t\bar{F}_t(\widehat{q}_t) - w_t + \widehat{\gamma}_t - \widehat{\lambda} = 0, \quad t = 1,\ldots,n; \qquad (6.49)$$

$$\widehat{q}_t \geq 0, \quad t = 1,\ldots,n;$$

$$k - \sum_{t=1}^{n} \widehat{q}_t \geq 0;$$

$$\widehat{\gamma}_t\widehat{q}_t = 0, \quad t = 1,\ldots,n; \qquad (6.50)$$

$$\widehat{\lambda}\left(k - \sum_{t=1}^{n} \widehat{q}_t\right) = 0; \tag{6.51}$$

$$\widehat{\lambda} \geq 0; \quad \widehat{\gamma}_t \geq 0, \quad t = 1, \ldots, n.$$

Because of the concavity of the objective function and the fact that the Slater condition is satisfied, the Karush-Kuhn-Tucker conditions are both necessary and sufficient conditions for any primal optimal vector $\widehat{q}$ and dual optimal vector $(\widehat{\gamma}, \widehat{\lambda})$. Consider a particular value for $\widehat{q}$, $\widehat{\gamma}$, and $\widehat{\lambda}$. In particular, suppose we set:

$$\widehat{q} = q,$$
$$\widehat{\gamma} = (\gamma_1 \cdot 1_{\{q_1=0\}}, \ldots, \gamma_n \cdot 1_{\{q_n=0\}}),$$
$$\widehat{\lambda} = \lambda.$$

It can be shown that these values for $\widehat{q}$, $\widehat{\gamma}$, and $\widehat{\lambda}$ satisfy the Karush-Kuhn-Tucker conditions so that $\lambda^r(w) = \widehat{\lambda} = \lambda$. (There are two main steps in seeing this. First, consider the cases $\lambda = 0$ and $\lambda > 0$ separately. Then, for each of those cases, confirm that the equations (6.49), (6.50), and (6.51) are satisfied for these values $\widehat{q}$, $\widehat{\gamma}$, $\widehat{\lambda}$ when the wholesale price for good t is $w_t = p_t \bar{F}_t(q_t) - \lambda + \gamma_t \cdot 1_{\{q_t=0\}}$.) Therefore, the wholesale price vector w induces the retailer to have shadow price λ for the capacity constraint k, i.e., $\lambda^r(w) = \lambda$.

Next, we prove the converse. Suppose a wholesale price vector w induces retailer shadow price $\widehat{\lambda}$ for capacity k, i.e., $\lambda^r(w) = \widehat{\lambda}$. Therefore, there exist vectors $\widehat{q}$ and $\widehat{\gamma}$ that along with $\widehat{\lambda}$ and wholesale price vector w satisfy the Karush-Kuhn-Tucker conditions. Suppose we set: $q = \widehat{q}$, $\gamma = \widehat{\gamma}$, and $\lambda = \widehat{\lambda}$. It can be shown that these values for q, γ, and λ enable w's membership in $\mathcal{W}(\lambda)$ using the conditions in (6.48). Therefore, $w \in \mathcal{W}(\lambda) = \mathcal{W}(\widehat{\lambda})$. □

6.6.9 Proof: Partitioning the 'binding' wholesale prices by 'induced allocation'.

Proof of Theorem 6.5. Suppose we have an order quantity vector $q \in \mathbb{R}_+^{|N|}$ such that the condition $\sum_{t \in N} q_t = k$ holds. Consider a wholesale price vector w from the set

$$\mathcal{A}(q) \overset{\text{def}}{=} \left\{ w \mid w_t = p_t \bar{F}_t(q_t) - \lambda + \gamma_t \cdot 1_{\{q_t = 0\}} \;\; \forall t \in N, \; \lambda \in \mathbb{R}_+, \; \gamma \in \mathbb{R}_+^{|N|} \right\}. \quad (6.52)$$

For this vector w there exists a scalar λ and a vector γ that satisfy the conditions in (6.52), guaranteeing w's membership in the set $\mathcal{A}(q)$. The Karush-Kuhn-Tucker conditions for the retailer's decision problem, *RETAILER-PRIMAL(k,w)*, written in the proof of Theorem 6.4, are both necessary and sufficient conditions for any primal optimal vector $\widehat{q}$ and dual optimal vector $(\widehat{\gamma}, \widehat{\lambda})$. Consider a particular value for $\widehat{q}$, $\widehat{\gamma}$, and $\widehat{\lambda}$. In particular, suppose we set:

$$\begin{aligned}
\widehat{q} &= q, \\
\widehat{\gamma} &= (\gamma_1 \cdot 1_{\{q_1=0\}}, \ldots, \gamma_n \cdot 1_{\{q_n=0\}}), \\
\widehat{\lambda} &= \lambda.
\end{aligned}$$

It can be shown that these values for $\widehat{q}$, $\widehat{\gamma}$, and $\widehat{\lambda}$ satisfy the Karush-Kuhn-Tucker conditions so that $q^r(w) = \widehat{q} = q$. (To see this: verify that the equations (6.49), (6.50), and (6.51) are satisfied for these values $\widehat{q}$, $\widehat{\gamma}$, $\widehat{\lambda}$ when the wholesale price for good t is $w_t = p_t \bar{F}_t(q_t) - \lambda + \gamma_t \cdot 1_{\{q_t=0\}}$.) Therefore, the wholesale price vector w induces the retailer to order according to the vector q, i.e., $q^r(w) = q$.

Next, we prove the converse. Suppose a wholesale price vector w induces retailer to order according to the vector $\widehat{q}$, i.e., $q^r(w) = \widehat{q}$. Therefore, there exists a vector $\widehat{\gamma}$ and a scalar $\widehat{\lambda}$ that along with $\widehat{q}$ and the wholesale price vector w satisfy the Karush-Kuhn-Tucker conditions. Suppose we set: $q = \widehat{q}$, $\gamma = \widehat{\gamma}$, and $\lambda = \widehat{\lambda}$. It can be shown that these values for q, γ, and λ enable w's membership in $\mathcal{A}(q)$ using the conditions in (6.52). Therefore, $w \in \mathcal{A}(q) = \mathcal{A}(\widehat{q})$. □

6.6.10 Proof: The shadow price for supplier Y's aggregate induced order.

Proof of Theorem 6.6. Recall the definition $g_i(x) \stackrel{\text{def}}{=} x \cdot f(x)/\bar{F}_i(x)$ of the generalized failure rate function. For each good $i \in Y$, we have that

$$\frac{\partial}{\partial q_i}\left(\left(p_i\bar{F}_i(q_i) - \lambda^r\left(x; w_{\overline{Y}}\right) - c_i\right)\cdot q_i\right) = p_i\bar{F}_i(q_i)\cdot(1-g_i(q_i)) - \lambda^r\left(x; w_{\overline{Y}}\right) - c_i.$$

Each c.d.f. F_i is strictly increasing over $[0, l_i]$, continuously differentiable, and has the IGFR property, so that $p_i\bar{F}_i(q_i)\cdot(1-g_i(q_i))$ is continuous, nonnegative, and strictly decreasing in q_i while $g_i(q_i) \leq 1$ (and negative when $g_i(q_i) > 1$). For good $i \in Y$, we define the order quantity $\widehat{q}_i$ in terms of q_i^e, the equilibrium induced order for good i in the unconstrained setting (see equation (6.4)), as follows: $\widehat{q}_i \stackrel{\text{def}}{=} \min\{q_i^e, k\}$. From equation (6.4), observe that $g_i(q_i^e) < 1$. Then, we have that $\left(p_i\bar{F}_i(q_i) - \lambda^r\left(x; w_{\overline{Y}}\right) - c_i\right)\cdot q_i$ is strictly concave for $q_i \in [0, \widehat{q}_i]$. Therefore, the objective function for *Y-SUPPLIER-PRICING-PRIMAL(x,$w_{\overline{Y}}$)* is strictly concave for $q_Y \in \{q \in \mathbb{R}_+^{|Y|} \mid 0 \leq q_i \leq \widehat{q}_i\}$ which is a superset of the feasible set for *Y-SUPPLIER-PRICING-PRIMAL(x,$w_{\overline{Y}}$)* (since $x \in \left(0, \min\left\{\sum_{t\in Y} q_t^e, k\right\}\right]$). Because the feasible set is convex and compact, *Y-SUPPLIER-PRICING-PRIMAL(x,$w_{\overline{Y}}$)* has a unique solution.

Consider the Lagrangian $\mathcal{L}(q_Y, \gamma_1, \ldots, \gamma_{|Y|}, \lambda)$ for *Y-SUPPLIER-PRICING-PRIMAL(x,$w_{\overline{Y}}$)*:

$$\mathcal{L}(q_Y, \gamma_1, \ldots, \gamma_{|Y|}, \lambda) = \sum_{i\in Y}\left(p_i\bar{F}_i(q_i) - \lambda^r\left(x; w_{\overline{Y}}\right) - c_i\right)\cdot q_i + \sum_{i\in Y}\gamma_i q_i + \lambda\left(x - \sum_{i\in Y} q_i\right).$$

The Karush-Kuhn-Tucker conditions for supplier Y's decision problem, *Y-SUPPLIER-PRICING-PRIMAL(x,$w_{\overline{Y}}$)*, are:

$$p_t\bar{F}_t(q_t)(1-g_t(q_t)) - \lambda^r\left(x; w_{\overline{Y}}\right) - c_t + \gamma_t - \lambda = 0, \quad t \in Y; \tag{6.53}$$

$$q_t \geq 0, \quad t \in Y;$$

$$x - \sum_{t\in Y} q_t = 0;$$

$$\gamma_t q_t = 0, \quad t \in Y; \tag{6.54}$$

$$\gamma_t \geq 0, \quad t \in Y.$$

Since $x \neq 0$, it can be shown that a constraint qualification condition on a particular matrix (each row of which is the gradient of an effective constraint at the optimal order vector) is satisfied. Briefly, the constraint qualification condition requires that the matrix have rank equal to the number of effective constraints. See Sundaram (1996, Chap. 6, Thm 6.10, p.165) for a detailed description of the constraint qualification condition. Therefore, the Karush-Kuhn-Tucker conditions are necessary for any primal optimal vector q_Y. Furthermore, because of the concavity of the objective function and the functions that define the constraints, the Karush-Kuhn-Tucker conditions are sufficient conditions for any primal optimal vector q_Y.

As a result, since the primal problem has a unique solution, it can be shown that the dual problem has a unique solution using these conditions. Let

$$\left(\left(q_i^Y(x; w_{\overline{Y}})\right)_{i \in Y}, \left(\gamma_i^Y(x; w_{\overline{Y}})\right)_{i \in Y}, \lambda^Y(x; w_{\overline{Y}}) \right)$$

denote the unique vector that satisfies the Karush-Kuhn-Tucker conditions.

When $j \in Y \backslash Z^Y(x; w_{\overline{Y}})$, from equation (6.54) we have $\gamma_j^Y(x; w_{\overline{Y}}) = 0$. Therefore, from equation (6.53) we have

$$\lambda^Y(x; w_{\overline{Y}}) = p_j \bar{F}_j \left(q_j^Y(x; w_{\overline{Y}})\right) \cdot \left(1 - g_j \left(q_j^Y(x; w_{\overline{Y}})\right)\right) - c_j - \lambda^r \left(x; w_{\overline{Y}}\right).$$

When $i \in Z^Y(x; w_{\overline{Y}})$, from equation (6.53), we have

$$\lambda^Y(x; w_{\overline{Y}}) = p_i - c_i - \lambda^r \left(x; w_{\overline{Y}}\right) + \gamma_i^Y(x; w_{\overline{Y}}) \geq p_i - c_i - \lambda^r \left(x; w_{\overline{Y}}\right).$$

Thus, the conditions in equations (6.16) and (6.17) hold.

Furthermore, if $x = \sum_{t \in Y} q_t^e \leq k$, we have $q_i^Y(x; w_{\overline{Y}}) = q_i^e$ for every good $i \in Y$.

Therefore, when $j \in Y \setminus Z^Y(x; w_{\overline{Y}})$, from equation (6.53) we have

$$\lambda^Y(x; w_{\overline{Y}}) + \lambda^r(x; w_{\overline{Y}}) = p_j \bar{F}_j(q_j^e) \cdot \left(1 - g_j(q_j^e)\right) - c_j = 0.$$

On the other hand, assume $\lambda^Y(x; w_{\overline{Y}}) + \lambda^r(x; w_{\overline{Y}}) = 0$ with an induced aggregate order $x \in \left(0, \min\left\{\sum_{t \in Y} q_t^e, k\right\}\right]$. When $j \in Y \setminus Z^Y(x; w_{\overline{Y}})$, we have

$$p_j \bar{F}_j\left(q_j^Y(x; w_{\overline{Y}})\right) \cdot \left(1 - g_j\left(q_j^Y(x; w_{\overline{Y}})\right)\right) - c_j = 0$$

from equation (6.53). Therefore, $q_j^Y(x; w_{\overline{Y}}) = q_j^e$ from equation (6.4). Assume $Z^Y(x; w_{\overline{Y}})$ is not empty. When $i \in Z(w)$, from equation (6.53), we have

$$p_i - c_i \leq p_i - c_i + \gamma_i^Y(x; w_{\overline{Y}}) = 0.$$

This is a contradiction because $c_i < p_i$. Therefore, the set $Z^Y(x; w_{\overline{Y}})$ is empty when the condition $\lambda^Y(x; w_{\overline{Y}}) + \lambda^r(x; w_{\overline{Y}}) = 0$ holds. And so we have $\sum_{t \in Y} q_t^e = \sum_{t \in Y} q_t^Y(x; w_{\overline{Y}}) = x \leq k$.

Suppose that x_1 and x_2 satisfy $0 < x_1 < x_2 \leq \min\left\{\sum_{t \in Y} q_t^e, k\right\}$. We show that $\lambda^Y(x_2; w_{\overline{Y}}) + \lambda^r(x_2; w_{\overline{Y}}) < \lambda^Y(x_1; w_{\overline{Y}}) + \lambda^r(x_1; w_{\overline{Y}})$. Assume that $\lambda^Y(x_1; w_{\overline{Y}}) + \lambda^r(x_1; w_{\overline{Y}}) \leq \lambda^Y(x_2; w_{\overline{Y}}) + \lambda^r(x_2; w_{\overline{Y}})$ holds, instead. From Equation (6.16) in Theorem 6.6, we have that for every good $j \in Y$ such that $q_j^Y(x_i; w_{\overline{Y}}) > 0$,

$$\lambda^Y(x_i; w_{\overline{Y}}) + \lambda^r(x_i; w_{\overline{Y}}) = p_j \bar{F}_j\left(q_j^Y(x_i; w_{\overline{Y}})\right) \cdot \left(1 - g_j\left(q_j^Y(x_i; w_{\overline{Y}})\right)\right) - c_j, \quad (6.55)$$

for $i = 1, 2$. Because of our assumption on the cumulative distribution functions and $\lambda^Y(x_1; w_{\overline{Y}}) + \lambda^r(x_1; w_{\overline{Y}}) \leq \lambda^Y(x_2; w_{\overline{Y}}) + \lambda^r(x_2; w_{\overline{Y}})$, from equation (6.55), we have $q_j^Y(x_2; w_{\overline{Y}}) \leq q_j^Y(x_1; w_{\overline{Y}})$ for every good $j \in Y$. So that

$$x_2 = \sum_{j \in Y} q_j^Y(x_2; w_{\overline{Y}}) \leq \sum_{j \in Y} q_j^Y(x_1; w_{\overline{Y}}) = x_1. \quad (6.56)$$

But this is a contradiction because $x_1 < x_2$ holds. Thus, the inequality $\lambda^Y(x_2; w_{\overline{Y}}) +$

$\lambda^r(x_2; w_{\overline{Y}}) < \lambda^Y(x_1; w_{\overline{Y}}) + \lambda^r(x_1; w_{\overline{Y}})$ follows. And so we have that the function $\lambda^Y(x; w_{\overline{Y}}) + \lambda^r(x; w_{\overline{Y}})$ is strictly decreasing as $x \in \left(0, \min\left\{\sum_{t\in Y} q_t^e, k\right\}\right]$ increases. □

6.6.11 Proof: Any induced aggregate order above $\bar{x}$ is not optimal.

Proof of Corollary 6.7. Suppose that for every good $t \in Y$, the c.d.f. F_t has the IGFR property. Assume that the conditions

$$\lambda^Y\left(\min\left\{\sum_{t\in Y} q_t^e, k\right\}; w_{\overline{Y}}\right) \le 0, \tag{6.57}$$

$$0 < \max\left\{p_i - c_i - \lambda^r\left(0; w_{\overline{Y}}\right) \mid i \in Y\right\}, \tag{6.58}$$

hold. From Corollary 6.5, we know $\lambda^r(x; w_{\overline{Y}})$ is continuous at $x = 0$. Furthermore, the cumulative distribution functions are continuously differentiable. Therefore, from equation (6.58), we have that there exists some small positive value $\delta < \min\left\{\sum_{t\in Y} q_t^e, k\right\}$ such that the condition

$$0 < \max\left\{p_i \bar{F}_i(\delta) \cdot (1 - g_i(\delta)) - c_i - \lambda^r\left(\delta; w_{\overline{Y}}\right) \mid i \in Y\right\}$$

holds. And so, from Theorem 6.6, it follows that there exists a small positive value $\hat{x} < \delta$ such that $\lambda^Y(\hat{x}; w_{\overline{Y}}) > 0$. Using a technique similar to our proof that $\lambda^r(x; w_{\overline{Y}})$ is continuous (i.e., Corollary 6.5), it can be shown that $\lambda^Y(x; w_{\overline{Y}})$ is continuous for $x \in \left(0, \min\left\{\sum_{t\in Y} q_t^e, k\right\}\right]$. And from Theorem 6.6, it follows that $\lambda^Y(x; w_{\overline{Y}})$ is strictly decreasing because we know that $\lambda^r(x; w_{\overline{Y}})$ is nondecreasing from Corollary 6.5. Therefore, from equation (6.57), we have that there exists a value $\bar{x}$, where $\hat{x} < \bar{x} \le \min\left\{\sum_{t\in Y} q_t^e, k\right\}$, that satisfies the equation $\lambda^Y(\bar{x}; w_{\overline{Y}}) = 0$. For any unit above $\bar{x}$ that supplier Y induces the retailer to order (in aggregate), the supplier incurs a loss because the marginal profit on the x^{th} unit is upper bounded by $\lambda^Y(x; w_{\overline{Y}})$, which is a negative number for any $x > \bar{x}$. Therefore, supplier Y would never induce the retailer to order more than $\bar{x}$ units of good(s) Y in aggregate. Furthermore, from Theorem 6.6, we have that $\lambda^Y(x; w_{\overline{Y}}) = -\lambda^r(x; w_{\overline{Y}}) \leq 0$ if and only if $x = \sum_{t\in Y} q_t^e \leq k$ and that $\lambda^Y(x; w_{\overline{Y}})$ is strictly decreasing. Therefore, we have $\bar{x} \leq \sum_{t\in Y} q_t^e$.

Now assume that the conditions

$$0 < \lambda^Y \left(\min \left\{ \sum_{t\in Y} q_t^e, k \right\}; w_{\overline{Y}} \right), \tag{6.59}$$

$$0 < \max \{ p_i - c_i - \lambda^r (0; w_{\overline{Y}}) \mid i \in Y \}, \tag{6.60}$$

hold instead. From Theorem 6.6, we have that $\lambda^Y(x; w_{\overline{Y}}) = -\lambda^r(x; w_{\overline{Y}}) \leq 0$ if and only if $x = \sum_{t\in Y} q_t^e \leq k$ and that $\lambda^Y(x; w_{\overline{Y}})$ is strictly decreasing. Therefore, from equation (6.59) we have $\min\{\sum_{t\in Y} q_t^e, k\} < \sum_{t\in Y} q_t^e$, implying $\min\{\sum_{t\in Y} q_t^e, k\} = k$. Suppose we define $\bar{x}$ to equal $\min\{\sum_{t\in Y} q_t^e, k\} = k$. It follows trivially that supplier Y would never induce the retailer to order more than $\bar{x}$ units of good(s) Y in aggregate because the retailer has a capacity constraint of k units.

Finally, assume the condition

$$\max \{ p_i - c_i - \lambda^r (0; w_{\overline{Y}}) \mid i \in Y \} \leq 0,$$

holds instead, and that we define $\bar{x} \stackrel{\text{def}}{=} 0$. Therefore, for any unit above $\bar{x}$ that supplier Y induces the retailer to order (in aggregate), the supplier incurs a loss because the marginal profit on the x^{th} unit is upper bounded by $\lambda^Y(x; w_{\overline{Y}})$, which is a negative number for any $x > \bar{x}$. And so, supplier Y would never induce the retailer to order more than $\bar{x}$ units of good(s) Y in aggregate. □

6.6.12 Proof: Optimal aggregate order for a fixed $w_{\overline{Y}}$.

Proof of Theorem 6.7. The (Weierstrass) Extreme Value Theorem says that for any continuous and real function f on a compact metric space X, there exists a point $x^* \in X$ such $f(x^*) = \sup_{x\in X} f(x)$. (Rudin 1976, Theorem 4.16) From Corollary 6.5, we have that $\lambda^r \left(\sum_{t\in Y} q_t; w_{\overline{Y}}\right)$ is continuous on the set of feasible order quantity vectors, $Q \stackrel{\text{def}}{=} \{q \in \mathbb{R}^n_+ \mid \sum_{t=1}^n q_t \leq k\}$. Therefore, we have that supplier Y's objective function, $\sum_{i \in Y} \left(p_i \bar{F}_i(q_i) - \lambda^r \left(\sum_{t \in Y} q_t; w_{\overline{Y}}\right) - c_i\right) \cdot q_i$, when solving *Y-SUPPLIER-INDUCING-AGGREGATE-ORDER($w_{\overline{Y}}$)*, is continuous on Q (because it consists of a finite sum of products of continuous functions). The set Q is closed and bounded (and hence compact) and is a metric space (under the Euclidean metric). Therefore, applying the (Weierstrass) Extreme Value Theorem, we have that an optimal aggregate order quantity $x^Y(w_{\overline{Y}})$ and optimal induced order vector $q^Y(w_{\overline{Y}})$ *exist* for the problem *Y-SUPPLIER-INDUCING-AGGREGATE-ORDER($w_{\overline{Y}}$)*.

From Theorem 6.6 we have that for any fixed value x, the optimal vector of goods that supplier Y induces the retailer to order is $q^Y(x; w_{\overline{Y}})$, the solution to the decision problem *Y-SUPPLIER-PRICING-PRIMAL(x,$w_{\overline{Y}}$)*. Therefore, we can re-express the objective function

$$\sum_{i \in Y} \left(p_i \bar{F}_i(q_i) - \lambda^r \left(\sum_{t \in Y} q_t; w_{\overline{Y}}\right) - c_i\right) \cdot q_i$$

for the problem *Y-SUPPLIER-INDUCING-AGGREGATE-ORDER($w_{\overline{Y}}$)* as

$$\sum_{i \in Y} \left(p_i \bar{F}_i\left(q_i^Y(x; w_{\overline{Y}})\right) - \lambda^r(x; w_{\overline{Y}}) - c_i\right) \cdot q_i^Y(x; w_{\overline{Y}})$$

so that the only decision variable we need to solve for is x.

Recall from Corollary 6.6 that $\lambda^r(x; w_{\overline{Y}})$ is not differentiable everywhere. However, from the proof of Corollary 6.6, we know that both the derivative from the right and left do exist for $\lambda^r(x; w_{\overline{Y}})$ (and are equal almost everywhere except at $|\overline{Y}| - 1$ points at most). Therefore, we can write

$$
\begin{aligned}
&\frac{\partial}{\partial x^-}\left(\sum_{i\in Y}\left(p_i\bar{F}_i\left(q_i^Y(x;w_{\overline{Y}})\right)-\lambda^r\left(x;w_{\overline{Y}}\right)-c_i\right)\cdot q_i^Y(x;w_{\overline{Y}})\right)\\
&=\frac{\partial}{\partial x^-}\left(\sum_{i\in Y\setminus Z^Y(x;w_{\overline{Y}})}\left(p_i\bar{F}_i\left(q_i^Y(x;w_{\overline{Y}})\right)-\lambda^r\left(x;w_{\overline{Y}}\right)-c_i\right)\cdot q_i^Y(x;w_{\overline{Y}})\right)\\
&=\sum_{i\in Y\setminus Z^Y(x;w_{\overline{Y}})}\frac{\partial}{\partial x^-}\left(\left(p_i\bar{F}_i\left(q_i^Y(x;w_{\overline{Y}})\right)-c_i\right)\cdot q_i^Y(x;w_{\overline{Y}})\right)-\frac{\partial}{\partial x^-}\left(\lambda^r\left(x;w_{\overline{Y}}\right)\cdot q_i^Y(x;w_{\overline{Y}})\right)\\
&=\sum_{i\in Y\setminus Z^Y(x;w_{\overline{Y}})}\left(\frac{\partial}{\partial q_i^Y}\left(\left(p_i\bar{F}_i\left(q_i^Y(x;w_{\overline{Y}})\right)-c_i\right)\cdot q_i^Y(x;w_{\overline{Y}})\right)-\lambda^r\left(x;w_{\overline{Y}}\right)\right)\cdot\frac{\partial q_i^Y}{\partial x^-}\\
&\qquad-\frac{\partial\lambda^r\left(x;w_{\overline{Y}}\right)}{\partial x^-}\cdot q_i^Y(x;w_{\overline{Y}})\\
&=\sum_{i\in Y\setminus Z(k,x,w_{\overline{Y}})}\left(p_i\bar{F}_i\left(q_i^Y(x;w_{\overline{Y}})\right)\cdot\left(1-g_i\left(q_i^Y(x;w_{\overline{Y}})\right)\right)-c_i-\lambda^r\left(x;w_{\overline{Y}}\right)\right)\cdot\frac{\partial q_i^Y}{\partial x^-}\\
&\qquad-\frac{\partial\lambda^r\left(x;w_{\overline{Y}}\right)}{\partial x^-}\cdot q_i^Y(x;w_{\overline{Y}})\\
&=\sum_{i\in Y\setminus Z(k,x,w_{\overline{Y}})}\lambda^Y(x;w_{\overline{Y}})\cdot\frac{\partial q_i^Y}{\partial x^-}-\frac{\partial\lambda^r\left(x;w_{\overline{Y}}\right)}{\partial x^-}\cdot q_i^Y(x;w_{\overline{Y}})\\
&=\left(\sum_{i\in Y\setminus Z(k,x,w_{\overline{Y}})}\frac{\partial q_i^Y}{\partial x^-}\right)\cdot\lambda^Y(x;w_{\overline{Y}})-\left(\sum_{i\in Y\setminus Z(k,x,w_{\overline{Y}})}q_i^Y(x;w_{\overline{Y}})\right)\cdot\frac{\partial\lambda^r\left(x;w_{\overline{Y}}\right)}{\partial x^-}\\
&=\lambda^Y(x;w_{\overline{Y}})-x\cdot\frac{\partial\lambda^r(x;w_{\overline{Y}})}{\partial x^-}
\end{aligned}
\tag{6.61}
$$

Since $\bar{x}\neq 0$ and the cumulative distribution function for demand of each good $y\in Y$ has the IGFR property, from Corollary 6.7 we have that $\lambda^Y(x;w_{\overline{Y}})$ is nonnegative for every $x\in[0,\bar{x}]$. Furthermore, from Theorem 6.6 we have that $\lambda^Y(x;w_{\overline{Y}})$ is strictly decreasing as x increases. Since the retailer is service constrained for good(s) $\overline{Y}$, from Corollary 6.5 and Corollary 6.6 we have that $x\cdot\frac{\partial\lambda^r(x;w_{\overline{Y}})}{\partial x^-}$ is nonnegative and nondecreasing. Therefore, from equation (6.61), we have that supplier Y's objective function is concave in the induced aggregate order x. And so, equation (6.19) holds. From the proof of Theorem 6.6, we have that equation (6.20) holds.

□

6.6.13 Proof: Characterizing $\mathcal{W}_Y^{\mathbf{br}}(w_{\overline{Y}})$.

First, we introduce some preliminary definitions. For the motivation for these definitions see See Sundaram (1996) (Chapter 9). Consider metric spaces Q and W and a correspondence $\mathcal{D} : W \to 2^Q$.

DEFINITION 6.1. The correspondence $\mathcal{D}$ is *upper-hemicontinuous* at a point $w \in W$, if for all open sets V such that $\mathcal{D}(w) \subset V$, there exists an open set U containing w, such that $w' \in U \cap W$ implies $\mathcal{D}(w') \subset V$. The correspondence $\mathcal{D}$ is *upper-hemicontinuous* on W, if $\mathcal{D}$ is upper-hemicontinuous at each point $w \in W$.

DEFINITION 6.2. The correspondence $\mathcal{D}$ is *lower-hemicontinuous* at a point $w \in W$, if for all open sets V such that $\mathcal{D}(w) \cap V \neq \emptyset$, there exists an open set U containing w, such that $w' \in U \cap W$ implies $\mathcal{D}(w') \cap V \neq \emptyset$. The correspondence $\mathcal{D}$ is *lower-hemicontinuous* on W, if $\mathcal{D}$ is lower-hemicontinuous at each point $w \in W$.

Finally, the correspondence $\mathcal{D}$ is *continuous* at a point $w \in W$, if $\mathcal{D}$ is both upper-hemicontinous and lower-hemicontinuous at w. And the correspondence $\mathcal{D}$ is *continuous* on W, if $\mathcal{D}$ is continuous at each point $w \in W$.

We apply Berge's Maximum Theorem (Proposition 6.1) in the proof of Lemma 6.1, below. Therefore, we state the Maximum Theorem for completeness.[9]

PROPOSITION 6.1 **(Berge's Maximum Theorem).** *Consider metric spaces Q and W, a continuous function $f : Q \times W \to \mathbb{R}$, and a compact-valued and continuous correspondence $\mathcal{D} : W \to 2^Q$. Suppose we define the function f^* and correspondence $\mathcal{D}^*$ by the equations*

$$f^*(w) \overset{def}{=} \max\{f(q, w) \mid q \in \mathcal{D}(w)\}, \tag{6.62}$$

$$\mathcal{D}^*(w) \overset{def}{=} \{q \in \mathcal{D}(w) \mid f(q, w) = f^*(w)\}. \tag{6.63}$$

Then, the function f^* is continuous on W, and the correspondence $\mathcal{D}^*$ is compact-valued and upper-hemicontinous on W.

[9] See Sundaram (1996) (Chapter 9) and Border (1989) (Chapter 12) for the proof of the Maximum Theorem.

Now we state and prove Lemma 6.1 (for use in the proof of Theorem 6.8 and Theorem 6.9).

LEMMA 6.1. *Supplier Y's objective function* $\pi_Y(w_Y, w_{\overline{Y}}) \overset{def}{=} \sum_{i \in Y}(w_i - c_i)q_i^r(w_Y, w_{\overline{Y}})$, *when solving Y-SUPPLIER($w_{\overline{Y}}$), is continuous in the vector* $(w_Y, w_{\overline{Y}})$.

Proof of Lemma 6.1. First, we show that for any good $t \in N$, the retailer's induced order quantity $q_t^r(w_Y, w_{\overline{Y}})$ is continuous in the vector $(w_Y, w_{\overline{Y}})$. Denote the set of feasible order quantity vectors by $Q \overset{\text{def}}{=} \{q \in \mathbb{R}_+^n \mid \sum_{t=1}^n q_t \leq k\}$ and the set of feasible wholesale price vectors by $W \overset{\text{def}}{=} \prod_{t \in N}[c_t, p_t]$. Consider the function $f : Q \times W \to \mathbb{R}$ defined by the equation $f(q, w) \overset{\text{def}}{=} \sum_{i=1}^n p_i E[S_i(q_i)] - w_i q_i$ and the correspondence $\mathcal{D} : W \to 2^Q$ defined by the equation $\mathcal{D}(w) = Q$. For any good $t \in N$, the expected sales $E[S_t(q_t)]$, when the retailer orders q_t units, equals $q_t \cdot \bar{F}_t(q_t) + \int_0^{q_t} x \cdot f_t(x)\, dx = \int_0^{q_t} \bar{F}_t(x)\, dx$ (by using integration by parts). Since $\bar{F}_t(x)$ is continuous on Q, we have that $E[S_t(q_t)] = \int_0^{q_t} \bar{F}_t(x)\, dx$ is continuous on Q (Rudin 1976, Theorem 6.20), so that the function f is continuous on $Q \times W$ (since f involves finite sums and products of continuous functions). Furthermore, the correspondence $\mathcal{D}$ is compact-valued and continuous, because for any wholesale price vector $w \in W$ the equation $\mathcal{D}(w) = Q$ holds. Therefore, from Proposition 6.1, we have that the correspondence $\mathcal{D}^*$ (as defined in Equation (6.63)) is compact-valued and upper-hemicontinous on W. However, every order quantity vector in the set $\mathcal{D}^*(w)$ is a solution to *RETAILER-PRIMAL(k,w)* and in the proof of Theorem 6.1, we showed that *RETAILER-PRIMAL(k,w)* has a unique solution, $q_t^r(w)$. Therefore, $\mathcal{D}^*(w)$ is single-valued (for any $w \in W$) and equals $q_t^r(w)$. Since $\mathcal{D}^*$ is single-valued and upper-hemicontinous on W, it must, therefore, be continuous on W, implying that the function q_t^r is continuous on W. Furthermore, since supplier Y's profit $\pi_Y(w)$ is a finite sum of products of continuous functions on W, the function $\pi_Y(w)$ is also continuous on W. □

Proof of Theorem 6.8. The (Weierstrass) Extreme Value Theorem says that for any continuous and real function f on a compact metric space X, there exists a point $x^* \in X$ such $f(x^*) = \sup_{x \in X} f(x)$. (Rudin 1976, Theorem 4.16) Since, the hypercube $\prod_{t\in Y}[c_t, p_t]$ is closed and bounded (and hence compact) as well as a metric space (under the Euclidean metric), and since the supplier's objective function is (real) continuous in its decision vector w_Y (from Lemma 6.1), we, therefore, have (by applying the Extreme Value Theorem) that supplier Y can attain the supremum of its objective function (over its constraint set) from a vector in its constraint set, i.e., the hypercube $\prod_{t\in Y}[c_t, p_t]$, implying that the solution set $\mathcal{W}_Y^{\text{br}}(w_{\overline{Y}})$ is non-empty.

From the proof of Theorem 6.7, we have that when supplier Y solves *Y-SUPPLIER-INDUCING-AGGREGATE-ORDER($w_{\overline{Y}}$)*, there exists an optimal (and unique) aggregate order quantity $x^Y(w_{\overline{Y}})$ and an optimal (and unique) induced order vector $q^Y(w_{\overline{Y}})$. Therefore, from equation (6.14), we have that the set $\mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$ is non-empty. Furthermore, for every good $j \in Y \setminus Z^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$, we must have

$$p_j\bar{F}_j(q_j^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})) - \lambda^r(x^Y(w_{\overline{Y}}); w_{\overline{Y}}) \geq c_j, \tag{6.64}$$

otherwise, the quantity $x^Y(w_{\overline{Y}})$ and the vector $q^Y(w_{\overline{Y}})$ would not be a solution for *Y-SUPPLIER-INDUCING-AGGREGATE-ORDER($w_{\overline{Y}}$)* (because supplier Y could increase the value of the objective function by choosing the induced order for good j to be zero, if equation (6.64) did not hold for good j). Therefore, for every wholesale price vector $w' \in \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$ and for any good $j \in Y \setminus Z^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$, we have $c_j \leq w_j' \leq p_j$ from equation (6.14). Also, from equation (6.14), we have that there always exists a wholesale price vector $w' \in \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$ such that for every good $i \in Z^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$, we have $w_i' = p_i$. Therefore, we have that the set $\mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}}) \cap \prod_{t\in Y}[c_t, p_t]$ is non-empty.

Next, we show that $\mathcal{W}_Y^{\text{br}}(w_{\overline{Y}}) \subseteq \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}}) \bigcap \prod_{t\in Y}[c_t, p_t]$. Consider any wholesale price vector $w_Y \in \mathcal{W}_Y^{\text{br}}(w_{\overline{Y}})$ for goods Y. From the constraints of *Y-SUPPLIER($w_{\overline{Y}}$)*, we know that $w_Y \in \prod_{t\in Y}[c_t, p_t]$. Assume that $w_Y \notin \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$. From Theorem 6.1, we have that the objective function for *Y-SUPPLIER($w_{\overline{Y}}$)* satisfies

$$\sum_{i\in Y}(w_i - c_i)\cdot q_i^r(w) = \sum_{i\in Y}\left(p_i\bar{F}_i\left(q_i^r(w)\right) - \lambda^r(w) - c_i\right)\cdot q_i^r(w). \tag{6.65}$$

And from Theorem 6.5 and equation (6.14), we have that for any induced order quantity vector q'_Y that maximizes the objective function of *Y-SUPPLIER-INDUCING-AGGREGATE-ORDER($w_{\overline{Y}}$)* subject to its constraints, there exists a wholesale price vector $w'_Y \in \mathcal{W}^Y(\sum_{i\in Y} q'_i; w_{\overline{Y}}) \cap \prod_{t\in Y}[c_t, p_t]$ that satisfies

$$\sum_{i\in Y}\left(p_i\bar{F}_i(q'_i) - \lambda^r\left(\sum_{t\in Y} q'_t; w_{\overline{Y}}\right) - c_i\right)\cdot q'_i = \sum_{i\in Y}(w'_i - c_i)\cdot q^r_i(w'_Y, w_Y). \quad (6.66)$$

From our assumption that $w_Y \notin \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$, we have $\sum_{i\in Y}(w_i - c_i)\cdot q^r_i(w) < \sum_{i\in Y}(w'_i - c_i)\cdot q^r_i(w'_Y, w_Y)$ so that $w_Y \notin \mathcal{W}^{\text{br}}_Y(w_{\overline{Y}})$. But this is a contradiction. Thus, we have $w_Y \in \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}}) \cap \prod_{t\in Y}[c_t, p_t]$.

Next, we show that $\mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}}) \bigcap \prod_{t\in Y}[c_t, p_t] \subseteq \mathcal{W}^{\text{br}}_Y(w_{\overline{Y}})$. Consider any wholesale price vector $w'_Y \in \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}}) \bigcap \prod_{t\in Y}[c_t, p_t]$ for goods Y. Then, there exists a vector q'_Y of order quantities such that for any good $i \in Y$, $q'_i = q^r_i(w'_Y, w_Y)$ and equation (6.66) holds. Assume that $w'_Y \notin \mathcal{W}^{\text{br}}_Y(w_{\overline{Y}})$. Then, there exists a $w_Y \in \prod_{t\in Y}[c_t, p_t]$ such that $\sum_{i\in Y}(w_i - c_i)\cdot q^r_i(w) > \sum_{i\in Y}(w'_i - c_i)\cdot q^r_i(w'_Y, w_Y)$. But, since $\lambda^r(w) = \lambda^r\left(\sum_{t\in Y} q^r_i(w); w_{\overline{Y}}\right)$ (due to Theorem 6.3), from equation (6.65) we have

$$\sum_{i\in Y}(w_i - c_i)\cdot q^r_i(w) = \sum_{i\in Y}\left(p_i\bar{F}_i\left(q^r_i(w)\right) - \lambda^r\left(\sum_{t\in Y} q^r_i(w); w_{\overline{Y}}\right) - c_i\right)\cdot q^r_i(w). \quad (6.67)$$

Therefore, from equation (6.66), we have

$$\sum_{i\in Y}\left(p_i\bar{F}_i\left(q^r_i(w)\right) - \lambda^r\left(\sum_{t\in Y} q^r_i(w); w_{\overline{Y}}\right) - c_i\right)\cdot q^r_i(w) > \sum_{i\in Y}\left(p_i\bar{F}_i(q'_i) - \lambda^r\left(\sum_{t\in Y} q'_t; w_{\overline{Y}}\right) - c_i\right)\cdot q'_i. \quad (6.68)$$

But, this is a contradiction because $w'_Y \in \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}}) \bigcap \prod_{t\in Y}[c_t, p_t]$ and the vector $q^r_Y(w)$ is in the feasible set of *Y-SUPPLIER-INDUCING-AGGREGATE-ORDER($w_{\overline{Y}}$)*. Thus, we have $w'_Y \in \mathcal{W}^{\text{br}}_Y(w_{\overline{Y}})$.

Finally, we show that the set $\mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}}) \bigcap \prod_{t\in Y}[c_t, p_t]$ (as defined in equation (6.14)) is convex. Consider any two wholesale price vectors $a, b \in \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$ and any real number $\eta \in [0, 1]$. For every good $j \in Y \setminus Z^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$, we have

$a_j = b_j$ (from equation (6.14)). Furthermore, for every good $i \in Z^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$, we have $\eta \cdot a_i + (1-\eta) \cdot b_i > \min\{a_i, b_i\}$. Therefore, $q_Y^r(\eta \cdot a_Y + (1-\eta) \cdot b_Y, w_{\overline{Y}}) = q_Y^r(a_Y, w_{\overline{Y}})$, implying that the wholesale price vector $\eta \cdot a_i + (1-\eta) \cdot b_i \in \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$. Therefore, the set $\mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$ is convex, and $\mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}}) \bigcap \prod_{t \in Y}[c_t, p_t]$ is also convex since it is the intersection of two convex sets. □

6.6.14 Proof: Existence of an equilibrium when s suppliers compete.

We apply Kakutani's Fixed Point Theorem (Proposition 6.2). Therefore, we state it here for completeness.[10]

PROPOSITION 6.2 **(Kakutani's Fixed Point Theorem).** *Consider a subset $W \subseteq \mathbb{R}^n$ that is non-empty, convex, and compact and a correspondence $\mathcal{B} : W \to 2^W$ that is upper hemicontinuous with the additional property that $\mathcal{B}(w)$ is non-empty and convex for every $w \in W$. Then, there exists a fixed point $w \in W$ for $\mathcal{B}$, i.e., there exists a point $w \in W$ such that $w \in \mathcal{B}(w)$.*

Proof of Theorem 6.9. Suppose there are $s \geq 2$ suppliers identified by the subsets of goods they offer: $Y_1, \ldots, Y_s$. Denote the set of feasible supplier wholesale price vectors by $W \stackrel{\text{def}}{=} \prod_{t \in N}[c_t, p_t]$. Consider the supplier best response correspondence $\mathcal{W}^{\text{br}} : W \to 2^W$ defined as the s–ary Cartesian product over the s supplier best response mappings. Namely, for any wholesale price vector $w \in W$, $\mathcal{W}^{\text{br}}(w) \stackrel{\text{def}}{=} (\mathcal{W}^{\text{br}}_{Y_1}(w_{\overline{Y}_1}), \ldots, \mathcal{W}^{\text{br}}_{Y_s}(w_{\overline{Y}_s})) = \{(w_{Y_1}, \ldots, w_{Y_s}) \mid w_{Y_1} \in \mathcal{W}^{\text{br}}_{Y_1}(w_{\overline{Y}_1}), \ldots, w_{Y_s} \in \mathcal{W}^{\text{br}}_{Y_s}(w_{\overline{Y}_s})\}$. In order to show that an equilibrium exists, we will apply Kakutani's fixed point theorem to the supplier best response correspondence $\mathcal{W}^{\text{br}} : W \to 2^W$. First, observe that the set $W \subset \mathbb{R}^n$ is non-empty, compact, and convex. Furthermore, from Theorem 6.8, we have that for any supplier $Y \subseteq N$ that faces competing wholesale prices $w_{\overline{Y}}$, the best response mapping $\mathcal{W}^{\text{br}}_Y(w_{\overline{Y}})$ is non-empty and convex[11], implying that

[10] See Border (1989) (Chapter 15) for the proof of Kakutani's Fixed Point Theorem and Cachon and Netessine (2004) for an overview of existence theorems applied in the supply chain literature.

[11] Convexity follows from our assumption that the retailer is service constrained for good(s) $\overline{Y}$, and the cumulative distribution function for demand of each good $y \in Y$ has the IGFR property.

$\mathcal{W}^{\text{br}}(w)$ is non-empty and convex.

Finally, we show that the correspondence $\mathcal{W}^{\text{br}}$ is upper hemicontinuous. Assume $\mathcal{W}^{\text{br}}$ is not upper hemicontinuous. Since W is compact, we have that the correspondence $\mathcal{W}^{\text{br}}$ is not closed.[12] Therefore, there exists wholesale price vectors $\bar{w}, \bar{z} \in W$ such that the sequence $\{w^l\}$ of wholesale price vectors converges to $\bar{w}$ and the sequence $\{z^l\}$ of wholesale price vectors satisfies $z^l \in \mathcal{W}^{\text{br}}(w^l)$ and converges to $\bar{z}$, yet $\bar{z} \notin \mathcal{W}^{\text{br}}(\bar{w})$. Therefore, if we denote supplier Y's profit by $\pi_Y(w_Y, w_{\overline{Y}}) \overset{\text{def}}{=} \sum_{i \in Y}(w_i - c_i)q_i^r(w_Y, w_{\overline{Y}})$, there exists some supplier Y_a ($a \in \{1, \ldots, s\}$) and some wholesale price vector $\hat{w}_{Y_a} \in \prod_{t \in Y_a}[c_t, p_t]$ such that $\pi_{Y_a}(\bar{z}_{Y_a}, \bar{w}_{\overline{Y}_a}) < \pi_{Y_a}(\hat{w}_{Y_a}, \bar{w}_{\overline{Y}_a})$. Therefore, there exists an $\epsilon > 0$ such that

$$\pi_{Y_a}(\bar{z}_{Y_a}, \bar{w}_{\overline{Y}_a}) + \epsilon < \pi_{Y_a}(\hat{w}_{Y_a}, \bar{w}_{\overline{Y}_a}) - \epsilon. \tag{6.69}$$

Supplier Y_a's objective function $\pi_{Y_a}(w)$ is continuous on W (see Lemma 6.1), therefore there exists an integer m such that for $l > m$, we have $|\pi_{Y_a}(\bar{z}_{Y_a}, \bar{w}_{\overline{Y}_a}) - \pi_{Y_a}(z^l_{Y_a}, w^l_{\overline{Y}_a})| < \epsilon$. So that the inequality $\pi_{Y_a}(z^l_{Y_a}, w^l_{\overline{Y}_a}) < \pi_{Y_a}(\bar{z}_{Y_a}, \bar{w}_{\overline{Y}_a}) + \epsilon$ holds. Therefore, from Equation (6.69), we have

$$\pi_{Y_a}(z^l_{Y_a}, w^l_{\overline{Y}_a}) < \pi_{Y_a}(\hat{w}_{Y_a}, \bar{w}_{\overline{Y}_a}) - \epsilon. \tag{6.70}$$

Because of the continuity of supplier Y_a's objective function, there also exists an integer o such that for $l > o$, we have $|\pi_{Y_a}(\hat{w}_{Y_a}, \bar{w}_{\overline{Y}_a}) - \pi_{Y_a}(\hat{w}_{Y_a}, w^l_{\overline{Y}_a})| < \epsilon$. So that we have $\pi_{Y_a}(\hat{w}_{Y_a}, \bar{w}_{\overline{Y}_a}) - \epsilon < \pi_{Y_a}(\hat{w}_{Y_a}, w^l_{\overline{Y}_a})$. Therefore, from Equation (6.70), for $l > \max\{m, o\}$, we have

$$\pi_{Y_a}(z^l_{Y_a}, w^l_{\overline{Y}_a}) < \pi_{Y_a}(\hat{w}_{Y_a}, w^l_{\overline{Y}_a}). \tag{6.71}$$

But, this is a contradiction because $z^l \in \mathcal{W}^{\text{br}}(w^l)$. Therefore, the correspondence $\mathcal{W}^{\text{br}}$ is upper hemicontinuous. And, therefore, by applying Kakutani's fixed point

[12] See Border (1989) for the following result: Consider sets $D \subset \mathbb{R}^l$, $R \subset \mathbb{R}^m$ and the correspondence $C : D \to 2^R$. If R is compact and C is closed, then C is upper hemicontinuous.

theorem to the supplier best response correspondence $\mathcal{W}^{\text{br}}$, we have that there exists a vector $w^{\text{eq}} \in W$ of wholesale prices for all n goods, such that $w^{\text{eq}} \in \mathcal{W}^{\text{br}}(w^{\text{eq}})$. □

6.6.15 Proof: Unique equilibrium shadow price for capacity when s suppliers compete.

Proof of Theorem 6.10. Suppose there is one supplier (i.e., $s = 1$) denoted by the set N of goods offered. The (Weierstrass) Extreme Value Theorem says that for any continuous and real function f on a compact metric space X, there exists a point $x^* \in X$ such $f(x^*) = \sup_{x \in X} f(x)$. (Rudin 1976, Theorem 4.16) Since, the hypercube $\prod_{t \in N}[c_t, p_t]$ is closed and bounded (and hence compact) as well as a metric space (under the Euclidean metric), and since supplier N's objective function $\pi_N(w) \stackrel{\text{def}}{=} \sum_{i \in N}(w_i - c_i)q_i^r(w)$ is (real) continuous in its decision vector w (from Lemma 6.1), we, therefore, have (by applying the Extreme Value Theorem) that supplier N can attain the supremum of its objective function (over its constraint set) from a vector in its constraint set, i.e., the hypercube $\prod_{t \in N}[c_t, p_t]$, implying that a solution exists.

Next, we show that when there is one supplier (i.e., $s = 1$) *every* solution w to the supplier's decision problem in the first stage induces the retailer to have a shadow price $\lambda^r(w) = 0$. Assume that some solution vector w' induces a positive retailer shadow price $\lambda^r(w') > 0$ instead. From Theorem 6.5, we have that there exists another wholesale price vector $\widehat{w}$ such that the retailer orders the same amount as under w' (i.e., $q^r(\widehat{w}) = q^r(w')$), but the shadow price $\lambda^r(\widehat{w}) = 0$. Therefore, from Theorem 6.1 we have that $w_i' < \widehat{w}_i$ for every good i that the retailer orders so that $\pi_N(w') < \pi_N(\widehat{w})$. But this is a contradiction because w' is a solution vector for the supplier's decision problem in the first stage. Thus, it follows that every solution w to the supplier's decision problem in the first stage induces the retailer to have a shadow price $\lambda^r(w) = 0$.

Suppose there is more than one supplier (i.e., $s \geq 2$). We denote supplier i by the subset Y_i of goods offered. Furthermore, suppose that the retailer is service constrained for goods N, the demand for each good $t \in N$ has the IGFR property, and Assumption 6.1 holds. We show that *every* equilibrium wholesale price vector w induces the retailer to have the same shadow price which we denote by λ^{eq} (i.e., $\lambda^r(w) = \lambda^{\text{eq}}$). Assume that instead we have two distinct equilibrium wholesale price vectors, w' and $\widehat{w}$, but that they induce different shadow prices for the retailer's capacity (i.e., $\lambda^r(w') \neq \lambda^r(\widehat{w})$). Without loss of generality, suppose

$$0 \leq \lambda^r(w') < \lambda^r(\widehat{w}). \tag{6.72}$$

Recall from Section 6.4.3 that the set

$$L(q^r(\widehat{w}), q^r(w')) \stackrel{\text{def}}{=} \{l \in \{1, \ldots, s\} \mid \sum_{i \in Y_l} q_i^r(\widehat{w}) > \sum_{i \in Y_l} q_i^r(w')\}$$

denotes the suppliers that have a larger share of the retailer's capacity under wholesale price vector $\widehat{w}$ when compared to the allocation under wholesale price vector w'. If the two distinct wholesale price vectors induce the retailer to make the same allocation, i.e., $q^r(\widehat{w}) = q^r(w')$, then the set $L(q^r(\widehat{w}), q^r(w'))$ is empty, otherwise the set $L(q^r(\widehat{w}), q^r(w'))$ must be nonempty because the equation

$$\sum_{i \in N} q_i^r(w') \leq \sum_{i \in N} q_i^r(\widehat{w}) = k \tag{6.73}$$

holds (which follows from equation (6.72) and Theorem 6.1).

Consider the case when the set $L(q^r(\widehat{w}), q^r(w'))$ is nonempty. For the purposes of this proof only, we define $\bar{x} \stackrel{\text{def}}{=} \min\left\{\sum_{t \in Y} q_t^e, k\right\}$. For every supplier $l \in L(q^r(\widehat{w}), q^r(w'))$, from Theorem 6.7, we have that

$$\sum_{i \in Y_l} q_i^r(\widehat{w}) = \sup\left\{x \in [0, \bar{x}] \mid \lambda^{Y_l}(x; \widehat{w}_{\overline{Y_l}}) - x \cdot \frac{\partial \lambda^r(x; \widehat{w}_{\overline{Y_l}})}{\partial x^-} \geq 0\right\}. \tag{6.74}$$

From Theorem 6.6 (which implies that the function $\lambda^{Y_l}(x; w'_{\overline{Y_l}})$ is strictly decreasing as $x \in \left(0, \min\left\{\sum_{t \in Y} q_t^e, k\right\}\right]$ increases) and Theorem 6.7, we have that for every supplier $l \in L(q^r(\widehat{w}), q^r(w'))$ the equation

$$\sum_{i \in Y_l} q_i^r(w') = \inf \left\{ x \in [0, \bar{x}] \mid \lambda^{Y_l}(x; w'_{\overline{Y_l}}) - x \cdot \frac{\partial \lambda^r(x; w'_{\overline{Y_l}})}{\partial x^+} \le 0 \right\} \tag{6.75}$$

holds. From Assumption 6.1 we have

$$\sum_{l \in L(q^r(\widehat{w}), q^r(w'))} \left(\sum_{i \in Y_l} q_i^r(\widehat{w}) \right) \cdot \left. \frac{\partial \lambda^r(x; \widehat{w}_{\overline{Y_l}})}{\partial x^-} \right|_{x = \sum_{i \in Y_l} q_i^r(\widehat{w})} \ge \tag{6.76}$$

$$\sum_{l \in L(q^r(\widehat{w}), q^r(w'))} \left(\sum_{i \in Y_l} q_i^r(w') \right) \cdot \left. \frac{\partial \lambda^r(x; w'_{\overline{Y_l}})}{\partial x^+} \right|_{x = \sum_{i \in Y_l} q_i^r(w')} .$$

Therefore, there exists a supplier $\hat{l} \in L(q^r(\widehat{w}), q^r(w'))$ such that the equation

$$\left(\sum_{i \in Y_{\hat{l}}} q_i^r(\widehat{w}) \right) \cdot \left. \frac{\partial \lambda^r(x; \widehat{w}_{\overline{Y_{\hat{l}}}})}{\partial x^-} \right|_{x = \sum_{i \in Y_{\hat{l}}} q_i^r(\widehat{w})} \ge \left(\sum_{i \in Y_{\hat{l}}} q_i^r(w') \right) \cdot \left. \frac{\partial \lambda^r(x; w'_{\overline{Y_{\hat{l}}}})}{\partial x^+} \right|_{x = \sum_{i \in Y_{\hat{l}}} q_i^r(w')} \tag{6.77}$$

holds. And so we have

$$0 \le \lambda^{Y_{\hat{l}}} \left(\sum_{i \in Y_{\hat{l}}} q_i^r(\widehat{w}); \widehat{w}_{\overline{Y_{\hat{l}}}} \right) - \left(\sum_{i \in Y_{\hat{l}}} q_i^r(\widehat{w}) \right) \cdot \left. \frac{\partial \lambda^r(x; \widehat{w}_{\overline{Y_{\hat{l}}}})}{\partial x^-} \right|_{x = \sum_{i \in Y_{\hat{l}}} q_i^r(\widehat{w})} \tag{6.78}$$

$$< \lambda^{Y_{\hat{l}}} \left(\sum_{i \in Y_{\hat{l}}} q_i^r(w'); w'_{\overline{Y_{\hat{l}}}} \right) - \left(\sum_{i \in Y_{\hat{l}}} q_i^r(w') \right) \cdot \left. \frac{\partial \lambda^r(x; w'_{\overline{Y_{\hat{l}}}})}{\partial x^+} \right|_{x = \sum_{i \in Y_{\hat{l}}} q_i^r(w')} \tag{6.79}$$

$$\le 0. \tag{6.80}$$

(Equation (6.78) follows from equation (6.74). Applying Theorem 6.6 and noting the equations (6.72) and (6.73), we have $\lambda^{Y_{\hat{l}}} \left(\sum_{i \in Y_{\hat{l}}} q_i^r(\widehat{w}); \widehat{w}_{\overline{Y_{\hat{l}}}} \right) < \lambda^{Y_{\hat{l}}} \left(\sum_{i \in Y_{\hat{l}}} q_i^r(w'); w'_{\overline{Y_{\hat{l}}}} \right)$ holds. Therefore, from equation (6.77) we have equation (6.79). And, equation (6.80) follows from equation (6.75).) Notice that equation (6.80) leads to a contradiction, $0 < 0$. Thus, it follows that *every* equilibrium wholesale price vector w induces the retailer to have the same shadow price which we denote by λ^{eq} (i.e., $\lambda^r(w) = \lambda^{eq}$). □

6.6.16 Proof: Unique capacity allocation when s suppliers compete.

Proof of Theorem 6.11. Consider the case when the retailer orders from at least two suppliers when facing wholesale price vector w'. Assume that the equation

$$\lambda^r(\widehat{w}) > \lambda^r(w') + \min\left\{m_{Y_j}(w') \mid j \in \{1,\ldots,s\},\ Y_j \cap (N \setminus Z(w')) \neq \emptyset\right\} \tag{6.81}$$

holds, instead, for two equilibrium wholesale price vectors $\widehat{w}$ and w' that induce the same retailer order (i.e., $q^r(\widehat{w}) = q^r(w')$) but induce shadow prices for capacity satisfying $\lambda^r(w') \leq \lambda^r(\widehat{w})$. Consider any supplier Y_d such that

$$d \in \arg \min_{j\in\{1,\ldots,s\},\ Y_j\cap(N\setminus Z(w'))\neq\emptyset} m_{Y_j}(w'). \tag{6.82}$$

We show that supplier Y_d will deviate from the wholesale price vector $\widehat{w}_{Y_d}$ when the wholesale price vector for the other goods $\overline{Y_d}$ is held fixed at $\widehat{w}_{\overline{Y_d}}$. Since the two equilibrium wholesale price vectors $\widehat{w}$ and w' induce the same retailer order (i.e., $q^r(\widehat{w}) = q^r(w')$), from the proof of Corollary 6.6 (see equation (6.47)) and Theorem 6.6 we have that the equation

$$\left(\sum_{i\in Y_d} q_i^r(\widehat{w})\right) \cdot \left.\frac{\partial\lambda^r(x;\widehat{w}_{\overline{Y_d}})}{\partial x^-}\right|_{x=\sum_{i\in Y_d} q_i^r(\widehat{w})} \geq \left(\sum_{i\in Y_d} q_i^r(w')\right) \cdot \left.\frac{\partial\lambda^r(x;w'_{\overline{Y_d}})}{\partial x^-}\right|_{x=\sum_{i\in Y_d} q_i^r(w')}. \tag{6.83}$$

holds. Furthermore, since $q^r(\widehat{w}) = q^r(w')$ holds, from Theorem 6.3 and Theorem 6.6 we have that the equation

$$\lambda^{Y_d}\left(\sum_{i\in Y_d} q_i^r(\widehat{w});\widehat{w}_{\overline{Y_d}}\right) + \lambda^r(\widehat{w}) = \lambda^{Y_d}\left(\sum_{i\in Y_d} q_i^r(w');w'_{\overline{Y_d}}\right) + \lambda^r(w'). \tag{6.84}$$

holds. Therefore, we have

$$\begin{aligned}
m_{Y_d}(\widehat{w}) &\stackrel{\text{def}}{=} \lambda^{Y_d}\left(\sum_{i\in Y_d} q_i^r(\widehat{w}); \widehat{w}_{\overline{Y_d}}\right) - \left(\sum_{i\in Y_d} q_i^r(\widehat{w})\right)\cdot \left.\frac{\partial\lambda^r(x;\widehat{w}_{\overline{Y_d}})}{\partial x^-}\right|_{x=\sum_{i\in Y_d} q_i^r(\widehat{w})} \\
&\leq \lambda^{Y_d}\left(\sum_{i\in Y_d} q_i^r(w'); w'_{\overline{Y_d}}\right) + \lambda^r(w') - \lambda^r(\widehat{w}) \\
&\qquad - \left(\sum_{i\in Y_d} q_i^r(w')\right)\cdot \left.\frac{\partial\lambda^r(x;w'_{\overline{Y_d}})}{\partial x^-}\right|_{x=\sum_{i\in Y_d} q_i^r(w')} \\
&= \lambda^r(w') + m_{Y_d}(w') - \lambda^r(\widehat{w}) \\
&< 0. \qquad (6.85)
\end{aligned}$$

Equation (6.85) follows from equation (6.81). But this is a contradiction because, according to Theorem 6.7, supplier Y_d would deviate from the wholesale price vector $\widehat{w}_{Y_d}$ when the wholesale price vector for the other goods $\overline{Y_d}$ is held fixed at $\widehat{w}_{\overline{Y_d}}$. Thus, equation (6.25) follows.

Consider the case when the retailer orders from only one supplier (i.e., supplier Y) when facing wholesale price vector w'. Assume that the equation

$$\lambda^r(\widehat{w}) > \lambda^r(w') \qquad (6.86)$$

holds, instead. Supplier Y's objective function is $\pi_Y(w) \stackrel{\text{def}}{=} \sum_{i\in Y}(w_i - c_i)q_i^r(w)$. Since $\sum_{i\in Y} q_i^r(\widehat{w}) = \sum_{i\in Y} q_i^r(w') = k$, from Theorem 6.7 we have $q_i^r(\widehat{w}) = q_i^r(w')$ for every good $i \in Y$. Therefore, from Theorem 6.1 we have that $\widehat{w}_i < w'_i$ for every good i that the retailer orders so that $\pi_Y(\widehat{w}) < \pi_Y(w')$. Observe that $\pi_Y(w') = \pi_Y(w'_Y, \widehat{w}_{\overline{Y}})$ since $\lambda^r(w') = \lambda^r(w'_Y, c_{\overline{Y}})$ (for an equilibrium w') implying $\lambda^r(w') = \lambda^r(w'_Y, \widehat{w}_{\overline{Y}})$ (using Theorem 6.4). And, so we have $\pi_Y(\widehat{w}) < \pi_Y(w'_Y, \widehat{w}_{\overline{Y}})$. Therefore, supplier Y prefers wholesale price vector w'_Y over $\widehat{w}_Y$ when the other good(s) $\overline{Y}$ have fixed their wholesale price vector to be $\widehat{w}_{\overline{Y}}$. But this is a contradiction because $\widehat{w}$ is an equilibrium wholesale price vector. Thus, it follows that $\lambda^r(\widehat{w}) \leq \lambda^r(w')$.

□

6.6.17 Proof: Supplier collusion.

Proof of Theorem 6.12. Denote supplier Y's objective function by the function $\pi_Y(w) \stackrel{\text{def}}{=} \sum_{i \in N}(w_i - c_i)q_i^r(w)$. From Theorem 6.5, we have that there exists another wholesale price vector $\widehat{w}$ such that the retailer orders the same amount as under w' (i.e., $q^r(\widehat{w}) = q^r(w')$), but the shadow price $\lambda^r(\widehat{w}) = 0$. Therefore, from Theorem 6.1 we have that $w'_i < \widehat{w}_i$ for every good i that the retailer orders so that for any supplier Y we have $\pi_Y(w') < \pi_Y(\widehat{w})$. Furthermore, from Theorem 6.10 we have that for the two-stage game with one supplier any equilibrium wholesale price vector induces a shadow price of zero units. Therefore, if w^* is a solution to $(Y_1 \cup \ldots \cup Y_s)$-*SUPPLIER* (see Section 6.1.3), then we have

$$\sum_{t=1}^{s} \pi_{Y_t}(\widehat{w}) \leq \sum_{t=1}^{s} \pi_{Y_t}(w^*).$$

So we can conclude that

$$\sum_{t=1}^{s} \pi_{Y_t}(w') < \sum_{t=1}^{s} \pi_{Y_t}(w^*).$$

□

CHAPTER 7

Multiple retailers buying from a newsvendor

The supply chain described in Chapter 6 operates in *push-mode* because the supplier 'pushes' the inventory to the retailer so that the retailer takes on the inventory risk for the supply channel.[1] On the other hand, when the supplier takes on the inventory risk for the channel, and the retailer replenishes (or 'pulls' inventory) from the supplier as demand materializes (e.g., by drop-shipping), the supply chain operates in *pull-mode.*(Cachon and Lariviere 2001, Cachon 2004)

In this chapter, we consider a 'pull'-version of the game in Chapter 6 where multiple retailers pull inventory from a capacity-constrained supplier during the sales season. We conduct comparative statics and analyze the equilibria of this game. Furthermore, we analyze the impact of retailer collusion on the equilibria of the game.

Chapter Outline

In Section 7.1 we explain the supply chain setting. In Section 7.2, we analyze the supplier's capacity allocation decision and derive the (endogenous) price for the supplier's capacity. We conduct comparative statics in Section 7.3. Then, in Section 7.4, we analyze the equilibrium setting, by providing conditions for the existence of an equilibrium in Section 7.4.2 and for uniqueness in Section 7.4.4. Finally, in

[1] See Cachon (2004) for a detailed discussion on 'push' and 'pull' modes of supply chain operation.

Section 7.5, we consider retailer collusion/integration and show that the supplier's shadow price for capacity decreases and that every retailer can achieve more profit.

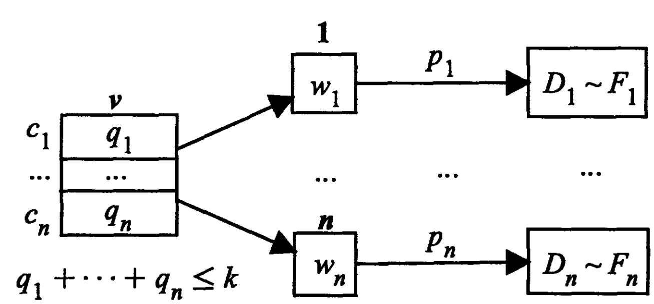
■ 7.1 Model.

As in our 'push' version of this model which we explained in Chapter 6, there are n different goods. Good i has a fixed marginal cost of c_i per unit supplied and a fixed retail price p_i. The demand for good i, D_i, is random with probability density function (p.d.f.) f_i and cumulative distribution function (c.d.f.) F_i. We assume that the distribution for demands D_i does not depend on the inventory $(q_1, q_2, \ldots, q_n)$ in the supply channel.

In contrast to our previous model, however, there is a single supplier offering all n goods with a total capacity that is constrained by some $k > 0$, so that the capacity constraint can be expressed in the form $q_1 + \cdots + q_n \leq k$. And there are s retailers, each 'pulling' from a subset of the n goods, such that no two retailers 'pull' the same good.

The models parameters are summarized in Figure 7-1, with the arrows denoting the direction of product flow.

Figure 7-1 **"n goods & 1 capacity constrained supplier" model.**



Note. A capacity-constrained supplier v offers n goods, each 'pulled' by exactly one retailer. The retailers are not depicted here. The supplier faces marginal cost c_i (per unit) for good i. The retailer for good i faces uncertain demand D_i downstream with c.d.f. F_i when the price for good i is fixed at p_i (per unit). Each retailer, for each good i the retailer sells, offers a wholesale price w_i to the capacity-constrained supplier for each unit of capacity dedicated to good i. The supplier must decide on a capacity allocation q for the goods $\{1, \ldots, n\}$.

7.1.1 Equilibrium setting

The game, again, consists of two stages. In the first stage, the retailers move simultaneously, and each retailer offers the supplier a wholesale price vector for the good(s) that the retailer would pull. In the second stage, the supplier guarantees the retailer for good $i \in N \stackrel{\text{def}}{=} \{1, \ldots, n\}$ an inventory level of q_i units (for good i) from its total capacity of k units.

7.1.2 Supplier's problem in the second stage

The supplier is faced with an offered wholesale price vector w (by the retailers) and uncertain sales $S_i(x) \stackrel{\text{def}}{=} \min\{x, D_i\}$ for product $i \in \{1, \ldots, n\}$ (when dedicating x units of capacity to good i). The supplier decides on a vector of capacity guarantees for the goods (and their respective retailers) in order to maximize expected profit $\pi_v(q) \stackrel{\text{def}}{=} E[\sum_{i=1}^n w_i S_i(q_i) - c_i q_i]$ while keeping in mind the capacity constraint k. Namely, the supplier solves the following convex program with linear constraints in the decision vector, q:

SUPPLIER-PRIMAL(k,w):

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i=1}^{n} (w_i E[S_i(q_i)] - c_i q_i) \\
\text{subject to} \quad & q_i \geq 0, \quad i = 1, \ldots, n \\
& k - \sum_{i=1}^{n} q_i \geq 0.
\end{aligned}
\tag{7.1}
$$

Because of our assumptions on the distribution of the demand D_i for each product, it can be shown that *SUPPLIER-PRIMAL(k,w)* has a unique solution (vector), which we denote by $q^v(w)$. We denote the unique solution, $\arg\max_{q \in \mathbb{R}_+^n} \pi_v(q)$, for the unconstrained supplier's problem by $q^u(w)$. Note that the unconstrained supplier's problem can be decomposed into n independent newsvendor problems, each of which decides on an order quantity for a single good. Therefore, $q_i^u(w)$ equals the optimal order quantity for a newsvendor ordering good i only, which is well known to be

$\bar{F}_i^{-1}(c_i/w_i)$ units (e.g., Cachon and Terwiesch (2006)).

The dual problem in the decision variables $\gamma_1, \gamma_2, \ldots, \gamma_n$ (the shadow prices for the nonnegativity constraints) and λ (the shadow price for the capacity constraint) is:

SUPPLIER-DUAL(k,w):

$$\text{minimize} \quad \max_{\{q \in \mathbb{R}_+^n \mid k - \sum_{t=1}^n q_t \geq 0\}} \sum_{i=1}^n (w_i E[S_i(q_i)] - c_i q_i) + \sum_{i=1}^n \gamma_i q_i + \lambda \left(k - \sum_{i=1}^n q_i \right) \tag{7.2}$$

$$\text{subject to} \quad \gamma_i \geq 0, \quad i = 1, \ldots, n$$

$$\lambda \geq 0.$$

Also, *SUPPLIER-DUAL(k,w)* has a unique solution which we denote by $(\gamma_1^v(w), \ldots, \gamma_n^v(w), \lambda^v(w))$.

7.1.3 Retailer's problem in the first stage

When the retailers offer wholesale price vector w and the supplier, in response, dedicates capacity $q^v(w)$, a retailer, 'pulling' from the set $Y \subseteq N \stackrel{\text{def}}{=} \{1, \ldots, n\}$ of goods, obtains profit $\pi_Y(w) \stackrel{\text{def}}{=} \sum_{i \in Y}(p_i - w_i) E\left[S_i\left(q_i^v(w)\right)\right]$. If there exist other good(s) $\overline{Y} \stackrel{\text{def}}{=} N \setminus Y$, then retailer Y's profit depends on the wholesale prices offered by the other retailer(s) (due to the terms $\{q_i^v(w)\}_{i \in Y}$).[2] And, therefore, retailer Y competes in a simultaneous-move *game* in the first-stage against the other retailer(s).

If there exist other good(s) $\overline{Y}$ and the corresponding wholesale price vector $w_{\overline{Y}}$ is held fixed, a retailer, pulling the good(s) Y, determines the vector of wholesale price(s) to offer for good(s) Y by solving the following program with linear constraints in the decision vector, w_Y:

[2]Retailer Y denotes the retailer that 'pulls' *only* from the set Y of goods.

Y-RETAILER($w_{\overline{Y}}$):

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in Y} (p_i - w_i) \cdot E\left[S_i\left(q_i^v(w)\right)\right] \\ \text{subject to} \quad & p_i - w_i \geq 0, \quad i \in Y, \\ & w_i - c_i \geq 0, \quad i \in Y. \end{aligned} \tag{7.3}$$

Similar to our 'push-mode' equilibrium setting, in this 'pull' setting we can characterize the solution set $\mathcal{W}_Y^{\text{br}}(w_{\overline{Y}})$ for *Y-RETAILER($w_{\overline{Y}}$)*, when the supplier is service constrained for good(s) $\overline{Y}$ and the c.d.f. for each good $y \in Y$ has the IGFR property. And, therefore, again, we denote the best response correspondence by $\mathcal{W}^{\text{br}} \stackrel{\text{def}}{=} (\mathcal{W}_{Y_1}^{\text{br}}, \ldots, \mathcal{W}_{Y_s}^{\text{br}})$, so that any (pure-strategy) equilibrium in the simultaneous-move game (and, thus, in the overall Stackelberg game) corresponds to some fixed point of the correspondence $\mathcal{W}^{\text{br}}$, i.e., a vector w^{eq} of wholesale prices for all n goods, such that $w^{\text{eq}} \in \mathcal{W}^{\text{br}}(w^{\text{eq}})$.

7.1.4 Equilibrium with an unconstrained supplier

Cachon and Lariviere (2001) and Cachon (2004) analyze this Stackelberg game, for an unconstrained channel with one supplier and one retailer. But, their equilibrium results are applicable in a setting with multiple retailers pulling multiple goods from one *unconstrained* supplier. In particular, since for any good $i \in N$, the quantity that the supplier prepares $q_i^v(w)$ equals $q_i^u(w) = \bar{F}_i^{-1}(c_i/w_i)$ when the supplier is unconstrained, we have that good i's profit, $(p_i - w_i) \cdot S_i(q_i^v(w))$, is not dependent on the wholesale price of any other good. Therefore, in the first stage, any retailer offering a wholesale price for only one good faces a 'buying from a newsvendor' problem and any retailer Y offering wholesale prices for more than one good can decompose its problem into $|Y|$ independent 'buying from the newsvendor' problems.

Applying Cachon and Lariviere (2001) and Cachon (2004) to our setting: when F_t has the IGFR property for every good $t \in N$ and the supplier is unconstrained (i.e., k is sufficiently large), the game results in a unique outcome (q^e, w^e) defined implicitly in terms of the equations

$$p_t\bar{F}_t(q_t^e) - c_t\left(1 + \left(E[S_t(q_t^e)]/\bar{F}_t(q_t^e)\right)\cdot\left(f_t(q_t^e)/\bar{F}_t(q_t^e)\right)\right) = 0, \quad t = 1, \ldots, n; \tag{7.4}$$

$$w_t^e\bar{F}_t(q_t^e) - c_t = 0, \quad t = 1, \ldots, n. \tag{7.5}$$

7.1.5 Definition: Valuation for capacity

In Section 7.2, we show that when the retailers offer wholesale price vector w in the first round of the Stackelberg game, they induce an *endogenous* valuation, $\sum_{i=1}^{n} E[S_i(q_i^v(w))] \cdot \lambda^v(w)/\bar{F}_i\,(q_i^v(w))$, for the supplier's capacity.[3] Furthermore, the shadow price $\lambda^v(w)$ in our optimization problem can also be interpreted, more traditionally, as the marginal value of the supplier's capacity. In this chapter, we are interested in understanding the valuations that are feasible in our equilibrium setting. In particular, if we denote the set of equilibrium wholesale price vectors for the Stackelberg game (when the retailer has a capacity of k units) by $\mathcal{W}^{\text{eq}}(k) \stackrel{\text{def}}{=} \{w \mid w \in \mathcal{W}^{\text{br}}(w)\}$, we are interested in determining properties of the set of *equilibrium* shadow prices for capacity, i.e., the set $\Lambda^{\text{eq}}(k) \stackrel{\text{def}}{=} \{\lambda \mid \lambda = \lambda^v(w), w \in \mathcal{W}^{\text{eq}}(k)\}$.

Again, although the analysis in Section 7.3 is specific to a newsvendor setting and a wholesale price contract, our analysis can be generalized for other equilibrium settings under different supply contracts.

■ 7.2 An endogenous valuation for the supplier's capacity k

Theorem 7.1, below, parallels Theorem 6.1 in that it implicitly defines the supplier's shadow price $\lambda^v(w)$ for capacity k and the supplier's allocation of capacity for the set N of goods in the second stage when the supplier is offered wholesale price vector w.

[3] So that any retailer that obtains $q_i^v(w)$ units of the supplier's capacity for good i, in effect, pays the supplier an extra amount $E[S_i(q_i^v(w))] \cdot \lambda^v(w)/\bar{F}_i\,(q_i^v(w))$ when compared to the amount that an unconstrained supplier would require for that same amount.

THEOREM 7.1. *Let* $Z(w) \stackrel{def}{=} \{i \mid q_i^v(w) = 0\} \subset N$ *be the set of products that are not stocked in the supplier's decision problem when offered wholesale price vector* $w = (w_1, w_2, ..., w_n)$. *For any wholesale price vector* w, *there exists some* $\lambda^v(w)$ *such that the following conditions hold:*

$$\lambda^v(w) = w_j \bar{F}_j \left(q_j^v(w)\right) - c_j, \qquad \forall\, j \in N \setminus Z(w), \tag{7.6}$$

$$\lambda^v(w) \geq w_i - c_i, \qquad \forall\, i \in Z(w). \tag{7.7}$$

Furthermore, $\lambda^v(w) = 0$ *if and only if* $\sum_{t=1}^{n} q_t^u(w) \leq k$.

Proof. See Section 7.6.1.

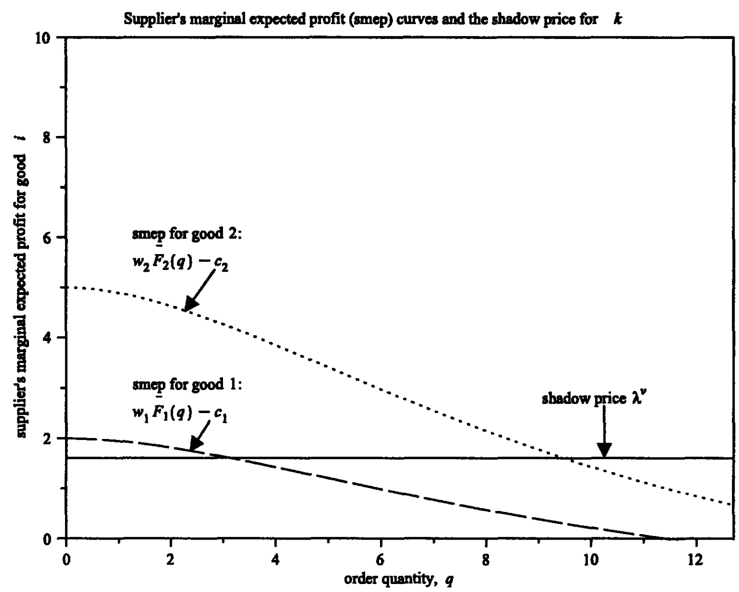
Equations (7.6) and (7.7) describe the shadow price's role as a 'threshold' of marginal profit that the supplier requires from each good. When the capacity constraint k is larger than or equal to the unconstrained supplier's total optimal order quantity, $\sum_{t=1}^{n} q_t^u(w)$, we have $\lambda^v(w) = 0$, so that equation (7.6) reduces to the 'classic' optimal order quantity result for a newsvendor: $\bar{F}_j \left(q_j^v(w)\right) = c_j/w_j$ for any ordered good j. Furthermore, equation (7.7) implies that the supplier dedicates capacity to every good, if $c < w$.

However, when the capacity constraint is binding for the supplier (i.e., $\sum_{t=1}^{n} q_t^u(w) > k$), the supplier's shadow price $\lambda^v(w)$ for the capacity constraint is strictly positive. And, therefore, equation (7.6), which can be reexpressed as $w_j = c_j/\bar{F}_j \left(q_j^v(w)\right) + \lambda^v(w)/\bar{F}_j \left(q_j^v(w)\right)$, implies that for every good j, for which the supplier dedicates capacity, the retailer *pays* the supplier $c_j/\bar{F}_j \left(q_j^v(w)\right)$ per unit of good j, a known result for unconstrained channels (Cachon and Lariviere 2001), but, in addition, the supplier *charges* the retailer $\lambda^v(w)/\bar{F}_j \left(q_j^v(w)\right)$ per unit of capacity dedicated to good j, when the retailer 'pulls' the good from the supplier. Thus, the supplier obtains an uncertain income, $\sum_{i=1}^{n} S_i(q_i^v(w)) \cdot \lambda^v(w)/\bar{F}_i \left(q_i^v(w)\right)$, from 'selling' capacity k, in addition to its usual uncertain income, $\sum_{i=1}^{n} S_i(q_i^v(w)) \cdot c_i/\bar{F}_i \left(q_i^v(w)\right)$.

In other words, the portfolio $q^v(w)$ of goods that the supplier prepares for the retailers, would have cost the supplier the extra amount $\sum_{i=1}^{n} E[S_i(q_i^v(w))] \cdot \lambda^v(w)/\bar{F}_i \left(q_i^v(w)\right)$, if the supplier was unconstrained (or k was large enough).

Figure 7-2 illustrates the supplier's 'threshold' capacity allocation rule when the capacity constraint is binding. Equation (7.8), below, uniquely determines the threshold $\lambda^v(w)$, and suggests a simple algorithm for calculating the shadow price $\lambda^v(w)$ when given a single plot displaying the supplier's marginal expected profit curve for each available good (e.g., Figure 7-2): start with initial threshold $\lambda = 0$ and increase λ until the sum of implied order quantities equals $\min\{\sum_{t=1}^n q_t^u(w), k\}$.

Figure 7-2 **The shadow price $\lambda^v(w)$ as a 'threshold rule' on the supplier's marginal expected profit (smep) curves.**



Note. Supplier's capacity is $k \approx 12.7$ units. The supplier considers two goods (dash & dot), as in Figure 6-2, when allocating capacity and faces the same demand distributions, retail prices, and wholesale prices. The cost (per unit) for each good is $c = 1$, resulting in a supplier capacity valuation $\lambda^v(w) \approx 1.6$. Compare this with the retailer's capacity valuation $\lambda^r(w) \approx 2$, in Figure 6-2, under the same wholesale prices.

COROLLARY 7.1. *For any wholesale price vector w, the supplier's shadow price $\lambda^v(w)$ satisfies*

$$\lambda^v(w) = \min\left\{\lambda \mid w_t\bar{F}_t(q_t) - c_t \le \lambda \ \ \forall t \in N, \ \sum_{t=1}^n q_t = \min\left\{\sum_{t=1}^n q_t^u(w), k\right\}, \ q \in \mathbb{R}_+^n\right\}. \tag{7.8}$$

Proof. See Section 7.6.2.

Regardless of whether or not the capacity constraint is binding, the supplier's optimal inventory level for any good can be expressed, more generally, as a function that depends on the shadow price $\lambda^v(w)$ as shown in Corollary 7.2. This result follows directly from equations (7.6) and (7.7).

COROLLARY 7.2. *For any wholesale price vector w, the supplier prepares $q_t^v(w) = \bar{F}_t^{-1}\left(\frac{\max\{\lambda^v(w)+c_t, w_t\}}{w_t}\right)$ units of good $t \in N$.*

As a result, under the 'threshold allocation rule' the ratio of the service levels (i.e., fill rates) for any two goods (that the supplier prepares) equals the corresponding ratio of the supplier's gross profit margins of those goods, not including income derived from the capacity constraint[4], as formalized in Corollary 7.3.

COROLLARY 7.3. *For good $t \in N$ and wholesale price vector w, define $u_t(w) \stackrel{def}{=} (w_t - c_t - \lambda^v(w))/w_t$ (the supplier's gross profit margin for the income derived only from the good, not the capacity). For any two goods $a, b \in N \backslash Z(w)$ that the supplier prepares, we have $F_a(q_a^v(w))/F_b(q_b^v(w)) = u_a(w)/u_b(w)$.*

■ 7.3 Comparative statics, and the game's geometry.

We show how changes in the offered wholesale prices effect the supplier's shadow price for capacity in Section 7.3.1. In Section 7.3.2, we derive a useful property that simplifies our analysis when considering retailers ordering more than one good. Then, in Section 7.3.3 we partition the set of wholesale prices into equivalence classes based on the supplier shadow price they induce or the supplier inventory vector they induce. So that, in Section 7.4.1, we can recast the retailer's problem *Y-RETAILER($w_{\overline{Y}}$)* into a (simpler) problem of choosing an aggregate quantity to induce the supplier to prepare (see Decision Problem (7.18)). Finally, in Section 7.4.4, we provide conditions for the existence and uniqueness of an equilibrium (endogenous) capacity price and conclude with a section analyzing a special case of the Stackelberg game, i.e., when the retailers collude on pricing.

[4]For each unit of good $t \in N$ that the supplier prepares when offered wholesale price vector w, the profit margin $w_t - c_t - \lambda^v(w)$ for good t does not include the uncertain income $\lambda^v(w)$ received from 'selling' a unit of capacity.

7.3.1 The supplier's shadow price for capacity when a wholesale price increases

In Theorem 7.2, we show that the shadow price for capacity is nondecreasing when one good's wholesale price increases (and provide conditions on when the shadow price is strictly increasing). In addition, we provide a simple upper bound on the increase in the shadow price.

THEOREM 7.2. *Consider two different wholesale price vectors w and w'. Suppose that w' differs from w on exactly one good $i \in N$ so that $w'_i > w_i$ and $w'_{-i} = w_{-i}$. Then, $\lambda^v(w) \leq \lambda^v(w')$. And, $\lambda^v(w) < \lambda^v(w')$ if and only if good i is included in the supplier's inventory under w' (i.e., $i \in N \setminus Z(w')$) and the capacity constraint is binding for the supplier under w' (i.e., $\sum_{t=1}^{n} q_t^u(w') > k$). Furthermore,*

$$(\lambda^v(w') + c_i) \,/\, (\lambda^v(w) + c_i) \leq w'_i/w_i. \tag{7.9}$$

And $(\lambda^v(w') + c_i) \,/\, (\lambda^v(w) + c_i) = w'_i/w_i$, if and only if, the supplier prepares k units of good i under w and w'.

Proof. See Section 7.6.3.

Therefore, when the supplier's capacity constraint is binding (so that the supplier 'charges' for capacity), a retailer that competes with other retailers on price (by increasing its wholesale price(s) offer(s)) creates two effects: the price-increasing retailer increases *every* retailer's cost λ^v in obtaining a unit of the supplier's capacity, and the price-increasing retailer increases its share of the supplier's capacity when the supplier prepares the good at the higher price (cf. Corollary 7.4).

COROLLARY 7.4. *Under the same assumptions as in Theorem 7.2, we have $q_i^v(w) \leq q_i^v(w')$ and $q_o^v(w') \leq q_o^v(w)$ for any other good $o \neq i$. Furthermore, the following two conditions are equivalent.*

(a) The supplier prepares more of good i under w', i.e., $q_i^v(w) < q_i^v(w')$, if $q_i^v(w) < k$. And the supplier prepares less of any other good $o \neq i$ under w', i.e., $q_o^v(w') < q_o^v(w)$, if $o \in N \setminus Z(w)$.

(b) *The supplier prepares good i under w', i.e., $i \in N \setminus Z(w')$, and the capacity constraint is binding, i.e., $\sum_{t=1}^{n} q_t^u(w') > k$.*

Proof. See Section 7.6.4.

7.3.2 An invariance property on the supplier's shadow price for capacity

As shown in Theorem 7.2 and Corollary 7.4, any retailer Y can induce a change in the supplier's shadow price λ^v for capacity by increasing the offered wholesale price(s) for good(s) Y, or, equivalently, taking away supplier capacity from competing goods $\overline{Y}$.[5] In particular, from Theorem 7.3, it follows that when retailer Y takes away $x < k$ units of capacity from competing retailers (when the offered wholesale prices $w_{\overline{Y}}$ for competing goods $\overline{Y}$ are fixed), retailer Y induces the supplier to have shadow price $\lambda^v(x; w_{\overline{Y}})$, as defined in equation (7.10) below, for capacity k.

THEOREM 7.3. *Consider a retailer $Y \subset N$ competing with good(s) $\overline{Y}$ for a supplier's capacity k. Suppose $w \overset{def}{=} (w_Y, w_{\overline{Y}})$ and the wholesale price vector $w_{\overline{Y}}$ is held fixed. If retailer Y's wholesale price vector w_Y induces the supplier to allocate $x < k$ units of capacity to retailer Y (i.e., $\sum_{t \in Y} q_t^v(w) = x$), then the supplier's shadow price $\lambda^v(w)$ equals*

$$\lambda^v(x; w_{\overline{Y}}) \overset{def}{=} \min \left\{ \lambda \mid w_t \bar{F}_t(q_t) - c_t \leq \lambda \ \ \forall t \in \overline{Y}, \ \sum_{t \in \overline{Y}} q_t = \min \left\{ \sum_{t \in \overline{Y}} q_t^u(w), k - x \right\}, \ q \in \mathbb{R}_+^{|\overline{Y}|} \right\}. \tag{7.10}$$

Furthermore, if $\lambda^v(w) = \lambda^v(x; w_{\overline{Y}}) > 0$, then $\sum_{t \in Y} q_t^v(w) = x$ holds.

Proof. See Section 7.6.5.

In other words, when $w_{\overline{Y}}$ is held fixed and $\sum_{t=1}^{n} q_t^v(w) = k$, the supplier's shadow price for capacity $\lambda^v(w)$ is invariant to changes in the offered wholesale price vector w_Y as long as the aggregate capacity allocation, $\sum_{t \in Y} q_t^v(w)$, remains the same. Fur-

[5]Sometimes, in order to affect a supplier's shadow price for capacity, a retailer Y may be required to increase the wholesale price(s) for good(s) Y beyond the retail price(s), but of course that would not occur in our formulation.

thermore, the supplier's shadow price $\lambda^v(x; w_{\overline{Y}})$ is a nondecreasing function of the aggregate stocking quantity x as formalized in Corollary 7.5.

COROLLARY 7.5. *Under the same assumptions as in Theorem 7.3, $\lambda^v(x; w_{\overline{Y}})$ is continuous. When x satisfies $0 \leq x \leq k - \sum_{t \in \overline{Y}} q_t^u(w)$, we have $\lambda^v(x; w_{\overline{Y}}) = 0$, and, when x satisfies $\max\{k - \sum_{t \in \overline{Y}} q_t^u(w), 0\} \leq x < k$, we have $\lambda^v(x; w_{\overline{Y}})$ is strictly increasing.*

Proof. See Section 7.6.6.

Therefore, when $\lambda^v(x; w_{\overline{Y}})$ is strictly positive and $x \in [0, k)$, the function $\lambda^v(x; w_{\overline{Y}})$ is strictly increasing. Furthermore, the average capacity cost that a single-good retailer (good i) incurs when inducing the supplier to allocate x units for good i equals $\lambda^v(x; w_{\overline{Y}}) \cdot S_i(x)/(x\bar{F}_i(x))$. From Corollary 7.5 and Theorem 3.1, we have that this average capacity cost is increasing in the induced aggregate order x when the c.d.f. F_i has the IGFR property. And, from Corollary 7.6, below, we have that the marginal capacity shadow price (i.e., $\frac{\partial \lambda^v(x; w_{\overline{Y}})}{\partial x^+}$) is also increasing (in the induced aggregate order x) when the retailer is service constrained for good(s) $\overline{Y}$.

COROLLARY 7.6. *Under the same assumptions as in Theorem 7.3, $\lambda^v(x; w_{\overline{Y}})$ is differentiable (i.e., $\frac{\partial \lambda^v(x; w_{\overline{Y}})}{\partial x^+} = \frac{\partial \lambda^v(x; w_{\overline{Y}})}{\partial x^-}$) and $\frac{\partial \lambda^v(x; w_{\overline{Y}})}{\partial x}$ is continuous at $x > 0$ when $\lambda^v(x; w_{\overline{Y}}) > 0$ and $\lambda^v(x; w_{\overline{Y}}) \neq w_i - c_i$ for any $i \in \overline{Y}$. If x satisfies the equation $\max\{k - \sum_{t \in \overline{Y}} q_t^u(w), 0\} \leq x < k$ and the retailer is service constrained for good(s) $\overline{Y}$, then $\frac{\partial \lambda^v(x; w_{\overline{Y}})}{\partial x^+}$ and $\frac{\partial \lambda^v(x; w_{\overline{Y}})}{\partial x^-}$ are strictly increasing.*

Proof. See Section 7.6.7.

7.3.3 Set of wholesale prices for a particular capacity price λ or capacity allocation q

What are the set of wholesale prices that retailer Y can offer the supplier in order to make the supplier shadow price $\lambda^v(x; w_{\overline{Y}})$ realizable? (Afterall, retailer Y chooses wholesale prices in the first stage, not order quantities.) Theorem 7.4, below, provides the set $\mathcal{W}(\lambda)$ of wholesale prices for good(s) N that induce the supplier to have shadow price λ. Therefore, from Theorem 7.4, we have that $\mathcal{W}(\lambda^v(x;w_{\overline{Y}})) \cap \{w' \mid w'_{\overline{Y}} = w_{\overline{Y}}\}$ is the set of all wholesale price vectors for good(s) Y that induce the supplier to have shadow price, $\lambda^v(x;w_{\overline{Y}})$, for allocated supplier capacity x, when the wholesale price(s) offer(s) for competing good(s) $\overline{Y}$ is held fixed at $w_{\overline{Y}}$.

THEOREM 7.4. *When $\lambda \geq 0$, any wholesale price vector in the set*

$$\mathcal{W}(\lambda) \stackrel{def}{=} \left\{ w \;\middle|\; \begin{array}{l} w_t = (c_t + \lambda - \gamma_t \cdot 1_{\{q_t=0\}})/\bar{F}_t(q_t) \quad \forall t \in N,\ q,\gamma \in \mathbb{R}_+^{|N|}, \\ \sum_{t\in N} q_t = \min\left\{\left(\sum_{t\in N} q_t^u(w)\right) \cdot 1_{\{\lambda=0\}} + k \cdot 1_{\{\lambda>0\}},\ k\right\} \end{array} \right\} \tag{7.11}$$

induces the supplier to have shadow price λ for the capacity constraint k. Furthermore, if a wholesale price vector w induces supplier shadow price λ for capacity k, then $w \in \mathcal{W}(\lambda)$.

Proof. See Section 7.6.8.

When $\lambda^v > 0$, Theorem 7.1 implies that $\min\left\{\sum_{t\in N} q_t^u(w), k\right\} = k$, and, thus, Theorem 7.4 suggests that the set $\mathcal{W}(\lambda^v)$ can be indexed by the simplex $\{q \mid \sum_{t\in N} q_t = k,\ q \in \mathbb{R}_+^{|N|}\}$ of stocking quantities. Furthermore, when $\lambda^v(x;w_{\overline{Y}}) > 0$, the set $\mathcal{W}(\lambda^v(x;w_{\overline{Y}})) \cap \{w' \mid w'_{\overline{Y}} = w_{\overline{Y}}\}$ can be indexed by the (lower dimensional) simplex $\{q \mid \sum_{t\in Y} q_t = x,\ q \in \mathbb{R}_+^{|Y|}\}$ of stocking quantities. Also, when $\lambda^v(x;w_{\overline{Y}}) > 0$, we have that $\lambda^v(x;w_{\overline{Y}})$ is invertible (from Corollary 7.5), so that for every $w \in \mathcal{W}(\lambda^v(x;w_{\overline{Y}})) \cap \{w' \mid w'_{\overline{Y}} = w_{\overline{Y}}\}$, we have $\sum_{t\in Y} q_t^v(w) = x$ (from Theorem 7.3).

Only wholesale prices in the set $\mathcal{W}(\lambda^v(x;w_{\overline{Y}})) \cap \{w' \mid w'_{\overline{Y}} = w_{\overline{Y}}\}$ induce the supplier to allocate x units of capacity for retailer Y's goods (in aggregate), when the offered wholesale price(s) for competing good(s) $\overline{Y}$ is held fixed at $w_{\overline{Y}}$. This set may be large, but Section 7.4.1 shows that there is a unique division of induced (aggregate) order x among retailer Y's goods that is optimal for retailer Y when the demand for every good $t \in Y$ has the IGFR property, so that the subset of wholesale price vectors of interest to retailer Y is much smaller. In particular, Theorem 7.1 and Theorem 7.3 imply that the optimal wholesale price vectors (for retailer Y) from the set $\mathcal{W}(\lambda^v(x;w_{\overline{Y}})) \cap \{w' \mid w'_{\overline{Y}} = w_{\overline{Y}}\}$ are identical in every good (component) $j \in Y$ included in the newsvendor's inventory. Therefore, if for the unique division of induced stocking quantity x we have $x_t > 0$ for every good $t \in Y$, then there is a unique maximizing wholesale price vector in the set $\mathcal{W}(\lambda^v(x; w_{\overline{Y}})) \cap \{w' \mid w'_{\overline{Y}} = w_{\overline{Y}}\}$ for retailer Y.

The set of wholesale price vectors $\{w' \mid \min\{\sum_{t\in N} q_t^u(w), k\} = k\}$ can, also, be partitioned according to the supplier's allocation (vector) q of capacity k (where $\sum_{t\in N} q_t = k$), as shown in Theorem 7.5.

THEOREM 7.5. *Suppose* $q \in \mathbb{R}_+^{|N|}$ *and* $\sum_{t\in N} q_t = k$. *Any wholesale price vector in the set*

$$\mathcal{A}(q) \stackrel{\text{def}}{=} \left\{ w \mid w_t = (c_t + \lambda - \gamma_t \cdot 1_{\{q_t=0\}})/\bar{F}_t(q_t) \;\; \forall t \in N, \;\; \lambda \in \mathbb{R}_+, \;\; \gamma \in \mathbb{R}_+^{|N|} \right\} \tag{7.12}$$

induces the supplier to allocate/stock the vector q. *Furthermore, if a wholesale price vector* w *induces the supplier to stock the vector* q, *then* $w \in \mathcal{A}(q)$.

Proof. See Section 7.6.9.

Figure 7-3 illustrates Theorem 7.4 and Theorem 7.5 for the example depicted in Figure 7-2. Notice, in Figure 7-3, that if the retailers offer wholesale prices farther along the ray of asterisks, their allocation (at the supplier) stays the same, but they end up being charged more for their allocated capacity.

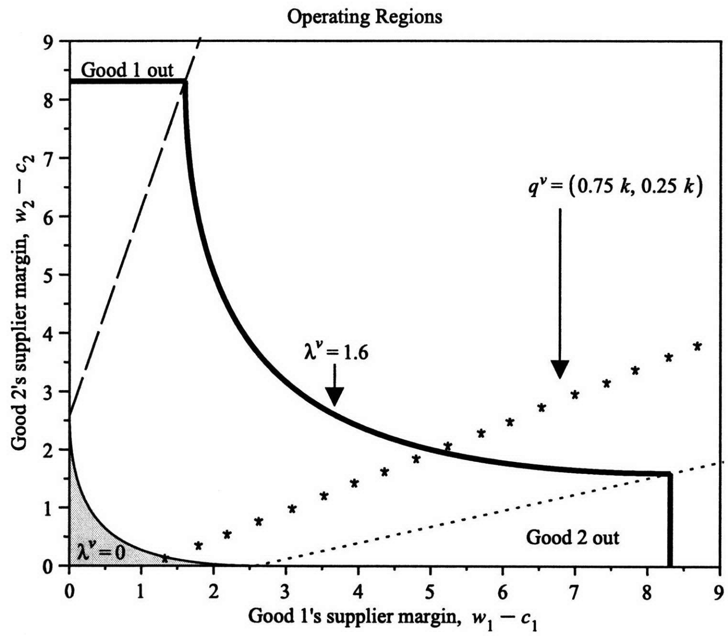
7.4 Analysis for the two-stage game.

In this section we analyze the equilibria for the two stage game. We start by reformulating the retailer's best response problem.

7.4.1 Recasting a retailer's problem & its shadow price for allocated supplier capacity

Consider a retailer $Y \subset N$ faced with the problem *Y-RETAILER($w_{\overline{Y}}$)* in the decision vector w_Y when competing with good(s) $\overline{Y}$ (whose wholesale price vector $w_{\overline{Y}}$ is held fixed) for a supplier's capacity k. From the proof of Theorem 7.1, we have that every

Figure 7-3 Wholesale price vectors that induce a particular capacity charge or capacity allocation.



Note. The supplier considers stocking two goods (dash & dot), as in Figure 7-2, and faces the same capacity, demand distributions, costs, and retail prices. The shaded region, near the origin, denotes the wholesale prices that induce the supplier to have a shadow price for capacity of zero. Whereas, the thickest line denotes the set of wholesale prices that induce the supplier to have a shadow price of 1.6 units for capacity. Also the region above the dashed ray represents the wholesale prices that induce the supplier to oust the good 'dash' from its inventory allocation (and stock only 'dot'), whereas, the region below the dotted ray denotes the wholesale prices that induce the supplier to oust 'dot' from its allocation. The ray denoted by asterisks represents the wholesale prices that induce the supplier to order $(.75 \cdot k, .25 \cdot k)$.

wholesale price vector w_Y is associated with some shadow price, $\lambda^v(w)$, for a supplier's capacity (where $w = (w_Y, w_{\overline{Y}})$) so that the set of wholesale prices can be partitioned into equivalence classes (i.e., $\{\mathcal{W}(\lambda)\}_{\lambda \geq 0}$) indexed by shadow price λ for a newsvendor's capacity (cf. Theorem 7.4). And from Theorem 7.3 and Corollary 7.5, we have that every positive supplier shadow price for capacity is associated with a unique aggregate capacity allocation/induced inventory (i.e., $\sum_{t \in Y} q_t$) by the supplier for retailer Y. Therefore, retailer Y has a simple algorithm for solving *Y-RETAILER($w_{\overline{Y}}$)* in order to maximize profit: 1) start with an initial aggregate number of units $x = 0$ to induce the supplier to stock, 2) if $\lambda^v(x; w_{\overline{Y}}) > 0$, find the wholesale price vector in the set $\mathcal{W}(\lambda^v(x; w_{\overline{Y}})) \cap \{w' \mid w'_{\overline{Y}} = w_{\overline{Y}}\}$ that maximizes profit (which, thereby, determines the optimal number of units q_t of each good $t \in Y$ the supplier is induced to stock/prepare, such that $\sum_{t \in Y} q_t = x$), otherwise, if $\lambda^v(x; w_{\overline{Y}}) = 0$, find the wholesale price vector in the set $\mathcal{W}(0) \cap \{w' \mid w'_{\overline{Y}} = w_{\overline{Y}},\ \sum_{t \in Y} q^v_t(w') = x\}$ that maximizes profit, 3) keep track of the maximum attainable profit, thus far, and the associated capacity allocation x and optimal wholesale price vector, 4) increase x and go to step two, if $x < \bar{x}$ where $\bar{x}$ is an upper bound on the aggregate quantity of goods that retailer Y would induce the supplier to stock. The upper bound $\bar{x}$ is formally defined later in this section (i.e., Corollary 7.7).

Retailer Y's optimal wholesale price(s) when inducing (aggregate) inventory x.

Suppose $x \in [0, \bar{x}]$. When the c.d.f. F_t has the IGFR property for every good $t \in Y$, step two of this algorithm can be described by a convex program with linear constraints in the decision vector q_Y to induce the supplier to stock. In particular, from Theorem 7.3 and Theorem 7.4, we have that maximizing the objective function $\sum_{i \in Y}(p_i - w_i)E[S_i(q_i^v(w))]$ of the program *Y-RETAILER($w_{\overline{Y}}$)* (i.e., equation (7.3)) over the set of wholesale prices $\mathcal{W}(\lambda^v(x; w_{\overline{Y}})) \cap \{w' \mid w'_{\overline{Y}} = w_{\overline{Y}}, \sum_{t \in Y} q_t^v(w') = x\}$ can be re-expressed as maximizing $\sum_{i \in Y}(p_i - (\lambda^v(\sum_{t \in Y} q_t; w_{\overline{Y}}) + c_i)/\bar{F}_i(q_i))E[S_i(q_i)]$ over the set of induced order vectors $\{q \mid q_t \geq 0 \ \forall t \in Y, \ \sum_{t \in Y} q_t = x\}$. Therefore, the convex program with linear constraints in the decision vector q_Y that solves step two of the algorithm is:

Y-RETAILER-PRICING-PRIMAL(x,$w_{\overline{Y}}$):

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in Y} \left(p_i - \frac{\lambda^v \left(\sum_{t \in Y} q_t; w_{\overline{Y}}\right) + c_i}{\bar{F}_i(q_i)} \right) \cdot E[S_i(q_i)] \qquad (7.13) \\ \text{subject to} \quad & q_i \geq 0, \quad i \in Y \\ & x - \sum_{i \in Y} q_i = 0. \end{aligned}$$

When the c.d.f. F_t has the IGFR property for each good $t \in Y$, it can be shown that *Y-RETAILER-PRICING-PRIMAL(x,$w_{\overline{Y}}$)* has a unique solution (vector), which we denote by $q^Y(x; w_{\overline{Y}})$. So that the set of wholesale prices $\mathcal{W}^Y(x; w_{\overline{Y}})$ that maximize retailer Y's profit when the retailer induces the supplier to stock x

units in aggregate and when the other retailers charge $w_{\overline{Y}}$ is

$$\mathcal{W}^Y(x; w_{\overline{Y}}) \stackrel{\text{def}}{=} \left\{ w' \;\middle|\; \begin{array}{l} w'_t = \frac{c_t + \lambda^v(x; w_{\overline{Y}}) - \gamma_t \cdot 1_{\{q_t^Y(x; w_{\overline{Y}})=0\}}}{\bar{F}_t(q_t^Y(x; w_{\overline{Y}}))} \quad \forall t \in Y, \\ w'_{\overline{Y}} = w_{\overline{Y}}, \;\; \gamma \in \mathbb{R}_+^{|Y|} \end{array} \right\}. \tag{7.14}$$

Note that every good $t \in Y$ that is included in the supplier's stocking decision has a unique wholesale price.

The dual problem in the decision variables $\gamma_1, \gamma_2, \ldots, \gamma_{|Y|}$ (the shadow prices for the nonnegativity constraints) and λ (the shadow price for the aggregate induced order) is:

Y-RETAILER-PRICING-DUAL(x,$w_{\overline{Y}}$):

$$\begin{array}{ll} \text{minimize} & \displaystyle\max_{\{q \in \mathbb{R}_+^{|Y|} | x - \sum_{i \in Y} q_i = 0\}} \sum_{i \in Y} \left(p_i - \frac{\lambda^v \left(\sum_{t \in Y} q_t; w_{\overline{Y}} \right) + c_i}{\bar{F}_i(q_i)} \right) \cdot E[S_i(q_i)] \quad (7.15) \\ & \displaystyle\qquad + \sum_{i \in Y} \gamma_i q_i + \lambda \left(x - \sum_{i \in Y} q_i \right) \\ \text{subject to} & \gamma_i \geq 0, \;\; i \in Y. \end{array}$$

Also, *Y-RETAILER-PRICING-DUAL(x,$w_{\overline{Y}}$)* has a unique solution which we denote by $(\gamma_1^Y(x; w_{\overline{Y}}), \ldots, \gamma_{|Y|}^Y(x; w_{\overline{Y}}), \lambda^Y(x; w_{\overline{Y}}))$.

Theorem 7.6 formalizes the idea that retailer Y's shadow price $\lambda^Y(x; w_{\overline{Y}})$ describes a threshold for the marginal profit of an additional unit of any good in the set Y (when inducing an aggregate stocking level x at the supplier and facing a fixed supplier shadow price $\lambda^v(x; w_{\overline{Y}})$ for capacity).

THEOREM 7.6. *Suppose that for every good $t \in Y$, the c.d.f. F_t has the IGFR property. Let $Z^Y(x; w_{\overline{Y}}) \stackrel{def}{=} \{i \in Y \mid q_i^Y(x; w_{\overline{Y}}) = 0\}$ be the set of products that are not stocked in retailer Y's decision problem when faced with wholesale price vector $w_{\overline{Y}}$ and inducing an aggregate stocking level x at the supplier. For any wholesale price vector $w_{\overline{Y}}$ and induced aggregate stocking level $x \in \left(0, \min\left\{\sum_{t \in Y} q_t^e, k\right\}\right]$, the* *following conditions hold:*

$$\lambda^Y(x;w_{\overline{Y}}) = p_j\bar{F}_j\left(q_j^Y(x;w_{\overline{Y}})\right) - (c_j + \lambda^v(x;w_{\overline{Y}}))\cdot\left(1+\frac{E[S_j(q_j^Y(x;w_{\overline{Y}}))]}{\bar{F}_j(q_j^Y(x;w_{\overline{Y}}))}\frac{f_j(q_j^Y(x;w_{\overline{Y}}))}{\bar{F}_j(q_j^Y(x;w_{\overline{Y}}))}\right), \quad \forall\, j \in Y\setminus Z^Y(x;w_{\overline{Y}}), \tag{7.16}$$

$$\lambda^Y(x;w_{\overline{Y}}) \ge p_i - c_i - \lambda^v(x;w_{\overline{Y}}), \qquad \forall\, i \in Z^Y(x;w_{\overline{Y}}). \tag{7.17}$$

Furthermore, $\lambda^Y(x;w_{\overline{Y}}) + \lambda^v(x;w_{\overline{Y}}) \le 0$ *when* $x = \sum_{t\in Y} q_t^e \le k$*. And, the function* $\lambda^Y(x;w_{\overline{Y}}) + \lambda^v(x;w_{\overline{Y}})$ *is strictly decreasing as* $x \in \left(0, \min\left\{\sum_{t\in Y} q_t^e, k\right\}\right]$ *increases.*

Proof. See Section 7.6.10.

From Equation (7.16), we have that retailer Y's shadow price $\lambda^Y(x;w_{\overline{Y}})$ represents an upper bound for the retailer's marginal profit on the x^{th} unit that the supplier stocks (when retailer Y chooses the optimal number of units of each good $y \in Y$ to induce the supplier to stock, so that the supplier stocks x units in aggregate) and accounts for the marginal cost of the good as well as the marginal cost for the supplier's capacity, $\lambda^v(x;w_{\overline{Y}})$. From Theorem 7.6, we have that the function $\lambda^Y(x;w_{\overline{Y}})$ is strictly decreasing in x, because the function $\lambda^Y(x;w_{\overline{Y}}) + \lambda^v(x;w_{\overline{Y}})$ is strictly decreasing and from Corollary 7.5 we know that $\lambda^v(x;w_{\overline{Y}})$ is nondecreasing. Therefore, retailer Y only considers inducing the supplier to stock up to some $\bar{x}$ units (in aggregate) where $\bar{x}$ is defined in Corollary 7.7.

COROLLARY 7.7. *Under the same assumptions as in Theorem 7.6, retailer* Y *would never induce the supplier to stock more than* $\bar{x}$ *units of good(s)* Y *in aggregate where* $\bar{x}$ *is defined according to the following conditions. If the conditions*

$$0 < \max\left\{p_i - c_i - \lambda^v(0;w_{\overline{Y}}) \mid i \in Y\right\} \text{ and } \lambda^Y\left(\min\left\{\sum_{t\in Y} q_t^e, k\right\};w_{\overline{Y}}\right) \le 0,$$

hold, then $\bar{x}$ *is the positive value that satisfies the equation* $\lambda^Y(\bar{x};w_{\overline{Y}}) = 0$*. But, if the conditions*

$$0 < \max\left\{p_i - c_i - \lambda^v(0;w_{\overline{Y}}) \mid i \in Y\right\} \text{ and } 0 < \lambda^Y\left(\min\left\{\sum_{t\in Y} q_t^e, k\right\};w_{\overline{Y}}\right),$$

hold, then $\bar{x}$ *equals* $\min\left\{\sum_{t\in Y} q_t^e, k\right\} = k$. *Finally, if the condition*

$$\max\left\{p_i - c_i - \lambda^v\left(0; w_{\overline{Y}}\right) \mid i \in Y\right\} \leq 0,$$

holds, then $\bar{x}$ *equals zero. Under any of these conditions,* $\bar{x} \leq \sum_{t\in Y} q_t^e$.

Proof. See Section 7.6.11.

Recall that via equation (7.14), the retailer can map any induced aggregate stocking level x to the set $\mathcal{W}^Y(x; w_{\overline{Y}})$ of wholesale prices that should be offered to achieve that aggregate stocking level. Next, we analyze the optimal aggregate capacity allocation that the retailer should induce (and hence the wholesale prices the retailer should offer) when faced with wholesale price vector $w_{\overline{Y}}$ from competing good(s) $\overline{Y}$.

Retailer Y's optimal induced stocking level x and best response to $w_{\overline{Y}}$.

Consider a retailer $Y \subset N$ competing with good(s) $\overline{Y}$ for a supplier's capacity k. Suppose $w = (w_Y, w_{\overline{Y}})$ and the wholesale price vector $w_{\overline{Y}}$ is held fixed. From Theorem 7.3 and Theorem 7.4, we have that the objective function $\sum_{i\in Y}(p_i - w_i)E[S_i(q_i^v(w))]$ of the program *Y-RETAILER($w_{\overline{Y}}$)* (i.e., equation (7.3)) can be re-expressed as $\sum_{i\in Y}(p_i - (\lambda^v(\sum_{t\in Y} q_t; w_{\overline{Y}}) + c_i)/\bar{F}_i(q_i))E[S_i(q_i)]$. Therefore, as suggested in the beginning of this section, retailer Y's problem of maximizing profit and deciding the optimal wholesale price vector $w_Y^{\text{br}}(w_{\overline{Y}})$ when solving *Y-RETAILER($w_{\overline{Y}}$)* can be recast as the equivalent problem of deciding upon an aggregate quantity x to induce the supplier to stock and then deciding how to split the aggregate stocking quantity x among the goods Y. Formally, the program with linear constraints in the decision quantity x and decision vector q_Y that solves *Y-RETAILER($w_{\overline{Y}}$)* is:

Y-RETAILER-INDUCING-AGGREGATE-ORDER(k, $w_{\overline{Y}}$*):*

$$\begin{aligned}
\text{maximize} \quad & \sum_{i \in Y} \left(p_i - \frac{\lambda^v\left(\sum_{t \in Y} q_t; w_{\overline{Y}}\right) + c_i}{\bar{F}_i(q_i)} \right) \cdot E[S_i(q_i)] \qquad (7.18) \\
\text{subject to} \quad & q_i \geq 0, \quad i \in Y \\
& x \geq 0, \\
& x - \sum_{i \in Y} q_i = 0, \\
& k - x \geq 0.
\end{aligned}$$

Theorem 7.7, below, provides sufficient conditions for *Y-RETAILER-INDUCING-AGGREGATE-ORDER(*$w_{\overline{Y}}$*)* to have a unique solution and, under those conditions, we denote the optimal aggregate order quantity by $x^Y(w_{\overline{Y}})$ and optimal induced order vector by $q^Y(w_{\overline{Y}})$. Therefore, from the proof of Theorem 7.6, we have that the optimal induced order quantity vector $q^Y(w_{\overline{Y}})$ must equal $q^Y(x^Y; w_{\overline{Y}})$. And, from equation (7.14), retailer Y's best response to competing wholesale prices $w_{\overline{Y}}$ is the set of wholesale prices $\mathcal{W}^Y(x^Y; w_{\overline{Y}})$. Furthermore, when retailer Y is faced with competing wholesale price vector $w_{\overline{Y}}$ and when it is optimal for retailer Y to induce the supplier to stock every good $y \in Y$ (i.e., $Z^Y(x^Y; w_{\overline{Y}}) = \emptyset$), from equation (7.14), we have that retailer Y's best response is unique (i.e., the set $\mathcal{W}^Y(x^Y; w_{\overline{Y}})$ has only one wholesale price vector).

THEOREM 7.7. *Y-RETAILER-INDUCING-AGGREGATE-ORDER(*$w_{\overline{Y}}$*) has a unique solution* $\left(x^Y(w_{\overline{Y}}), q^Y(w_{\overline{Y}})\right)$ *defined implicitly by the conditions*

$$x^Y(w_{\overline{Y}}) = \sup \left\{ x \in [0, \bar{x}] \mid \lambda^Y(x; w_{\overline{Y}}) - \left(\sum_{i \in Y} \frac{E[S(q_i^Y(x; w_{\overline{Y}}))]}{\bar{F}_i(q_i^Y(x; w_{\overline{Y}}))} \right) \cdot \frac{\partial \lambda^v(x; w_{\overline{Y}})}{\partial x^-} \geq 0 \right\}, \tag{7.19}$$

$$q^Y(w_{\overline{Y}}) = q^Y\left(x^Y(w_{\overline{Y}}); w_{\overline{Y}}\right), \tag{7.20}$$

when $\bar{x} \neq 0$*, the retailer is service constrained for good(s)* $\overline{Y}$*, and the cumulative distribution function for demand of each good* $y \in Y$ *has the IGFR property.*

Proof. See Section 7.6.12.

Now we can state the main result of this section.

THEOREM 7.8. *Consider a retailer $Y \subset N$ that faces the problem Y-RETAILER$(w_{\overline{Y}})$ (when the vector $w_{\overline{Y}}$ of wholesale price offers from its competitors is held fixed) and, therefore, decides upon an optimal set $\mathcal{W}_Y^{br}(w_{\overline{Y}})$ of wholesale price vectors from the hypercube $\prod_{t \in Y}[c_t, p_t]$. Suppose the retailers are service constrained for good(s) $\overline{Y}$, and the cumulative distribution function for demand of each good $y \in Y$ has the IGFR property. Then, the solution set $\mathcal{W}_Y^{br}(w_{\overline{Y}})$ is non-empty, convex, and satisfies*

$$\mathcal{W}_Y^{br}(w_{\overline{Y}}) = \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}}) \bigcap \prod_{t \in Y}[c_t, p_t]. \tag{7.21}$$

Proof. See Section 7.6.13.

7.4.2 Existence of equilibrium

In Theorem 7.9, we provide sufficient conditions so that the two-stage game described in Section 7.1.1 has at least one equilibrium wholesale price vector, and, therefore, resulting supplier capacity allocation vector and shadow price for capacity.

THEOREM 7.9. *With more than one retailer (i.e., when $s \geq 2$), an equilibrium wholesale price vector exists when the retailers are service constrained for goods N and the demand for each good $t \in N$ has the IGFR property.*

Proof. See Section 7.6.14.

Denote the set of equilibrium wholesale price vectors for the Stackelberg game (when the supplier has a capacity of k units) by $\mathcal{W}^{\mathrm{eq}}(k) \stackrel{\mathrm{def}}{=} \{w \mid \text{ for every retailer } Y, w_Y \in \mathcal{W}_Y^{\mathrm{br}}(w_{\overline{Y}})\}$. Furthermore, denote the set of (resulting) equilibrium capacity prices by $\Lambda^{\mathrm{eq}}(k) \stackrel{\mathrm{def}}{=} \{\lambda \mid \lambda = \lambda^v(w), w \in \mathcal{W}^{\mathrm{eq}}(k)\}$. From Theorem 7.9, we know that the set $\mathcal{W}^{\mathrm{eq}}(k)$ is non-empty, so that the set $\Lambda^{\mathrm{eq}}(k)$ is, also, non-empty. Therefore, two values that allow us to bound the valuation for the supplier's capacity are $\lambda^{\min}(k) \stackrel{\mathrm{def}}{=} \inf \Lambda^{\mathrm{eq}}(k)$ and $\lambda^{\max}(k) \stackrel{\mathrm{def}}{=} \sup \Lambda^{\mathrm{eq}}(k)$. But, often times, we can do better, and give an exact valuation for the supplier's capacity. In the next section, we describe an economic assumption that guarantees a unique equilibrium capacity price, so that the set $\Lambda^{\mathrm{eq}}(k)$ has exactly one element.

7.4.3 An economic assumption

When retailers have larger allocations of the newsvendor's capacity, the newsvendor's capacity price is more sensitive to attempts to increase that allocation. Formally, consider s retailers and two different (aggregate) capacity allocations to those retailers (represented by the vectors $a = (a_1, \ldots, a_s)$ and $b = (b_1, \ldots, b_s)$) induced by two optimal wholesale price vectors w^a and w^b (respectively) for those aggregate allocations (see Section 7.4.1) that cause the newsvendor to allocate his entire capacity k (i.e., $\sum_{t=1}^{s} a_t = k$ and $\sum_{t=1}^{s} b_t = k$). From Theorem 7.3, we have that retailer competition induces some virtual 'charge' for capacity (involving λ_a^v and λ_b^v, respectively), paid to the newsvendor. From Theorem 7.6, we have that the i^{th} retailer (i.e., retailer Y_i) has thresholds $\lambda_a^{Y_i}$ and $\lambda_b^{Y_i}$ for its marginal profit when faced with competing wholesale price vectors $w^a_{\overline{Y_i}}$ and $w^b_{\overline{Y_i}}$, respectively. Denote the subset of retailers that have a larger share of the newsvendor's capacity under allocation a when compared to allocation b by $L(a,b)$ (i.e., formally, $L(a,b) \stackrel{\text{def}}{=} \{i \in \{1, \ldots, s\} \mid a_i > b_i\}$).[6]

ASSUMPTION 7.1. *Consider the retailers $L(a,b)$ that have a higher allocation under allocation a when compared to allocation b. The marginal increase in the price of capacity, $\lambda^v(w^a)$, for a percent increase in the induced average sales for each good (by each retailer in $L(a,b)$) is larger than the marginal increase in the price of capacity, $\lambda^v(w^b)$, for a percent increase in the induced average sales for each good (by each retailer in $L(a,b)$), i.e.,*

$$\sum_{i \in L(a,b)} \left(\sum_{j \in Y_i} \frac{E[S(q_j^{Y_i}(a_i; w^a_{\overline{Y_i}}))]}{\bar{F}_j(q_j^{Y_i}(a_i; w^a_{\overline{Y}}))} \right) \cdot \frac{\partial \lambda^v(a_i; w^a_{\overline{Y_i}})}{\partial a_i^-} \geq \sum_{i \in L(a,b)} \left(\sum_{j \in Y_i} \frac{E[S(q_j^{Y_i}(b_i; w^b_{\overline{Y_i}}))]}{\bar{F}_j(q_j^{Y_i}(b_i; w^b_{\overline{Y}}))} \right) \cdot \frac{\partial \lambda^v(b_i; w^b_{\overline{Y_i}})}{\partial b_i^+}. \tag{7.22}$$

[6] From the 'pigeon-hole principle', we have that the subset, $L(a,b)$, is not empty because the allocation vectors are not equal.

In the case of two retailers each with one good, if the newsvendor is serviced constrained for both goods, we can show that inequality (7.22) is a necessary condition.

7.4.4 Uniqueness of equilibrium shadow price

In Theorem 7.10, we provide sufficient conditions so that the two-stage game described in Section 7.1.1 has a unique equilibrium shadow price for capacity.

THEOREM 7.10. *With one retailer (i.e., when $s = 1$), any equilibrium wholesale price vector results in the supplier having a shadow price for capacity of zero units. Furthermore, with more than one retailer (i.e., when $s \geq 2$), if the retailers are service constrained for goods N, the demand for each good $t \in N$ has the IGFR property, and Assumption 7.1 holds, then for any two equilibrium wholesale price vectors $\widehat{w}$ and w' that induce different allocations (i.e., $q^v(\widehat{w}) \neq q^v(w')$) the induced shadow prices for capacity ($\lambda^v(\widehat{w})$ and $\lambda^v(w')$) are the same (i.e., there is a unique equilibrium shadow price for the supplier's capacity).*

Proof. See Section 7.6.15.

When there is more than one retailer (i.e., $s \geq 2$), this theorem implies that if there are two equilibrium wholesale price vectors inducing different allocations of the supplier's capacity, then there is a unique equilibrium shadow price which we denote by λ^{eq}. And so, geometrically, the equilibrium wholesale price vectors are a subset of the set $\mathcal{W}(\lambda^{\text{eq}})$ as defined in Theorem 7.4 and depicted in Figure 7-3. In Theorem 7.11, we consider the scenario when there are two equilibrium wholesale price vectors that induce the supplier to stock the same vector of goods.

THEOREM 7.11. *Consider the two-stage game described in Section 7.1.1 with more than one retailer (i.e., $s \geq 2$). Suppose the retailers are service constrained for goods N, the demand for each good $t \in N$ has the IGFR property, and that there are two equilibrium wholesale price vectors $\widehat{w}$ and w' that induce the same supplier capacity allocation (i.e., $q^v(\widehat{w}) = q^v(w')$) but induce shadow prices for capacity satisfying $\lambda^v(w') \leq \lambda^v(\widehat{w})$. Denote retailer Y_j's marginal profit for inducing the supplier to dedicate an extra unit of capacity (when the supplier faces wholesale* *price vector w') by the function*

$$m_{Y_j}(w') \stackrel{def}{=} \lambda^{Y_j}\left(\sum_{i\in Y_j} q_i^v(w'); w'_{\overline{Y_j}}\right) - \left(\sum_{i\in Y_j} \frac{E[S(q_i^v(w'))]}{\bar{F}_i(q_i^v(w'))}\right) \cdot \left.\frac{\partial \lambda^v(x; w'_{\overline{Y_j}})}{\partial x^-}\right|_{x=\sum_{i\in Y_j} q_i^v(w')}.$$

Then, we have the following upper bound on the shadow price $\lambda^v(\widehat{w})$ when the supplier allocates capacity from two or more retailers when offered wholesale price vector w':

$$\lambda^v(\widehat{w}) \le \lambda^v(w') + \min\left\{m_{Y_j}(w') \mid j \in \{1,\ldots,s\},\ Y_j \cap (N \setminus Z(w')) \neq \emptyset\right\}. \quad (7.23)$$

And, we have the following upper bound on the shadow price $\lambda^v(\widehat{w})$ when the supplier allocates capacity for exactly one retailer when offered wholesale price vector w':

$$\lambda^v(\widehat{w}) \le \lambda^v(w'). \quad (7.24)$$

Proof. See Section 7.6.16.

Therefore, under the same assumptions as in Theorem 7.10, Theorem 7.11 (in conjunction with Theorem 7.10) implies there are three possible scenarios in an equilibrium setting: either there is a unique equilibrium shadow price (with multiple equilibrium allocations), or there is a unique equilibrium allocation (with multiple equilibrium shadow prices), or there is a unique equilibrium allocation and shadow price. These two theorems rule out the possibility of having two different equilibrium wholesale price vectors, $\widehat{w}$ and w', that simultaneously induce different supplier allocations and different supplier shadow prices for capacity (i.e., such that both $q^v(\widehat{w}) \neq q^v(w')$ and $\lambda^v(\widehat{w}) \neq \lambda^v(w')$ hold).

■ 7.5 Retailer collusion.

Theorem 7.12 formalizes the idea that if there are more than two retailers and they collude by offering pricing as if they were one firm, then they'd make more profit in aggregate than they would from any equilibrium that induces a positive equilibrium supplier shadow price for capacity. Consequently, there exists a division of the collusion profit such that every retailer would receive more profit than they would from the equilibrium wholesale price that induces a positive shadow price.

THEOREM 7.12. *Consider the two-stage game described in Section 7.1.1 with more than one retailer (i.e., $s \geq 2$). Suppose there is an equilibrium wholesale price vector w' that induces a positive shadow price $\lambda^v(w') > 0$. If the retailers collude on pricing by setting prices as if they were one firm, then the aggregate retailer profit would be larger than the aggregate retailer profit from price vector w'.*

Proof. See Section 7.6.17.

However, does the supply chain's profits increase or decrease when the retailers collude? And will the supplier receive lower wholesale prices for every good when the retailers collude? We leave these and other questions for future work.

7.6 Proofs

In order to not disrupt the flow of presentation, the proofs for our results in this chapter are contained here.

7.6.1 Proof: The shadow price for capacity and the goods ordered.

Proof of Theorem 7.1. First, we write the Lagrangian $\mathcal{L}(q, \gamma_1, \ldots, \gamma_n, \lambda)$ for *SUPPLIER-PRIMAL(k,w)*:

$$\mathcal{L}(q, \gamma_1, \ldots, \gamma_n, \lambda) = \sum_{i=1}^{n} (w_i E[\min(q_i, D_i)] - c_i q_i) + \sum_{i=1}^{n} \gamma_i q_i + \lambda \left(k - \sum_{i=1}^{n} q_i \right).$$

Note that $\pi_v(q)$ is strictly concave for $q \in [0, l_1) \times \cdots [0, l_n)$ because each c.d.f. F_i is strictly increasing over $[0, l_i)$. Because the feasible set is convex and compact, *SUPPLIER-PRIMAL(k,w)* has a unique solution.

The Karush-Kuhn-Tucker conditions for the retailer's decision problem, *SUPPLIER-*

PRIMAL(k,w), are:

$$w_t\bar{F}_t(q_t) - c_t + \gamma_t - \lambda = 0, \quad t = 1, \ldots, n; \tag{7.25}$$

$$q_t \geq 0, \quad t = 1, \ldots, n;$$

$$k - \sum_{t=1}^{n} q_t \geq 0;$$

$$\gamma_t q_t = 0, \quad t = 1, \ldots, n; \tag{7.26}$$

$$\lambda \left(k - \sum_{t=1}^{n} q_t \right) = 0; \tag{7.27}$$

$$\lambda \geq 0; \quad \gamma_t \geq 0, \quad t = 1, \ldots, n.$$

Because of the concavity of the objective function and the fact that the Slater condition is satisfied, the Karush-Kuhn-Tucker conditions are both necessary and sufficient conditions for any primal optimal vector q and dual optimal vector (γ, λ). As a result, since the primal problem has a unique solution, it can be shown that the dual problem also has a unique solution, using these conditions. Let $(q^v(w), \gamma^v(w), \lambda^v(w))$ denote the unique vector that satisfies the Karush-Kuhn-Tucker conditions.

When $j \in N \setminus Z(w)$, from equation (7.26) we have $\gamma_j^v(w) = 0$. Therefore, from equation (7.25) we have $\lambda^v(w) = w_j\bar{F}_j\left(q_j^v(w)\right) - c_j$. When $i \in Z(w)$, from equation (7.25), we have $\lambda^v(w) = w_i - c_i + \gamma_i^v(w) \geq w_i - c_i$. Thus, the conditions in equations (7.6) and (7.7) hold.

Furthermore, if $\sum_{t=1}^{n} q_t^u(w) \leq k$, we have $q^v(w) = q^u(w)$. Therefore, when $j \in N \setminus Z(w)$, from equation (7.25) we have $\lambda^v(w) = w_j\bar{F}_j\left(q_j^u(w)\right) - c_j = 0$.

On the other hand, assume $\lambda^v(w) = 0$. When $j \in N \setminus Z(w)$, we have $w_j\bar{F}_j\left(q_j^v(w)\right) - c_j = 0$ from equation (7.25). Therefore, $q_j^v(w) = q_j^u(w)$. When $i \in Z(w)$, from equation (7.25), we have $w_i - c_i \leq w_i - c_i + \gamma_i^v(w) = 0$. Thus, $q_i^u(w) = 0 = q_i^v(w)$. And so we have $\sum_{t=1}^{n} q_t^u(w) = \sum_{t=1}^{n} q_t^v(w) \leq k$. $\square$

7.6.2 Proof: The shadow price for capacity as the minimum of some set.

Proof of Corollary 7.1. Let $\Lambda \stackrel{\text{def}}{=} \{\lambda \mid w_t\bar{F}_t(q_t) - c_t \leq \lambda \ \ \forall t \in N,\ \sum_{t=1}^n q_t = \min\{\sum_{t=1}^n q_t^u(w), k\},\ q \in \mathbb{R}_+^n\}$.

The vector $q^v(w) \in \mathbb{R}_+^n$ satisfies $\sum_{t=1}^n q_t^v(w) = \min\{\sum_{t=1}^n q_t^u(w), k\}$. Furthermore, from equation (7.25), we have $\lambda^v(w) = w_t\bar{F}_t(q_t^v(w)) - c_t + \gamma_t^v(w) \geq w_t\bar{F}_t(q_t^v(w)) - c_t$ when $t = 1, \dots, n$. Therefore, we have $\lambda^v(w) \in \Lambda$.

Assume there exists a $\lambda' < \lambda^v(w)$ such that $\lambda' \in \Lambda$. Then there must exist a vector $q' \in \mathbb{R}_+^n$ such that $\sum_{t=1}^n q'_t = \min\{\sum_{t=1}^n q_t^u(w), k\}$ and $w_t\bar{F}_t(q'_t) - c_t \leq \lambda'$ when $t = 1, \dots, n$. When $j \in N \setminus Z(w)$, from equation (7.25) we have $\lambda^v(w) = w_j\bar{F}_j\left(q_j^v(w)\right) - c_j$. Since $\lambda' < \lambda^v(w)$, when $j \in N \setminus Z(w)$, we have $w_j\bar{F}_j(q'_j) - c_j < w_j\bar{F}_j\left(q_j^v(w)\right) - c_j$, implying $\bar{F}_j(q'_j) < \bar{F}_j\left(q_j^v(w)\right)$ and, thus, $q'_j > q_j^v(w)$. Therefore, $\sum_{j\in N\setminus Z(w)} q'_j > \sum_{j\in N\setminus Z(w)} q_j^v(w) = \sum_{t\in N} q_t^v(w) = \min\{\sum_{t=1}^n q_t^u(w), k\}$, implying $\sum_{t=1}^n q'_t > \min\{\sum_{t=1}^n q_t^u(w), k\}$. But this is a contradiction. Thus, $\lambda^v(w) = \min_{\lambda\in\Lambda} \lambda$. □

7.6.3 Proof: $\lambda^v(w)$ is nondecreasing as w_t increases, and the increase is bounded.

Proof of Theorem 7.2. Let $\Lambda(w) \stackrel{\text{def}}{=} \{\lambda \mid w_t\bar{F}_t(q_t) - c_t \leq \lambda \ \ \forall t \in N,\ \sum_{t=1}^n q_t = \min\{\sum_{t=1}^n q_t^u(w), k\},\ q \in \mathbb{R}_+^n\}$. Since $w_i < w'_i$ and $w'_{-i} = w_{-i}$, we have $q_i^u(w) < q_i^u(w')$ and $q_o^u(w) = q_o^u(w')$ for any other good $o \neq i$. Therefore, $\min\{\sum_{t=1}^n q_t^u(w), k\} \leq \min\{\sum_{t=1}^n q_t^u(w'), k\}$. If $\min\{\sum_{t=1}^n q_t^u(w), k\} = k$, then $\Lambda(w') \subseteq \Lambda(w)$ so that we have $\min \Lambda(w) \leq \min \Lambda(w')$. And, from Corollary 7.1, we have $\lambda^v(w) \leq \lambda^v(w')$. Otherwise, if $\min\{\sum_{t=1}^n q_t^u(w), k\} < k$, then from Theorem 7.1 we have $\lambda^v(w) = 0$ so that $\lambda^v(w) \leq \lambda^v(w')$ because the shadow prices are nonnegative.

Next, we show that $\lambda^v(w) < \lambda^v(w')$ holds if and only if the conditions $i \in N\setminus Z(w')$ and $\sum_{t=1}^n q_t^u(w') > k$ hold by proving the statement: $\lambda^v(w) = \lambda^v(w')$ holds if and only if $i \in Z(w')$ or $\sum_{t=1}^n q_t^u(w') \leq k$ holds.

First, we prove the direction: if $i \in Z(w')$ or $\sum_{t=1}^n q_t^u(w') \leq k$ holds, then

$\lambda^v(w) = \lambda^v(w')$ holds. If $\sum_{t=1}^n q_t^u(w') \le k$ holds, then from Theorem 7.1 we have $\lambda^v(w') = 0$. Since $\lambda^v(w) \le \lambda^v(w')$ and the shadow prices are nonnegative, we have $\lambda^v(w) = \lambda^v(w')$. When $i \in Z(w')$ and $\sum_{t=1}^n q_t^u(w') > k$ hold, assume $\lambda^v(w) < \lambda^v(w')$ holds, instead. Then, for any $j \in N \setminus Z(w)$, we have $q_j^v(w') < q_j^v(w)$ when either $j \in N \setminus Z(w')$ (from equation (7.6)) or $j \in Z(w')$. Since $\lambda^v(w) < \lambda^v(w')$, from equation (7.7), we have $Z(w) \subseteq Z(w')$ so that $N \setminus Z(w') \subseteq N \setminus Z(w)$. Therefore, $\sum_{j\in N\setminus Z(w')} q_j^v(w') < \sum_{j\in N\setminus Z(w)} q_j^v(w) \le k$. From equation (7.27), we have $\lambda^v(w') = 0$, implying $\sum_{t=1}^n q_t^u(w') \le k$ (by Theorem 7.1). But this is a contradiction since $\sum_{t=1}^n q_t^u(w') > k$ holds. Thus, $\lambda^v(w) = \lambda^v(w')$.

Next, we show that $\lambda^v(w) = \lambda^v(w')$ implies $i \in Z(w')$ or $\sum_{t=1}^n q_t^u(w') \le k$ holds. Assume $i \in N\setminus Z(w')$ and $\sum_{t=1}^n q_t^u(w') > k$ hold, instead. Therefore, we have $\lambda^v(w) = \lambda^v(w') > 0$ from Theorem 7.1. And equation (7.27) implies $\sum_{j\in N\setminus Z(w)} q_j^v(w) = k$. If $i \in N \setminus Z(w)$, then $q_i^v(w') > q_i^v(w)$ from equation (7.6) since $w_i < w_i'$. If $i \in Z(w)$, we also have $q_i^v(w') > q_i^v(w) = 0$. Furthermore, equation (7.7) implies that for any good $t \ne i$: $t \in Z(w')$ if and only if $t \in Z(w)$ (because $w_t = w_t'$ and $\lambda^v(w) = \lambda^v(w')$). Thus, the sets $Z(w)$ and $Z(w')$ are identical unless $i \in Z(w)$. Therefore, $N \setminus Z(w') = (N \setminus Z(w)) \cup \{i\}$. And for any good $j \ne i$ such that $j \in N \setminus Z(w')$ we have $q_j^v(w') = q_j^v(w)$ (from equation (7.6)). Therefore, $\sum_{j\in N\setminus Z(w')} q_j^v(w') > \sum_{j\in N\setminus Z(w)} q_j^v(w) = k$. But this is a contradiction. Thus, $i \in Z(w')$ or $\sum_{t=1}^n q_t^u(w') \le k$ holds.

Next, we prove inequality (7.9) holds. If $\lambda^v(w) = \lambda^v(w')$, inequality (7.9) follows. If $\lambda^v(w) < \lambda^v(w')$, then $i \in N\setminus Z(w')$ and $\sum_{t=1}^n q_t^u(w') > k$ hold, as proven. Therefore, we have $\lambda^v(w') > 0$ from Theorem 7.1 and equation (7.27) implies $\sum_{j\in N\setminus Z(w')} q_j^v(w') = k$. Assume the inequality $(\lambda^v(w')+c_i)/(\lambda^v(w)+c_i) > w_i'/w_i$ holds, instead. Therefore, rearranging terms, $(\lambda^v(w') + c_i)/w_i' > (\lambda^v(w) + c_i)/w_i$ holds. Corollary 7.2, then, implies $q_i^v(w') < q_i^v(w)$ and that for any $j \in N \setminus Z(w)$ such $j \ne i$, we have $q_j^v(w') < q_j^v(w)$ (because $(\lambda^v(w') + c_j)/w_j' > (\lambda^v(w) + c_j)/w_j$ holds). The inequalities $q_i^v(w') < q_i^v(w)$ and $\lambda^v(w) < \lambda^v(w')$ imply $Z(w) \subseteq Z(w')$ (from equation (7.7)). Therefore, $N \setminus Z(w') \subseteq N \setminus Z(w)$. And we have $k = \sum_{j\in N\setminus Z(w')} q_j^v(w') < \sum_{j\in N\setminus Z(w)} q_j^v(w)$. But this is a contradiction. Thus, inequality (7.9) follows.

If the supplier prepares k units of good i under w and w', then, from equation (7.6), we have that $(\lambda^v(w')+c_i)/w'_i = (\lambda^v(w)+c_i)/w_i$ holds. If $(\lambda^v(w')+c_i)/w'_i = (\lambda^v(w)+c_i)/w_i$ holds, then, from Corollary 7.2, we have that $q_i^v(w') = q_i^v(w)$. Furthermore, since $w_i < w'_i$, we have $\lambda^v(w) < \lambda^v(w')$. Therefore, as proven, we have that good i is included in the supplier's inventory under w' (and thus w) and that the capacity constraint is binding for the supplier under w'. From Theorem 7.1, we have $\lambda^v(w') > 0$ and equation (7.27) implies $\sum_{j\in N\setminus Z(w')} q_j^v(w') = k$. Assume the inequality $q_i^v(w') < k$ holds, instead of the equality $q_i^v(w') = k$. Then, there exists at least one other good $o \in N \setminus Z(w')$, where $o \neq i$. Good i is included in the retailer's order under both w and w', and $\lambda^v(w) < \lambda^v(w')$, therefore, we have $Z(w) \subseteq Z(w')$ (from equation (7.7)), implying $N \setminus Z(w') \subseteq N \setminus Z(w)$. And for any good $j \neq i \in N \setminus Z(w)$, from Corollary 7.2, we have $q_j^v(w') < q_j^v(w)$ because $(\lambda^v(w)+c_j)/w_j < (\lambda^v(w')+c_j)/w'_j$ holds. Therefore, we have $k = \sum_{j\in N\setminus Z(w')} q_j^v(w') < \sum_{j\in N\setminus Z(w)} q_j^v(w)$. But this is a contradiction. Thus, the equality $q_i^v(w') = k$ follows. □

7.6.4 Proof: The effect of a wholesale price increase on the supplier's inventory.

Proof of Corollary 7.4. From Theorem 7.2, we have $\lambda^v(w) \leq \lambda^v(w')$. Since $w'_o = w_o$, we have $(\lambda^v(w)+c_o)/w_o \leq (\lambda^v(w')+c_o)/w'_o$. The inequality $q_o^v(w') \leq q_o^v(w)$, then, follows from Corollary 7.2. Furthermore, from inequality (7.9), we have $(\lambda^v(w')+c_i)/w'_i \leq (\lambda^v(w)+c_i)/w_i$. Therefore, from Corollary 7.2, we have $q_i^v(w) \leq q_i^v(w')$.

First, we show condition (b) implies condition (a). If $q_i^v(w) < k$, then, from Theorem 7.2, we have the strict inequality $(\lambda^v(w')+c_i)/w'_i < (\lambda^v(w)+c_i)/w_i$. Since the retailer orders good i under w', from Corollary 7.2, we have $\max\{\lambda^v(w')+c_i, w'_i\} < w'_i$. Therefore, $\max\{\lambda^v(w')+c_i, w'_i\}/w'_i < \max\{\lambda^v(w)+c_i, w_i\}/w_i$ holds. And, since the c.d.f. F_i is strictly increasing over $[0, l_i)$, we have $q_i^v(w) < q_i^v(w')$ (from Corollary 7.2). Furthermore, condition (b) and Theorem 7.2 imply $\lambda^v(w) < \lambda^v(w')$. Since $w'_o = w_o$, we have $(\lambda^v(w)+c_o)/w_o < (\lambda^v(w')+c_o)/w'_o$. If $o \in N \setminus Z(w)$, then, from Corollary 6.2, we have $\max\{\lambda^v(w)+c_o, w_o\} < w_o$. Therefore, $\max\{\lambda^v(w)+c_o, w_o\} < \max\{\lambda^v(w')+c_o, w'_o\}$. And, from Corollary 7.2, we have $q_o^v(w') < q_o^v(w)$ because the c.d.f. F_o is strictly increasing over $[0, l_o)$.

Next, we show condition (a) implies condition (b). Assume $i \in Z(w')$ or $\sum_{t=1}^{n} q_t^u(w') \leq k$ holds, instead. From Theorem 7.2, we have $\lambda^v(w) = \lambda^v(w')$. Therefore, for any good $o \neq i$, we have $(\lambda^v(w) + c_o)/w_o = (\lambda^v(w') + c_o)/w'_o$ because $w_o = w'_o$. From Corollary 7.2, then, we have $q_o^v(w) = q_o^v(w')$. But this contradicts condition (a) when $o \in N \setminus Z(w)$. It can be shown that the set $N \setminus Z(w)$ includes some good $o \neq i$ when $\sum_{t=1}^{n} q_t^u(w') \leq k$ holds. When $\sum_{t=1}^{n} q_t^u(w') > k$ and $i \in Z(w')$ hold, we have $q_i^v(w) = q_i^v(w') = 0$ because $q_i^v(w) \leq q_i^v(w')$ holds and $q_i^v(w)$ must be nonnegative. But this contradicts condition (a) since $q_i^v(w) < k$ holds, yet $q_i^v(w) = q_i^v(w')$. Thus, condition (b) holds. $\square$

7.6.5 Proof: A retailer effects the price for capacity via its induced allocation.

Proof of Theorem 7.3. Since the wholesale price vector w induces the supplier to prepare x units of goods for retailer Y, we have that when the supplier solves the convex program *SUPPLIER-PRIMAL(k,w)* in the decision vector q, the optimal stocking quantity vector $q^v(w)$ is also the unique solution to the following convex program in the decision vector q:

SUPPLIER-WITH-Y-GUARANTEE(k,x,w):

$$\begin{aligned}
\text{maximize} \quad & \sum_{i=1}^{n} (w_i E[S_i(q_i)] - c_i q_i) \qquad (7.28)\\
\text{subject to} \quad & q_i \geq 0, \quad i = 1, \ldots, n\\
& \sum_{i \in Y} q_i = x\\
& k - \sum_{i=1}^{n} q_i \geq 0.
\end{aligned}$$

Therefore, since the objective function in (7.28) is separable into the sum of two independent expressions,

$$\sum_{i=1}^{n}(w_iE[S_i(q_i)]-c_iq_i)=\sum_{i\in Y}(w_iE[S_i(q_i)]-c_iq_i)+\sum_{i\in\overline{Y}}(w_iE[S_i(q_i)]-c_iq_i)\,, \quad (7.29)$$

the order quantity vector $q^{v}_{\overline{Y}}(w)$ is the solution to the following convex program in the decision vector q:

SUPPLIER-RESTRICTED-TO-$\overline{Y}$-PRIMAL(k,x,$w_{\overline{Y}}$):

$$\begin{aligned}\text{maximize} \quad & \sum_{i\in\overline{Y}}(w_iE[S_i(q_i)]-c_iq_i) \\ \text{subject to} \quad & q_i\geq 0,\quad i\in\overline{Y} \\ & (k-x)-\sum_{i\in\overline{Y}}q_i\geq 0.\end{aligned} \quad (7.30)$$

The dual problem in the decision variables $\gamma_{\overline{Y}}$ (the shadow price vector for the nonnegativity constraints) and λ (the shadow price for the capacity constraint) is:

SUPPLIER-RESTRICTED-TO-$\overline{Y}$-DUAL(k,x,$w_{\overline{Y}}$):

$$\text{minimize} \quad \max_{\{q\in\mathbb{R}_+^{|\overline{Y}|}\,|\,k-\sum_{t\in\overline{Y}}q_t\geq 0\}}\sum_{i\in\overline{Y}}(w_iE[S_i(q_i)]-c_iq_i)+\sum_{i\in\overline{Y}}\gamma_iq_i+\lambda\left((k-x)-\sum_{i\in\overline{Y}}q_i\right) \quad (7.31)$$

$$\begin{aligned}\text{subject to} \quad & \gamma_i\geq 0,\quad i\in\overline{Y} \\ & \lambda\geq 0.\end{aligned}$$

Note that *SUPPLIER-RESTRICTED-TO-$\overline{Y}$-DUAL(k,x,$w_{\overline{Y}}$)* is identical to *SUPPLIER-DUAL($k-x$,$w_{\overline{Y}}$)*, when $N=\overline{Y}$. Therefore, we have from Theorem 7.1 that *SUPPLIER-RESTRICTED-TO-$\overline{Y}$-DUAL(k,x,$w_{\overline{Y}}$)* has a unique solution which we denote by

$(\gamma^v_{|\overline{Y}|}(x;w_{\overline{Y}}), \lambda^v(x;w_{\overline{Y}}))$. Furthermore, from Corollary 7.1 we have that

$$\lambda^v(x;w_{\overline{Y}}) = \min\left\{\lambda \mid w_t\bar{F}_t(q_t) - c_t \le \lambda \;\; \forall t \in \overline{Y},\; \sum_{t\in\overline{Y}} q_t = \min\left\{\sum_{t\in\overline{Y}} q^u_t(w), k - x\right\},\; q \in \mathbb{R}_+^{|\overline{Y}|}\right\}. \tag{7.32}$$

Since $x < k$, there exists at least one good $j \in \overline{Y}$ such that $q^v_j(w) > 0$. Since the vector $q^v_{\overline{Y}}(w)$ is the solution to *SUPPLIER-RESTRICTED-TO-$\overline{Y}$-PRIMAL(k,x,$w_{\overline{Y}}$)*, from Equation (7.6) in Theorem 7.1, we have that

$$\lambda^v(x;w_{\overline{Y}}) = w_j\bar{F}_j\left(q^v_j(w)\right) - c_j. \tag{7.33}$$

Since the vector $q^v(w)$ is the solution to *SUPPLIER-PRIMAL(k,w)*, from Equation (7.6) in Theorem 7.1, we also have that

$$\lambda^v(w) = w_j\bar{F}_j\left(q^v_j(w)\right) - c_j. \tag{7.34}$$

Therefore, from Equations (7.33) and (7.34), we have that $\lambda^v(w) = \lambda^v(x;w_{\overline{Y}})$.

Next, we prove the partial converse. Denote the solution to *SUPPLIER-RESTRICTED-TO-$\overline{Y}$-PRIMAL(k,x,$w_{\overline{Y}}$)* by the vector $q^v_{\overline{Y}}(x;w_{\overline{Y}})$. From Equation (7.6) in Theorem 7.1, we have that for every good $j \in \overline{Y}$ such that $q^v_j(x;w_{\overline{Y}}) > 0$,

$$\lambda^v(x;w_{\overline{Y}}) = w_j\bar{F}_j\left(q^v_j(x;w_{\overline{Y}})\right) - c_j. \tag{7.35}$$

Furthermore, from Equation (7.6), we also have that for every good $j \in N$ such that $q^v_j(w) > 0$,

$$\lambda^v(w) = w_j\bar{F}_j\left(q^v_j(w)\right) - c_j. \tag{7.36}$$

Therefore, since $\lambda^v(w) = \lambda^v(x;w_{\overline{Y}})$, from Equations (7.35) and (7.36), we have that $q^v_j(w) = q^v_j(x;w_{\overline{Y}})$ for every $j \in \overline{Y}$, implying that

$$\sum_{j\in\overline{Y}} q^v_j(w) = \sum_{j\in\overline{Y}} q^v_j(x;w_{\overline{Y}}) \tag{7.37}$$

holds. Since $\lambda^v(x;w_{\overline{Y}}) > 0$, from Theorem 7.1 it follows that $\sum_{j\in\overline{Y}} q_j^v(x;w_{\overline{Y}}) = k-x$. So that from Equation (7.37), we can conclude that $\sum_{j\in\overline{Y}} q_j^v(w) = k-x$. Since $\lambda^v(w) > 0$, from Theorem 7.1 we also have that $\sum_{j\in N} q_j^v(w) = k$. Therefore, since $\sum_{j\in N} q_j^v(w) = \sum_{t\in\overline{Y}} q_t^v(w) + \sum_{t\in Y} q_t^v(w)$, we have $\sum_{j\in Y} q_j^v(w) = x$. □

7.6.6 Proof: The supplier's price for capacity is continuous and increasing.

Proof of Corollary 7.5. Suppose $0 \le k - \sum_{t\in\overline{Y}} q_t^u(w)$. Then, there exists an x that satisfies $0 \le x \le k - \sum_{t\in\overline{Y}} q_t^u(w)$. For any such x, we have $\sum_{t\in\overline{Y}} q_t^u(w) \le k - x$. Therefore, since *SUPPLIER-RESTRICTED-TO-$\overline{Y}$-DUAL(k,x,$w_{\overline{Y}}$)* is identical to *SUPPLIER-DUAL(k − x,$w_{\overline{Y}}$)*, when $N = \overline{Y}$, we have from Theorem 7.1 that $\lambda^v(x;w_{\overline{Y}}) = 0$ when x satisfies $0 \le x \le k - \sum_{t\in\overline{Y}} q_t^u(w)$.

Suppose that x_1 and x_2 satisfy $\max\left\{k - \sum_{t\in\overline{Y}} q_t^u(w), 0\right\} \le x_1 < x_2 < k$. We show that $\lambda^v(x_1;w_{\overline{Y}}) < \lambda^v(x_2;w_{\overline{Y}})$. Assume that $\lambda^v(x_2;w_{\overline{Y}}) \le \lambda^v(x_1;w_{\overline{Y}})$ holds, instead. Denote the solution to *SUPPLIER-RESTRICTED-TO-$\overline{Y}$-PRIMAL(k,x_i,$w_{\overline{Y}}$)* by the vector $q_{\overline{Y}}^v(x_i;w_{\overline{Y}})$ for $i = 1,2$. From Equation (7.6) in Theorem 7.1, we have that for every good $j \in \overline{Y}$ such that $q_j^v(x_i;w_{\overline{Y}}) > 0$,

$$\lambda^v(x_i;w_{\overline{Y}}) = w_j \bar{F}_j\left(q_j^v(x_i;w_{\overline{Y}})\right) - c_j, \tag{7.38}$$

for $i = 1,2$. Because of our assumption on the cumulative distribution functions and $\lambda^v(x_2;w_{\overline{Y}}) \le \lambda^v(x_1;w_{\overline{Y}})$, from equation (7.38), we have

$$\sum_{j\in\overline{Y}} q_j^v(x_1;w_{\overline{Y}}) \le \sum_{j\in\overline{Y}} q_j^v(x_2;w_{\overline{Y}}). \tag{7.39}$$

Since $k - x_2 < \sum_{t\in\overline{Y}} q_t^u(w)$, from Theorem 7.1, we have $0 < \lambda^v(x_2;w_{\overline{Y}})$, implying that $0 < \lambda^v(x_1;w_{\overline{Y}})$ holds too. Therefore both $\sum_{j\in\overline{Y}} q_j^v(x_2;w_{\overline{Y}}) = k - x_2$ and $\sum_{j\in\overline{Y}} q_j^v(x_1;w_{\overline{Y}}) = k - x_1$ hold. But then equation (7.39) implies that $k - x_1 \le k - x_2$ so that $x_2 \le x_1$ holds. But this is a contradiction. Thus, the inequality $\lambda^v(x_1;w_{\overline{Y}}) < \lambda^v(x_2;w_{\overline{Y}})$ follows.

Next, we prove $\lambda^v(x; w_{\overline{Y}})$ is continuous when $x \in [0, k)$. Since $\lambda^v(x; w_{\overline{Y}}) = 0$ when x satisfies $0 \leq x \leq k - \sum_{t \in \overline{Y}} q_t^u(w)$, we need to show that $\lambda^v(x; w_{\overline{Y}})$ is continuous when x satisfies $\max\left\{k - \sum_{t \in \overline{Y}} q_t^u(w), 0\right\} \leq x < k$. Suppose that x satisfies $\max\left\{k - \sum_{t \in \overline{Y}} q_t^u(w), 0\right\} \leq x < k$. Denote the inverse of $\lambda^v(x; w_{\overline{Y}})$ by $\lambda^{-1} : [0, \max_{t \in \overline{Y}} w_t - c_t) \to [\max\left\{k - \sum_{t \in \overline{Y}} q_t^u(w), 0\right\}, k)$. (Note this exists since λ^v is strictly increasing and onto the set $[0, \max_{t \in \overline{Y}} w_t - c_t)$). Pick any number $\epsilon > 0$. Consider the neighborhoods around x defined by the radiuses $\delta_1 \stackrel{\text{def}}{=} x - \lambda^{-1}(\max\{\lambda^v(x; w_{\overline{Y}}) - \epsilon, 0\}) \geq 0$ and $\delta_2 \stackrel{\text{def}}{=} \lambda^{-1}(\min\{\lambda^v(x; w_{\overline{Y}}) + \epsilon, \max_{t \in \overline{Y}} w_t - c_t\}) - x \geq 0$. It can be shown that both δ_1 and δ_2 can not be zero. If either δ_1 or δ_2 is zero, consider the radius $\delta \stackrel{\text{def}}{=} \max\{\delta_1, \delta_2\} > 0$, otherwise we set $\delta \stackrel{\text{def}}{=} \min\{\delta_1, \delta_2\} > 0$. Denote the δ neighborhood around a number z by $N_\delta(z)$. It can be shown that if $x' \in N_\delta(x) \cap [\max\left\{k - \sum_{t \in \overline{Y}} q_t^u(w), 0\right\}, k)$, then $\lambda^v(x'; w_{\overline{Y}}) \in N_\epsilon(\lambda^v(x; w_{\overline{Y}}))$. □

7.6.7 Proof: The marginal price for capacity is increasing.

Proof of Corollary 7.6. From Corollary 7.5, when x satisfies $0 \leq x < k - \sum_{t \in \overline{Y}} q_t^u(w)$ we have $\frac{\partial \lambda^v(x; w_{\overline{Y}})}{\partial x^+} = 0$ and when x satisfies $0 < x \leq k - \sum_{t \in \overline{Y}} q_t^u(w)$ we have $\frac{\partial \lambda^v(x; w_{\overline{Y}})}{\partial x^-} = 0$.

Denote the solution to *SUPPLIER-RESTRICTED-TO-$\overline{Y}$-PRIMAL(k,x,$w_{\overline{Y}}$)* by the vector $q_{\overline{Y}}^v(x; w_{\overline{Y}})$. Let $Z(k, x, w_{\overline{Y}}) \stackrel{\text{def}}{=} \{i \mid q_i^v(x; w_{\overline{Y}}) = 0\} \subset \overline{Y}$ be the set of products that are not ordered in the supplier's decision problem. Suppose x satisfies $\max\left\{k - \sum_{t \in \overline{Y}} q_t^u(w), 0\right\} \leq x < k$. Observe that the equation $\sum_{t \in \overline{Y}} q_t^v(x; w_{\overline{Y}}) = k - x$ holds. Therefore, we can express x via the equation $x = k - \sum_{t \in \overline{Y}} q_t^v(x; w_{\overline{Y}})$. Denote the inverse of $\lambda^v(x; w_{\overline{Y}})$ by $x^v(\lambda; w_{\overline{Y}})$. Since $\lambda^v(x; w_{\overline{Y}})$ is strictly increasing and continuous, when $\frac{\partial x^v(\lambda; w_{\overline{Y}})}{\partial \lambda^+}$ exists it must be the case that $\frac{\partial \lambda^v(x; w_{\overline{Y}})}{\partial x^+} = \left(\frac{\partial x^v(\lambda; w_{\overline{Y}})}{\partial \lambda^+}\right)^{-1}$

where the equations $\lambda^v(x;w_{\overline{Y}}) = \lambda$ and $x^v(\lambda;w_{\overline{Y}}) = x$ hold. Furthermore,

$$\begin{aligned}
\frac{\partial x^v(\lambda;w_{\overline{Y}})}{\partial\lambda^+} &= \frac{\partial}{\partial\lambda^+}\left(k - \sum_{t\in\overline{Y}} q_t^v(x;w_{\overline{Y}})\right) \\
&= \frac{\partial k}{\partial\lambda^+} - \sum_{t\in\overline{Y}} \frac{\partial q_t^v(x;w_{\overline{Y}})}{\partial\lambda^+} \\
&= -\sum_{t\in\overline{Y}\setminus Z(k,x,w_{\overline{Y}})} -\frac{1}{w_t\cdot f_t\left(q_t^v(x;w_{\overline{Y}})\right)} \qquad (7.40)\\
&= \sum_{t\in\overline{Y}\setminus Z(k,x,w_{\overline{Y}})} \frac{1}{w_t\cdot f_t\left(q_t^v(x;w_{\overline{Y}})\right)}. \qquad (7.41)
\end{aligned}$$

(Equation 7.40 follows from Theorem 7.1 because from Equation (7.6) we have that for every good $j\in\overline{Y}$ such that $q_j^v(x_i;w_{\overline{Y}}) > 0$, the equation

$$\lambda^v(x;w_{\overline{Y}}) = w_j\bar{F}_j\left(q_j^v(x;w_{\overline{Y}})\right) - c_j,$$

holds.) Therefore, we have that $\frac{\partial\lambda^v(x;w_{\overline{Y}})}{\partial x^+} = \left(\sum_{t\in\overline{Y}\setminus Z(k,x,w_{\overline{Y}})} \frac{1}{w_t\cdot f_t\left(q_t^v(x;w_{\overline{Y}})\right)}\right)^{-1}$ when x satisfies $\max\left\{k - \sum_{t\in\overline{Y}} q_t^u(w), 0\right\} \le x < k$.

Suppose x satisfies $\max\left\{k - \sum_{t\in\overline{Y}} q_t^u(w), 0\right\} < x < k$. Consider the supplier's problem *SUPPLIER-RESTRICTED-TO-$\overline{Y}$-PRIMAL(k,x,$w_{\overline{Y}}$)*. Let $A(k,x,w_{\overline{Y}}) \stackrel{\text{def}}{=} \{i\in Z(k,x,w_{\overline{Y}}) \mid \lambda^v(x;w_{\overline{Y}}) = w_i - c_i\} \subset \overline{Y}$ be the set of products that were almost ordered by supplier. Then, we have

$$\begin{aligned}
\frac{\partial x^v(\lambda; w_{\overline{Y}})}{\partial \lambda^-} &= \frac{\partial}{\partial \lambda^-}\left(k - \sum_{t\in\overline{Y}} q_t^v(x; w_{\overline{Y}})\right) \\
&= \frac{\partial k}{\partial \lambda^-} - \sum_{t\in\overline{Y}} \frac{\partial q_t^v(x; w_{\overline{Y}})}{\partial \lambda^-} \\
&= -\sum_{t\in A(k,x,w_{\overline{Y}})\cup\left(\overline{Y}\setminus Z(k,x,w_{\overline{Y}})\right)} -\frac{1}{w_t \cdot f_t\left(q_t^v(x; w_{\overline{Y}})\right)} \qquad (7.42)\\
&= \sum_{t\in A(k,x,w_{\overline{Y}})\cup\left(\overline{Y}\setminus Z(k,x,w_{\overline{Y}})\right)} \frac{1}{w_t \cdot f_t\left(q_t^v(x; w_{\overline{Y}})\right)} \\
&= \sum_{t\in \overline{Y}\setminus Z(k,x,w_{\overline{Y}})} \frac{1}{w_t \cdot f_t\left(q_t^v(x; w_{\overline{Y}})\right)} + \sum_{t\in A(k,x,w_{\overline{Y}})} \frac{1}{w_t \cdot f_t(0)}. \qquad (7.43)
\end{aligned}$$

(Equation 7.42 follows from Theorem 7.1 because from Equation (7.6) we have that for every good $j \in A(k, x, w_{\overline{Y}}) \cup \left(\overline{Y} \setminus Z(k, x, w_{\overline{Y}})\right)$, the equation

$$\lambda^v(x; w_{\overline{Y}}) = w_j \bar{F}_j\left(q_j^v(x; w_{\overline{Y}})\right) - c_j, \qquad (7.44)$$

holds.) Therefore, we have that

$$\frac{\partial \lambda^v(x; w_{\overline{Y}})}{\partial x^-} = \left(\sum_{t\in A(k,x,w_{\overline{Y}})\cup\left(\overline{Y}\setminus Z(k,x,w_{\overline{Y}})\right)} \frac{1}{w_t \cdot f_t\left(q_t^v(x; w_{\overline{Y}})\right)}\right)^{-1} \qquad (7.45)$$

when x satisfies $\max\left\{k - \sum_{t\in\overline{Y}} q_t^u(w), 0\right\} < x < k$.

Suppose $\lambda^v(x; w_{\overline{Y}}) > 0$ and $\lambda^v(x; w_{\overline{Y}}) \neq w_i - c_i$ for any $i \in \overline{Y}$. From Corollary 7.5, we have that x satisfies $\max\left\{k - \sum_{t\in\overline{Y}} q_t^u(w), 0\right\} < x < k$. And from the definition of $A(k, x, w_{\overline{Y}})$ we have that $A(k, x, w_{\overline{Y}}) = \emptyset$. Therefore, from Equations (7.41) and (7.43), we have that $\frac{\partial x^v(\lambda; w_{\overline{Y}})}{\partial \lambda^+} = \frac{\partial x^v(\lambda; w_{\overline{Y}})}{\partial \lambda^-}$ which implies that $\frac{\partial \lambda^v(x; w_{\overline{Y}})}{\partial x^+} = \frac{\partial \lambda^v(x; w_{\overline{Y}})}{\partial x^-}$. From Corollary 7.5, we have $\lambda^v(x; w_{\overline{Y}})$ is continuous at x, implying that $x^v(\lambda; w_{\overline{Y}})$ is continuous at λ (where the equations $\lambda^v(x; w_{\overline{Y}}) = \lambda$ and $x^v(\lambda; w_{\overline{Y}}) = x$ hold). Therefore, from equation (7.44), we have that $q_t^v\left(x^v(\lambda; w_{\overline{Y}}); w_{\overline{Y}}\right)$ is continuous at λ. Since $\frac{\partial x^v(\lambda; w_{\overline{Y}})}{\partial \lambda} = \sum_{t\in\overline{Y}\setminus Z(k,x,w_{\overline{Y}})} \frac{1}{w_t \cdot f_t\left(q_t^v(x; w_{\overline{Y}})\right)}$ and the p.d.f. f_t is continuous for every good $t \in Y$, we have that $\frac{\partial x^v(\lambda; w_{\overline{Y}})}{\partial \lambda}$ is continuous at λ since the set of goods $\overline{Y} \backslash Z(k, x, w_{\overline{Y}})$ do not change for small changes in λ. Therefore, $\frac{\partial \lambda^v(x; w_{\overline{Y}})}{\partial x}$ is continuous at x.

Suppose x satisfies the equation $\max\left\{k - \sum_{t \in \overline{Y}} q_t^u(w), 0\right\} \leq x < k$ and the retailer is service constrained for good(s) $\overline{Y}$. From equation (6.46), we have that for every good $t \in \overline{Y} \backslash Z(k, x, w_{\overline{Y}})$, the order quantity $q_t^v\left(x^v(\lambda; w_{\overline{Y}}); w_{\overline{Y}}\right)$ is strictly decreasing in λ. Since the retailer is service constrained for good t, we have $f_t\left(q_t^v\left(x^v(\lambda; w_{\overline{Y}}); w_{\overline{Y}}\right)\right)$ is strictly increasing in λ. Therefore, $\frac{\partial x^v(\lambda; w_{\overline{Y}})}{\partial \lambda^+}$ and $\frac{\partial x^v(\lambda; w_{\overline{Y}})}{\partial \lambda^-}$ are strictly decreasing in λ. So that we have $\frac{\partial \lambda^v(x; w_{\overline{Y}})}{\partial x^+}$ and $\frac{\partial \lambda^v(x; w_{\overline{Y}})}{\partial x^-}$ are strictly increasing in x. □

7.6.8 Proof: Partitioning the set of wholesale prices by 'capacity charge'.

Proof of Theorem 7.4. Suppose $\lambda \geq 0$. Consider a wholesale price vector w from the set

$$\mathcal{W}(\lambda) \stackrel{\text{def}}{=} \left\{ w \;\middle|\; \begin{array}{l} w_t = (c_t + \lambda - \gamma_t \cdot 1_{\{q_t = 0\}})/\bar{F}_t(q_t) \quad \forall t \in N, \;\; q, \gamma \in \mathbb{R}_+^{|N|}, \\ \sum_{t \in N} q_t = \min\left\{\left(\sum_{t \in N} q_t^u(w)\right) \cdot 1_{\{\lambda = 0\}} + k \cdot 1_{\{\lambda > 0\}}, \;\; k\right\} \end{array} \right\}. \quad (7.46)$$

For this vector w there exist vectors q and γ that satisfy the conditions in (7.46) which guarantee w's membership in the set $\mathcal{W}(\lambda)$. From the proof of Theorem 7.1, we have that The Karush-Kuhn-Tucker conditions for the supplier's decision problem, *SUPPLIER-PRIMAL(k,w)*, are:

$$w_t \bar{F}_t(\widehat{q}_t) - c_t + \widehat{\gamma}_t - \widehat{\lambda} = 0, \quad t = 1, \ldots, n; \quad (7.47)$$

$$\widehat{q}_t \geq 0, \quad t = 1, \ldots, n;$$

$$k - \sum_{t=1}^{n} \widehat{q}_t \geq 0;$$

$$\widehat{\gamma}_t \widehat{q}_t = 0, \quad t = 1, \ldots, n; \quad (7.48)$$

$$\widehat{\lambda}\left(k - \sum_{t=1}^{n} \widehat{q}_t\right) = 0; \tag{7.49}$$

$$\widehat{\lambda} \geq 0; \quad \widehat{\gamma}_t \geq 0, \quad t = 1, \ldots, n.$$

Because of the concavity of the objective function and the fact that the Slater condition is satisfied, the Karush-Kuhn-Tucker conditions are both necessary and sufficient conditions for any primal optimal vector $\widehat{q}$ and dual optimal vector $(\widehat{\gamma}, \widehat{\lambda})$. Consider a particular value for $\widehat{q}$, $\widehat{\gamma}$, and $\widehat{\lambda}$. In particular, suppose we set:

$$\widehat{q} = q,$$
$$\widehat{\gamma} = (\gamma_1 \cdot 1_{\{q_1=0\}}, \ldots, \gamma_n \cdot 1_{\{q_n=0\}}),$$
$$\widehat{\lambda} = \lambda.$$

It can be shown that these values for $\widehat{q}$, $\widehat{\gamma}$, and $\widehat{\lambda}$ satisfy the Karush-Kuhn-Tucker conditions so that $\lambda^v(w) = \widehat{\lambda} = \lambda$. (There are two main steps in seeing this. First, consider the cases $\lambda = 0$ and $\lambda > 0$ separately. Then, for each of those cases, confirm that the equations (7.47), (7.48), and (7.49) are satisfied for these values $\widehat{q}$, $\widehat{\gamma}$, $\widehat{\lambda}$ when the wholesale price for good t is $w_t = (c_t + \lambda - \gamma_t \cdot 1_{\{q_t=0\}})/\bar{F}_t(q_t)$.) Therefore, the wholesale price vector w induces the retailer to have shadow price λ for the capacity constraint k, i.e., $\lambda^v(w) = \lambda$.

Next, we prove the converse. Suppose a wholesale price vector w induces retailer shadow price $\widehat{\lambda}$ for capacity k, i.e., $\lambda^v(w) = \widehat{\lambda}$. Therefore, there exist vectors $\widehat{q}$ and $\widehat{\gamma}$ that along with $\widehat{\lambda}$ and wholesale price vector w satisfy the Karush-Kuhn-Tucker conditions. Suppose we set: $q = \widehat{q}$, $\gamma = \widehat{\gamma}$, and $\lambda = \widehat{\lambda}$. It can be shown that these values for q, γ, and λ enable w's membership in $\mathcal{W}(\lambda)$ using the conditions in (7.46). Therefore, $w \in \mathcal{W}(\lambda) = \mathcal{W}(\widehat{\lambda})$. $\square$

7.6.9 Proof: Partitioning the 'binding' wholesale prices by 'induced allocation'.

Proof of Theorem 7.5. Suppose we have an order/stocking quantity vector $q \in \mathbb{R}_+^{|N|}$ such that the condition $\sum_{t \in N} q_t = k$ holds. Consider a wholesale price vector w from the set

$$\mathcal{A}(q) \stackrel{\text{def}}{=} \left\{ w \mid w_t = (c_t + \lambda - \gamma_t \cdot 1_{\{q_t=0\}})/\bar{F}_t(q_t) \ \ \forall t \in N, \ \lambda \in \mathbb{R}_+, \ \gamma \in \mathbb{R}_+^{|N|} \right\}. \tag{7.50}$$

For this vector w there exists a scalar λ and a vector γ that satisfy the conditions in (7.50), guaranteeing w's membership in the set $\mathcal{A}(q)$. The Karush-Kuhn-Tucker conditions for the supplier's decision problem, *SUPPLIER-PRIMAL(k,w)*, written in the proof of Theorem 7.4, are both necessary and sufficient conditions for any primal optimal vector $\widehat{q}$ and dual optimal vector $(\widehat{\gamma}, \widehat{\lambda})$. Consider a particular value for $\widehat{q}$, $\widehat{\gamma}$, and $\widehat{\lambda}$. In particular, suppose we set:

$$\begin{aligned} &\widehat{q} = q, \\ &\widehat{\gamma} = (\gamma_1 \cdot 1_{\{q_1=0\}}, \ldots, \gamma_n \cdot 1_{\{q_n=0\}}), \\ &\widehat{\lambda} = \lambda. \end{aligned}$$

It can be shown that these values for $\widehat{q}$, $\widehat{\gamma}$, and $\widehat{\lambda}$ satisfy the Karush-Kuhn-Tucker conditions so that $q^v(w) = \widehat{q} = q$. (To see this: verify that the equations (7.47), (7.48), and (7.49) are satisfied for these values $\widehat{q}$, $\widehat{\gamma}$, $\widehat{\lambda}$ when the wholesale price for good t is $w_t = (c_t + \lambda - \gamma_t \cdot 1_{\{q_t=0\}})/\bar{F}_t(q_t)$.) Therefore, the wholesale price vector w induces the retailer to order according to the vector q, i.e., $q^v(w) = q$.

Next, we prove the converse. Suppose a wholesale price vector w induces retailer to order according to the vector $\widehat{q}$, i.e., $q^v(w) = \widehat{q}$. Therefore, there exists a vector $\widehat{\gamma}$ and a scalar $\widehat{\lambda}$ that along with $\widehat{q}$ and the wholesale price vector w satisfy the Karush-Kuhn-Tucker conditions. Suppose we set: $q = \widehat{q}$, $\gamma = \widehat{\gamma}$, and $\lambda = \widehat{\lambda}$. It can be shown that these values for q, γ, and λ enable w's membership in $\mathcal{A}(q)$ using the conditions in (7.50). Therefore, $w \in \mathcal{A}(q) = \mathcal{A}(\widehat{q})$. □

7.6.10 Proof: The shadow price for retailer Y's aggregate induced order.

Proof of Theorem 7.6. Recall the definition $g_i(x) \stackrel{\text{def}}{=} x \cdot f(x)/\bar{F}_i(x)$ of the generalized failure rate function. For each good $i \in Y$, we have that

$$\frac{\partial}{\partial q_i}\left(\left(p_i - \frac{\lambda^v\left(x; w_{\overline{Y}}\right) + c_i}{\bar{F}_i(q_i)}\right) \cdot E[S_i(q_i)]\right) = p_i\bar{F}_i\left(q_i\right) - \left(c_i + \lambda^v\left(x; w_{\overline{Y}}\right)\right) \cdot \left(1 + \frac{E[S_i(q_i)]}{\bar{F}_i(q_i)}\frac{f_i(q_i)}{\bar{F}_i(q_i)}\right).$$

Each c.d.f. F_i is strictly increasing over $[0, l_i]$, continuously differentiable, and has the IGFR property, so that $E[S_i(q_i)] \cdot f_i(q_i)/(\bar{F}_i(q_i))^2$ is continuous, nonnegative, and increasing in q_i (see Lemma 1 in Cachon (2004) for the proof). For good $i \in Y$, we define the order quantity $\widehat{q}_i$ in terms of q_i^e, the equilibrium induced order for good i in the unconstrained setting (see equation (7.4)), as follows: $\widehat{q}_i \stackrel{\text{def}}{=} \min\{q_i^e, k\}$. Observe that the function

$$\left(p_i - \frac{\lambda^v\left(x; w_{\overline{Y}}\right) + c_i}{\bar{F}_i(q_i)}\right) \cdot E[S_i(q_i)]$$

is strictly concave for $q_i \in [0, \widehat{q}_i]$ (and that any value that maximizes the function and respects the capacity constraint must be in the set $[0, \widehat{q}_i]$). Therefore, the objective function for *Y-RETAILER-PRICING-PRIMAL(x,$w_{\overline{Y}}$)* is strictly concave for $q_Y \in \{q \in \mathbb{R}_+^{|Y|} \mid 0 \le q_i \le \widehat{q}_i\}$ which is a superset of the feasible set for *Y-RETAILER-PRICING-PRIMAL(x,$w_{\overline{Y}}$)* (since $x \in \left(0, \min\left\{\sum_{t\in Y} q_t^e, k\right\}\right]$). Because the feasible set is convex and compact, *Y-RETAILER-PRICING-PRIMAL(x,$w_{\overline{Y}}$)* has a unique solution.

Consider the Lagrangian $\mathcal{L}(q_Y, \gamma_1, \ldots, \gamma_{|Y|}, \lambda)$ for *Y-RETAILER-PRICING-PRIMAL(x,$w_{\overline{Y}}$)*:

$$\mathcal{L}(q_Y, \gamma_1, \ldots, \gamma_{|Y|}, \lambda) = \sum_{i\in Y}\left(p_i - \frac{\lambda^v\left(x; w_{\overline{Y}}\right) + c_i}{\bar{F}_i(q_i)}\right) \cdot E[S_i(q_i)] + \sum_{i\in Y}\gamma_i q_i + \lambda\left(x - \sum_{i\in Y} q_i\right).$$

The Karush-Kuhn-Tucker conditions for retailer Y's decision problem, *Y-RETAILER-*

PRICING-PRIMAL$(x, w_{\overline{Y}})$, are:

$$p_t\bar{F}_t(q_t) - (c_t + \lambda^v(x; w_{\overline{Y}})) \cdot \left(1 + \frac{E[S_t(q_t)]}{\bar{F}_t(q_t)} \frac{f_t(q_t)}{\bar{F}_t(q_t)}\right) + \gamma_t - \lambda = 0, \quad t \in Y; \quad (7.51)$$

$$q_t \geq 0, \quad t \in Y;$$

$$x - \sum_{t \in Y} q_t = 0;$$

$$\gamma_t q_t = 0, \quad t \in Y; \quad (7.52)$$

$$\gamma_t \geq 0, \quad t \in Y.$$

Since $x \neq 0$, it can be shown that a constraint qualification condition on a particular matrix (each row of which is the gradient of an effective constraint at the optimal order vector) is satisfied. Briefly, the constraint qualification condition requires that the matrix have rank equal to the number of effective constraints. See Sundaram (1996, Chap. 6, Thm 6.10, p.165) for a detailed description of the constraint qualification condition. Therefore, the Karush-Kuhn-Tucker conditions are necessary for any primal optimal vector q_Y. Furthermore, because of the concavity of the objective function and the functions that define the constraints, the Karush-Kuhn-Tucker conditions are sufficient conditions for any primal optimal vector q_Y.

As a result, since the primal problem has a unique solution, it can be shown that the dual problem has a unique solution using these conditions. Let

$$\left(\left(q_i^Y(x; w_{\overline{Y}})\right)_{i \in Y}, \left(\gamma_i^Y(x; w_{\overline{Y}})\right)_{i \in Y}, \lambda^Y(x; w_{\overline{Y}})\right)$$

denote the unique vector that satisfies the Karush-Kuhn-Tucker conditions.

When $j \in Y \setminus Z^Y(x; w_{\overline{Y}})$, from equation (7.52) we have $\gamma_j^Y(x; w_{\overline{Y}}) = 0$. Therefore, from equation (7.51) we have

$$\lambda^Y(x; w_{\overline{Y}}) = p_j\bar{F}_j\left(q_j^Y(x; w_{\overline{Y}})\right) - (c_j + \lambda^v(x; w_{\overline{Y}})) \cdot \left(1 + \frac{E[S_j(q_j^Y(x; w_{\overline{Y}}))]}{\bar{F}_j(q_j^Y(x; w_{\overline{Y}}))} \frac{f_j(q_j^Y(x; w_{\overline{Y}}))}{\bar{F}_j(q_j^Y(x; w_{\overline{Y}}))}\right).$$

When $i \in Z^Y(x; w_{\overline{Y}})$, from equation (7.51), we have

$$\lambda^Y(x; w_{\overline{Y}}) = p_i - c_i - \lambda^v(x; w_{\overline{Y}}) + \gamma_i^Y(x; w_{\overline{Y}}) \geq p_i - c_i - \lambda^v(x; w_{\overline{Y}}).$$

Thus, the conditions in equations (7.16) and (7.17) hold.

Furthermore, suppose $x = \sum_{t \in Y} q_t^e \leq k$. Assume that the inequality

$$\lambda^Y(x; w_{\overline{Y}}) + \lambda^v(x; w_{\overline{Y}}) > 0$$

holds, instead. Therefore, for good $t \in Y$, if we substitute $q_t^Y(x; w_{\overline{Y}})$ into equation (7.4) we have

$$p_t \bar{F}_t(q_t^Y(x; w_{\overline{Y}})) - c_t \left(1 + \frac{E[S_t(q_t^Y(x; w_{\overline{Y}}))]}{\bar{F}_t(q_t^Y(x; w_{\overline{Y}}))} \cdot \frac{f_t(q_t^Y(x; w_{\overline{Y}}))}{\bar{F}_t(q_t^Y(x; w_{\overline{Y}}))}\right) > 0, \quad t = 1, \ldots, n. \tag{7.53}$$

But this is a contradiction, because equation (7.53) implies that the unconstrained supplier should stock more than $\sum_{t \in N} q_t^e$ units of good in aggregate. And, so we have that the inequality

$$\lambda^Y(x; w_{\overline{Y}}) + \lambda^v(x; w_{\overline{Y}}) \leq 0$$

holds when $x = \sum_{t \in Y} q_t^e \leq k$.

Suppose that x_1 and x_2 satisfy $0 < x_1 < x_2 \leq \min\left\{\sum_{t \in Y} q_t^e, k\right\}$. We show that $\lambda^Y(x_2; w_{\overline{Y}}) + \lambda^v(x_2; w_{\overline{Y}}) < \lambda^Y(x_1; w_{\overline{Y}}) + \lambda^v(x_1; w_{\overline{Y}})$. Assume that $\lambda^Y(x_1; w_{\overline{Y}}) + \lambda^v(x_1; w_{\overline{Y}}) \leq \lambda^Y(x_2; w_{\overline{Y}}) + \lambda^v(x_2; w_{\overline{Y}})$ holds, instead. From Equation (7.16) in Theorem 7.6, we have that for every good $j \in Y$ such that $q_j^Y(x_i; w_{\overline{Y}}) > 0$,

$$\lambda^Y(x_i; w_{\overline{Y}}) + \lambda^v(x_i; w_{\overline{Y}}) = p_j \bar{F}_j\left(q_j^Y(x_i; w_{\overline{Y}})\right) - (c_j + \lambda^v(x_i; w_{\overline{Y}})) \frac{E[S_j(q_j^Y(x_i; w_{\overline{Y}}))]}{\bar{F}_j(q_j^Y(x_i; w_{\overline{Y}}))} \frac{f_j(q_j^Y(x_i; w_{\overline{Y}}))}{\bar{F}_j(q_j^Y(x_i; w_{\overline{Y}}))} - c_j, \tag{7.54}$$

for $i = 1, 2$. Because of our assumption on the cumulative distribution functions and $\lambda^Y(x_1; w_{\overline{Y}}) + \lambda^v(x_1; w_{\overline{Y}}) \leq \lambda^Y(x_2; w_{\overline{Y}}) + \lambda^v(x_2; w_{\overline{Y}})$, from equation (7.54), we must have $q_j^Y(x_2; w_{\overline{Y}}) \leq q_j^Y(x_1; w_{\overline{Y}})$ for every good $j \in Y$. So that

$$x_2 = \sum_{j \in Y} q_j^Y(x_2; w_{\overline{Y}}) \leq \sum_{j \in Y} q_j^Y(x_1; w_{\overline{Y}}) = x_1. \tag{7.55}$$

But this is a contradiction because $x_1 < x_2$ holds. Thus, the inequality $\lambda^Y(x_2; w_{\overline{Y}}) + \lambda^v(x_2; w_{\overline{Y}}) < \lambda^Y(x_1; w_{\overline{Y}}) + \lambda^v(x_1; w_{\overline{Y}})$ follows. And so we have that the function $\lambda^Y(x; w_{\overline{Y}}) + \lambda^v(x; w_{\overline{Y}})$ is strictly decreasing as $x \in \left(0, \min\left\{\sum_{t \in Y} q_t^e, k\right\}\right]$ increases. □

7.6.11 Proof: Any induced aggregate order above $\bar{x}$ is not optimal.

Proof of Corollary 7.7. Suppose that for every good $t \in Y$, the c.d.f. F_t has the IGFR property. Assume that the conditions

$$\lambda^Y\left(\min\left\{\sum_{t \in Y} q_t^e, k\right\}; w_{\overline{Y}}\right) \leq 0, \tag{7.56}$$

$$0 < \max\left\{p_i - c_i - \lambda^v\left(0; w_{\overline{Y}}\right) \mid i \in Y\right\}, \tag{7.57}$$

hold. From Corollary 7.5, we know $\lambda^v(x; w_{\overline{Y}})$ is continuous at $x = 0$. Furthermore, the cumulative distribution functions are continuously differentiable. Therefore, from equation (7.57), we have that there exists some small positive value $\delta < \min\left\{\sum_{t \in Y} q_t^e, k\right\}$ such that the condition

$$0 < \max\left\{p_i \bar{F}_i(\delta) - \left(c_i + \lambda^v\left(\delta; w_{\overline{Y}}\right)\right) \cdot \left(1 + \frac{E[S_i(\delta)]}{\bar{F}_i(\delta)} \frac{f_i(\delta)}{\bar{F}_i(\delta)}\right) \mid i \in Y\right\}$$

holds. And so, from Theorem 7.6, it follows that there exists a small positive value $\hat{x} < \delta$ such that $\lambda^Y(\hat{x}; w_{\overline{Y}}) > 0$. Using a technique similar to our proof that $\lambda^v(x; w_{\overline{Y}})$ is continuous (i.e., Corollary 7.5), it can be shown that $\lambda^Y(x; w_{\overline{Y}})$ is continuous for $x \in \left(0, \min\left\{\sum_{t \in Y} q_t^e, k\right\}\right]$. And from Theorem 7.6, it follows that $\lambda^Y\left(x; w_{\overline{Y}}\right)$ is strictly decreasing because we know that $\lambda^v(x; w_{\overline{Y}})$ is nondecreasing from Corollary 7.5. Therefore, from equation (7.56), we have that there exists a value $\bar{x}$, where $\hat{x} < \bar{x} \le \min\left\{\sum_{t\in Y} q_t^e, k\right\}$, that satisfies the equation $\lambda^Y(\bar{x}; w_{\overline{Y}}) = 0$. For any unit above $\bar{x}$ that retailer Y induces the supplier to stock (in aggregate), the retailer incurs a loss because the marginal profit on the x^{th} unit is upper bounded by $\lambda^Y(x; w_{\overline{Y}})$, which is a negative number for any $x > \bar{x}$. Therefore, retailer Y would never induce the supplier to stock more than $\bar{x}$ units of good(s) Y in aggregate. Furthermore, from Theorem 7.6, we have that $\lambda^Y(x; w_{\overline{Y}}) \le 0$ when $x = \sum_{t\in Y} q_t^e \le k$ and that $\lambda^Y(x; w_{\overline{Y}})$ is strictly decreasing. Therefore, we have $\bar{x} \le \sum_{t\in Y} q_t^e$.

Now assume that the conditions

$$0 < \lambda^Y\left(\min\left\{\sum_{t\in Y} q_t^e, k\right\}; w_{\overline{Y}}\right), \tag{7.58}$$

$$0 < \max\left\{p_i - c_i - \lambda^v\left(0; w_{\overline{Y}}\right) \mid i \in Y\right\}, \tag{7.59}$$

hold instead. From Theorem 7.6, we have that $\lambda^Y(x; w_{\overline{Y}}) \le 0$ when $x = \sum_{t\in Y} q_t^e \le k$ and that $\lambda^Y(x; w_{\overline{Y}})$ is strictly decreasing. Therefore, from equation (7.58) we have $\min\left\{\sum_{t\in Y} q_t^e, k\right\} < \sum_{t\in Y} q_t^e$, implying $\min\left\{\sum_{t\in Y} q_t^e, k\right\} = k$. Suppose we define $\bar{x}$ to equal $\min\left\{\sum_{t\in Y} q_t^e, k\right\} = k$. It follows trivially that retailer Y would never induce the supplier to stock more than $\bar{x}$ units of good(s) Y in aggregate because the supplier has a capacity constraint of k units.

Finally, assume the condition

$$\max\left\{p_i - c_i - \lambda^v\left(0; w_{\overline{Y}}\right) \mid i \in Y\right\} \le 0,$$

holds instead, and that we define $\bar{x} \overset{\text{def}}{=} 0$. Therefore, for any unit above $\bar{x}$ that retailer Y induces the supplier to stock (in aggregate), the retailer incurs a loss because the marginal profit on the x^{th} unit is upper bounded by $\lambda^Y(x; w_{\overline{Y}})$, which is a negative number for any $x > \bar{x}$. And so, retailer Y would never induce the supplier to stock more than $\bar{x}$ units of good(s) Y in aggregate. □

7.6.12 Proof: Optimal aggregate order for a fixed $w_{\overline{Y}}$.

Proof of Theorem 7.7. The (Weierstrass) Extreme Value Theorem says that for any continuous and real function f on a compact metric space X, there exists a point $x^* \in X$ such $f(x^*) = \sup_{x \in X} f(x)$. (Rudin 1976, Theorem 4.16) From Corollary 7.5, we have that $\lambda^v \left(\sum_{t \in Y} q_t; w_{\overline{Y}}\right)$ is continuous on the set of feasible capacity allocation vectors, $Q \stackrel{\text{def}}{=} \{q \in \mathbb{R}^n_+ \mid \sum_{t=1}^n q_t \leq k\}$. Therefore, we have that retailer Y's objective function, $\sum_{i \in Y} \left(p_i - \left(\lambda^v(\sum_{t \in Y} q_t; w_{\overline{Y}}) + c_i\right)/\bar{F}_i(q_i)\right) \cdot E[S_i(q_i)]$, when solving *Y-RETAILER-INDUCING-AGGREGATE-ORDER($w_{\overline{Y}}$)*, is continuous on Q (because it consists of a finite sum of products of continuous functions). The set Q is closed and bounded (and hence compact) and is a metric space (under the Euclidean metric). Therefore, applying the (Weierstrass) Extreme Value Theorem, we have that an optimal aggregate order quantity $x^Y(w_{\overline{Y}})$ and optimal induced order vector $q^Y(w_{\overline{Y}})$ *exist* for the problem *Y-RETAILER-INDUCING-AGGREGATE-ORDER($w_{\overline{Y}}$)*.

From Theorem 7.6 we have that for any fixed value x, the optimal vector of goods that retailer Y induces the supplier to stock is $q^Y(x; w_{\overline{Y}})$, the solution to the decision problem *Y-RETAILER-PRICING-PRIMAL(x,$w_{\overline{Y}}$)*. Therefore, we can re-express the objective function

$$\sum_{i \in Y} \left(p_i - \frac{\lambda^v \left(\sum_{t \in Y} q_t; w_{\overline{Y}}\right) + c_i}{\bar{F}_i(q_i)} \right) \cdot E[S_i(q_i)]$$

for the problem *Y-RETAILER-INDUCING-AGGREGATE-ORDER($w_{\overline{Y}}$)* as

$$\sum_{i \in Y} \left(p_i - \frac{\lambda^v (x; w_{\overline{Y}}) + c_i}{\bar{F}_i(q_i^Y(x; w_{\overline{Y}}))} \right) \cdot E[S_i(q_i^Y(x; w_{\overline{Y}}))]$$

so that the only decision variable we need to solve for is x.

Recall from Corollary 7.6 that $\lambda^v(x; w_{\overline{Y}})$ is not differentiable everywhere. However, from the proof of Corollary 7.6, we know that both the derivative from the right and left do exist for $\lambda^v(x; w_{\overline{Y}})$ (and are equal almost everywhere except at $|\overline{Y}| - 1$ points at most). Therefore, we can write

$$
\begin{aligned}
&\frac{\partial}{\partial x^-}\left(\sum_{i\in Y}\left(p_i-\frac{\lambda^v(x;w_{\overline{Y}})+c_i}{\bar{F}_i(q_i^Y(x;w_{\overline{Y}}))}\right)\cdot E[S_i(q_i^Y(x;w_{\overline{Y}}))]\right)\\
&=\frac{\partial}{\partial x^-}\left(\sum_{i\in Y\setminus Z^Y(x;w_{\overline{Y}})}\left(p_i-\frac{\lambda^v(x;w_{\overline{Y}})+c_i}{\bar{F}_i(q_i^Y(x;w_{\overline{Y}}))}\right)\cdot E[S_i(q_i^Y(x;w_{\overline{Y}}))]\right)\\
&=\sum_{i\in Y\setminus Z^Y(x;w_{\overline{Y}})}\frac{\partial}{\partial x^-}\left(\left(p_i-\frac{c_i}{\bar{F}_i(q_i^Y(x;w_{\overline{Y}}))}\right)\cdot E[S_i(q_i^Y(x;w_{\overline{Y}}))]\right)\\
&\qquad-\frac{\partial}{\partial x^-}\left(\lambda^v(x;w_{\overline{Y}})\cdot\frac{E[S_i(q_i^Y(x;w_{\overline{Y}}))]}{\bar{F}_i(q_i^Y(x;w_{\overline{Y}}))}\right)\\
&=\sum_{i\in Y\setminus Z^Y(x;w_{\overline{Y}})}\frac{\partial}{\partial q_i^Y}\left(\left(p_i-\frac{\lambda^v(x;w_{\overline{Y}})+c_i}{\bar{F}_i(q_i^Y(x;w_{\overline{Y}}))}\right)\cdot E[S_i(q_i^Y(x;w_{\overline{Y}}))]\right)\cdot\frac{\partial q_i^Y}{\partial x^-}\\
&\qquad-\frac{\partial\lambda^v(x;w_{\overline{Y}})}{\partial x^-}\cdot\frac{E[S_i(q_i^Y(x;w_{\overline{Y}}))]}{\bar{F}_i(q_i^Y(x;w_{\overline{Y}}))}\\
&=\sum_{i\in Y\setminus Z(k,x,w_{\overline{Y}})}\left(p_i\bar{F}_i\left(q_i^Y(x;w_{\overline{Y}})\right)-\left(c_i+\lambda^v(x;w_{\overline{Y}})\right)\cdot\frac{\partial}{\partial q_i^Y}\left(\frac{E[S_i(q_i^Y(x;w_{\overline{Y}}))]}{\bar{F}_i(q_i^Y(x;w_{\overline{Y}}))}\right)\right)\cdot\frac{\partial q_i^Y}{\partial x^-}\\
&\qquad-\frac{\partial\lambda^v(x;w_{\overline{Y}})}{\partial x^-}\cdot\frac{E[S_i(q_i^Y(x;w_{\overline{Y}}))]}{\bar{F}_i(q_i^Y(x;w_{\overline{Y}}))}\\
&=\sum_{i\in Y\setminus Z(k,x,w_{\overline{Y}})}\lambda^Y(x;w_{\overline{Y}})\cdot\frac{\partial q_i^Y}{\partial x^-}-\frac{\partial\lambda^v(x;w_{\overline{Y}})}{\partial x^-}\cdot\frac{E[S_i(q_i^Y(x;w_{\overline{Y}}))]}{\bar{F}_i(q_i^Y(x;w_{\overline{Y}}))}\\
&=\left(\sum_{i\in Y\setminus Z(k,x,w_{\overline{Y}})}\frac{\partial q_i^Y}{\partial x^-}\right)\cdot\lambda^Y(x;w_{\overline{Y}})-\left(\sum_{i\in Y\setminus Z(k,x,w_{\overline{Y}})}\frac{E[S_i(q_i^Y(x;w_{\overline{Y}}))]}{\bar{F}_i(q_i^Y(x;w_{\overline{Y}}))}\right)\cdot\frac{\partial\lambda^v(x;w_{\overline{Y}})}{\partial x^-}\\
&=\lambda^Y(x;w_{\overline{Y}})-\left(\sum_{i\in Y}\frac{E[S_i(q_i^Y(x;w_{\overline{Y}}))]}{\bar{F}_i(q_i^Y(x;w_{\overline{Y}}))}\right)\cdot\frac{\partial\lambda^v(x;w_{\overline{Y}})}{\partial x^-}
\end{aligned}
\tag{7.60}
$$

Since $\bar{x}\neq 0$ and the cumulative distribution function for demand of each good $y\in Y$ has the IGFR property, from Corollary 7.7 we have that $\lambda^Y(x;w_{\overline{Y}})$ is nonnegative for every $x\in[0,\bar{x}]$. Furthermore, from Theorem 7.6 we have that $\lambda^Y(x;w_{\overline{Y}})$ is strictly decreasing as x increases. Since the retailer is service constrained for good(s) $\overline{Y}$, from Corollary 7.5 and Corollary 7.6 we have that $\frac{\partial\lambda^v(x;w_{\overline{Y}})}{\partial x^-}$ is nonnegative and nondecreasing. Furthermore, from Theorem 7.6 we have that for any good $i\in Y$, the function $q_i^Y(x;w_{\overline{Y}})$ is nondecreasing as x increases, when the c.d.f. F_i has the IGFR property, so that the function $E[S_i(q_i^Y(x;w_{\overline{Y}}))]/\bar{F}_i(q_i^Y(x;w_{\overline{Y}}))$ is nondecreasing as x increases. Therefore, from equation (7.60), we have that retailer Y's objective function is concave in the induced aggregate order x. And so, equation (7.19) holds. From the proof of Theorem 7.6, we have that equation (7.20) holds. □

7.6.13 Proof: Characterizing $\mathcal{W}_Y^{\mathbf{br}}(w_{\overline{Y}})$.

We apply Berge's Maximum Theorem (Proposition 6.1) in the proof of Lemma 6.1, below. See Section 6.6.13 for a statement of that Theorem. Now we state and prove Lemma 7.1 (for use in the proof of Theorem 7.8 and Theorem 7.9).

LEMMA 7.1. *Retailer Y's objective function,*

$$\pi_Y(w_Y, w_{\overline{Y}}) \stackrel{def}{=} \sum_{i \in Y} (p_i - w_i) \cdot E[S_i(q_i^v(w_Y, w_{\overline{Y}}))],$$

when solving Y-RETAILER($w_{\overline{Y}}$), is continuous in the vector $(w_Y, w_{\overline{Y}})$.

Proof of Lemma 7.1. First, we show that for any good $t \in N$, the supplier's induced stocking quantity $q_t^v(w_Y, w_{\overline{Y}})$ is continuous in the vector $(w_Y, w_{\overline{Y}})$. Denote the set of feasible order quantity vectors by $Q \stackrel{\text{def}}{=} \{q \in \mathbb{R}_+^n \mid \sum_{t=1}^n q_t \leq k\}$ and the set of feasible wholesale price vectors by $W \stackrel{\text{def}}{=} \prod_{t \in N}[c_t, p_t]$. Consider the function $f : Q \times W \to \mathbb{R}$ defined by the equation $f(q, w) \stackrel{\text{def}}{=} \sum_{i=1}^n w_i E[S_i(q_i)] - c_i q_i$ and the correspondence $\mathcal{D} : W \to 2^Q$ defined by the equation $\mathcal{D}(w) = Q$. For any good $t \in N$, the expected sales $E[S_t(q_t)]$, when the supplier stocks q_t units, equals $q_t \cdot \bar{F}_t(q_t) + \int_0^{q_t} x \cdot f_t(x)\, dx = \int_0^{q_t} \bar{F}_t(x)\, dx$ (by using integration by parts). Since $\bar{F}_t(x)$ is continuous on Q, we have that $E[S_t(q_t)] = \int_0^{q_t} \bar{F}_t(x)\, dx$ is continuous on Q (Rudin 1976, Theorem 6.20), so that the function f is continuous on $Q \times W$ (since f involves finite sums and products of continuous functions). Furthermore, the correspondence $\mathcal{D}$ is compact-valued and continuous, because for any wholesale price vector $w \in W$ the equation $\mathcal{D}(w) = Q$ holds. Therefore, from Proposition 6.1, we have that the correspondence $\mathcal{D}^*$ (as defined in Equation (6.63)) is compact-valued and upper-hemicontinous on W. However, every order quantity vector in the set $\mathcal{D}^*(w)$ is a solution to *SUPPLIER-PRIMAL(k,w)* and in the proof of Theorem 7.1, we showed that *SUPPLIER-PRIMAL(k,w)* has a unique solution, $q_t^v(w)$. Therefore, $\mathcal{D}^*(w)$ is single-valued (for any $w \in W$) and equals $q_t^v(w)$. Since $\mathcal{D}^*$ is single-valued and upper-hemicontinous on W, it must, therefore, be continuous on W, implying that the function q_t^v is continuous on W. Therefore, the function $E[S_t(q_t^v(w_Y, w_{\overline{Y}}))]$ is continuous on W. Furthermore, since retailer Y's profit $\pi_Y(w)$ is a finite sum of products of continuous functions on W, the function $\pi_Y(w)$ is also continuous on W. □

Proof of Theorem 7.8. The (Weierstrass) Extreme Value Theorem says that for any continuous and real function f on a compact metric space X, there exists a point $x^* \in X$ such $f(x^*) = \sup_{x \in X} f(x)$. (Rudin 1976, Theorem 4.16) Since, the hypercube $\prod_{t \in Y}[c_t, p_t]$ is closed and bounded (and hence compact) as well as a metric space (under the Euclidean metric), and since the retailer's objective function is (real) continuous in its decision vector w_Y (from Lemma 7.1), we, therefore, have (by applying the Extreme Value Theorem) that retailer Y can attain the supremum of its objective function (over its constraint set) from a vector in its constraint set, i.e., the hypercube $\prod_{t \in Y}[c_t, p_t]$, implying that the solution set $\mathcal{W}_Y^{\text{br}}(w_{\overline{Y}})$ is non-empty.

From the proof of Theorem 7.7, we have that when retailer Y solves *Y-RETAILER-INDUCING-AGGREGATE-ORDER($w_{\overline{Y}}$)*, there exists an optimal (and unique) aggregate stocking quantity $x^Y(w_{\overline{Y}})$ and an optimal (and unique) induced capacity allocation vector $q^Y(w_{\overline{Y}})$. Therefore, from equation (7.14), we have that the set $\mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$ is non-empty. Furthermore, for every good $j \in Y \backslash Z^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$, we must have

$$(c_j + \lambda^v(x^Y(w_{\overline{Y}}); w_{\overline{Y}}))/\bar{F}_j(q_j^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})) \le p_j, \tag{7.61}$$

otherwise, the quantity $x^Y(w_{\overline{Y}})$ and the vector $q^Y(w_{\overline{Y}})$ would not be a solution for *Y-RETAILER-INDUCING-AGGREGATE-ORDER($w_{\overline{Y}}$)* (because retailer Y could increase the value of the objective function by choosing the induced stocking quantity for good j to be zero, if equation (7.61) did not hold for good j). Therefore, for every wholesale price vector $w' \in \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$ and for any good $j \in$

$Y \setminus Z^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$, we have $c_j \leq w'_j \leq p_j$ from equation (7.14). Also, from equation (7.14), we have that there always exists a wholesale price vector $w' \in \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$ such that for every good $i \in Z^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$, we have $w'_i = c_i$. Therefore, we have that the set $\mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}}) \cap \prod_{t \in Y}[c_t, p_t]$ is non-empty.

Next, we show that $\mathcal{W}_Y^{\text{br}}(w_{\overline{Y}}) \subseteq \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}}) \bigcap \prod_{t \in Y}[c_t, p_t]$. Consider any wholesale price vector $w_Y \in \mathcal{W}_Y^{\text{br}}(w_{\overline{Y}})$ for goods Y. From the constraints of *Y-RETAILER($w_{\overline{Y}}$)*, we know that $w_Y \in \prod_{t \in Y}[c_t, p_t]$. Assume that $w_Y \notin \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$. From Theorem 7.1, we have that the objective function for *Y-RETAILER($w_{\overline{Y}}$)* satisfies

$$\sum_{i \in Y} (p_i - w_i) \cdot E[S_i(q_i^v(w))] = \sum_{i \in Y} \left(p_i - (\lambda^v(w) + c_i)/\bar{F}_i(q_i^v(w))\right) \cdot E[S_i(q_i^v(w))]. \tag{7.62}$$

And from Theorem 7.5 and equation (7.14), we have that for any induced order quantity vector q'_Y that maximizes the objective function of *Y-RETAILER-INDUCING-AGGREGATE-ORDER($w_{\overline{Y}}$)* subject to its constraints, there exists a wholesale price vector $w'_Y \in \mathcal{W}^Y(\sum_{i \in Y} q'_i; w_{\overline{Y}}) \cap \prod_{t \in Y}[c_t, p_t]$ that satisfies

$$\sum_{i \in Y} \left(p_i - \left(\lambda^v\left(\sum_{t \in Y} q'_t; w_{\overline{Y}}\right) + c_i\right) / \bar{F}_i(q'_i)\right) \cdot E[S_i(q'_i)] = \sum_{i \in Y} (p_i - w'_i) \cdot E[S_i(q_i^v(w'_Y, w_Y))]. \tag{7.63}$$

From our assumption that $w_Y \notin \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$, we have

$$\sum_{i \in Y} (p_i - w_i) \cdot E[S_i(q_i^v(w))] < \sum_{i \in Y} (p_i - w'_i) \cdot E[S_i(q_i^v(w'_Y, w_Y))]$$

so that $w_Y \notin \mathcal{W}_Y^{\text{br}}(w_{\overline{Y}})$. But this is a contradiction. Thus, we have

$$w_Y \in \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}}) \cap \prod_{t \in Y}[c_t, p_t].$$

Next, we show that $\mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}}) \bigcap \prod_{t \in Y}[c_t, p_t] \subseteq \mathcal{W}_Y^{\text{br}}(w_{\overline{Y}})$. Consider any wholesale price vector $w'_Y \in \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}}) \bigcap \prod_{t \in Y}[c_t, p_t]$ for goods Y. Then, there exists a vector q'_Y of order quantities such that for any good $i \in Y$, $q'_i =$

$q_i^v(w'_Y, w_Y)$ and equation (7.63) holds. Assume that $w'_Y \notin \mathcal{W}_Y^{\text{br}}(w_{\overline{Y}})$. Then, there exists a $w_Y \in \prod_{t \in Y}[c_t, p_t]$ such that

$$\sum_{i \in Y} (p_i - w_i) \cdot E[S_i(q_i^v(w))] > \sum_{i \in Y} (p_i - w'_i) \cdot E[S_i(q_i^v(w'_Y, w_Y))].$$

But, since $\lambda^v(w) = \lambda^v\left(\sum_{t \in Y} q_i^v(w); w_{\overline{Y}}\right)$ (due to Theorem 7.3), from equation (7.62) we have

$$\sum_{i \in Y} (p_i - w_i) \cdot E[S_i(q_i^v(w))] = \sum_{i \in Y} \left(p_i - \left(\lambda^v\left(\sum_{t \in Y} q_i^v(w); w_{\overline{Y}}\right) + c_i\right) / \bar{F}_i\left(q_i^v(w)\right)\right) \cdot E[S_i(q_i^v(w))]. \tag{7.64}$$

Therefore, from equation (7.63), we have

$$\sum_{i \in Y} \left(p_i - \left(\lambda^v\left(\sum_{t \in Y} q_i^v(w); w_{\overline{Y}}\right) + c_i\right) / \bar{F}_i\left(q_i^v(w)\right)\right) \cdot E[S_i(q_i^v(w))]$$

is strictly larger than

$$\sum_{i \in Y} \left(p_i - \left(\lambda^v\left(\sum_{t \in Y} q'_t; w_{\overline{Y}}\right) + c_i\right) / \bar{F}_i(q'_i)\right) \cdot E[S_i(q'_i)].$$

But, this is a contradiction because $w'_Y \in \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}}) \bigcap \prod_{t \in Y}[c_t, p_t]$ and the vector $q_Y^v(w)$ is in the feasible set of *Y-RETAILER-INDUCING-AGGREGATE-ORDER*($w_{\overline{Y}}$). Thus, we have $w'_Y \in \mathcal{W}_Y^{\text{br}}(w_{\overline{Y}})$.

Finally, we show that the set $\mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}}) \bigcap \prod_{t \in Y}[c_t, p_t]$ (as defined in equation (7.14)) is convex. Consider any two wholesale price vectors $a, b \in \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$ and any real number $\eta \in [0, 1]$. For every good $j \in Y \setminus Z^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$, we have $a_j = b_j$ (from equation (7.14)). Furthermore, for every good $i \in Z^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$, we have $\eta \cdot a_i + (1-\eta) \cdot b_i > \min\{a_i, b_i\}$. Therefore, $q_Y^v(\eta \cdot a_Y + (1-\eta) \cdot b_Y, w_{\overline{Y}}) = q_Y^v(a_Y, w_{\overline{Y}})$, implying that the wholesale price vector $\eta \cdot a_i + (1-\eta) \cdot b_i \in \mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$. Therefore, the set $\mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}})$ is convex, and $\mathcal{W}^Y(x^Y(w_{\overline{Y}}); w_{\overline{Y}}) \bigcap \prod_{t \in Y}[c_t, p_t]$ is also convex since it is the intersection of two convex sets. □

7.6.14 Proof: Existence of an equilibrium when s retailers compete.

In this proof we apply Kakutani's Fixed Point Theorem (Proposition 6.2). See Section 6.6.14 for a statement of that theorem.

Proof of Theorem 7.9. Suppose there are $s \geq 2$ retailers identified by the subsets of goods they offer: $Y_1, \ldots, Y_s$. Denote the set of feasible retailer wholesale price vectors by $W \stackrel{\text{def}}{=} \prod_{t \in N}[c_t, p_t]$. Consider the retailer best response correspondence $\mathcal{W}^{\text{br}} : W \rightarrow 2^W$ defined as the s-ary Cartesian product over the s retailer best response mappings. Namely, for any wholesale price vector $w \in W$, $\mathcal{W}^{\text{br}}(w) \stackrel{\text{def}}{=} (\mathcal{W}^{\text{br}}_{Y_1}(w_{\overline{Y}_1}), \ldots, \mathcal{W}^{\text{br}}_{Y_s}(w_{\overline{Y}_s})) = \{(w_{Y_1}, \ldots, w_{Y_s}) \mid w_{Y_1} \in \mathcal{W}^{\text{br}}_{Y_1}(w_{\overline{Y}_1}), \ldots, w_{Y_s} \in \mathcal{W}^{\text{br}}_{Y_s}(w_{\overline{Y}_s})\}$. In order to show that an equilibrium exists, we will apply Kakutani's fixed point theorem to the retailer best response correspondence $\mathcal{W}^{\text{br}} : W \rightarrow 2^W$. First, observe that the set $W \subset \mathbb{R}^n$ is non-empty, compact, and convex. Furthermore, from Theorem 7.8, we have that for any retailer $Y \subseteq N$ that faces competing wholesale prices $w_{\overline{Y}}$, the best response mapping $\mathcal{W}^{\text{br}}_Y(w_{\overline{Y}})$ is non-empty and convex[7], implying that $\mathcal{W}^{\text{br}}(w)$ is non-empty and convex.

Finally, we show that the correspondence $\mathcal{W}^{\text{br}}$ is upper hemicontinuous. Assume $\mathcal{W}^{\text{br}}$ is not upper hemicontinuous. Since W is compact, we have that the correspondence $\mathcal{W}^{\text{br}}$ is not closed.[8] Therefore, there exists wholesale price vectors $\bar{w}, \bar{z} \in W$ such that the sequence $\{w^l\}$ of wholesale price vectors converges to $\bar{w}$ and the sequence $\{z^l\}$ of wholesale price vectors satisfies $z^l \in \mathcal{W}^{\text{br}}(w^l)$ and converges to $\bar{z}$, yet $\bar{z} \notin \mathcal{W}^{\text{br}}(\bar{w})$. Therefore, if we denote retailer Y's profit by $\pi_Y(w_Y, w_{\overline{Y}}) \stackrel{\text{def}}{=} \sum_{i \in Y}(p_i - w_i)E[S_i(q_i^v(w_Y, w_{\overline{Y}}))]$, there exists some retailer Y_a ($a \in \{1, \ldots, s\}$) and some wholesale price vector $\hat{w}_{Y_a} \in \prod_{t \in Y_a}[c_t, p_t]$ such that $\pi_{Y_a}(\bar{z}_{Y_a}, \bar{w}_{\overline{Y}_a}) < \pi_{Y_a}(\hat{w}_{Y_a}, \bar{w}_{\overline{Y}_a})$. Therefore, there exists an $\epsilon > 0$ such that

$$\pi_{Y_a}(\bar{z}_{Y_a}, \bar{w}_{\overline{Y}_a}) + \epsilon < \pi_{Y_a}(\hat{w}_{Y_a}, \bar{w}_{\overline{Y}_a}) - \epsilon. \tag{7.65}$$

[7] Convexity follows from our assumption that the retailers are service constrained for good(s) $\overline{Y}$, and the cumulative distribution function for demand of each good $y \in Y$ has the IGFR property.

[8] See Border (1989) for the following result: Consider sets $D \subset \mathbb{R}^l$, $R \subset \mathbb{R}^m$ and the correspondence $\mathcal{C} : D \rightarrow 2^R$. If R is compact and $\mathcal{C}$ is closed, then $\mathcal{C}$ is upper hemicontinuous.

Retailer Y_a's objective function $\pi_{Y_a}(w)$ is continuous on W (see Lemma 7.1), therefore there exists an integer m such that for $l > m$, we have $|\pi_{Y_a}(\bar{z}_{Y_a}, \bar{w}_{\overline{Y}_a}) - \pi_{Y_a}(z^l_{Y_a}, w^l_{\overline{Y}_a})| < \epsilon$. So that the inequality $\pi_{Y_a}(z^l_{Y_a}, w^l_{\overline{Y}_a}) < \pi_{Y_a}(\bar{z}_{Y_a}, \bar{w}_{\overline{Y}_a}) + \epsilon$ holds. Therefore, from Equation (7.65), we have

$$\pi_{Y_a}(z^l_{Y_a}, w^l_{\overline{Y}_a}) < \pi_{Y_a}(\hat{w}_{Y_a}, \bar{w}_{\overline{Y}_a}) - \epsilon. \tag{7.66}$$

Because of the continuity of retailer Y_a's objective function, there also exists an integer o such that for $l > o$, we have $|\pi_{Y_a}(\hat{w}_{Y_a}, \bar{w}_{\overline{Y}_a}) - \pi_{Y_a}(\hat{w}_{Y_a}, w^l_{\overline{Y}_a})| < \epsilon$. So that we have $\pi_{Y_a}(\hat{w}_{Y_a}, \bar{w}_{\overline{Y}_a}) - \epsilon < \pi_{Y_a}(\hat{w}_{Y_a}, w^l_{\overline{Y}_a})$. Therefore, from Equation (7.66), for $l > \max\{m, o\}$, we have

$$\pi_{Y_a}(z^l_{Y_a}, w^l_{\overline{Y}_a}) < \pi_{Y_a}(\hat{w}_{Y_a}, w^l_{\overline{Y}_a}). \tag{7.67}$$

But, this is a contradiction because $z^l \in \mathcal{W}^{\text{br}}(w^l)$. Therefore, the correspondence $\mathcal{W}^{\text{br}}$ is upper hemicontinuous. And, therefore, by applying Kakutani's fixed point theorem to the retailer best response correspondence $\mathcal{W}^{\text{br}}$, we have that there exists a vector $w^{\text{eq}} \in W$ of wholesale prices for all n goods, such that $w^{\text{eq}} \in \mathcal{W}^{\text{br}}(w^{\text{eq}})$. □

7.6.15 Proof: Unique equilibrium shadow price for capacity when s retailers compete.

Proof of Theorem 7.10. Suppose there is one retailer (i.e., $s = 1$) denoted by the set N of goods offered. The (Weierstrass) Extreme Value Theorem says that for any continuous and real function f on a compact metric space X, there exists a point $x^* \in X$ such $f(x^*) = \sup_{x \in X} f(x)$. (Rudin 1976, Theorem 4.16) Since, the hypercube $\prod_{t \in N}[c_t, p_t]$ is closed and bounded (and hence compact) as well as a metric space (under the Euclidean metric), and since retailer N's objective function $\pi_N(w) \stackrel{\text{def}}{=} \sum_{i \in N}(p_i - w_i)E[S_i(q^v_i(w))]$ is (real) continuous in its decision vector w (from Lemma 7.1), we, therefore, have (by applying the Extreme Value Theorem) that retailer N can attain the supremum of its objective function (over its constraint set) from a vector in its constraint set, i.e., the hypercube $\prod_{t \in N}[c_t, p_t]$, implying that a solution exists.

Next, we show that when there is one retailer (i.e., $s = 1$) *every* solution w to the retailer's decision problem in the first stage induces the supplier to have a shadow price $\lambda^v(w) = 0$. Assume that some solution vector w' induces a positive retailer shadow price $\lambda^v(w') > 0$ instead. From Theorem 7.5, we have that there exists another wholesale price vector $\widehat{w}$ such that the supplier allocates the same amount as under w' (i.e., $q^v(\widehat{w}) = q^v(w')$), but the shadow price $\lambda^v(\widehat{w}) = 0$. Therefore, from Theorem 7.1 we have that $\widehat{w}_i < w'_i$ for every good i that the supplier stocks so that $\pi_N(w') < \pi_N(\widehat{w})$. But this is a contradiction because w' is a solution vector for the retailer's decision problem in the first stage. Thus, it follows that every solution w to the retailer's decision problem in the first stage induces the supplier to have a shadow price $\lambda^v(w) = 0$.

Suppose there is more than one retailer (i.e., $s \geq 2$). We denote retailer i by the subset Y_i of goods offered. Furthermore, suppose that the retailers are service constrained for goods N, the demand for each good $t \in N$ has the IGFR property, and Assumption 7.1 holds. We show that *every* equilibrium wholesale price vector w induces the supplier to have the same shadow price which we denote by λ^{eq} (i.e., $\lambda^v(w) = \lambda^{\mathrm{eq}}$). Assume that instead we have two distinct equilibrium wholesale price vectors, w' and $\widehat{w}$, but that they induce different shadow prices for the supplier's capacity (i.e., $\lambda^v(w') \neq \lambda^v(\widehat{w})$). Without loss of generality, suppose

$$0 \leq \lambda^v(w') < \lambda^v(\widehat{w}). \tag{7.68}$$

Recall from Section 7.4.3 that the set

$$L(q^v(\widehat{w}), q^v(w')) \stackrel{\text{def}}{=} \{l \in \{1, \ldots, s\} \mid \sum_{i \in Y_l} q_i^v(\widehat{w}) > \sum_{i \in Y_l} q_i^v(w')\}$$

denotes the retailers that have a larger share of the supplier's capacity under wholesale price vector $\widehat{w}$ when compared to the allocation under wholesale price vector w'. If the two distinct wholesale price vectors induce the supplier to make the same allocation, i.e., $q^v(\widehat{w}) = q^v(w')$, then the set $L(q^v(\widehat{w}), q^v(w'))$ is empty, otherwise the set $L(q^v(\widehat{w}), q^v(w'))$ must be nonempty because the equation

$$\sum_{i\in N} q_i^v(w') \le \sum_{i\in N} q_i^v(\widehat{w}) = k \tag{7.69}$$

holds (which follows from equation (7.68) and Theorem 7.1).

Consider the case when the set $L(q^v(\widehat{w}), q^v(w'))$ is nonempty. For the purposes of this proof only, we define $\bar{x} \stackrel{\text{def}}{=} \min\left\{\sum_{t\in Y} q_t^e, k\right\}$. For every retailer $l \in L(q^v(\widehat{w}), q^v(w'))$, from Theorem 7.7, we have that

$$\sum_{i\in Y_l} q_i^v(\widehat{w}) = \sup\left\{x \in [0,\bar{x}] \mid \lambda^{Y_l}(x;\widehat{w}_{\overline{Y_l}}) - \left(\sum_{i\in Y_l} \frac{E[S_i(q_i^{Y_l}(x;\widehat{w}_{\overline{Y_l}}))]}{\bar{F}_i(q_i^{Y_l}(x;\widehat{w}_{\overline{Y}}))}\right) \cdot \frac{\partial\lambda^v(x;\widehat{w}_{\overline{Y_l}})}{\partial x^-} \ge 0\right\}. \tag{7.70}$$

From Theorem 7.6 (which implies that the function $\lambda^{Y_l}(x; w'_{\overline{Y_l}})$ is strictly decreasing as $x \in \left(0, \min\left\{\sum_{t\in Y} q_t^e, k\right\}\right]$ increases) and Theorem 7.7, we have that for every retailer $l \in L(q^r(\widehat{w}), q^r(w'))$ the equation

$$\sum_{i\in Y_l} q_i^v(w') = \inf\left\{x \in [0,\bar{x}] \mid \lambda^{Y_l}(x;w'_{\overline{Y_l}}) - \left(\sum_{i\in Y_l} \frac{E[S_i(q_i^{Y_l}(x;w'_{\overline{Y_l}}))]}{\bar{F}_i(q_i^{Y_l}(x;w'_{\overline{Y}}))}\right) \cdot \frac{\partial\lambda^v(x;w'_{\overline{Y_l}})}{\partial x^+} \le 0\right\} \tag{7.71}$$

holds. From Assumption 7.1 we have

$$\sum_{l\in L(q^v(\widehat{w}),q^v(w'))} \left(\sum_{i\in Y_l} \frac{E[S_i(q_i^v(\widehat{w}))]}{\bar{F}_i(q_i^v(\widehat{w}))}\right) \cdot \left.\frac{\partial\lambda^v(x;\widehat{w}_{\overline{Y_l}})}{\partial x^-}\right|_{x=\sum_{i\in Y_l} q_i^v(\widehat{w})} \ge \tag{7.72}$$
$$\sum_{l\in L(q^v(\widehat{w}),q^v(w'))} \left(\sum_{i\in Y_l} \frac{E[S_i(q_i^v(w'))]}{\bar{F}_i(q_i^v(w'))}\right) \cdot \left.\frac{\partial\lambda^v(x;w'_{\overline{Y_l}})}{\partial x^+}\right|_{x=\sum_{i\in Y_l} q_i^v(w')}.$$

Therefore, there exists a retailer $\hat{l} \in L(q^r(\widehat{w}), q^r(w'))$ such that the equation

$$\left(\sum_{i\in Y_{\hat{l}}} \frac{E[S_i(q_i^v(\widehat{w}))]}{\bar{F}_i(q_i^v(\widehat{w}))}\right) \cdot \left.\frac{\partial\lambda^v(x;\widehat{w}_{\overline{Y_{\hat{l}}}})}{\partial x^-}\right|_{x=\sum_{i\in Y_{\hat{l}}} q_i^v(\widehat{w})} \ge \left(\sum_{i\in Y_{\hat{l}}} \frac{E[S_i(q_i^v(w'))]}{\bar{F}_i(q_i^v(w'))}\right) \cdot \left.\frac{\partial\lambda^v(x;w'_{\overline{Y_{\hat{l}}}})}{\partial x^+}\right|_{x=\sum_{i\in Y_{\hat{l}}} q_i^v(w')} \tag{7.73}$$

holds. And so we have

$$0 \le \lambda^{Y_{\hat{l}}}\left(\sum_{i\in Y_{\hat{l}}} q_i^v(\widehat{w}); \widehat{w}_{\overline{Y_{\hat{l}}}}\right) - \left(\sum_{i\in Y_{\hat{l}}} \frac{E[S_i(q_i^v(\widehat{w}))]}{\bar{F}_i(q_i^v(\widehat{w}))}\right) \cdot \left.\frac{\partial \lambda^v(x; \widehat{w}_{\overline{Y_{\hat{l}}}})}{\partial x^-}\right|_{x=\sum_{i\in Y_{\hat{l}}} q_i^v(\widehat{w})} \tag{7.74}$$

$$< \lambda^{Y_{\hat{l}}}\left(\sum_{i\in Y_{\hat{l}}} q_i^v(w'); w'_{\overline{Y_{\hat{l}}}}\right) - \left(\sum_{i\in Y_{\hat{l}}} \frac{E[S_i(q_i^v(w'))]}{\bar{F}_i(q_i^v(w'))}\right) \cdot \left.\frac{\partial \lambda^v(x; w'_{\overline{Y_{\hat{l}}}})}{\partial x^+}\right|_{x=\sum_{i\in Y_{\hat{l}}} q_i^v(w')} \tag{7.75}$$

$$\le 0. \tag{7.76}$$

(Equation (7.74) follows from equation (7.70). Applying Theorem 7.6 and noting the equations (7.68) and (7.69), we have $\lambda^{Y_{\hat{l}}}\left(\sum_{i\in Y_{\hat{l}}} q_i^v(\widehat{w}); \widehat{w}_{\overline{Y_{\hat{l}}}}\right) < \lambda^{Y_{\hat{l}}}\left(\sum_{i\in Y_{\hat{l}}} q_i^v(w'); w'_{\overline{Y_{\hat{l}}}}\right)$ holds. Therefore, from equation (7.73) we have equation (7.75). And, equation (7.76) follows from equation (7.71).) Notice that equation (7.76) leads to a contradiction, $0 < 0$. Thus, it follows that *every* equilibrium wholesale price vector w induces the supplier to have the same shadow price which we denote by λ^{eq} (i.e., $\lambda^v(w) = \lambda^{\text{eq}}$). $\square$

7.6.16 Proof: Unique capacity allocation when s suppliers compete.

Proof of Theorem 7.11. Consider the case when the supplier allocates capacity for at least two retailers when offered wholesale price vector w'. Assume that the equation

$$\lambda^v(\widehat{w}) > \lambda^v(w') + \min\left\{m_{Y_j}(w') \mid j \in \{1,\ldots,s\},\ Y_j \cap (N \setminus Z(w')) \neq \emptyset\right\} \tag{7.77}$$

holds, instead, for two equilibrium wholesale price vectors $\widehat{w}$ and w' that induce the same supplier capacity allocation (i.e., $q^v(\widehat{w}) = q^v(w')$) but induce shadow prices for capacity satisfying $\lambda^v(w') \le \lambda^v(\widehat{w})$. Consider any retailer Y_d such that

$$d \in \arg \min_{j\in\{1,\ldots,s\},\ Y_j\cap(N\setminus Z(w'))\neq\emptyset} m_{Y_j}(w'). \tag{7.78}$$

We show that retailer Y_d will deviate from the wholesale price vector $\widehat{w}_{Y_d}$ when the wholesale price vector for the other goods $\overline{Y_d}$ is held fixed at $\widehat{w}_{\overline{Y_d}}$. Since the two equilibrium wholesale price vectors $\widehat{w}$ and w' induce the same capacity allocation (i.e., $q^v(\widehat{w}) = q^v(w')$), from the proof of Corollary 7.6 (see equation (7.45)) and Theorem 7.6 we have that the equation

$$\left(\sum_{i \in Y_d} \frac{E[S_i(q_i^v(\widehat{w}))]}{\bar{F}_i(q_i^v(\widehat{w}))}\right) \cdot \frac{\partial \lambda^v(x; \widehat{w}_{\overline{Y_d}})}{\partial x^-}\Bigg|_{x=\sum_{i \in Y_d} q_i^v(\widehat{w})} \geq \left(\sum_{i \in Y_d} \frac{E[S_i(q_i^v(w'))]}{\bar{F}_i(q_i^v(w'))}\right) \cdot \frac{\partial \lambda^v(x; w'_{\overline{Y_d}})}{\partial x^-}\Bigg|_{x=\sum_{i \in Y_d} q_i^v(w')} . \tag{7.79}$$

holds. Furthermore, since $q^v(\widehat{w}) = q^v(w')$ holds, from Theorem 7.3 and Theorem 7.6 we have that the equation

$$\lambda^{Y_d}\left(\sum_{i \in Y_d} q_i^v(\widehat{w}); \widehat{w}_{\overline{Y_d}}\right) + \lambda^v(\widehat{w}) \leq \lambda^{Y_d}\left(\sum_{i \in Y_d} q_i^v(w'); w'_{\overline{Y_d}}\right) + \lambda^v(w'). \tag{7.80}$$

holds. Therefore, we have

$$\begin{aligned}
m_{Y_d}(\widehat{w}) &\stackrel{\text{def}}{=} \lambda^{Y_d}\left(\sum_{i \in Y_d} q_i^v(\widehat{w}); \widehat{w}_{\overline{Y_d}}\right) - \left(\sum_{i \in Y_d} \frac{E[S_i(q_i^v(\widehat{w}))]}{\bar{F}_i(q_i^v(\widehat{w}))}\right) \cdot \frac{\partial \lambda^v(x; \widehat{w}_{\overline{Y_d}})}{\partial x^-}\Bigg|_{x=\sum_{i \in Y_d} q_i^v(\widehat{w})} \\
&\leq \lambda^{Y_d}\left(\sum_{i \in Y_d} q_i^v(w'); w'_{\overline{Y_d}}\right) + \lambda^v(w') - \lambda^v(\widehat{w}) \\
&\qquad - \left(\sum_{i \in Y_d} \frac{E[S_i(q_i^v(w'))]}{\bar{F}_i(q_i^v(w'))}\right) \cdot \frac{\partial \lambda^v(x; w'_{\overline{Y_d}})}{\partial x^-}\Bigg|_{x=\sum_{i \in Y_d} q_i^v(w')} \\
&= \lambda^v(w') + m_{Y_d}(w') - \lambda^v(\widehat{w}) \\
&< 0.
\end{aligned} \tag{7.81}$$

Equation (7.81) follows from equation (7.77). But this is a contradiction because, according to Theorem 7.7, retailer Y_d would deviate from the wholesale price vector $\widehat{w}_{Y_d}$ when the wholesale price vector for the other goods $\overline{Y_d}$ is held fixed at $\widehat{w}_{\overline{Y_d}}$. Thus, equation (7.23) follows.

Consider the case when the supplier allocates capacity to only one retailer (i.e., retailer Y) when facing wholesale price vector w'. Assume that the equation

$$\lambda^v(\widehat{w}) > \lambda^v(w') \tag{7.82}$$

holds, instead. Retailer Y's objective function is $\pi_Y(w) \stackrel{\text{def}}{=} \sum_{i\in Y}(p_i - w_i)E[S_i(q_i^v(w))]$. Since $\sum_{i\in Y} q_i^v(\widehat{w}) = \sum_{i\in Y} q_i^v(w') = k$, from Theorem 7.7 we have $q_i^v(\widehat{w}) = q_i^v(w')$ for every good $i \in Y$. Therefore, from Theorem 7.1 we have that $\widehat{w}_i > w'_i$ for every good i that the supplier stocks so that $\pi_Y(\widehat{w}) < \pi_Y(w')$. Observe that $\pi_Y(w') = \pi_Y(w'_Y, \widehat{w}_{\overline{Y}})$ since $\lambda^v(w') = \lambda^v(w'_Y, c_{\overline{Y}})$ (for an equilibrium w') implying $\lambda^v(w') = \lambda^v(w'_Y, \widehat{w}_{\overline{Y}})$ (using Theorem 7.4). And, so we have $\pi_Y(\widehat{w}) < \pi_Y(w'_Y, \widehat{w}_{\overline{Y}})$. Therefore, retailer Y prefers wholesale price vector w'_Y over $\widehat{w}_Y$ when the other good(s) $\overline{Y}$ have fixed their wholesale price vector to be $\widehat{w}_{\overline{Y}}$. But this is a contradiction because $\widehat{w}$ is an equilibrium wholesale price vector. Thus, it follows that $\lambda^v(\widehat{w}) \leq \lambda^v(w')$. $\square$

7.6.17 Proof: Retailer collusion.

Proof of Theorem 7.12. Denote retailer Y's objective function by the function $\pi_Y(w) \stackrel{\text{def}}{=} \sum_{i\in N}(p_i - w_i)E[S_i(q_i^v(w))]$. From Theorem 7.5, we have that there exists another wholesale price vector $\widehat{w}$ such that the supplier stocks the same amount as under w' (i.e., $q^v(\widehat{w}) = q^v(w')$), but the shadow price $\lambda^v(\widehat{w}) = 0$. Therefore, from Theorem 7.1 we have that $\widehat{w}_i < w'_i$ for every good i that the supplier stocks so that for any retailer Y we have $\pi_Y(w') < \pi_Y(\widehat{w})$. Furthermore, from Theorem 7.10 we have that for the two-stage game with one retailer any equilibrium wholesale price vector induces a shadow price of zero units. Therefore, if w^* is a solution to $(Y_1 \cup \ldots \cup Y_s)$-*RETAILER* (see Section 7.1.2), then we have

$$\sum_{t=1}^{s} \pi_{Y_t}(\widehat{w}) \leq \sum_{t=1}^{s} \pi_{Y_t}(w^*).$$

So we can conclude that

$$\sum_{t=1}^{s} \pi_{Y_t}(w') < \sum_{t=1}^{s} \pi_{Y_t}(w^*). \quad \square$$

Chapter 8

Conclusions and future work

The "accepted wisdom" in the supply contracts literature is that in single supplier/single retailer situations, the supplier has no incentive to set a wholesale price that will maximize the channel's profits. Thus, simple contracts based on wholesale price are considered inefficient. This observation has motivated the study of (the harder to implement) risk-sharing contracts which allow for flexible allocation of the (optimal) profit. The first half of our thesis focuses on coordination and demonstrates that when the supply channel is resource constrained, wholesale price contracts can be as efficient as risk-sharing contracts and even somewhat flexible in allocating channel profit. We show that this efficiency result holds regardless of the supply channel's mode of operation (push or pull). Intuitively, the efficiency and profit-allocation flexibility of risk-sharing contracts are derived, in part, from additional contract parameters. However, when a supply channel is resource constrained, the inherent resource parameter (e.g., capacity or budget), is sufficient for enriching a wholesale price contract to have the benefits of a risk-sharing contract. More generally, we find that resource parameters can enhance the efficiency properties of risk-sharing contracts too.

The second half of our thesis focuses on competition rather than coordination. We consider two supply channels (one operating in push-mode and the other operating in pull-mode) each with one newsvendor with limited capacity and firms that compete for that capacity in an equilibrium setting. After conducting some comparative statics and analyzing each game's 'geometry', we show that the equilibrium setting oftentimes creates an endogenous valuation for the newsvendor's capacity. And we show that when the firms collude against the newsvendor, they will decrease the value of the newsvendor's capacity to zero.

Before we describe some paths for future work, we highlight some of the ideas from this thesis.

8.1 Resource parameters versus contract parameters

One of the reasons that revenue-sharing, buyback, and an assortment of other contracts are able to coordinate the retailer in an unconstrained setting is because those contracts have two or more parameters. Intuitively, the 'flexibility' of those parameters creates contracts where the retailer has an incentive to order the system-optimal amount and that allows the supplier to earn a profit. Interestingly, our model also introduces another 'parameter', capacity. But capacity is not part of the contract. Rather it is part of the system. So instead of introducing complexity into the contract (with another contract parameter) one should check if an inherent resource parameter (such as capacity) can lead to the use of simpler contracts.

In particular, if demand is large enough relative to capacity for the *channel's* problem, then wholesale price contracts that coordinate the channel and allow both the supplier and retailer to profit *exist.* Consequently the potential to reach a channel optimal outcome in a negotiation setting exists. Also, demand and capacity are both levers in practice. Therefore, if demand is not large enough relative to capacity for a wholesale price contract to be efficient, demand can increased (e.g., through marketing) so that a simple wholesale price contract is efficient.

Resource constraints are a part of many supply channels. The first part of this thesis shows that taking them into consideration in the analysis is important in assessing the actual efficiency of contracts for constrained channels.

8.2 Extra flexibility in allocating profit

One of the problems with wholesale price contracts in an unconstrained setting is that they do not provide any flexibility in allocating the channel-optimal profit. In Section 3.2, we show that in a constrained setting wholesale price contracts offer some flexibility in allocating the channel profit without sacrificing coordination. Another lesson for *constrained* channels is that buyback and revenue sharing contracts *still* coordinate the channel (see Section 3.3.1). And those contracts coordinate the constrained channel for a larger set of parameters than for the unconstrained case, gaining some extra flexibility in allocating the channel-optimal profit for a given level of risk.

8.3 Efficiency loss

Furthermore, in the Stackelberg game (Section 2.3) where the supplier acts as the 'leader', if the capacity constraint is tight, the equilibrium outcome is channel optimal. Otherwise, when the equilibrium is not efficient (because the capacity k is not small enough), we provide a distribution-free worst-case characterization of the efficiency loss, as measured by $\text{Eff}(k, \beta)$ (see Section 2.5).

8.4 Coordination when there are multiple goods

Chapter 4 shows that when a supply channel (operating in push-mode) has more than one good (which most do) coordination is possible with wholesale price contracts (depending on the capacity of the channel) but much care needs to be taken so that the wholesale prices satisfy a very particular relationship with one another (as specified in Theorem 4.1). If a manager negotiates prices blindly without considering this relationship, the channel is forgoing profit (even though his firm may be better off financially with those terms). Furthermore, the difficulty in maintaining the relationship between these wholesale prices is further exacerbated by the fact that a firm may have multiple divisions/silos each responsible for procuring and negotiating different goods (while maintaining an overall firm budget or capacity level). In other words, coordination with multiple goods is difficult/tricky (due to the relationship between the wholesale prices that needs to be preserved), but very much possible.

8.5 Capacity valuation and collusion

Chapter 6 introduced a model where suppliers compete for a retailer's capacity. We showed that competition creates an endogenous valuation for the retailer's capacity and that supplier collusion can eliminate the value of that capacity entirely. Interestingly, in practice some retailers can overcome this effect of collusion by owning or contracting with a 'private label' supplier that has extremely low prices and enough demand for its goods. By doing this, the retailer is adding an exogenous supplier with fixed low wholesale prices to the game, thereby artificially creating a lower bound for the retailer's shadow price for capacity.

8.6 Future work

Cachon (2003) mentions that coordination in multiple supplier settings has not been explored. Chapter 6 constitutes initial steps in that direction. A particularly interesting question that we are pursuing as future work is the impact of supplier collusion on the supply channel profit. In particular, when suppliers collude is the channel profit larger in equilibrium when compared to a setting where suppliers compete against one another? In other words, will the market operate more efficiently if suppliers collude or, rather, if some subsets of suppliers collude?

Also as we showed in Chapter 4, coordination when there are multiple goods requires a particular relationship to be satisfied between the wholesale price contracts. As we've remarked, in practice this is difficult because a firm may carry out its procurement function in a decentralized fashion. It seems that by merely choosing a different contract (e.g., a buyback or revenue sharing contract) from the literature, we will not make the coordination problem for multiple goods any easier in practice. Therefore, are there non-traditional contracts that can incentivize different silos within a firm to allocate the firm's entire capacity more optimally?

In Section 2.5 and Section 5.5, we calculate the worst-case efficiency loss for a one-supplier/one-retailer supply channel with a capacity constraint, operating in push-mode and pull-mode, respectively. But how does this efficiency loss change as we add more firms and goods as in Chapters 6 and 7? How is the efficiency loss affected by collusion?

In Section 2.3 and Section 5.3, we considered equilibrium settings for a one-supplier/one-retailer supply channel with a capacity constraint. The constraint was exogenous. What happens if we allow the constraint to be endogenous? Consider the game with the newsvendor moving first to choose capacity, the other firm(s) moving second offering wholesale prices for that capacity, and finally the newsvendor making a capacity allocation/ordering decision. In such a game, will every equilibrium outcome have the property that the endogenous capacity constraint is binding? How efficient will the channel operate in equilibrium?

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