On Approximate Dynamic Inversion

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***Abstract*—Approximate Dynamic Inversion has been established as a method to control minimum-phase, nonaffine-in-control systems [1]. In this report, we re-state the main results of [1], clarify some minor notational errors, and prove the same results in an expanded form. In the large, the main results of [1] still stand. The development follows [1] closely, and no novelty is claimed herein. The purpose of this report is mainly to supplement our existing results in [2]–[4] that rely heavily on the results of [1].**

***Index Terms*—Dynamic inversion, feedback linearization, approximate.**

I. Introduction

IN [1], an Approximate Dynamic Inversion (ADI) control law was proposed that drives a given minimum-phase nonaffine-in-control system towards a chosen *stable* reference model. The control signal was defined as a solution of "fast" dynamics, and Tikhonov's Theorem [5, Theorem 11.2, pp. 439 – 440] in singular perturbation theory was used to show that the control signal approaches the exact dynamic inversion solution, and that the system state approaches and maintains within an arbitrarily close neighborhood of the state of a chosen reference model when the controller dynamics are made sufficiently fast.

Previous results in [2]–[4] rely heavily on the results of [1]. This report re-state the main results of [1], clarify some minor notational errors, and prove the same results in an expanded form. The main purpose is to supplement previous results in [2]–[4]. Importantly, *no novelty of any form is claimed herein*. The main results of [1] are Theorems 2 and 3 (in [1]), which establish the ADI method for single-input-single-output (SISO) and multi-input-multi-output (MIMO) nonlinear systems respectively. These correspond in the present report to Theorems 4 and 5 respectively. The primary differences in the statement of these theorems between [1] and the present report are the first and third technical assumptions, arising from some (minor) notational errors, but nonetheless can lead to confusion or erroneous interpretations. We will show explicitly that the assumptions stated in Theorems 4 and 5 herein leads correctly to the desired results. Another key difference between [1] and the present report is in the part of the proof which verifies the second technical assumption. It was claimed in [1] that the second technical assumption implies, together with [5, Lemma 4.6, pp. 176], that the reduced system of the associated singular perturbation model is input-to-state *exponentially* stable, which is stronger than the conclusion of [5, Lemma 4.6, pp. 176]. We will prove a stronger version of [5, Lemma 4.6, pp. 176] as Lemma 1 in Section II-B, to justify the claim. However, to establish Lemma 1, some required intermediate results which are strengthened versions of corresponding results in [5] must be established, which are presented in Section II-B. Subtle differences between [1] and the present report will be mentioned in passing, together with some clarifications.

The report will proceed as follows. We recall Tikhonov's Theorem from singular perturbation theory, which is the basis of the ADI method, in Section II-A. Strengthened versions of corresponding results in [5] are presented in Section II-B, leading to the sought Lemma 1. The main result for SISO systems and its extension to MIMO systems are presented in Section III and IV respectively. The final section concludes this report.

II. Preliminaries

Here, we present Tikhonov's Theorem from singular perturbation theory and some strengthened versions of corresponding results in [5]. These will be needed to establish the main results of ADI.

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A. Tikhonov's Theorem from Singular Perturbation Theory

Consider the *standard singular perturbation model* [5, Chapter 11]

$$\begin{aligned} \dot{x} &= f(t, x, z, \epsilon), \quad x(0) = \xi(\epsilon), \\ \epsilon \dot{z} &= g(t, x, z, \epsilon), \quad z(0) = \eta(\epsilon), \end{aligned} \tag{1}$$

where ϵ is a small positive parameter, and $\xi(\epsilon)$, $\eta(\epsilon)$ depend smoothly on ϵ. Assume that f and g are continuously differentiable in their arguments for all $(t, x, z, \epsilon) \in [0, \infty) \times D_x \times D_z \times [0, \epsilon_0]$, where $D_x \subset \mathbb{R}^n$ and $D_z \subset \mathbb{R}^m$ are domains, and $\epsilon_0 > 0$. By the *standard singular perturbation model*, it is meant that the equation

$$0 = g(t, x, z, 0) \tag{2}$$

has $k \geq 1$ isolated real roots

$$z = h_i(t, x), \quad i \in \{1, 2, \dots, k\},$$

for each $(t, x) \in [0, \infty) \times D_x$. We fix one particular i, and henceforth omit the subscript i. Define

$$y = z - h(t, x).$$

The *reduced system* is then obtained by setting $\epsilon = 0$, $z = h(t, x)$ in the first equation of (1) to get

$$\dot{x} = f(t, x, h(t, x), 0), \qquad x(0) = \xi_0 = \xi(0). \tag{3}$$

Let $\tau = t/\epsilon$. The *boundary layer system* in the y coordinates in the τ time scale is then given by

$$\frac{dy}{d\tau} = g(t, x, y + h(t, x), 0), \quad y(0) = \eta_0 - h(0, \xi_0), \tag{4}$$

where $\eta_0 = \eta(0)$. The following is the main result needed.

Theorem 1 (Tikhonov [5, Theorem 11.2, pp. 439 – 440])**.** *Consider the singular perturbation problem of* (1) *and let* $z = h(t, x)$ *be an isolated root of* (2)*. Assume that the following conditions hold for all*

$$(t, x, z - h(t, x), \epsilon) \in [0, \infty) \times D_x \times D_y \times [0, \epsilon_0],$$

for some domains $D_x \subset \mathbb{R}^n$ *and* $D_y \subset \mathbb{R}^m$ *which contain their respective origins:*

1) *On any compact subset of* $D_x \times D_y$*, the functions* f*,* g*, their first partial derivatives with respect to* (x, z, ϵ)*, and the first partial derivative of* g *with respect to* t *are continuous and bounded,* $h(t, x)$ *and* $\frac{\partial g}{\partial z}(t, x, z, 0)$ *have bounded first partial derivatives with respect to their arguments,* $\frac{\partial f}{\partial x}(t, x, h(t, x), 0)$ *is Lipschitz in* x*, uniformly in* t*, and the initial data* $\xi(\epsilon)$ *and* $\eta(\epsilon)$ *are smooth functions of* ϵ*.*
2) *The origin is an exponentially stable equilibrium point of the reduced system* (3)*. There exists a continuously differentiable Lyapunov function* $V\colon [0, \infty) \times D_x \to [0, \infty)$ *such that*

$$\begin{gathered} W_1(x) \leq V(t, x) \leq W_2(x), \\ \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, x, h(t, x), 0) \leq -W_3(x), \end{gathered}$$

holds for all $(t, x) \in [0, \infty) \times D_x$*, where* W_1*,* W_2 *and* W_3 *are continuous positive definite functions on* D_x*. Let* $c > 0$ *be chosen so that* $\{x \in D_x \mid W_1(x) \leq c\}$ *is a compact subset of* D_x*.*
3) *The origin is an exponentially stable equilibrium point of the boundary layer system* (4)*, uniformly in* (t, x)*.*

Let $R_y \subset D_y$ *be the region of attraction of the autonomous system*

$$\frac{dy}{d\tau} = g(0, \xi_0, y + h(0, \xi_0), 0),$$

and Ω_y *be a compact subset of* R_y*. Then, for each compact set* $\Omega_x \subset \{x \in D_x \mid W_2(x) \leq \rho c, \rho \in (0, 1)\}$*, there is a positive constant* ϵ^* *such that for all* $t > 0$*,* $\xi_0 \in \Omega_x$*,* $\eta_0 - h(0, \xi_0) \in \Omega_y$*, and* $\epsilon \in (0, \epsilon^*)$*, the singular perturbation problem of* (1) *has a unique solution* $x(t, \epsilon)$*,* $z(t, \epsilon)$ *on* $[0, \infty)$*, and*

$$x(t, \epsilon) - \bar{x}(t) = O(\epsilon)$$

holds uniformly for all $t \in [0, \infty)$*, where* $\bar{x}(t)$ *is the solution of the reduced system* (3)*.*

Proof: See [5, Appendix C.18, pp. 706 – 708]. ■

Proposition 1 (See also [5, pp. 433], and [1, Remark 1])**.** *If the eigenvalue condition*

$$\operatorname{Re}\left(\lambda\left(\frac{\partial g}{\partial z}(t, x, h(t, x), 0)\right)\right) \leq -k < 0 \tag{5}$$

holds for some positive constant k *and for all* $(t, x) \in [0, \infty) \times D_x$*, then the origin* $y = 0$ *of the boundary layer system* (4) *is exponentially stable, uniformly in* $(t, x) \in [0, \infty) \times D_x$*, for sufficiently small initial conditions,* $\|y(0)\|$*.*

Proof: Since $z = h(t, x)$ is the solution of (2), we have $g(t, x, h(t, x), 0) = 0$, which shows that $y = 0$ is an equilibrium point of (4). It remains to show that it is exponentially stable. Define

$$\tilde{g}(\tau, y) = g(\epsilon\tau, x(\epsilon\tau), y + h(\epsilon\tau, x(\epsilon\tau)), 0),$$

so that the boundary layer system (4) can be rewritten as

$$\frac{dy}{d\tau} = \tilde{g}(\tau, y), \tag{6}$$

with $x(\epsilon\tau)$ viewed as an exogenous time-varying signal. Then

$$A(\tau) = \frac{\partial \tilde{g}}{\partial y}(\tau, y)\bigg|_{y=0} = \frac{\partial g}{\partial z}(\epsilon\tau, x(\epsilon\tau), h(\epsilon\tau, x(\epsilon\tau)), 0).$$

When (5) holds, all eigenvalues of $A(\tau)$ have strictly negative real parts for all $(\tau, x) \in [0, \infty) \times D_x$, so that the origin is an exponentially stable equilibrium point of the linear system

$$\frac{d\tilde{y}}{d\tau} = A(\tau)\tilde{y}.$$

By [5, Theorem 4.13], the origin is an exponentially stable equilibrium point of the nonlinear system (6), which translates directly to exponential stability of the origin of the boundary layer system (4). ■

B. Other Auxiliary Results

Some other intermediate results that will be needed are established here. All of these are strengthened versions of corresponding results in [5]. The main result needed is Lemma 1, but to establish this, the following are needed. Define the closed ball B_r as

$$B_r = \{x \in \mathbb{R}^n \mid \|x\| \leq r\},$$

and system

$$\dot{x} = f(t, x), \tag{7}$$

where $f\colon [0, \infty) \times D \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[0, \infty) \times D$, and $D \subset \mathbb{R}^n$ is a domain that contains the origin.

Theorem 2 (See also [5, Theorem 4.18, pp. 172])**.** *Let $D \subset \mathbb{R}^n$ be a domain that contains the origin and $V\colon [0, \infty) \times D \to \mathbb{R}$ be a continuously differentiable function such that*

$$\begin{aligned} c_1\|x\|^2 \leq V(t, x) \leq c_2\|x\|^2, \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -c_3\|x\|^2, \quad \forall \|x\| \geq \mu > 0, \end{aligned} \tag{8}$$

$\forall t \geq 0$ and $\forall x \in D$, where c_1, c_2 and c_3 are positive constants, and $c_1 < c_2$. Take $r > 0$ such that $B_r \subset D$ and suppose that

$$0 < \mu < \sqrt{\frac{c_1}{c_2}} r. \tag{9}$$

Then, for every initial state $x(t_0)$ satisfying $\|x(t_0)\| \leq \sqrt{\frac{c_1}{c_2}} r$, there exists $T \geq 0$ (dependent on $x(t_0)$ and μ) such that the solution of (7) satisfies

$$\|x(t)\| \leq \sqrt{\frac{c_2}{c_1}} \|x(t_0)\| e^{-\frac{c_3}{2c_2}(t - t_0)}, \quad \forall t \in [t_0, t_0 + T), \tag{10}$$

$$\|x(t)\| \leq \sqrt{\frac{c_2}{c_1}} \mu, \quad \forall t \in [t_0 + T, \infty). \tag{11}$$

Moreover, if $D = \mathbb{R}^n$, then (10) and (11) hold for any initial state $x(t_0)$, with no restriction on how large μ is.

Proof: Let $\rho = c_1 r^2$ and $\eta = c_2 \mu^2$ and define

$$\Omega_{t,\eta} = \{x \in B_r \mid V(t, x) \leq \eta\},$$
$$\Omega_{t,\rho} = \{x \in B_r \mid V(t, x) \leq \rho\}.$$

Since $\eta < \rho$ by (9), we have $\Omega_{t,\eta} \subset \Omega_{t,\rho}$. A boundary point x of $\Omega_{t,\eta}$ satisfies either $\|x\| = r > \sqrt{c_2/c_1}\mu > \mu$ by (9), or $c_2\mu^2 = \eta = V(t, x) \leq c_2\|x\|^2$ which implies $\|x\| \geq \mu$. Similarly, a boundary point x of $\Omega_{t,\rho}$ satisfies either $\|x\| = r > \mu$ or $c_1 r^2 = \rho = V(t, x) \leq c_2\|x\|^2$ which implies $\|x\| \geq \sqrt{c_1/c_2} r > \mu$ by (9). Hence on all boundary points of $\Omega_{t,\eta}$ and $\Omega_{t,\rho}$, we have $\|x\| \geq \mu$ so that $\dot{V}(t, x)$ is negative by (8), and all solutions starting in $\Omega_{t,\eta}$ or $\Omega_{t,\rho}$ cannot leave them. Since $c_2\|x(t_0)\|^2 \leq \rho$ by assumption, we have

$$V(t_0, x(t_0)) \leq c_2\|x(t_0)\|^2 \leq \rho \Rightarrow x(t_0) \in \Omega_{t_0,\rho}.$$

Then, $x(t) \in \Omega_{t,\rho}$ for all $t \geq t_0$. A solution starting in $\Omega_{t,\rho}$ must enter $\Omega_{t,\eta}$ in finite time because in the set $\Omega_{t,\rho} \setminus \Omega_{t,\eta}$, $\dot{V}$ satisfies

$$\dot{V}(t, x) \leq -c_3\mu^2 < 0.$$

The foregoing inequality implies that

$$V(t, x(t)) \leq V(t_0, x(t_0)) - c_3\mu^2(t - t_0) \leq \rho - c_3\mu^2(t - t_0),$$

which shows that $V(t, x(t))$ reduces to η within the time interval $[t_0, t_0 + (\rho - \eta)/(c_3\mu^2)]$. For a solution starting inside $\Omega_{t,\eta}$, inequality (11) holds for all $t \geq t_0$, since for any $x(t_0) \in \Omega_{t,\eta}$, inequality $c_1\|x(t)\|^2 \leq V(t, x(t)) \leq \eta = c_2\mu^2$ holds for all $t \geq t_0$, which implies (11) with $T = 0$. For a solution starting inside $\Omega_{t,\rho}$ but outside $\Omega_{t,\eta}$, let $t_0 + T$ be the first time it enters $\Omega_{t,\eta}$. For all $t \in [t_0, t_0 + T]$,

$$\dot{V} \leq -c_3\|x\|^2 \leq -\frac{c_3}{c_2} V.$$

Hence, by the Comparison Lemma [5, Lemma 3.4, pp. 102], $V(t, x(t))$ satisfies

$$V(t, x(t)) \leq V(t_0, x(t_0)) e^{-\frac{c_3}{c_2}(t - t_0)}, \quad \forall t \in [t_0, t_0 + T],$$

which gives for all $t \in [t_0, t_0 + T]$,

$$c_1\|x(t)\|^2 \leq V(t, x(t)) \leq V(t_0, x(t_0)) e^{-\frac{c_3}{c_2}(t - t_0)},$$

$$\leq c_2\|x(t_0)\|^2 e^{-\frac{c_3}{c_2}(t-t_0)},$$

yielding (10). If $D = \mathbb{R}^n$, then r can be chosen arbitrarily large, and any initial state $x(t_0)$ can be included in the set $\{x \in \mathbb{R}^n \mid \|x\| \leq \sqrt{\frac{c_1}{c_2}}r\}$. ■

We will need the definition of *input-to-state exponential stability*. Consider the system

$$\dot{x} = f(t, x, u), \tag{12}$$

where $f\colon [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x and u.

Definition 1 (See also [6] and [5, Definition 4.7, pp. 175])**.** *The system* (12) *is said to be input-to-state* exponentially *stable if there exist a class $\mathcal{K}$ function γ and positive constants k and λ such that for any initial state $x(t_0)$ and any bounded input $u(t)$, the solution $x(t)$ exists for all $t \geq t_0$ and satisfies*

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)} + \gamma\left(\sup_{t_0\leq\tau\leq t}\|u(\tau)\|\right). \tag{13}$$

For definitions and properties of class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions, see [5, Section 4.4].

Theorem 3 (See also [5, Theorem 4.19, pp. 176])**.** *Let $V\colon [0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that*

$$c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2,$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x,u) \leq -c_3\|x\|^2, \quad \forall \|x\| \geq c_4\|u\| > 0,$$

for all $(t,x,u) \in [0,\infty)\times\mathbb{R}^n\times\mathbb{R}^m$, where c_1, c_2, c_3, c_4 are positive constants with $c_1 < c_2$. Then the system (12) *is input-to-state exponentially stable, and its solution satisfies* (13) *with*

$$k = \sqrt{\frac{c_2}{c_1}}, \qquad \lambda = \frac{c_3}{2c_2}, \qquad \gamma(r) = \sqrt{\frac{c_2}{c_1}}c_4 r.$$

Proof: By applying the global version of Theorem 2, we find that the solution $x(t)$ exists and satisfies

$$\|x(t)\| \leq \sqrt{\frac{c_2}{c_1}}\left(\|x(t_0)\|e^{-\frac{c_3}{2c_2}(t-t_0)} + \sup_{\tau\geq t_0} c_4\|u(\tau)\|\right),$$

for all $t \geq t_0$. Since $x(t)$ depends only on $u(\tau)$ for $\tau \in [t_0, t]$, the supremum on the right-hand side of the above inequality can be taken over $[t_0, t]$, which yields (13). ■

Lemma 1 (See also [5, Lemma 4.6, pp. 176])**.** *Suppose $f(t,x,u)$ is continuously differentiable and globally Lipschitz in (x,u), uniformly in t. If the unforced system of* (12)*, namely*

$$\dot{x} = f(t,x,0), \tag{14}$$

has a globally exponentially stable equilibrium point at the origin $x = 0$, then the system (12) *is input-to-state* exponentially *stable. Its solution satisfies* (13)*, and γ can be chosen to be a linear function*

$$\gamma(r) = cr,$$

for some positive constant c.

Remark 1. *Observe that all assumptions of Lemma 1 are identical to those of Lemma 4.6 in [5, pp. 176], but the conclusion is* stronger*, namely of input-to-state* exponential *stability, with γ of* (13) *being a linear function. Note that not all class $\mathcal{K}$ functions can be bounded above by a (class $\mathcal{K}_\infty$) linear function, e.g. $\gamma(r) = \tan(r)$ for $r \in [0, \frac{\pi}{2})$.*

Proof: View the system (12) as a perturbation of the unforced system (14). The Converse Lyapunov Theorem [5, Theorem 4.14, pp. 162 – 163] shows that the unforced system (14) has a Lyapunov function $V(t,x)$ that satisfies

$$\begin{aligned} \tilde{c}_1\|x\|^2 \leq V(t,x) \leq \tilde{c}_2\|x\|^2, \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x,0) \leq -\tilde{c}_3\|x\|^2, \\ \left\|\frac{\partial V}{\partial x}\right\| \leq \tilde{c}_4\|x\|, \end{aligned} \tag{15}$$

for some positive constants $\tilde{c}_1$, $\tilde{c}_2$, $\tilde{c}_3$, $\tilde{c}_4$, with $\tilde{c}_1 < \tilde{c}_2$, globally. Due to the uniform global Lipschitz property of f, the perturbation term satisfies

$$\|f(t,x,u) - f(t,x,0)\| \leq L\|u\|,$$

for some Lipschitz constant $L > 0$, for all $t \geq t_0$ and all (x,u). The derivative of V along solutions of (12) satisfies

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x,0) + \frac{\partial V}{\partial x}(f(t,x,u) - f(t,x,0)), \\ &\leq -\tilde{c}_3\|x\|^2 + \tilde{c}_4 L\|x\|\|u\|. \end{aligned}$$

To use the term $-\tilde{c}_3\|x\|^2$ to dominate $\tilde{c}_4 L\|x\|\|u\|$ for large $\|x\|$, we rewrite the foregoing inequality as

$$\dot{V} \leq -\tilde{c}_3(1-\theta)\|x\|^2 - \tilde{c}_3\theta\|x\|^2 + \tilde{c}_4 L\|x\|\|u\|,$$

where $\theta \in (0,1)$. Then,

$$\dot{V} \leq -\tilde{c}_3(1-\theta)\|x\|^2, \qquad \forall\|x\| \geq \frac{\tilde{c}_4 L\|u\|}{\tilde{c}_3\theta},$$

for all (t,x,u). Hence, the conditions of Theorem 3 are satisfied with

$$c_1 = \tilde{c}_1, \quad c_2 = \tilde{c}_2, \quad c_3 = \tilde{c}_3(1-\theta), \quad c_4 = \frac{\tilde{c}_4 L}{\tilde{c}_3\theta}.$$

We conclude that the system is input-to-state *exponentially* stable with solution satisfying (13), and

$$k=\sqrt{\frac{\tilde{c}_2}{\tilde{c}_1}},\qquad \lambda=\frac{\tilde{c}_3(1-\theta)}{2\tilde{c}_2},\qquad \gamma(r)=\frac{\tilde{c}_4 L}{\tilde{c}_3\theta}\sqrt{\frac{\tilde{c}_2}{\tilde{c}_1}}r.$$

Hence γ can be chosen to be a linear function $\gamma(r)=cr$, with $c\geq\frac{\tilde{c}_4 L}{\tilde{c}_3\theta}\sqrt{\frac{\tilde{c}_2}{\tilde{c}_1}}$. ■

III. Tracking Design for Minimum-phase Nonaffine-in-control SISO Systems

Consider an n-th order SISO *nonaffine-in-control* system of (constant and well-defined) relative degree ρ, expressed in normal form

$$\begin{aligned}\phi^{(\rho)}&=f(x,z,u),\quad x(0)=x_0,\\ \dot{z}&=g(x,z,u),\quad z(0)=z_0,\end{aligned}\tag{16}$$

defined for all $(x,z,u)\in D_x\times D_z\times D_u$ with $D_x\subset\mathbb{R}^\rho$, $D_z\subset\mathbb{R}^{n-\rho}$ and $D_u\subset\mathbb{R}$ being domains containing the origins. The (partial) state x is defined as $x=[\phi,\dot{\phi},\phi^{(2)},\ldots,\phi^{(\rho-1)}]^{\mathrm{T}}$, and $\phi^{(q)}$ denotes the q-th time derivative of ϕ. The state vector of the system is $[x^{\mathrm{T}},z^{\mathrm{T}}]^{\mathrm{T}}$, u is the control input, and $f\colon D_x\times D_z\times D_u\to\mathbb{R}$, $g\colon D_x\times D_z\times D_u\to\mathbb{R}^{n-\rho}$ are continuously differentiable functions of their arguments. To ensure that its relative degree is constant and well-defined, assume that $\frac{\partial f}{\partial u}$ is bounded away from zero for all $(x,z,u)\in D_x\times D_z\times D_u$. That is, there exists $b_0>0$ such that $\left|\frac{\partial f}{\partial u}\right|\geq b_0$ for all $(x,z,u)\in D_x\times D_z\times D_u$. This implies that $\operatorname{sign}\left(\frac{\partial f}{\partial u}\right)\in\{-1,+1\}$ is a constant, and that $\psi\colon u\mapsto f(x,z,u)$ is a bijection for every fixed $(x,z)\in D_x\times D_z$. Additionally, assume that the function f cannot be explicitly inverted with respect to u.

Remark 2. *That ψ is a bijection means that the inverse of f with respect to u* exist *for every fixed $(x,z)\in D_x\times D_z$. By f being not explicitly invertible with respect to u, it is meant that an analytical expression for u in terms of x, z, and the evaluation of f at (x,z,u) cannot be written. This happens for example, when f is a transcendental equation in u, like*

$$f(x,z,u)=\sin(u)+2u.$$

The problem is to design a controller so that x tracks the state of a chosen ρ-th order *stable* linear reference model

$$\phi_r^{(\rho)}+a_{r(\rho-1)}\phi_r^{(\rho-1)}+\cdots+a_{r1}\dot{\phi}_r+a_{r0}\phi_r=b_r r,\tag{17}$$

where $x_r=[\phi_r,\dot{\phi}_r,\phi_r^{(2)},\ldots,\phi_r^{(\rho-1)}]^{\mathrm{T}}\in\mathbb{R}^\rho$ is its state, r is a continuously differentiable reference input signal with bounded time derivative $\dot{r}$, and $x_r(0)=x_{r0}$ is some chosen initial state, possibly with $x_{r0}=x_0$. Stability of the reference model requires that all roots of the characteristic equation

$$s^\rho+a_{r(\rho-1)}s^{\rho-1}+\cdots+a_{r1}s+a_{r0}=0$$

lie in the open left half complex plane, denoted by $\mathbb{C}_-$.

Define the tracking error $\phi_e=\phi-\phi_r$ and error vector $e=x-x_r=[\phi_e,\dot{\phi}_e,\phi_e^{(2)},\ldots,\phi_e^{(\rho-1)}]^{\mathrm{T}}\in\mathbb{R}^\rho$, and choose the desired *stable* error dynamics

$$\phi_e^{(\rho)}+a_{e(\rho-1)}\phi_e^{(\rho-1)}+\cdots+a_{e1}\dot{\phi}_e+a_{e0}\phi_e=0,\tag{18}$$

with initial condition defined by $e(0)=e_0=x_0-x_{r0}$. Similarly, stability of the desired error dynamics requires that all roots of

$$s^\rho+a_{e(\rho-1)}s^{\rho-1}+\cdots+a_{e1}s+a_{e0}=0$$

lie in $\mathbb{C}_-$. Observe that in [1], a_{ei} was set equal to a_{ri} for $i\in\{0,1,\ldots,\rho-1\}$. This is a minor extension of [1] that allows the error dynamics to be specified *independently* of the reference model dynamics.

For notational convenience in the sequel, define

$$\begin{aligned}a_r&=[a_{r0},a_{r1},\ldots,a_{r(\rho-1)}]^{\mathrm{T}},\\ a_e&=[a_{e0},a_{e1},\ldots,a_{e(\rho-1)}]^{\mathrm{T}},\end{aligned}\qquad \alpha=\operatorname{sign}\left(\frac{\partial f}{\partial u}\right).$$

As observed above, $\alpha\in\{-1,+1\}$ is a constant. The open-loop (time-varying) error dynamics are then given by the system

$$\begin{aligned}\phi_e^{(\rho)}&=f(e+x_r(t),z,u)+a_r^{\mathrm{T}}x_r(t)-b_r r(t),\\ \dot{z}&=g(e+x_r(t),z,u),\end{aligned}\tag{19}$$

with initial conditions $e(0)=e_0$, $z(0)=z_0$. Observe that time variance in (19) is induced by the external signals $x_r(t)$ and $r(t)$ only.

We want to apply Theorem 1 to the system (19) with an appropriate controller to be specified. The ideal dynamic inversion control is found by solving

$$f(e+x_r(t),z,u)+a_r^{\mathrm{T}}x_r(t)-b_r r(t)=-a_e^{\mathrm{T}}e\tag{20}$$

for u, resulting in the exponentially stable closed-loop tracking error dynamics (18). Since (20) cannot (in general) be solved explicitly for u, an approximation of the dynamic inversion controller is constructed by introducing fast dynamics

$$\epsilon\dot{u}=-\alpha\tilde{f}(t,e,z,u),\quad u(0)=u_0,\tag{21}$$

where

$$\tilde{f}(t,e,z,u)=f(e+x_r(t),z,u)+a_r^{\mathrm{T}}x_r(t)-b_r r(t)+a_e^{\mathrm{T}}e.$$

Here, ϵ is a positive controller design parameter, chosen sufficiently small to achieve closed-loop stability. Observe that (21) relaxes the requirement for exact dynamic inversion while increasing the control in a direction to reduce the discrepancy (20) so as to approach the exact dynamic inversion solution.

Let $u = h(t, e, z)$ be an isolated root of $\tilde{f}(t, e, z, u) = 0$. In accordance with the theory of singular perturbations [5, Chapter 11], the reduced system for (19), (21), obtained by setting $\epsilon = 0$ and $u = h(t, e, z)$ is

$$\phi_e^{(\rho)} = -a_e^{\mathrm{T}} e, \qquad e(0) = e_0, \tag{22}$$

$$\dot{z} = g(e + x_r(t), z, h(t, e, z)), \qquad z(0) = z_0. \tag{23}$$

With $v = u - h(t, e, z)$ and $\tau = t/\epsilon$, the boundary layer system is

$$\frac{dv}{d\tau} = -\alpha \tilde{f}(t, e, z, v + h(t, e, z)). \tag{24}$$

Applying Theorem 1 to (19) and (21) yields the following.

Theorem 4 (Hovakimyan et al. [1, Theorem 2])**.** *Consider the system* (19) *and* (21)*, and let $u = h(t, e, z)$ be an isolated root of $\tilde{f}(t, e, z, u) = 0$. Assume that the following conditions hold for all*

$$(t, e, z, u - h(t, e, z), \epsilon) \in [0, \infty) \times D_{e,z} \times D_v \times [0, \epsilon_0],$$

for some domains $D_{e,z} \subset \mathbb{R}^n$ and $D_v \subset \mathbb{R}$ which contain their respective origins:

1) *On any compact subset of $D_{e,z} \times D_v$, the functions f, g, their first partial derivatives with respect to (x, z, u), and $r(t)$, $\dot{r}(t)$ are continuous and bounded, $h(t, e, z)$ and $\frac{\partial f}{\partial u}(x, z, u)$ have bounded first partial derivatives with respect to their arguments, and $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial z}$, $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial z}$ as functions of $(e + x_r(t), z, h(t, e, z))$, are Lipschitz in e and z uniformly in t.*
2) *The origin is an exponentially stable equilibrium of the system*
$$\dot{z} = g(x_r(t), z, h(t, 0, z)).$$
The map $(e, z) \mapsto g(e + x_r(t), z, h(t, e, z))$ is continuously differentiable and Lipschitz in (e, z) uniformly in t.
3) *$(t, e, z) \mapsto \left|\frac{\partial f}{\partial u}(e + x_r(t), z, h(t, e, z))\right|$ is bounded from below by some positive number for all $(t, e, z) \in [0, \infty) \times D_{e,z}$.*

Then the origin of (24) *is exponentially stable. Let $R_v \subset D_v$ be the region of attraction of the autonomous system*

$$\frac{dv}{d\tau} = -\alpha \tilde{f}(0, e_0, z_0, v + h(0, e_0, z_0)),$$

and Ω_v be a compact subset of R_v. Then, for each compact subset $\Omega_{e,z} \subset D_{e,z}$, there exists positive constants ϵ^* and T such that for all $t > 0$, $(e_0, z_0) \in \Omega_{e,z}$, $u_0 - h(0, e_0, z_0) \in \Omega_v$, and $\epsilon \in (0, \epsilon^*)$, the system (16), (17), (21) *has a unique solution $x(t, \epsilon)$, $z(t, \epsilon)$, $x_r(t)$, $u(t, \epsilon)$ on $[0, \infty)$, and*

$$x(t, \epsilon) - x_r(t) = O(\epsilon)$$

holds uniformly for all $t \in [T, \infty)$.

Remark 3. *The primary differences between Theorem 4 and Theorem 2 of [1] is the first and third technical assumptions. For comparison, we recall here these assumptions from [1]:*

1) *On any compact subset of $D_{e,z} \times D_v$, the functions f and g and their first partial derivatives with respect to (e, z, u), and the first partial derivative of f with respect to t are continuous and bounded, $h(t, e, z)$ and $\frac{\partial f}{\partial u}(t, e, z, u)$ have bounded first derivatives with respect to their arguments, $\frac{\partial f}{\partial e}$ and $\frac{\partial f}{\partial z}$ as functions of $(t, e, z, h(t, e, z))$ are Lipschitz in e and z, uniformly in t.*
3) *$(t, e, z, v) \mapsto \frac{\partial f}{\partial u}(t, e, z, v + h(t, e, z))$ is bounded below by some positive number for all $(t, e, z) \in [0, \infty) \times D_{e,z}$.*

Note that f and g are not explicitly functions of t and e.

Proof: The proof proceeds by showing that satisfaction of the assumptions above implies satisfaction of the assumptions of Theorem 1, whose result can be translated to the stated conclusions. We identify x, z, y, and $h(t, x)$ of Theorem 1, (denoted here by x_s, z_s, y_s, and $h_s(t, x_s)$ respectively for distinction) with quantities in (19) and (21) by

$$x_s \sim [e^{\mathrm{T}}, z^{\mathrm{T}}]^{\mathrm{T}}, \quad z_s \sim u, \quad y_s \sim v, \quad h_s(t, x_s) \sim h(t, e, z).$$

Also, f and g of Theorem 1 (denoted here by f_s and g_s) are identified with quantities in (19) and (21) as

$$f_s \sim \begin{bmatrix} \dot{\phi}_e \\ \phi_e^{(2)} \\ \vdots \\ \phi_e^{(\rho-1)} \\ f(e + x_r(t), z, u) + a_r^{\mathrm{T}} x_r(t) - b_r r(t) \\ g(e + x_r(t), z, u) \end{bmatrix} \in \mathbb{R}^n,$$

$$g_s \sim -\alpha \tilde{f}(t, e, z, u) \in \mathbb{R}.$$

Now, translate the first assumption of Theorem 1. Since $x_r(t)$ is the state of the exponentially stable system (17), $x_r(t)$ and $\dot{x}_r(t)$ are both continuous and bounded if $r(t)$ and $\dot{r}(t)$ are continuous and bounded. To have f_s and g_s continuous and bounded for any compact subset of $D_{x_s} \times D_{y_s}$ requires that f,

g and $r(t)$ be continuous and bounded for any compact subset of $D_{e,z} \times D_v$. Similarly, to have the first partial derivatives of f_s and g_s with respect to (x_s, z_s, ϵ) continuous and bounded, we require that the first partial derivatives of f and g with respect to (x, z, u) be continuous and bounded. The first partial derivative of g_s with respect to t, corresponds in the present section to the first partial derivative of $-\alpha\tilde{f}(t, e, z, u)$ with respect to t given by

$$-\alpha \left(\frac{\partial f}{\partial x}(e + x_r(t), z, u)\dot{x}_r(t) + a_r^{\mathrm{T}}\dot{x}_r(t) - b_r\dot{r}(t) \right).$$

Hence we require $\frac{\partial f}{\partial x}$ and $\dot{r}(t)$ to be continuous and bounded. The requirement that $h_s(t, x_s)$ have bounded first partial derivatives with respect to its arguments translates directly to requiring the same for $h(t, e, z)$. Since $\frac{\partial g_s}{\partial z_s}(t, x_s, z_s, 0)$ corresponds to $-\alpha\frac{\partial \tilde{f}}{\partial u}(t, e, z, u)$, and given by

$$-\alpha\frac{\partial f}{\partial u}(e + x_r(t), z, u),$$

we require that $\frac{\partial f}{\partial u}(x, z, u)$ have bounded first partial derivatives with respect to its arguments, and that $\dot{r}$ is bounded. The remaining Lipschitz conditions of Theorem 1 on x_s are straightforward. Also, since the initial conditions are independent of ϵ, the smoothness conditions are automatically satisfied. Summarizing these gives assumption 1 above.

Next, we show that the second assumption of Theorem 1 holds. To show that the origin is an exponentially stable equilibrium point of the reduced system (22), (23), we proceed in a manner similar to the proof of Lemma 4.7 in [5, pp. 180]. Let $t_0 \geq 0$ be the initial time. Clearly, $e = 0$ is an exponentially stable equilibrium point of (22), and its solution satisfies

$$\|e(t)\| \leq k_e\|e(t_0)\| \exp(-\lambda_e(t - t_0)), \tag{25}$$

for some positive constants k_e, λ_e, and for all $t \geq t_0$. With assumption 2 above, Lemma 1 shows that system (23) with e as input, is input-to-state exponentially stable, and its solution satisfies

$$\|z(t)\| \leq k_z\|z(s)\| \exp(-\lambda_z(t - s)) + \sup_{s\leq\zeta\leq t} c_z\|e(\zeta)\|, \tag{26}$$

for some positive constants k_z, λ_z, c_z, and for all $t \geq s \geq t_0$. Substituting $s = (t + t_0)/2$ into (26) yields

$$\|z(t)\| \leq k_z \left\| z\left(\frac{t + t_0}{2}\right) \right\| \exp\left(-\frac{\lambda_z(t - t_0)}{2}\right) + \sup_{\frac{t+t_0}{2}\leq\zeta\leq t} c_z\|e(\zeta)\|. \tag{27}$$

To estimate $\left\|z\left(\frac{t+t_0}{2}\right)\right\|$, substitute $s = t_0$ and replace t by $\frac{t+t_0}{2}$ in (26) to obtain

$$\left\| z\left(\frac{t + t_0}{2}\right) \right\| \leq k_z\|z(t_0)\| \exp\left(-\frac{\lambda_z(t - t_0)}{2}\right) + \sup_{t_0\leq\zeta\leq\frac{t+t_0}{2}} c_z\|e(\zeta)\|. \tag{28}$$

Using (25), we have

$$\sup_{t_0\leq\zeta\leq\frac{t+t_0}{2}} c_z\|e(\zeta)\| \leq c_z k_e\|e(t_0)\|, \tag{29}$$

$$\sup_{\frac{t+t_0}{2}\leq\zeta\leq t} c_z\|e(\zeta)\| \leq c_z k_e\|e(t_0)\| \exp\left(-\frac{\lambda_e(t - t_0)}{2}\right). \tag{30}$$

Let the composite state be $x_{ez} = [e^{\mathrm{T}}, z^{\mathrm{T}}]^{\mathrm{T}}$. Using (27), we obtain

$$\begin{aligned}\|x_{ez}(t)\| &\leq \|e(t)\| + \|z(t)\|,\\ &\leq \|e(t)\| + k_z \left\| z\left(\frac{t + t_0}{2}\right) \right\| \exp\left(-\frac{\lambda_z(t - t_0)}{2}\right)\\ &\quad + \sup_{\frac{t+t_0}{2}\leq\zeta\leq t} c_z\|e(\zeta)\|,\end{aligned}$$

which, on substitution of (25) and (30), and using the fact that $\exp(-|a|) \leq \exp(-\frac{|a|}{2})$ for all $a \in \mathbb{R}$ yields

$$\begin{aligned}\|x_{ez}(t)\| &\leq (1 + c_z)k_e\|e(t_0)\| \exp\left(-\frac{\lambda_e(t - t_0)}{2}\right)\\ &\quad + k_z \left\| z\left(\frac{t + t_0}{2}\right) \right\| \exp\left(-\frac{\lambda_z(t - t_0)}{2}\right).\end{aligned}$$

Substitution of (28) into the preceding gives

$$\begin{aligned}\|x_{ez}(t)\| &\leq (1 + c_z)k_e\|e(t_0)\| \exp\left(-\frac{\lambda_e(t - t_0)}{2}\right)\\ &\quad + k_z \exp\left(-\frac{\lambda_z(t - t_0)}{2}\right) \left(\sup_{t_0\leq\zeta\leq\frac{t+t_0}{2}} c_z\|e(\zeta)\| \right.\\ &\quad \left. + k_z\|z(t_0)\| \exp\left(-\frac{\lambda_z(t - t_0)}{2}\right) \right),\end{aligned}$$

and using (29) yields

$$\begin{aligned}\|x_{ez}(t)\| &\leq (1 + c_z)k_e\|e(t_0)\| \exp\left(-\frac{\lambda_e(t - t_0)}{2}\right)\\ &\quad + c_z k_e k_z\|e(t_0)\| \exp\left(-\frac{\lambda_z(t - t_0)}{2}\right)\\ &\quad + k_z^2\|z(t_0)\| \exp(-\lambda_z(t - t_0)).\end{aligned}$$

Finally, defining $\lambda_{ez} = \frac{1}{2}\min\{\lambda_e, \lambda_z\}$, and using the facts that $\|e(t_0)\| \leq \|x_{ez}(t_0)\|$, $\|z(t_0)\| \leq \|x_{ez}(t_0)\|$, we obtain

$$\|x_{ez}(t)\| \leq k_{ez}\|x_{ez}(t_0)\| \exp(-\lambda_{ez}(t - t_0)),$$

where $k_{ez} = (1+c_z)k_e + c_z k_e k_z + k_z^2$, valid for all $t \geq t_0 \geq 0$. This shows that $x_{ez} = 0$ is an exponentially stable equilibrium point of the reduced system (22), (23).

Hence by a Converse Lyapunov Theorem [5, Theorem 4.14, pp. 162 – 163], there exists a Lyapunov function $V\colon [0,\infty)\times D_{e,z}\to[0,\infty)$ that satisfies

$$\begin{aligned} c_1\|x_{ez}\|^2 \le V(t,x_{ez}) \le c_2\|x_{ez}\|^2,\\ \frac{\partial V}{\partial t}+\frac{\partial V}{\partial x_{ez}}\hat{f}(t,x_{ez}) \le -c_3\|x_{ez}\|^2, \end{aligned}$$

where

$$\hat{f}(t,x_{ez}) = \begin{bmatrix} \dot{\phi}_e, \\ \phi_e^{(2)}, \\ \vdots, \\ \phi_e^{(\rho-1)}, \\ -a_e^{\mathrm{T}} e, \\ g(e+x_r(t),z,h(t,e,z)) \end{bmatrix} \in \mathbb{R}^n.$$

It can be seen that the Lyapunov function condition of assumption 2 in Theorem 1 is satisfied with $W_1(r)=c_1\|r\|^2$, $W_2(r)=c_2\|r\|^2$, and $W_3(r)=c_3\|r\|^2$. Further, by choosing c sufficiently small, the set $\{x_{ez}\in D_{e,z} \mid W_1(x_{ez}) = c_1\|x_{ez}\|^2 \le c\}$ can be made compact. We conclude that satisfaction of assumption 2 in the current theorem implies satisfaction of assumption 2 in Theorem 1.

We will use Proposition 1 to show that the origin is an exponentially stable equilibrium point of the boundary layer system (24), uniformly in (t,e,z), so that assumption 3 of Theorem 1 is satisfied. Define

$$\tilde{g}(t,e,z,u) = -\alpha\tilde{f}(t,e,z,u),$$

so that the boundary layer system (24) can be rewritten as

$$\frac{dv}{d\tau} = \tilde{g}(t,e,z,v+h(t,e,z)). \tag{31}$$

Then, using the definitions of $\tilde{f}$ in (21) and α, we have

$$\begin{aligned} \frac{\partial\tilde{g}}{\partial u}(t,e,z,u) &= -\operatorname{sign}\left(\frac{\partial f}{\partial u}\right)\frac{\partial\tilde{f}}{\partial u}(t,e,z,u),\\ &= -\operatorname{sign}\left(\frac{\partial f}{\partial u}\right)\frac{\partial f}{\partial u}(e+x_r(t),z,u),\\ &= -\left|\frac{\partial f}{\partial u}(e+x_r(t),z,u)\right|, \end{aligned}$$

and hence,

$$\frac{\partial\tilde{g}}{\partial u}(t,e,z,h(t,e,z)) = -\left|\frac{\partial f}{\partial u}(e+x_r(t),z,h(t,e,z))\right|.$$

From the preceding, assumption 3 implies that the eigenvalue condition (5) holds. Proposition 1 then applies to show that the boundary layer system (31) or (24), has the origin as an exponentially stable equilibrium, uniformly in $(t,e,z)\in[0,\infty)\times D_{e,z}$.

Thus all assumptions of Theorem 1 are implied by the current assumptions. Observe that the set

$$\Omega_{e,z}\subset\{x_{ez}\in D_{e,z} \mid W_2(x_{ez}) = c_2\|x_{ez}\|^2 \le \rho c, \rho\in(0,1)\}$$

is compact by the choice of c above. Then, for each such compact set $\Omega_{e,z}$, there exists a positive constant ϵ^* such that for all $t>0$, $(e_0,z_0)\in\Omega_{e,z}$, $u_0-h(0,e_0,z_0)\in\Omega_v$, and $\epsilon\in(0,\epsilon^*)$, the system (19), (21) has a unique solution $e(t,\epsilon)$, $z(t,\epsilon)$, $u(t,\epsilon)$ on $[0,\infty)$, and

$$\begin{aligned} e(t,\epsilon)-\bar{e}(t) = O(\epsilon),\\ z(t,\epsilon)-\bar{z}(t) = O(\epsilon), \end{aligned}$$

holds uniformly for all $t\in[0,\infty)$, where $\bar{e}(t)$ and $\bar{z}(t)$ are the solutions of the reduced system (22) and (23) respectively. Since $\bar{e}(t)$ is the solution of the exponentially stable system (22), and $x_r(t)$ is the solution of system (17), using the definition of $e=x-x_r$ in the above yields

$$\begin{aligned} x(t,\epsilon)-x_r(t)-\bar{e}(t) = O(\epsilon),\\ x(t,\epsilon)-x_r(t)-\exp(At)e_0 = O(\epsilon), \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 1\\ -a_{e0} & -a_{e1} & \dots & -a_{e(\rho-1)} \end{bmatrix} \in \mathbb{R}^{\rho\times\rho}.$$

Since (22) is exponentially stable, A is Hurwitz, so that for any $\epsilon>0$, there exists $T<\infty$ such that

$$\|\exp(At)e_0\| \le \epsilon, \qquad \forall t\ge T.$$

Then for all $t\ge T$, we reach the desired conclusion

$$x(t,\epsilon)-x_r(t) = O(\epsilon).$$

■

Remark 4. *Observe that if $e_0=0$, then T can be chosen to be 0, and $x(t,\epsilon)-x_r(t)=O(\epsilon)$ holds for all $t>0$. This can be achieved by setting $x_{r0}=x_0$. See also [1] for further discussions.*

IV. Extension to Minimum-phase Nonaffine-in-control MIMO Systems

Consider the n-th order MIMO *nonaffine-in-control* system expressed in normal form

$$\begin{aligned} \phi_1^{(\rho_1)} &= f_1(x,z,u), & x_1(0) &= x_{10},\\ \phi_2^{(\rho_2)} &= f_2(x,z,u), & x_2(0) &= x_{20},\\ &\vdots & &\vdots\\ \phi_m^{(\rho_m)} &= f_m(x,z,u), & x_m(0) &= x_{m0},\\ \dot{z} &= g(x,z,u), & z(0) &= z_0, \end{aligned} \tag{32}$$

defined for all $(x,z,u) \in D_x \times D_z \times D_u$ with $D_x \subset \mathbb{R}^{\rho}$, $D_z \subset \mathbb{R}^{n-\rho}$, and $D_u \subset \mathbb{R}^m$ being domains containing the origins. The (partial) state x is defined as $x = [x_1, x_2, \ldots, x_m]^{\mathrm{T}} \in \mathbb{R}^{\rho}$, $\rho = \sum_{i=1}^{m} \rho_i$, with each $x_i = [\phi_i, \dot{\phi}_i, \ldots, \phi_i^{(\rho_i-1)}]^{\mathrm{T}} \in \mathbb{R}^{\rho_i}$ for $i \in \{1,2,\ldots,m\}$, and $\phi^{(q)}$ denotes the q-th time derivative of ϕ. The state vector of the system is $[x^{\mathrm{T}}, z^{\mathrm{T}}]^{\mathrm{T}}$, $u = [u_1, u_2, \ldots, u_m]^{\mathrm{T}} \in \mathbb{R}^m$ is the control input, and $f_i \colon D_x \times D_z \times D_u \to \mathbb{R}$ for $i \in \{1,2,\ldots,m\}$, $g \colon D_x \times D_z \times D_u \to \mathbb{R}^{n-\rho}$ are continuously differentiable functions of their arguments. Define

$$f(x,z,u) = [f_1(x,z,u), f_2(x,z,u), \ldots, f_m(x,z,u)]^{\mathrm{T}}. \tag{33}$$

Assume that the inverse of the function $u \mapsto f(x,z,u)$ exists for each fixed $(x,z) \in D_x \times D_z$, but that it cannot be written in closed form.

The problem is to design a controller so that x tracks the state of a chosen ρ-th order *stable* linear reference model described by the following set of linear ordinary differential equations

$$\begin{aligned} \phi_{r1}^{(\rho_1)} + a_{r1}^{\mathrm{T}} x_{r1} &= b_{r1} r_1, & x_{r1}(0) &= x_{r10},\\ \phi_{r2}^{(\rho_2)} + a_{r2}^{\mathrm{T}} x_{r2} &= b_{r2} r_2, & x_{r2}(0) &= x_{r20},\\ &\vdots & &\vdots\\ \phi_{rm}^{(\rho_m)} + a_{rm}^{\mathrm{T}} x_{rm} &= b_{rm} r_m, & x_{rm}(0) &= x_{rm0}, \end{aligned} \tag{34}$$

where for each $i \in \{1,2,\ldots,m\}$, the corresponding vectors are $a_{ri} = [a_{ri0}, a_{ri1}, \ldots, a_{ri(\rho_i-1)}]^{\mathrm{T}} \in \mathbb{R}^{\rho_i}$ and $x_{ri} = [\phi_{ri}, \dot{\phi}_{ri}, \ldots, \phi_{ri}^{(\rho_i-1)}]^{\mathrm{T}} \in \mathbb{R}^{\rho_i}$, r_i is a continuously differentiable reference input signal with bounded time derivative $\dot{r}_i$. Let $r = [r_1, r_2, \ldots, r_m]^{\mathrm{T}}$. Here, ρ_i corresponds to those defined for system (32) so that x_{ri} is of the same dimension as x_i in (32), and the state of the reference model $x_r = [x_{r1}^{\mathrm{T}}, x_{r2}^{\mathrm{T}}, \ldots, x_{rm}^{\mathrm{T}}]^{\mathrm{T}}$ is of the same dimension as x in (32). Stability of the reference model requires that for each $i \in \{1,2,\ldots,m\}$, all roots of the characteristic equation

$$s^{\rho_i} + a_{ri(\rho_i-1)} s^{\rho_i-1} + \cdots + a_{ri1} s + a_{ri0} = 0$$

lie in $\mathbb{C}_-$.

Define the tracking error $e = x - x_r$, which can be decomposed as

$$e = [e_1^{\mathrm{T}}, e_2^{\mathrm{T}}, \ldots, e_m^{\mathrm{T}}]^{\mathrm{T}},$$
$$e_i = x_i - x_{ri} = [\phi_{ei}, \dot{\phi}_{ei}, \ldots, \phi_{ei}^{(\rho_i-1)}]^{\mathrm{T}}, \quad i \in \{1,2,\ldots,m\}.$$

Choose the desired *stable* error dynamics as described by the following set of linear ordinary differential equations

$$\begin{aligned} \phi_{e1}^{(\rho_1)} + a_{e1}^{\mathrm{T}} e_1 &= 0, & e_1(0) &= e_{10} = x_{10} - x_{r10},\\ \phi_{e2}^{(\rho_2)} + a_{e2}^{\mathrm{T}} e_2 &= 0, & e_2(0) &= e_{20} = x_{20} - x_{r20},\\ &\vdots & &\vdots\\ \phi_{em}^{(\rho_m)} + a_{em}^{\mathrm{T}} e_m &= 0, & e_m(0) &= e_{m0} = x_{m0} - x_{rm0}, \end{aligned} \tag{35}$$

where for each $i \in \{1,2,\ldots,m\}$, the corresponding vectors are $a_{ei} = [a_{ei0}, a_{ei1}, \ldots, a_{ei(\rho_i-1)}]^{\mathrm{T}} \in \mathbb{R}^{\rho_i}$. Similarly, ρ_i corresponds to those defined for system (32) so that e_i is of the same dimension as x_i in (32), and the state of the desired error dynamics e is of the same dimension as x in (32). Stability of the desired error dynamics requires that for each $i \in \{1,2,\ldots,m\}$, all roots of the characteristic equation

$$s^{\rho_i} + a_{ei(\rho_i-1)} s^{\rho_i-1} + \cdots + a_{ei1} s + a_{ei0} = 0$$

lie in $\mathbb{C}_-$. Similar to the SISO case, observe that this is a minor extension of [1], wherein a_{eij} is set equal to a_{rij} for $i \in \{1,2,\ldots,m\}$, $j \in \{0,1,\ldots,\rho_i-1\}$.

The open-loop (time-varying) error dynamics are then given by the system

$$\begin{aligned} \phi_{e1}^{(\rho_1)} &= f_1(e + x_r(t), z, u) + a_{r1}^{\mathrm{T}} x_{r1}(t) - b_{r1} r_1(t),\\ \phi_{e2}^{(\rho_2)} &= f_2(e + x_r(t), z, u) + a_{r2}^{\mathrm{T}} x_{r2}(t) - b_{r2} r_2(t),\\ &\vdots\\ \phi_{em}^{(\rho_m)} &= f_m(e + x_r(t), z, u) + a_{rm}^{\mathrm{T}} x_{rm}(t) - b_{rm} r_m(t),\\ \dot{z} &= g(e + x_r(t), z, u), \end{aligned} \tag{36}$$

with initial conditions $e(0) = e_0$, $z(0) = z_0$. Similarly, observe that time variance in (36) is induced by the external signals $x_r(t)$ and $r(t)$ only.

The ideal dynamic inversion control is found by solving the system of m equations

$$
\begin{aligned}
f_1(e+x_r(t),z,u)+a_{r1}^{\mathrm{T}}x_{r1}(t)-b_{r1}r_1(t)&=-a_{e1}^{\mathrm{T}}e_1,\\
f_2(e+x_r(t),z,u)+a_{r2}^{\mathrm{T}}x_{r2}(t)-b_{r2}r_2(t)&=-a_{e2}^{\mathrm{T}}e_2,\\
&\vdots \\
f_m(e+x_r(t),z,u)+a_{rm}^{\mathrm{T}}x_{rm}(t)-b_{rm}r_m(t)&\\
=-a_{em}^{\mathrm{T}}e_m,&
\end{aligned}
\tag{37}
$$

for $u \in \mathbb{R}^m$, resulting in the exponentially stable closed-loop tracking error dynamics (35). Since (37) cannot (in general) be solved explicitly for u, an approximation of the dynamic inversion controller is constructed by introducing fast dynamics

$$
\epsilon\dot{u}=P\tilde{f}(t,e,z,u), \quad u(0)=u_0, \tag{38}
$$

where $P\in\mathbb{R}^{m\times m}$ is a chosen constant matrix, and with (33),

$$
\tilde{f}(t,e,z,u)=f(e+x_r(t),z,u)+A_rx_r(t)-B_rr(t)+A_ee,
$$

$$
A_r=\begin{bmatrix} a_{r1}^{\mathrm{T}} & 0 & \dots & \dots & 0\\ 0 & a_{r2}^{\mathrm{T}} & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & \dots & 0 & a_{rm}^{\mathrm{T}}\end{bmatrix}\in\mathbb{R}^{m\times\rho},
$$

$$
B_r=\begin{bmatrix} b_{r1} & 0 & \dots & \dots & 0\\ 0 & b_{r2} & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & \dots & 0 & b_{rm}\end{bmatrix}\in\mathbb{R}^{m\times m},
$$

$$
A_e=\begin{bmatrix} a_{e1}^{\mathrm{T}} & 0 & \dots & \dots & 0\\ 0 & a_{e2}^{\mathrm{T}} & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & \dots & 0 & a_{em}^{\mathrm{T}}\end{bmatrix}\in\mathbb{R}^{m\times\rho}.
$$

Let $u=h(t,e,z)$ be an isolated root of $\tilde{f}(t,e,z,u)=0$. The reduced system for (36), (38), obtained by setting $\epsilon=0$ and $u=h(t,e,z)$ is

$$
\begin{aligned}
\phi_{e1}^{(\rho_1)}&=-a_{e1}^{\mathrm{T}}e_1, & e_1(0)&=e_{10},\\
\phi_{e2}^{(\rho_2)}&=-a_{e2}^{\mathrm{T}}e_2, & e_2(0)&=e_{20},\\
&\vdots & &\vdots\\
\phi_{em}^{(\rho_m)}&=-a_{em}^{\mathrm{T}}e_m, & e_m(0)&=e_{m0},\\
\dot{z}&=g(e+x_r(t),z,h(t,e,z)), & z(0)&=z_0.
\end{aligned}
\tag{39}
$$

With $v=u-h(t,e,z)$ and $\tau=t/\epsilon$, the boundary layer system is

$$
\frac{dv}{d\tau}=P\tilde{f}(t,e,z,v+h(t,e,z)). \tag{40}
$$

Applying Theorem 1 to (36), (38), and noting the definition of f in (33) yields the following.

Theorem 5 (Hovakimyan et al. [1, Theorem 3])**.** *Consider the system* (36) *and* (38)*, and let $u=h(t,e,z)$ be an isolated root of $\tilde{f}(t,e,z,u)=0$. Assume that the following conditions hold for all*

$$
(t,e,z,u-h(t,e,z),\epsilon)\in[0,\infty)\times D_{e,z}\times D_v\times[0,\epsilon_0],
$$

for some domains $D_{e,z}\subset\mathbb{R}^n$ and $D_v\subset\mathbb{R}^m$ which contain their respective origins:

1) *On any compact subset of $D_{e,z}\times D_v$, the functions f, g, their first partial derivatives with respect to (x,z,u), and $r(t)$, $\dot{r}(t)$ are continuous and bounded, $h(t,e,z)$ and $\frac{\partial f}{\partial u}(x,z,u)$ have bounded first partial derivatives with respect to their arguments, and $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial z}$, $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial z}$ as functions of $(e+x_r(t),z,h(t,e,z))$, are Lipschitz in e and z uniformly in t.*
2) *The origin is an exponentially stable equilibrium of the system*
$$\dot{z}=g(x_r(t),z,h(t,0,z)).$$
The map $(e,z)\mapsto g(e+x_r(t),z,h(t,e,z))$ is continuously differentiable and Lipschitz in (e,z) uniformly in t.
3) *For every $(t,e,z)\in[0,\infty)\times D_{e,z}$, all the eigenvalues of*
$$P\frac{\partial f}{\partial u}(e+x_r(t),z,h(t,e,z))$$
have negative real parts bounded away from zero.

Then the origin of (40) *is exponentially stable. Let $R_v\subset D_v$ be the region of attraction of the autonomous system*

$$
\frac{dv}{d\tau}=P\tilde{f}(0,e_0,z_0,v+h(0,e_0,z_0)),
$$

and Ω_v be a compact subset of R_v. Then, for each compact subset $\Omega_{e,z}\subset D_{e,z}$, there exists positive constants ϵ^* and T such that for all $t>0$, $(e_0,z_0)\in\Omega_{e,z}$, $u_0-h(0,e_0,z_0)\in\Omega_v$, and $\epsilon\in(0,\epsilon^*)$, the system (32)*,* (34)*,* (38) *has a unique solution $x(t,\epsilon)$, $z(t,\epsilon)$, $x_r(t)$, $u(t,\epsilon)$ on $[0,\infty)$, and*

$$
x(t,\epsilon)-x_r(t)=O(\epsilon)
$$

holds uniformly for all $t\in[T,\infty)$.

Proof: Similar to the proof of Theorem 4, we show that satisfaction of the above assumptions imply satisfaction of those of Theorem 1 to get the desired conclusions. In the same way as the proof of Theorem 4, it can be shown that the first assumption of Theorem 1 is implied by assumption 1 above.

For the second assumption, note that the first m equations of (39) represents m decoupled exponentially stable linear time invariant systems with composite state $e = [e_1^{\mathrm{T}}, e_2^{\mathrm{T}}, \ldots, e_m^{\mathrm{T}}]^{\mathrm{T}}$. Hence for each $i \in \{1, 2, \ldots, m\}$, the solutions satisfy

$$\|e_i(t)\| \leq k_{ei} \|e_i(t_0)\| \exp(-\lambda_{ei}(t - t_0)),$$

for some positive constants k_{ei} and λ_{ei}, for all $t \geq t_0$. Using the facts that

$$\|e(t)\| \leq \sum_{i=1}^{m} \|e_i(t)\|, \qquad \|e_i(t)\| \leq \|e(t)\|,$$

and for all $c_1 > c_2 > 0$,

$$\exp(-c_1(t - t_0)) \leq \exp(-c_2(t - t_0)), \qquad \forall t \geq t_0,$$

we have for all $t \geq t_0$,

$$\begin{aligned}
\|e(t)\| &\leq \sum_{i=1}^{m} \|e_i(t)\| \leq \sum_{i=1}^{m} k_{ei} \|e_i(t_0)\| \exp(-\lambda_{ei}(t - t_0)), \\
&\leq \|e(t_0)\| \sum_{i=1}^{m} k_{ei} \exp(-\lambda_{ei}(t - t_0)), \\
&\leq \|e(t_0)\| \exp(-\lambda_e(t - t_0)) \sum_{i=1}^{m} k_{ei}, \\
&= k_e \|e(t_0)\| \exp(-\lambda_e(t - t_0)),
\end{aligned}$$

where $0 < \lambda_e = \min\{\lambda_{e1}, \lambda_{e2}, \ldots, \lambda_{em}\}$ and $k_e = \sum_{i=1}^{m} k_{ei}$. Hence the verification of assumption 2 proceeds as in the proof of Theorem 4.

We will use Proposition 1 to show that the origin of the boundary layer system (40) is an exponentially stable equilibrium point, uniformly in (t, e, z). Define

$$\tilde{g}(t, e, z, u) = P\tilde{f}(t, e, z, u),$$

so that the boundary layer system (40) can be rewritten as

$$\frac{dv}{d\tau} = \tilde{g}(t, e, z, v + h(t, e, z)).$$

Then, taking derivatives,

$$\frac{\partial \tilde{g}}{\partial u}(t, e, z, u) = P \frac{\partial \tilde{f}}{\partial u}(t, e, z, u) = P \frac{\partial f}{\partial u}(e + x_r(t), z, u),$$

and hence,

$$\frac{\partial \tilde{g}}{\partial u}(t, e, z, h(t, e, z)) = P \frac{\partial f}{\partial u}(e + x_r(t), z, h(t, e, z)).$$

Hence assumption 3 implies that the eigenvalue condition (5) holds, and Proposition 1 applies to show that the boundary layer system has the origin as an exponentially stable equilibrium, uniformly in $(t, e, z) \in [0, \infty) \times D_{e,z}$.

The stated conclusions follow immediately from Theorem 1 in a similar manner to the proof of Theorem 4. ■

Conclusions

The statements of the ADI method are re-stated with some minor notational corrections, and the proofs are expanded. This is to supplement our existing results in [2]–[4]. As such, Theorem 1 and 2 in [3] should be replaced by Theorem 4 and 5 in the present report respectively. Also, Theorem 1 in [4] should be replaced by Theorem 4 in the present report.

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References

[1] N. Hovakimyan, E. Lavretsky, and A. Sasane, "Dynamic inversion for nonaffine-in-control systems via time-scale separation. part I," *J. Dyn. Control Syst.*, vol. 13, no. 4, pp. 451 – 465, Oct. 2007.

[2] J. Teo and J. P. How, "Equivalence between approximate dynamic inversion and proportional-integral control," in *Proc. 47th IEEE Conf. Decision and Control*, Cancun, Mexico, Dec. 2008, pp. 2179 – 2183.

[3] ——, "Equivalence between approximate dynamic inversion and proportional-integral control," MIT, Cambridge, MA, Tech. Rep. ACL08-01, Sep. 2008, Aerosp. Controls Lab. [Online]. Available: http://hdl.handle.net/1721.1/42839

[4] J. Teo, J. P. How, and E. Lavretsky, "On approximate dynamic inversion and proportional-integral control," in *Proc. American Control Conf.*, St. Louis, MO, Jun. 2009, to appear.

[5] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ: Prentice Hall, 2002.

[6] L. Grüne, E. D. Sontag, and F. R. Wirth, "Assymptotic stability equals exponential stability, and iss equals finite energy gain – if you twist your eyes," *Syst. Control Lett.*, vol. 38, no. 2, pp. 127 – 134, Oct. 1999.