A rank 2 theory for constrained Willmore tori in the 3-dimensional sphere

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Abstract

The main subject of this thesis are constrained Willmore tori in the 3-dimensional sphere S^3. It is known that constrained Willmore tori in the 4-sphere come with an associated $\mathbb{C}^*$-family of flat $\mathrm{SL}(4,\mathbb{C})$-connections ∇^λ. This allows to study constrained Willmore tori as an integrable system. The initial surface can be reconstructed by holomorphic data on the spectral curve Σ, which is the riemann surface on which the eigenlines of ∇^λ are well-defined. If the constrained Willmore torus lies in S^3, there is a further symmetry on the spectral curve, a holomorphic involution σ. In this thesis we show that this involution allows to reduce the family of $\mathrm{SL}(4,\mathbb{C})$-connections into a family ∇^x of $\mathrm{SL}(2,\mathbb{C})$-connections. We achieve this by pushing forward the eigenline bundle of ∇^λ on the quotient Σ/σ. Therefore, the parameter x takes values in a hyperelliptic surface. The rank 2 family of connections then allows to give a Sym-Bobenko formula, similiar to the case of constant mean curvature surfaces in S^3. Further, if the quotient surface $\Sigma/\sigma = \mathbb{C}P^1$, the surface is of constant mean curvature in a space form.

Kurzzusammenfassung

In dieser Arbeit betrachten wir, wie der Titel bereits vermuten lässt, constrained Willmore Tori in der 3-dimensionalen Sphäre S^3. Ein constrained Willmore Torus in der 4-dimensionalen Sphäre definiert eine assoziierte $\mathbb{C}^*$-Familie flacher $\mathrm{SL}(4,\mathbb{C})$-Zusammenhänge ∇^λ. Mittels dieser bilden constrained Willmore Tori ein integrables System. Man kann einen constrained Willmore Torus aus holomorphen Daten über der Spektralkurve zurückgewinnen. Die Spektralkurve ist die Riemannsche Fläche, welche die Eigenlinien der Zusammenhänge ∇^λ parametrisiert. Ist ein constrained Willmore Torus in einer 3-dimensionalen Sphäre enthalten, so erhält man eine Symmetrie auf der Spektralkurve, genauer, eine holomorphe Involution σ. In dieser Arbeit zeigen wir, dass diese Symmetrie es ermöglicht eine assoziierte Familie von flachen $\mathrm{SL}(2,\mathbb{C})$-Zusammenhängen zu betrachten. Wir erreichen dies durch den Pushforward der Eigenlinienbündel von ∇^λ auf die Quotientenfläche Σ/σ. Der Parameter x kommt dann aus dieser hyperelliptischen Fläche. Damit haben wir eine Beschreibung von constrained Willmore Tori welche näher an der integrablen System Theorie für CMC Tori liegt. Beispielsweiße erlaubt die Rang 2 Formulierung eine Sym-Bobenko Formel um einen constrained Willmore Torus zu erhalten. Ist die hyperelliptische Fläche Σ/σ bereits $\mathbb{C}P^1$, so ist der constrained Willmore Torus bereits von konstanter mittlerer Krümmung in einer Raumform.

Keywords
Constrained Willmore torus, Integrable system, Quaternionic surface theory, Sym-Bobenko formula

Contents

Introduction

Optimization is one of the major tasks in mathematics (and life). This also includes differential geometry, where one can ask the question whether there is an optimal realisation of a manifold in a given target space, i.e, whether there is an immersion

$$f : M \to N$$

of a manifold M into a target manifold N that is particularly round, minimizing some functional, very symmetric or just a beautiful surface[1]. In our case M is a surface and the target space is a simply connected 3-dimensional space form, i.e., the Euclidean space $\mathbb{R}^3$, the hyperbolic space $\mathbb{H}^3$ or the round sphere S^3 of constant curvature 0, -1, and 1, respectively. The investigation and construction of constant mean curvature (CMC) surfaces into a 3-dimensional space form is classical. A surface of constant mean curvature H is a critical point of the area under enclosed volume constraint. Hopf conjectured in [Hop50] that the only CMC surfaces in $\mathbb{R}^3$ are round spheres. This conjecture was partly proven by Alexandrov [Ale58] under the additional assumption that the surface is embedded. Over 30 years later Wente [Wen86] gave an example for an immersed CMC torus in $\mathbb{R}^3$, i.e., disproved the Hopf-conjecture. The case of CMC spheres is trivial in any space form. They are given by round spheres. As the curvature of the hyperbolic 3 space has the correct sign to use the Alexandrov maximum principle, all embedded CMC surfaces in $\mathbb{H}^3$ are round spheres. In contrary Lawson [Law70] gave examples of embedded minimal surfaces in S^3 for any genus. He proposed that the only embedded minimal torus in S^3 is the Clifford torus, which was then shown by Brendle [Bre13]. Based on Brendles work Andrews and Li [AL15] proved that all embedded CMC tori in S^3 are tori of revolution.

A CMC surface in S^3 defines an associated holomorphic $\mathbb{C}^*$-family of flat $\mathrm{SL}(2,\mathbb{C})$-connections ∇^λ on the trivial bundle $\underline{\mathbb{C}}^2 \to M$. The family satisfies three conditions:

i) Asymptotic: ∇^λ has a first order pole in $\lambda = \infty$. The residue $\Phi \in \Gamma(K\mathrm{SL}(2,\mathbb{C}))$ is non-vanishing and nilpotent.

ii) Intrinsic closing: The familiy is unitary for $\lambda \in S^1 \subset \mathbb{C}$.

iii) Extrinsic closing: There are 2 points λ_1, $\lambda_2 \in S^1$ such that ∇^{λ_i} is trivial.

Let F_λ be a parallel frame of ∇^λ. Then

$$f = F_{\lambda_2}^{-1} F_{\lambda_1} \tag{1}$$

is (under some further assumptions) the initial CMC immersion into $S^3 = \mathrm{SU}(2)$. The points λ_1, λ_2 are called Sym-points and (1) is the Sym-Bobenko formula. Therefore, constructing CMC surfaces is tantamaunt to writing down such families of connections. The flat connections approach translates the problem of solving the PDE

[1] Clearly a surface is beautiful if it is CMC or Willmore, just see the pictures of Nick Schmitt on http://service.ifam.uni-hannover.de/~geometriewerkstatt/

$$H = \text{const.}$$

into an ODE of finding parallel frames to ∇^λ. The complexity of the family ∇^λ depends on the genus of M. For spheres, ∇^λ is trivial for all λ and contains no further information. In the early 90's, Hitchin [Hit90] and Bobenko [Bob91b] used the fact that the fundamental group of a torus M is abelian in order to prove that CMC tori in S^3 form an integrable system. The holonomy representation of the connections ∇^λ is determined by the eigenlines of the connection. The commutativity of the fundamental group implies that a simple eigenline of the holonomy along one path is an eigenline for every path in the fundamental group. Therefore there is a hyperelliptic curve Σ on which the eigenlines of the holonomy of ∇^λ are well defined. The surface Σ is called the spectral curve of the immersion f and the eigenlines of the holonomy define a group homomorphism

$$\Psi : M \to \mathrm{Jac}(\Sigma)$$

from the torus into the Jacoby variety of Σ. This map, together with further holomorphic data on Σ, determines the family of flat connections and consequently the CMC immersion. As shown in [Bob91a], this approach works, with some adjustments to the Sym-Bobenko formula (1), also for $\mathbb{R}^3$ and $\mathbb{H}^3$. Therefore all CMC tori in 3-dimensional space form can be written in terms of theta functions.

There are two natural ways to generalize the CMC tori integrable system theory. One possibility is to consider surfaces of higher genus. As the fundamental group is non-abelian, it is not possible to define the spectral curve via the eigenline of the holonomy, cf. [Hel12b]. Integrable system theory for high genus CMC surfaces with symmetries, just like the famous Lawson minimal surfaces [Law70], was considered in [Hel14b, HHS18, HHS15, HHT19]. In this thesis we will follow the second way. We will stay in the genus 1 case, but instead of CMC tori, we will take a look on constrained Willmore tori in S^3 which include CMC tori in all space forms.

The Willmore functional is given by

$$\mathcal{W} = \int_M (H^2 + 1)\, dA,$$

and it measures the roundness of a surface and is considered to be the bending energy in physics. The Willmore energy is invariant under conformal changes of the metric of the ambient space. Hence there is no reason to distinguish between the space forms. A Willmore surface in S^3 is an immersion $f : M \to S^3$ which is a critical point of $\mathcal{W}$. If f is critical among conformal variations, the immersion is called constrained Willmore (cW). The Euler Lagrange equation for constrained Willmore tori is given by

$$\Delta H + 2H(H^2 - K) = \langle q, \mathring{\mathrm{I\!I}} \rangle. \tag{2}$$

Here K denotes the Gaussian curvature, $\mathring{\mathrm{I\!I}}$ is the trace free part of the second fundamental form, and the Lagrange multiplier $q \in H^0(K^2)$ is a holomorphic quadratic differential. When the immersion is isothermic, for example for CMC surfaces, the Hopf differential takes values in a real line in $\mathbb{C}$, so does $\mathring{\mathrm{I\!I}}$. Therefore, (2) degenerates for such tori. Kuwert and Schätzle [KS13] used higher derivatives of the projection into the Teichmüller space and showed that isothermic constrained Willmore tori satisfy (2).

In the case of genus 0, Bryant [Bry84] classified all Willmore spheres in S^3 as compactifications of minimal surfaces in $\mathbb{R}^3$ with planar ends. The fourth-order Euler Lagrange equation reduces to the second-order equation of minimal surfaces. A constrained Willmore 2-sphere is, since all spheres are conformally equivalent, already Willmore. In particular all cW spheres are isothermic. This is not true for higher genus: There are examples by Pinkall [Pin85], and Ferus and Pedit [FP90] of Willmore tori which are not isothermic and therefore not CMC in a space form.

As in the CMC case there is a family ∇^λ of flat connections. The family ∇^λ associated to a cW surface in S^4 consists of flat $\mathrm{SL}(4,\mathbb{C})$-connections on the trivial bundle $\underline{\mathbb{C}}^4 \to M$. In contrast to the CMC case there is no Sym-Bobenko-formula. Bohle showed in [Boh10], that the spectral curve parametrising the eigenlines of the holonomy is a compact 4-fold cover of $\mathbb{C}P^1$. In both cases, CMC and cW, the family ∇^λ is given by the pushforward of connections on the eigenlinebundles. Applying the integrable system techniques, Heller [Hel14a] constructed examples of constrained Willmore tori. Those tori are equivariant, i.e., there is a 1-parameter family of Möbius symmetries. The examples by Ferus and Pedit, and Pinkall are equivariant as well.

When the immersion maps into a 3-sphere $S^3 \subset S^4$, the spectral curve Σ has a further symmetry, a holomomorphic involution σ. The involution σ enables to split the 4-fold covering $\Sigma \to \mathbb{C}P^1$ into 2 double covers

$$\Sigma \longrightarrow X = \Sigma/\sigma \longrightarrow \mathbb{C}P^1.$$

In this thesis we will show that pushing forward the eigenlines of ∇^λ defines a rank 2 bundle $\hat{E}$ on the in between surface X which is invariant under ∇^λ. Therefore there exists an associated meromorphic X-family $\hat{\nabla}^x$ of rank 2 connections. The bundle $\hat{E}$ is not trivial as a bundle over X. Let $x_1, x_2 \in X$ be the two points over $\lambda = 1 \in \mathbb{C}P^1$. As $\nabla^{\lambda=1}$ is trivial, the connections $\hat{\nabla}^{x_1}$ and $\hat{\nabla}^{x_2}$ are trivial as well. By trivializing with a meromorphic frame, we get a meromorphic family of connections on the trivial $\underline{\mathbb{C}}^2$ bundle. The main theorem of the thesis is the following:

> **Theorem**
> Let T^2 be a torus, X be a hyperelliptic surface, and ∇^x an admissible meromorphic X-family of flat connections on $\underline{\mathbb{C}}^2 \to T^2$. The surface
>
> $$f = F_{x_2}^{-1} F_{x_1} : T^2 \to S^3 \qquad \text{(Sym-Bob)}$$
>
> is a constrained Willmore immersion. Here F_{x_1}, F_{x_2} are parallel frames of the trivial connections ∇^{x_1} and ∇^{x_2}, respectively.

Similiar to the CMC case, a family is admissible if it satisfies a intrinsic and extrinsic closing condition as well as some certain asymptotic behaviour. While it was possible before to reconstruct the surface by the theory developed in [BLPP12], a Sym-Bobenko reconstruction is easier to control when doing deformation theory. Therefore, it may be helpful for a higher genus flow of cW surfaces or when studying the Whitham flow.

The thesis is organized as follows. It consists of two parts. The first part (Section 1-3) contains fundamentals and preliminarys. Deeper theory on constrained Willmore surfaces and the integrable system theory, as well as the main result of the thesis can be found in the second part (Section 4-6).

As the Willmore functional and the constraint are invariant under conformal changes of the target space it is more natural to use a Möbius geometric framework like the quaternionic approach to surface theory in the 4-sphere developed in [PP98], [BFL$^+$02], and [FLPP01]. Section 1 establishes this framework. Another quite useful aproach is the lightcone model, e.g. used in [BPP02]. A useful introduction into constrained Willmore surfaces in the lightcone setting can be found in [Qui09]. In Section 2, some important facts and theorems on Riemann surface theory are recalled. We will end the first part with Section 3, where the push forward bundle is introduced. This bundle will later naturally appear in the integrable system approach.

Section 4 opens the second part with the description of constrained Willmore surfaces in the quaternionic language. In Section 5, we discuss the spectral curve for CMC tori in S^3 and cW tori in S^4, but also for conformal immersions in S^4, developed in [BLPP12]. The last is interesting as it enables a reconstruction of the initial surface from the spectral data. The spectral curve of a conformal immersion is not defined via the holonomy of a family of connections. Instead a conformal immersion defines a (quaternionic) holomorphic structure on a trivial rank 2 bundle. The spectral curve parametrizes the monodromy of the possible holomorphic sections with monodromy. For a general conformal immersed torus, the spectral curve is not necessarily compact. If the spectral curve is compact, the torus is called of finite type and it is possible to reconstruct the initial immersion. A constrained Willmore torus is of finite type since, as we will see in Section 5.5, both spectral curves coincide.

In the last Section 6, we will restrict ourselves to the case of constrained Willmore tori in a 3-sphere. The in-between surface X is a sphere if and only if the immersion is already CMC in space form. This is a result of Heller [Hel15]. We will subsequently look at the general case and proof the main theorem of the thesis. Note that most of the proofs only work for simple tori, which is a mild assumption though.

Part I

FUNDAMENTALS

1. Quaternionic theory

We will study constrained Willmore tori in the tradition of the Berlin school of Franz Pedit, Ulrich Pinkall, et. al., i.e., in the setting of quaternionic surface theory. We will follow [BFL$^+$02] to give an introduction into this theory. The quaternionic description of surfaces in the 4-sphere S^4, or some space form contained in S^4, has the advantage that conformal maps are given by quaternionic Möbius transformations and many of the objects that occur are Möbius invariant. As we will see the Willmore energy of a surface is invariant under conformal changes of the ambient space, and therefore under conformal transformations of S^4. Since S^4 can be viewed as the projective space $\mathbb{H}P^1$ these are given by $\mathrm{PGL}(2, \mathbb{H})$. The non-commutativity of the quaternions changes things compared to the complex case of Riemann surface theory, but still many of the results carry over to the quaternionic set-up.

1.1. The quaternions

Definition 1.1 (Quaternions)
The space of quaternions $\mathbb{H}$ is the 4-dimensional $\mathbb{R}$-vector space, with basis $1, \mathbb{i}, \mathbb{j}, \mathbb{k}$ with the (unique) associative multiplication defined by the neutrality of 1 and

$$\mathbb{i}^2 = \mathbb{j}^2 = \mathbb{k}^2 = \mathbb{i}\mathbb{j}\mathbb{k} = -1$$
$$\text{and} \quad \mathbb{i}\mathbb{j} = -\mathbb{j}\mathbb{i}.$$

The product is by definition non-commutative. The quaternionic conjugate of $a = a_0 + a_1\mathbb{i} + a_2\mathbb{j} + a_3\mathbb{k}$ is

$$\bar{a} = a_0 - a_1\mathbb{i} - a_2\mathbb{j} - a_3\mathbb{k}.$$

The conjugation obeys

$$\overline{ab} = \bar{b}\,\bar{a}.$$

The real part of a quaternion is

$$\mathrm{Re}(a) = \frac{1}{2}(a + \bar{a}),$$

the imaginary part is $\mathrm{Im}(a) = \frac{1}{2}(a - \bar{a})$. The subset of imaginary quaterions,

$$\mathrm{Im}(\mathbb{H}) = \{a \in \mathbb{H} | a = -\bar{a}\},$$

is the real vector space spanned by $\mathbb{i}, \mathbb{j}, \mathbb{k}$ and therefore isomorphic to $\mathbb{R}^3$. The standard metric from $\mathbb{R}^4$ can be written as

$$\langle a, b \rangle = \mathrm{Re}(a\bar{b}).$$

Every non-zero element $a \in \mathbb{H}$ has an inverse

$$a^{-1} = \frac{1}{a\bar{a}}\bar{a} = \frac{1}{\langle a, a \rangle}\bar{a},$$

so $\mathbb{H}$ is a skew-field.

Lemma 1.2

(i) Two quaternions commute if and only if the imaginary parts of both are linearly dependent.

(ii) Let a be a quaternion, then
$$a^2 = -1$$
if and only if
$$|a|^2 = 1 \quad \text{and} \quad a \in \mathrm{Im}(\mathbb{H}).$$
The set of quaternions satisfying $a^2 = -1$ is therefore a 2-sphere in $\mathrm{Im}(\mathbb{H})$.

Proof

Follows from

$$\begin{aligned} ab = &(a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) \\ &+ (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2)\mathbb{i} \\ &+ (a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1)\mathbb{j} \\ &+ (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)\mathbb{k}, \end{aligned}$$

where $a, b \in \mathbb{H}$. $\square$

Fixing an $i \in \mathbb{H}$ with $i^2 = -1$ yields a splitting of $\mathbb{H}$ into two complex vector spaces $\mathbb{H} = \mathrm{span}\{1, i\} \oplus \mathrm{span}\{1, i\}^{\perp}$. In the following, if not stated otherwise, we will always identify $\mathbb{H}$ and $\mathbb{C}^2$ by
$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}\mathbb{j},$$
i.e., fix i as the multiplication by the quaternion $\mathbb{i}$. From now on we will always identify $\mathbb{H}$ and $\mathbb{C}^2$ by this, if not stated otherwise.

Definition 1.3 (Quaternionic vector space)

A quaternionic vector space V is a real vector space equipped with a multiplication by quaternions from the right.

Due to the non-commutativity of quaternions there is no natural quaternionic structure on tensor products of quaternionic vector spaces. In particular,

$$\mathrm{Hom}_{\mathbb{H}}(V, W) = \{\text{quaternionic linear maps: } V \to W\}$$

is not a quaternionic space. If we define $A \cdot \lambda$, for $A \in \mathrm{Hom}_{\mathbb{H}}(V, W)$ and $\lambda \in \mathbb{H}$, as multiplication from right, i.e., $A\lambda(v) = A(v)\lambda$, one gets

$$(A \cdot \lambda)(v\mu) = A(v\mu)\lambda = A(v)\mu\lambda$$

for $v \in V$ and $\mu \in \mathbb{H}$, which is generally not equal to

$$A(v)\lambda\mu = (A\lambda)(v)\mu = (A\lambda)(v\mu).$$

In the special case of $V^* = \mathrm{Hom}_{\mathbb{H}}(V, \mathbb{H})$, one can define a scalar multiplication, by multiplying the conjugated quaternion from the left, i.e.,

$$\alpha \cdot \lambda := \bar{\lambda}\alpha.$$

Since there is no natural identification of $\mathbb{H}$ and $\mathbb{C}^2$, it is helpfull to fix a complex structure, i.e. $J \in \mathrm{End}_{\mathbb{H}}(V)$ with $J^2 = -\,\mathrm{id}$. For $v \in V$ and $x, y \in \mathbb{R}$ we set

$$(x + iy)v = xv + yJv.$$

Definition 1.4
The pair (V, J) is called a complex quaternionic (bi-)vector space, since it has a quaternionic multiplication from the right and a complex multiplication from the left which are compatible.

V splits into the $\pm\mathbb{i}$-eigenspaces of J as every element can be written as $v = v_+ + v_-$, where $v_\pm = \frac{1}{2}(v \mp Jv\mathbb{i})$ is a $\pm\mathbb{i}$-eigenvector. Therefore

$$V = V_+ \oplus V_-$$

for

$$V_\pm = \{v \in V \mid Jv = \pm v\mathbb{i}\}.$$

The eigenspaces are complex vector spaces with respect to J. Since $Jv_+\mathbb{j} = v_+\mathbb{i}\mathbb{j} = -v_+\mathbb{j}\mathbb{i}$, we have

$$V_- = V_+\mathbb{j}$$

Vice Versa, every complex vector space $\hat{V}$ gives rise to a complex quaternionic vector space defined as

$$V = \hat{V} \oplus \hat{V}\mathbb{j},$$

i.e. $V = \hat{V} \oplus \hat{V}$ with $(v, w)\mathbb{i} = (vi, -wi)$, $(v, w)\mathbb{j} = (-w, v)$, and $J(v, w) = (vi, wi)$.

Remark
The splitting of complex quaternionic vector spaces generalizes the splitting of $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}\mathbb{j}$ by choosing J to be the left multiplication with the quaternion $\mathbb{i}$.

A real 2-dimensional linear subspace in $\mathbb{H}^2$ is equipped with two normal vectors. We will later use them for the 2-dimensional tangent space of a Riemann surface M.

Lemma 1.5 (Fundamental lemma)

1. Let $U \subset \mathbb{H}$ be a real subspace of dimension 2 . Then, there exist $N, R \in \mathbb{H}$ with the following three properties:

$$\begin{aligned} &N^2 = -1 = R^2, \\ &NU = U = UR, \\ &U = \{x \in \mathbb{H} | NxR = x\}. \end{aligned} \tag{3}$$

The pair (N, R) is unique up to sign. To be more specific, N and R are already determined by the first two conditions. If U is oriented, there is only one such pair such that N is compatible with the orientation.

2. If U, N and R satisfy (3) and $U \subset \operatorname{Im} \mathbb{H}$, then $N = R$, and N is a Euclidean unit normal vector of U in $\operatorname{Im} \mathbb{H} = \mathbb{R}^3$.
3. Given $N, R \in \mathbb{H}$ with $N^2 = -1 = R^2$, then the sets

$$U := \{x \in \mathbb{H} | NxR = x\} \quad \text{and} \quad U^\perp := \{x \in \mathbb{H} | NxR = -x\}$$

are orthogonal real subspaces of dimension 2.

N and R are called the left and right normal vector of U, respectively.

For a proof see Lemma 2 of [BFL$^+$02].

Definition 1.6 (Quaternionic vector bundle)
A quaternionic vector bundle over a Riemann surface M is a real vector bundle with a fiberwise multiplication by quaternions from the right.

A complex quaternionic vector bundle (V, J) is a quaternionic vector bundle V endowed with a section $J \in \Gamma(\operatorname{End}(V))$, such that $J^2 = -1$.

If M is compact, the degree of (V, J) is defined as the degree of the underlying complex vector bundle $\hat{V}$. If (V, J) is a complex quaternionic vector bundle, V^* is also complex quaternionic with complex structure J^*.

The space of endomorphisms $\operatorname{End}_{\mathbb{H}}(V)$ may not be an quaternionic vector bundle, but it is still a real vector bundle. Furthermore it can be decomposed into two complex vector bundles

$$\operatorname{End}_{\mathbb{H}}(V) = \operatorname{End}_+(V) \oplus \operatorname{End}_-(V),$$

given by the J-commuting and the J-anticommuting endomorphisms. For $\alpha \in \operatorname{End}_{\mathbb{H}}(V)$ the $\pm$-part is given by

$$\alpha_\pm = \frac{1}{2}(\alpha \mp J\alpha J).$$

Let K, $\bar{K}$ be the canonical and anti-canonical line bundle on M and

$$KV = K \otimes_{\mathbb{C}} V = \{\omega \in \Omega^1(V) | *\omega = J\omega\},$$
$$\bar{K}V = \bar{K} \otimes_{\mathbb{C}} V = \{\omega \in \Omega^1(V) | *\omega = -J\omega\}.$$

The $*$-operator is the usual operator on Riemann surfaces, i.e., $\omega \circ \tilde{J}$, where $\tilde{J}$ is the complex structure on TM. Any 1-form $\omega \in \Omega^1(V)$ splits into its K and $\bar{K}$-part

$$\omega = \omega' + \omega''.$$

This yields

$$\Omega^1(V) = KV \oplus \bar{K}V.$$

The $'$ and $''$-parts are given by

$$\omega' = \frac{1}{2}(\omega - J*\omega) \in KV, \quad \text{and} \quad \omega'' = \frac{1}{2}(\omega + J*\omega) \in \bar{K}V.$$

Definition 1.7 (Quaternionic holomorphic structure)
Let (V, J) be a quaternionic bundle. A quaternionic linear operator

$$D : \Gamma(V) \to \Gamma(\bar{K}V)$$

satisfying the Leibniz rule

$$D(\psi\lambda) = (D\psi)\lambda + (\psi d\lambda)'',$$

for $\psi \in \Gamma(V)$ and $\lambda : M \to \mathbb{H}$, is called a quaternionic holomorphic structure. The triple (V, J, D) is called a holomorphic quaternionic bundle. A section $\psi \in \Gamma(V)$ satisfying $D\psi = 0$ is called D-holomorphic.

Although the definition is quite similiar, there is a difference to a complex holomorphic structure $\bar{\partial}$-operator. A quaternionic holomorphic structure is in general not J-linear. But one can split D into its J-commuting and J-anticommuting part. The J-linear part

$$\bar{\partial} = \frac{1}{2}(D - JDJ)$$

is a holomorphic structure. The J-anticommuting part, the so called Hopf field,

$$Q = \frac{1}{2}(D + JDJ),$$

is not a first order differential operator but a section in $\bar{K}\mathrm{End}_-V$. This can be seen by the short calculation

$$\begin{aligned}
2Q(\psi\lambda) &= D(\psi\lambda) + JD(J\psi\lambda) \\
&= (D\psi)\lambda + \frac{1}{2}(\psi d\lambda + J\psi * d\lambda) + JD(J\psi)\lambda + \frac{1}{2}J(J\psi d\lambda + J^2\psi * d\lambda) \\
&= (D\psi)\lambda + (JDJ\psi)\lambda = 2Q(\psi)\lambda,
\end{aligned}$$

where ψ is a section in V and λ a $\mathbb{H}$-valued function. The operator $\bar{\partial}$ can be seen as a complex holomorphic structure on the complex vector bundle $\hat{V}$, and is therefore elliptic. $D = \bar{\partial} + Q$ differs by a zero order operator, so D is elliptic and on compact surfaces the space of holomorphic sections $H^0(V) = \ker(D)$ is finite dimensional. From now on we will often omit the "quaternionic" and just speak of holomorphic structures.

Definition 1.8
A quaternionic linear operator $\bar{D} : \Gamma(V) \to \Gamma(KV)$, satisfying

$$\bar{D}(\psi\lambda) = (\bar{D}\psi)\lambda + (\psi d\lambda)'$$

is called an anti-holomorphic structure on the complex quaternionic vector bundle (V, J).

Analogously to the decomposition of a holomorphic structure, an anti-holomorphic structure splits into the J-linear anti-holomorphic structure ∂ and the J-anti-linear part A. The operator $A \in \Gamma(K\mathrm{End}_-(V))$ is called the Hopf field of $\bar{D}$.

Remark
Any quaternionic linear connection ∇ on (V, J) splits into an anti-holomorphic structure $\nabla' = \frac{1}{2}(\nabla - J * \nabla)$ and a holomorphic structure $\nabla'' = \frac{1}{2}(\nabla + J * \nabla)$.

Quaternionic linebundles are real 4-dimensional, therefore it is always possible to find a non-vanishing section by transversality theory.

Lemma 1.9 (see Section 4.1 of [BFL$^+$02])
Any quaternionic line bundle L over a Riemann surface is isomorphic to the trivial $\mathbb{H}$-bundle.

Example
Let L be a quaternionic line bundle. Then, by Lemma 1.9, there exists an non-vanishing section ψ. There exists a unique holomorphic structure D, such that ψ is holomorphic and as L has rank 1, there exists a quaternionic valued function N with $N^2 = -1$ such that $J\psi = \psi N$. The Hopf field Q is fully determined by the value on ψ. We have

$$\begin{aligned} 4Q(\psi) = 2(D + JDJ)\psi &= 2JD(\psi N)) \\ &= 2(J(D\psi)N + J(\psi dN)'') \\ &= J(\psi dN + J\psi * dN) \\ &= \psi(NdN - *dN). \end{aligned}$$

1.2. The quaternionic projective space

The n-dimensional quaternionic space $\mathbb{H}P^n$ is defined, analogously to the real projective space, as the space of quaternionic lines in the quaternionic vector space $\mathbb{H}^{n+1}$. Both, S^4 and $\mathbb{H}P^1$ are 1-point compactifications of $\mathbb{R}^4$ and therefore diffeomorphic. Since we are interested in surface theory in the round sphere $S^3 \subset S^4$, it is useful to study $\mathbb{H}P^1$.

The manifold structure of

$$\mathbb{H}P^1 = \{[x] \,|\, 0 \neq x \in \mathbb{H}^2\},$$

where $[x] = x\mathbb{H}$, is given by affine coordinates. The atlas is given by two charts (U_1, g_1), (U_2, g_2) with

$$U_1 = \mathbb{H}P^1 \setminus \{[1,0]\} = \{[a,1] \,|\, a \in \mathbb{H}\}, \quad U_2 = \mathbb{H}P^1 \setminus \{[0,1]\},$$

and

$$\begin{aligned} g_1 : U_1 \to \mathbb{H}, &\quad [x,y] \mapsto xy^{-1}, \\ g_2 : U_2 \to \mathbb{H}, &\quad [x,y] \mapsto yx^{-1}. \end{aligned}$$

Thus the transition function is given by $g_2 \circ g_1^{-1}(x) = x^{-1}$.

Let $\pi : \mathbb{H}^2 \setminus \{0\} \to \mathbb{H}P^1$ be the canonical projection, so $\pi(x) = [x]$. The Fubini-Study metric at a point $[x]$ is defined by

$$\langle d_x\pi(v), d_x\pi(w)\rangle_{\mathbb{H}P^1} = \frac{1}{\langle x, x\rangle} \operatorname{Re}(\langle (v)^N, (w)^N\rangle),$$

where $\langle \cdot\,, \cdot\rangle$ is the standard metric on $\mathbb{H}^2$ and $(v)^N$ is the orthogonal projection on $(x\mathbb{H})^\perp$.

Proposition 1.10
The tangent space $T_l\mathbb{H}P^1$ can be identified with $\operatorname{Hom}(l, \mathbb{H}^2/l)$ via

$$F \in \operatorname{Hom}(l, \mathbb{H}^2/l) \mapsto d_x\pi(F(x)),$$

where $l = x\mathbb{H}$.

Proof
Let $x = (x_1, x_2) \in \mathbb{H}^2 \setminus \{0\}$ and $x\mathbb{H} \in U_1$, then

$$g_1 \circ \pi(x) = x_1 x_2^{-1} = x_1\langle (0, 1), x\rangle$$

and therefore

$$d_x(g_1 \circ \pi)(v) = v_1 x_2^{-1} - x_1 x_2^{-1} v_2 x_2^{-1}.$$

Since g_1 is a chart we get

$$\ker(d_x\pi) = x\mathbb{H} \quad \text{and} \quad d_{x\lambda}\pi(v\lambda) = d_x\pi(v)$$

and therefore the map

$$F \longmapsto d_x\pi(F(x))$$

is a well-defined isomorphism between $\operatorname{Hom}(l, \mathbb{H}^2/l)$ and $T_l\mathbb{H}P^1$, independent of the chosen $x \in l$. □

For $y = x\lambda$ and $(w)^N = (v)^N\lambda$, it is

$$d_x\pi(v) = d_y\pi(w),$$

therefore the Fubini-Study-metric is well-defined. We will see now that the Fubini-Study metric is in fact the round metric on the 4-sphere. Working with the chart g_1 it is

$$g_1^{-1}(a) = [a, 1] = \pi(a, 1)$$

and

$$d_a(g_1^{-1})(v) = d_{(a,1)}\pi(v, 0).$$

At $l = (a, 1)\mathbb{H}$ the perpendicular part can be calculated as

$$(v, 0)^\perp = (v, 0) - (a, 1)\frac{\bar{a}v}{1 + |a|^2} = (v, -\bar{a}v)\frac{1}{1 + |a|^2}.$$

Therefore, with respect to the chart g_1, the metric looks like

$$\begin{aligned} g_{1*}(\langle \cdot\,, \cdot \rangle)_a(v,w) &= \langle d_{(a,1)}\pi(v,0), d_{(a,1)}\pi(w,0)\rangle \\ &= \frac{1}{\langle (a,1),(a,1)\rangle}\,\mathrm{Re}(\langle (v,-\bar{a}v)\frac{1}{1+|a|^2}, (w,-\bar{a}w)\frac{1}{1+|a|^2}\rangle) \\ &= \frac{1}{(1+|a|^2)^3}\,\mathrm{Re}(\bar{v}w+\bar{v}a\bar{a}w) \\ &= \frac{1}{(1+|a|^2)^2}\,\mathrm{Re}(\bar{v}w). \end{aligned}$$

The last term is just the metric of the round sphere with curvature 4 in a stereographic projection to $\mathbb{R}^4 = \mathbb{H}$.

1.3. Conformal immersions into the 4-sphere

The identification of $\mathbb{H}P^1$ and S^4 yields a natural identification of maps into S^4 and quaternionic line subbundles of the trivial bundle

$$V = M \times \mathbb{H}^2.$$

The projective space $\mathbb{H}P^1$ is equipped with a canonical quaternionic line bundle τ, assigning every point $p \in M$ the corresponding line in $\mathbb{H}^2$, so

$$\tau = \{(l,v)|v \in l \in \mathbb{H}P^1\}.$$

The line bundle corresponding to a map $f : M \to S^4$ is then given by the pullback of the trivial bundle

$$L = f^*\tau.$$

For a bundle $L \subset V$, the map f is defined by

$$f(x) = L_x.$$

Therefore we obtain

Lemma 1.11
Let M be a Riemann surface. There is a 1:1-correspondence between maps $f : M \to S^4$ and quaternionic line subbundles of $V = M \times \mathbb{H}^2$.

Instead of investigating maps to S^4, we can consider quaternionic line subbundles of V. If the map f maps into S^3, we can consider f as

$$f : M \to S^3 \subset \mathbb{H} = \{[x,1] \,|\, x \in \mathbb{H}\} = \mathbb{H}P^1 \setminus \{[1,0]\}.$$

This implies, that a trivializing section of L is given by

$$\psi(p) = \begin{pmatrix} f(p) \\ 1 \end{pmatrix} \in (f^*\tau)_p.$$

Definition 1.12
A map $f : (M, g) \to (N, h)$ between Riemannian manifolds (M, g) and (N, h) is conformal if the metric induced by h is conformally equivalent with g, i.e.,

$$g = e^{\lambda} f^* h.$$

In dimension 2, i.e., if M is a Riemann surface with complex structure J on TM, this is equivalent to

$$|df(X)|^2 = |df(JX)|^2 \quad \text{and} \quad \langle df(X), df(JX)\rangle = 0$$

for all sections $X \in \Gamma(TM)$.

Proposition 1.13
A map $f : M \to \mathbb{H}$ of a Riemann surface M is conformal, if and only if there exists $N, R : M \to \operatorname{Im}\mathbb{H}$ with $N^2 = R^2 = -1$ and

$$df \circ J = *df = N df = -df R.$$

N and R are called the left and right normal vector of f, respectively.

Remark
Conformality is obviously a generalisation of holomorphicity of a map $f : M \to \mathbb{C}$, i.e.,

$$*df = i df.$$

Further, if f is an immersion, the existence of either N or R is sufficient. If the targetspace is $\mathrm{SU}(2) = S^3 = \{x \in \mathbb{H} | |x|^2 = 1\}$ then N and R are the right and left translation of the normal vector of f to the Lie algebra of $SU(2)$, i.e. $\operatorname{Im}\mathbb{H}$.

Proof (of Proposition 1.13)
We will only restrict on the case that f is an immersion, as we, in this thesis, will always assume this property to hold. Let N be the left normal vector of f, then we have

$$|df(JX)| = |N df(X)| = |N||df(X)| = |df(X)|, \tag{4}$$

and

$$\langle df(JX), df(X)\rangle = \langle N df(X), df(X)\rangle = \operatorname{Re}(N df(X)\overline{df(X)}) = |df(X)|^2 \operatorname{Re}(N) = 0, \tag{5}$$

so f is conformal. On the other hand, if f is conformal, by the Fundamental lemma 1.5 the normal vectors N, R of $df(TM) \subset \mathbb{H}$ exist. By (4) and (5), $N df(X)$ and $df(X)$ are perpendicular and have the same length. Since $df(TM)$ is two dimensional it is

$$N df = \pm * df.$$

Due to $N df(Y) R = df(Y)$, we have $df R = \mp * df$. The sign depends on the chosen orientation. $\square$

For a conformal coordinate $z = x + iy$ on M, we have

$$df = f_x dx + f_y dy = f_x dx + N f_x dy = f_x dx - f_x R dy,$$

for some $\mathbb{H}$-valued function f_x. Therefore $N = df(JX)df(X)^{-1}$ for all non-vanishing local sections $X \in \Gamma(TM)$.

Since we want to work with the associated line bundle L instead of the map f , we need a way to define the differential in terms of L. As mentioned in Proposition 1.10, this can be done via

$$\text{(6)} \qquad \operatorname{Hom}(l, \mathbb{H}^2/l) = T_l\mathbb{H}P^1, \qquad F \mapsto d_x\hat{p}(F(x)),$$

where $\hat{p} : \mathbb{H}^2 \setminus \{0\} \to \mathbb{H}P^1$ is the canonical projection.

Lemma and Definition 1.14
Under the identification (6) the differential of a map $f : M \to \mathbb{H}P^1$, or equivalently $L \subset V$, is given by the 1-form $\delta \in \Omega^1(\operatorname{Hom}(L, V/L)$, defined by

$$\delta = \pi d_{|L},$$

where $\pi : \mathbb{H}^2 \to V/L$ is the canonical projection. The bundle L defines an immersion f if its differential $\delta : TM \to \operatorname{Hom}(L, V/L)$ is injective at every point. In that case, we call L immersed.

Proof
Indeed δ defines a 1-form, since it is tensorial: Let $\psi \in \Gamma(L)$, $\lambda : M \to \mathbb{H}$, then

$$\pi(d_{|L}(\psi\lambda)) = \pi((d_L\psi)\lambda) + \pi(\underbrace{\psi d\lambda}_{\in L}) = \pi(d_{|L}\psi)\lambda.$$

Take a nowhere vanishing section $\psi \in \Gamma(L)$. The following diagramm commutes

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[dr, "f"'] & V \setminus \{0\} \arrow[d, "\hat{p}"] \\ & \mathbb{H}P^1 \end{tikzcd}$$

By (6), df is identified with $\xi \in \Omega^1(\operatorname{Hom}(L, V/L))$, for which

$$df_p = d\hat{p}_x(\xi_p(x)) \qquad \forall\, 0 \neq x \in L_p.$$

Using the diagramm we get

$$df_p = d\hat{p}_{\psi(p)} d\psi_p \qquad \forall\, p \in M.$$

Comparing those two equations, one sees that

$$\xi(\psi(q)) = \pi_q d\psi_q = \delta\psi,$$

which implies that δ corresponds to the differential of f under the identification (6). $\square$

Example
The differential δ of the conformal embedding $\mathbb{H} \ni x \mapsto [x, 1]$ is given by

$$\delta : T_x\mathbb{H} = \mathbb{H} \ni v \mapsto \left(\begin{pmatrix} x \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} v \\ 0 \end{pmatrix} \quad \text{mod} \quad \begin{pmatrix} x \\ 1 \end{pmatrix} \right). \tag{7}$$

After choosing affine coordinates of $\mathbb{H}P^1$, a map to S^4 can be seen as a map to $\mathbb{H}$. Using the chain rule and (7), the differential of $f : M \to \mathbb{H} \subset \mathbb{H}P^1$ is

$$\delta : TM \ni v \mapsto \left(\begin{pmatrix} f(p) \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} d_pf(v) \\ 0 \end{pmatrix} \quad \text{mod} \quad \begin{pmatrix} f(p) \\ 1 \end{pmatrix} \right).$$

Using the identification (6), the Fundamental lemma 1.5 can be reformulated.

Proposition 1.15
There is a 1:1-correspondence between oriented planes in $\mathrm{Hom}(l, \mathbb{H}^2/l)$, with l a quaternionic line in $\mathbb{H}^2$, and pairs $(J, \tilde{J})$, where J is a complex structure on l and $\tilde{J}$ on $\mathbb{H}^2/l$. The subspace $U \subset \mathrm{Hom}(l, \mathbb{H}^2/l)$ is given as the Hom_+-part, i.e.,

$$U = \{A \in \mathrm{Hom}(l, \mathbb{H}^2/l) \mid \tilde{J}A = AJ\}.$$

The space $U^\perp = \{A \mid \tilde{J}A = -AJ\}$ is the orthogonal complement of U.

Proof
Let $l = \begin{pmatrix} x \\ 1 \end{pmatrix}\mathbb{H}$. The affine coordinates $\mathbb{H} \ni x \mapsto [x, 1]$ identifies $T_l\mathbb{H}P^1$ with $\mathbb{H}$. The differential δ, see (7), then identifies $\mathbb{H}$ and $\mathrm{Hom}(l, \mathbb{H}^2/l)$. An oriented 2-dimensional plane in $\mathrm{Hom}(l, \mathbb{H}^2/l)$ corresponds to a plane in $\mathbb{H}$. Because of Lemma 1.5 this plane is characterized by a pair of normal vectors N, R. The complex structure J then can be defined by

$$J \begin{pmatrix} x \\ 1 \end{pmatrix} = -\begin{pmatrix} x \\ 1 \end{pmatrix} R$$

and $\tilde{J}$ by

$$\tilde{J}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{mod}\, l \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} N \,\mathrm{mod}\, l.$$

Now $NvR = \pm v$ is equivalent to $\tilde{J}\delta(v) = \pm\delta(v)J$. Since the affine coordinates are conformal, $U^\perp$ is the orthogonal complement. □

Definition 1.16
A subbundle $L \subset V$ with $J \in \mathrm{End}(L)$, such that the differential δ satisfies

$$*\delta = \delta J,$$

is called a holomorphic curve. If L is immersed, there is a unique complex structure $\tilde{J}$ on V/L such that

$$*\delta = \delta J = \tilde{J}\delta,$$

since δ is nowhere vanishing.

Proposition 1.17
Let L be an immersed holomorphic curve. Then the corresponding map $f: M \to \mathbb{H}P^1$ is conformally immersed. Vice versa, every conformal immersion f corresponds to an immersed holomorphic curve.

Proof
Since we only look at immersed maps it is enough to show the existence of J on L and of the left normal R, respectively. We choose a point $\infty \in \mathbb{H}P^1$ which does not lie in L. Working in affine coordinates, the corresponding map f is then a map into $\mathbb{H}$. Then, $\psi = (f, 1)$ is a nonvanishing section of L, and therefore $L = \psi\mathbb{H}$. Let now f be conformal. Then there is a $\mathrm{Im}(\mathbb{H})$-valued function R, such that $*df = -df R$. Define the complex structure by $J\psi = -\psi R$. Then

$$\delta J\psi = \pi\left(-d\left(\begin{pmatrix} f \\ 1 \end{pmatrix} R\right)\right) = \pi(\underbrace{\psi dR}_{\in L}) + \pi\begin{pmatrix} -dfR \\ 0 \end{pmatrix} = \pi\begin{pmatrix} *df \\ 0 \end{pmatrix} = *\pi d\begin{pmatrix} f \\ 1 \end{pmatrix} = *\delta\psi$$

and L is a holomorphic curve.

Starting with a holomorphic curve L, the complex structure J defines $R \in \mathbb{H}$ as in the proof of Proposition 1.15 by

$$J\psi = -\psi R.$$

Then $*\delta = \delta J$ implies $*df = -dfR$ and f is conformal. □

1.4. The mean curvature sphere congruence

For a surface $f: M \to S^4$, the mean curvature sphere congruence is a map S assigning to every point $p \in M$ a 2-sphere approximating f in p as good as possible. The mean curvature sphere congruence S contains $f(p)$ and is tangent to the surface in $f(p)$, so S approximates f up to first order. By forcing S to have the same mean curvature as $f(M)$ in p, the sphere is uniquely determined and approximates f to second order. By identification of S^4 and $\mathbb{H}P^1$, every 2-sphere in S^4 corresponds to a complex structure on the trivial bundle $V = M \times \mathbb{H}^2$. We will work with S definded as a map into the set of complex structures. With respect to the inner product defined by the trace, see Definition 4.1, S is a conformal map, cf. [BFL$^+$02, Proposition 4]. Therefore, the mean curvature sphere congruence is also known as the conformal Gauß map.

We denote the set of complex structures by

$$\mathcal{Z} = \{S \in \mathrm{End}(\mathbb{H}^2) | S^2 = -1\}.$$

As mentioned, we identify $\mathcal{Z}$ with the set of oriented 2-spheres in $\mathbb{H}P^1 = S^4$. The corresponding 2-sphere to $S \in \mathcal{Z}$ is

$$S' := \{l \in \mathbb{H}P^1 | Sl = l\}.$$

The following statement holds true.

Proposition 1.18

1. S' is a 2-sphere in $\mathbb{H}P^1$.
2. Each 2 sphere can be obtained in this way.
3. Every $S \in \mathcal{Z}$ corresponds to a unique oriented 2-sphere S'.

PROOF
First of all, S' is non-empty. Identifying $\mathbb{H}^2$ as $\mathbb{C}^2$, S is $\mathbb{C}$-linear and therefore has an eigenvector v with eigenvalue N. Then

$$S(v\mathbb{H}) = vN\mathbb{H} = v\mathbb{H},$$

so $v\mathbb{H} \in S'$. We will see that there are suitable affine coordinates such that S' is a plane, i.e. a sphere in $\mathbb{H}P^1$. Take a basis v, w of $\mathbb{H}^2$ with $v\mathbb{H} \in S'$, then

$$Sv = vN \quad \text{and} \quad Sw = -vH - wR,$$

for some $N, R, H \in \mathbb{H}$. Since $S^2 = -1$ we have

$$N^2 = R^2 = -1, \quad \text{and} \quad NH = HR.$$

Take affine coordinates $x \mapsto (vx + w)\mathbb{H}$, then

$$\begin{aligned}(vx+w)\mathbb{H} \in S' &\Leftrightarrow \exists \lambda \in \mathbb{H} : S(vx+w) = (vx+w)\lambda \\ &\Leftrightarrow v(Nx - H) - wR = vx\lambda + w\lambda \\ &\Leftrightarrow Nx - H = -xR \\ &\Leftrightarrow Nx + xR = H.\end{aligned}$$

This is an inhomogenous (real) linear equation to the homogenous equation

$$Nx + xR = 0 \quad \Leftrightarrow \quad NxR = x,$$

which defines a 2-dimensional plane by Lemma 1.5, i.e., a sphere in $\mathbb{H}P^1$.

For a given 2-sphere S' take affine coordinates, such that S' is a plane (i.e. such that $\infty \in S'$). This plane is, by Lemma 1.5, defined by an equation $Nx + xR = H$. Define S by N, R and H. Since the pair (N, R) is unique for an oriented S', so is S. □

By the previous proposition the mean curvature sphere congruence of a conformal immersion $f : M \to \mathbb{H}P^1$ can be defined as a complex structure on the trivial bundle $V = M \times \mathbb{H}^2$.

Theorem 1.19 (Theorem 2 of [BFL+02])
Let $f : M \to S^4$ be the conformal immersion of a Riemann surface and $L \subset V$ the associated (quaternionic) line subbundle. Then, there exists a unique complex structure $S : M \to \mathcal{Z}$ with the following properties:

1. $SL = L$

2. $*\delta = \delta S = S\delta$, where S on V/L is defined by projection. This is equivalent to $(dS)L \subset L$
3. $Q_{|L} = 0$, where Q is the Hopf field (the S-anti-linear part) of ∇'', and ∇ is the trivial connection on V.

This S is called the mean curvature sphere congruence.

Remark
The first condition ensures that $f(p) \in S'_p$. The second condition means that the (oriented) tangent space of S' and the tangent plane of the surface coincide in $f(p)$. The last one ensures that the mean curvature of S' equals the one of the surface in p. The equations

$$Q_{|L} = 0, \quad \text{and} \quad \text{image}(A) \subset L \tag{8}$$

are equivalent, cf. [BFL$^+$02, Lemma 5].

Remark
With respect to the complex structure S, δ is an isomorphism

$$\delta : L \to K(V/L).$$

Let $A \in \Gamma(K\mathrm{End}_-(V))$ and $Q \in \Gamma(\bar{K}\mathrm{End}_-(V))$ be the Hopf fields of the trivial connection ∇, i.e. the S-anticommuting parts of ∇' and ∇''. As shown above, ∇ decomposes as $\nabla = \partial + A + \bar{\partial} + Q$.

Lemma 1.20
The Hopf fields A and Q can be expressed in terms of S, via

$$A = \frac{1}{4}(SdS + *dS) \quad \text{and} \quad Q = \frac{1}{4}(SdS - *dS) \tag{9}$$

Proof
Let $\psi \in \Gamma(V)$. Then

$$\begin{aligned}
\frac{1}{2}((d - S * d)S)\psi &= (d'S)\psi \\
&= \nabla'(S\psi) - S\nabla'\psi \\
&= \partial S\psi + AS\psi - S\partial\psi - SA\psi \\
&= S\partial\psi - SA\psi - S\partial\psi - SA\psi \\
&= -2SA\psi
\end{aligned}$$

Hence $A = \frac{1}{4}(SdS + *dS)$. Analogously, by taking the d''-part one gets the result for Q. $\square$

Remark
As seen in 1.10 the tangent space $T_l\mathbb{H}P^1$ is isomorphic to $\mathrm{Hom}(l, V/l)$. Working in the latter model, the tangent space of an conformally immersed surface $f : M \to \mathbb{H}P^1$ is given as

$$TM = \mathrm{Hom}_+(L, V/L),$$

and the normal bundle is given by

$$\mathrm{Hom}_-(L, V/L).$$

The $\pm$-parts are taken with respect to the complex structure S. This follows from Proposition 1.15 and $\delta S = S\delta$.

Lemma 1.21
Let $L \subset V$ be an immersed holomorphic line bundle, S the corresponding mean curvature sphere, and ∇ the trivial connection on V, then L is ∇''-invariant. In particular the restriction $\nabla''_{|L}$ gives a well-defined holomorphic structure on L.

PROOF
Let $\pi : V \to V/L$ be the canonical projection. It is

$$\pi\left(\nabla''_{|L}\right) = \pi\left(\frac{1}{2}\nabla_{|L} + S * \nabla_{|L}\right) = \frac{1}{2}\left(\delta + S * \delta\right) = 0,$$

which means that $\nabla''_{|L}$ is L-valued. □

1.5. The mean curvature sphere in affine coordinates

So far we defined S coordinate free, but sometimes it will be usefull to describe the situation in coordinates. Choosing a point which is not contained in the holomorphic curve $L \subset \mathbb{H}^2$ we can view the corresponding map f as a map into $\mathbb{H}$. The canonical frame of L is

$$\begin{pmatrix} f \\ 1 \end{pmatrix}$$

and the canonical point representing ∞ is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since S preserves L, it is plausible to work with a basis vector of L to create a basis of $\mathbb{H}^2$. As we have already choosen a linear independent vector for ∞, it is reasonable to use the above vectors as a basis. The calculations of this section can be found in [BFL$^+$02]. By revisiting those we get a bit more insight how S and the mean curvature of f relate.

Suppose f is a map to $\mathbb{H} \subset \mathbb{H}P^1$. We describe S relative to the frame $(1,0), (f,1)$. Since $SL \subset L$, we have

$$S = G\begin{pmatrix} N & 0 \\ -H & -R \end{pmatrix} G^{-1}$$

for $G = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$. $S^2 = -1$ and $*\delta = \delta S = S\delta$ then imply

$$\begin{aligned} &N^2 = R^2 = -1 \\ &N df = *df = -df R \\ &RH = HN, \end{aligned}$$

i.e., N and R are the left and right normal vector of f, and H is perpendicular to TM.

Furthermore, since $dG = GdG = G^{-1}dG = \begin{pmatrix} 0 & df \\ 0 & 0 \end{pmatrix}$ holds, we have

$$\begin{aligned} dS &= dG \begin{pmatrix} N & 0 \\ -H & -R \end{pmatrix} G^{-1} + Gd \begin{pmatrix} N & 0 \\ -H & -R \end{pmatrix} G^{-1} + G \begin{pmatrix} N & 0 \\ -H & -R \end{pmatrix} dG^{-1} \\ &= G \left(\begin{pmatrix} 0 & df \\ 0 & 0 \end{pmatrix} \begin{pmatrix} N & 0 \\ -H & -R \end{pmatrix} + \begin{pmatrix} dN & 0 \\ -dH & -dR \end{pmatrix} - \begin{pmatrix} N & 0 \\ -H & -R \end{pmatrix} \begin{pmatrix} 0 & df \\ 0 & 0 \end{pmatrix} \right) G^{-1} \\ &= G \begin{pmatrix} dN - dfH & 0 \\ -dH & -dR + Hdf \end{pmatrix} G^{-1}, \end{aligned}$$

$$\begin{aligned} SdS &= G \begin{pmatrix} N & 0 \\ -H & -R \end{pmatrix} \begin{pmatrix} dN - dfH & 0 \\ -dH & -dR + Hdf \end{pmatrix} G^{-1} \\ &= G \begin{pmatrix} NdN - NdfH & 0 \\ -HdN + HdfH + RdH & RdR - RHdf \end{pmatrix} G^{-1} \\ &= G \begin{pmatrix} -*dfH + NdN & 0 \\ HdfH + RdH - HdN & -H*df + RdR \end{pmatrix} G^{-1}. \end{aligned}$$

Using (9) we get

$$\begin{aligned} 4Q &= SdS - *dS \\ &= G \begin{pmatrix} NdN - *dN & 0 \\ HdfH + RdH - HdN + *dH & RdR + *dR - 2H*df \end{pmatrix} G^{-1} \end{aligned}$$

as well as

$$\begin{aligned} 4A &= SdS + *dS \\ &= G \begin{pmatrix} NdN + *dN - 2*dfH & 0 \\ HdfH + RdH - HdN - *dH & RdR - *dR \end{pmatrix} G^{-1}. \end{aligned} \tag{10}$$

The equations $Q_{|L} = 0$ and $\operatorname{im} A \subset L$ from (8) yield

$$RdR + *dR - 2H*df = 0, \quad NdN + *dN - 2*dfH = 0. \tag{11}$$

Lemma 1.22
Let $\mathcal{H}$ be the mean curvature vector of a conformal map $f : M \to \mathbb{H}$, and R the right normal vector. Then

$$\bar{\mathcal{H}}df = \frac{1}{2}(RdR + *dR) = RdR'.$$

Lemma 1.22 and (11) imply

$$2Hdf = R*dR - dR = R(*dR + RdR) = 2R\bar{\mathcal{H}}df$$

and therefore

$$H = -R\bar{\mathcal{H}}.$$

Analogously one can show $H = -\bar{\mathcal{H}}N$.

Notation
On a Riemann surface a 2-form η is defined by the value $\eta(X, JX)$ for some $0 \neq X \in \Gamma(TM)$. For many calculations we will use the quadratic form and we denote it by the same symbol, i.e. $\eta(X) := \eta(X, JX)$. Let α, β be 1-forms, then $\alpha\wedge\beta(X) = \alpha\wedge\beta(X, JX) = \alpha(X)\beta(JX) - \alpha(JX)\beta(X) = (\alpha * \beta - *\alpha\beta)(X)$.

PROOF (OF LEMMA 1.22)
By the Fundamental lemma 1.5 the normal bundle of $f : M \to \mathbb{H}$ is given as the set of vectors satisfying

$$NvR = -v,$$

and the tangent vectors satisfy

$$NvR = v. \tag{12}$$

The second fundamental form is the normal part of $X(df(Y))$, and therefore given by

$$\mathrm{I\!I}(X, Y) = \frac{1}{2}(X(df(Y)) - N(X(df(Y))R).$$

Differentiating (12) with $v = df(Y)$ in direction of X yields

$$dN(X)df(Y)R + NX(df(Y))R + Ndf(Y)dR(X) = X(df(Y)),$$

which implies

$$\mathrm{I\!I}(X, Y) = \frac{1}{2}(*df(Y)dR(X) - dN(X) * df(Y)R).$$

Taking the trace, and recall that X, JX are orthogonal, we get

$$\begin{aligned} 4\mathcal{H}|df|^2 &= *df dR - dN * df + * * df * dR - *dN * *df \\ &= -df(RdR + *dR) - (dNN - *dN)df. \end{aligned} \tag{13}$$

This can be further simplified by observing

$$\begin{aligned} (dNN - *dN)df &= dN * df - *dNdf = dN \wedge df = d(Ndf) \\ &= -d(dfR) = df \wedge dR = df * dR - *df dR = df(*dR + RdR). \end{aligned}$$

Further, using $|df|^2 = d\bar{f}df$, (13) yields

$$4\mathcal{H}d\bar{f} = -2df(*dR + RdR).$$

Conjugating and using that $R \in \operatorname{Im}\mathbb{H}$, i.e.

$$\overline{*dR + RdR} = - * dR + dRR = -(RdR + *dR),$$

then gives the desired result. $\square$

1.6. The bundle V/L

Another interesting bundle for conformal maps is the quotient bundle V/L. The bundle has a canonical holomorphic structure which is Möbius invariant, i.e., does not depend on the chosen point at ∞. The initial immersion f is given as the quotient of holomorphic sections of V/L. If the space of holomorphic sections is of minimal dimension all quotients of holomorphic sections give a Möbius transformation of f.

Denote the canonical projection by π. The mean curvature sphere S is a canonical complex structure on V. It is $SL \subset L$ and $dSL \subset L$, therefore the projection of S on V/L yields a complex structure on V/L. Since L is ∇''-invariant by Lemma 1.21, the holomorphic structure

$$D = \pi\nabla'' \tag{14}$$

is well defined on V/L, i.e. for $\psi \in \Gamma(V/L)$,

$$D\psi = \pi\nabla''\hat{\psi}$$

does not depend on the choice of the lift $\hat{\psi} \in \Gamma(V)$. By construction D is the unique holomorphic structure such that ∇ parallel sections project to holomorphic sections. But there is also a reverse construction:

Lemma and Definition 1.23
Let D be the holomorphic structure on V/L from (14). Then every (local) D-holomorphic section ψ of V/L gives rise to a lift $\hat{\psi} \in \Gamma(V)$ with $\nabla\hat{\psi} \in \Omega^1(L)$ and $\pi(\hat{\psi}) = \psi$. The section $\hat{\psi}$ is unique and is called the prolongation of ψ.

Proof
Let ψ be a (local) holomorphic section of V/L. Let $\tilde{\psi} \in \Gamma(V)$ be a lift of ψ, i.e., $\pi(\tilde{\psi}) = \psi$. For another lift $\hat{\psi}$, we have $\hat{\psi} = \tilde{\psi} + \varphi$ with $\varphi \in \Gamma(L) \subset \Gamma(V)$. Now $\nabla(\hat{\psi}) = \nabla\tilde{\psi} + \nabla\varphi$ is L-valued if and only if

$$0 = \pi(\nabla(\hat{\psi})) = \pi\nabla\tilde{\psi} + \delta\varphi.$$

$D\psi = 0$ implies $\pi\nabla\tilde{\psi}$ is KV/L valued, as well as δ. Since δ is non-vanishing, i.e., an isomorphism, there is a unique φ satisfying $\nabla\tilde{\psi} + \nabla\varphi = 0$. □

For surfaces in the unit 3-sphere $S^3 \subset \mathbb{H}$ the sections $\pi(1,0)$ and $\pi(0,1)$ are trivializing sections of V/L and holomorphic. Moreover, f is given as the quotient of these sections, since

$$\pi(1,0)f + \pi(0,1) = \pi(f,1) = 0.$$

The quotient $\tilde{f}$ of any two (linear independent) holomorphic sections $\pi(a,b)$ and $\pi(c,d)$ of V/L, coming from constant sections $(a,c), (b,d)$ of V, will be a Möbius transformation of f, i.e.

$$\tilde{f} = (af+b)(cf+d)^{-1}.$$

Since $\pi(1,0)$ and $\pi(0,1)$ are linear independent, the dimension of the space of holomorphic sections $H^0(V/L,D)$ is at least 2. The dimension of $H^0(V/L,D)$ is an important invariant of f.

Definition 1.24
The dual curve of a holomorphic curve L is

$$L^{\perp} = \{\alpha \in V^* \mid L \subset \ker \alpha\}.$$

The dual curve is itself a holomorphic curve, equipped with the dual complex structure. A given $\alpha \in \Gamma(V^*)$ defines a section in the dual bundle L^{-1} of L by restricting α on L. This makes L^{-1} canonically isomorphic to $V^*/L^{\perp}$ and $\left(L^{\perp}\right)^{-1}$ is isomorphic to V/L. For maps into S^3, there is a further symmetry:

Lemma 1.25
If f maps to $S^3 \subset \mathbb{H}$, then L^{-1} and V/L are holomorphic isomorphic by

$$\alpha_{|L} \mapsto \pi(\begin{pmatrix} 1 \\ 0 \end{pmatrix}),$$

where $\alpha, \beta \in V^*$ is the dual basis of $(1,0), (0,1) \in V$.

Proof
The complex structure on L^{-1} is given by the dual of S. The map f is S^3-valued, therefore $\bar{f}f = 1$, which implies

$$\overline{df}f = -\bar{f}df.$$

The left and right normal vectors satisfy

$$\begin{aligned} \overline{df}Nf &= -\overline{Ndf}f = -\overline{*df}f \\ &= \bar{f} * df = -\bar{f}dfR = \overline{df}fR, \end{aligned}$$

and therefore $fR = Nf$ (which fits the description of N and R as right and left translation of the normal vector of f in $\mathfrak{su}(2) = \mathrm{Im}(\mathbb{H})$). So, for $\psi = \begin{pmatrix} f \\ 1 \end{pmatrix} \in \Gamma(L)$,

$$S\alpha(\psi) = \alpha(S\psi) = \alpha(-\psi R) = -\alpha(\psi)R = -fR = -Nf = \bar{N}\alpha(\psi) = (\alpha N)(\psi),$$

and the isomorphism induced by $\alpha \mapsto \pi(1,0)$ is compatible with the complex structure, since $S\pi(1,0) = \pi(1,0)N$ holds as well. The isomorphism is holomorphic since the sections α and $\pi(1,0)$ are both holomorphic. □

1.7. The mean curvature sphere of surfaces in the 3-sphere

Every 3-sphere in $\mathbb{H}P^1$ is given as the set of isotropic lines of an indefinite hermitian product on $\mathbb{H}^2$, see [Pet04, Section 13] for further details. The 3-sphere of radius 1 in $\mathbb{H} \subset \mathbb{H}P^1$ is given by the isotropic lines of the inner product (,) on $V = M \times \mathbb{H}^2$, defined by

$$(v, w) = \bar{v}_1 w_1 - \bar{v}_2 w_2. \tag{15}$$

$[x, 1]$ is a isotropic line of $(\, , \,)$ if

$$0 = \left(\begin{pmatrix} x \\ 1 \end{pmatrix}, \begin{pmatrix} x \\ 1 \end{pmatrix} \right) = \bar{x}x - 1.$$

The product $(\, , \,)$ is non-degenerate, so it induces an isomorphism $V \cong V^*$. If V is a complex quaternionic bundle with structure i, V^* is a complex quaternionic bundle with respect to the complex stucture $-i$. For a subbundle $L \subset V$, we denote the annihilator bundle by $L^\perp$, i.e. $(L^\perp, L) = 0$. If L defines a surface in S^3, we have $L^\perp = L$ by the isomorphism $(\, , \,)$.

Remark
The canonical linear system is the subspace of $H^0(L^{-1})$, given by restricting constant sections on L. The isomporphism of $L \subset V$ and $L^\perp \subset V^*$ is compatible with the canonical linear systems of L and $L^\perp$, i.e., it maps holomorphic section of $L^{-1} = V^*/L^\perp$ coming from constant sections in V^* onto sections in $(L^\perp)^{-1} = V/L$ coming from constant sections. Again, see [Pet04, Section 13] for further details.

Lemma 1.26
The mean curvature sphere congruence of $L^\perp \subset V^* = \mathbb{H}^2$ is given by $S^\perp = S^*$.

Proof
Since S stabilizes L, we have

$$0 = \langle L^\perp, L \rangle = \langle L^\perp, SL \rangle = \langle S^* L^\perp, L \rangle$$

and therefore S^* stabilizes $L^\perp$. Using that dS stabilizes L as well, dS^* also leaves $L^\perp$ invariant. Furthermore, let $\tilde{Q}$ be the Hopf field with respect to S^*, then

$$\begin{aligned} 4\tilde{Q} &= S^* dS^* - *dS \\ &= (dSS - *dS)^* \\ &= -(SdS + *dS)^* \\ &= -4A^*. \end{aligned}$$

We have $\operatorname{im} A \subset L$, which yields $L^\perp \subset \ker A^* = \ker \tilde{Q}$. So S^* satisfies the conditions of a mean curvature sphere congruence. □

Proposition 1.27
Let L be a immersed holomorphic curve in $\mathbb{H}P^1$. Then L is a surface in S^3 if and only if $S = S^*$. Furthermore $A^* = -Q$ and $Q^* = -A$.

See [BFL+02, Proposition 18] for a proof.

Lemma 1.28
For conformal surfaces in S^3, it is

$$\ker A = \operatorname{im} Q$$

at points where A and Q are not vanishing. In particular, $AQ = 0$.

Proof
By definition, $SQ = -QS$, so for $\psi = Q\varphi \in \operatorname{im} Q$ we get

$$S\psi = -QS\varphi \in \operatorname{im} Q,$$

i.e., $\operatorname{im} Q$ is S-invariant. In particular, $S\psi = \psi\lambda$ for some section $\psi \in \Gamma(\operatorname{im} Q)$ and $\mathbb{H}$-valued function λ with $\lambda^2 = -1$. Then, by Proposition 1.27, we obtain

$$-\lambda(\psi, \psi) = \bar{\lambda}(\psi, \psi) = (S\psi, \psi) = (\psi, S\psi) = (\psi, \psi)\lambda.$$

This implies, since (ψ, ψ) is real valued,

$$(\psi, \psi) = 0,$$

in other words $\operatorname{im} Q = (\operatorname{im} Q)^{\perp}$. Using $A = -Q^*$, we get

$$\operatorname{im} Q = (\operatorname{im} Q)^{\perp} = \ker Q^* = \ker A$$

as desired. □

1.8. Pairings

In a complex setting, holomorphic structures on complex line bundles L induce complex structures on L^{-1}. While a connection ∇ on a complex quaternionic vector bundle W still defines a connection on the dual W^*, this is generally not true for holomorphic structures D. Let ∇ be a quaternionic linear connection on the complex quaternionic bundle (W, J) with decomposition $\nabla = \partial + \bar{\partial} + A + Q$, i.e., the holomorphic structure on W is given by $D = \bar{\partial} + Q$. By

$$d(\alpha(v)) = \nabla\alpha(v) + \alpha(\nabla(v)),$$

for $\alpha \in W^*$ and $v \in W$, ∇ induces a connection ∇ on W^*. The induced holomorphic structure on (W^*, J^*) is $D = \bar{\partial} - A^*$, where $\bar{\partial}$ is the holomorphic structure induced by $\bar{\partial}$ on W. The holomorphic structure $D = \bar{\partial} + Q$ induces a structure $\tilde{D} = \bar{\partial} - Q^*$ by

$$\langle Dv, \alpha\rangle + \langle v, \tilde{D}\alpha\rangle = \frac{1}{2}(d\langle v, \alpha\rangle + *d\langle v, J^*\alpha\rangle),$$

a so called mixed structure. While the holomorphic structure D on W does not give a holomorphic structure on W^*, it defines a holomorphic structure on a paired bundle:

Definition 1.29
A pairing between two complex quaternionic vector bundles (W, J), $(\tilde{W}, \tilde{J})$ is a map

$$(\cdot, \cdot) : \tilde{W} \times W \to T^*M \otimes_{\mathbb{R}} \mathbb{H}$$

such that for $\psi \in \Gamma(W)$, $\varphi \in \Gamma(\tilde{W})$ and quaternionic functions λ, μ, it holds

$$(\varphi\mu, \psi\lambda) = \bar{\mu}(\varphi, \psi)\lambda,$$

and

$$*(\varphi, \psi) = (\varphi, J\psi) = (\tilde{J}\varphi, \psi).$$

If $(\, , \,)$ is a pairing of $\tilde{W}$ and W, then $\overline{(\, , \,)}$ is a pairing of W and $\tilde{W}$.

Two holomorphic structures D and $\tilde{D}$ on (W, J), $(\tilde{W}, \tilde{J})$ respectively, are called compatible if

$$d(\varphi, \psi) = (\tilde{D}\varphi \wedge \psi) + (\varphi \wedge D\psi) \tag{16}$$

holds for all sections $\varphi \in \Gamma(\tilde{W})$ and $\psi \in \Gamma(W)$. Here, d denotes the exterior derivative and

$$(\tilde{D}\varphi \wedge \psi)(X, Y) = (\tilde{D}\varphi(X), \psi)(Y) - (\tilde{D}\varphi(Y), \psi)(X),$$
$$(\varphi \wedge D\psi)(X, Y) = (\varphi, D\psi(X))(Y) - (\varphi, D\psi(Y))(X).$$

Example
Let (W, J) be a complex bundle. Then the pairing of KW^* and W is given by evaluation of the W^* part on W, i.e.,

$$(\omega, \psi) := \omega(\psi)$$

for $\omega \in \Gamma(KW^*)$ and $\psi \in \Gamma(W)$.

Any pairing $(\, , \,)$ between $\tilde{W}$ and W yields an isomorphism between $\tilde{W}$ and KW^{-1} by

$$\varphi \mapsto (\varphi, \cdot).$$

This isomorphism preserves the complex structure.

As stated before, a given connection ∇ on W defines a connection on W^*. Using the identification $\bar{K}K = \Lambda^2 T^* M_{\mathbb{C}}$ by $d\bar{z} \otimes dz \mapsto d\bar{z} \wedge dz = -dz \wedge d\bar{z}$ the exterior derivative yields a holomorphic structure $\tilde{D} = d^\nabla$ on KW^*. This follows from the Leibniz rule:

$$d^\nabla(\omega\lambda) = d^\nabla\omega\lambda - \omega \wedge d\lambda = (d^\nabla\omega)\lambda + \frac{1}{2}(\omega d\lambda + J^*\omega * d\lambda).$$

The holomorphic structures d^∇ and $D = \nabla''$ are compatible, since

$$d\langle\omega, \psi\rangle = \langle d^\nabla\omega, \psi\rangle - \langle\omega \wedge \nabla\psi\rangle = \langle d^\nabla\omega, \psi\rangle - \langle\omega \wedge D\psi\rangle.$$

Lemma 1.30
Let L be a complex quaternionic bundle with holomorphic structure D. Then there is a unique holomorphic structure $\tilde{D}$ on KL^{-1}, compatible with D.

Proof
After identifying $\bar{K}K$ with $\Lambda^2(T^*M)$, we have $(\tilde{D}\omega \wedge \psi) = (\tilde{D}\omega, \psi)$, where the L^{-1} part of $\tilde{D}$ is evaluatet at ψ. By linearity, $\tilde{D}\omega$ is determined by $(\tilde{D}\omega, \psi)$, which then can be calculated by the compatibility equation (16). $\square$

The pairing between W and KV^* can be used to get the Riemann Roch theorem in the quaternionic setting, see [FLPP01, Theorem2.2] for further information.

1.9. The Weierstraß representation

The classical Weierstrass representation for minimal surfaces in $\mathbb{R}^3$ allows to write a minimal surface $f : M \to \mathbb{R}^3$ in terms of a meromorphic function and a holomorphic function. Eisenhart [Eis09], Konopelchenko [Kon96], and Taimanov [Tai97] (with a global approach) generalised the Weierstrass representation for conformal immersions in $\mathbb{R}^3$ by adding some noise to the holomorphic structure. Using the quaternionic theory, Pedit and Pinkall [PP98] were able to give a coordinate free, more intrinsic version of the Weierstrass representation for conformal immersed surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$.

Theorem 1.31
For an immersion $f : M \to \mathbb{H}$, there exist paired quaternionic line bundles KL^{-1} and L with holomorphic sections $\varphi \in H^0(KL^{-1})$ and $\psi \in H^0(L)$ such that

$$(\varphi, \psi) = df. \tag{17}$$

The holomorphic bundles and sections are uniquely determined by f up to isomorphism.

Definition 1.32
For a conformal immersion $f : M \to \mathbb{H}$, the representation $df = (\varphi, \psi)$ is called the Weierstraß representation.

Remark
The holomorphicity of φ and ψ is with respect to a quaternionic holomorphic structure. Let $\hat{\psi} \in \Gamma(L)$ and $\hat{\varphi} \in \Gamma(KL^{-1})$ with

$$\bar{\partial}\hat{\psi} = 0 \quad \text{and} \quad \bar{\partial}\hat{\varphi} = 0,$$
$$S\hat{\psi} = \psi\mathbb{i} \quad \text{and} \quad S * \hat{\varphi} = \hat{\varphi}\mathbb{i}.$$

Here $\bar{\partial}$ is the complex part of the quaternionic holomorphic structure D on L. Writing

$$\psi = \hat{\psi}(\psi_1 + \psi_2\mathbb{j}) \quad \text{and} \quad \varphi = \hat{\varphi}(\varphi_1 + \varphi_2\mathbb{j})$$

for complex valued functions $\psi_1, \psi_2, \varphi_1, \varphi_2$, it becomes visible that (17) is a generalisation of the Weierstrass representation for conformal immersions in $\mathbb{R}^3$.

Proof (of Theorem 1.31)
To show existence take the tautological line bundle defined by f, i.e. $L = \psi\mathbb{H}$ with $\psi = (f, 1)$. Let α, β be the dual basis of $(1, 0), (0, 1) \in \mathbb{H}^2$ and let $\beta_L \in \Gamma(L^{-1})$ be the restriction of β on L. Take $\varphi = \beta_L d\bar{f}$, then $(\varphi, \psi) = df$ holds. Note that ψ is only holomorphic with respect to $\nabla''_{|L}$ if f is minimal, but one can define a unique holomorphic structure on L by demanding ψ to be holomorphic.
We have $S\psi = -\psi R$, and therefore $S\beta = \beta R = -\beta\bar{R}$, which implies

$$*\beta\overline{df} = -\beta\overline{dfR} = -\beta\overline{R}d\bar{f} = S\beta df,$$

i.e. $\varphi \in \Gamma(KL^{-1})$. Since $0 = d(\varphi, \psi)$, φ is holomorphic with respect to the holomorphic structure induced by the pairing.

To see that the bundles and sections are unique, assume there is another line bundle $\tilde{L}$ with holomorphic sections $\tilde{\varphi} \in \Gamma(K\tilde{L}^{-1})$, $\tilde{\psi} \in \Gamma(\tilde{L})$, satisfying

$$(\tilde{\varphi}, \tilde{\psi}) = df.$$

The isomorphism defined by $\psi \mapsto \tilde{\psi}$ is compatible with the complex structure, since by

$$(\tilde{\varphi}, \tilde{S}\tilde{\psi}) = *(\tilde{\varphi}, \tilde{\psi}) = *df = -dfR = (\tilde{\varphi}, -\psi R).$$

The complex structure on $\tilde{L}$ is also given by the mean curvature congruence of f, and so

$$S\psi = -\psi R \mapsto -\tilde{\psi}R = \tilde{S}\tilde{\psi}.$$

Since ψ and $\tilde{\psi}$ are both holomorphic, the holomorphic structure is also preserved. □

Remark
The holomorphic structure on L is given by taking a connection on L^{-1} such that the non-vanishing ∞-section β is flat. The holomorphic structure on L is then the $''$-part of the dual connection. The complex part $\bar{\partial}$ of the holomorphic stucture does not depend on the chosen point at ∞ and is given by $\nabla''_{|L}$, where $\nabla = d$ on V, see [Boh03, Lemma 17].

Lemma 1.33
Let f be a conformal immersion into S^3. Then, the bundles L and KL^{-1} are isomorphic as complex bundles.

Proof
Consider the pairing $-\overline{(\,,\,)}$ of L and KL^{-1}. Take the sections $\tilde{\psi} = \beta\overline{df}f \in \Gamma(KL^{-1})$ and $\tilde{\varphi} = (f, 1)\bar{f} \in \Gamma(L)$. Then, using $f\bar{f} = 1$, and therefore $fd\bar{f} = -df\bar{f}$, we obtain

$$-\overline{(\tilde{\psi}, \tilde{\varphi})} = -\overline{\bar{f}df\bar{f}} = -fd\bar{f}f = f\bar{f}df = df.$$

Therefore we get an complex isomorphism of L and KL^{-1} by mapping ψ to $\tilde{\psi}$. Note that $\tilde{\varphi}$ and $\tilde{\psi}$ are not holomorphic with respect to the holomorphic structures ∇'' and the one compatible with the pairing, since

$$\nabla''\tilde{\varphi} = \nabla''(\psi\bar{f}) = \frac{1}{2}(\psi d\bar{f} + S\psi\overline{*df}) = \frac{1}{2}(\psi d\bar{f} + \psi R\overline{dfR}) = \psi d\bar{f} \neq 0,$$

so the isomorphism is complex, i.e., compatible with the complex stucture S, but not holomorphic with respect to the induced structures. □

Lemma 1.34
Let f be a conformal immersion into S^3 and

$$L = E \oplus E\mathbb{j}$$

be the splitting of L into the $\pm\mathbb{i}$ eigenspaces of S. Then, E is a spin bundle, i.e., $E^2 = K$.

Proof
By Lemma 1.33 the bundles L and KL^{-1} are isomorphic as complex bundles. The splitting of KL^{-1} into $\pm\mathbb{i}$-eigenspaces is given by

$$KL^{-1} = KE^{-1} \oplus KE^{-1}.$$

Thus $E = KE^{-1}$, which is equivalent to $E^2 = K$. $\square$

2. Riemann surface theory

Riemann surface theory is the study of complex one dimensional surfaces M or equivalently a real 2-dimensional surface equipped a class of conformal metrics or equivalently with a 90°-rotation $J : TM \to TM$. In this chapter, we will collect some important facts about Riemann surfaces and properties of vector bundles on Riemann surfaces.

2.1. Riemann Roch and Riemann Hurwitz

The Riemann Roch and Riemann Hurwitz Theorem are fundamental theorems of Riemann surface theory. For further details, for example a proof of Riemann Roch, see e.g. [Don11]. Working with coverings, i.e., non-constant holomorphic maps between Riemann surfaces, the Riemann Hurwitz theorem is quite useful, relating the genuses of the Riemann surfaces with the number of branchpoints.

Theorem 2.1 (Riemann Hurwitz)
Let M, N be Riemann surfaces of genus g_Y, and g_X, respectively. Let $f : N \to M$ be a branched n-fold covering. Then

$$2g_N - 2 = b + n(2g_M - 2),$$

where b is the total branch order, i.e., $b = \sum_{p \in Y} \operatorname{ord}_p(df)$.

Proof
The map f is holomorphic, therefore $df : TN \to TM$ can be interpreted as a holomorphic section

$$df \in \Gamma(M, K_N f^*(TM))$$

The number of zeroes of df is by definition b. On the other hand, for the degree of $K_N f^*(T_M)$, we calculate

$$\deg(K_N f^*(TM)) = \deg(K_N) - \deg f \deg TM.$$

Using $TM = K_M^*$ and (19) then finishes the proof. □

In the case of a 2-fold cover, which is the case we will be most interested in, the total branch order is just the number of branch points, which, by Riemann Hurwitz is only dependant on the genus of the surfaces M and N.

Theorem 2.2 (Riemann Roch)
Let E be a holomorphic vector bundle over a compact Riemann surface M of genus g. Then

$$h^0(E) - h^0(KE^*) = \deg E - (g-1) \operatorname{rank} E,$$

where $h^0(E)$ denotes the dimension of the space of holomorphic sections $H^0(E)$.

Remark
Riemann Roch still holds in the quaternionic setting, see [FLPP01, Section 2], and relates the dimension of holomorphic sections of the paired bundles E and KE^*.

Example
Let $E = \mathbb{C}$ be the trivial line bundle, then Riemann Roch implies

$$h^0(K) = h^0(\mathbb{C}) - \deg \mathbb{C} + (g-1) \operatorname{rank} \mathbb{C} = g. \tag{18}$$

Therefore, using $E = K$, leads to

$$\deg K = (g-1) \operatorname{rank} K + h^0(K) - h^0(KK^*) = 2g - 2. \tag{19}$$

For a general line bundle L, Riemann Roch yields estimates for the degree.

Corollary 2.3
Let L be a holomorphic line bundle, then, cf. figure 1,

1. If $\deg(L) < 0$ then $h^0(L) = 0$. This follows from degree formula.
2. $0 \le h^0(L) \le \deg(L) + 1$
3. $\deg(L) - g + 1 \le h^0(L)$. This becomes useful for $\deg(L) \ge g - 1$.
4. $h^0(L) = \deg(L) - g + 1$ for $\deg(L) > 2g - 2$.
5. In fact there is a better upper bound than in 2., namely:

$$h^0(L) \le \frac{\deg(L)}{2} + 1.$$

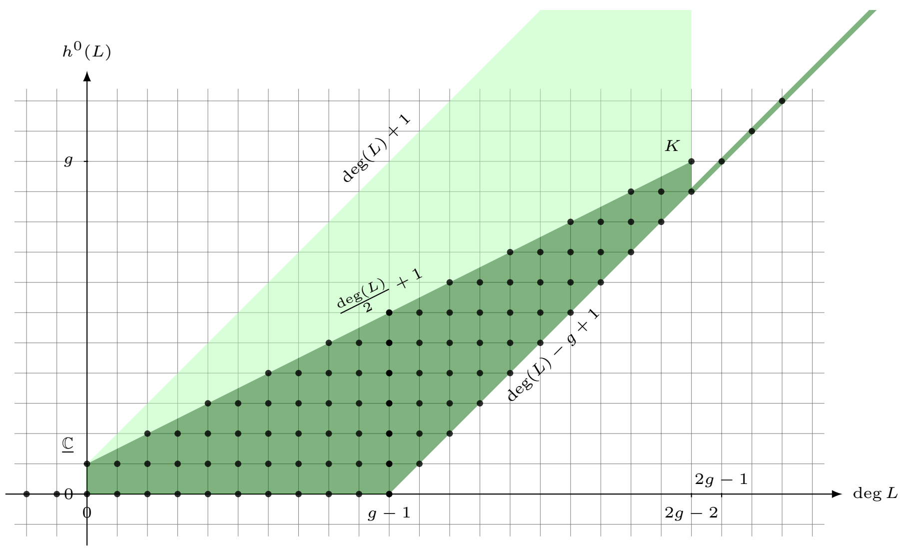

Figure 1: The possible dimensions of holomorphic sections for line bundles

2.2. Hyperelliptic surfaces

The easiest example class of Riemann surfaces are the hyperelliptic surfaces. A hyperelliptic surface Σ can be defined as a 2-fold covering of $\mathbb{C}P^1$. Due to this covering many objects, e.g. the holomorphic differentials, can be written down quite explicitly in terms of a global chart of $\mathbb{C}P^1$.

Definition 2.4
We define a hyperelliptic surface Σ as the compactification of the set

$$\Sigma_0 = \{(y, z) \in \mathbb{C}^2 \,|\, y^2 = p(z)\},$$

where $p(z) = \prod_{i=1}^{d}(z - z_i)$ is a polynomial with only simple zeroes.

Since

$$2y\,dy - p'(z)\,dz \neq 0$$

the implicit function theorem yields that Σ_0 is a submanifold. If $y \neq 0$, the map

$$z : (z, y) \mapsto z$$

is a local chart of Σ_0. y is a local chart in neighborhoods around the points with $y = 0$. The map $z : \Sigma_0 \to \mathbb{C}$ is a 2-fold covering. The manifold Σ_0 can be compactified by adding one or two points over $\infty \in \mathbb{C}P^1$ depending on whether the degree of p is odd or even. Assume that the degree is even and let $w = \frac{1}{z}$. Then $y^2 = p(z)$ becomes

$$\begin{aligned} y^2 &= \prod_{i=1}^{d}(\frac{1}{w} - z_i) \\ \Leftrightarrow \quad y^2 w^d &= \prod_{i=1}^{d}(1 - z_i w) \\ \Leftrightarrow \quad \tilde{y}^2 &= \prod_{i=1}^{d}(1 - z_i w), \end{aligned}$$

with $\tilde{y} = yz^{-d/2}$. The set

$$\Sigma_\infty = \{(\tilde{y}, w) \,|\, \tilde{y}^2 = \prod_{i=1}^{d}(1 - z_i w)\}$$

is again a manifold and can be glued together with Σ_0 by

$$(y, z) \in \Sigma_0 \mapsto (yz^{-d/2}, \frac{1}{z}) \in \Sigma_\infty.$$

The glued manifold is then the hyperelliptic surface Σ. Since $w = 0$ gives $\tilde{y} \neq 0$, there are two points ∞_1, ∞_2 over $z = \infty$. Charts around those points are given by w. The sets

$$\{(y, z) \in \Sigma_0 \,|\, |z| \leq 2\} \quad \text{and} \quad \{(\tilde{y}, w) \in \Sigma_\infty \,|\, |w| \leq 2\}$$

are compact and cover Σ, which is therefore compact.

If d is odd we get

$$\tilde{y}^2 = w \prod_{i=1}^{d} (1 - z_i w),$$

with $\tilde{y} = yz^{-(d+1)/2}$ and $w = 1/z$. Then, $w = 0$ is a branchpoint, and there is only one point ∞ to compactify Σ_0.

The map $z : \Sigma_0 \to \mathbb{C}$ extends to a double cover $z : \Sigma \to \mathbb{C}P^1$. Let $d = 2g + 2$ or $2g + 1$, then $z : \Sigma \to \mathbb{C}P^1$ has $2g + 2$ branchpoints. By Riemann Hurwitz 2.1 the genus of Σ is g.

Remark
We further get uniqueness in the following sense. By Riemann's existence theorem a 2-fold covering is determined up to biholomorphy by its branchpoints, e.g. see [Don11, Theorem 2]. Therefore every 2-fold cover of $\mathbb{C}P^1$ is given by the construction above.

Definition 2.5
The map

$$\sigma : \Sigma \to \Sigma, \quad (y, z) \mapsto (-y, z)$$

is called the hyperelliptic involution.

Proposition 2.6
Every surface M of genus g equipped with an involution σ with $2g + 2$ fixpoints is a hyperelliptic surface.

PROOF
M is a 2-fold cover of M/σ. By Riemann Hurwitz M/σ has genus 0, i.e., is $\mathbb{C}P^1$. □

Let $d = 2g + 2$ be even. Then the meromorphic function y on Σ has exactly $2g + 2$ zeroes in the branch points $(0, z_i)$ and two poles, each of order $g + 1$ in ∞_1, ∞_2. The differential dz also has zeroes of order 1 in the branch points and has a double pole in the two points over ∞. Therefore the meromorphic differential

$$\frac{dz}{y}$$

has no poles at all, i.e., is holomorphic, with a zero of order $g - 1$ in ∞_1 and ∞_2. Since z has only simple poles in ∞ we get

$$\frac{z^l dz}{y} \in H^0(K) \quad \text{for } 0 \leq l \leq g - 1.$$

It holds $h^0(K) = g$ by (18), therefore those sections form a basis of $H^0(K)$. This is also true for $d = 2g + 1$.

2.3. The Picard group and the Jacobian

The degree is the only obstruction for two complex line bundles to be isomorphic as real vector bundles. For holomorphic vector bundles this does not hold. Therefore one can study the set of isomorphism classes of holomorphic line bundles, the Picard group.

Definition 2.7 (Picard group)
The Picard group of a Riemann surface M is defined as

$$\mathrm{Pic}(M) = \{\text{holomorphic line bundles}\} / \{\text{holomorphic isomorphisms}\}.$$

Equipped with the tensor product $\otimes$ the Picard group is actually a group. The equivalence classes of line bundles of degree d are denoted by

$$\mathrm{Pic}_d(M).$$

The degree 0 line bundles $\mathrm{Pic}_0(M)$ are a subgroup of $\mathrm{Pic}(M)$. Let L and $\tilde{L}$ be line bundles of degree d. Since

$$\deg(L \otimes \tilde{L}^{-1}) = 0,$$

the subsets $\mathrm{Pic}_d(M) \subset \mathrm{Pic}(M)$ are affine subspaces with underlying group $\mathrm{Pic}_0(M)$. Therefore it is enough to study the degree 0 line bundles.

Let L be a line bundle of degree 0. Since L is smooth isomorphic to the trivial bundle $\underline{\mathbb{C}}$ we can view L as the trivial bundle equipped with some holomorphic structure $\bar{\partial}$ which in general is not the canonical holomorphic structure $\bar{\partial}^{\mathbb{C}}$ on $\mathbb{C}$. With respect to a trivializing frame, we obtain

$$\bar{\partial} = \bar{\partial}^{\mathbb{C}} + \alpha$$

for some $\alpha \in \Gamma(\bar{K})$. Let $g \in \Gamma(M, \mathrm{End}(\underline{\mathbb{C}}))$, i.e., a $\mathbb{C} \setminus \{0\}$-valued function. Then the gauged holomorphic structure is given by

$$\bar{\partial}_{\cdot g} = g^{-1} \circ \bar{\partial} \circ g = \bar{\partial} + g^{-1}(\bar{\partial}^{\mathbb{C}} g).$$

Definition 2.8 (Jacobian)
The Jacobian is the set of possible gauge classes of holomorphic structures on $\underline{\mathbb{C}}$, i.e.,

$$\mathrm{Jac}(M) = \left\{\text{holomorphic structures } \bar{\partial} : \underline{\mathbb{C}} \to \bar{K}\right\} / \Gamma(M, \mathrm{End}(\underline{\mathbb{C}}))$$

Two line bundles $L = (\underline{\mathbb{C}}, \bar{\partial}^L)$ and $\tilde{L} = (\underline{\mathbb{C}}, \bar{\partial}^{\tilde{L}})$ are isomorphic as holomorphic vector bundles if and only if the holomorphic structures are gauge equivalent, i.e., if there is a gauge g satisfying

$$\bar{\partial}^L_{\cdot g} = \bar{\partial}^{\tilde{L}}.$$

Therefore we have the isomorphy

$$\mathrm{Jac}(M) \cong \mathrm{Pic}_0(M).$$

Theorem 2.9
Let M be a Riemann surface of genus g. Then the Jacobian is a (complex) g-dimensional torus. More detailed $\mathrm{Jac}(M)$ is given as

$$\mathrm{Jac}(M) = \overline{H^0(K)}/\Lambda,$$

where

$$\Lambda = \left\{\bar{\omega} \in \overline{H^0(K)} \mid \int_\gamma -\omega + \bar{\omega} \in 2\pi i\mathbb{Z} \text{ for all closed } \gamma\right\}$$

is a lattice of full rank.

A useful tool to work with $\bar{\partial}$-operators is the Serre duality.

Theorem 2.10 (Serre duality)
Let M be a Riemann surface and $E \to M$ be a vector bundle equipped with a holomorphic structure $\bar{\partial}$. Then

i) $H^0(E) = \ker(\bar{\partial})$ is finite dimensional.

ii) $H^1(E)^* = (\Gamma(KE)/\operatorname{im}(\bar{\partial}))^* \cong H^0(KE^*)$

iii) There exists an solution ψ of $\bar{\partial}\psi = \varphi$ if and only if $\varphi \in \Gamma(\bar{K}E)$ satisfies

$$\int_M \langle \varphi, \omega \rangle = 0, \quad \forall \omega \in H^0(KE^*).$$

A proof of both theorems can be found in [Don11]. In order to give some insight to Theorem 2.9, we will demonstrate that for every holomorphic structure $\bar{\partial}$ on $\underline{\mathbb{C}}$ there is a gauge $g = e^f$, where $f : M \to \mathbb{C}$, such that

$$\bar{\partial}.g = \bar{\partial}^{\mathbb{C}} + \bar{\eta}$$

for some $\bar{\eta} \in H^0(\bar{K})$. Every holomorphic structure can be written as

$$\bar{\partial} = \bar{\partial}^{\mathbb{C}} + \alpha$$

for some section $\alpha \in \Gamma(\bar{K})$. The gauged connection then looks like

$$\bar{\partial}.g = \bar{\partial} + \bar{\partial}^{\mathbb{C}} f = \bar{\partial}^{\mathbb{C}} + \alpha + \bar{\partial}^{\mathbb{C}} f.$$

By the Serre duality, there is a f such that

$$\bar{\partial}^{\mathbb{C}} f = \bar{\eta} - \alpha,$$

if and only if

$$\int_M \langle \alpha - \bar{\eta}, \omega \rangle = 0, \quad \forall \omega \in H^0(K).$$

Since the pairing

$$\overline{H^0(K)} \times H^0(K) \to \mathbb{C}, \quad (\bar{\eta}, \omega) \mapsto \int_M \bar{\eta} \wedge \eta$$

is non-degenerated, there is an $\bar{\eta}$ with

$$\int_M \alpha \wedge \omega = \int_M \bar{\eta} \wedge \omega, \quad \forall \omega \in H^0(K).$$

Remark
The lattice Λ of Theorem 2.9 comes from gauges

$$g(z) = \exp\left(\int_{z_0}^{z} -\omega + \bar{\omega} \right),$$

with $\bar{\omega} \in \overline{H^0(K)}$. Two holomorphic structures $\bar{\partial}^{\mathbb{C}} + \bar{\eta}$, $\bar{\partial}^{\mathbb{C}} + \bar{\nu}$ with $\bar{\eta}$, $\bar{\nu} \in \overline{H^0(K)}$ are gauge equivalent if and only if

$$g(z) = \exp\left(\int_{z_0}^{z} -(\eta - \nu) + \bar{\eta} - \bar{\nu} \right)$$

is well defined, i.e., if the periods are all $2\pi i\mathbb{Z}$-valued.

2.4. Kodaira embedding

The Kodaira correspondence gives an answer to the question if a line bundle is a holomorphic subbundle of the trivial $\mathbb{C}^k$ bundle.

Let

$$L^{-1} \to M$$

be a holomorphic line bundle on a Riemann surface M and

$$H \subset H^0(L^{-1})$$

a $k+1$-dimensional linear subset of the space of holomorphic sections. A point $p \in M$ is called a base point of H, if all holomorphic sections of H vanish in p. Assume H is base point free, then the evaluation map

$$\mathrm{ev}_p : \mathbb{C}^{k+1} \cong H \to L_p^{-1}, \quad \psi \mapsto \psi(p)$$

is surjective. This implies that the dual map

$$\mathrm{ev}_p^* : L_p \to H^* \cong \mathbb{C}^{k+1}$$

is injective, i.e. realises L as a subbundle of $\mathbb{C}^{k+1}$. In other words ev^* defines a map f from M to $\mathbb{C}P^k$ by $f(p) = \mathrm{ev}_p^*(L_p)$. More explicitly, in a basis $s_0, s_1, \ldots, s_k$ of H and for a local non-vanishing holomorphic section φ of L^{-1} (in H), f is given by

$$f(p) = \left[\begin{pmatrix} \frac{s_0}{\varphi} \\ \vdots \\ \frac{s_k}{\varphi} \end{pmatrix} \right]. \tag{20}$$

This can be seen as follows. Let $\langle\,,\,\rangle$ be the pairing between L and L^*. It is

$$\langle s_i, f\rangle = \langle s_i, \mathrm{ev}_p^*(\alpha)\rangle = \langle \mathrm{ev}_p(s_i), \alpha\rangle = \frac{s_i}{\varphi}\langle \varphi(p), \alpha\rangle.$$

Therefore the identification of $P(H^*)$ and $\mathbb{C}P^k$ is given by

$$[\alpha] \in P(H^*) \mapsto \left[\begin{pmatrix}\langle s_0, \alpha\rangle \\ \vdots \\ \langle s_k, \alpha\rangle\end{pmatrix}\right],$$

which implies (20).

Example
Let L be a line bundle with $h^0(L) = 2$ and no base points. Then the map $f : M \to \mathbb{C}\cup\{\infty\}$ is just given as the quotient of two basis vectors s_0, s_1, i.e.

$$f = \frac{s_0}{s_1}.$$

The procedure can be reversed. A map $f : M \to \mathbb{C}P^k$ defines a line bundle $L \subset \mathbb{C}^{k+1} \to M$ and the restriction of linear forms $\alpha \in (\mathbb{C}^{k+1})^*$ yield holomorphic sections $\alpha_{|L} \in H^0(L^{-1})$. The subspace $H \subset H^0(L^{-1})$ given by restricting constant sections of $\mathbb{C}^{k+1}$ is called the canonical linear system.

The above procedure is known as the Kodaira correspondence, which yields obstructions for a line bundle to be realisable as a holomorphic subbundle of $\mathbb{C}^{k+1}$:

Theorem 2.11 (Kodaira correspondence)
A base point free $k+1$ dimensional subspace H of $H^0(L^{-1})$ defines a unique (up to a $\mathrm{PSL}(k+1)$-action) realisation of L as a subbundle of $\mathbb{C}^{k+1}$. A subbundle L of $\mathbb{C}^{k+1}$ defines a unique subspace of $H^0(L^{-1})$ by restricting the constant sections of $(\mathbb{C}^{k+1})^*$ on L. A line bundle L can be uniquely (up to $\mathrm{PSL}(k+1)$ action) embedded if and only if $H = H^0(L^{-1})$, i.e. $h^0(L^{-1}) = k+1$.

The Kodaira embedding can also be generalized to the quaternionic setting, see [FLPP01, Section 2.9] for further details.

3. Pushforward bundle

Given a covering $N \to M$ of Riemann surfaces and a line bundle L on N, there is a construction to define a higher rank bundle, the so called pushforward bundle, on M such that the holomorphic sections of L and the pushforward bundle can be identified. The dual bundle L^{-1} is a canonical line subbundle of the pushforward bundle. The idea of the construction is to define the pushforward bundle by holomorphic sections of L. In a regular point the pushforward bundle is then given by $L_p^k = L_{p_1} \oplus \cdots \oplus L_{p_k}$, where $f^{-1}(\{p\} = \{p_1, \ldots, p_k\}$. In order to get a well defined bundle the frames of L^k need to have a pole in the branchpoints. We will first demonstrate the basic construction with an easy example, the pushforward bundle of a line bundle on a hyperelliptic surface.

3.1. Pushforward bundle of a line bundle on a hyperelliptic surface

Let Σ be a hyperelliptic surface given by the equation $y^2 = p(z)$ for some polynomial p. We denote the $2:1$-covering from $\Sigma \to \mathbb{C}P^1$ by z. The pushforward bundle

$$z_*L \to \mathbb{C}P^1$$

of a holomorphic line bundle $L \to \Sigma$ is defined using local frames.

Let $U \subset \mathbb{C}P^1$ be an open subset, such that

$$z^{-1}(U) = U^+ \dot\cup U^-$$

splits into two disjoint open subsets of Σ. Further let U be small enough such that there are non-vanishing sections

$$t_+ \in H^0(U^+, L) \quad \text{and} \quad t_- \in H^0(U^-, L).$$

Then we set

$$(s_+, s_-) := (s_+ \oplus 0, 0 \oplus s_-) \tag{21}$$

as a local holomorphic frame of z_*L. The bundle, generated by these frames, does not depend on the choice of the sections $t_\pm$: Let $U, \tilde{U}$ be two such open subsets with frames $(t_+, t_-), (\tilde{t}_+, \tilde{t}_-)$. On $U \cap \tilde{U}$ the cocycle with respect to these frames is given by

$$g = \begin{pmatrix} g_+ & 0 \\ 0 & g_- \end{pmatrix},$$

where g_+ is the gauge between t_+ and $\tilde{t}_+$,i.e., $t_+ = \tilde{t}_+ g_+$, and g_- between t_- and $\tilde{t}_-$. The functions $g_\pm : U^\pm \cap \tilde{U}^\pm \to \mathbb{C}$ are holomorphic and without zeroes. Since $U^\pm \cap \tilde{U}^\pm$ can be identified with $U \cap \tilde{U}$, g is a holomorphic cocycle on $U \cap \tilde{U} \subset \mathbb{C}P^1$. If $z \in \mathbb{C}P^1$ is a regular point there is always a neighborhood such that we can find a frame (t_+, t_-).

Let us now focus on the case that z is a branchpoint. At a branchpoint z_0, there is a centered chart $(\tilde{z}, U)$ of $\mathbb{C}P^1$, and a chart $(y, U_0 = z^{-1}(U))$ of Σ, such that the double covering is given by $y \mapsto y^2 = \tilde{z}$. Let

$$t \in H^0(U_0, L)$$

be a local non-vanishing holomorphic section. Every function $f \in H^0(U_0, \mathbb{C})$ splits into a σ-invariant and a σ-anti-invariant part

$$f(y) = f^+(\tilde{z}) + y f^-(\tilde{z}).$$

Therefore every section s of L on U_0 can be written as $s = t(f^+ + y f^-)$. While y is only defined on U_0, the functions $f_\pm$ can be seen as well-defined functions on U. So we take

$$(t, yt) \tag{22}$$

as a frame of $z_* L_{|U}$. As in the regular case choosing another section $\tilde{t} \in H^0(U_0, L)$ gives an holomorphic cocycle: It is

$$\tilde{t} = f^+ t + f^- yt \quad \text{and} \quad y\tilde{t} = y^2 f^- t + f^+ yt.$$

Therefore the cocycle between (t, yt) and $(\tilde{t}, y\tilde{t})$ is

$$g = \begin{pmatrix} f^+ & \tilde{z} f^- \\ f^- & f^+ \end{pmatrix}$$

with $f^+(p) \neq 0$ and therefore $\det g \neq 0$ around z_0. As $f^\pm$ and $\tilde{z}$ can be seen as well-defined functions on U, g is a well-defined cocycle on U.

Next we'll calculate the cocycle between frames at regular points, see (21), and frames at branch points as in (22). Let U be the neighborhood of a branch point with frame (t, yt) for some nonvanishing $t \in H^0(U_0, L)$, $\tilde{U}$ a subset with $z^{-1}(\tilde{U}) = \tilde{U}^+ \dot\cup \tilde{U}^-$ and $U \cap \tilde{U} \neq \emptyset$. Assume that $\tilde{U} \subset U$, then $t^+ = t_{|\tilde{U}_+}$ is a nonvanishing section in $H^0(\tilde{U}_+, L)$, similiarly $t^- = t_{|\tilde{U}_-} \in H^0(\tilde{U}_-, L)$. On $U \cap \tilde{U}$, the function

$$y = \sqrt{\tilde{z}}$$

is well-defined. Choose

$$y_{|\tilde{U}} = y_{|\tilde{U}^+} = -y_{|\tilde{U}^-}.$$

Then

$$t^+ = \frac{1}{2y_{|\tilde{U}}}(y_{|\tilde{U}} t + yt) \quad \text{and} \quad t^- = \frac{1}{2y_{|\tilde{U}}}(y_{|\tilde{U}} t - yt).$$

Note that yt is the basis vector in $z_* L$. The cocycle with respect to the frames is therefore given by

$$g_{\tilde{U}U} = \begin{pmatrix} 1 & y_{|U\cap\tilde{U}} \\ 1 & -y_{|U\cap\tilde{U}} \end{pmatrix},$$

and the determinant on $U \cup \tilde{U}$ is $-2y \neq 0$.

Proposition 3.1 (Properties of the pushforward bundle)
There is a one-to-one-correspondence between holomorphic sections of L and holomorphic sections of z_*L, i.e. $H^0(\Sigma, L) = H^0(\mathbb{C}P^1, z_*(L))$. This holds also for meromorphic sections.

The degree of the pushforward bundle of a line bundle L on a hyperelliptic surface of genus g is

$$\deg(z_*L) = \deg(L) - (g+1).$$

Proof
Let $s \in H_0(\Sigma, L)$ be a holomorphic section of L. Then s defines a holomorphic section of z_*L in the following way: Over $U \subset \mathbb{C}P^1$ with $z^{-1}(U) = U^+ \dot\cup U^-$ we have

$$s = s_+ \oplus s_- = s_{|U^+} \oplus s_{|U^-}.$$

We use $\oplus$ to emphasize that in z_*L the sections $s_\pm$ are linearly independent. If U contains a branch point, and t is a nonvanishing section, then $s = (f^+ + f^- y)t$ defines a section

$$s = f^+ t \oplus f^-\, yt.$$

Both definitions coincide where they are both defined since

$$s_{|U^+} \oplus s_{|U^-} = \frac{1}{2y}(y \cdot s + ys) + \frac{1}{2y}(y \cdot s - ys) = s.$$

Vice versa, a holomorphic section $s \in H^0(\mathbb{C}P^1, z_*L)$ is away from branch points locally given as $s_+ \oplus s_-$ and at branch points as $s = f^+ t \oplus f^- yt$. The sections $s_\pm$ and s again agree where both are defined and therefore form a well defined holomorphic section in L. For meromorphic sections the construction works analogously.

For the degree formula, we consider

$$s \wedge ys \in \mathcal{M}(\mathbb{C}P^1, \Lambda^2(z_*L)),$$

where $s, ys \in \mathcal{M}(\Sigma, L) = \mathcal{M}(\Sigma, L)$ are meromorphic sections.

We will now calculate the zeroes and poles of $s \wedge ys$. Let $z \in \mathbb{C}P^1$ be a regular point and (t_+, t_-) be a local frame of z_*L as in (21). Again we write $s = s_+ + s_-$. Then

$$s_\pm = t_\pm f^\pm.$$

and therefore

$$s \wedge ys = (s_+ + s_-) \wedge y(s_+ + s_-) = -2yf^+ f^-\, t_+ \wedge t_-,$$

which implies that, if $z \neq \infty$, $s \wedge ys$ has the same order of zero or pole in z as the sum of orders in the points above z. If $z = \infty$ then y adds a pole of order $g+1$.

In the case that $z \in \mathbb{C}$ is a branch point, we take a basis (t, yt) as in (22). Then

$$s = f^+ t + f^- yt,$$

and we get
$$s \wedge ys = (f^{+2} - y^2 f^{-2}) t \wedge yt.$$
Since $\tilde{z} = y^2$, the order of f counts twice in $\xi \in \Sigma$. Thus the order of $s \wedge ys$ in z equals the order of s in ξ.

If $z = \infty$ is a branchpoint, then $\tilde{y} = \frac{y}{z^{g+1}}$ is a centered chart at $\infty \in \Sigma$ and therefore $t, \tilde{y}t$ a frame for some non-vanishing section t. For easier notation we assume that s is nonvanishing at ∞ and take $s = t$.

$$s \wedge ys = s \wedge \tilde{y} z^{g+1} s = z^{g+1} s \wedge \tilde{y} s$$

and the single pole of z in ∞ then imply that $s \wedge ys$ has a pole of order $g+1$.

Summarizing, the divisor of $s \wedge ys$ is

$$(s \wedge ys) = (s) - (g+1)\infty,$$

where the divisor of s has to be pushed forward. Thus, we obtain the degree of $z_* L$. □

Theorem 3.2
Let Σ be a hyperelliptic surface of genus g, and $L \in \mathrm{Pic}_{g+1}$, i.e., $\deg z_* L = 0$. Then $z_* L$ is trivial if and only if $L \otimes z^*(\mathcal{O}(-1))$ has no non-trivial holomorphic sections.

Proof
One can check, that
$$z_*(L \otimes z^*(\mathcal{O}(-1)) = z_* L \otimes \mathcal{O}(-1).$$
Due to Birkhoff-Grothendick, the degree 0 bundle $z_* L$ has the form

$$z_* L = \mathcal{O}(k) \oplus \mathcal{O}(-k),$$

and is trivial if and only if $k = 0$. This is the case if and only if

$$\mathcal{O}(k-1) \oplus \mathcal{O}(-k-1) = z_* L \otimes \mathcal{O}(-1)$$

has no non-trivial holomorphic section. □

Remark
Line bundles which are not covered by Theorem 3.2 form an $(g-1)$-dimensional subset in $\mathrm{Pic}_{g+1}(\Sigma)$.

The pushforward bundle carries a polynomial Killing field ξ, i.e. a meromorphic section in $\mathrm{End}(z_* L)$ with only one pole in $z = \infty$. We define a Killing field ξ by

$$\xi : s \mapsto ys. \tag{23}$$

To be more explicit: Let U be some neighborhood with $z^{-1}(U) = U^+ \dot{\cup} U^-$. For $s \in H^0(U^+, L) \subset H^0(U, z_* L)$, we set
$$\xi(s_p) = y(p) s_p.$$

Analogously, for $s \in H^0(U^-, L) \subset H^0(U, z_* L)$, we set

$$\xi(s_p) = y(p)s_p = -y_{|U_+}(p)s_p.$$

This yields a well-defined map

$$\xi : z_* L_{|\mathbb{C}P^1 \setminus \{branchpoints\}} \to z_* L_{|\mathbb{C}P^1 \setminus \{branchpoints\}}.$$

The map ξ can be extended through the branchpoints: Take U with a branchpoint and centered chart y of Σ, with y^2 a chart on U. Let $s \in H^0(z^{-1}(U), L)$ be a holomorphic section without zeroes and set

$$\xi(s) = ys \quad \text{and} \quad \xi(ys) = y^2\, s.$$

On the overlap both definitions coincide since

$$s_{U^+} \oplus 0 = \frac{1}{2y}(ys + (ys) \overset{\xi}{\mapsto} \frac{1}{2}(ys + y(ys)) = ys \oplus 0.$$

This works analoguesly for s on U^-.

Proposition 3.3
The in (23) constructed $\xi \in \mathcal{M}(\mathbb{C}P^1, \mathrm{End}(z_* L))$ is a Killing field with

$$\det \xi = -y^2 = -p(z).$$

Proof
By construction, ξ maps meromorphic sections on meromorphic sections, and is therefore meromorphic itself. Since y only has poles at ∞, the same holds for ξ. With respect to a basis (s_+, s_-), ξ is given by

$$\xi = \begin{pmatrix} y_{|U_+} & 0 \\ 0 & -y_{|U_+} \end{pmatrix},$$

and with respect to a basis (s, ys), ξ is given by

$$\xi = \begin{pmatrix} 0 & y^2 \\ 1 & 0 \end{pmatrix}.$$

Therefore $\det \xi = -y^2$. $\square$

Theorem 3.4
The eigenlines of ξ are given by L^* and $\sigma^* L^*$, where σ is the hyperelliptic involution.

Proof
Let $p_1, \dots p_{2g+2} \in \Sigma$ be the ramification points of $\Sigma \to \mathbb{C}P^1$. The dual bundle is given by

$$L^* = \sigma^*(L) \otimes L(-p_1 \dots - p_{2g+2}).$$

This holds since for $L = L(q_1 + \cdots q_{2g+2+h} - r_1 - \cdots - r_h)$, we have

$$
\begin{aligned}
&L \otimes \sigma^*(L) \otimes L(-p_1 \cdots - p_{2g+2}) \\
&\quad = L(\sum_{i=1}^{2g+2+h} q_i + \sum_{i=1}^{2g+2+h} \sigma(q_i) - \sum_{i=1}^{h}(r_i + \sigma(r_i)) - p_1 \cdots - p_{2g+2})
\end{aligned}
$$

and via the meromorphic function

$$\frac{y\Pi_{i=1}^{h}(z - z(r_i))}{\Pi_{i=1}^{2g+2+h}(z - z(q_i))}$$

this is the trivial bundle.

The eigenlines at a regular point z are by construction given by the span of s_+ and s_-. We now use the pullback of the pushforward bundle $z^*(z_*L) \to \Sigma$. On a subset $U \subset \Sigma$, with no ramification points, a frame is given by s_+, s_-, where $s_+ \in H^0(U, L)$, is nonvanishing and $s_- \in H^0(U, \sigma^*L) = H^0(\sigma(U), L)$. For U around a ramification point p, we have the frame (s, ys). However on the pullback s_+, s_- is a frame on $U \setminus \{p\}$, since y is defined on the whole set U. The map

$$
\begin{aligned}
&L \otimes L(-p_1 - \cdots - p_{2g+2}) \to z^*(z_*L), \\
&\qquad s \otimes s_{-p_1-\cdots-p_{2g+2}} \mapsto \begin{cases} s(p) \in L_p & \text{if } p \text{ is not a ramification point} \\ \frac{1}{2y}(y(p)s(p) + (ys)(p)) & \text{if } p \text{ is a ramification point} \end{cases}
\end{aligned}
$$

realises $\sigma^*(L^*) = L \otimes L(-p_1 - \cdots - p_{2g+2})$ as a subbundle of $z^*(z_*(L))$ which spans an eigenline of ξ at every point p. □

3.2. The general case

We will now generalize the above construction for an arbitrary branched covering between Riemann surfaces. The construction is similiar to the hyperelliptic case. The main difference is that a branch point is not necessarily totally branched. For a definition in an algebraic geometric point of view see [HSW99].

Let $\pi : N \to M$ be a branched k-fold covering of two Riemann surfaces M and N. Let E be a rank l bundle on N. The pushforward bundle π_*E is of rank kl. We will procede as in the hyperelliptic case and first define frames at regular points and afterwards in branchpoints. Let $p \in M$ be a regular point. Take a neighborhood U of p, such that $\pi^{-1}(U)$ are k disjoint copies U_i of U and $E_{|U_i}$ trivializes for every $1 \leq i \leq k$. Let s_i be a frame of E_{U_i}, then set

$$\underline{s}_{\text{reg}} = (s_1, \ldots s_k) \tag{24}$$

as a frame of $\pi_*E_{|U}$. The cocycle between two frames constructed this way is

$$g = \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ 0 & g_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_k \end{pmatrix},$$

where $g_i : U \cong U_i \to \mathbb{C}$ is the cocycle of two frames of $E_{|U_i}$.

Next, we will write down frames at branch points. If π is totally branched in p, we are in a similiar situation as in the hyperelliptic case. There are centered charts (z, U) of M and $(y, \tilde{U} = \pi^{-1}(U))$ of N, such that

$$z = y^k.$$

We write holomorphic functions on $\tilde{U}$ as $\sum_{i=0}^{k-1} f_i(z)y^i$. Let s be a holomorphic frame of E on $\tilde{U}$ (we assume that $\tilde{U}$ is small enough). Then every holomorphic section can be written as

$$\sum_{i=0}^{k-1} s f_i(z) y^i,$$

and we take the frame

$$\underline{s}_{\mathrm{tot}} = (s, ys, \ldots, y^{k-1}s). \tag{25}$$

As in the hyperelliptic case, choosing another frame $\tilde{s}$ gives an holomorphic equivalent frame: The cocycle between $\underline{s}_{\mathrm{tot}}$ and $\underline{\tilde{s}}_{\mathrm{tot}}$ is

$$g = \begin{pmatrix} f_1 & zf_k & zf_{k-1} & \cdots & zf_2 \\ f_2 & f_1 & zf_k & \cdots & zf_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{k-1} & f_{k-2} & f_{k-3} & \cdots & zf_k \\ f_k & f_{k-1} & zf_{k-2} & \cdots & f_1 \end{pmatrix}, \tag{26}$$

where $\tilde{s} = \sum_{j=1}^{k} s g_i y^i$. The functions f_i, and therefore g, can be interpreted as functions on U.

In the case that p is not totally branched, there are $q_1, \ldots, q_m$ ramification points over p. For every q_i there is a centered chart $(y_i, \tilde{U}_i)$ around q_i and a chart z around p such that

$$y_i^{k_i} = z$$

for some k_i with $\sum k_i = k$. We choose holomorphic frames $s_i \in H^0(\tilde{U}_i, E)$ and get a frame of $\pi_* E$ via

$$\underline{s}_{\mathrm{br}} = (s_1, y_1 s_1, \ldots, y^{k_1 - 1}, s_2, \ldots, y_2^{k_2-1} s_2, \ldots, s_m, \ldots y_m^{k_m-1} s_m).$$

So we basically take m versions of the totally branched case. Taking other holomorphic frames $\tilde{s}_i$ gives a cocycle

$$\begin{pmatrix} g_1 & 0 & \cdots & 0 \\ 0 & g_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_m \end{pmatrix} \quad \text{with} \quad g_i = \begin{pmatrix} f_1^i & zf_{k_i}^i & zf_{k_i-1}^i & \cdots & zf_2^i \\ f_2^i & f_1^i & zf_{k_i}^i & \cdots & zf_3^i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{k_i-1}^i & f_{k_i-2}^i & f_{k_i-3}^i & \cdots & zf_{k_i}^i \\ f_{k_i}^i & f_{k_i-1}^i & zf_{k_i-2}^i & \cdots & f_1^i \end{pmatrix}.$$

So g_i is just a rank k_i version of the totally branched cocycle (26). Note that $\underline{s}_{\text{br}}$ is the general formula for the construction of frames since if $m = k$ we just fall into the regular case.

The last part in the construction of the pushforward bundle is the cocycle between frames $\underline{s}_{\text{reg}}$ at regular points and frames at branchpoints $\underline{s}_{\text{br}}$. We will assume that we have a totally branched point, and therefore a frame $\underline{s}_{\text{tot}}$. If the point is not totally branched, it will only split in matrices on the diagonal that look like the totally branched case.

Let $\hat{U}$ be a neigbourhood where $\underline{s}_{\text{tot}}$ is defined. Further, let $U \subset \hat{U}$ such that there is a frame $\underline{s}_{\text{reg}}$ on $\hat{U}$ and

$$\pi^{-1}U = \dot{\bigcup}_{i=1}^{k} U_i.$$

Since we know the cocycle between frames $\underline{s}_{\text{reg}}$ at regular points, it is enough to look at the case where

$$s_i = s_{|U_i}.$$

For s and s_i being as in (24) and (25). We sort the U_i such that

$$y_{|U_i} = \epsilon^{i-1} y_{|U_1}$$

for some k-th root ϵ of 1. This leads to

$$y^j s = \sum_{i=1}^{k} \epsilon^{(i-1)j} y^j_{|U_1} s_i.$$

Therefore the cocycle is given by

$$g = \begin{pmatrix} 1 & y & \cdots & y^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \epsilon^{k-1} y & \cdots & \epsilon^{(k-1)^2} y^{k-1} \end{pmatrix} = \left((\epsilon^{i-1} y)^{j-1} \right)_{i,j}, \tag{27}$$

which can be interpreted as a holomorphic cocycle on $U \cong U_1$.

Proposition 3.5 (Properties of the pushforward bundle)
The following statements hold true.

(i) If a covering is a composition of two, so $\pi = \pi_2 \circ \pi_1$, then $\pi_*(E) = \pi_{1*}(\pi_{2*}(E))$

(ii) $\deg \pi_* E = \deg E - (g_N - 1) \operatorname{rank} E + (g_M - 1) \deg \pi \operatorname{rank} E$

(iii) $H^0(M, \pi_* E) = H^0(N, E)$

(iv) $\pi_* E \otimes \tilde{L} = \pi_*(E \otimes \pi^* \tilde{L})$ for every line bundle $\tilde{L}$ on M.

(v) If $M = \mathbb{C}P^1$ then $\pi_* E$ is trivial if and only if $H^0(N, E) \otimes \pi^*(\mathcal{O}(-1))$ has no non-trivial holomorphic section.

Proof
As in the hyperelliptic case, the statements (i), (iii), (iv), and (v) follow by going through the definininitions. So we only show (ii): Let n be big enough such that

$$h^0(N, KE^* f^* L(-np)) = 0$$

and

$$h^0(M, K(\pi_* E)^* L(-np)) = 0.$$

Using $\operatorname{rank} \pi_* E = \deg \pi \operatorname{rank} E$, and the Riemann Roch theorem 2.2 we compute

$$\begin{aligned}\deg \pi_* E &= \deg \pi_* E L(np) - \operatorname{rank} \pi^* E \deg(L(np)) \\ &= h^0(\pi_*(E\pi^* L(np)) + (g_M - 1) \operatorname{rank}(\pi_* E L(np)) - n \deg \pi \operatorname{rank} E\end{aligned}$$

(iii) then gives

$$\deg \pi_* E = h^0(E\pi^*(L(np)) + (g_M - 1) \deg \pi \operatorname{rank} E - n \deg \pi \operatorname{rank} E,$$

and again using Riemann Roch implies

$$\begin{aligned}\deg \pi_* E &= \deg(E\pi^*(L(np))) - (g_N - 1) \operatorname{rank} E + (g_M - 1) \deg \pi \operatorname{rank} E - n \deg \pi \operatorname{rank} E \\ &= \deg E + \operatorname{rank} E \deg(\pi^*(L(np)) - (g_N - 1) \operatorname{rank} E + (g_M - 1) \deg \pi \operatorname{rank} E \\ &\quad - n \deg \pi \operatorname{rank} E \\ &= \deg E - (g_N - 1) \operatorname{rank} E + (g_M - 1) \deg \pi \operatorname{rank} E,\end{aligned}$$

which prooves (ii). □

Let us restrict to the case $M = \mathbb{C}P^1$. By multiplication with a suitable meromorphic section, one can again define a Killing field ξ: Let

$$w \in H^0(\Sigma, L \otimes \pi^*(L(-n\infty)),$$

be a non-trivial section, which exists for n large enough. We can view w as a meromorphic section in L with poles only in the points above ∞. The Killing field ξ is defined by

$$\xi : s \mapsto ws.$$

By construction, ξ is a meromorphic section and has a pole only in ∞. For more details, we refer to the construcion in the hyperelliptic case.

With respect to a basis $\underline{s}_{\text{reg}}$, the Killing field ξ is diagonal with entrys $\omega_{|U_i}$, where U_i is defined as in the notation before (24). Therefore at every $q \in N$ we have

$$\det(w(q) - \xi(z(q))) = 0,$$

i.e., N is the spectral curve of ξ.

Assume now, that $\pi_* L = \underline{\mathbb{C}}^k$. Then, there is a basis $t_1, \dots, t_k$ of holomorphic sections of $\pi_* L$. Since $H^0(N, L) = H^0(\mathbb{C}P^1, \pi_* L)$, the map

$$\Sigma \times \mathbb{C}^k \to L,$$
$$(\xi, v) \mapsto \sum_{i=1}^{k} v_i t_i(\xi)$$

is a surjective homomorphism of vector bundles. The dual map is therefore injective and gives

$$L^* \hookrightarrow \Sigma \times \mathbb{C}^{k*} = \Sigma \times \mathbb{C}^k$$

as a subbundle which by construction is the eigenspace corresponding to the eigenvalue w of the dual Killing field ξ^t on $(z_* L)^*$.

Remark
In the hyperelliptic surface the Killing field ξ is trace free, therefore ξ is gauge equivalent to ξ^t. By this, L^* is an eigenline of ξ, and one does not have to use the dual Killing field.

3.3. An alternative description

Instead of looking at the pushforward of the dual bundle, we now want to examine the dual construction, i.e. how to write down the dual of the pushforward bundle such that the initial bundle is a subbundle. See [Hel14b, Section 3.2] for the hyperelliptic surface case.

For the most part, the construction is the same as before, but instead of taking the basis

$$\underline{s}_{\text{tot}} = (s, ys, \dots y^{k-1}s)$$

in branch points, the basis consists of

$$s, \frac{1}{y}s, \dots y^{-k+1}s.$$

Then the lines spanned by the vectors s_i of a regular frame

$$\underline{s}_{\text{reg}} = (s_1, \dots, s_k)$$

coalesce in such points, since

$$s_i = \frac{1}{k}(s + \epsilon_i y_{|U_1}(y^{-1}s) + \dots + \frac{1}{\epsilon_i^{k-1} y_{|U_1}^{k-1}}(y^{k-1}s) \underset{y\to 0}{\longrightarrow} \frac{1}{k}s.$$

Definition 3.6
Let $\pi : N \to M$ be a k-fold covering between Riemann surfaces and E be a vector bundle on N. We denote the bundle constructed by the above frames by E_E^π.

Remark
As for the pushforward bundle, we have

$$E_E^{\pi_2 \circ \pi_1} = E^{\pi_2}_{E_E^{\pi_1}}$$

if a covering is the composition of two.

Proposition 3.7
The bundle E_E^π is the dual bundle of the pushforward of the dual bundle $\pi_* E^*$.

Proof
Let

$$\underline{s}^*_{\text{reg}} = (s_1^*, \dots, s_k^*)$$

be a basis of $\pi_* E^*$ at a regular point with frames $s_i^* \in H^0(U_i, E^*)$. Further, let s_i be the dual frame. Then

$$\underline{s}_{\text{reg}} = (s_1, \dots, s_k)$$

is a frame in E_E^π. Since the frames have the correct cocycle, one can choose them to be the dual frame of $\underline{s}^*$. For example let

$$\underline{s}^*_{\text{tot}} = (s^*, ys^*, \dots, y^{k-1}s^*)$$

be a frame at a total branched point and assume $s_i = s_{|U_i}$ then the cocycle between $\underline{s}^*_{\text{reg}}$ and $\underline{s}^*_{\text{tot}}$ is given by g from (27). The cocycle between $\underline{s}_{\text{reg}}$ and

$$\underline{s}_{\text{tot}} = (s, \frac{1}{y}s, \dots, y^{-k+1}s)$$

is given by

$$\tilde{g} = \begin{pmatrix} 1 & y^{-1} & \cdots & y^{-k+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \epsilon^{k-1}y & \cdots & (\epsilon^{k-1}y)^{-k+1} \end{pmatrix} = (g^{-1})^t$$

Which is the inverse and transpose of the g from (27). After checking the cocycles between other frames, one sees that the frames $\underline{s}_{\text{tot}}$ and $\underline{s}_{\text{reg}}$ have the correct behaviour to be choosen as the dual basis of $\underline{s}^*_{\text{tot}}$ and $\underline{s}^*_{\text{reg}}$. □

If π is given by an involution σ on N, i.e.,

$$\pi : N \to M = N/\sigma,$$

one can construct E_E^π as a σ-invariant bundle on N which therefore can be seen as a bundle on M. Denote the set of ramification points of π by D_π. On $N \setminus D_\pi$ the bundle E_E^π is given by

$$E_E^\pi = E \oplus \sigma^* E \to N \setminus D_\pi.$$

Let (y, U) be a centered chart around a ramification points $q \in N$ such that $\sigma(y) = -y$. Then, a local coordinate on $\pi(U)$ is given by

$$z = y^2.$$

Take a holomorphic frame

$$s_1 \in H^0(U, L) \quad \text{and} \quad s_2 = \sigma^*(s_1) \in H^0(U, \sigma^* L).$$

By definition, (s_1, s_2) is a frame on $U \setminus \{q\}$. Define a frame (t_1, t_2) on U by

$$s_1 = \frac{1}{2}(t_1 + y t_2), \quad s_2 = \frac{1}{2}(t_1 - y t_2),$$

or equivalently

$$t_1 = s_1 + s_2 \quad \text{and} \quad t_2 = \frac{1}{y}(s_1 - s_2).$$

The last equation is σ-invariant. So we can extend

$$L \oplus \sigma^* L \to N \setminus D_\pi$$

by such frames to a σ-invariant line bundle

$$E_E^\pi \to N.$$

The initial bundle is a subbundle of the pullback $\pi^* E_E^\pi$ by

$$s \in L_q \mapsto \begin{cases} s(q) & \text{if q is not a ramification point} \\ \frac{1}{2}\left(s(q) + y(q)(y^{-1}s)(q)\right) & \text{if q is a ramification point} \end{cases}$$

This also holds if π is an arbitrary covering.

If we already start with a subline bundle we may know something about the dual of the pushforward bundle. By construction we know that in regular points frames are given by L, which gives, by comparing frames, the following proposition.

Proposition 3.8
Let $L \subset \mathbb{C}^4 \to \Sigma$ be an eigenline of a polynomial Killing field, such that the eigenlines only coalesce by the order of the branch point, then

$$V = E_L^\pi = \underline{\mathbb{C}}^4.$$

Part II

CONSTRAINED WILLMORE SURFACES AND SPECTRAL CURVES

4. Constrained Willmore surfaces

Let $f : M \to \mathbb{R}^3$ be an oriented closed surface in $\mathbb{R}^3$. Willmore [Wil65] defined the Willmore energy as the L^2-norm of the mean curvature H

$$\mathcal{W}(f) = \int_M H^2 \, dA.$$

Let κ_1, κ_2 be the principal curvatures of f. By Gauß-Bonnet the Willmore energy

$$\mathcal{W}(f) = \int_M (\kappa_1 + \kappa_2)^2 \, dA$$

is, up to a topological constant, given by

$$\int_M (\kappa_1^2 + \kappa_2^2) \, dA$$

as well as

$$\int_M (\kappa_1 - \kappa_2)^2 \, dA.$$

Therefore $\mathcal{W}$ measures the roundness of f. Let $(N, \langle\, , \rangle)$ be a Riemannian manifold. The Willmore functional can be generalised for surfaces $f : M \to N$ as

$$W(f) = \int_M |\mathring{\mathrm{I\!I}}| \, dA,$$

where $\mathring{\mathrm{I\!I}}$ is the trace free part of the second fundamental form. Let $\mathrm{I\!I}$ be the second fundamental form and

$$\mathcal{H} = \frac{1}{2} \operatorname{tr} \mathrm{I\!I} = \frac{1}{2} \sum_i \mathrm{I\!I}(X_i, X_i)$$

the mean curvature vector, then the trace free part is given by

$$\mathring{\mathrm{I\!I}} = \mathrm{I\!I} - \mathcal{H}|df|^2.$$

The Willmore functional is invariant under conformal changes of the ambient metric: The trace free part of the second fundamental form is itself invariant among conformal changes. Therefore, changing the metric g to $e^{2u}g$, the absolute value $|\mathring{\mathrm{I\!I}}|$ scales by e^{-2u}. Since the volume form scales by e^{2u}, the Willmore functional is invariant. For more details, we refer to [Qui09, Section 4.1].

In the case that N has constant sectional curvature $\bar{K}$ the Willmore energy can be calculated, again see [Qui09, Section 4.1] for details, to be

$$\mathcal{W}(f) = 2 \int_M (H^2 + \bar{K} - K) \, dA.$$

So, for surfaces in the 3 sphere $f : M \to S^3$, the Willmore functional is, up to a topological constant, given by

$$\int_M (H^2 + 1) \, dA.$$

In physics the Willmore functional is the bending energy of f. Critical points of $\mathcal{W}$ are called Willmore surfaces. All Minimal surfaces are Willmore.

4.1. The Willmore functional in the quaternionic setting

In this section, we want to define the Willmore functional in the quaternionic setting. We will do this in accordance to [BFL+02].

Definition 4.1
Let V be a quaternionic vector space of dimension n. Set

$$\langle A\rangle := \frac{1}{4n}\,\mathrm{tr}_{\mathbb{R}}(A),$$

where $\mathrm{tr}_{\mathbb{R}}$ is the trace of A as a real endomorphism.

$$\langle A, B\rangle = \langle AB\rangle$$

defines an indefinite inner product on $\mathrm{End}(V)$.

Example
Let $A : \mathbb{H} \to \mathbb{H}$ be defined by $1 \mapsto a = a_0 + ia_1 + ja_2 + ka_3$. Working in the basis $1, i, j, k$ we have

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ -a_1 & a_0 & -a_3 & a_2 \\ -a_2 & a_3 & a_0 & -a_1 \\ -a_3 & -a_2 & a_1 & a_0 \end{pmatrix},$$

and therefore $\langle A\rangle = \mathrm{Re}\, a = a_0$ and $\langle A, A\rangle = \mathrm{Re}\, a^2 = a_0^2 - a_1^2 - a_2^2 - a_3^2$.

Definition 4.2
Let f be a conformal map into S^4 and $\nabla = \partial + \bar{\partial} + A + Q$ be the splitting of $\nabla = d$ on V with respect to the mean curvature sphere congruence S of f. The Willmore functional of f is given by

$$\mathcal{W}(f) = \int_M \langle A \wedge *A\rangle. \tag{28}$$

Clearly this definition of the Willmore functional does not depend on the chosen metric in $\mathbb{H}P^1$, i.e. is invariant among conformal changes of the ambient space.

Remark
In quaternionic theory the Willmore functional is defined for a holomorphic bundle $(\tilde{L}, D = \bar{\partial} + Q)$ as

$$\mathcal{W}(\tilde{L}) = \int_M \langle Q \wedge *Q\rangle.$$

The Willmore functional of the map f is not the Willmore functional of L, but the Willmore functional of L^{-1} which has a canonical (Möbius invariant) holomorphic structure. See [FLPP01] for further information.

Next we will see that the quaternionic Willmore functional indeed is equivalent to the classical definition. We will assume that there is a fixed point ∞ in $\mathbb{H}P^1$ which does not lie on the surface, i.e., the surface is a surface in $\mathbb{H}$.

Proposition 4.3
The Willmore energy of $f : M \to \mathbb{H} = \mathbb{R}^4$ defined in (28) equals, up to a topological constant, the classical Willmore energy

$$\mathcal{W}(f) = \int |\mathcal{H}|^2 dA.$$

Proof
As we have seen in (10), the Hopf field A is given by

$$A = \frac{1}{4}\begin{pmatrix} 0 & 0 \\ HdfH + RdH - HdN - *dH & RdR - *dR \end{pmatrix}$$

with respect to the frame $(1,0),(f,1)$. In particular,

$$\begin{aligned} \langle A \wedge *A \rangle &= \langle -AA - *A * A\rangle = -2\langle A^2 \rangle \\ &= -\frac{1}{4}\operatorname{Re}(RdR - *dR)(RdR - *dR) = \frac{1}{4}\operatorname{Re}(RdR - *dR)\overline{(RdR - *dR)} \\ &= \frac{1}{4}|RdR - *dR|^2 = \frac{1}{4}|dR + R * dR|^2 = |dR''|^2. \end{aligned}$$

Note that

$$dR = dR' + dR''$$

is the splitting of dR into its K and $\bar{K}$-parts with respect to the complex structure defined by the multiplication with R from the left. They are given by

$$dR' = \frac{1}{2}(dR - R * dR) \quad \text{and} \quad dR'' = \frac{1}{2}(dR + R * dR).$$

The right normal vector R is S^2-valued. The pullback of the volume form on S^2 is given by

$$\begin{aligned} R^*\omega(X,Y) &= \langle dR(X), RdR(Y)\rangle \\ &= \frac{1}{2}(\langle dR(X), RdR(Y)\rangle - \langle dR(Y), RdR(X)\rangle) \\ &= \frac{1}{2}\langle dR \wedge RdR\rangle(X,Y). \end{aligned}$$

By the splitting of $dR = dR' + dR''$, we get

$$dR \wedge RdR = dR' \wedge *dR' - dR' \wedge *dR'' + dR'' \wedge *dR' - dR'' \wedge *dR''.$$

Since $dRR = -RdR$, we have that dR' is a right-$\bar{K}$-form, i.e. $*dR' = -dR'R$. Therefore

$$dR' \wedge *dR'' = -dR'dR'' - *dR' * dR'' = -dR'dR'' - dR'RRdR'' = 0.$$

Analogously, $dR'' \wedge *dR' = 0$, which yields

$$R^*\omega = \langle dR' \wedge *dR' - dR'' \wedge *dR''\rangle = |dR'|^2 - |dR''|^2.$$

The mapping degree of R is defined as

$$\deg(R) = \frac{1}{4\pi}\int_M R^*\omega,$$

which implies, using the quaternionic Willmore definition (28), that

$$\mathcal{W}(f) = \int_M \frac{1}{4}|dR''|^2 = \int_M \frac{1}{4}|dR'|^2 - \pi\deg(R).$$

Lemma 1.22 states that dR' is related to the mean curvature vector by

$$R\,dR' = \bar{\mathcal{H}}df.$$

Therefore

$$\int_M |dR'|^2 = \int_M |\mathcal{H}|^2|df|^2 = \int_M H^2 dA,$$

which is the classical Willmore energy of $\mathbb{R}^4$. □

4.2. Willmore surfaces and constrained Willmore surfaces

Definition 4.4
A surface $f : M \to \mathbb{H}P^1$ is called a Willmore surface if it is a critical point of $\mathcal{W}$, i.e.,

$$\frac{\partial W(f_t)}{\partial t}_{|t=0} = 0$$

for all variations $f_t : M \times [-\varepsilon, \varepsilon] \to \mathbb{H}P^1$ of $f_0 = f$.

If f is a critical point of $\mathcal{W}$ among variations which fix the conformal class of M, f is called constrained Willmore.

Examples for Constrained Willmore surfaces are constant mean curvature (CMC) surfaces, see [BPP08, Corollary 16]. If f is isothermic, the Euler Lagrange equation (32) (see below) degenerates. For points, where the projection to the Teichmüller space is not submersiv it was not clear whether a critical point among conformal variations is a solution to (32), see [BPP08, Appendix]. As the constrained Willmore tori should be defined by the Euler Lagrange equation they were defined as critical points among infinitesimal conformal variations. In [KS13] Kuwert and Schätzle proved, using the second order of the projection into the Teichmüller space, that all critical points under conformal variations are solutions to the Euler Lagrange equation, i.e., it is not necessary to consider infinitesimal conformal variations.

Theorem 4.5 (Theorem 3 in [BFL+02])
An immersed holomorphic curve L is Willmore if and only if

$$d * Q = 0.$$

Proposition 4.6
$d * Q = 0$ is equivalent to any of the following:

i) $d * A = 0$.

ii) $d(S * dS) = 0$.

iii) S is harmonic, i.e., $(d * dS)^T = 0$, where $(\,\cdot\,)^T$ means the $\mathcal{Z}$ tangential part of the $\operatorname{End}(V)$ valued form.

Proof (of the Proposition)
Recall (9), by which we obtain

$$4d * A = d(S * dS - dS) = d(S * dS) = d(S * dS + dS) = 4d * Q. \tag{29}$$

Therefore i) and iii) are equivalent to $d * Q = 0$. Since $T\mathcal{Z} = \operatorname{Hom}_-(V)$, the $\mathcal{Z}$-tangential part of $B \in \operatorname{End}(V)$ is given by

$$(B)^T = \frac{1}{2}(B + SBS),$$

which implies

$$\begin{aligned} 2S(d * dS)^T &= Sd * dS - (d * dS)S \\ &= d(S * dS) - dS \wedge *dS - d((*dS)S) + *dS \wedge dS \\ &= d(S * dS) + dSdS - *dS * dS - d(-S * dS) + *dS * dS - dSdS \\ &= 2d(S * dS). \end{aligned}$$

□

Theorem 4.7
The map $f : M \to S^4$ is constrained Willmore if and only if there exists $\eta \in \Omega^1(\mathcal{R})$ with

$$d(2 * A + \eta) = 0, \tag{30}$$

where A is the Hopf field of f, L the associated line bundle, and

$$\mathcal{R} = \{B \in \operatorname{End}(V) \,|\, \operatorname{im} B \subset L \subset \ker B\}. \tag{31}$$

The equation (30) is the Euler-Lagrange equation for constrained Willmore surfaces, with Lagrange multiplier η.

For a proof of Theorem 4.7 (in the Lightcone setting) see [Qui09, Theorem 5.6]. A comparison of the lightcone model with the quaternionic model can be found in [HHN19, Section 2.3].

Remark

a) The Euler Lagrange equation (30) is written in divergence form. Therefore $2 * A + \eta$ is a conserved quantity of a constrained Willmore surface. A constrained Willmore surface is Willmore if η is 0.

b) The Euler Lagrange equation of $\mathcal{W}$ can also be calculated to be, see [Sch13], the fourth order PDE

$$\Delta H + 2H(H^2 - K) = \langle q, \mathring{\mathrm{I\!I}} \rangle. \tag{32}$$

where the Lagrange multiplier $q \in H^0(K^2)$ is a holomorphic quadratic differential.

Due to (29), the constrained Willmore condition (30) is equivalent to

$$d(2 * Q + \eta) = 0.$$

Furthermore (30) yields

$$\begin{aligned} 0 &= \pi(d(2 * A + \eta)) \\ &= \delta \wedge (2 * A + \eta) \\ &= \delta * (2 * A + \eta) - *\delta(2A + \eta) \\ &= \delta(2S * A + *\eta) - \delta S(2A + \eta) \\ &= \delta(*\eta) - \delta(S\eta). \end{aligned}$$

Therefore, since δ is non-vanishing,

$$*\eta = S\eta.$$

Also, for $\psi \in \Gamma(L)$,

$$((2 * Q + \eta) \wedge \delta)\psi = (2 * Q + \eta) \wedge d\psi = -d(\underbrace{(2 * Q + \eta)\psi}_{=0}) - d(2 * Q + \eta)\psi = 0.$$

Thus

$$0 = (2 * Q + \eta) \wedge \delta = (2 * Q + \eta) * \delta - *(2 * Q + \eta)\delta = (2 * Q + \eta)S\delta - (2 * QS + *\eta)\delta,$$

concluding $\eta \in \Gamma(K\mathrm{End}(V)_+)$, i.e.,

$$*\eta = S\eta = \eta S.$$

Decomposing $\nabla = \hat{\nabla} + A + Q$ one obtains,

$$0 = d^{\nabla}(2 * A + \eta) = d^{\hat{\nabla}}\mu + d^{\hat{\nabla}}2 * A + 2[A \wedge *A] + 2[Q \wedge *A] + [A \wedge \eta] + [Q \wedge \eta]$$

Using that L lies in the kernel of Q and η, and A and η are L-valued as well as $*A \wedge Q = 0$, we get

$$0 = \underbrace{d^{\hat{\nabla}}\mu}_{+} + \underbrace{2Sd^{\hat{\nabla}}A + A \wedge \eta + \eta \wedge Q}_{-}.$$

The S commuting part is $d^{\hat{\nabla}}\eta$ and $2Sd^{\hat{\nabla}}A + A \wedge \eta + \eta \wedge Q$ is S-anti-commuting. Therefore $d^{\hat{\nabla}}\eta$ vanishes. This means, see [Boh03, Lemma 52], that $\eta\delta \in \Gamma(K^2\mathrm{End}_+(L)) = \Gamma(K^2)$ is a holomorphic quadratic differential. In particular, a non zero η vanishes at isolated points only.

Definition 4.8
A map f is called isothermic if there is a non-trivial 1-form $\omega \in \Omega^1(\mathcal{R})$ such that $d^{\nabla}\omega = 0$.

If f is not isothermic, η occuring in (30) is unique.

5. The integrable system approach

In order to construct constrained Willmore surfaces, one has to solve the Euler Lagrange equation (32) of the Willmore functional, i.e. solve a non-linear elliptic PDE. Solving non-linear PDEs is a rather hard problem. Some non-linear elliptic PDEs are directly related to a system of linear differential equations. Those are called integrable systems. The prototype of an integrable system is the Arnold-Liouville theorem, cf. [Arn78]. The Arnold-Liouville theorem states, that if a Hamilton dynamical system has enough conserved quantities, there are coordinates such that the equation of motion is a linear equation. Those coordinates are called the action-angle coordinates. Furthermore the system can be solved up to an integration.

A conformal immersion of a torus into the 4-sphere gives rise to a Riemann surface, the so called spectral curve Σ. The spectral curve carries a T^2-family $\mathcal{L}_p$ of line bundles such that the map

$$\Psi : T^2 \to \mathrm{Jac}(\Sigma), \quad p \mapsto \mathcal{L}_p \otimes \mathcal{L}_{p_0}^{-1}$$

is a group homomorphism. If the spectral curve Σ has finite genus, using the so called finite gap method, one can reconstruct the immersion from the spectral data. In this work we will discuss two approaches to obtain the spectral curve.

In the case of minimal, CMC, and constrained Willmore surfaces, the problem of solving the non-linear Euler Lagrange equation can be replaced by looking for a $\mathbb{C}^*$ family of parallel frames F^λ of a holomorphic family of flat connections ∇^λ. The Euler-Lagrange equation is then equivalent to the flatness of the connections. The spectral curve is the Riemann surface parametrising the eigenvalues of the holonomy H^λ of ∇^λ. The eigenlines of the connections define the group homomorphism Ψ. This map can be seen as the linearisation of the problem comparable to the action angle coordinates in the case of the Arnold-Liouville theorem.

Instead of searching parallel sections, one can look for Darboux transforms, see Definition 5.2, of the conformally immersed torus or, as it will turn out, for holomorphic sections with monodromy in the quotient bundle V/L. Since a conformally immersed torus gives a unique holomorphic structure on V/L, this spectral curve can be defined for any conformally immersed torus in S^3. Generally, the spectral curve will not be of finite genus, but for constrained Willmore tori it is. This holds since the spectral curves coincide, when both are defined.

5.1. The multiplier spectral curve

Let $f : T^2 = \mathbb{C}/\Gamma \to S^4$ be the conformal immersion of a torus and $L \subset V = \underline{\mathbb{H}}^2$ be the associated line bundle. Further denote by

$$\delta = \pi \circ \nabla_{|L} \in \Omega^1(\mathrm{Hom}(L, V/L)$$

the derivative of f from Definition 1.14, where $\pi : V \to V/L$ is the canonical projection.

Definition 5.1
A sphere congruence

$$\tilde{S} : T^2 \to \{S \in \mathrm{End}(\mathbb{H}^2) | S^2 = -1\}$$

of f is a map into the space of 2-spheres in $\mathbb{H}P^1$, such that $\tilde{S}_p$ contains $f(p)$ and is tangent to f, i.e.

$$\tilde{S}L = L, \quad *\delta = \delta\tilde{S} = \tilde{S}\delta. \tag{33}$$

In contrast to the mean curvature sphere congruence, cf. Theorem 1.19, the mean curvature does not need to coincide.

Remark
Recall that the set of oriented 2 spheres in $\mathbb{H}P^1$ is the set of complex structures in $\mathbb{H}^2$ by Proposition 1.18. The $\tilde{S}_p$-invariance of L_p means that the sphere $\tilde{S}_p$ contains $f(p)$. In particular, $\tilde{S}$ defines a complex structure on V/L. If $*\delta = \tilde{S}\delta$ we say that $\tilde{S}$ left-envelops f. Analogously, $\tilde{S}$ is right enveloping if $*\delta = \delta\tilde{S}$. The sphere $\tilde{S}_p$ being tangent to $f(T^2)$ in $f(p)$ is equivalent to $\tilde{S}_p$ being left and right enveloping.

Originially two conformal maps $f, f^\sharp : T^2 \to S^4$ are called Darboux transforms of each other if there is a sphere correspondence that suits f as well as $f^\sharp$. This already implies, see [HJ03], that f and $f^\sharp$ are isothermic surfaces. For a given isothermic surface f there is a 1 parameter family of Darboux transforms, see for example [HJP97] for further information. In order to generalize Darboux transforms for general conformal maps, one needs to loosen the condition of being tangent to both surfaces.

Definition 5.2
A map $f^\sharp : T^2 \to S^4$ is called a (generalized) Darboux transform of the conformal map f, if f and $f^\sharp$ are pointwise distinct and there is a sphere correspondence $\tilde{S}$ of f which left-envelops $f^\sharp$. Let L and $L^\sharp$ be the corresponding line bundles, then those conditions translate to

$$V = L \oplus L^\sharp, \quad \tilde{S}L = L, \quad *\delta = \tilde{S}\delta = \delta\tilde{S}, \quad \tilde{S}L^\sharp = L^\sharp, \quad \text{and } *\delta^\sharp = \tilde{S}\delta^\sharp, \tag{34}$$

where $\delta^\sharp = \pi^\sharp \circ \nabla_{|L^\sharp}$ for the canonical projection $\pi^\sharp : V \to V/L^\sharp$.

If f and $f^\sharp$ coincide at isolated points then $f^\sharp$ is called a singular Darboux transform.

Definition 5.3
Let W be a quaternionic vector bundle over $T^2 = \mathbb{C}/\Gamma$ and let $\widetilde{W}$ denote the pullback of W on the universal cover $\mathbb{C}$ of T^2. Further, let

$$h : \Gamma = \pi_1(T^2) \to \mathbb{H}_*$$

be a group homomorphism. A section with monodromy h is defined as a section $\psi \in \Gamma(\widetilde{W})$, satisfying

$$\gamma^*\psi = \psi h(\gamma)$$

for all $\gamma \in \Gamma$.

Remark
Let γ_1 and γ_2 be generators of Γ. Then h is already defined by the values h_{γ_1} and h_{γ_2}.

Lemma 5.4 (see Lemma 2.3 of [BLPP12])
Let $f : T^2 \to S^4$ be a conformal immersion and $f^\sharp : T^2 \to S^4$ such that $V = L \oplus L^\sharp$. Identifying V/L and $L^\sharp$ as well as $V/L^\sharp$ and L induces a splitting of the trivial connection d on V

$$d = \begin{pmatrix} \nabla^L & \delta^\sharp \\ \delta & \nabla^\sharp \end{pmatrix}.$$

The map $f^\sharp$ is a Darboux transform of f if and only if the connection $\nabla^\sharp$ is flat.

Proof
The Flatness of d implies

$$F^{\nabla^\sharp} = -\delta \wedge \delta^\sharp.$$

If $f^\sharp$ is a Darboux transform there is a sphere congruence $\tilde{S}$ such that, see (34), $\delta^\sharp$ is left K and δ right K with respect to $\tilde{S}$. This implies the flatness of $\nabla^\sharp$.

The map f being a conformal immersion together with the conditions of a sphere congruence (33) imply that with respect to the splitting $V = L \oplus L^\sharp$ every sphere congruence of f can be written as

$$\tilde{S} = \begin{pmatrix} J & 0 \\ \alpha & \tilde{J} \end{pmatrix},$$

with fixed complex structures J and $\tilde{J}$ on L and $V/L = L^\sharp$, respectively. Therefore, the unique complex structure, which leaves $L^\sharp$ invariant, is given by

$$\tilde{S} = \begin{pmatrix} J & 0 \\ 0 & \tilde{J} \end{pmatrix}.$$

Using the quadratic form notation from Section 4.1 we get

$$F^{\nabla^\sharp} = -\delta \wedge \delta^\sharp = -\delta * \delta^\sharp + *\delta\delta^\sharp = -\delta * \delta^\sharp + \delta\tilde{S}\delta^\sharp.$$

Since δ is nowhere vanishing, $\nabla^\sharp$ being flat implies that $\tilde{S}$ is left enveloping and therefore $f^\sharp$ is a Darboux transform. □

The flatness of $\nabla^\sharp$ is equivalent to the existence of local parallel frames. Those extend to non-trivial $\nabla^\sharp$-parallel sections

$$\hat{\psi} \in \Gamma(\widetilde{L^\sharp}) \subset \Gamma(\widetilde{V})$$

with monodromy, i.e., sections $\hat{\psi} \in \Gamma(\widetilde{V})$ with monodromy that satisfy

$$d\hat{\psi} \in \Omega^1(L).$$

Recall the canonical holomorphic structure $D = \pi \circ d''$ on V/L from Section 1.6. In Lemma 1.23 we have seen that there is a one-to-one-correspondence between D-holomorphic sections ψ and their prolongations, i.e. sections $\hat{\psi}$ in V with L valued derivative. A prolongation $\hat{\psi} \in \Gamma(\widetilde{V})$ is a section with monodromy h if and only if the corresponding holomorphic section $\psi = \pi\hat{\psi} \in \Gamma(\widetilde{V/L})$ has monodromy h. Therefore, every D-holomorphic section ψ with monodromy gives rise to a Darboux transform

$$L^\sharp = \hat{\psi}\mathbb{H},$$

where $\hat{\psi}$ is the prolongation of ψ. Since $\hat{\psi}$ is a section with monodromy, the map

$$f^\sharp : T^2 \longrightarrow \mathbb{H}P^1, \quad p \longmapsto \hat{\psi}(z)\mathbb{H}$$

is independent of the point $z \in \mathbb{C}$ above $p \in T^2$, i.e., is well defined as a map from T^2. This construction also works for singular Darboux transforms, where the holomorphic section ψ has zeroes at the points where $L^\sharp_p = L_p$. See [BLPP12, Lemma 2.7] for further details. The correspondence between D-holomorphic sections with monodromy, up to scaling, and Darboux transforms is bijective. Let ψ be a holomorphic section with monodromy h. Scaling the holomorphic section by a constant quaternion λ gives the same Darboux transform, but conjugates the monodromy h since

$$\psi\lambda(z+\gamma) = \psi(z)h_\gamma\lambda = \psi\lambda(z)\lambda^{-1}h_\gamma\lambda.$$

Definition 5.5
The quaternionic spectrum of V/L is the subset

$$\mathrm{Spec}_{\mathbb{H}}(V/L, D) \subset \mathrm{Hom}(\Gamma, \mathbb{H}_*)/\mathbb{H}_*$$

of conjugacy classes of possible monodromies $h : \Gamma \to \mathbb{H}_*$ of holomorphic sections, i.e., $[h] \in \mathrm{Spec}_{\mathbb{H}}(V/L)$ if and only if there exists a D holomorphic section with monodromy h.

Let h be a representative of a point in $\mathrm{Spec}_{\mathbb{H}}(V/L, D)$. Since we are working on a torus, there is a conjugation of h which is complex valued: The fundamental group Γ of a torus is abelian, and therefore

$$h_{\gamma_1}h_{\gamma_2} = h_{\gamma_2}h_{\gamma_1},$$

for generators γ_1, γ_2 of Γ. By Lemma 1.2, this means that the imaginary parts of h_{γ_1} and h_{γ_2} are linearly dependent, and there is a $\lambda \in \mathbb{H}_*$ such that simultaneously $\lambda^{-1}h_{\gamma_1}\lambda$ and $\lambda^{-1}h_{\gamma_2}\lambda$ is complex valued, i.e., lies in the span of 1 and $\mathbb{i}$. This means that $\lambda^{-1}h\lambda$ is complex valued.

Definition 5.6
The (complex) spectrum of D is

$$\mathrm{Spec}(V/L, D) \subset \mathrm{Hom}(\Gamma, \mathbb{C}_*).$$

$h \in \mathrm{Spec}(V/L, D)$ holds if and only if there is a D-holomorphic section with monodromy h. It is $\mathrm{Hom}(\Gamma, \mathbb{C}_*) \cong \mathbb{C}_* \times \mathbb{C}_*$ by choosing 2 generators of Γ.

Let $\psi \in \Gamma(\widetilde{V/L})$ be a section with complex monodromy h. Then, the section $\psi\mathbb{j}$ has monodromy $\bar{h}$, since $\mathbb{i}$ and $\mathbb{j}$ anti-commute. So there is an involution ρ on $\mathrm{Spec}(V/L)$, defined by

$$\rho(h) = \bar{h}. \tag{35}$$

It holds

$$\mathrm{Spec}(V/L, D)/\rho = \mathrm{Spec}_{\mathbb{H}}(V/L, D).$$

In order to see that the (complex) spectrum is an analytic set, one can define it using a family D_ω of elliptic operators. We will only sketch this approach, for further details we refer to [BPP09].

Let $\psi \in \Gamma(\widetilde{V/L})$ be a holomorphic section with monodromy h. There is a harmonic 1-form ω on T^2 such that

$$e^{\int_\gamma \omega} = h(\gamma).$$

The form ω is unique up to adding a point of the dual lattice

$$\Gamma^* = \left\{ \eta \in \mathrm{Harm}(T^2, \mathbb{C}) \mid \int_\gamma \eta \in 2\pi i\mathbb{Z} \forall\gamma \right\}.$$

Set

$$D_\omega = e^{\int -\omega} \circ D \circ e^{\int \omega}.$$

The section $\varphi = \psi e^{-\int \omega}$ is a D_ω-holomorphic section without monodromy. Note that the multiplication by $e^{\int \omega}$ is from the right. Although, in general $e^{\int \omega}$ is only defined on the universal cover, we have

$$\begin{aligned} D_\omega(\varphi) &= (D(\varphi e^{\int \omega}))e^{\int -\omega} \\ &= D(\varphi)e^{\int \omega}e^{\int -\omega} + \varphi(de^{\int \omega})'' e^{-\int \omega} \\ &= D\varphi + \varphi\omega'' e^{\int \omega} e^{-\int \omega} \\ &= (D + \omega'')\varphi. \end{aligned}$$

Therefore D_ω is a well defined holomorphic structure on T^2. Define

$$\widetilde{\mathrm{Spec}}(V/L, D) = \{\omega \in \mathrm{Harm}(T^2, \mathbb{C}) | \ker D_\omega \neq \{0\}\}.$$

Then $\omega \in \widetilde{\mathrm{Spec}}(V/L, D)$ if and only if there is a section $\varphi \in \Gamma(V/L)$ such that

$$D_\omega \varphi = 0,$$

which is equivalent to

$$D(\varphi e^{\int \omega}) = 0.$$

Therefore $\omega \in \widetilde{\mathrm{Spec}}(V/L, D)$ if and only if $h_\gamma = e^{\int_\gamma \omega} \in \mathrm{Spec}(V/L, D)$, which implies

$$\mathrm{Spec}(V/L) = \widetilde{\mathrm{Spec}}(V/L)/\Gamma^*.$$

Since the elliptic operators D_ω are Fredholm, we can use the following proposition to conclude that $\widetilde{\mathrm{Spec}}(V/L, D)$ is an analytic subset of $\mathrm{Harm}(T^2, \mathbb{C})$. Thus $\mathrm{Spec}(V/L, D)$ is an analytic subset of $\mathbb{C}_* \times C_*$.

Proposition 5.7 (Proposition 3.1. in [BPP09])
Let $F(\lambda) : E_1 \to E_2$ be a holomorphic family of Fredholm operators between Banach spaces E_1 and E_2 parameterized over a connected complex manifold M. Then the minimal kernel dimension of $F(\lambda)$ is attained on the complement of an analytic subset $N \subset M$.

If M is 1-dimensional, the holomorphic vector bundle $V_\lambda = \ker(F(\lambda))$ over $M \setminus N$ extends through the set N of isolated points to a holomorphic vector subbundle of the trivial E_1-bundle over M .

The dimension of Spec(V/L) depends on the degree of V/L. It holds

$$\begin{aligned}
&\dim(\mathrm{Spec}(V/L)) = 2 && \text{if } \deg(V/L) > 0\\
&\dim(\mathrm{Spec}(V/L)) = 1 && \text{if } \deg(V/L) = 0\\
&\dim(\mathrm{Spec}(V/L)) = 0 && \text{if } \deg(V/L) < 0.
\end{aligned}$$

For conformal maps from tori into S^3 we have $\deg(V/L) = 0$ and therefore the spectrum is one-dimensional.

Definition 5.8
The multiplier spectral curve Σ_{mult} is the normalisation of Spec$(V/L, D)$.

The involution ρ from (35) defines an anti-holomorphic involution on Σ_{mult} denoted by the same symbol.

The multiplier spectral curve carries a canonical line bundle: Let $h \in$ Spec(V/L) and $\psi \in \Gamma(\widetilde{V/L})$ be a holomorphic section with holonomy h. Then ψ spans a holomorphic subbundle $\mathcal{L}_{h,\psi} \subset \widetilde{V/L}$. Since ψ has monodromy, the bundle $L_{h,\psi}$ is well defined on the torus T^2. We use the following theorem, proven in [BPP09].

Theorem 5.9
The space of holomorphic sections with monodromy $h \in$ Spec(V/L) is finite dimensional. Generically it is 1-dimensional.

Hence, generically $\mathcal{L}_{h,\psi}$ does only depend on h, and $\mathcal{L}_h = \ker D_\omega$ for ω with $h = e^{\int \omega}$. Therefore by Proposition 5.7 $\mathcal{L}_h$ depends holomorphically on h, and the line bundle

$$\mathcal{L} \longrightarrow \Sigma_{mult}$$

extends through the points with higher dimensional space of holomorphic sections. Since $\mathcal{L}$ is defined by $\mathcal{L} \subset \ker D_\omega$, it is called the kernel bundle. The kernel bundle is compatible with the involution ρ, it is $\rho^*\mathcal{L} = \mathcal{L}\mathbb{j}$, because $\psi\mathbb{j}$ is the holomorphic section with holonomy $\bar{h}$.

Note that Σ_{mult} is not necessarily compact.

Theorem 5.10
The spectral curve Σ_{mult} has finite genus, and therefore can be compactified if and only if it has two ends interchanged by ρ. The 2 points needed to compactify Σ_{mult} are called 0 and ∞.

This theorem was prooven in [BPP09]. We now restrict ourselves to the case that Σ_{mult} is compactifiable and has finite genus. We will see in Section 5.5 that this is the case for constrained Willmore tori in S^4. We denote the compactification of Σ_{mult} by Σ.

The kernel bundle $\mathcal{L}$ does not extend to $0, \infty \in \Sigma$, since the multiplier function $h : \Sigma_{mult} \to \mathbb{C}_* \times \mathbb{C}_*$ has an essential singularity in the points 0 and ∞, see [BPP09, Section 5.1]. But if we evaluate the holomorphic section with monodromy spanning $\mathcal{L}^h$ at p, we get the line $\mathcal{L}^h_p \subset (V/L)_p$. This way we get the line bundle

$$\mathcal{L}_p \longrightarrow \Sigma_{mult}$$

which extends to Σ. The lines $\mathcal{L}^0_p, L^\infty_p \subset (V/L)_p$ at 0 and $\infty \in \Sigma$ are given by the $\pm\mathbb{i}$ eigenspaces of the mean curvature sphere congruence S (or any other sphere congruence of f).

Theorem 5.11 (Theorem 5.6 in [BPP09])
Fix $p_0 \in T^2$. The map

$$\Psi : T^2 \to \mathrm{Jac}(\Sigma), \quad p \mapsto \mathcal{L}_p \mathcal{L}_{p_0}^{-1}$$

is a group homomorphism.

By taking the prolongation of holomorphic sections one can lift the bundle $\mathcal{L}$ to a bundle $\hat{\mathcal{L}} \subset V$. This gives the map

$$F : T^2 \times \Sigma_{mult} \to \mathbb{C}P^3, \quad (p, \cdot) \mapsto \hat{\mathcal{L}}_p. \tag{36}$$

This map extends on Σ and yields the following main theorem.

Theorem 5.12 (Theorem 4.2. in [BLPP12])
Let $f : T^2 \to S^4$ be a conformal immersion with trivial normal bundle whose spectral curve Σ has finite genus. Then there exists a map

$$F : T^2 \times \Sigma \to \mathbb{C}P^3,$$

such that

(i) $F(p, \cdot) : \Sigma \to \mathbb{C}P^3$ is an algebraic curve for all $p \in T^2$.

(ii) The original conformal immersion $f : T^2 \to S^4$ is obtained by the twistor projection of the evaluation of F at the points at infinity:

$$f = \pi_{\mathbb{H}} F(\cdot, 0) = \pi_{\mathbb{H}} F(\cdot, \infty).$$

The twistor projection $\pi_{\mathbb{H}} : \mathbb{C}P^3 \to \mathbb{H}P^1$ is the map induced by identifying $\mathbb{C}^4$ and $\mathbb{H}^2$ via the choice of a quaternionic structure $\mathbb{j}$. A quaternionic structure on $\mathbb{C}^4$ is a complex anti-linear endomorphism $\mathbb{j}$ with $\mathbb{j}^2 = -1$ and anti-commuting with $\mathbb{i}$. Choosing such a structure makes $\mathbb{C}^4$ to a quaternionic vector space and $\pi_{\mathbb{H}}$ is then given as

$$\psi\mathbb{C} \subset \mathbb{C}^4 \mapsto \psi\mathbb{H} \subset \mathbb{C}^4 = \mathbb{H}^2.$$

In the proof of Theorem 5.12 it is shown that as ξ tends to ∞ the lines $\hat{\mathcal{L}}_p^\xi$ converge to the $\mathbb{i}$-eigenline of the mean curvature correspondence S that lies in $L \subset V$. The line $\hat{\mathcal{L}}_p^0$ is given by the $-\mathbb{i}$-eigenspace.

The Kodaira embedding theorem 2.11 states that a line bundle is uniquely embedded in $\mathbb{C}^4$ if the space of holomorphic sections is 4-dimensional. Since F is defined by line bundles, F can be uniquely (up to a $\mathrm{PSL}(4), \mathbb{C})$-action) determined by the spectral curve Σ and the 0-, 1- or 2-dimensional torus $Z = \mathrm{im}\,\Psi$, where Ψ is the group homomorphism

$$\Psi : T^2 \to \mathrm{Jac}(\Sigma)$$

from Theorem 5.11. Note that the line bundles of $Z \otimes \mathcal{L}_{p_0}$ must be compatible with the quaternionic structure $\mathbb{j}$, which is equivalent to $Z \subset \mathrm{Jac}(\Sigma)$ consisting of ρ^* invariant bundles. Using the $\mathbb{j}$-compatibility F is determined up to $\mathrm{PSL}(n{+}1, \mathbb{C})$-actions, compatible with $\mathbb{j}$. Those act as Möbius transformations on $\mathbb{H}P^1 = S^4$.

Definition 5.13
An immersion $f : M \to S^4$ is called simple, if the map F is, up to Möbius transformations, uniquely determined by Σ and Z.

This definition immediately leads to the following statements from [Hel15].

Proposition 5.14
Let $f : T^2 \to S^3$ be a simple conformal immersion and V/L the associated quotient bundle. Then the space of holomorphic sections $H^0(V/L)$ needs to be quaternionic 2-dimensional.

Proof
As we have seen in Section 1.6, the constant sections in V always yield holomorphic sections of V/L and Möbius transformations of f are given by the quotient of such sections. The space of sections coming from constant sections is 2-dimensional. Suppose there are 3 quaternionic independent holomorphic sections, then one of them does not come from a constant section in V. So the quotient of this section with a constant section yields an $\tilde{f} : T^2 \to \mathbb{H}P^1$, not Möbius equivalent to f. Let $\tilde{L}$ be the line bundle to $\tilde{f}$. The bundles V/L and $V/\tilde{L}$ are holomorphic isomorphic and so the spectral curves as well as the Z's are the same but the corresponding maps do differ not only by a Möbius transformation. Therefore f is not simple. □

Proposition 5.15
A simple immersion (not necessarily constrained Willmore) of spectral genus 1 is equivariant, i.e. has a 1-parameter group of Möbius symmetries.

Proof

For the $\mathbb{j}$-invariance the subset $Z \subset \mathrm{Jac}(\Sigma) = \Sigma$ needs to be ρ^*-invariant. As ρ is an anti-holomorphic involution on a torus, the fixpoint set, and therefore Z, is at most real 1-dimensional. Thus $\Psi : T^2 \to \mathrm{Jac}(\Sigma)$ has at least a 1-dimensional kernel. Let $z = x + iy$ be a holomorphic chart of T^2, such that the x direction lies in the kernel of Ψ. Then the maps $f(x, y)$ and $\tilde{f}(x, y) = f(x + x_0, y)$ will have the same spectral curve and Z. Since f is simple there is a Möbius-transformation M_{x_0} such that $f = M_{x_0}\tilde{f}$. Since f is unique, $M_t M_s \tilde{f}$ and M_{t+s} have to agree, i.e., the map $M : \mathbb{R} \to \mathrm{Möb}(S^4)$ is a group homomorphism, so f is equivariant. □

5.2. The spectral curve of a CMC torus

In this section we will give a short overview on the integrable system approach to constant mean curvature (CMC) surfaces. The theory was developed by Nigel Hitchin [Hit90] and Pinkall and Sterling [PS89] in the late 80's. We will follow the summaries of this theory in [Hel15] and [Hel13].

Let $f : M \to S^3 = \mathrm{SU}(2)$ be a conformally immersed surface with mean curvature function H. Further let

$$W = M \times \mathbb{H} \cong M \times \mathbb{C}^2$$

be the trivial bundle. The identification of $\mathbb{H}$ with $\mathbb{C}^2$ is given by restricting the scalar multiplication on $\mathbb{C} \cong \mathrm{span}(1, \mathbb{i})$. The complex multiplication with i is given by right multiplication with the quaternion $\mathbb{i}$.

The 3-sphere SU(2) is a Lie group. The left and right translation give two canonical ways to trivialize the tangent bundle. We use left trivialisation to identify

$$TS^3 \cong S^3 \times \mathrm{Im}(\mathbb{H}).$$

The bundle W can then be identified as the pull back of the spinor bundle on S^3 with Clifford multiplication

$$TS^3 \times W \cong S^3 \times \mathrm{Im}(\mathbb{H}) \times \mathbb{H} \longrightarrow W, \quad (a, w) \mapsto aw.$$

This construction identifies the tangent vectors as the skew symmetric trace free endomorphisms on $W = M \times \mathbb{C}^2$. The bundle W is a complex rank 2 vector bundle with quaternionic structure j, i.e., an anti-linear endomomorphism with $j^2 = -1$, given by right multiplication with $\mathbb{j}$. The standard complex hermitian metric $(\cdot, \cdot)$ on $\mathbb{C}^2$ then gives rise to a symplectic form

$$\hat{\omega} = -(\cdot, j\cdot).$$

Denote the trivial connections on TS^3 coming from left and right trivialisation by ∇^L and ∇^R, respectively. Working in left trivialisation they are given by

$$\nabla^L = d \qquad \text{and} \qquad \nabla^R = d + \alpha,$$

where α is the Maurer-Cartan form. The Levi-Civita connection on TS^3 is given by the average of those, i.e., by

$$\nabla^{LC} = d + \frac{1}{2}\alpha.$$

The Levi-Civita connection induces an connection on W with the Maurer Cartan form acting by the Clifford multiplication, i.e., the $\mathrm{Im}(\mathbb{H})$-valued form acts by left multiplication on the quaternions. The induced connection is unitary, since α is $\mathfrak{su}(2)$-valued. The pullback of α by f is

$$f^*\alpha = f^{-1}df,$$

so f is the gauge between the two trivial connections

$$\nabla^L = d \quad \text{and} \quad \nabla^R = d + \alpha = d + f^{-1}df$$

on $W \to M$. Let $\alpha = \alpha' + \alpha''$ be the splitting of α into its K and $\bar{K}$-part. Since α is $\mathfrak{su}(2)$ valued, it is

$$\alpha'' = -\alpha'^*. \tag{37}$$

We associate with f the $\mathbb{C}_*$-family of connections

$$\nabla^\lambda = d + \frac{1}{2}(1+\lambda^{-1})(1+iH)\alpha' + \frac{1}{2}(1+\lambda)(1-iH)\alpha''.$$

The flatness of the family ∇^λ is equivalent to f being a CMC surface. For the case of minimal surfaces see [Hel13] for more detailed informations. By (37) the connections ∇^λ are unitary for $\lambda \in S^1$. Further

$$j^{-1}\alpha' j = -\alpha^*$$

implies the symmetry

$$\nabla^{\bar{\lambda}^{-1}} = j^{-1}\nabla^\lambda j. \tag{38}$$

Let F_λ be a parallel frame of ∇^λ, defined on the universal cover $\tilde{M}$ of M. For $\lambda \in S^1$ the frame F^λ can be choosen as a unitary frame. For $\lambda_1 = -1$ and $\lambda_2 = \frac{1+iH}{1-iH}$ the connections are trivial, since

$$\nabla^{\lambda_1} = d = \nabla^L \quad \text{and} \quad \nabla^\lambda = d + \alpha = \nabla^R.$$

Therefore the unitary parallel frames $F_{\lambda_2}, F_{\lambda_1}$ are well defined on M. The SU(2)-valued gauge $\tilde{f} = F_{\lambda_2}^{-1}F_{\lambda_1}$ between ∇^{λ_1} and ∇^{λ_2} is a well defined map $\tilde{f} : M \to S^3$. The initial map f is the gauge between the constant frames in left and right trivialisation. Those frames are in particular parallel and unitary. This implies that the map $\tilde{f}$ is a Möbius transformation of f.

We will now assume that $M = T^2$ is a torus. In [Bob91b], Bobenko showed that all CMC tori can be reconstructed from such families of connections.

Theorem 5.16
Let ∇^λ be a family of flat connections on W of the form

$$\nabla^\lambda = \nabla + \frac{1}{2}(1+\lambda^{-1})\Phi' + \frac{1}{2}(1+\lambda)\Phi'', \quad \lambda \in \mathbb{C}_*, \tag{39}$$

that is unitary along S^1 and has the symmetry (38). Let F_λ be a parallel frame of ∇^λ, unitary along S^1 and holomorphically depending on λ and $\det F_\lambda = 1$.

For two distinct points $\lambda_1, \lambda_2 \in S^1$ the map

$$f = F_{\lambda_2}^{-1} F_{\lambda_1} \tag{40}$$

defined on the universal cover $\mathbb{C}$ of T^2 has constant mean curvature $H = -i\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}$ in S^3.

If $\lambda_1 = \lambda_2 \in S^1$ then

$$f = F_{\lambda_1}^{-1} \frac{\partial F_\lambda}{\partial \lambda}_{|\lambda_1} \tag{41}$$

is CMC in $\mathbb{R}^3$. For $\lambda_1 = \bar{\lambda_2}^{-1}$ and $|\lambda_2|^2 \leq 1$

$$f = F_{\lambda_2}^{-1} F_{\lambda_1} \tag{42}$$

is CMC in H^3 with mean curvature $H = \frac{1+|\lambda_2|^2}{1-|\lambda_2|^2}$.

All CMC tori in simply connected 3-dimensional space forms are given in this way. The points λ_1, λ_2 used to reconstruct the surface are called the sym points

Remark
The $\mathbb{R}^3$ case is the limit of the S^3 case. The CMC torus in the sphere $1/(\lambda_2-\lambda_1)S^3 \subset \mathbb{C}^2$ of radius $1/|\lambda_2-\lambda_1|$ (i.e. with mean curvature $|\lambda_2-\lambda_1|$), instead of the sphere with radius 1, is given by

$$\frac{1}{\lambda_2-\lambda_1} F_{\lambda_2}^{-1} F_{\lambda_1}.$$

In the limit this tends to

$$F_{\lambda_1}^{-1} \frac{\partial F_\lambda}{\partial \lambda}_{|\lambda_1},$$

as well as the spheres converge to the flat $\mathbb{R}^3$. The so obtained torus has mean curvature $H = 1$. By scaling, every value can be obtained.

The mean curvature of a CMC torus in $\mathbb{H}^3$, described by (42), is greater than 1. A CMC-surface in $\mathbb{H}^3$ with mean curvature $H < 1$ is non compact, and will intersect the boundary of $\mathbb{H}^3$. Babich and Bobenko [BB93] showed, that constrained Willmore tori can be obtained by glueing two such components over the infinity boundary. Those tori are also given by families of flat connections. But those are not unitary over S^1.

In order to parametrice CMC tori, one needs to write down families of connections of the form (39). A helpful tool is the spectral curve defined by Hitchin [Hit90].

Let $T^2 = \mathbb{C}/\Gamma$ be a torus and ∇^λ be a family of flat connections of the form (39) with symmetry (38). Further let H_p^λ be the holonomy representation of the $\mathrm{SL}(2,\mathbb{C})$-valued connection ∇^λ in the base point $p \in T^2$. Choose $\gamma \in \Gamma$ as one of the generators of Γ. The spectral curve Σ of ∇^λ is then defined as the normalization and compactification of the 1-dimensional analytic variety

$$\{(\mu, \lambda) \in \mathbb{C}_* \times \mathbb{C}_* | \det(H_p^\lambda(\gamma) - \mu) = 0\}.$$

Note that in order to see that this set is compactifiable one has to study the behavior of the holonomy as λ tends to 0 and ∞. This is done in [Hit90, Section 3].

Since the fundamental group Γ is abelian, the holonomy along different paths commute as well. In particular simple eigenspaces along one path are eigenspaces for the whole holonomy representation: Let v be an eigenvector to the simple eigenvalue $\mu \neq 0$ of $H^\lambda(\gamma_0)$, then

$$\mu H^\lambda(\gamma)v = H^\lambda(\gamma)\mu v = H^\lambda(\gamma)H^\lambda(\gamma_0)v = H^\lambda(\gamma_0)H^\lambda(\gamma)v, \tag{43}$$

i.e., $H^\lambda(\gamma)v$ is an eigenvector to the eigenvalue μ. If μ is a simple eigenvalue, v must also be an eigenvector of $H^\lambda(\gamma)$. Hitchin showed, that the holonomy is generically diagonalizable with distinct eigenvalues. Therefore changing γ does change the eigenvalues of $H_p^\lambda(\gamma)$ but does not change the spectral curve. Changing the basepoint p leads to a conjugation of H^λ via parallel transport. The eigenvalues are not affected by this, so the spectral curve does not depend on p.

Since $H_p^\lambda(\gamma)$ is $\mathrm{SL}(2,\mathbb{C})$-valued the projection on the second component $\lambda : \Sigma \to \mathbb{C}P^1$ is a 2-fold covering. So Σ is an hyperelliptic surface, with hyperelliptic involution σ originating in

$$(\mu, \lambda) \mapsto (\mu^{-1}, \lambda).$$

The symmetry (38) of ∇^λ yields that if λ is a branchpoint, so is $\bar{\lambda}^{-1}$. By [Hit90] the spectral curves of minimal surfaces are branched over 0 and ∞, so Σ is given by

$$\mu^2 = \lambda \prod_{i=1}^{g} (\lambda - q_i)(\lambda - \bar{q}_i),$$

where q_i are the odd order zeroes of $\det(H_p^\lambda(\gamma) - \mu)$ (without multiplicity). The function $\xi \in \Sigma \mapsto \mu$, mapping (μ, λ) to the eigenvalue μ is well defined on $\mathring{\Sigma} = \Sigma \setminus \{0, \infty\}$ and has an essential singularity in the points over 0 and ∞, see [Hit90, Section 3]. The eigenspace $\ker(H_p^{\lambda(\xi)}(\gamma) - \mu(\xi))$ is generically one-dimensional, and therefore defines a line bundle $\mathcal{L}_p \to \mathring{\Sigma}$ with

$$\mathcal{L}_p^\xi \subset \ker(H_p^{\lambda(\xi)}(\gamma) - \mu(\xi))$$

The line bundle $\mathcal{L}_p$ extends holomorphically on Σ. On $\mathring{\Sigma}$ the line $\mathcal{L}_p^\xi$ can be extended on T^2 by parallel transport, such that one gets a linebundle

$$\mathcal{L} \to \mathring{\Sigma} \times T^2.$$

This does not work in the points at infinity, but, by the following proposition, changing the base point gives a linear map into the Jacobian. For further details see [Hit90, Section 7].

Proposition 5.17
Fix $p_0 \in T^2$. The map

$$\Psi : T^2 \to \mathrm{Jac}(\Sigma), \quad p \mapsto \mathcal{L}_{p_0}^{-1} \otimes \mathcal{L}_p$$

is a group homomorphism.

The eigenline bundle $\mathcal{L}_p$ is a subbundle of the trivial bundle $\underline{\mathbb{C}}^2 = W_p \to \Sigma$. We can use this to calculate the degree of $\mathcal{L}_p$ by counting how often $\mathcal{L}_p$ and $\sigma^*\mathcal{L}_p$ coalesce: Recall the symplectic form $\hat{\omega}$ on W_p. The evaluation of $\hat{\omega}$ on $\mathcal{L}_p$ (in one entry) and $\sigma^*\mathcal{L}_p$ (in the other) is a map

$$\mathcal{L}_p \otimes \sigma^*\mathcal{L}_p \longrightarrow \mathbb{C}.$$

So $\hat{\omega}$ defines a holomorphic section in $\mathcal{L}_p^{-1} \otimes \sigma^*\mathcal{L}_p^{-1}$, which we denote by the same symbol. The section $\hat{\omega} \in H^0(\mathcal{L}_p^{-1} \otimes \sigma^*\mathcal{L}_p^{-1})$ vanishes exactly at points where $\mathcal{L}_p$ and $\sigma^*\mathcal{L}_p$ coincide. Since $\mathcal{L}_p$ and $\sigma^*\mathcal{L}_p$ clearly have the same degree,

$$\deg \mathcal{L}_p = \deg(\hat{\omega}).$$

Since $\hat{\omega}$ has zeroes in the ramification points, the degree of $\mathcal{L}_p$ is at least $g+1$. The zeroes of $\hat{\omega}$ do not depend on $p \in T^2$ since the two line bundles just differ by parallel transport.

Definition 5.18
Let $f : T^2 \to S^3$ be a conformal immersion of constant mean curvature and let Σ be its spectral curve. The genus g of Σ is called the geometric genus of f. The spectral genus p is the genus of the (possibly singular) curve $\tilde{\Sigma}$ defined by the zeroes of $\hat{\omega}$. Thus $\tilde{\Sigma}$ is given by

$$\mu^2 = \lambda \prod_{i=1}^{g} \bar{q}_i^{-1} (\lambda - q_i)^{r_i} (\lambda - \bar{q}_i)^{r_i},$$

where the $q_i, \bar{q}_i$ mark the zeroes of $\hat{\omega}$ counted with multiplicity r_i.

Proposition 5.19 (see [Hit90, Section 7])
The degree of $\mathcal{L}_p$ is $p+1$, where p is the arithmetic spectral genus of f. Further $\mathcal{L}_p$ is non-special for all $p \in T^2$, i.e., $h^0(KL_p^{-1})) = 0$.

Hitchin did not only show that a CMC immersion yields the spectral curve and the eigenline bundle, but also that the process can be reversed. A hyperelliptic curve Σ with a suitable family of line bundles gives a family of connections of the form (39), and therefore a CMC surface such that the spectral curve is given by Σ.

Theorem 5.20 ([Hit90, Theorem 8.1])
Let $\tilde{\Sigma}$ be a (singular) hyperelliptic surface, given by

$$\mu^2 = \lambda \prod_{i=1}^{p} \bar{q}_i^{-1} (\lambda - q_i)(\lambda - \bar{q}_i)$$

with $q_i \in \mathbb{C}_* \setminus S^1$. Let Σ be the normalization of $\tilde{\Sigma}$ and $\mathcal{L}_p \to \Sigma$ be a T^2 family of line bundles of degree $p+1$, such that

$$\Psi : T^2 \to \mathrm{Pic}_{p+1}, \quad p \mapsto \mathcal{L}_p$$

is a group homomorphism. If $\mathcal{L}_p$ are non-special and $\rho^*\mathcal{L}_p = \mathcal{L}_p\mathbb{j}$, then one gets a $\mathbb{C}_*$-family ∇^λ of connections of the form (39) on a rank 2 bundle W with trivial determinant bundle, such that $\mathcal{L}_p$ is the eigenline bundle of H_p^λ.

5.3. The associated family of a constrained Willmore torus

For constrained Willmore tori C. Bohle [Boh10] showed that there also is an associated family of connections. In contrast to the CMC-case it is no longer a complex rank 2 theory but instead a family of connections on $V = \mathbb{H}^2$, i.e. a complex rank 4 theory. As in the CMC case we view $\mathbb{H}^2$ as $\mathbb{C}^4$ by restricting the scalar multiplicaion on $\mathbb{C} = \mathrm{span}(1, \mathbb{i})$. The multiplication with i is therefore given by right multiplication with the quaternion $\mathbb{i}$.

Let $f : T^2 \to S^3$ be a constrained Willmore torus and L the corresponding line bundle, see Section 1. Recall the notation from Section 4.2, in particular

$$\mathcal{R} = \{B \in \mathrm{End}(V) | \operatorname{im} B \subset L \subset \ker B\}. \tag{31}$$

By Theorem 4.7 there exists a 1-form $\eta \in \Omega^1(\mathcal{R})$ satisfying the Euler Lagrange equation

$$d(2 * A + \eta) = 0. \tag{30}$$

Set $2 * A_0 := 2 * A + \eta$ and define the $\mathbb{C}^*$ family of connections on V as

$$\nabla^\lambda = \nabla + (\lambda - 1)\frac{1 - iS}{2}A_0 + (\lambda^{-1} - 1)\frac{1 + iS}{2}A_0.$$

Since η and A are left K-forms with respect to S, the term $\frac{1-iS}{2}A_0$ is the $(1,0)$ part of A_0, i.e.

$$*\frac{1 - iS}{2}A_0 = \frac{1 - iS}{2}A_0 i,$$

and $\frac{1+iS}{2}A_0$ is the $(0,1)$ part of A_0. Therefore, we write

$$\nabla^\lambda = \nabla + (\lambda - 1)A_0^{(1,0)} + (\lambda^{-1} - 1)A_0^{(0,1)} \tag{44}$$

Lemma 5.21
For all $\eta \in \Omega^1(\mathcal{R})$ the family ∇^λ is a family of $\mathrm{SL}(4, \mathbb{C})$-connections.

Proof
Let ψ, φ be a basis of the $\mathbb{i}$-eigenspace of S, then they are also a (quaternionic) basis of $\mathbb{H}^2$. Since A is a $K\mathrm{End}_-(V)$-form we have

$$A\psi = \psi dza + \varphi dzb$$

with $\mathbb{H}$-valued functions $a, b,$. Further A anti-commutes with S, so

$$SA\psi = -AS\psi = -A\psi\mathbb{i}.$$

Using the quaternionic linearity of A we get

$$SA\psi = -\psi dza\mathbb{i} - \varphi dzb\mathbb{i}. \tag{45}$$

On the other hand we have

$$SA\psi = S\psi dza + \varphi dzb = \psi dz\mathbb{i}a + \varphi dz\mathbb{i}b. \tag{46}$$

Combining (45) and (46), we obtain

$$\mathbb{i}a = -a\mathbb{i} \quad \text{and} \quad \mathbb{i}b = -b\mathbb{i}.$$

This implies that a is $\mathbb{C}\mathbb{j}$ valued or in other words $A\psi$ takes values in the $-\mathbb{i}$ eigenspace of S. The same holds for φ. For the $-\mathbb{i}$-eigenspace everything is conjugated, so we get

$$A = \begin{pmatrix} 0 & \alpha \\ \bar{\alpha} & 0 \end{pmatrix}$$

with respect to the complex splitting of V in $\pm\mathbb{i}$-eigenspaces. Thus

$$\operatorname{tr} A = 0.$$

The terms SA, η as well as $S\eta$ are also tracefree. For SA it follows analogously to A, and for η and $S\eta$ this follows from $\eta \in \Omega^1(\mathcal{R})$ and the definition (31) of $\mathcal{R}$. Concluding ∇^λ is a $\mathrm{SL}(4, \mathbb{C})$-connection for every λ. □

Theorem 5.22
Let $\eta \in \Omega^1(\mathcal{R})$. Then the family ∇^λ from (44) is a flat family of connections if and only if the Euler Lagrange equation (30) holds, i.e., if f is constrained Willmore.

Proof
The curvature of ∇^λ is

$$\begin{aligned} F^{\nabla^\lambda} &= F^\nabla + (\lambda - 1)d^\nabla A_0^{(1,0)} + (\lambda^{-1} - 1)d^\nabla A_0^{(0,1)} \\ &\qquad + ((\lambda - 1)A_0^{(1,0)} + (\lambda^{-1} - 1)A_0^{(0,1)}) \wedge ((\lambda - 1)A_0^{(1,0)} + (\lambda^{-1} - 1)A_0^{(0,1)}) \\ &= (\lambda - 1)d^\nabla A_0^{(1,0)} + (\lambda^{-1} - 1)d^\nabla A_0^{(0,1)} \\ &\qquad + (\lambda - 1)(\lambda^{-1} - 1)(A_0^{(1,0)} \wedge A_0^{(0,1)} + A_0^{(0,1)} \wedge A_0^{(0,1)}) \\ &= (\lambda - 1)d^\nabla A_0^{(1,0)} + (\lambda^{-1} - 1)d^\nabla A_0^{(0,1)} + (2 - \lambda - \lambda^{-1})(A_0 \wedge A_0) \\ &= (\lambda - 1)(d^\nabla A_0^{(1,0)} - A_0 \wedge A_0) + (\lambda^{-1} - 1)(d^\nabla A_0^{(0,1)} - A_0 \wedge A_0). \end{aligned}$$

The terms $A_0^{(1,0)} \wedge A_0^{(1,0)}$ and $A_0^{(0,1)} \wedge A_0^{(0,1)}$ vanish, since they are both $(1,0)$ forms or $(0,1)$ forms, respectively. If $F^{\nabla^\lambda} = 0$ for all λ, then $d^\nabla A_0^{1,0} = A_0 \wedge A_0 = d^\nabla A_0^{(0,1}$), and since $*A_0 = -i(A_0^{(1,0)} - A_0^{(0,1)})$, we get

$$d^\nabla * A_0 = 0.$$

Vice versa, if f is constrained Willmore we start with the last equation and deduce that

$$d^\nabla A_0^{(1,0)} = d^\nabla \frac{1}{2}(A_0 - i * A_0) = \frac{1}{2} d^\nabla A_0 = d^\nabla A_0^{(0,1)}.$$

Since $\nabla S = 2 * Q - 2 * A = 2 * Q_0 - 2 * A_0$ we thus obtain

$$\begin{aligned}\frac{1}{2} d^\nabla A_0 &= \frac{1}{2} d^\nabla S * A_0 = \frac{1}{2}(\nabla S \wedge *A_0 + S d^\nabla * A_0)\\ &= (*Q - *A) \wedge A_0.\end{aligned}$$

Making use of $\operatorname{image} \eta \subset L \subset \ker Q$ and $\operatorname{image} A \subset \ker Q$, we get

$$\begin{aligned}\frac{1}{2} d^\nabla A_0 &= *A \wedge *A_0\\ &= -AS \wedge SA_0 = A \wedge A_0\\ &= A_0 \wedge A_0,\end{aligned}$$

and therefore $F^{\nabla^\lambda} = 0$. □

The right multiplication with the quaternion $\mathbb{j}$ is an complex anti-linear map on the complex vector space (V, i). As in the CMC case ∇^λ has the symmetry

$$\nabla^{\bar\lambda^{-1}} = \mathbb{j}^{-1} \nabla^\lambda \mathbb{j} \tag{47}$$

In particular ∇^λ is quaternionic for $\lambda \in S^1 \subset \mathbb{C}^*$. (47) holds since A is quaternionic linear and $\mathbb{i}$ and $\mathbb{j}$ anti-commute.

5.4. The spectral curve of constrained Willmore tori in the 4-sphere

In this section, we define the spectral curve Σ of the associated familiy of connections ∇^λ for maps $f : T^2 \to S^4$ from a torus T^2 into the 4-sphere. For the most part this will be an summary of the properties shown by C. Bohle, [Boh10].

Let $f : T^2 \to S^4$ be a constrained Willmore torus and ∇^λ the associated flat family of $\mathrm{SL}(4, \mathbb{C})$-connections from (44). Fix $p \in T^2$ and denote by $H_p^\lambda(\gamma)$ the holonomy of ∇^λ in p along the closed curve γ. Since ∇^λ is flat, we get the holonomy representation

$$H_p^\lambda : \Gamma \to \mathrm{SL}(4, \mathbb{C})$$

of the fundamental group $\pi_1(T^2) = \Gamma$. Let $\gamma_0 \in \Gamma$. As in the CMC case, see (43), every simple eigenspace of $H_p^\lambda(\gamma_0)$ is an eigenspace of $H_p^\lambda(\gamma)$ for every $\gamma \in \Gamma$. For a different point $\tilde p \in T^2$ the Holonomies H_p^λ and $H_{\tilde p}^\lambda$ differ by conjugation, i.e.

$$P_{\tilde p p}^\lambda H_p^\lambda = H_{\tilde p}^\lambda P_{\tilde p p}^\lambda,$$

where $P_{\tilde p p}^\lambda$ is the parallel translation of ∇^λ along a path from p to $\tilde p$. In particular the eigenvalues do not depend on the chosen base point. For a fixed γ, the set

$$\{(\lambda, \mu) \in \mathbb{C}_* \times \mathbb{C}_* | \det(H_p^\lambda(\gamma) - \mu \,\mathrm{id})\}$$

is a 1-dimensional analytic subset of $\mathbb{C}_* \times \mathbb{C}_*$ and independent of p.

Theorem 5.23 (Prop. 3.1 in [Boh10])
Let $f : T^2 \to S^4$ be a constrained Willmore torus, and $p \in T^2$. The holonomy representation H_p^λ of ∇^λ belongs to one of the following cases

I There is a $\gamma \in \Gamma$ such that the eigenvalues of $H_p^\lambda(\gamma)$ are generically distinct. Away from finitely many points those eigenvalues are locally given as non-constant holomorphic functions of λ.

II For all $\gamma \in \Gamma$ there are 2 non-constant eigenvalues and one common 2 dimensional eigenspace corresponding to the eigenvalue 1. The restriction on the non-constant eigenvalues yields a 2-fold covering of $\mathbb{C}_*$.

III The only eigenvalue is 1, and either H_p^λ is the identity matrix for all λ or splits into two 2×2 Jordan blocks.

Remark
We are only interested in the first case. If the holonomy H_p^λ of ∇^λ has only 2 non-constant eigenvalues, i.e., we are in Case II, the immersion f is isothermic, see [Boh10, Section 6]. Then we can choose a $\hat{\eta} \in \Omega^1(\mathcal{R})$ satisfying the Euler-Lagrange equatuion (30) such that the corresponding connection $\hat{\nabla}^\lambda$ has Holonomy $\hat{H}_p^\lambda$ which belongs to I, i.e. has 4 non-constant eigenvalues. Case III only appears in the case of a super conformal immersion with planar ends in $\mathbb{R}^4$.

The holonomy spectral curve Σ_{hol} is, if we are in the first case, the normalization of the set

$$\{(\lambda, \mu) \in \mathbb{C}_* \times \mathbb{C}_* | \det(H_p^\lambda(\gamma) - \mu \,\mathrm{id})\}.$$

The projection onto the first coordinate, which we denote by $\lambda : \Sigma_{hol} \to \mathbb{C}^*$, yields a branched 4-fold covering of $\mathbb{C}_*$. In the second case, Σ_{hol} is given as the normalization of the 2-fold covering of $\mathbb{C}_*$. In the third case there is no spectral curve.

The surface Σ_{hol} is the (up to isomorphism) unique Riemann surface, such that there is a line subbundle

$$\hat{\mathcal{L}}_p \subset V = \underline{\mathbb{C}}^4 \to \Sigma$$

such that at every $\xi \in \Sigma_{hol}$ the line $\hat{\mathcal{L}}_p^\xi$ is an eigenline (of a non-constant eigenvalue) of $H^{\lambda(\xi)}$.

The quaternionic symmetry (47) of ∇^λ gives an anti-holomorphic involution ρ on Σ_{hol}: If $\hat{\psi}$ is an eigenvector of H_p^λ to the eigenvalue μ, $\psi\mathbb{j}$ is an eigenvector of $H_p^{\bar{\lambda}^{-1}}$ with respect to the eigenvalue $\bar{\mu}$. Therefore we obtain

$$\rho(\lambda, \mu) = (\bar{\lambda}^{-1}, \bar{\mu}), \tag{48}$$

and furthermore

$$\rho^*(\hat{\mathcal{L}}_p)^\xi = \hat{\mathcal{L}}_p^{\rho(\xi)} = \hat{\mathcal{L}}_p^\xi \mathbb{j}.$$

By parallel transport with ∇^λ one can extend $\hat{\mathcal{L}}_p$ to a line bundle $\hat{\mathcal{L}} \to \Sigma_{hol} \times T^2$.

The main theorem concerning the holonomy spectral curve is the following theorem from [Boh10].

Theorem 5.24 ([Boh10, Theorem 5.1])
Let $f : T^2 \to S^4$ be a constrained Willmore immersion such that the associated family of connections ∇^λ has a holonomy representation H_p^λ which belongs to case I of Theorem 5.23. Then the spectral curve Σ_{hol} can be compactified to a 4-fold branched covering of $\mathbb{C}P^1$ with branchpoints at 0 and ∞.

Definition 5.25
We call the compactification Σ of Σ_{hol} the spectral curve of f.

The proof of Theorem 5.24 is done by the construction of polynomial Killing fields. Polynomial Killing fields ξ are polynomial sections in $\mathrm{End}_{\mathbb{C}}(V)$ which are ∇^λ parallel, i.e.

$$\xi(\lambda, p) = \sum_{k=0}^{d} \xi_k(p)\lambda^k$$

and $\xi(\lambda, \cdot)$ is ∇^λ-parallel. For generic λ (if the 4 eigenvalues of H_p^λ are pairwise distinct) the eigenlines of ∇^λ-parallel sections are the eigenlines $\hat{\mathcal{L}}$ of the holonomy. Therefore there is a vector bundle W over $\mathbb{C}^*$ whose fiber over generic λ coincides with the space of ∇^λ-parallel sections. While the family of connections ∇^λ does not extend over ∞, the family of holomorphic structures

$$\bar{\partial}^\lambda = \nabla^{\lambda(0,1)} = \nabla^{(0,1)} + (\lambda^{-1} - 1)A_0^{(0,1)}$$

does. The space of $\bar{\partial}^\lambda$ holomorphic sections in $\mathrm{End}(V)$ is, for generic λ, 4-dimensional and therefore coincides with the space of ∇^λ-parallel sections. Therefore, W extends over ∞. The interested reader can find more details in [Boh10, Lemma 5.5]. The anti-holomorphic structures

$$\partial^\lambda = \nabla^{\lambda(1,0)} = \nabla^{(1,0)} + (\lambda - 1)A_0^{(1,0)},$$

or analogously the symmetry (47), imply that W extends over 0. Concluding W is a vector bundle over $\mathbb{C}P^1$. A killing field ξ is given as a meromorphic section in W whose only pole is at ∞.

The existence of an polynomial Killing field ξ implies, due to its polynomial structure, that there can be only finitely many points where the eigenlines coalesce. Therefore the holonomy spectral curve has finite genus and is compactifiable. Further, the line bundle $\hat{\mathcal{L}}_p \to \Sigma_{hol}$ extends on Σ, since it is parametrizing the eigenline of the Killing field.

5.5. Multiplier v Holonomy

The multiplier spectral curve Σ_{mult} is defined by D-holomorphic sections in V/L and is equipped with the kernel bundle $\mathcal{L} \to T^2 \times \Sigma_{mult}$. The fibre $\mathcal{L}_h$ coincides for generic multipliers $h \in \Sigma_{mult}$ with the space of D-holomorphic sections with monodromy h, see Section 5.1. The holonomy spectral curve Σ_{hol} is given by the holonomy representation of a flat connection ∇^λ as defined in (44). The fact that every D-holomorphic section ψ corresponds to its prolongation $\hat{\psi} \in \Gamma(V)$ with $\nabla\hat{\psi} \in \Omega^1(L)$, see Lemma 1.23, now enables us to identify Σ_{mult} and Σ_{hol}. As before let $f : T^2 \to S^4$ be a constrained Willmore torus and ∇^λ the associated family of flat connections, then we can establish the following result.

Proposition 5.26
Every (local) ∇^λ-parallel section is the prolongation of a (local) D-holomorphic section in V/L. Therefore every fibre $\hat{\mathcal{L}}^\xi \to T^2$ of the eigenline bundle $\hat{\mathcal{L}} \to T^2 \times \Sigma_{hol}$ at $\xi \in \Sigma_{hol}$ is given by the prolongation of a kernel bundle $\mathcal{L}^h \to T^2$ for some multiplier $h \in \Sigma_{mult}$.

Proof
Let $\hat{\psi}$ be a ∇^λ parallel section with monodromy. Then $\nabla^\lambda \hat{\psi} = 0$ implies

$$\nabla \hat{\psi} = -(\lambda - 1) A_0^{(1,0)} \hat{\psi} - (\lambda^{-1} - 1) A_0^{(0,1)} \hat{\psi}.$$

Since A_0 is L-valued we have $\nabla \hat{\psi} \in \Omega^1(L)$. By Lemma 1.23 $\hat{\psi}$ is the prolongation of a D-holomorphic section ψ with monodromy. $\square$

Recall that by prolongating the kernel bundle we get a map $F : T^2 \times \Sigma_{hol} \to \mathbb{C}P^3$, see (36). Proposition 5.26 implies a natural map

$$\iota : \Sigma_{hol} \to \Sigma_{mult}$$

with

$$\hat{\mathcal{L}}^\xi_p = F(p, \iota(\xi)).$$

The involutions ρ on Σ_{mult} (35) and Σ_{hol} (48) commute with ι because we have

$$\rho^*(\hat{\mathcal{L}}^\xi_p) = \hat{\mathcal{L}}^\xi_p \mathbb{j} \quad \text{as well as} \quad \rho^* F(p, h) = F(p, h)\mathbb{j}.$$

As Bohle has shown, the map ι is almost bijective. Therefore we can identify Σ_{mult} and Σ_{hol}.

Theorem 5.27 ([Boh10, Theorem 4.5])
Let $f : T^2 \to S^3$ be a constrained Willmore torus. Then ι is injective and the image is Σ_{mult} with at most 4 points removed. The missing points correspond to prolongations $\hat{\psi}$ which satisfy the equations

$$\nabla \hat{\psi} \in \Omega^1(\widetilde{L}) \quad \text{and} \quad (A_0 \hat{\psi})^{(1,0)} = 0,$$

or

$$\nabla \hat{\psi} \in \Omega^1(\widetilde{L}) \quad \text{and} \quad (A_0 \hat{\psi})^{(0,1)} = 0.$$

Those equations are the asymptotic equations of

$$\nabla \hat{\psi} + (\lambda - 1) A_0^{(1,0)} \hat{\psi} + (\lambda^{-1} - 1) A_0^{(0,1)} \hat{\psi} = \nabla^\lambda \hat{\psi} = 0$$

for $\lambda \to \infty$ or $\lambda \to 0$ respectively.

By Theorem 5.24 the spectral curve has finite genus, which implies by Theorem 5.10 that the spectral curve Σ_{mult} can be compactified to the spectral curve Σ by adding two points 0 and ∞, which are interchanged by ρ.

6. The rank 2 theory for cW Tori in the 3-sphere

If the constrained Willmore immersion f maps into the 3-sphere, the spectral curve has a further symmetry, an involution σ. In Section 6.1 we will introduce the holomorphic involution σ which enables us to define a rank 2 bundle which is invariant under the holonomy in Section 6.2. Therefore the rank 4 $\mathbb{C}_*$-family of connections ∇^λ can be restricted to this subbundle, and defines a family of rank 2 connections. The construction of $\hat{E}$ is done by the pushforward construction from Section 3. Section 6.3 treats the case where Σ/σ is a sphere. In this case the immersion is already CMC, which was shown by L. Heller [Hel15]. An investigation of the general case and a way to reconstruct constrained Willmore tori by a Sym-Bobenko formula can be found in the Main theorem 6.18 in Section 6.4.

6.1. The spectral curve of constrained Willmore tori in the 3-sphere

Let $f : T^2 \to S^3$ be a constrained Willmore torus in S^3 and L be the associated line bundle, cf. Lemma 1.11. We will use the notation established in Section 5 and always assume, that we are in the case where the spectral curve Σ is a 4-fold cover of $\mathbb{C}P^1$. Let $L^\perp \subset V^*$ be the dual curve from Definition 1.24. The identification of L and $L^\perp$ as seen in Section 1.7 gives a further symmetry on the spectral curve Σ of f: By Lemma 1.26 we have

$$S^\perp = S^*, \qquad \text{and} \qquad A^\perp = -Q^*.$$

Therefore the dual curve $L^\perp$ itself is constrained Willmore with

$$\eta^\perp = -\eta^*.$$

So it holds $A_0^\perp = -Q_0^*$ and the associated family of connections of $L^\perp$ is given by

$$\begin{aligned}\nabla^{\perp\lambda} &= \nabla + (\lambda - 1)\frac{1 + iS^\perp}{2}A_0^\perp + (\lambda^{-1} - 1)\frac{1 - iS^\perp}{2}A_0^\perp \\ &= \nabla - (\lambda - 1)\frac{1 + iS^*}{2}Q_0^* - (\lambda^{-1} - 1)\frac{1 - iS^*}{2}Q_0^*.\end{aligned}$$

Here $\nabla = d$ on V^* is the dual connection to d on V. The dual connection of $\nabla^{\perp\lambda}$ is then given by

$$\begin{aligned}\hat{\nabla}^\lambda &= \nabla + (\lambda - 1)Q_0\frac{1 - iS}{2} + (\lambda^{-1} - 1)Q_0\frac{1 + iS}{2} \\ &= \nabla + (\lambda - 1)Q_0^{(1,0)} + (\lambda^{-1} - 1)Q_0^{(0,1)}.\end{aligned}$$

This connection is gauge equivalent to ∇^λ,i.e.,

$$\nabla^\lambda = g \circ \hat{\nabla}^\lambda \circ g^{-1},$$

with gauge

$$g = ((\lambda + 1) - i(\lambda - 1)S)\,.$$

In Section 1.7, we have seen that conformal immersions into S^3 satisfy

$$L = L^\perp, \qquad S = S^*, \quad \text{and} \quad A^\perp = -Q^* = A,$$

due to the identification of V and V^* via the indefinite hermitian product (15). This implies $\nabla^\lambda = \nabla^{\perp\lambda}$ and therefore ∇^λ is gauge equivalent to its dual connection $\hat{\nabla}^\lambda$. Let now μ be an eigenvalue of $H_p^\lambda(\gamma)$. Then μ^{-1} is an eigenvalue of the Holonomy $\hat{H}_p^\lambda(\gamma)$ of the dual connection $\hat{\nabla}_p^\lambda$. As the connections are gauge equivalent both holonomies only differ by conjugation. Therefore μ^{-1} must again be an eigenvalue of $H^\lambda(\gamma)$. This implies the existence of an holomorphic map on Σ_{hol}

$$\sigma : (\lambda, \mu) \longmapsto (\lambda, \mu^{-1}).$$

This map extends to Σ and we get the following lemma.

Lemma 6.1
Let $f : T^2 \to S^3$ be a constrained Willmore torus. Then there is a holomorphic involution σ on Σ such that if $\hat{\mathcal{L}}_p(\xi)$ is the eigenline of the holonomy $H_p^{\lambda(\xi)}$ to the eigenvalue μ, then $\hat{\mathcal{L}}_p(\sigma(\xi)) = \sigma^*(\hat{\mathcal{L}}_p)(\xi)$ is the eigenline of $H_p^{\lambda(\xi)} = H_p^{\lambda(\sigma(\xi))}$ to the eigenvalue μ^{-1}. In particular σ fixes λ.

Remark
The symmetry σ can also be defined via the multiplier spectral curve by pairing V/L and $K(V/L)^{-1}$ and using that for the 3-sphere these bundles are holomorphic isomorphic. Therefore, the existence of a holomorphic section with monodromy h implies that h^{-1} as well lies in the spectrum Spec(V/L) of V/L. See the theorem on page 47 of [Hel12a], as well as [Boh03, Theorem 1] for further details.

Definition 6.2
We will denote the quotient surface by

$$X := \Sigma/\sigma.$$

As σ is a holomorphic involution, X is a Riemann surface which is naturally doubly covered by Σ. The map $\lambda : \Sigma \to \mathbb{C}P^1$ is a 4-fold cover. Since σ leaves λ invariant, i.e.,

$$\lambda = \sigma^*\lambda = \lambda \circ \sigma,$$

the map λ is well defined on X. By construction $\lambda : X \to \mathbb{C}P^1$ is a double cover of $\mathbb{C}P^1$ and therefore X a hyperelliptic surface. The hyperelliptic involution on X is denoted by χ. As ρ and σ commute, ρ defines an anti-holomorphic involution on X, denoted by the same symbol.

Remark
The points $0, \infty \in \Sigma$, belonging to the ends of Σ_{mult}, are ramification points of $\Sigma \to X$. This holds since h^{ξ} has essential singularities in those points, whereas in the other points lying above 0 and $\infty \in \mathbb{C}P^1$, the multiplier is a well-defined holomorphic function, cf. Theorem 5.27. Since σ maps $h \mapsto h^{-1}$, the point $\infty \in \Sigma$ can't be mapped to another point over $[\infty] \in X$.

The surface X has four noteworthy points. The two points 0 and ∞ which belong to the ends of $\Sigma_{mult} \subset \Sigma$, as well as the points x_1, x_2 over $\lambda = 1$, which belong to the trivial connection.

Definition 6.3
A hyperelliptic surface X with marked blue points p, q is a surface X with an anti-holomorphic involution ρ commuting with the hyperelliptic involution χ, such that the two points p and $q \in X$ satisfy

$$p = \rho(q) \quad \text{and} \quad \chi(p) \neq q.$$

Two points $p, q \in X$ are called green points if

$$p = \chi(q)$$

and the equivalence class $[p] = \{p, q\} \in X/\chi = \mathbb{C}P^1$ is preserved by ρ.

Example
The points $0, \infty \in X$ are blue points, and the points x_1, x_2 are green points of the hyperelliptic surface X.

Example 6.4
Let X be a hyperelliptic surface with marked blue points $0, \infty$ and marked green points $x_1, x_2 \in X$. After a Möbius transformation, we can always assume that $0, \infty \in X$ lie over $0, \infty \in \mathbb{C}P^1$, respectively. The anti-holomorphic involution on $\mathbb{C}P^1$ induced by ρ interchanges $0, \infty \in \mathbb{C}P^1$ and is therefore given by

$$\lambda \mapsto a\bar{\lambda}^{-1},$$

for some $a \in \mathbb{R}$. Since $[x_1] \in \mathbb{C}P^1$ is a fixpoint, we obtain $a > 0$. Changing the coordinate to $a^{-1/2}\lambda$ the real-structure is given by

$$\lambda \mapsto \bar{\lambda}^{-1}.$$

After a rotation we obtain that x_1, x_2 lie over $1 \in \mathbb{C}P^1$. We can therefore always assume that $\lambda(x_1) = \lambda(x_2) = 1$ and $0, \infty \in X$ lie over $\lambda = 0$ and $\lambda = \infty$, respectively. The surface X is then given by one of the equations

$$y^2 = \prod_{i=1}^{g_X+1} \bar{\lambda}_i(\lambda - \lambda_i)(\lambda - \bar{\lambda}_i^{-1}), \quad \text{or} \quad y^2 = \lambda \prod_{i=1}^{g_X} \bar{\lambda}_i(\lambda - \lambda_i)(\lambda - \bar{\lambda}_i),$$

when there are no branchpoints over S^1, and

$$y^2 = \prod_{i=1}^{k} \bar{\lambda}_i(\lambda - \lambda_i)(\lambda - \bar{\lambda}_i^{-1}) \prod_{j=1}^{2g_X+2-2k} i\mu_j^{-1/2}(\lambda - \mu_j),$$

or,

$$y^2 = \lambda \prod_{i=1}^{k} \bar{\lambda}_i(\lambda - \lambda_i)(\lambda - \bar{\lambda}_i) \prod_{j=1}^{2g_X+2-2k} i\mu_j^{-1/2}(\lambda - \mu_j),$$

if there are branchpoints $\mu_j \in S^1$. The anti-holomorphic involution covering $\lambda \mapsto \bar{\lambda}^{-1}$ is given by

$$\rho : (y, \lambda) \mapsto (\pm\bar{y}\bar{\lambda}^{-g-1}, \bar{\lambda}^{-1}).$$

6.2. The rank 2 bundle E

While the holomorphic involution $\sigma : \Sigma \to \Sigma$ leaves $\lambda : \Sigma \to \mathbb{C}P^1$ invariant, the anti-holomorphic involution ρ satisfies, by (48),

$$\rho^*\lambda = \bar{\lambda}^{-1}.$$

Hence the involution

$$\rho \circ \sigma : \Sigma \to \Sigma$$

fixes $\lambda \in S^1$. Still it may be fixpoint free on Σ. For CMC tori in a 3-dimensional space form the involution $\rho\circ\sigma$ has fixpoints. Examples for constrained Willmore surfaces with fixpoint free $\rho \circ \sigma$ are the Willmore surfaces studied by Babich and Bobenko [BB93]. It should be true that a constrained Willmore torus with fixpoint free $\rho \circ \sigma$ is never embedded.

We are more interested in the case where $\rho \circ \sigma$ has fixpoints.

Let $\xi = (\lambda, \mu) \in \Sigma$ be a fixpoint of $\rho \circ \sigma$. Therefore

$$(\bar{\lambda}^{-1}, \bar{\mu}) = \rho(\xi) = \sigma(\xi) = (\lambda, \mu^{-1}), \tag{49}$$

which implies that $\mu \in S^1$ is unitary. For generic λ the 4 eigenvalues of the holonomy H_p^λ are distinct. In this case there exists a basis such that the holonomy has the form

$$H_p^\lambda = \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu^{-1} & 0 & 0 \\ 0 & 0 & \tilde{\mu} & 0 \\ 0 & 0 & 0 & \tilde{\mu}^{-1} \end{pmatrix}.$$

Therefore H_p^λ (generically) splits into 2×2-blocks of $\mathrm{SL}(2, \mathbb{C})$ matrices. The idea is to define a rank 2 bundle which is generically given by

$$\hat{E}_p^\xi = \hat{\mathcal{L}}_p^\xi \oplus \sigma^*\hat{\mathcal{L}}_p^\xi, \tag{50}$$

in order to restrict to only one block

$$\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}.$$

For fixpoints ξ the holonomy is then unitary by (49). This makes this approach a similiar theory to the case of CMC tori, cf. Section 5.2. An advantage of this constrained Willmore tori rank 2 theory is, that it includes both, the Hitchin and Bobenko theory, as well as the Babich Bobenko tori. Since the equation (50) is σ-invariant we will naturally work on $X = \Sigma/\sigma$.

Definition 6.5
We define $\hat{E}_p \to \Sigma/\sigma$ as the dual of the pushforward bundle $\pi_* \hat{\mathcal{L}}_p^{-1}$, where

$$\pi : \Sigma \to \Sigma/\sigma$$

is the canonical projection.

It is easier to define $\hat{E}$ on Σ as a σ-invariant bundle. Away from branchpoints of $\Sigma \to X$ we set

$$\hat{E}_p = \hat{\mathcal{L}}_p \oplus \sigma^* \hat{\mathcal{L}}_p.$$

At a branchpoint ξ we take the frame

$$t_1 = s_1 + s_2, \qquad t_2 = \frac{1}{y}(s_1 - s_2), \tag{51}$$

where s_1 is a nonvanishing holomorphic section of $\hat{\mathcal{L}}$ in a neighbourhood of ξ, $s_2 = \sigma^* s_1$ and y is a centered chart around ξ with $\sigma(y) = -y$. Since this frame is σ-invariant, $\hat{E}$ is σ-invariant and therefore defines a bundle $\hat{E}_p \to X$, for further details see Section 3.3.

Proposition 6.6
If $f : T^2 \to S^3$ is a simple constrained Willmore torus, the bundle $\hat{E}_p \to X$ is canonically a subbundle of $\mathbb{C}^4$, and has degree $\deg \hat{E}_p = -2g_X - 2$, where g_X is the genus of X.

In order to prepare the proof, we will firstly establish the following lemma.

Lemma 6.7
The degree of $\hat{\mathcal{L}}_p$ satisfies

$$\deg \hat{\mathcal{L}}_p \leq -g_\Sigma - 3,$$

where g_Σ is the genus of Σ. Equality holds if and only if the constrained Willmore torus is simple. In that case the lines $\hat{\mathcal{L}}_p$ and $\sigma^* \hat{\mathcal{L}}_p$ only coalesce at branch points of $\Sigma \to X$. There they coalesce with first order.

Proof (of Proposition 6.6)
Knowing the degree of $\hat{\mathcal{L}}_p$, we can calculate the degree of $\hat{E}$ using Proposition 3.5. By construction, $\hat{\mathcal{L}}_p$ is a subbundle of $\mathbb{C}^4$. As the linebundles $\hat{\mathcal{L}}_p$ and $\sigma^* \hat{\mathcal{L}}_p$ only coalesce with first order, the frame (t_1, t_2) from (51) gives well defined linear independent local sections in $\mathbb{C}^4$. Therefore $\hat{E}_p$ is a subbundle of $\mathbb{C}^4$. $\square$

Proof (of Lemma 6.7)
We will estimate the degree of $\hat{\mathcal{L}}_p$ by counting the points in which the (generically) four eigenlines coincide. The bundle $\hat{\mathcal{L}}_p \otimes \sigma^*\hat{\mathcal{L}}_p$ is σ-invariant (identifying $\hat{\mathcal{L}}_p^\xi \otimes \hat{\mathcal{L}}_p^{\sigma(\xi)}$ and $\hat{\mathcal{L}}_p^{\sigma(\xi)} \otimes \hat{\mathcal{L}}_p^\xi$), and therefore is well defined as a line bundle on X. By Riemann Hurwitz, see Theorem 2.1, there are $2g_X+2$ branch points of $X \to \mathbb{C}P^1$ and $2g_\Sigma - 4g_X + 2$ ramification points of $\Sigma \to X$. Denote the ramification points by

$$x_1, \cdots, x_{2g_\Sigma - 4g_X+2} \in X.$$

As $\hat{\mathcal{L}}_p \to \Sigma$ is a subbundle of $\mathbb{C}^4$, the determinant of $\mathbb{C}^4$ defines a (local) holomorphic section in

$$(\hat{\mathcal{L}}_p \otimes \sigma^*\hat{\mathcal{L}}_p)^{-1} \otimes \chi^*(\hat{\mathcal{L}}_p \otimes \sigma^*\hat{\mathcal{L}}_p)^{-1}$$

by evaluation. For $x \in X$ we evaluate the determinant at $\hat{\mathcal{L}}_p^{\xi_1}, \hat{\mathcal{L}}_p^{\xi_2}, \hat{\mathcal{L}}_p^{\sigma(\xi_1)}, \hat{\mathcal{L}}_p^{\sigma(\xi_2)}$, where ξ_1 is a point above x and ξ_2 a point above $\chi(x)$. This does not depend on the choice of ξ_1 and ξ_2, since the determinant is alternating. Further, det is multilinear, and therefore defines a linear map

$$\omega^x : (\hat{\mathcal{L}}_p \otimes \sigma^*\hat{\mathcal{L}}_p)^x \otimes \chi^*(\hat{\mathcal{L}}_p \otimes \sigma^*\hat{\mathcal{L}}_p)^x \longrightarrow \mathbb{C}.$$

The map ω^x is zero exactly in those points where the 4 lines $\hat{\mathcal{L}}_p^{\xi_1}, \hat{\mathcal{L}}_p^{xi_2}, \hat{\mathcal{L}}_p^{\sigma(\xi_1)}, \hat{\mathcal{L}}_p^{\sigma(\xi_2)}$ do not span $\mathbb{C}^4$. Away from the points

$$x_1, \cdots x_{2g_\Sigma - 4g_X+2}, \chi(x_1), \cdots, \chi(x_{2g_\Sigma-4g_X+2})$$

this map locally depends holomorphically on x. As the lines $\mathcal{L}_p$ and $\sigma^*\mathcal{L}_p$ may coincide by an odd order in a point over x_i, ω is maybe not well defined on X. Instead, we look at the double covering $\tilde{X}$ of X branched over the points $x_1, \cdots x_{2g_\Sigma-4g_X+2}, \chi(x_1), \cdots, \chi(x_{2g_\Sigma-4g_X+2})$. Then ω is a well defined global holomorphic section

$$\omega \in H^0(\tilde{X}, (\hat{\mathcal{L}}_p \otimes \sigma^*\hat{\mathcal{L}}_p)^* \otimes \chi^*(\hat{\mathcal{L}}_p \otimes \sigma^*\hat{\mathcal{L}}_p)^*).$$

We can estimate the zeroes of ω, which are counted with multiplicity. In the points over x_i there are at least zeroes of order 1, as the lines $\mathcal{L}_p$ and $\sigma^*\mathcal{L}_p$ coincide there. Similiar we have zeroes over $\chi(x_i)$. In the $4g_X+4$ many points over branch points of $X \to \mathbb{C}P^1$ the four eigenlines only span a 2 dimensional space. Therefore, we have at least a double zero there. In sum we get

$$2\deg(\hat{\mathcal{L}}_p \otimes \sigma^*(\hat{\mathcal{L}}_p))^* = \deg(\hat{\mathcal{L}}_p \otimes \sigma^*\hat{\mathcal{L}}_p)^* \otimes \chi^*(\hat{\mathcal{L}}_p \otimes \sigma^*\hat{\mathcal{L}}_p)^* = \deg\omega \geq 4g_\Sigma + 12$$

as linebundles over $\tilde{X}$. The degree of $\hat{\mathcal{L}}_p \otimes \sigma^*(\hat{\mathcal{L}}_p)$ as a linebundle over Σ is therefore

$$\deg \hat{\mathcal{L}}_p \otimes \sigma^*\hat{\mathcal{L}}_p \leq -2g_\Sigma - 6,$$

which implies

$$\deg \hat{\mathcal{L}}_p \leq -g_\Sigma - 3.$$

If the torus is simple, we have by Kodaira embedding Theorem 2.11

$$h^0(\hat{\mathcal{L}}_p^{-1}) = 4.$$

Using Riemann-Roch, see Corollary 2.3, we get the estimate

$$\deg(\hat{\mathcal{L}}_p) \geq -g_\Sigma - 3.$$

Therefore equality holds and the only zeroes of ω come from the branch and ramification points in X. In particular, the only points where $\hat{\mathcal{L}}_p$ and $\sigma^*\hat{\mathcal{L}}_p$ coincide are the branch points of $\Sigma \to X$. $\square$

Remark
If f is simple, the set

$$\{(\lambda, \mu) \in \mathbb{C}_* \times \mathbb{C}_* | \det(H_p^\lambda(\gamma) - \mu \,\mathrm{id})\}$$

is already a smooth set and is the spectral curve up to compactification.

Knowing the degree of $\hat{\mathcal{L}}_p$, we get an estimate for $\deg \mathcal{L}_p$:

Proposition 6.8
Let f be a simple constrained Willmore torus. Then we get the estimate

$$\deg \mathcal{L}_p \geq -g_\Sigma - 1$$

for the kernel bundle $\mathcal{L}$. Equality holds if and only if the holomorphic sections with monodromy spanning $\mathcal{L}$ have no zeroes.

Remark
If the Willmore energy of f is bounded by 8π then the holomorphic sections with monodromy spanning $\mathcal{L}$ are nonvanishing, see [BLPP12, Lemma 2.11] and [BPP09, Remark 4.10].

Proof (of Proposition 6.8)
Let $\hat{\psi} \in \mathcal{M}(\hat{\mathcal{L}}_p)$ be a meromorphic section of $\hat{\mathcal{L}}_p \to \Sigma$. Then

$$\psi = \pi(\hat{\psi}) \in \mathcal{M}(\mathcal{L}_p)$$

is a meromorphic section of $\mathcal{L}_p$. Since π can only add zeroes, the degree of $\mathcal{L}_p$ will be higher than the degree of $\hat{\mathcal{L}}_p$. As it is shown in the proof of [BLPP12, Theorem 4.2] we have for ξ near infinity

$$\hat{\mathcal{L}}_p^\xi = (1 + \delta^{-1}\eta^\xi)\mathcal{L}_p^\xi. \tag{52}$$

Here $\delta : L \to K(V/L)$ is the derivative of f from Lemma 1.14, and $\eta^\xi \in \Omega^1(\mathrm{End}(V/L))$ is a local meromorphic 1-form with simple pole at $\xi = \infty$. With the splitting

$$V = V/L \oplus L$$

(52) gives a well defined subbundle in V. The simple pole of η implies that ψ has a zero of first order at $\xi = \infty$ (if $\hat{\psi}$ is nonvanishing in ∞). The same argument holds at $\xi = 0$. Therefore

$$\deg \mathcal{L}_p \geq \deg \hat{\mathcal{L}}_p + 2 = -g_\Sigma - 1.$$

If all holomorphic sections with monodromy spanning $\mathcal{L}$ have no zeroes, i.e., the corresponding Darboux transforms are non-singular, then

$$\mathcal{L}_p^\xi \not\subseteq L_p$$

for $\xi \neq 0, \infty$ and therefore π can't add further zeroes. □

Proposition 6.9
Let f be a simple constrained Willmore torus. The bundle $\hat{E}_p \subset \mathbb{C}^4$ is H_p^λ-invariant.

Proof
In order to work with the eigenline bundles, we view $\hat{E}_p$ as a bundle over Σ. At generic ξ the lines $\hat{\mathcal{L}}_p^\xi$ and $\sigma^*\hat{\mathcal{L}}_p^\xi$ are linear independent eigenlines of $H_p^{\lambda(\xi)}$. Therefore $\hat{E}_p^\xi$ is H_p^λ-invariant, since it is spanned by the two eigenlines. Around a branch point ξ_0 consider the frame (t_1, t_2) from (51). We have

$$\begin{aligned} H^{\lambda(\xi)}t_1 &= H^{\lambda(\xi)}(s_1+s_2) = \mu s_1 + \mu^{-1}s_2 \\ &= \frac{1}{2}(\mu(t_1+yt_2)+\mu^{-1}(t_1-yt_2)) \\ &= \frac{1}{2}((\mu+\mu^{-1})t_1 + y(\mu-\mu^{-1})t_2), \end{aligned}$$

and

$$\begin{aligned} H^{\lambda(\xi)}t_2 &= H^{\lambda(\xi)}(\frac{1}{y}(s_1-s_2)) = \frac{1}{y}(\mu s_1 - \mu^{-1}s_2) \\ &= \frac{1}{2}(\frac{1}{y}(\mu-\mu^{-1})t_1 + (\mu+\mu^{-1})t_2. \end{aligned}$$

Since

$$\mu(\xi_0) = \mu^{-1}(\xi_0),$$

the function $\frac{1}{y}(\mu-\mu^{-1})$ is a well defined holomorphic function and $\hat{E}_p$ is H_p^λ-invariant in branchpoints. □

With respect to the basis t_1, t_2 we have

$$H_{\hat{E}}^{\lambda(\xi)} = \frac{1}{2}\begin{pmatrix} \mu+\mu^{-1} & \frac{1}{y}(\mu-\mu^{-1}) \\ y(\mu-\mu^{-1}) & \mu+\mu^{-1} \end{pmatrix}.$$

In particular, at branchpoints it is

$$H_{\hat{E}}^\lambda = \begin{pmatrix} \mu & c \\ 0 & \mu \end{pmatrix}.$$

Varying p we get a bundle

$$\hat{E} \longrightarrow T^2 \times X.$$

The bundle $\hat{E}^x \to T^2$ is given by parallel transport of $\hat{E}^x_p$ with respect to $\nabla^{\lambda(x)}$. Let

$$X^* = X \setminus \{\textit{points over } 0, \infty\}.$$

By construction, we can restrict the family ∇^λ on $\hat{E}^x$ to obtain a X^*-family of connections $\hat{\nabla}^x$ on $\hat{E}$. Generically we have

$$\nabla^{\lambda(x)} = \begin{pmatrix} \hat{\nabla}^x & 0 \\ 0 & \hat{\nabla}^{\chi(x)} \end{pmatrix}$$

with respect to the splitting $V = E^x \oplus E^{\chi(x)}$. Since ∇^λ is flat, so is $\hat{\nabla}^x$. In particular,

$$\deg \hat{E}^x = 0$$

as a bundle over the torus. Therefore we have the following theorem

Theorem 6.10
Let $f : T^2 \to S^3$ be a simple constrained Willmore torus. Then there is a hyperelliptic surface X and a holomorphic X^*-family of conncections $\hat{\nabla}^x$ on a family of rank 2 bundles $E \to T^2 \times X^*$ with $\deg \hat{E}_p = -2g_X - 2$, and such that $\hat{E}$ is uniquely a subbundle of $V = \mathbb{C}^4$. Let $\tilde{x}_1$ and $\tilde{x}_2$ be the points over 1 and $\underline{s}_1 \Gamma(\hat{E}^{x_1}) \subset \Gamma(V)$, $\underline{s}_2 \in \Gamma(\hat{E}^{x_2}) \subset \Gamma(V)$ be parallel frames of $\nabla^{\tilde{x}_1}$ and $\nabla^{\tilde{x}_2}$, respectively. Then the quotient of the projections $\pi(\underline{s}_1)$ and $\pi(\underline{s}_2)$ is a Möbiustransformation of f, where $\pi : V \to V/L$.

Remark
For a non-simple torus it is possible to define a similiar rank 2 bundle $\hat{E}$. Similiar to the CMC-case [Hit90], we would need to use a (singular) spectral curve which contains information to which order $\hat{\mathcal{L}}$ and $\chi^* \hat{\mathcal{L}}$ coalesce.

In order to be able to reconstruct the surface from the family of connections we need to be able to compare the fibres (over the sym points). Therefore we want to write down the family of connections $\hat{\nabla}^x$ on a trivial bundle. The first idea is to use the canonical projection

$$\pi : V \to V/L$$

to get a connection on V/L. As we will see we need to improve this approach for the general case. If the restriction $\pi_{|\hat{E}^x_p} : \hat{E}^x_p \subset V \to V/L_p$ is an isomorphism, we can define a connection

$$\pi \circ \hat{\nabla}^x \circ \pi^{-1}_{|E}$$

on V/L. At a point p where the holomorphic section spanning $\mathcal{L}^\xi$ has a zero, π will not be an isomorphism. Let us look at the case that all Darboux transforms parametrized by $\mathcal{L}$ are nonsingular. If neither $\mathcal{L}_p$ and $\sigma^* \mathcal{L}_p$ nor $\hat{\mathcal{L}}_p$ and $\sigma^* \hat{\mathcal{L}}_p$ coalesce in $\xi \in \Sigma$, then π is clearly an isomorphism. If both pairs coalesce in a branch point of $\Sigma \to X$ by first order, π is an isomorphism as well. Clearly coalescation of $\hat{\mathcal{L}}_p$ and $\sigma^* \hat{\mathcal{L}}_p$ implies coalescation of $\mathcal{L}_p$ and $\sigma^* \mathcal{L}_p$. The following lemma deals with the converse.

Lemma 6.11
If $\mathcal{L}^{\xi}_p = \mathcal{L}^{\tilde{\xi}}_p$ for all $p \in T^2$, then $\hat{\mathcal{L}}^{\xi}_p = \hat{\mathcal{L}}^{\tilde{\xi}}_p$. In particular, if f is simple then $\mathcal{L}^{\xi} = \sigma^*\mathcal{L}^{\xi}$ for all p implies that ξ is a branchpoint of $\Sigma \to X$.

PROOF
Let $\xi, \tilde{\xi}$ be points away from infinity and $\mathcal{L}^{\xi} = \mathcal{L}^{\tilde{\xi}}$ for all p. Then the holomorphic sections φ, ψ in $\widetilde{\mathcal{L}}^{\xi}$ and $\widetilde{\mathcal{L}}^{\tilde{\xi}}$ are complex linearly dependent, i.e. $\varphi = \psi\lambda$, for some $\mathbb{C}_*$ valued function λ on the universal cover $\mathbb{C}$ of T^2. We want to show that λ must be constant. Look at

$$0 = D(\psi\lambda) = D(\psi)\lambda + \frac{1}{2}(\psi d\lambda + S\psi * d\lambda) = \frac{1}{2}(\psi d\lambda + S\psi * d\lambda) \tag{53}$$

Assume $d\lambda \neq 0$. As λ is complex, (53) can only hold if $S\psi = \pm\psi\mathbb{i}$ and $d\lambda$ is either holomorphic or anti-holomorphic. If $d\lambda$ is non zero everywhere, we get, that ψ lies in the $\pm\mathbb{i}$ eigenspace everywhere, and so D is a complex connection with respect to S, i.e. $DS = SD$ (note that we are working on V/L , a quaternionic 1 dimensional bundle). This implies that the mixed structure on L, see Section 1.8, is complex as well, i.e., $A_{|L} = 0$. By [BFL+02, Lemma 22] this implies $A = 0$ which gives $A_0 = 0$. Therefore $A_{|L} \neq 0$ holds on an open set and the (anti-)holomorphic function λ must be constant there. But then λ has to be constant on whole $\mathbb{C}$. Therefore we can assume that $\psi = \varphi$ holds, in particular the monodromy satisfies $h = h^{-1}$. By the uniqueness of the prolongations, the line bundles $\hat{\mathcal{L}}^{\xi}$ and $\hat{\mathcal{L}}^{\tilde{\xi}}$ coalesce as well. □

6.3. The "$\mathrm{X} = \mathbb{C}\mathrm{P}^1$"-case

The easiest case is if the σ-quotient surface X is a sphere. The degree of the kernel bundle then implies that we can define the family of connections on V/L. By the Hitchin CMC-theory, cf. Section 5.2, f will then be a constant mean curvature torus in a 3-dimensional space form.

Theorem 6.12 (Theorem 7 of [Hel15])
Let $f : T^2 \to S^3$ be simple constrained Willmore Torus. If $\rho \circ \sigma$ has fixpoints and

$$X = \mathbb{C}P^1,$$

then f is a constant mean curvature Torus in a 3 dimensional space form.

PROOF
Take a global chart x of $X = \mathbb{C}P^1$ such that the ends $0, \infty \in \Sigma$ of the multiplier spectral curve are the points over 0 and ∞. As, by assumption, ρ has fixpoints on X, we can repeat the arguments from Example 6.4, and choose x, such that, ρ is given by

$$x \mapsto \bar{x}^{-1},$$

and the fixpoint set is given by $S^1 \subset \mathbb{C}P^1$.

Let ω be the symplectic form on $(V/L)_p \to \Sigma$, given by the identification of V/L and $\mathbb{H}$ by the section

$$\phi = \pi\left(\begin{pmatrix}1\\0\end{pmatrix}\right).$$

The evaluation of ω on $\mathcal{L}_p$ and $\sigma^*\mathcal{L}_p$ defines a holomorphic section

$$\omega \in H^0(\mathcal{L}_p^{-1} \otimes \sigma^*\mathcal{L}_p^{-1}), \tag{54}$$

denoted by the same symbol. The section ω vanishes in ξ if and only if $\mathcal{L}_p^\xi$ and $\sigma^*\mathcal{L}_p^\xi$ coincide, in particular in the $2g+2$ ramification points. Therefore

$$2\deg\mathcal{L}^{-1} = \deg\mathcal{L}^{-1} \otimes \sigma^*\mathcal{L}^{-1} \geq 2g_\Sigma + 2.$$

By Proposition 6.8 equality holds and there cannot be further points in which $\mathcal{L}_p$ and $\sigma^*\mathcal{L}_p$ coalesce. This does not depend on $p \in T^2$ since the branch points of $\Sigma \to X$ do not depend on p.

Using the CMC theory, see Theorem 5.20, there is a $\mathbb{C}^*$-family of $\mathrm{SL}(2,\mathbb{C})$-connections $\tilde{\nabla}^x$ on V/L of the form

$$\tilde{\nabla}^x = d + \frac{1}{2}(1+x^{-1})\alpha' + \frac{1}{2}(1+x)\alpha'', \tag{55}$$

such that $\mathcal{L}$ is the eigenline bundle. In Theorem 5.20, $\mathcal{L}$ is not already realised as a subbundle. But going through the proof of the theorem, see Section 8 of [Hit90], we can choose the realisation of $\mathcal{L}$. Also we can choose the $\hat{\nabla}^x$ parallel sections to be the holomorphic sections with monodromy spanning $\mathcal{L}$, since they are non-vanishing by Proposition 6.8.

Using Proposition 5.14, simplicity of f implies that the only holomorphic sections with trivial monodromy are the ones with constant prolongation. These are exactly the $\nabla^{\lambda=1}$ parallel sections. So the only trivial connections of $\tilde{\nabla}^x$ are the at most 2 trivial connections $\tilde{\nabla}^{\tilde{x}_1}$, $\tilde{\nabla}^{\tilde{x}_2}$, where $\tilde{x}_1$, $\tilde{x}_2$ satisfy $\lambda(\tilde{x}_1) = \lambda(\tilde{x}_2) = 1$. The Sym-Bobenko formulas (40) and (41) then give a CMC immersion $\tilde{f}$ in 3-dimensional space form . The map $\tilde{f}$ is

(a) CMC in S^3, if $\tilde{x}_1$, $\tilde{x}_2$ are fixpoints of ρ on X, i.e. $\tilde{x}_1, \tilde{x}_2 \in S^1 \subset \mathbb{C}P^1 = X$.

(b) CMC in H^3, if $\tilde{x}_1$, $\tilde{x}_2$ are no fixpoints. This can happen if the branchpoints of $X \to \mathbb{C}P^1$ are on S^1 and 1 is not on that part of S^1, where the fixpoints lie.

(c) CMC in $\mathbb{R}^3$, if $\tilde{x}_1 = \tilde{x}_2 \in S^1$ This happens if 1 is a branchpoint of $X \to \mathbb{C}P^1$.

By construction $\tilde{f}$ is given as gauge between holomorphic sections coming from constant sections. Since f is given as the gauge between $[1,0]$ and $[0,1]$ (or $([1,0],[j,0])$ and $([0,1],[0,j])$ if written complex) the maps f and $\tilde{f}$ only differ by a Möbius transformation. $\square$

Remark

By construction we have

$$\tilde{\nabla}^x = \pi\hat{\nabla}^x\pi^{-1}_{|\hat{E}}$$

for the family of connections $\hat{\nabla}^x$ on $\hat{E}^x$. This holds because the parallel sections of $\hat{\nabla}^x$ are the prolongations of the parallel sections of $\tilde{\nabla}^x$. The connection $\hat{\nabla}^x$ extends over the points at zero and infinity which are not the ends of Σ_{mult}, since $\tilde{\nabla}^x$ is defined there. This is not suprising since there E^x is spanned by sections lying in the kernel of $A^{(1,0)}$ or $A^{(0,1)}$, therefore the restriction of the limit "$\nabla^\infty = d + \infty A^{(1,0)} - A_0^{(0,1)}$" makes sense.

6.4. The general case

If X has genus higher than 0, the bundle V/L is not given as the push forward of $\mathcal{L}$. Assume that $\deg \mathcal{L}_p = g_\Sigma + 1$. Then the holomorphic section $\omega \in H^0(\mathcal{L}_p^{-1} \otimes \sigma^* L_p^{-1})$ from (54) has $2g_\Sigma + 2$ many zeroes. But there are only $2g_\Sigma - 4g_X + 2$ branchpoints of $\Sigma \to X$. Therefore there need to be $4g_X$ additional points (counted with multiplicity) in which ω has a zero, i.e., where $\hat{\mathcal{L}}_p$ and $\sigma^*\hat{\mathcal{L}}_p$ coalesce. By Lemma 6.11 these points need to be p-dependent or lie over 0 and ∞. Via π the connection $\hat{\nabla}^x$ induces connections on V/L. If the lines $\hat{\mathcal{L}}^\xi$ and $\hat{\mathcal{L}}^{\sigma(\xi)}$ coalesce in $p \in T^2$ (and are generically distinct) the induced connection $\nabla^{x(\xi)}$ has a pole in p. The basic idea we are following is, that it should be possible to gauge the ξ-dependant poles in p to poles in ξ. For this reason the Main Theorem starts with a family of connections, with additional poles in ξ, and then defines a constrained Willmore immersion via a Sym-Bobenko formula. Firstly we will workout some further properties of the bundle $\hat{E}$.

Proposition 6.13
Denote by $x_1, \dots x_{2g_X+2} \in X$ the branch points of $X \to \mathbb{C}P^1$. The determinant bundle $\Lambda^2 \hat{E}_p$ of $\hat{E}_p$ satisfies

$$\Lambda^2 \hat{E}_p^* \otimes \Lambda^2 \chi^* \hat{E}_p^* = L((4g_X + 4)x_1) = L(2x_1 + \cdots + 2x_{2g+2}).$$

Proof

$$\Lambda^2 \chi^* \hat{E}^* = \chi^* \Lambda^2 \hat{E}^*$$

implies that the bundle $L = \Lambda^2 \hat{E}^* \otimes \Lambda^2 \chi^* \hat{E}^*$ is χ-invariant. Therefore, using

$$\deg \hat{E}^* = 2g_X + 2,$$

the bundle L is given by the pullback of a linebundle of degree $2g_X + 2$. □

Proposition 6.14
It even holds

$$\Lambda^2 E_p^* = L(x_1 + \cdots x_{2g_X+2})$$

Proof
It is

$$\Lambda^2 E^* \wedge \Lambda^2 \chi^* E = \Lambda^4 \mathbb{C}^4.$$

Therefore there are $\omega \in \Gamma(\Lambda^2 E^*)$ and $\eta \in \Gamma(\Lambda^2 \chi^* E^*)$ such that

$$\omega \wedge \eta = \det$$

using that det is χ-invariant we get

$$\eta = \chi^* \omega$$

as wished. $\square$

Denote by $\det_{\hat{E}}$ the holomorphic section of $\Lambda^2 \hat{E}_p^*$ with simple poles in the x_i. By parallel transport with $\hat{\nabla}^x$ we can extend $\det_{\hat{E}}$ on T^2.

The idea is to trivialize $\hat{E}$ by choosing a meromorphic frame F^x with $\det_{\hat{E}}(F^x) = 1$. This frame will have poles in the x_i, therefore the connection ∇^x on $\mathbb{C}^2$, induced by $\hat{\nabla}^x$, has singularities there. Further we have the quaternionic structure $\mathbb{j}$ on $\mathbb{C}^4$ which defines an anti-linear map from $\hat{E}^x$ to $\hat{E}^{\rho(x)}$. For a fixpoint x, the bundle $\hat{E}^x$ is preserved by the multiplication with $\mathbb{j}$. By this we obtain a complex metric on $\hat{E}^x$ by

$$(\cdot, \cdot) = \det_{\hat{E}}(\mathbb{j}\cdot, \cdot).$$

If we choose F_x such that F_x is orthonormal with respect to $(\ ,\ )$, then ∇^x will be an unitary connection at the fixpoints of ρ.

Philosophically speaking, the associated family of a constrained Willmore torus is the pushforward by the 4-fold covering $\Sigma \to \mathbb{C}P^1$ of a family of linebundle connections, such that the obtained family is a holomorphic family. For a CMC torus the pushforward to the punctured quotient surface $X^* = X \setminus \{0, \infty\}$ is already a holomorphic family of connections. For a cW torus we only have a holomorphic family on a negative degree bundle. Therefore we can only expect to obtain a meromorphic family of connections, by taking a global frame.

We will not deeper investigate this trivialisation approach and instead try to reconstruct cW tori. First we will combine the main results of [Boh10] and [BLPP12] to obtain the following Theorem.

Theorem 6.15
Let ∇^λ be a $\mathbb{C}^*$ family of flat $\mathrm{SL}(4,\mathbb{C})$ connections of the form

$$\nabla^\lambda = d + (\lambda - 1)\Phi + (\lambda^{-1} - 1)\Psi, \tag{56}$$

with $\Phi \in \Gamma(K\mathfrak{sl}(4,\mathbb{C}))$ and $\Psi = \mathbb{j}^{-1}\Psi\mathbb{j} \in \Gamma(\bar{K}\mathfrak{sl}(4,\mathbb{C}))$, where $\mathbb{j}$ is a quaternionic structure such that $\mathbb{C}^4 = \mathbb{H}^2$. Let Σ be the spectral curve of ∇^λ, which is a 4-fold cover of $\mathbb{C}P^1$ and let $\hat{\mathcal{L}}_p$ be the eigenlinebundle. Let the image of Φ and Ψ be complex 1-dimensional. Equivalently there are 2 points $0, \infty \in \Sigma$ lying over $0, \infty$ such that the monodromy or eigenvalue function has only singularities in 0 and ∞ and there essential singularities.

Then if

$$L = \mathrm{Image}(\Phi + \Psi)$$

defines a conformal surface in S^4, then L is constrained Willmore.

Remark
The exact form (56) of the family ∇^λ can be replaced by the asymptotics, the symmetry (47)

$$\nabla^{\bar{\lambda}^{-1}} = \mathbb{j}^{-1}\nabla^\lambda \mathbb{j}, \tag{47}$$

and the triviality of ∇^1. Let ∇^λ be a $\mathbb{C}^*$ family of connections with simple pole in 0 and ∞, such that the residue at 0 is a $\bar{K}$-form. Then

$$\nabla^\lambda = \tilde{\nabla} + \lambda^{-1}\Psi + a(\lambda, p)$$

for $\Psi \in \Gamma(\bar{K}\mathfrak{sl}(4,\mathbb{C}))$ and some a varying holomorphically in λ. The constant part of a can be included in $\tilde{\nabla}$, therefore we can assume $a = \sum_{k=1}^\infty \lambda^k \phi_k$. By the symmetry (47), and since ∇^λ has a pole of first order in ∞ we get

$$\nabla^\lambda = \tilde{\nabla} + \lambda^{-1}\Psi + \lambda\Phi,$$

with

$$\Psi = \mathbb{j}^{-1}\Phi\mathbb{j}.$$

As ∇^1 is trivial we can trivialize V using this connection and obtain the form (56). Then $\mathbb{j}$ becomes a multiplication with a constant and we get $V = \mathbb{H}^2$ with $\mathbb{j}$ a unitary quaternion anticommuting with $\mathbb{i}$.

Proof (of Theorem 6.15)
In this proof we will only look at the Willmore case. The general case is computationally more evolved. Set $B_0 = \Phi + \Psi$. We want to show that B_0 is strongly related to the Hopf field A of L. A ∇^λ parallel section Ψ is a section with monodromy that satisfies

$$d\Psi \in \Omega^1(L).$$

Therefore it defines a Darboux transform of L. This defines a map

$$\iota : \Sigma \to \Sigma_{mult}$$

from the spectral curve of ∇^λ to the multiplier spectral curve of L, which is well defined as L is conformal. Let F be the map from Theorem 5.12. By construction

$$F(p, \iota(\xi)) = \hat{\mathcal{L}}_p^\xi.$$

As the monodromy has a singularity for $\xi \to \infty$ on Σ, we see that $\iota(\xi)$ needs to go to one of the ends of the multiplier spectral curve as well. By Theorem 5.12 the limit of the eigenlines is then given by the $\pm\mathbb{i}$ eigenspaces (contained in L) of the mean curvature sphere S of L. On the other hand, $\mathcal{L}$ approaches the image $\operatorname{im}\Psi$ when $\xi \to 0$ and $\operatorname{im}\Phi$ when going to ∞. Therefore we have

$$*\Phi = S\Phi \quad \text{and} \quad *\Psi = S\Psi,$$

i.e, B_0 is a K-form with respect to S.

The family ∇^λ is flat, therefore, cf. the proof of Theorem 5.22,

$$0 = F^{\nabla^\lambda} = (\lambda - 1)(d^\nabla \Phi - B_0 \wedge B_0) + (\lambda^{-1} - 1)(d^\nabla \Psi - B_0 \wedge B_0).$$

This implies

$$d * B_0 = 0.$$

We split

$$B_0 = B + *\eta$$

into an S anticommuting part $B = B_0 + SB_0S$ and a S commuting part $*\eta = B_0 - SB_0S$. We want to show that

$$B = cA$$

for some real constant c. Then we have

$$d^\nabla \left(2 * A + \frac{2}{c}\eta\right) = 0,$$

i.e., A satisfies the Euler Lagrange equation (30) and therefore L is indeed constrained Willmore. As mentioned, we will here only discuss the Willmore case, i.e., $\eta = 0$ and $d * B = 0$. Using affine coordinates, i.e., with respect to the frame

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} f \\ 1 \end{pmatrix},$$

we have

$$*B = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}.$$

As seen in Section 1.5, the mean curvature sphere is given by

$$S = \begin{pmatrix} N & 0 \\ -H & -R \end{pmatrix},$$

with N and R the left, respectively right, normal vector of f. By $B \in \Gamma(K\mathrm{End}(V)_-)$, we get

$$*b_1 = -Rb_1 = -b_1 N + b_2 H \quad \text{and} \quad * b_2 = -Rb_2 = b_2 R. \tag{57}$$

Using the same arguments as in Section 1.5 we have

$$0 = d * B = \begin{pmatrix} 0 & df \\ 0 & 0 \end{pmatrix} * B + d \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix} + *B \begin{pmatrix} 0 & df \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} df \wedge b_1 & df \wedge b_2 \\ db_1 & db_2 + b_1 \wedge df \end{pmatrix}. \tag{58}$$

The upper part vanishes as B is a K-form. The second equation of (57) implies, using the Fundamental lemma (see lemma 1.5), that b_2 takes values in the orthogonal space of R in $\mathrm{im}(\mathbb{H})$. Therefore, we can write

$$b_2 = e\, dR + f\, * dR + gRdR + hR * dR,$$

for $\mathbb{R}$ valued functions e, f, g, h. Again using (57) we get

$$b_2 = (e + fR)(dR + R * dR).$$

By (10), or [BFL+02, (7.15)], the Hopf field is given by

$$*A = \frac{1}{4}\begin{pmatrix} 0 & 0 \\ dH + R * dH + \frac{1}{2}(NdN - *dN) & dR + R * dR \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a_1 & a_2 \end{pmatrix}.$$

Let $d = \hat{\nabla} + A + Q$ be the splitting of d in $+$ and $-$ parts with respect to S. Then we have

$$\begin{aligned} 0 = d * B &= d^{\hat{\nabla}} * B + [A \wedge *B] + [Q \wedge *B] \\ &= Sd^{\hat{\nabla}} B + (A \wedge *B + *B \wedge A) \\ &= \underbrace{Sd^{\hat{\nabla}} B}_{-} + \underbrace{2 * B * A - 2 * A * B}_{+}. \end{aligned} \tag{59}$$

The term $[Q \wedge *B]$ vanishes by type considerations. The $\pm$ parts have to vanish individually, therefore

$$\begin{pmatrix} 0 & 0 \\ b_2 a_1 & b_2 a_2 \end{pmatrix} = *B * A = *A * B = \begin{pmatrix} 0 & 0 \\ a_2 b_1 & a_2 b_2 \end{pmatrix}. \tag{60}$$

As a_2 anti-commutes with R, we deduce $f = 0$, i.e.,

$$b_2 = e(dR + R * dR).$$

Then

$$a_2 b_1 = b_2 a_1 = 4e a_2 a_1$$

implies

$$b_1 = 4e a_1.$$

As $d * A \in \Omega^2(\mathcal{R})$, we have $da_2 + a_1 \wedge df = 0$ and therefore the lower right part of (58) gives

$$0 = db_2 + b_1 \wedge df = 4e(da_2 + a_1 \wedge df) + 4de \wedge a_2 = 4de \wedge a_2.$$

Therefore e must be constant and L is Willmore. □

As mentioned above, the idea is to trivialize the bundle $\hat{E}^x$ by a global meromorphic frame. Doing this we get a meromorphic family of flat connections ∇^x on the trivial $\underline{\mathbb{C}}^2$ bundle. If the frame is compatible and $\lambda = 1$ is not a ramification point of $X \to \mathbb{C}P^1$ the family comes with some properties.

Definition 6.16
Let X be a hyperelliptic surface with marked blue points 0 and ∞ and marked green points $\tilde{x}_1$, $\tilde{x}_2$, see Definition 6.3. We call a meromorphic X-family of connections ∇^x on $\mathbb{C}^2 \to T^2$ admissible, if it satisfies the following conditions.

(i) **Extrinsic closing:** The connections $\nabla^{\tilde{x}_1}$ and $\nabla^{\tilde{x}_2}$ are trivial, where $\tilde{x}_1, \tilde{x}_2$ are the points over $\lambda = 1$.

(ii) **Intrinsic closing:** The involution ρ has fixpoints and ∇^x is unitary for fixpoints x of ρ.

(iii) **Symmetry:** The family satisfies

$$\nabla^{\rho(x)} = \mathbb{j}^{-1} \nabla^x \mathbb{j}, \tag{61}$$

where $\mathbb{j}$ is a quaternionic structure on $\mathbb{C}^2 = \mathbb{H}$.

(iv) **Assymptotics:** The family has a simple pole in 0 and $\infty \in X$, such that with respect to a centered chart η at ∞,

$$\nabla^\eta = \nabla + \eta^{-1}\phi + \eta\psi_1 + \cdots \tag{62}$$

for some nilpotent nonvanishing $\phi \in \Gamma(K\mathfrak{sl}(2,\mathbb{C}))$, and such that the $(1,0)$-part of the terms of higher order ψ_i are linear dependant on ϕ.

(v) The family has singularities in the branchpoints $x_1 \dots x_{2g_X+2}$ of $X \to \mathbb{C}P^1$ which can be removed by going to a bundle $\hat{E}^x$ of degree $-2g_X - 2$. The rank 2 bundle $\hat{E}$ is quaternionic with respect to ρ, i.e., there is an isomorphism $\mathbb{j} : \overline{\rho^* \hat{E}} \to \hat{E}$ which squares to -1. Further the pushforward of $\hat{E}^*$ by $\lambda : X \to X/\chi = \mathbb{C}P^1$ shall be trivial.

Definition 6.17
Analogous to Definition 5.13 we call a family ∇^x of connections simple if the T^2-family of eigenlines of the holonomy has a unique, up to Möbius transformations, realisation as smooth family of holomorphic subbundles in $\mathbb{C}^4$.

The main theorem is now that a constrained Willmore immersion can be reconstructed from such families of connections.

Main Theorem 6.18
Let X be a hyperelliptic surface with marked blue points 0 and ∞ and marked green points $\tilde{x}_1$ and $\tilde{x}_2$ and ∇^x be a simple admissible meromorphic X-family of flat connections on a torus T^2 with parallel frames F_x, see Definitions 6.3, 6.16, and 6.17. Then

$$f = F_{\tilde{x}_2}^{-1} F_{\tilde{x}_1}$$

is a constrained Willmore tori in S^3.

Proof
We will first show that ∇^x defines a constrained Willmore immersion by the push forward construction. As in Example 6.4 we choose a chart λ of $\mathbb{C}P^1 = X/\chi$, such that ∞ and $0 \in X$ lie over $\infty, 0 \in \mathbb{C}P^1$, respectively, and $\lambda(\tilde{x}_1) = \lambda(\tilde{x}_2) = 1$. By assumption the push forward of the dual bundle of $\hat{E}$ is trivial. Therefore the dual bundle

$$V_p = \left(\lambda_* \left(\hat{E}_p\right)^*\right)^*,$$

where $\hat{E}_p$ is canonically a subbundle, see Section 3.3, is trivial as well. Away from branchpoints we have

$$V_p = \hat{E}_p^x \oplus \chi^* \hat{E}_p^x, \tag{63}$$

where x is a point over $\lambda \in \mathbb{C}P^1$. The connection

$$\nabla^\lambda = \begin{pmatrix} \nabla^x & 0 \\ 0 & \nabla^{\tilde{x}} \end{pmatrix},$$

with respect to the splitting (63), is then a flat connection. By the extrinsic closing condition ∇^1 trivialises V along T^2.

By the push forward construction the connections extend to the ramification points. Let λ_0 be a branchpoint and y a centered chart at the point x_0 over λ_0 with $y^2 = (\lambda - \lambda_0)$. Further let $\underline{s}^x$ and $\underline{\tilde{t}}^x = \chi^*(\underline{s}^x)$ be a local family of ∇^x-parallel frames holomorphically depending on x. The connection ∇^{λ_0} is then the unique connection such that the frame

$$\underline{t} = \left(\underline{s} + \underline{\tilde{s}}, \frac{1}{y}(\underline{s} - \underline{\tilde{s}})\right)$$

of V is parallel at λ_0.

The assymptotics: By construction the residue of ∇^λ in ∞ is a K-form and has a 1 dimensional image, as otherwise there would be a second line where the eigenvalue has a singularity. Denote this K-form by Φ. By construction Φ is trace free. Denote the residue at 0 by Ψ. Then the family is given by

$$\nabla^\lambda = \nabla + \lambda\Phi + \lambda^{-1}\Psi.$$

By the triviality of ∇^1 we get

$$\nabla^\lambda = d + (\lambda - 1)\Phi + (\lambda^{-1} - 1)\Psi.$$

The symmetry: The quaternionic structure $\mathbb{j}$ on $\mathbb{C}^2$ defines a anti-holomorphic map $\mathbb{j} : \hat{E}^x \to \hat{E}^{\rho(x)}$ which makes $\hat{E}$ an quaternionic bundle, i.e., $\overline{\rho * (\hat{E}_p)} \cong \hat{E}_p$ with an isomorphism that squares to -1. In particular, the space of holomorphic sections $H^0(\hat{E}_p^*)$ is a quaternionic space with quaternionic structure compatible with $\mathbb{j}$. This implies that there is a $\mathbb{j}_V : V \to V$ which is compatible with $\mathbb{j}$ on $\hat{E}^x$, i.e., for $\psi \in \hat{E}_p^x$

$$\mathbb{j}_V(\psi) = \mathbb{j}_x(\psi) \in \hat{E}^{\rho(x)}.$$

In other words the map $\mathbb{j}^x : V \to V$ defined by $\mathbb{j}_x : \hat{E}_p^x \to \hat{E}_p^{\rho(x)}$ via the splitting (63) does not depend on x. As $\nabla^1 = d$, we can view $\mathbb{j}$ as right multiplication with the quaternion $\mathbb{j}$. We have

$$\nabla^{\bar{\lambda}^{-1}} = \mathbb{j}^{-1}\nabla^\lambda\mathbb{j},$$

and therefore

$$\Psi = \mathbb{j}^{-1}\Phi\mathbb{j}.$$

By Theorem 6.15, the twistor projection L of the eigenline $\hat{\mathcal{L}}$ at ∞ gives a constrained Willmore torus. We still need to show that L is the same map as the one given by the gauge between $\nabla^{\tilde{x}_1}$ and $\nabla^{\tilde{x}_2}$.

The assymptotics (62) of ∇^x imply, by similiar arguments as in [Hit90, Section 7], that the eigenline $\mathcal{L}^x$ of ∇^x goes to $\ker\phi$ when $x \to \infty$. The symmetry (61) then ensures that

$$\mathcal{L}^x \longrightarrow \ker \mathbb{j}^{-1}\phi\mathbb{j} \quad \text{for} \quad x \longrightarrow 0 \in X.$$

Let $\tilde{J}$ be the unique complex structure with $\ker\phi$ as $\mathbb{i}$ eigenspace and $\ker\phi\mathbb{j}$ as $-\mathbb{i}$ eigenspace. Let x be a fixpoint of ρ and $D = \nabla^{x''}$ with respect to $\tilde{J}$. The holomorphic structure does not depend on x, since the $(0,1)$-part of ∇^x (as a complex connection) changes by a $\mathrm{Hom}(\ker\phi, \ker\phi\mathbb{j})$ valued 1-form and the difference of the $(1,0)$-part takes values in $\mathrm{Hom}(\ker\phi\mathbb{j}, \ker\phi\mathbb{j})$. By the unitarity both parts are conjugated and therefore D is preserved.

All ∇^x-parallel sections define D-holomorphic sections with monodromy. The gauge between ∇^{x_1} and ∇^{x_2} is given by the quotient of 2 holomorphic sections φ_1, φ_2 (given by a parallel section of ∇^{x_i}), i.e., $\varphi_1 = \varphi_2 f$. By

$$p \in T^2 \mapsto \begin{bmatrix} f(p) \\ 1 \end{bmatrix} \in \mathbb{H}P^1,$$

f defines a conformal map such that $\mathbb{C}^2 = V/L$ and $D = \pi \circ d''$.

By construction the eigenlines of ∇^λ share the same spectral curve as well as the same subset in the Jacobian defined by the eigenline bundle as well as the kernel bundle. As ∇^x is simple, the map F from Theorem 5.12 is unique and must be the one (up to a Möbius transformation) that parametrizes the eigenlines of ∇^λ. Therefore f and L only differ by a Möbius transformation. □

6.5. A return to the CMC case

We will now again take a look at the CMC-case. We will investigate how the rank 4 family ∇^λ look like, and especially look at the case where there are no fixpoints of ρ on X. Therefore we will see, how the constrained Willmore theory includes both, the CMC theory of Hitchin [Hit90], and Bobenko [Bob91b] as well as the Babich-Bobenko tori [BB93], obtained by glueing CMC surfaces of mean curvature $H < 1$ over the boundary of H^3.

Let ∇^λ be the rank 4 family of flat connections associated to a simple constraint Willmore torus in S^3, such that $X = \mathbb{C}P^1$. As in the proof of Theorem 6.12, we choose a coordinate x on X, such that $x = 0, \infty$ are the points belonging to the ends of the spectral curve lying over $\lambda = 0, \infty$, respectively. Further, if ρ has fixpoints, x is chosen in a way such that ρ is given by

$$x \mapsto \bar{x}^{-1}.$$

If ρ has no fixpoints on X, then $\chi \circ \rho$ has fixpoints, and we can choose x such that

$$\rho \circ \chi(x) = \bar{x}^{-1}.$$

In both cases $S^1 \subset X$ consists of points lying over $S^1 \subset \mathbb{C}P^1$. Firstly, we will look at the easiest case, where the branchpoints of $X \to \mathbb{C}P^1$ are given by $x = 0, \infty$. The hyperelliptic involution, commuting with ρ or $\rho \circ \chi$ and fixing $x = 0, \infty$, is then given by

$$\chi : x \mapsto -x.$$

As $S^1 \subset X$ lies over $S^1 \subset \mathbb{C}P^1$, we obtain $x^2 = a^2\lambda$, for some $a \in S^1$. After a rotation

$$x^2 = \lambda$$

and the points over $\nabla^\lambda = 1$ are $x = \pm 1$, i.e., the trivial connections of the induced family ∇^x are $\nabla^{\pm 1}$. In particular, f is (Möbius equivalent to) a minimal torus in S^3. If ρ has fixpoints, we have seen in Theorem 6.12, that there is a global frame of $\hat{E}$ on $X \setminus \{0, \infty\}$ such that $\hat{\nabla}^x$ is given by

$$\hat{\nabla}^x = d + \omega(x),$$

with $\omega(x) = x^{-1}\omega_{-1} + \omega_0 + x\omega_1$, and $*\omega_{\pm 1} = \pm i\omega_{\pm 1}$. This does hold as well for the fixpointfree case, as the kernel bundle $\mathcal{L}_p$ still is a linear family of quaternionic line bundles with $\deg(\mathcal{L}_p) = g_\Sigma + 1$. Therefore the arguments of [Hit90, Section 7 and 8] are still valid and the pushforward defines a family of connections with the above asymptotics. As we assume f to be simple, the bundles $\hat{E}$ and $\chi^*\hat{E}$ do only intersect in branch points. Therefore, away from 0 and ∞, we do have $\mathbb{C}^4 = \hat{E}^x \oplus \hat{E}^{-x}$. Hence ∇^λ is gauge equivalent to

$$\tilde{\nabla}^x = d + \begin{pmatrix} \omega(x) & 0 \\ 0 & \omega(-x) \end{pmatrix}$$

with $x^2 = \lambda$. The involution ρ is compatible with the multiplication with the quaternion $\mathbb{j}$. If ρ has fixpoints this implies, that we can get the symmetry

$$\hat{\nabla}^x = -\mathbb{j}\bar{\hat{\nabla}}^{\bar{x}^{-1}}\mathbb{j},$$

in particular, $d + \omega(x)$ and $d + \overline{\omega(\bar{x}^{-1})}$ are gauge equivalent. When ρ has no fixpoints we obtain that

$$d + \omega(-x) \qquad \text{and} \qquad d + \overline{\omega(\bar{x}^{-1})}$$

are gauge equivalent as $\chi \circ \rho$ has fixpoints. The Babich-Bobenko tori [BB93] with $H = 0$ are of the latter type, see [HHN19, Proposition 3].

Let now $\lambda_0 \neq 0, \infty$ and $\bar{\lambda}_0^{-1}$ be the ramification points of $X \to \mathbb{C}P^1$. Again we choose a coordinate x on X such that ρ, or $\rho \circ \chi$ in the fixpoint free case, is given by $x \mapsto \bar{x}^{-1}$, and $x = 0, \infty$ are the points belonging to the ends of the spectral curve, lying over $\lambda = 0, \infty$. The hyperelliptic involution χ commutes with ρ and is therefore given by

$$\chi(x) = \frac{x - a}{\bar{a}x - 1}$$

for some $a = \chi(0) \in \mathbb{C}_* \setminus S^1$. If $|a| = 1$, χ maps to one point only and hence does not define an involution. The fixpoints of χ are given by

$$x_1 = \frac{1 + \sqrt{1 - a\bar{a}}}{\bar{a}} \qquad \text{and} \qquad x_2 = \frac{1 - \sqrt{1 - a\bar{a}}}{\bar{a}}.$$

If $|a| > 1$ they lie on S^1, and for $|a| < 1$ we obtain $x_1 = \bar{x}_2^{-1}$. The function λ can be seen as a map on X with poles in $x = \infty$ and $x = \bar{a}^{-1} = \chi(\infty)$, and zeroes in $x = 0$ and $x = \chi(0) = a$. Therefore

$$\lambda(x) = c\frac{x(x-a)}{\bar{a}x - 1} = c\, x\, \chi(x),$$

for some constant c. As $S^1 \subset X$ is preserved by χ, and consists of points lying over $\lambda \in S^1 \subset \mathbb{C}P^1$, the constant must be unitary as well, i.e., $c \in S^1$. After a rotation of the x coordinate we obtain

$$\lambda(x) = x\,\chi(x) = x\frac{x-a}{\bar{a}x-1}.$$

The fixpoints x_1, x_2, and hence a, are (up to a sign) determined by

$$x_1^2 = \lambda\left(x_1\right) = \lambda_0 \qquad \text{and} \qquad x_2^2 = \lambda(x_2) = \bar{\lambda}_0^{-1}.$$

As the constrained Willmore torus is simple, the CMC-theory again implies, that the connection $\hat{\nabla}^x$ is again gauge equivalent to

$$\nabla^x = d + \omega(x)$$

with

$$\omega(x) = x^{-1}\omega_{-1} + \omega_0 + x\omega_1 \in \Omega^1(\mathfrak{sl}(2,\mathbb{C}),$$

such that $d + \omega\left(x_i\right)$ is trivial, for $i = 1, 2$, and $*\omega_{\pm 1} = \pm i\omega_{\pm 1}$. Further the bundles $\hat{E}^x$ and $\hat{E}^{\chi(x)} \subset \mathbb{C}^4$ do only intersect in x_1, x_2. Hence on $\mathbb{C}_* \setminus \{\lambda_0, \bar{\lambda}_0^{-1}\}$ the connection ∇^λ is gauge equivalent to

$$\tilde{\nabla}^x = d + \begin{pmatrix} \omega(x) & 0 \\ 0 & \omega\left(\frac{x+a}{\bar{a}x+1}\right) \end{pmatrix}$$

with $\lambda = \dfrac{x(x-a)}{\bar{a}x-1}$. As before, dependant on whether ρ has fixpoints or not,

$$d + \omega(x) \qquad \text{and} \qquad d + \overline{\omega(\bar{x}^{-1})},$$

or

$$d + \omega\left(\frac{x+a}{\bar{a}x+1}\right) \qquad \text{and} \qquad d + \overline{\omega(\bar{x}^{-1})}$$

are gauge equivalent, respectively.

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