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by
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Abstract

We consider the impact of sequential investment and active management on the value of a portfolio of real options. The options are assumed to be interdependent, in that exercise of any one is assumed to produce, in addition to some intrinsic value based on an underlying asset, further information regarding the values of other options based on related assets. We couch the problem in terms of oil exploration, where a discrete number of related geological prospects are available for drilling, and management's objective is to maximize the expected value of the combined exploration campaign. Management's task is complex because the expected value of the investment sequence depends on the order in which options are exercised.

A basic conclusions is that, although dependence increases the variance of potential outcomes, it also increases the expected value of the embedded portfolio of options and magnifies the value of optimal management. Stochastic dynamic programming techniques may be used to establish the optimal sequence. Given certain restrictions on the risk structure, however, we demonstrate that the optimal dynamic program can be implemented by policies that are relatively simple to execute. In other words, we provide sufficient conditions for the optimality of intuitive decision rules, like "biggest first," "most likely first," or "greatest intrinsic value first," and we develop exact analytic expressions for the implied value of the portfolio. This permits the value of active management to be assessed directly. Finally, the sufficient conditions we identify are shown to be consistent with plausible exploration risk structures.

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Managing a Portfolio of Real Options:
Sequential Exploration of Dependent Prospects

1. Introduction

We consider the impact of sequential investment and active management on the value of a portfolio of real options. The options are assumed to be dependent, in that exercise of any one is assumed to produce, in addition to some intrinsic value based on its underlying asset, further information regarding the values of other options based on related assets. We take the values of the underlying assets to be positively related; a high value on any one tends to increase the likelihood of high values elsewhere. Valuation of such portfolios is complex in that the combined value of the entire portfolio may depend on the order in which options are exercised, and the optimal order is not always obvious (and sometimes counterintuitive) when the number of options exceeds two.

As a frame of reference, we couch the problem in terms of oil exploration, where a discrete number of related geological "prospects" are available for drilling and management's objective is to maximize the net present value of the entire exploration campaign. Such prospects typically differ in size and probability of success, and are said to be "dependent" or "associated" if success on one increases the conditional probability of success on others.[1] Each prospect represents a real option, which if successfully exercised (via drilling) conveys the intrinsic value of the underlying oil, plus information regarding the value of remaining prospects. How much should management be willing to pay to acquire such a portfolio? Certainly more than the sum of the intrinsic values, because that measure ignores the value created by using intervening information to

[1] Tong (1980, pp. 78-90) discusses the "association" of random variables (a property equivalent to "positive quadrant dependence" in the bivariate case) and reviews many of the relevant statistical implications.

actively manage the exploration sequence. Holding all else equal, the existence of dependence among prospects adds value to the portfolio of options.

Our results highlight an important difference between real option applications and the standard financial option paradigm: in many applications of real options, the value of the underlying asset will not be revealed until after the option has been exercised. The true value of the asset will often depend on additional sources of uncertainty that can be resolved only through investment and exploitation of the asset. Paddock, Siegel, and Smith (1988) demonstrated in the case of petroleum exploration (where the number of "shares" to be acquired via drilling an exploratory well is uncertain) that, where only a single asset is involved, the basic analogy to financial options is preserved and standard techniques based on the risk-neutral valuation principle may be applied. When several dependent assets are involved, however, the valuation problem changes in a fundamental way. The flow of available information is endogenized—subject to managment's decision to exercise one option that could reveal information regarding the values of others. The flow of information has to be managed in concert with investment in the underlying assets, and the value of the portofolio as a whole will reflect managment's skill in combining these two functions.

A simple illustration shows the importance sequencing dependent investments optimally. Consider three prospects, each requires the expenditure of \$80 million to test, and returns a gross value of \$100 million if successful. Joint and marginal probabilities of success for the three prospects are shown below:

$$p_1 = .820$$
$$p_2 = .810$$
$$p_3 = .803$$

$$
\begin{array}{ll}
p_{1\cap 2\cap 3} = .640 & p_{1\cap 2\cap \bar{3}} = .080 \\
p_{1\cap \bar{2}\cap 3} = .078 & p_{\bar{1}\cap 2\cap 3} = .075 \\
p_{1\cap \bar{2}\cap \bar{3}} = .022 & p_{\bar{1}\cap \bar{2}\cap 3} = .010 \\
p_{\bar{1}\cap 2\cap \bar{3}} = .015 & p_{\bar{1}\cap \bar{2}\cap \bar{3}} = .080
\end{array}
$$

Prospect 1 is the most likely, and prospect 3 the least likely to succeed. It is nonetheless optimal to test the third prospect first because it generates valuable information that more than compensates for its lesser intrinsic value. The value of the portfolio if prospect 3 is tested first amounts to $15.12 million.[2] In contrast, the value of the portfolio if all three prospects are tested simultaneously, or without regard for intervening outcomes, amounts to only $3.3 million. Thus, dependence among individual prospects quadruples the value of this portfolio, but only if the investments are made in proper sequence and the resulting information acted upon in an optimal manner.

Subject to certain regularity conditions, stochastic dynamic programming techniques may be applied to identify the optimal order of trials in problems of this type—and to ascertain portfolio value. That approach relies heavily on computational power but does not contribute much economic insight regarding the elements of a successful sequential investment strategy. Of course, as the size of the portfolio grows, dynamic programming sollutions impose ever larger computational demands and information requirements, as well.[3]

We show that, given certain plausible constraints on the structure of dependent risks, the solution to this portfolio management problem reduces to a form that is much simpler and easier for management to execute. In the extreme, for example, is the case where the value of each asset is independent of the others, which implies that the value of

[2] This value is obtained by solving a simple decision tree.
[3] For example, see Haugen (1996) and Jorgenson (1999).

the portfolio is independent of the order in which the options are exercised. But that example throws out the baby with the bath water. Our goal is to identify, where possible, simple rules for managing a portfolio of *dependent* options.

We also show that the risk structure most commonly used to describe dependent petroleum exploration outcomes is sufficient for the optimality of simple decision rules, like “biggest first,” “most likely first,” or “greatest intrinsic value first”; and we develop exact analytic expressions for the value of the portfolio in such cases. This permits the incremental value of active management to be assessed simply and directly.

We are just as much interested in establishing the limits beyond which simplified decision rules would fail to optimize the value of the portfolio. Positive association is not, by itself, sufficient for our results—as the preceding example already demonstrated. A special form of association among the underlying assets is required to achieve much in the way of simplification.

2. Related Literature

Relatively few papers have considered the impact of sequential investment and project interdependence on the value of a portfolio of real options. Trigeorgis (1993) was among the first to consider the implications of interdependence and establish the non-additivity of real option values. His analysis, however, pertains to a collection of options all written on the same underlying asset, whereas we have in mind applications where distinct assets underlay each option in the portfolio. In addition, the sequence in which options might be exercised is predetermined in his analysis, whereas flexibility in determining this sequence is paramount in our study.

Vishwanath (1992) derives, as we do, sufficient conditions for the application of relatively simple rules to solve the problem of optimal sequential investment in a collection of projects. She also shares our view regarding the inherent practical limitations of the dynamic programming approach, which in her words is likely to shed "little economic insight (and) would be a complex brute force task." Unlike us, however, she confines her analysis to projects whose payoffs are mutually independent, all of which must be exercised.[4] Thus, two crucial aspects of our framework are missing in her work.

Cortazar, Schwartz, and Casassus (2001) investigate the impact of geologic and price risk on the value of a collection of interrelated natural resource options. As in Trigeorgis (1993), however, the analysis pertains to multiple options written on the same underlying real asset, and the investment sequence is predetermined.

Childs, Ott, and Triantis (1998) undertake what is perhaps the most comprehensive study of the impact of interdependence on real option valuation and investment sequence. They describe problems wherein the form of interdependence ranges from mutual exclusivity to perfect complimentarity. Their analysis is limited to the two-prospect case, however, and only the "mutually exclusive" case is discussed in the text. Their results anticipate one of our main observations: that it is not always advisable to exercise the most valuable option first; and they describe the conditions that work for and against such an outcome. Our work pertains to options whose values tend to be positively correlated, rather than mutually exclusive, and we have found that some results obtained easily for the two-prospect case fail to generalize even to the three-

[4] The sequence of investments matters in that framework, but for reasons that relate to risk preferences.

prospect case—the difference coming from the extra degrees of freedom that are inherent in multivariate distributions.[5]

Several papers in the practitioners' arena are also pertinent to our work. Murtha (1996) has investigated the impact of dependence among petroleum prospects, but within a rigid investment structure that would require all prospects to be drilled. He finds (correctly within that framework) that expected reserve volumes are unaffected by dependence among prospects, although the variance of reserves (payoffs) must increase. His conclusion, that "dependence increases the riskiness" of exploration, overlooks the value of flexible management—a value that we conclude can be very high. Delfiner (2000) adheres closely to Murtha's approach to portfolio analysis, to the point of repeating Murtha's potentially misleading conclusion, that "dependencies increase the exploration risk." What the real options approach incorporates is what both Murtha and Delfiner leave out: the ability of managerial flexibility to turn inflated variance into enhanced return. Wang, et. al. (2000) do recognize that dependence creates managerial options to sequence petroleum exploration prospects optimally based on updated information, but they provide no analysis of the value of such options.

3. Preliminaries: The *N*=2 Case:

We start by assuming there are two prospects, with intrinsic values:

$p_1V_1 - C$ and $p_2V_2 - C,$

where p_i represents the probability of success on the i^{th} prospect, V_i is the expected value of that prospect conditional on success, and C is the cost of performing the trial (we assume identical drilling costs over all prospects).[6]

[5] It is impossible with only two prospects, for example, to construct the type of illustration we presented in the introduction; i.e., where the lower-probability prospect should go first.

Without loss of generality, we order prospects in terms of decreasing intrinsic value, thus:

$p_1 V_1 > p_2 V_2$.

To simplify the presentation, we will assume that all prospects are initially “in the money,” thus: $p_2 V_2 - C > 0$.[7] As noted already, association of outcomes implies:

$$p_{i.j} \geq p_i \quad and \quad p_{j.i} \geq p_j \,;$$

that is, success on either prospect increases the chance of success on the other. Due to the impact of the specific information generated by the first trial, the value of the portfolio depends on which prospect goes first. The expected value of starting with prospect 1 is given by:

$$\Pi_1 \quad = \quad (p_1 V_1 - C) + (p_{2.1} V_2 - C) p_1 + m\left(p_{2.\bar{1}} V_2 - C\right) p_{\bar{1}} \,,$$

where for convenience we define: $m(x) = \max(x, 0)$, $p_{\bar{1}} = 1 - p_1$, and $p_{2.\bar{1}} = p_{\bar{1} \cap 2} / p_{\bar{1}}$, etc. The term $m\left(p_{2.\bar{1}} V_2 - C\right) p_{\bar{1}}$ embodies the value of the option not to test the second prospect after failing on the first.

Likewise, the expected value of starting with prospect 2 is given by:

$$\Pi_2 \quad = \quad (p_2 V_2 - C) + (p_{1.2} V_1 - C) p_2 + m\left(p_{1.\bar{2}} V_1 - C\right) p_{\bar{2}} \,.$$

The *premium* earned by starting with prospect one is given by the difference:

$$\begin{aligned} \Delta \quad = \quad & (p_1 V_1 - C) - (p_2 V_2 - C) \quad + \quad (p_{2.1} V_2 - C) p_1 - (p_{1.2} V_1 - C) p_2 \\ & \quad + \quad m\left(p_{2.\bar{1}} V_2 - C\right) p_{\bar{1}} - m\left(p_{1.\bar{2}} V_1 - C\right) p_{\bar{2}} \\ = \quad & (p_1 V_1 - p_2 V_2) + p_{1 \cap 2}(V_2 - V_1) - (p_1 - p_2) C + m\left(p_{2.\bar{1}} V_2 - C\right) p_{\bar{1}} - m\left(p_{1.\bar{2}} V_1 - C\right) p_{\bar{2}} \,. \end{aligned}$$

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[6] In the vernacular of the petroleum industry, V_i is sometimes referred to as the “unrisked value” of a prospect, whereas $p_i V_i$ is the “risked” value.

[7] The value of the portfolio may be positive even if $p_1 V_1 < 0$, and we believe it is possible to derive similar results for such cases, but that goes beyond the scope of the present work.

We evaluate this expression, distinguishing three cases:

Case A: Neither prospect is condemned by failure of the other

In this case, both expressions of the form $m\left(p_{i.\bar{j}}V_i - C\right)$ are positive and we have:

$$\Delta_A = \left(p_1V_1 - p_2V_{2)}\right) + p_{1\cap 2}(V_2 - V_1) - (p_1 - p_2)C + p_{2\cap\bar{1}}V_2 - Cp_{\bar{1}} - p_{1\cap\bar{2}}V_1 + Cp_{\bar{2}}$$

$$= (p_1V_1 - p_2V_2) + (p_{1\cap 2} + p_{\bar{1}\cap 2})V_2 - (p_{1\cap 2} + p_{1\cap\bar{2}})V_1 - (p_1 + 1 - p_1)C - (-1 + p_2 - p_2)C$$

$$= \quad (p_1V_1 - p_2V_2) + p_2V_2 - p_1V_1 - C + C \quad = \quad 0. \tag{1}$$

Thus, order doesn't matter in Case A since management would test both prospects regardless of intervening outcomes.

Case B: Either prospect is condemned by failure of the other

In this case, both expressions of the form $m\left(p_{i.\bar{j}}V_i - C\right)$ are zero and we have:

$$\Delta_B \quad = \quad (p_1V_1 - p_2V_2) + p_{1\cap 2}(V_2 - V_1) - (p_1 - p_2)C\,.$$

$$= \quad (p_1 - p_{1\cap 2})V_1 - (p_2 - p_{1\cap 2})V_2 - (p_1 - p_2)C \tag{2}$$

But, $V_2 < p_1V_1/p_2$ *(by assumption,)* so:

$$\Delta_B \quad \geq \quad (p_1 - p_{1\cap 2})V_1 - (p_2 - p_{1\cap 2})V_1 p_1 / p_2 - (p_1 - p_2)C$$

$$= \quad V_1 p_{1\cap 2}\left(\frac{p_1 - p_2}{p_2}\right) - (p_1 - p_2)C$$

$$(p_1 - p_2)(p_{1.2}V_1 - C),$$

which takes the sign of $(p_1 - p_2)$ since $p_{1.2}V_1 - C > 0$ (recall that $p_{1.2} \geq p_1$). Thus, in Case B it is optimal to test first the prospect with higher intrinsic value if that prospect also has the higher probability of success. If the two prospects have equal values conditional on success (i.e., $V_1 = V_2$), you would always test the more likely prospect first. If they have equal probabilities of success, then the one with the greater conditional value must go first.

On the other hand, you would test the 2nd prospect (lower intrinsic value) first if its failure conveys enough information to compensate for its lower intrinsic value. Specifically, the condition for testing the second prospect first is (from Eq. 2):

$$(p_1 - p_{1\cap 2})V_1 - (p_2 - p_{1\cap 2})V_2 - (p_1 - p_2)C \quad < \quad 0$$

Equivalently:

$$p_{1\cap\bar{2}}(V_1 - C) \quad < \quad p_{\bar{1}\cap 2}(V_2 - C);$$

which implies that you would test the lower intrinsic value first if and only if:

$$\frac{p_{1\cap\bar{2}}}{p_{\bar{1}\cap 2}} \quad < \quad \frac{V_2 - C}{V_1 - C}. \tag{3}$$

In terms of the primitive parameters, lower values of the ratio $p_{1\cap\bar{2}} / p_{\bar{1}\cap 2}$ make it more likely that the lower intrinsic value prospect should go first. Intuitively, low values of $p_{1\cap\bar{2}} / p_{\bar{1}\cap 2}$ means that the odds are against prospect two generating many false negatives, at least relative to prospect one, which enhances the value of information gleaned from it.

<u>Case C: Only one prospect is condemned by failure of the other</u>

It is easy to show Δ_C and Δ_B have the same sign. To see this, examine the difference:

$$(p_{1.\bar{2}}V_1 - C)p_{\bar{2}} - (p_{2.\bar{1}}V_2 - C)p_{\bar{1}} = (p_1V_1 - p_2V_2) + p_{1\cap 2}(V_2 - V_1) - (p_1 - p_2)C = \Delta_B$$

where the first equality is implied by Eq. 1, and the second by Eq. 2. Thus:

$$\Delta_B \gtrless 0 \quad \Leftrightarrow \quad (p_{1.\bar{2}}V_1 - C)p_{\bar{2}} - (p_{2.\bar{1}}V_2 - C)p_{\bar{1}} \gtrless 0\,;$$

which, since $p_{\bar{1}}$ and $p_{\bar{2}}$ are non-negative, is equivalent to:

$$\Delta_B \gtrless 0 \quad \Leftrightarrow \quad \left(p_{1.\bar{2}}V_1 - C\right) \gtrless 0 \quad and \quad \left(p_{2.\bar{1}}V_2 - C\right) \lessgtr 0\,,$$

where we have used the fact that in Case C these two expressions must differ in sign. Therefore, if $\Delta_B > 0$, we know:

$$\begin{aligned}\Delta_C &= \left(p_1V_1 - p_2V_2\right) + p_{1\cap 2}\left(V_2 - V_1\right) - \left(p_1 - p_2\right)C - \left(p_{1.\bar{2}}V_1 - C\right)p_{\bar{2}} \\ &= \Delta_A - \left(p_{2.\bar{1}}V_2 - C\right)p_{\bar{1}} \quad > \quad 0\,.\end{aligned}$$

Likewise, for $\Delta_B < 0$, we have:

$$\Delta_C = \Delta_A + \left(p_{1.\bar{2}}V_1 - C\right)p_{\bar{2}} \quad < \quad 0\,.$$

<u>Implications</u>:

Some preliminary results can now be summarized. If neither prospect has the power to condemn the other, both may be tested simultaneously.[8] Otherwise, the necessary and sufficient condition for testing the i^{th} first is given by (cf. Eq. 3):

Test the i^{th} prospect first if and only if: $\dfrac{p_{i\cap\bar{j}}}{p_{\bar{i}\cap j}} > \dfrac{V_j - C}{V_i - C}$.

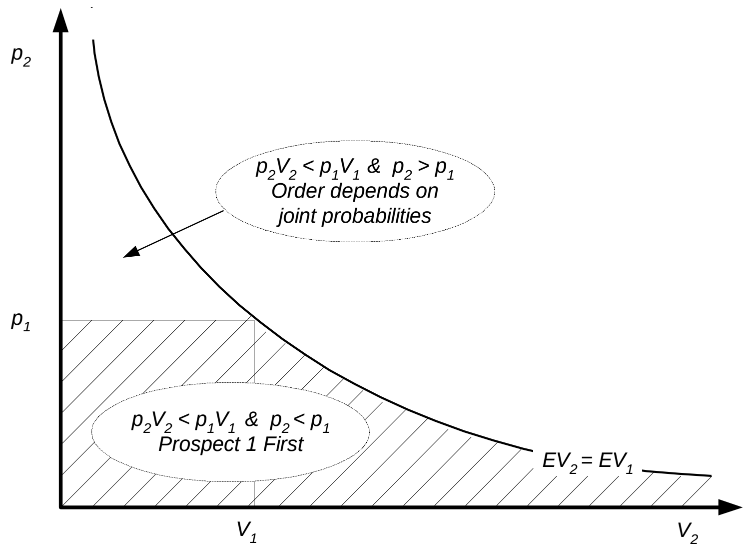
For i=1, this condition is assured if, in addition to $p_1V_1 > p_2V_2$, we have $p_1 > p_2$. On the other hand, if we have $V_1 = V_2$, the condition implies testing prospect one first since we have assumed $p_1 > p_2$. Our conclusions regarding the optimal investment sequence is summarized in the following diagram.

[8] Throughout this paper, we neglect the time value of money in order to emphasize the option-value component of portfolio value.

Optimal Sequence with Common Risk Structure:

If Prospect 2 Falls In Shaded Region, Test Prospect 1 First;

In Unshaded Region, Order Depends on Degree of Dependence



<u>Option Value</u>

We define the "static value" of the portfolio (Π_0) to be the sum of intrinsic values; i.e., the expected value of the portfolio if management ignores the information content of previous outcomes:

$$\Pi_0 = (p_1V_1 - C) + (p_2V_2 - C).$$

We may then define the "option value" of the i^{th} prospect (OV_i) as the additional value that comes by testing it first and using the resulting information to make subsequent investment decisions:

$$OV_i = \Pi_i - \Pi_0 = (p_{j.i}V_j - C)p_i +_m (p_{j.\bar{i}}V_j - C)p_{\bar{i}} - (p_jV_j - C).$$

Now, if failure on the i^{th} prospect does not condemn the j^{th}, we have:

$$OV_i = (p_{j.i}V_j - C)p_i + (p_{j.\bar{i}}V_j - C)p_{\bar{i}} - (p_jV_j - C)$$

$$= \quad p_{j\cap i}V_j - p_iC + p_{j.\bar{i}}V_j - p_{\bar{i}}C - p_jV_j + C \quad = \quad 0\,.$$

Thus, if the i^{th} prospect has no power to condemn the other, it has no option value. Alternatively, if failure on the i^{th} prospect does condemn the j^{th}, we have:

$$OV_i = (p_{j.i}V_j - C)p_i - (p_jV_j - C) = (p_{j\cap i} - p_j)V_j + (1 - p_i)C$$

$$= p_{\bar{i}}C - p_{\bar{i}\cap j}V_j > 0\,. \tag{4}$$

This option value has the natural interpretation of being the expected cost savings (in terms of deferred testing cost) less the foregone revenue due to the occurrence of a false negative (i.e., the i^{th} prospect wrongly deferring the j^{th}). Partial differentiation of Eq. (4) gives:

$$\frac{\partial OV_i}{\partial V_j} = -p_{\bar{i}\cap j} < 0 \qquad \frac{\partial OV_i}{\partial C} = p_{\bar{i}} > 0 \qquad \frac{\partial OV_i}{\partial p_{\bar{i}\cap j}} = -V_j\,.$$

Thus, option value falls as the intrinsic value of the other prospect rises since there is less chance that failure will defer it. Option value rises as the cost of trials rises since the potential cost savings is larger. Finally, option value falls as the probability of false negatives rises.

4. The General Case: $N \geq 2$

Before these results can be generalized to the case of multiple prospects, it is necessary to place further restrictions (beyond association) on the risk structure. In this section, we restrict attention to what is referred to in petroluem exploration as the "shared risk" information structure.[9]

"Shared Risk" Information Structures

We let: $p(F_i) = q_i$; and $p(\overline{F}_i) = 1\text{-}q_i$; for $i = 0, 1, ..., N$;

where the F_i represent independent events. Then, define:

[9] The name comes from Stabell (2000), although applications of this type have a much longer history in the petroleum industry. See Megill (1979) for example.

$$S_i = F_0 \cap F_i; \quad i = 1, 2, ..., N.$$

Intuitively, F_0 denotes the presence of a *common* factor that is necessary for success on each of the N prospects (e.g., the original deposition of carboniferous sediments in a prospective petroleum basin). For $i = 1, ..., N$, each of the F_i represents the presence of an additional prospect-specific factor that is necessary for success (e.g., a local trapping mechanism) on that specific prospect. The prospect-specific factors are assumed to be independent of each other and independent of the common factor. Thus:

$$p_i = p(S_i) = q_0 q_i; \quad i = 1, 2, ..., N.\text{; and:}$$

$$p_{i.j} = p(S_i|S_j) = p(S_i \cap S_j)/p(S_j) = p(F_0 \cap F_i \cap F_j)/p(F_0 \cap F_j) = q_0 q_i q_j / q_0 q_j = q_i$$

Also: $$p_{i.j\bar{k}} = q_0 q_i q_j q_{\bar{k}} / q_0 q_j q_{\bar{k}} = p_{i.j},$$

$$p_{i.jk} = q_0 q_i q_j q_k / q_0 q_j q_k = p_{i.j}, \qquad \text{etc.}$$

In the shared risk structure, relative probabilities of success among remaining prospects are not affected by previous outcomes, since:

$$\frac{p_{i.\{k\}}}{p_{j.\{k\}}} = \frac{p_i}{p_j} = \frac{q_i}{q_j},$$

where the set $\{k\}$ represents any set of outcomes on other prospects. From this, it also follows:

$$p_{i.\{k\}} V_i \underset{<}{>} p_{j.\{k\}} V_j \quad \Leftrightarrow \quad p_i V_i \underset{<}{>} p_j V_j$$

I.e., the ranking of remaining common-risk prospects by intrinsic value is not affected by the outcomes of previous trials. Within this framework, we can now prove:

Theorem 1: Given N prospects such that $p_iV_i \geq p_jV_j$ and $p_i \geq p_j$, for all i and j such that $i < j$, then at each stage in the investment sequence it is optimal to test next the prospect with highest intrinsic value. (Proof—see Appendix)

Option Values

We now extend the definition of option value to the N-prospect case. The specific results to follow are based on Theorem 1, and therefore presume that $p_i \geq p_j$, for all i and j such that $i < j$. Moreover, for the time being, we will assume that failure on prospect one would condemn prospect two.

The option value of the 1^{st} prospect as it affects the j^{th} can be defined as in the N=2 case:

$$OV_{1j} \quad = \quad p_{\bar{1}}C - p_{\bar{1}\cap j}V_j, \qquad \text{for } j = 2, ..., N.$$

The value of the portfolio can then be computed as the sum of these elementary option values (cf. Eq. 4):

Theorem 2: If failure on prospect one would condemn prospect two, then the value of the portfolio is given by:

$$\Pi^* \qquad \Pi_1 \quad = \quad \Pi_0 + OV_{12} + OV_{13} + ... + OV_{1N}\,.$$

$$= \quad \Pi_0 + p_{\bar{1}} \sum_{j=2}^{N} \left(C - p_{j.\bar{1}}V_j\right). \tag{5}$$

(Proof—see Appendix)

The value of actively managing the portfolio is therefore:

$$\Pi_1 - \Pi_0 \quad = \quad p_{\bar{1}} \sum_{j=2}^{N} \left(C - p_{j.\bar{1}}V_{\bar{1}}\right).$$

Portfolio Value—Comparative Statics

We differentiate Eq. (5) to observe the impact of parameter changes on the portfolio value, and on the value of active management. First, with respect to the cost of testing the prospects:

$$\frac{\partial \Pi_1}{\partial C} = -N + (N-1)p_{\bar{1}} = -Np_1 - p_{\bar{1}} < 0\,;$$

$$\frac{\partial (\Pi_1 - \Pi_0)}{\partial C} = (N-1)p_{\bar{1}} > 0\,;$$

which means that a higher testing cost decreases the value of the portfolio, but increases the value of active management.

With respect to the conditional value of each prospect:

$$\frac{\partial \Pi_1}{\partial V_j} = p_j - p_{\bar{1}\cap j} = p_{1\cap j} > 0\,;$$

$$\frac{\partial (\Pi_1 - \Pi_0)}{\partial V_j} = -p_{\bar{1}\cap j} = -(p_j - p_{1\cap j}) < 0\,;$$

which means that higher prospect value (conditional on success) increases the value of the portfolio, but decreases the value of active management.

Finally, with respect to the probability of obtaining a “false negative from each prospect (while holding constant the marginal probabilities of success):

$$\frac{\partial \Pi_1}{\partial p_{j.\bar{1}}} = \frac{\partial (\Pi_1 - \Pi_0)}{\partial p_{j.\bar{1}}} = -V_j < 0\,.$$

But, note that $p_{j.\bar{1}} = 1 - p_{\bar{j}.\bar{1}}$; thus:

$$\frac{\partial \Pi_1}{\partial p_{\bar{j}.\bar{1}}} = \frac{\partial (\Pi_1 - \Pi_0)}{\partial p_{\bar{j}.\bar{1}}} = V_j > 0\,;$$

which means that, holding other things equal, greater dependence increases the term $p_{\bar{j}.\bar{1}}$, and therefore increases both the value of the portfolio and the value of active managment; whereas greater probability of a false negative regarding any prospect decreases both the value of the portfolio and the value of active management.

Next we account for the case where failure on the *1st* prospect may not condemn the *2nd*. To make an interesting problem, some prospect must be condemned by one or more prior failures, else all prospects would be tested and the value of the portfolio would be given simply by the static value, Π_0. We let prospect m+1 (where $1 \leq m < N$) represent the most valuable "condemnable" prospect (i.e., the prospect of lowest index that could possibly be condemned by prior failures). We can then establish:

Theorem 3: If prospect m+1 is the condemnable prospect of highest intrinsic value, then the value of the portfolio is given by:

$$\Pi^* \quad = \quad \Pi_1 \quad = \quad \Pi_0 + p_{0(m)} C(N-m) - \sum_{i=m+1}^{N} p_{i \cap 0(m)} V_i \qquad (6)$$

where $p_{0(m)}$ is the probability of no success among the first m trials, and $p_{i \cap 0(m)}$ is the probability that the i^{th} prospect ($i \geq m$) succeeds and there are no successes among the first m trials. Proof: (see appendix).

Our previous eq. (5) represents the special case of (6) obtained by setting m=1. The same natural interpretation of option values applies here as in that case, but where the decision to test the first m prospects simultaneously is treated as a single act. The form of the expression is otherwise entirely analogous. Finally, the value of actively managing the portfolio is given by:

$$\Pi_1 - \Pi_0 \quad = \quad p_{0(m)} C(N-m) - \sum_{i=m+1}^{N} p_{i \cap 0(m)} V_i \,,$$

from which comparative static properties can be derived similar to those given above.

5. Summary

At this juncture we are able to organize the following simplifications to the general problem of managing a portfolio of dependent options.

1. When choosing between two prospects, it is optimal to test both simultaneously if neither has the power to condemn the other.

2. When choosing between two prospects, it is optimal to test first the prospect with *larger* intrinsic value if that prospect also has the larger probability of success. However, it is optimal in some cases to test first the prospect with *smaller* intrinsic value if it has the larger probability of success.

3. When choosing between two prospects, knowledge of the ratio of success probabilities conditional on failure of the other prospect is sufficient to order the prospects as a function of unrisked valuations.

4. If prospect dependence conforms to the "common-risk" risk structure, then these results generalize to comparisons among *N* prospects:

   a. When choosing among *N* prospects, it is optimal to test first the prospect with the largest intrinsic value if it also has the largest probability of success.

   b. When choosing among *N* prospects, it is also optimal to simultaneously test any prospects that would not be deferred (condemned) by failure on the prospect identified in part a, regardless of its intrinsic value and/or risk.

c. If all prospects are of the same "size" and conform to the "common-risk" structure, then it is optimal to test first the one with the largest probability of success.

d. If all prospects have the same probability of success and conform to the "common-risk" structure, then it is optimal to test first the one with the largest size.

5. The option value of a prospect measures the extent to which information revealed via a test of that prospect enhances the value of the rest of the portfolio of prospects.

6. The option value of a prospect increases directly with that prospect's degree of affiliation with other prospects.

7. The option value of any prospect varies directly with the cost of testing, but inversely with the intrinsic value of other prospects. In these two respects, circumstances that are associated with a decrease in the static value of the portfolio are associated with an increase in the value of managing the portfolio actively.

Our preliminary inquiry encourages us into further contemplation of how the structure underlying a portfolio of real options interplays with the optimal option exercise. We suggest broadening the search for probability spaces wherein simple decision rules are optimal and characterizing these rules in the vernacular of real options. Will it be possible to state sufficient and necessary conditions under which specific simple rules are optimal? What will these conditions look like and how closely will they conform to meaningful applications? How much value is contributed by the optimal exercise of imbedded options? If research outcomes are indeed associated, our results to date leave us optimistic about obtaining useful answers to these and similar questions.

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Appendix

Proof of Theorem 1:

The proof is by induction. The result has already been established for the case of N=2, so begin now with N=3. Pick any prospect j other than the first ($j \neq 1$) to test first. Among the N-1 prospects that remain, we have already shown that it is optimal to test the highest intrinsic value first. Since the order is preserved, the prospect with highest intrinsic value (and highest probability of success) after j has been tested is the same as before j was tested. There is no ambiguity therefore in the labeling of prospects. The maximal expected value of all N prospects, given that you start with j, may then be written:

$$\Pi_j = (p_j V_j - C) + (p_{1.j} V_1 - C) p_j + (p_{1.j} E_{j1} + p_{\bar{1}.j} E_{j\bar{1}}) p_j$$
$$+ \; m\left[(p_{1.\bar{j}} V_1 - C) + p_{1.\bar{j}} E_{\bar{j}1} + p_{\bar{1}.\bar{j}} E_{\bar{j}\bar{1}}\right] p_{\bar{j}} \quad ; \qquad \text{(A1)}$$

where: $p_{1.\bar{j}}$ = conditional probability of success on prospect 1 after failing on j.

$E_{1\bar{j}}$ = expected value of remaining prospects after success on 1 and failure on j.

$m[x]$ = maximum of (0,x).

If a negative value appears in the square bracket, then you would choose to not test 1 after failing on j. But, in that case you would test no further (else prospect 1 would not have been the optimal choice to follow $\bar{j}$) and the series ends.

We claim that a value not less than Π_j could be obtained by starting with the first instead of the j^{th}. The maximal expected value, given that you start with prospect 1, can be written as:

$$\Pi_1 = (p_1V_1 - C) + (p_{j.1}V_j - C)p_1 + (p_{j.1}E_{1j} + p_{\bar{j}.1}E_{1\bar{j}})p_1$$

$$+ \quad m\left[\left(p_{j.\bar{1}}V_j - C\right) + p_{j.\bar{1}}E_{\bar{1}j} + p_{\bar{j}.\bar{1}}E_{\bar{1}\bar{j}} - E_{\bar{1}}\right]p_{\bar{1}} + E_{\bar{1}}p_{\bar{1}} \qquad \text{(A2)}$$

where: $E_{\bar{1}}$ = expected value of remaining prospects after failure on 1 and not permitting *j* to go next. Equation (2) differs in form from (1) only because there is no assurance that *j* (which was chosen arbitrarily) should optimally follow 1. If the value in square brackets is non-negative, it should follow 1; otherwise not.

To prove our claim, we must show $\Pi_1 - \Pi_j \geq 0$. Using (A1) and (A2), we have:

$$\Delta = \Pi_1 - \Pi_j = (p_1V_1 - C) + (p_{j.1}V_j - C)p_1 - (p_jV_j - C) - (p_{1.j}V_1 - C)p_j \qquad (T_1)$$

$$+ \quad (p_{j.1}E_{1j} + p_{\bar{j}.1}E_{1\bar{j}})p_1 - (p_{1.j}E_{j1} + p_{\bar{1}.j}E_{j\bar{1}})p_j \qquad (T_2)$$

$$+ \quad m[d_{j.\bar{1}}]p_{\bar{1}} + E_{\bar{1}}p_{\bar{1}} - m[d_{1.\bar{j}}]p_{\bar{j}} \; ; \qquad (T_3)$$

where: $d_{j.\bar{1}} = (p_{j.\bar{1}}V_j - C) + p_{j.\bar{1}}E_{\bar{1}j} + p_{\bar{j}.\bar{1}}E_{\bar{1}\bar{j}} - E_{\bar{1}}$

and: $d_{1.\bar{j}} = (p_{1.\bar{j}}V_1 - C) + p_{1.\bar{j}}E_{\bar{j}1} + p_{\bar{1}.\bar{j}}E_{\bar{j}\bar{1}}$.

The terms $d_{j.\bar{1}}$ and $d_{1.\bar{j}}$ show the impact on portfolio value if each prospect is tested, rather than deferred, after the failure of the other. A negative value indicates that deferral is optimal.

We evaluate each of these three components separately, then combine results. It is straightforward to show (cf. the *N*=2 case):

$$T_1 = (p_{1\cap\bar{j}}V_1 - p_{\bar{1}\cap j}V_j) - (p_1 - p_j)C \quad > \quad 0.$$

We proceed to T_2, where due to the common-risk structure, we have:

$$E_{1j} = E_{1\bar{j}} = E_{j\bar{1}} \equiv E \geq 0 .$$

I.e., confirmation of any prospect confirms the common factor on all remaining prospects. After substituting these into T_2, we get:

$$T_2 = (p_1 - p_j)E \qquad \geq \qquad 0,$$

since $p_1 > p_j$ by assumption.

Regarding T_3, there are three possible cases to consider.

<u>Case A</u>: Neither $d_{j.\bar{1}}$ nor $d_{1.\bar{j}}$ is negative (neither is deferred by failure of the other). T_3 then takes the form:

$$T_3 \quad = \quad (p_{j\cap\bar{1}}V_j - p_{\bar{1}}C) + p_{j\cap\bar{1}}E_{\bar{1}j} + p_{\bar{j}\cap\bar{1}}E_{\bar{1}\bar{j}} - p_{\bar{1}}E_{\bar{1}} + p_{\bar{1}}E_{\bar{1}}$$

$$- p_{1\cap\bar{j}}V_j + p_{\bar{j}}C - p_{1\cap\bar{j}}E_{\bar{j}1} - p_{\bar{1}\cap\bar{j}}E_{\bar{j}\bar{1}} .$$

But, the common-risk structure implies:

$$E_{1\bar{j}} = E_{j\bar{1}} \equiv E > 0 .$$

After making this substitution and cancelling like terms, T_3 reduces to:

$$T_3 \quad = \quad -(p_{1\cap\bar{j}}V_1 - p_{\bar{1}\cap j}V_j) - (p_1 - p_j)(E - C) \; = \; -(T_1+T_2) \; < \; 0$$

Thus, in Case A:

$$\Delta_A \quad = \quad T_1 + T_2 + T_3 \quad = \quad 0$$

Thus, if failure of neither prospect would cause the other to be deferred, the order is of no consequence; they could be tested simultaneously.

<u>Case B</u>: Both $d_{j.\bar{1}}$ and $d_{1.\bar{j}}$ are negative (either is deferred by failure of the other). T_3 then takes the simple form: $T_3 = E_{\bar{1}}p_{\bar{1}} \geq 0$ (since $E_{\bar{1}}$ cannot be negative), which when combined with T_1 and T_2 gives:

$$\Delta_B \quad = \quad T_1 + T_2 + p_{\bar{1}} E_{\bar{1}} \quad > \quad 0 \, .$$

Thus, if failure on each prospect would cause the other to be deferred, the highest expected value (and most likely) should be tested first.

Case C: Only $d_{j.\bar{1}}$ is negative (only one is deferred by failure of the other). The fact that it is the *1st* prospect that would defer the *jth* can be deduced from the Case A result, where we showed:

$$T_3 \quad = \quad d_{j.\bar{1}} p_{\bar{1}} + E_{\bar{1}} p_{\bar{1}} - d_{1.\bar{j}} p_{\bar{j}} \quad = \quad -\left(T_1 + T_2\right) \quad < \quad 0 \, ,$$

which implies:

$$d_{j.\bar{1}} p_{\bar{1}} - d_{1.\bar{j}} p_{\bar{j}} \quad < \quad - E_{\bar{1}} p_{\bar{1}} \quad \leq \quad 0 \, .$$

Now, if $d_{j.\bar{1}}$ and $d_{\bar{1}.j}$ are to differ in sign (as Case C requires), then it must be that $d_{j.\bar{1}} < 0$ while $d_{\bar{1}.j} > 0$. Thus, it must be the *1st* prospect that has the power to defer the *jth*.

We can now easily evaluate T_3 in Case C by reference to Case A: what entered there into T_3 (and therefore Δ) as $d_{j.\bar{1}} p_{\bar{1}}$ enters here as 0. All else remains the same. Thus, we can simply subtract this term from the Case A result to obtain:

$$\Delta_C \quad = \quad \Delta_A - d_{j.\bar{1}} p_{\bar{1}} > 0 \, .$$

Thus, if failure on only one of the prospects is informative enough to cause deferral of the other, the most likely (and highest intrinsic value) prospect would be the informative one, and it should be tested.

Proof of Theorem 2:

Theorem 1 established that the value of the portfolio is given by Π_1, which can be computed directly using the decision tree approach. We keep in mind that if the *1st* prospect succeeds, then all prospects will be tested, and if the *1st* prospect fails, no more will be tested. (We assumed that the *1st* would condemn the *2nd*, but the *2nd* would optimally follow the *1st* under Theorem 1, thus no other prospect could follow the *1st* but the *2nd*. In other words, if the *1st* has the power to condemn the *2nd*, then it has the power to condemn them all.

Thus, we can compute the value of the entire portfolio as follows:

$$
\begin{aligned}
\Pi_1 &= p_1V_1 - C + p_1\sum_{j=2}^{N}\left(p_{j.1}V_j - C\right) \\
&= p_1V_1 - C + \sum_{j=2}^{N}\left(p_{j\cap 1}V_j - p_1C\right) \\
&= p_1V_1 - C + \sum_{j=2}^{N}\left(p_{j\cap 1}V_j + p_{j\cap\bar{1}}V_j - p_1C - p_{j\cap\bar{1}}V_j\right) \\
&= p_1V_1 - C + \sum_{j=2}^{N}\left(p_jV_j - p_1C - p_{j\cap\bar{1}}V_j\right) \\
&= p_1V_1 - C + \sum_{j=2}^{N}\left(p_jV_j - p_1C - (1-p_1)C - p_{j\cap\bar{1}}V_j + (1-p_1)C\right) \\
&= p_1V_1 - C + \sum_{j=2}^{N}\left(p_jV_j - C - p_{j\cap\bar{1}}V_j + (1-p_1)C\right) \\
&= p_1V_1 - C + \sum_{j=2}^{N}\left(p_jV_j - C\right) + \sum_{j=2}^{N}\left(p_{\bar{1}}C - p_{j\cap\bar{1}}V_j\right) \\
&= \Pi_0 + \sum_{j=2}^{N}OV_{1j}\ .
\end{aligned}
$$

<u>Proof of Theorem 3</u>:

The first m prospects will be tested simultaneously, *en block*. Prospect m+1 (and all remaining prospects) would be condemned unless at least one success occurs among the first m trials—in which case prospect m+1 (and all remaining prospects) would be tested. Thus, we can write the value of the portofolio as:

$$
\Pi^* \;=\; \Pi_1 \;=\; \sum_{i=1}^{m}\left(p_iV_i - C\right) + p_{\bar{0}(m)}\sum_{i=m+1}^{N}\left(p_{i.\bar{0}(m)}V_i - C\right),
$$

where $p_{\bar{0}(m)}$ is the probability of at least one success among the first *m* trials, and $p_{i.\bar{0}(m)}$ represents the probability that the i^{th} prospect ($i>m$) succeeds given that there was at least one success among the first *m* trials.

This expression can be simplified as follows:

$$\begin{aligned}
\Pi_1 &= \sum_{i=1}^{m}\left(p_i V_i - C\right) + \sum_{i=m+1}^{N}\left(p_i V_i - p_{i\cap 0(m)} V_i - C + \left(1 - p_{\bar{0}(m)}\right) C\right) \\
&= \sum_{i=1}^{N}\left(p_i V_i - C\right) + \sum_{i=m+1}^{N}\left(p_{0(m)} C - p_{i\cap 0(m)} V_i\right) \\
&= \Pi_0 + p_{0(m)} C (N - m) - \sum_{i=m+1}^{N} p_{i\cap 0(m)} V_i \,.
\end{aligned}$$