GENERALIZATIONS OF KEMPE'S UNIVERSALITY THEOREM

by

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Abstract

In 1876, A. B. Kempe presented a flawed proof of what is now called Kempe's Universality Theorem: that the intersection of a closed disk with any curve in $\mathbb{R}^2$ defined by a polynomial equation can be drawn by a linkage. Kapovich and Millson published the first correct proof of this claim in 2002, but their argument relied on different, more complex constructions. We provide a corrected version of Kempe's proof, using a novel contraparallelogram bracing. The resulting historical proof of Kempe's Universality Theorem uses simpler gadgets than those of Kapovich and Millson.

We use our two-dimensional proof of Kempe's theorem to give simple proofs of two extensions of Kempe's theorem first shown by King: a generalization to d dimensions and a characterization of the drawable subsets of $\mathbb{R}^d$. Our results improve King's by proving better continuity properties for the constructions.

We prove that our construction requires only $O(n^d)$ bars to draw a curve defined by a polynomial of degree n in d dimensions, improving the previously known bounds of $O(n^4)$ in two dimensions and $O(n^6)$ in three dimensions. We also prove a matching $\Omega(n^d)$ lower bound in the worst case.

We give an algorithm for computing a configuration above a given point on a given polynomial curve, running in time polynomial in the size of the dense representation of the polynomial defining the curve. We use this algorithm to prove the coNP-hardness of testing the rigidity of a given configuration of a linkage. While this theorem has long been assumed in rigidity theory, we believe this to be the first published proof that this problem is computationally intractable.

This thesis is joint work with Reid W. Barton and Erik D. Demaine.

Thesis Supervisor: Erik D. Demaine
Title: Esther & Harold Edgerton Associate Professor

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Chapter 1

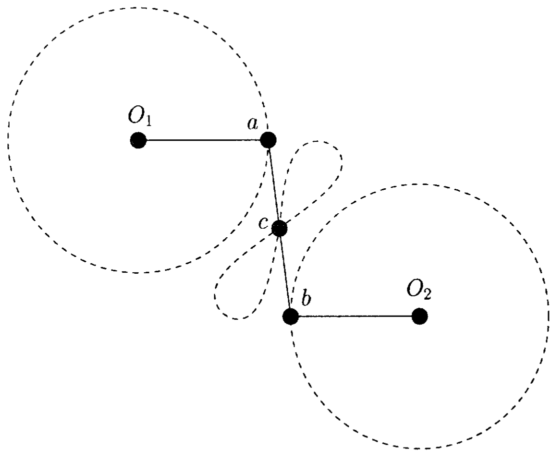
Introduction

A linkage is an idealized mechanical framework consisting of rigid bars attached to each other by hinges at their endpoints. We additionally fix some vertices of bars in place, to factor out rigid motions. We ignore issues of bars crossing because we are primarily modeling 2-dimensional linkages in a 3-dimensional world, and thus they are not a practical constraint to linkage design.

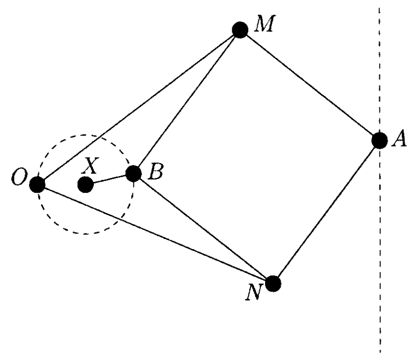
Early work on mechanical linkages was motivated by the design of locomotives [DO]. The goal was to build a device that would convert the linear motion of a piston into the circular motion of a wheel. An early discovery was James Watt's "parallel motion" linkage, invented in 1784. Watt's linkage converts approximate linear motion into circular motion. See Figure 1-1(a).

During the first half of the 19th century, classical geometry problems such as squaring the circle and trisecting an angle using a straightedge and compass were proved unsolvable. This eventually led to the widespread belief that exactly converting linear motion into circular motion was impossible [KM].

However, in 1864, Charles-Nicolas Peaucellier designed the first linkage that perfectly converted circular motion into linear motion. See Figure 1-1(b). Because the key vertices of his linkage are related by geometric inversion, the Peaucellier linkage is often called the Peaucellier inversor. In 1875, Harry Hart proposed another linkage solving the same problem using fewer bars [DO]. While the ideal inversors are theoretically superior, Watt's linkage continues to be preferred for practical applications.



(a) Watt's linkage. Vertices O_1 and O_2 are fixed, and c is the midpoint of ab. The figure-eight curve that is the locus of c approximates a straight line near the point where it crosses itself.



(b) Peaucellier's Linkage. Vertices O and X are fixed, and the locus of A is a straight line, because A is the inversion of B about a circle.

Figure 1-1: Linkages designed to draw a straight line.

In 1876, Alfred Bray Kempe (best known for his insightful but faulty proof of the Four Color Theorem in 1879) published a surprising proof that one could build a linkage such that a pen placed at a single vertex could draw the intersection of any algebraic curve with any closed disk [Kempe]. Kempe's Universality Theorem, as this result is now called, can be formalized as follows:

Theorem 1.1 (Kempe's Universality Theorem [KM]). *Let $f \in \mathbb{R}[x, y]$ be a polynomial, and let B be a closed disk in the plane. Then there exists a planar linkage that draws the set $B \cap \{ (x, y) \in \mathbb{R}^2 : f(x, y) = 0 \}$.*

Kempe's proof was flawed, however, because his constructions had additioanl configurations beyond those he intended them to have. Several more recent works reproduce versions of Kempe's argument, but also do not correctly address the additional configurations (e.g. [HJW], [GZCG]). The earliest rigorous proof of Theorem 1.1 we are aware of is the work of Kapovich and Millson, which was distributed as a preprint for several years before being published in 2002 [KM]. Henry King published a rigorous proof of this result in 1999, as a corollary of his work on cabled linkages, based on the (at the time unpublished) work of Kapovich and Millson [King].

However, Kapovich and Millson (and King following them) use different arguments from those presented in Kempe's paper. In particular, their universality is based on algebra over the complex plane, rather than trigonometry over the reals. As a result, the techniques of Kapovich and Millson do not generalize easily to d dimensions for d odd.

One might wonder whether Kempe's paper was fundamentally flawed, or whether there are simple changes that could be made to his proof to correct the flaws. In Chapters 3 and 4, we prove Theorem 1.1 using an argument that closely follows Kempe's original argument, bracing constructions where necessary. In contrast with [KM] and [King], our presentation is elementary. While there have been other elementary accounts of this result [JS], they use substantially more complex gadgets than those required for Kempe's original approach.

This thesis also addresses several interesting extensions and applications of Kempe's Universality Theorem. In order to state these results we must introduce some notation. If $\mathcal{L}$ is a linkage with n vertices, then $\mathrm{Conf}(\mathcal{L}) \subset (\mathbb{R}^d)^n$ is the configuration space of $\mathcal{L}$. These objects are defined precisely in Chapter 2.

1.1 Rigidity is Hard

In the field of rigidity theory, it has long been assumed that deciding whether a given configuration of a linkage is rigid (i.e. has no nontrivial motions starting from that configuration) is computationally intractable [DO]. In Chapter 5, we prove that this assumption is indeed correct:

Theorem 1.2. $\textsc{Rigidity}_d$, *the problem of deciding whether a given configuration of a d-dimensional linkage is rigid, is coNP-hard for all $d \geq 2$.*

Our reduction uses an efficient algorithm for computing a configuration of a linkage that projects to a given point of S to reduce from the coNP-hard problem of testing whether a given point in an algebraic variety defined by homogeneous equations of total degree 2 is isolated [Koiran]. While Gao et al. [GZCG] have previously presented an algorithm for computing the linkages of Kempe's Universality Theorem, their paper is not rigorous, having the same flaws as Kempe's paper, and they do not give a running time for their algorithm or work within a clear computational model.

Chapters 1–5 build up to the proof of Theorem 1.2. The remaining chapters contain a number of interesting results that follow relatively quickly from the lemmas needed to prove Theorem 1.2.

1.2 Signing Your Name

By the Weierstrauss Approximation Theorem, any continuous curve is well-approximated by an algebraic curve. Thus, Thurston summarized Kempe's Universality Theorem with the statement "there is a linkage that signs your name" [King]. Thurston's elegant statement is stronger than Theorem 1.1, because it suggests that if your signature is connected, one could draw your entire signature in a single continuous motion of the linkage. In Chapter 4, we prove the following stronger version of Theorem 1.1, which also follows from Kapovich and Millson's work:

Theorem 1.3 (Kapovich & Millson, [KM, Theorem E]). *Let $f \in \mathbb{R}[x, y]$ be a polynomial, and $S := \{ (x, y) \mid f(x, y) = 0\}$ be an algebraic curve. Let U be an open bounded subset of S. Then there is a linkage $\mathcal{L}$ so that the projection $p : \mathrm{Conf}(\mathcal{L}) \to \mathbb{R}^2$ onto the coordinates of a single vertex defines a covering space over U.*

By the lifting property of covering spaces, Theorem 1.3 implies that any path within the target set S lifts to a path in $\mathrm{Conf}(\mathcal{L})$. Thus you can indeed draw each connected component of your signature using a continuous motion of a linkage.

1.3 Higher Dimensions

In Chapter 6, we generalize Kempe's Universality Theorem to arbitrary dimensions d. Though this result was previously shown in a preprint by King [King3], our argument avoids reproving the properties of many 2-dimensional linkages in d dimensions by rotating the relevant distances into the x_1x_2-plane, and then appealing to our work in two dimensions. We prove the following theorem, generalized further to constraining the m points in $\mathbb{R}^d$:

Theorem 1.4 (King, [King3]). *Let $d \geq 2$. Let $f \in \mathbb{R}[x_{1,1}, \ldots, x_{m,d}]$ be a polynomial with real coefficients in dm variables of total degree n, and let B be a closed ball in $\mathbb{R}^d$. Then there exists a linkage over $\mathbb{R}^d$ such that the projection of $\mathrm{Conf}(\mathcal{L})$ onto the coordinates of m vertices is $B^m \cap \mathcal{Z}(f)$, where $\mathcal{Z}(f) = \{ (x_{1,1}, \ldots, x_{m,d}) : f(x_{1,1}, \ldots, x_{m,d}) = 0\}$ is the zero set of f.*

Theorem 1.4 follows fairly easily from Kempe's Universality Theorem in two dimensions, along with a d-dimensional Peaucellier inversor and a d-dimensional translator.

1.4 Characterization of Drawable Sets

Curves are not the only sets that can be drawn by a linkage. In Chapter 7, we prove the following characterization of drawable sets, first proved in another preprint by King:

Theorem 1.5 (King, [King2, King3]). *Let $d \geq 2$. Then a set $S \subset \mathbb{R}^d$ is drawable if and only if $S = \mathbb{R}^d$ or S is compact and semi-algebraic.*

Our proof of Theorem 1.5 is essentially a corollary of our proof of Kempe's Universality Theorem, using a modified Peaucellier inversor that constructs a half-space and a gadget constructing the union of two sets.

1.5 Construction Complexity

An important linkage design consideration is how complex a linkage needs to be in order to draw a given set. There are several natural complexity metrics for linkages: the number of bars, the ratio of the longest bar in the linkage to the shortest bar, and the maximum number of bars meeting at any vertex. We focus on the number of bars in the linkage.

Gao et al. [GZCG] obtained an $O(n^4)$ bound for the number of bars needed to draw a curve defined by a polynomial of degree n in two dimensions, and an $O(n^6)$ bound for drawing a curve defined by a polynomial of degree n in three dimensions. In our proof of Kempe's Universality Theorem (and its generalization to d dimensions), we show that drawing an algebraic curve of degree n in d dimensions can be done using $O(n^d)$ bars. We prove a more general result, that reduces to $O(n^d)$ in the case $m = 1$:

Theorem 1.6. *The linkage of Theorem 1.4 can be chosen to have* $O\left(\binom{n+dm}{n}\right)$ *bars.*

In Chapter 8, we prove a matching $\Omega(n^d)$ worst-case lower bound on the number of bars needed to draw the zero set of a single polynomial of degree n in d dimensions. We prove the following:

Theorem 1.7. *Drawing the zero-set of a polynomial function of total degree n in d variables requires $\Omega(n^d)$ bars in the worse case.*

Chapter 8 also obtains tight $\Theta(n)$ bounds on the number of bars required to draw an n-point set in the worst case.

1.6 Kempe's Strategy

In order to motivate the various linkages that we detail in this chapter, we now sketch Kempe's strategy for proving Theorem 1.1. Through a clever application of the trigonometric product-to-sum identities, rewrite the polynomial $f(x, y)$ within the ball B as a trigonometric expression of the form

$$f(x,y) = \sum_{|r|+|s|\leq n} f_{r,s}\cos(r\alpha + s\beta + \gamma_{r,s}),$$

where the $f_{r,s}$ and $\gamma_{r,s}$ are constants, the sum is over all pairs of integers (r, s) such that $|r| + |s| \leq n$, and

$$x = R\cos\alpha + R\cos\beta,$$
$$y = R\sin\alpha + R\sin\beta.$$

Then, use a series of clever gadgets to

1. construct the angles α and β from a point (x, y);

2. multiply angles by an integer, to construct $r\alpha$ and $s\beta$ (the "multiplicator");

3. add angles, to construct $r\alpha + s\beta + \gamma_{r,s}$ (the "additor"); a bar of length $f_{r,s}$ at this angle then has x-coordinate $f_{r,s}\cos(r\alpha + s\beta + \gamma_{r,s})$;

4. add the vectors constructed in the last step (the "translator"), to construct a point with x-coordinate $f(x, y)$; and

5. restrict a point to lie on a given line, to force the point whose x-coordinate is $f(x, y)$ onto $x = 0$ (a Peaucellier Inversor).

Our proof of Kempe's Universality Theorem closely follows Kempe's argument, deviating only where necessary to correct the proof or prove the continuity and rigidity properties that we need to prove $\textsc{Rigidity}_d$ is coNP-hard.

1.7 Generality of Kempe's Technique

Kapovich and Millson's techniques allow them to prove the following related universality theorem, first stated in oral lectures by W. Thurston in the late 1970s [KM] (though Thurston did not publish a proof):

Theorem 1.8 (Kapovich & Millson [KM, Corollary C]). *Let M be a smooth compact manifold. Then there is a linkage $\mathcal{L}$ such that* $\mathrm{Conf}(\mathcal{L})$ *is diffeomorphic to the disjoint union of a finite number of copies of M.*

We do not prove a result similar to Theorem 1.8 in this thesis. Note that unlike the results of this thesis, the relevant diffeomorphism is not in general a projection onto some of the coordinates of a configuration.

However, we observe that Kapovich and Millson describe several obstructions to using Kempe's techniques to obtain Theorem 1.8 and related results:

1. Some of Kempe's constructions have additional degenerate configurations (e.g. parallelograms can continuously deform into each other; see Section 3.1).

2. The projection $p : \mathrm{Conf}(\mathcal{L}) \to S$ might not be a covering.

3. The projection p might not be an analytically trivial covering.

This thesis directly addresses the first two of these obstructions in Kempe's constructions. However, the third obstruction cannot be avoided while following Kempe's constructions, because the additor's angle bisector construction requires a nontrivial cover. It was resolving this obstruction that caused Kapovich and Millson to use a multiplication construction based on some clever algebra in the complex plane. One could not easily draw a picture of the resulting linkage [KM].

The later chapters of this thesis rely on some results from real algebraic geometry. We have included the relevant definitions in the body of this thesis as needed, and we have collected the statements of required theorems from [BCR] in Appendix A.

Chapter 2

Linkages

We define linkages embedded in d-dimensional space, where $d \geq 2$. The definitions are also valid when $d = 1$, but the theorems are not; we consider the special case $d = 1$ in Section 6.4.

Definition 2.1. An *abstract linkage* is a pair $\mathcal{L} = (G, \ell)$ consisting of a graph $G = (V(G), E(G))$ and a function $\ell : E(G) \to \mathbb{R}_{\geq 0}$ that defines the lengths of the edges. We refer to the edges of G as *bars*.

Definition 2.2. A *linkage* in d dimensions is an abstract linkage $\mathcal{L} = (G, \ell)$ together with a function $f : W \to \mathbb{R}^d$ defined on a subset W of $V(G)$, assigning these vertices fixed locations in $\mathbb{R}^d$.

Definition 2.3. A *configuration* C of a linkage $\mathcal{L} = (G, \ell)$ in d dimensions is a map $C : V(\mathcal{L}) \to \mathbb{R}^d$ obeying the length and fixing constraints, i.e., C extends the fixing assignment f and if $(v, w) \in \mathcal{E}(G)$ then $|C(v) - C(w)| = \ell(v, w)$. The set of all such configurations is called the *configuration space* $\mathrm{Conf}(\mathcal{L})$ of $\mathcal{L}$.

Drawable sets formalize the idea of drawing a set with a pen attached to one vertex of a linkage.

Definition 2.4. The *trace* of a vertex v of a linkage $\mathcal{L}$ is the image of $\mathrm{Conf}(\mathcal{L})$ under the projection $\mathrm{tr}_v : \mathrm{Conf}(\mathcal{L}) \to \mathbb{R}^d$, $\mathrm{tr}_v(C) = C(v)$. Equivalently, it is the locus of positions of the vertex v in the configurations of $\mathcal{L}$. A linkage $\mathcal{L}$ *draws* a set $S \subset \mathbb{R}^d$ if there is a vertex $v \in \mathcal{L}$ whose trace is S. A set $S \subset \mathbb{R}^d$ is *drawable* if there exists a linkage $\mathcal{L}$ that draws S.

2.1 Constructible Sets

We construct linkages from simple components, each of which computes a function or imposes a relation on some subset of its vertices. For example, we would like to impose the relation that x is a point of some line ℓ. However, no single linkage can impose precisely this relation:

Proposition 2.5. *A drawable set S is either bounded or is $\mathbb{R}^d$.*

PROOF. Because S is drawable, S is the trace of a vertex v in a linkage $\mathcal{L}$. Suppose the connected component of $\mathcal{L}$ containing v contains a fixed vertex u. Then in any configuration, the vertex cannot be further from u than the sum of the lengths of all bars in that component, which is clearly finite, hence S is bounded. If v's component contains no fixed vertex, then given any configuration, we can obtain a new one by translating the connected component of $\mathcal{L}$ containing v so that v is at point in $\mathbb{R}^d$. Thus in this case the trace of v either is $\mathbb{R}^d$ or is empty, and thus bounded. ∎

The Peaucellier linkage is advertised as "drawing a straight line", but technically, it draws a straight line segment. Now, one can draw an arbitrarily large line segment by using a sufficiently large Peaucellier linkage. However, it would be extremely tedious to formulate our arguments in terms of line segments rather than lines. Thus, in order to prove theorems about linkages constructing unbounded objects such as lines, we must work with families of linkages, where for any bounded set $U \subset \mathbb{R}^d$, there is a linkage that with the desired property within U. We formalize this idea with *constructible sets*:

Definition 2.6. For an integer n, a closed set $S \subset (\mathbb{R}^d)^n$ is *constructible using N bars* if, for every bounded open subset U of $\mathbb{R}^d$, there is a linkage $\mathcal{L}$ with at most N bars and an n-tuple $(v_1, \ldots, v_n)$ of vertices of $\mathcal{L}$ such that

$$p(\mathrm{Conf}(\mathcal{L})) \cap U^n = S \cap U^n$$

where $p : \mathrm{Conf}(\mathcal{L}) \to (\mathbb{R}^d)^n$ is the projection onto the coordinates of vertices $(v_1, \ldots, v_n)$. In this situation, we say that $(\mathcal{L}, v_1, \ldots, v_n)$ *constructs* S inside the set U, or simply that $\mathcal{L}$ constructs S inside U.

One might worry that the structure of $\mathcal{L}$ could depend on the set U. In our constructions, only the bar lengths ℓ depend on U. There is no loss of generality: every bounded set U is contained in some ball, so consider the linkages corresponding to an increasing sequence of balls U_i. Because there are only finitely many linkage structures with at most N bars, some structure appears infinitely often in the list of linkages corresponding to the U_i's. Thus without loss of generality, we may assume the structure of $\mathcal{L}$ does not depend on U.

Knowing that a set is constructible does not guarantee that connected components of the set can be drawn continuously with a pen attached to a vertex of a linkage. To do this, we need stronger notions of constructibility. Because the formal definitions are somewhat technical, we begin with motivation and examples.

Continuous constructibility is the property needed to continuously draw a set using a pen. Informally, it requires that we can choose the linkage $\mathcal{L}$ so that starting from any configuration of $\mathcal{L}$, vertices $(v_1, \ldots, v_n)$ can move continuously within $S \cap U^n$ without having to suddenly reconfigure the linkage.

Rigid contructibility is the other property needed to prove the coNP-hardness of testing the rigidity of a given configuration of a linkage. Informally, rigid constructibility requires that we can choose $\mathcal{L}$ so that for any point x, $p^{-1}(x)$ is a finite set, so that $\mathcal{L}$ has no motions other than those necessary to continuously construct S.

Continuously constructible and rigidly constructible are orthogonal notions. The linkage shown in Figure 2-1 continuously constructs the annulus (as the trace of vertex C), but it does not rigidly construct the annulus because there are infinitely many configurations projecting down to any point on the circle drawn by vertex A. Watt's linkage (recall Figure 1-1(a)) is an example of a linkage that rigidly, but not continuously constructs its trace: vertex c cannot change directions at the point where its locus crosses itself without moving a and b discontinuously.

We often prove that sets are "continuously and rigidly constructible"; this means that the same linkage $\mathcal{L}$ has both properties.

Before we can state the formal definitions, we need to introduce some standard mathematical terminology.

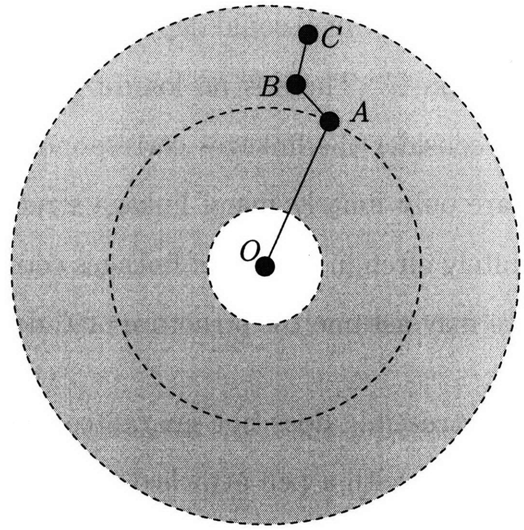


Figure 2-1: A linkage that does not rigidly construct its trace.

Definition 2.7. If $f : A \to B$ is a map, then the *fibre* of f over $b \in B$ is the set $f^{-1}(\{b\})$.

Definition 2.8. A set A is a *covering space* of a set B if there is a surjective continuous map $f : A \to B$ such that every $x \in B$ has a neighborhood U whose inverse image $f^{-1}(U)$ is a disjoint union of open sets, each mapped homeomorphically onto U by f.

In this situation we say that f is a *covering space map*, and that B is the *base* of the covering.

A covering space has N *sheets* if the fibres of f each have cardinality N.

Some simple examples of covering space maps include the map $z \mapsto z^n$ on $\mathbb{C} \setminus \{0\}$ and the projection onto the first coordinate of the set $\mathbb{R} \times \{0, 1\}$.

We now precisely define the various stronger notions of constructibility:

Definition 2.9. A set S is *continuously constructible* if, in the definition of constructibility, $\mathcal{L}$ can additionally be chosen so that for any path γ in $S \cap U^n$ starting from a point P, and point $Q \in p^{-1}(P)$, there is a path γ' in $\mathrm{Conf}(\mathcal{L})$ starting at Q lifting γ, so that $p \circ \gamma' = \gamma$. In this case we say that γ' *lifts* γ starting from Q.

A set S is *rigidly constructible* if, in the definition of constructibility, $\mathcal{L}$ can additionally be chosen so that fibres of $p : \mathrm{Conf}(\mathcal{L}) \cap p^{-1}(U^n) \to S \cap U^n$ are all finite.

A set S is *nicely constructible* if, in the definition of constructibility, $\mathcal{L}$ can additionally be chosen so that $\mathrm{Conf}(\mathcal{L}) \cap p^{-1}(U^n)$ is a covering space of $p(\mathrm{Conf}(\mathcal{L})) \cap U^n = S \cap U^n$.

Theorem 1.3 can be restated as the claim that $\{(x, y) \in \mathbb{R}^d : f(x, y) = 0\}$ is nicely constructible.

In Section 2.2, we prove nicely constructible implies continuously and rigidly constructible. One might hope for an even stronger notion, where $\mathrm{Conf}(\mathcal{L}) \cap p^{-1}(U^n)$ is isomorphic to $S \cap U^n$. However, this is in general impossible, because the disk is a 2-manifold with boundary, and thus cannot be diffeomorphic to an algebraic set ([KM], Remark 1.5).

The following proposition gives an example of a linkage that continuously and rigidly constructs its trace, but does not nicely construct its trace.

Proposition 2.10. *For any* $0 \leq r \leq R$, *the annulus*

$$A = \left\{(O, X) \in (\mathbb{R}^d)^2 : r \leq |OX| \leq R\right\}$$

is constructible. If $r > 0$, *it is continuously constructible. If* $r > 0$ *and* $d = 2$, *it is rigidly constructible.*

Proof. Construct the linkage $\mathcal{L}$ with a bar of length $\frac{R-r}{2}$ connecting O to new vertex Y, and a bar of length $\frac{R+r}{2}$ connecting X to Y. See Figure 2-2. If O and X lie at two points at distance d, then by the triangle inequality, we can find a location for Y satisfying the bar length constraints if and only if $r \leq d \leq R$. Thus A is constructible.

If $r > 0$, we claim then $\mathcal{L}$ continuously constructs A. If U is the interior of A, $\mathrm{Conf}(\mathcal{L}) \cap p^{-1}(U^2)$ is homeomorphic to $S^{d-2} \times (A \cap U^2)$, where S^k is the unit sphere in $\mathbb{R}^{k+1}$. On the boundary of A, the points of the S^{d-2} component are identified. This defines a continuous surjective map $g : S^{d-2} \times A \to \mathrm{Conf}(\mathcal{L})$. Given a path h in A starting at a given point $p \in \mathrm{Conf}(\mathcal{L})$, let $q \in g^{-1}(p)$. We can continuously lift the path in A to $S^{d-2} \times A$ starting at q; simply keep the first component fixed. Since g is continuous, we can now map this lifted path over to $\mathrm{Conf}(\mathcal{L})$, giving a continuous lift of h, starting at p, as desired.

If $r > 0$ and $d = 2$, then given the locations of O and X there are at most two possible locations for Y at the intersections of the distinct circles of radius $\frac{R+r}{2}$ about X and of radius $\frac{R-r}{2}$ about O, so the fibres are always finite. ∎

Following Kapovich and Millson, we call this linkage a "hook". If $r = 0$, $\mathcal{L}$ does not continuously construct the disk. Consider, for example, a path that maps each interval

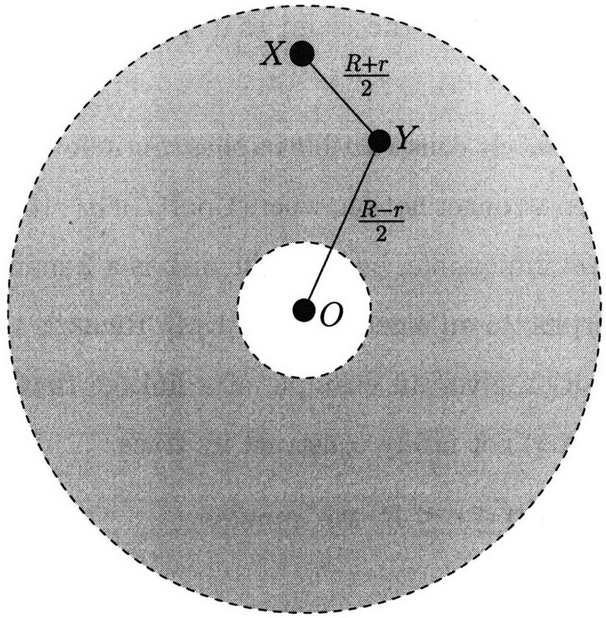


Figure 2-2: The hook linkage; the trace of B is shown in gray.

$[2^{-(i+1)}, 2^{-i}]$ to a path of length 2^{-i} going out from the origin and back alternating between two orthogonal directions. This path is continuous but no lift to $\mathrm{Conf}(\mathcal{L})$ can be continuous. In dimension $d > 2$, $\mathcal{L}$ is not rigidly constructible, because Y can rotate about the line connecting O and X. Even if $r > 0$ and $d = 2$, $\mathcal{L}$ does not nicely construct the annulus, because any neighborhood of a point on the boundary of the annulus contains interior points with 2-element fibres and boundary points with 1-element fibres.

One simple construction we use frequently is to construct a point at a given fixed distance along an existing bar, using a degenerate triangle linkage.

Proposition 2.11. *For any positive reals L_1 and L_2,*

$$S = \left\{ (O, P, Q) \in (\mathbb{R}^d)^3 : \frac{\overrightarrow{OP}}{|OP|} = \frac{\overrightarrow{OQ}}{|OQ|}, |OP| = L_1 \text{ and } |OQ| = L_2 \right\}$$

is nicely constructible.

PROOF. Without loss of generality, suppose $L_2 \geq L_1$. Construct the degenerate triangle linkage $\mathcal{L}$ with bars OP, PQ, OQ such that $|OP| = L_1, |OQ| = L_2, |PQ| = L_2 - L_1$. See Figure 2-3. In any configuration of this linkage, we have $|PQ| = |OQ - OP| = |OQ| - |OP|$, so by the triangle inequality O, P, and Q are colinear, and $\frac{\overrightarrow{OP}}{|OP|} = \frac{\overrightarrow{OQ}}{|OQ|}$. Thus $\mathrm{Conf}(\mathcal{L}) \subset S$.

Conversely, if $(O, P, Q) \in S$, then O, P, and Q are colinear, so $|PQ| = |OQ| - |OP|$. Thus $S \subset \mathrm{Conf}(\mathcal{L})$. Thus with the trivial projection p and any set U, $S \cap U^3 = p(\mathrm{Conf}(\mathcal{L})) \cap U^3$, and S is constructible. $\mathrm{Conf}(\mathcal{L}) \cap p^{-1}(U^3) = \mathrm{Conf}(\mathcal{L}) \cap U^3 = S \cap U^3$, so p is a homeomorphism and S is nicely constructible. ■

Figure 2-3: The degenerate triangle linkage

2.2 Properties of Constructible Sets

We should remark on a few properties of these stronger notions of constructibility. The product of two covering spaces is a covering space of the product of the bases. The composition of two covering space maps is a covering space map. The preimage of a subspace of the base under a covering space map is a covering space of that subspace of the base. It is easy to check that both maps with the path-lifting property and maps with finite fibres are preserved under product, composition, and restriction to a subspace of the domain and its preimage.

Proposition 2.12. *If S is nicely constructible, then it is continuously and rigidly constructible.*

Proof. That a nicely constructible set is continuously constructible is the lifting property of covering spaces: if X is a covering space over S, then any path $f : [0, 1] \to S$ in S lifts uniquely to a path in X starting at any lift of $f(0)$.

Covering spaces of semi-algebraic sets with semi-algebraic covering space maps always have finite fibres. The fibres of a covering space are always discrete, but they are also semi-algebraic, as the pre-images of a semi-algebraic set (a point) under a semi-algebraic map. By Theorem 2.4.5 of [BCR], they are finite. ■

In designing complex linkages, it is often convenient to "forget about" some vertices of a constructible set S by projecting down to a subset of the vertices of S. However, the projection of a constructible set might not be constructible, because it is not necessarily closed.

For example, $\{((x_1, y_1), (x_2, y_2)) \in (\mathbb{R}^2)^2 : x_1 x_2 = 1\}$ is constructible, but its projection onto the first coordinate is an open half-plane, which is not closed. The following proposition shows that this is the only possible obstruction:

Proposition 2.13. *Let $S \subset (\mathbb{R}^d)^m \times (\mathbb{R}^d)^n$ be constructible using N bars, and let $p : (\mathbb{R}^d)^m \times (\mathbb{R}^d)^n \to (\mathbb{R}^d)^m$ be the projection onto the first factor. If $p(S)$ is closed, then $p(S)$ is also constructible using N bars.*

If any path in $p(S)$ lifts to a path in S and S is continuously constructible, then $p(S)$ is continuously constructible.

If $p : S \to p(S)$ has finite fibres, and S is rigidly constructible, then $p(S)$ is also rigidly constructible.

If S is a covering space of $p(S)$ and S is nicely constructible, then $p(S)$ is nicely constructible.

PROOF. Let U be a bounded open subset of $\mathbb{R}^d$. Choose a compact set $K \subset \mathbb{R}^d$ containing U. Let $V_1 \subset V_2 \subset \cdots \subset \mathbb{R}^d$ be an increasing chain of bounded open sets whose union is $\mathbb{R}^d$. Then the compact set $p(S) \cap K^m$ is the union of its open subsets $p(S \cap (K^m \times V_i^n))$, so by compactness $p(S) \cap K^m = p(S \cap (K^m \times V_{i_0}^n))$ for some i_0. Let W be an bounded open subset of V containing both K and V_{i_0}, and let $\mathcal{L}$ be a linkage with vertices $(v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+n})$ such that the projection of $\mathrm{Conf}(\mathcal{L})$ to the coordinates of these vertices meets W^{m+n} in $S \cap W^{m+n}$. Then the projection of $\mathrm{Conf}(\mathcal{L})$ to the coordinates of $(v_1, \ldots, v_m)$ meets K^m in $p(S \cap (W^m \times W^n)) \cap K^m = p(S) \cap K^m$. In particular, $(\mathcal{L}, v_1, \ldots, v_n)$ constructs $p(S)$ inside U.

Because the path-lifting property is preserved under composition, if any path in $p(S) \cap U^m$ lifts to a path in $S \cap U^m \times U^n$, and S is continuously constructible, then $p(S)$ is continuously constructible.

Because the property of having finite fibres is preserved under composition, if $p : S \to p(S)$ has finite fibres and S is rigidly constructible, then $p(S)$ is rigidly constructible.

Because the composition of covering space maps where the second map has finitely many sheets is a covering space map, if S is nicely constructible and S is a covering space of $p(S)$, then $p(S)$ is nicely constructible. ∎

We often combine several linkages together by gluing some vertices together. The following proposition shows that gluing preserves our various notions of constructibility.

Proposition 2.14. *If $S \subset (\mathbb{R}^d)^m \times (\mathbb{R}^d)^n$ and $T \subset (\mathbb{R}^d)^n \times (\mathbb{R}^d)^p$ are constructible using N_S and N_T bars, respectively, then $R = \{ (x,y,z) \in (\mathbb{R}^d)^m \times (\mathbb{R}^d)^n \times (\mathbb{R}^d)^p : (x,y) \in S \text{ and } (y,z) \in T \}$ is constructible using $N_S + N_T$ bars.*

If S and T are both continuously constructible, then R is continuously constructible.

If S and T are both rigidly constructible, then R is rigidly constructible.

If S and T are both nicely constructible, then R is nicely constructible.

Proof. Given a bounded open set $U \subset \mathbb{R}^d$, let $\mathcal{L}_S$ be a linkage that constructs S inside U with projection p_S and $\mathcal{L}_T$ be a linkage that constructs T inside U with projection p_T.

Let $R' = S \times T \subset ((\mathbb{R}^d)^m \times (\mathbb{R}^d)^n) \times ((\mathbb{R}^d)^n \times (\mathbb{R}^d)^p)$, and $\mathcal{L}_{R'} = \mathcal{L}_S \sqcup \mathcal{L}_T$. Let $Z = \{ (x,y,y',z) \in (\mathbb{R}^d)^m \times (\mathbb{R}^d)^n \times (\mathbb{R}^d)^n \times (\mathbb{R}^d)^p : y = y' \}$, so that R is canonically isomorphic to $Z \cap R'$. Let $\mathcal{L}_{R' \cap Z}$ be $\mathcal{L}_{R'}$ along with additional 0-length bars connecting the corresponding points of $(\mathbb{R}^d)^n$, and let $\mathcal{L}_R$ be the linkage obtained by deleting the 0-length bars of $\mathcal{L}_{R' \cap Z}$ and identifying their endpoints. Let $p_{R'} = (p_S, p_T)$, $p_{R' \cap Z}$ be the restriction of $p_{R'}$ to $Z \cap R'$, and p_R be the projection from $\operatorname{Conf}(\mathcal{L}_R)$ to R that agrees with p_S and p_T.

We claim $p_R(\operatorname{Conf}(\mathcal{L}_R)) \cap U^{m+n+p} = R \cap U^{m+n+p}$, so that R is constructible using $N_S + N_T$ bars. If $(x,y,z) \in R \cap U^m \times U^n \times U^p$, then $(x,y) \in S \cap U^{m+n}$, $(y,z) \in T \cap U^{n+p}$. Because $\mathcal{L}_S$ and $\mathcal{L}_T$ construct S and T, respectively, $p_S^{-1}(x,y) \in \operatorname{Conf}(\mathcal{L}_S) \cap p_S^{-1}(U^{m+n})$, $p_T^{-1}(y,z) \in \operatorname{Conf}(\mathcal{L}_T) \cap p_T^{-1}(U^{n+p})$. Thus $p_{R'}^{-1}(x,y,y,z) \in \operatorname{Conf}(\mathcal{L}_{R'}) \cap p_{R'}^{-1}(U^{m+2n+p})$. Because the middle two coordinates are equal, $p_{R'}^{-1}(x,y,y,z) \in \operatorname{Conf}(\mathcal{L}_{R' \cap Z}) \cap p_{R'}^{-1}(U^{m+2n+p})$. Thus $p_R^{-1}(x,y,z) \in \operatorname{Conf}(\mathcal{L}_R) \cap p_R^{-1}(U^{m+n+p})$. The proof of the converse is similar.

Now suppose that $\mathcal{L}_S$ and $\mathcal{L}_T$ nicely construct S and T, respectively. Because a product of covering spaces is a covering space of the product of the bases, $\operatorname{Conf}(\mathcal{L}_{R'}) \cap p_{R'}^{-1}(U^{m+2n+p}) = \operatorname{Conf}(\mathcal{L}_S \sqcup \mathcal{L}_T) \cap p_{R'}^{-1}(U^{m+2n+p}) = \operatorname{Conf}(\mathcal{L}_S) \cap p_S^{-1}(U^{m+n}) \times \operatorname{Conf}(\mathcal{L}_T) \cap p_T^{-1}(U^{n+p})$ is a covering space of $S \cap U^{m+n} \times T \cap U^{n+p} = R' \cap U^{m+2n+p}$. Because $Z \cap R' \cap U^{m+2n+p}$ is a subspace of $R' \cap U^{m+2n+p}$ and $\operatorname{Conf}(\mathcal{L}_{R' \cap Z}) \cap p_{R'}^{-1}(U^{m+2n+p})$ is its preimage under $p_{R'}$, $Z \cap R' \cap U^{m+2n+p}$ is a covering space of $\operatorname{Conf}(\mathcal{L}_{R' \cap Z}) \cap p_{R'}^{-1}(U^{m+2n+p})$. Because $R \cong Z \cap R'$ and $\operatorname{Conf}(\mathcal{L}_R) \cong \operatorname{Conf}(\mathcal{L}_{R' \cap Z})$, it follows that $\operatorname{Conf}(\mathcal{L}_R) \cap p_R^{-1}(U^{m+n+p})$ is a covering space of $R \cap U^{m+n+p}$, so that R is nicely constructible.

Because both continuous constructibility and rigid constructibility are also preserved under products and restrictions to subspaces, the proof that the nicely constructible property is preserved under gluing works for these properties as well. ■

The following result extends Proposition 2.10.

Proposition 2.15. *Suppose T is a continously constructible set with vertices O and X (among others) such that at any point of T, $r < |OX| < R$. Then the set S obtained by gluing the linkage constructing T to a hook linkage OYX constructing*

$$A = \{(O, X) \in (\mathbb{R}^d)^2 : r \leq |OX| \leq R\}$$

is continuously constructible. If $d = 2$ and T is nicely constructible, then S is nicely constructible.

PROOF. Let $\mathcal{L}$ be the hook linkage. Let $U \subset \mathbb{R}^d$ be any open set contained in the interior of A and containing $p(T)$, where p is the projection down to vertices O and X of T. Then $\mathrm{Conf}(\mathcal{L}) \cap p^{-1}(U^2)$ is homeomorphic to $(A \cap U^2) \times S^{d-2}$. Then by the lifting argument from the $r > 0$ case of Proposition 2.10, S is continuously constructible.

If $d = 2$, then $\mathrm{Conf}(\mathcal{L}) \cap p^{-1}(U^2)$ is homeomorphic to $(A \cap U^2) \times \{0, 1\}$. Thus $\mathrm{Conf}(\mathcal{L}) \cap p^{-1}(U^2)$ is the union of two disjoint sets, each homeomorphic to $A \cap U^2$, and the projection is a covering space map. ■

It is not possible to strengthen Proposition 2.15 to an isomorphism, rather than a covering space. This is essentially the only reason why the best we can obtain is a covering space rather than an isomorphism. King's work on cabled linkages shows this in a formal sense. He proves that every compact algebraic set is analytically isomorphic to the configuration space of a cabled linkage via a polynomial map, and derives Theorem 1.3 from this by replacing each cabled bar with a hook [King].

In Chapter 7, we give a complete characterization of the possible drawable sets. To do this, we transfer results about bounded constructible sets to drawable sets using the following proposition:

Proposition 2.16. *A bounded set $S \subset \mathbb{R}^d$ is drawable if and only if it is constructible.*

PROOF. First, suppose that S is drawable. Then there is a linkage $\mathcal{L}$ and a projection p such that $p(\mathrm{Conf}(\mathcal{L})) = S$. Then for any bounded set U, $p(\mathrm{Conf}(\mathcal{L})) \cap U = S \cap U$, and S is constructible.

Conversely, suppose that S is constructible. Let U be an open ball of radius R containing S (this must exist because S is bounded). Because S is constructible, there is a linkage $\mathcal{L}$ and a projection p such that $p(\mathrm{Conf}(\mathcal{L})) \cap U = S \cap U = S$. We can create a linkage $\mathcal{L}'$ by attaching to $\mathcal{L}$'s output vertex v a hook defined by Proposition 2.10 constraining v to have a distance from the origin between 0 and R. Then $p(\mathrm{Conf}(\mathcal{L}')) = p(\mathrm{Conf}(\mathcal{L})) \cap U = S$, and S is drawable. ■

While there is a unique unbounded drawable set in d dimensions ($\mathbb{R}^d$), there are many unbounded constructible sets.

Chapter 3

Elementary Linkages

In this chapter, we show how to build gadgets constructing various simple sets that are useful for proving Kempe's Universality Theorem. For this chapter and the next, we work in two dimensions ($d = 2$) as our goal is to prove Kempe's Universality Theorem in the plane.

3.1 Parallelograms and Contraparallelograms

Let $\mathcal{L} = ABCD$ be a rectangle linkage, with $|AB| = |CD|$ and $|BC| = |DA|$. The configurations of $\mathcal{L}$ fall into three classes:

(i) *parallelograms*, where $AB \parallel CD$ and $BC \parallel DA$;

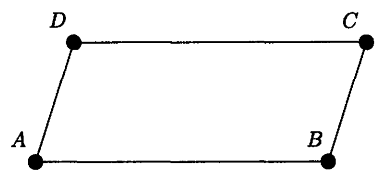
(ii) *degenerate*, where A, B, C, D are colinear;

(iii) *contraparallelograms*, where $AC \parallel BD$.

Kempe's paper used both parallelograms and contraparallelograms to construct various gadgets. Recall that the primary error in his flawed proof of Theorem 1.1 was his failure to consider the continuous motions between the different types of configurations of the rectangle linkage (see Figure 3-1).

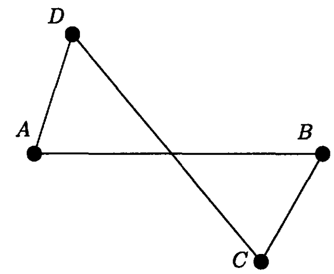
In this section, we show how one can "brace" a rectangular linkage by adding vertices and bars to produce linkages that construct the parallelogram configuration space and the contraparallelogram configuration space (each configuration space contains the degenerate configurations). Then we prove that, with these bracings, the gadgets in Kempe's original design rigidly construct various useful sets.

The parallelogram bracing was previously known [KM], but the contraparallelogram bracing is novel.



(a) parallelogram

(b) degenerate



(c) contraparallelogram

Figure 3-1: The various configurations of a rectangle linkage.

3.1.1 Parallelograms

Proposition 3.1. *For any $a, b > 0$, the parallelogram configuration space*

$$S = \{(A, B, C, D) \in (\mathbb{R}^2)^4 : |AB| = |CD| = a, |BC| = |AD| = b, AB \parallel CD, BC \parallel AD\}$$

is nicely constructible.

PROOF. We brace a rectangle linkage $\mathcal{L}$ to remove the contraparallelogram configurations as follows. By gluing degenerate triangle linkages, construct a new vertex M at the midpoint of AB and a new vertex N at the midpoint of CD. Then add a new bar MN of length $|BC|$ to obtain a new linkage $\mathcal{L}'$, as shown in Figure 3-2. Let p be the projection from configurations of $\mathcal{L}'$ to configurations of $\mathcal{L}$ that forgets about M and N.

Because S is defined by closed conditions, it is closed. We show that $p(\text{Conf}(\mathcal{L}')) = S$. From this it follows that, for any U, $p(\text{Conf}(\mathcal{L}')) \cap U^4 = S \cap U^4$, and thus S is constructible.

In a parallelogram configuration, the distance between the midpoints of AB and CD is always equal to $|BC|$, so any degenerate or parallelogram configuration of $\mathcal{L}$ can be extended to a configuration of $\mathcal{L}'$. Thus $S \subset p(\text{Conf}(\mathcal{L}'))$.

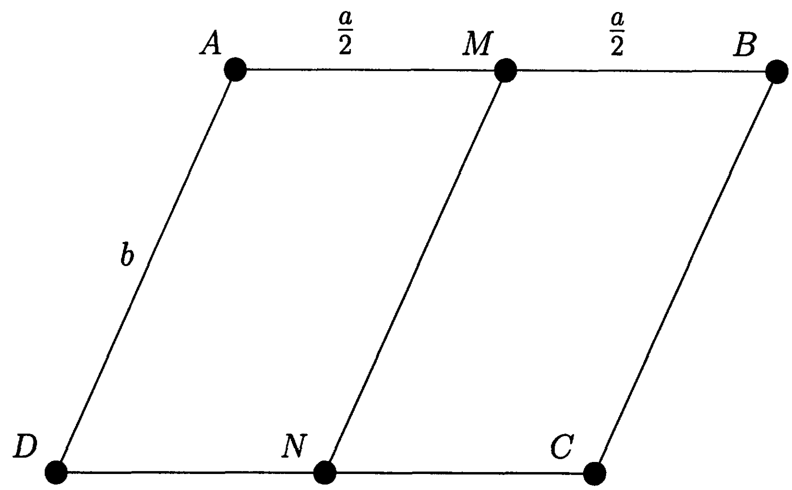


Figure 3-2: A braced parallelogram.

We now show $p(\mathrm{Conf}(\mathcal{L}')) \subset S$, so that a (nondegenerate) contraparallelogram configuration of $\mathcal{L}$ cannot be extended to a configuration of $\mathcal{L}'$. Note that the locations of M, N are determined by the locations of A, B, C, D, so we need only check whether the new bar has the right length. In a nondegenerate contraparallelogram configuration, $MN = \frac{AC+BD}{2}$. Let X be the intersection of AD and BC. Then by the triangle inequality,

$$\begin{aligned} 2|MN| = |AC| + |BD| &< (|AX| + |XC|) + (|BX| + |XD|) \\ &= (|AX| + |XD|) + (|BX| + |XC|) \\ &= |AD| + |BC| = 2|BC|, \end{aligned}$$

so $|MN| < |BC|$, a contradiction. Thus the only contraparallelogram configurations are degenerate.

To see that S is nicely constructible, notice that given $P = (A, B, C, D) \in S$, $M = \frac{A+B}{2}$ and $N = \frac{C+D}{2}$, so S is in fact homeomorphic to $\mathrm{Conf}(\mathcal{L}')$. Thus S is nicely constructible. ∎

Corollary 3.2. *For $R_1 > R_2 > 0$, the parallelogram configuration space*

$$S = \left\{(O, M, N, V) \in (\mathbb{R}^2)^4 : \overrightarrow{OV} = \overrightarrow{OM} + \overrightarrow{ON}, |OM| = R_1, |ON| = R_2\right\}$$

is nicely constructible.

PROOF. This is simply an equivalent description of the parallelogram configuration space. ■

3.1.2 Contraparallelograms

Proposition 3.3. *For any distinct $a, b > 0$, the contraparallelogram configuration space*

$$S = \{(A, B, C, D) \in (\mathbb{R}^2)^4 : |AB| = |CD| = a, |BC| = |AD| = b, AC \parallel BD\}$$

is nicely constructible.

PROOF. Because $|AB| = a \neq |AD| = b$, there is just one degenerate configuration up to a rigid motion. Without loss of generality, assume $a > b$. We brace the linkage $\mathcal{L}$ as follows. Let K, L, M, N be vertices at the midpoints of bars AB, BC, CD, DA respectively, and add a vertex X connected to K and M by bars of length R_1 and to L and N by bars of length R_2, where R_1 and R_2 are large and satisfy $R_2^2 - R_1^2 = \frac{1}{4}(a^2 - b^2)$. Call this new linkage $\mathcal{L}'$. See Figure 3-3. We must show that $\mathcal{L}'$ is a braced contraparallelogram constructing S. It suffices to show $p(\mathrm{Conf}(\mathcal{L}')) = S$, and thus for any U, $p(\mathrm{Conf}(\mathcal{L}')) \cap U^4 = S \cap U^4$, so that the contraparallelogram S is constructible.

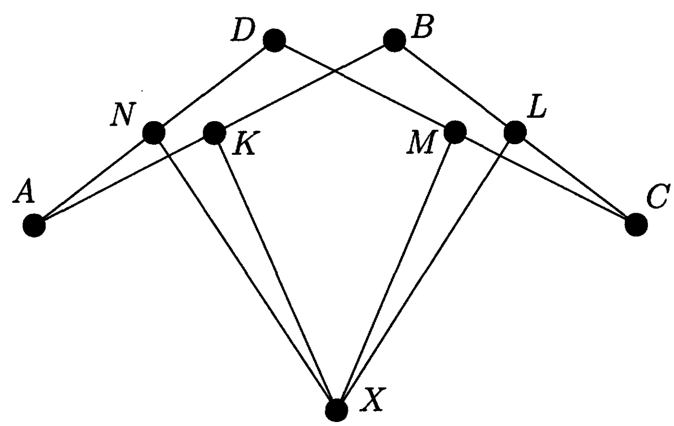


Figure 3-3: A braced contraparallelogram.

First we show $p(\mathrm{Conf}(\mathcal{L}')) \subset S$, or equivalently that a (nondegenerate) parallelogram configuration of $\mathcal{L}$ cannot be extended to a configuration of $\mathcal{L}'$. Let $ABCD$ be a nondegenerate parallelogram configuration of $\mathcal{L}$. Then $KLMN$ is also a parallelogram. Suppose X is a point such that $XK = XM = R_1$ and $XL = XN = R_2$. Then X lies on the perpendicular bisectors of KM and LN. Since $KLMN$ is a parallelogram, these perpendicular bisectors intersect in a single point, the center O of parallelogram $KLMN$. But X cannot lie at O because $R_1 > OK$ and $R_2 > OL$ (because we took R_1 and R_2 to be large). Hence there is no extension to a configuration of $\mathcal{L}'$.

It now suffices to show $p(\mathrm{Conf}(\mathcal{L}')) \supset S$. To do this, we need some results on the geometry of the pieces of $\mathcal{L}'$.

Lemma 3.4. *Let $ABCD$ be a contraparallelogram (possibly degenerate) with $AB = CD > AD = BC$ and let K, L, M, N be the midpoints of sides AB, BC, CD, DA, respectively. Then K, L, M, N are colinear, with K and M between L and N, $NK = ML$, and $NK \cdot NM = \frac{1}{4}(AB^2 - AD^2)$.*

Proof. The only statement not clear by inspection is the last one. Let P be the midpoint of BD and let H be the foot of the altitude from A to BD. Then $NK = \frac{1}{2}BD = DP$ and $NM = \frac{1}{2}AC = HP$, so $NK \cdot NM = DP \cdot HP$. But $DP \cdot HP$ is just the power of P with respect to the circumcircle of triangle ADH. This circle has center at N and radius $AN = \frac{1}{2}AD$, so $NK \cdot NM = DP \cdot HP = PN^2 - AN^2 = \frac{1}{4}(AB^2 - AD^2)$ because PN is a midline in triangle ABD. This proof requires $ABCD$ to be non-degenerate, but by continuity, the claim holds in the degenerate case too. ∎

Lemma 3.5. *Let K, L, M, N be colinear points ($N \neq L$) with K and M between L and N and $NK = ML$. Let X be a point on the perpendicular bisector of segment NL. Then $XN^2 - XK^2 = NK \cdot NM$.*

Proof. Let Q be the midpoint of segment NL, so XQ is perpendicular to the line through K, L, M, N. Since $NK = ML$, Q is also the midpoint of segment KM. Thus

$$\begin{aligned} XN^2 - XK^2 &= (XQ^2 + NQ^2) - (XQ^2 + KQ^2) \\ &= (NQ - KQ)(NQ + KQ) \\ &= NK \cdot NM \end{aligned}$$

as claimed. ∎

Lemma 3.5 implies that a contraparallelogram can be used to perform geometric inversion; Hart's Inversor was based on this observation.

Now we are ready to show that $p(\mathrm{Conf}(\mathcal{L}')) \supset S$, or equivalently that any degenerate or contraparallelogram configuration of $\mathcal{L}$ can be extended to a configuration of $\mathcal{L}'$.

Given any configuration of $\mathcal{L}$, we must find a point X such that $XK = XM = R_1$ and $XL = XN = R_2$. Choose X to be a point on the perpendicular bisector of NL such that $XL = XN = R_2$; we can do this because we chose R_2 to be large. Now, by Lemmas 1 and 2, $XN^2 - XK^2 = \frac{1}{4}(AB^2 - AD^2) = R_2^2 - R_1^2$, so $XK = XM = R_1$.

To see that S is nicely constructible, notice that K, L, M, N are each linear combinations of A, B, C, D, and X can be at either intersection of the circles of radius R_1 about K and M (giving two disconnected components of $\mathrm{Conf}(\mathcal{L})$, each homeomorphic to S). Thus $\mathrm{Conf}(\mathcal{L})$ is a 2-sheeted covering space of S, and S is nicely constructible. ■

To avoid cluttering our diagrams, we omit bracings of parallelograms and contraparallelograms from diagrams in subsequent sections.

3.2 Multiplying and Adding Angles

One can represent an angle θ by a pair of bars with a common vertex O. Often one bar is fixed, pointed away from O in the positive x direction. Following Kempe, we build gadgets to manipulate angles so that we can construct bars of length f_{rs} at angles $r\alpha + s\beta + \gamma_{rs}$ (see Section 1.6 for a reminder on our strategy). To do this, we need gadgets to negate angles (Kempe's *reversor*) and add angles (Kempe's *additor*, a clever combination of two reversors).

3.2.1 Multiplying Angles by an Integer

Proposition 3.6. *Let $a, b, c > 0$. Then the multiplicator*

$$S = \left\{(O, X, Y, Z) \in (\mathbb{R}^2)^4 \text{ such that } |OX| = a, |OY| = b, |OZ| = c, \angle XOY = \angle YOZ\right\}$$

is nicely constructible.

PROOF. After gluing a degenerate triangle linkage to OX, OY, and OZ, and projecting away the original vertices (this projection is clearly a homeomorphism), we can assume $a > b$ and $ac = b^2$. We construct $\mathcal{L}$ by gluing together similar contraparallelograms $YOXP$ and $ZOYW$ and degenerate triangle YWP, as shown in Figure 3-4.

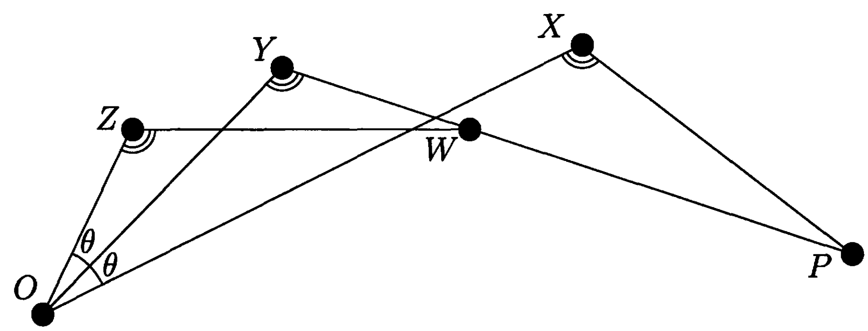


Figure 3-4: Kempe's reversor.

In any configuration of $\mathcal{L}$, angles XOY and YOZ are equal, since they are corresponding angles of similar contraparallelograms. Thus $p(\mathrm{Conf}(\mathcal{L})) \subset S$.

The coordinates of W and P are a continuous function of (O, X, Y, Z): W is the reflection of O across the perpendicular bisector of YZ, and P is the reflection of O across the perpendicular bisector of YX. Thus $S \subset p(\mathrm{Conf}(\mathcal{L}))$, and S is constructible. Further, S is a gluing of nicely constructible sets followed by a homeomorphic projection forgetting W and P, so S is nicely constructible. ■

By choosing which two of OX, OY, and OZ define the angle θ, the reversor allows us to construct the angles $-\theta$, 2θ, and $\theta/2$ (though for $\theta/2$, the configuration space contains both $\theta/2$ and $\theta/2 + \pi$). Kempe called this linkage the "reversor", because it can be used to negate angles. This construction easily generalizes to Kempe's "multiplicator" for multiplying angles by arbitrary integers k; simply glue together $k - 1$ reversors along adjacent contraparallelograms. In Chapter 4 we use this technique to (nicely) construct the angles $k\theta$, for all $-n \leq k \leq n$, using $O(n)$ bars.

3.2.2 Adding Angles

Proposition 3.7. *Let $a, b, c, d > 0$. Then the additor*

$$S = \{(O, W, X, Y, Z) \in (\mathbb{R}^2)^5 : |OW| = a, |OX| = b, |OY| = c, |OZ| = d, \\ \angle WOZ = \angle WOX + \angle WOY\}$$

is nicely constructible.

PROOF. Construct $\mathcal{L}$ by gluing two reversors: one enforcing $\angle XOM = \angle MOY$ and another enforcing $\angle WOM = \angle MOZ$, where $|OM| = 1$ and all other lengths are set according to the definition of S. See Figure 3-5. The first reversor ensures that $2\angle WOM = \angle WOX + \angle WOY$, and the second that $\angle WOZ = 2\angle WOM$, so that $\angle WOZ = \angle WOX + \angle WOY$ in any configuration of $\mathcal{L}$. Conversely, given any set of points (O, W, X, Y, Z) satisfying $\angle WOZ = \angle WOX + \angle WOY$, one can construct a configuration of $\mathcal{L}$ by placing M on either bisector of $\angle WOZ$.

This construction is a gluing of nicely constructible sets followed by a projection deleting vertex M. The projection deleting M is a 2-sheeted covering space map, because $M \mapsto Z$ is essentially the map $z \mapsto dz^2$ in the complex unit circle. Thus S is nicely constructible. ■

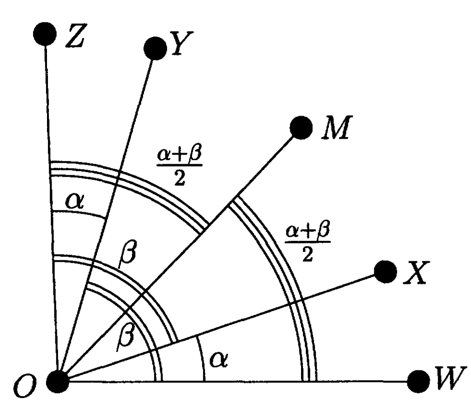


Figure 3-5: Kempe's additor.

3.3 Translating

The angle manipulation techniques described in the last section can be used to construct vertices $v_{r,s}$ with polar coordinates $f_{r,s}, r\alpha + s\beta + \gamma_{r,s}$. Next we need to be able to add together these vertices $v_{r,s}$.

Proposition 3.8. *Let $a > 0$. Then the translation by a bar OX,*

$$S = \left\{(O, X, Y, Z) \in (\mathbb{R}^2)^4 : |OX| = |YZ| = a, \overrightarrow{OX} + \overrightarrow{OY} = \overrightarrow{OZ}\right\}$$

is constructible. If $T \subset (\mathbb{R}^2)^k$ is a nicely constructible set containing vertices O, X, and Y such that inside T, $|C(O) - C(Y)|$ is bounded away from zero, then the set obtained by gluing S and T along vertices O, X, and Y is nicely constructible.

PROOF. Fix a bounded open set U, and pick $R > a$ such that the ball of radius R centered at O contains U.

By gluing together two parallelograms $OXMN$ and $NMZY$ with short sides OX, MN, NM, and ZY of length a and other sides of length R, we obtain a linkage $\mathcal{L}'$ that can translate the vector $\overrightarrow{OX}$ to any location in a large disk of radius $2R$. See Figure 3-6. Let p be the projection that forgets about all vertices except O, X, Y, and Z.

The conditions defining S are closed, so S is constructible if $p(\mathrm{Conf}(\mathcal{L})) \cap U^4 = S \cap U^4$.

We claim $p(\mathrm{Conf}(\mathcal{L})) \cap U^4 \subset S \cap U^4$. By the definitions of $OXMN$ and $NMZY$, $|OX| = |MN| = |YZ| = a$, and $OX \parallel NM \parallel YZ$. Thus $\overrightarrow{OX} = \overrightarrow{YZ} \Rightarrow \overrightarrow{OZ} = \overrightarrow{OY} + \overrightarrow{YZ} = \overrightarrow{OY} + \overrightarrow{OX}$. Thus $p(\mathrm{Conf}(\mathcal{L})) \subset S$ and so $p(\mathrm{Conf}(\mathcal{L})) \cap U^4 \subset S \cap U^4$.

We claim $p(\mathrm{Conf}(\mathcal{L})) \cap U^4 \supset S \cap U^4$. Pick $(O, X, Y, Z) \in S \cap U^4$. Since $Y \in U$, $|OY| \leq R$. Thus the circles of radius R about O and Y must intersect; place N at any such point of intersection, and set $M = N + \overrightarrow{OX}$. Now, $\overrightarrow{OZ} = \overrightarrow{OY} + \overrightarrow{OX}$, so $\overrightarrow{YZ} = \overrightarrow{OZ} - \overrightarrow{OY} = \overrightarrow{OX} = \overrightarrow{NM}$. Thus $|XM| = |MZ| = R$, and we have extended (O, X, Y, Z) to a configuration of $\mathcal{L}$.

If additionally S is glued along vertices O, X, and Y to a set T nicely constructed by a linkage $\mathcal{L}_T$ where, within T, $d(O, Y)$ is bounded from below, then N is being projected away from hook ONY, which is glued to a set on which $2R > |OY| > 0$. By Proposition 2.15, the projection forgetting N is a covering space map. Given N, $M = N + \overrightarrow{OX}$. It follows that

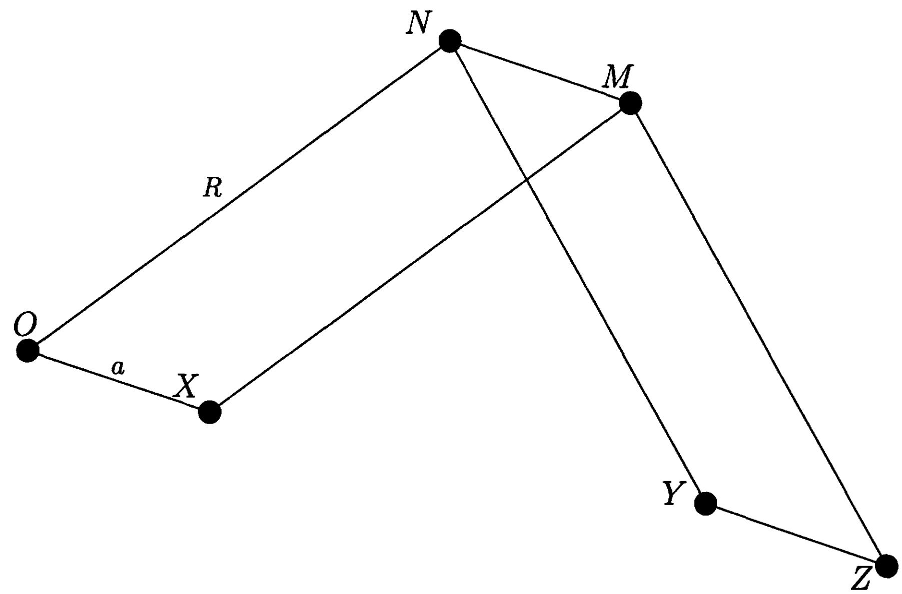


Figure 3-6: Kempe's translator.

the set obtained by gluing S and T is nicely constructible. ■

Observe that if $O = Y$, then N is free to move in a circle about O of radius R, so that S is not rigidly constructed by the translator.

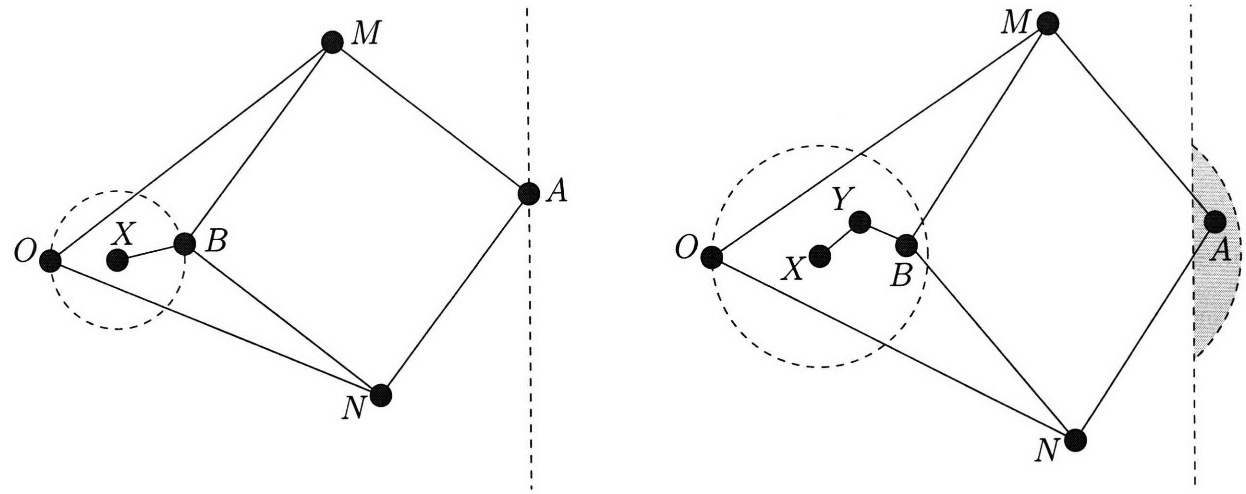
3.4 Drawing a Straight Line or Half-Plane

The Peaucellier linkage constructs a straight line. See Figure 3-7(a). Like the rectangle linkage, the Peaucellier linkage has some extra configurations, in this case those where the two vertices M and N coincide or the two vertices A and B coincide. Because these are the non-degenerate contraparallelogram configurations of the rhombus $BMAN$, bracing this rhombus as a parallelogram suffices to remove the extra configurations. Some prior work on the subject neglects or incorrectly treats this bracing issue (see for example [HJW] and [GZCG]).

The Peaucellier inversor can be modified to construct a half-plane by changing the constraint on B from a bar constructing a circle to a hook constructing a disk. See Figure 3-7(b).

Proposition 3.9. *Every line L is nicely constructible. Every half-plane H is continuously and rigidly constructible.*

PROOF. It suffices to show that the result holds for a particular small U that contains an



(a) Original Peaucellier inversor.

(b) Modified Peaucellier inversor.

Figure 3-7: The Peaucellier Linkage can be used to construct a line segment (a) or a half–plane (b).

interval of the line ℓ or an interval of the boundary of H; the general result follows after applying a suitable affine transformation.

Pick $D > C$, and construct the linkage $\mathcal{L}$ by fixing a vertex O and attaching bars constraining $|OM| = |ON| = D$ and $|BM| = |MA| = |AN| = |BN| = C$. Points O, B, and A are colinear because they all lie on the perpendicular bisector of MN.

Let Z be the center of rhombus $BMAN$ (because of our bracing, $BMAN$ must be in a rhombus configuration). Then $|OB| \cdot |OA| = (|OZ| - |BZ|)(|OZ| + |BZ|) = |OZ|^2 - |BZ|^2$. By the Pythagorean Theorem, $|OB| \cdot |OA| = (|OM|^2 - |ZM|^2) - (|BM|^2 - |ZM|^2) = D^2 - C^2$, a constant. It follows that A and B are related by geometric inversion f about O. Any configuration of the hook OMA lifts to a configuration of $\mathcal{L}$, so there are configurations of $\mathcal{L}$ where B is at any point in an annulus of radii $D - C$ and $D + C$. Thus $R = \{ (O, B, A) : |OB| \cdot |BA| = D^2 - C^2, D - C \leq |OB| \leq D + C$ and O, B, A are conlinear$\}$ is constructible.

Let P be the center of an open ball W contained in this annulus. Construct a linkage $\mathcal{L}_L$ by fixing a vertex X at the midpoint of OP and adding a bar attaching X to B of length equal to $|OX|$. Construct a linkage $\mathcal{L}_H$ by fixing a vertex X at the midpoint of OP and attaching X to B by a hook of positive inner radius and outer radius equal to $|OX|$.

Linkage $\mathcal{L}_L$ constrains B to lie on a circle through O, thus by inversion this constrains

A to a line. Linkage $\mathcal{L}_H$ constrains B to an annulus with O on the boundary, and thus constrains A to a vertical strip. Thus $p(\mathrm{Conf}(\mathcal{L}_L)) \subset L$ and $p(\mathrm{Conf}(\mathcal{L}_H)) \subset H$. Let $U = f(W)$. The bar attached to a fixed point draws a circle and the hook draws an annulus, so $L \cap U \subset p(\mathrm{Conf}(\mathcal{L}_L)) \cap U$ and $H \cap U \subset p(\mathrm{Conf}(\mathcal{L}_H)) \cap U$. Further, because the circle about X of radius $|OX|$ passes through W, for $\mathcal{L}_L$, L passes through U; and for $\mathcal{L}_H$, the boundary of H passes through U. We now can handle any U by applying an appropriate affine transformation to our linkage and observing that any bounded region of a half-plane is contained in some vertical strip containing the boundary of the half-plane. Thus L and H are constructible.

For both $\mathcal{L}_L$ and $\mathcal{L}_H$, the projection forgetting M is forgetting the middle vertex of hook OMA. Because no configuration of either linkage has this hook touch the boundary of the annulus, by Proposition 2.15, the projection forgetting M is a covering space map. Given M, N is the reflection of M across OA.

Line L is nicely constructible, because the projection forgetting O and B is a homeomorphism (B is the inversion of A about O with some fixed radius, and O is fixed), so L is nicely constructible.

Half-plane H is continuously and rigidly constructible by a similar argument, but the projection from $\mathrm{Conf}(\mathcal{L}_H)$ to H also forgets about the middle vertex of the hook XYB. Since this hook's annulus has nonzero inner radius, this projection has the path-lifting property and has finite fibres, and H is continuously and rigidly constructible. ∎

Linkage $\mathcal{L}_H$ does not nicely construct H, because the hook connecting X to B does not nicely construct the annulus.

Chapter 4

Proof of Kempe's Theorem

We now have the tools needed to prove Kempe's Universality Theorem (Theorem 1.1). By Proposition 2.16, Theorem 1.1 is equivalent to showing that, for any closed disk $B = \{\,(x,y) : (x-a)^2 + (y-b)^2 \le R\}$ and any $f \in \mathbb{R}[x,y]$ of degree n, $S = B \cap \{\,(x,y) : f(x,y) = 0\}$ is constructible. We prove that S is in fact continuously and rigidly constructible. Further, because it adds no new techniques to the argument, we prove the natural generalization to m output points:

Theorem 4.1. *Let $f \in \mathbb{R}[x_1, y_1, \ldots, x_m, y_m]$ be a polynomial of total degree n, and let B be a closed disk in the plane. Then $S = B^m \cap \{\,((x_1,y_1),\ldots,(x_m,y_m)) \in (\mathbb{R}^2)^m : f(x_1,y_1,\ldots,x_m,y_m) = 0\,\}$ is continuously and rigidly constructible using $O\left(\binom{n+2m}{2m}\right)$ bars.*

Theorem 1.1 clearly follows from Theorem 4.1. In order to prove Theorem 4.1, we first prove a version that replaces B with an annulus:

Theorem 4.2. *Let $f \in \mathbb{R}[x_1, y_1, \ldots, x_m, y_m]$ be a polynomial of total degree n, and let A be a closed annulus in the plane. Then $S = A^m \cap \{\,((x_1,y_1),\ldots,(x_m,y_m)) \in (\mathbb{R}^2)^m : f(x_1,y_1,\ldots,x_m,y_m) \ge 0\,\}$ is continuously and rigidly constructible using $O\left(\binom{n+2m}{2m}\right)$ bars. Further, $T = A^m \cap \{\,((x_1,y_1),\ldots,(x_m,y_m)) \in (\mathbb{R}^2)^m : f(x_1,y_1,\ldots,x_m,y_m) = 0\,\}$ is nicely constructible using $O\left(\binom{n+2m}{2m}\right)$ bars.*

Lemma 4.3. *Theorems 4.1 and Theorem 1.3 follow from Theorem 4.2.*

PROOF. Theorem 1.3 follows immediately from the case $m = 1$ of the second part of Theorem 4.2; simply pick an annulus A containing the desired bounded open set U.

To prove Theorem 4.1, pick A to be an annulus containing B. Then by Theorem 4.2, the linkage defined by gluing the output points of the linkages for $f(x_1, y_1, \ldots, x_m, y_m) \geq 0$, $f(x_1, y_1, \ldots, x_m, y_m) \leq 0$, and $(x_k - a)^2 + (y_k - b)^2 \leq R$ for each $1 \leq k \leq m$ continuously and rigidly constructs S. This construction uses $O\left(\binom{n+2m}{2m} + m\binom{4}{2}\right) = O\left(\binom{n+2m}{2m}\right)$ bars. ∎

To prove Theorem 4.2, we first convert f into a trigonometric expression. Without loss of generality, assume that A is centered at the origin. Let $R_1 + R_2$ and $R_1 - R_2$ be the radii of A. We begin by gluing together (braced) parallelogram linkages $OM_ku_kN_k$ with $|OM_k| = R_1$ and $|ON_k| = R_2$ for each k along common vertex O, which is fixed at the origin. Define angles α_k and β_k as in Figure 4-1. Set each *output vertex* $u_k = (x_k, y_k)$. Each vertex u_k can trace out the entire annulus A. Kempe's original construction used a rhombus rather than a parallelogram. We cannot do this, because with one vertex O of a rhombus fixed, the opposite vertex u_k does not rigidly construct the disk (the projection forgetting M and N has an infinite fibre over the degenerate configuration where $u_k = O$). So we use a parallelogram.

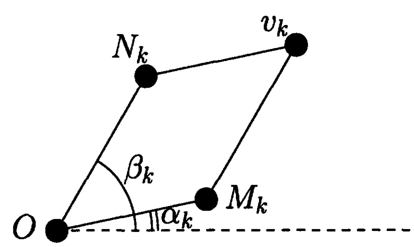


Figure 4-1: The initial parallelogram.

4.1 Trigonometric Algebra

We use $\mathbf{i}, \mathbf{j}$ to denote vectors in $\mathbb{Z}^m_{\geq 0}$, $\mathbf{r}, \mathbf{s}$ to denote vectors in $\mathbb{Z}^m$, and $\boldsymbol{\alpha}, \boldsymbol{\beta}$ to denote vectors in $(\mathbb{R}/(2\pi\mathbb{Z}))^m$. For brevity of notation, we often write sums of the form:

$$\sum_{|\mathbf{r}|+|\mathbf{s}|\leq n} g(\mathbf{r}, \mathbf{s})$$

where the sum is taken to be over all pairs of index vectors $(\mathbf{r}, \mathbf{s}) \in (\mathbb{Z}^m)^2$ satisfying the stated constraint.

The following lemma shows how inside A we can transform the polynomial f into a trigonometric function of the angles α_k and β_k defined as in Figure 4-1.

Lemma 4.4. *Given a polynomial $f \in \mathbb{R}[x_1, y_1, \ldots, x_m, y_m]$ and an annulus A of radii $R_1 - R_2$ and $R_1 + R_2$ centered at the origin, there exist constants $f_{\mathbf{r},\mathbf{s}}$, $\gamma_{\mathbf{r},\mathbf{s}}$ such that inside A,*

$$\begin{aligned} x_k &= R_1 \cos \alpha_k + R_2 \cos \beta_k. && (4.1)\\ y_k &= R_1 \sin \alpha_k + R_2 \sin \beta_k. && (4.2)\\ f(x_1, y_1, \ldots, x_m, y_m) &= \sum_{|\mathbf{r}|+|\mathbf{s}|\leq n} f_{\mathbf{r},\mathbf{s}} \cos(\mathbf{r}\cdot\boldsymbol{\alpha} + \mathbf{s}\cdot\boldsymbol{\beta} + \gamma_{\mathbf{r},\mathbf{s}}). && (4.3) \end{aligned}$$

PROOF. The x- and y-coordinates of u_k satisfy equations (4.1) and (4.2). Substitute these expressions into the polynomial $f(x_1, y_1, \ldots, x_m, y_m)$ to obtain a trigonometric expression of the form

$$f(x_1, y_1, \ldots, x_m, y_m) = \\ \sum_{|\mathbf{i}|+|\mathbf{j}|\leq n} c_{\mathbf{i},\mathbf{j}} \prod_{k=1}^{m} (R_1 \cos\alpha_k + R_2 \cos\beta_k)^{i_k} (R_1 \sin\alpha_k + R_2 \sin\beta_k)^{j_k} \qquad (4.4)$$

where $c_{\mathbf{i},\mathbf{j}}$ is the coefficient of $\prod_{k=1}^{m} x^{i_k} y^{j_k}$ in f. After expanding and repeated use of the trigonometric product-to-sum formulas

$$\begin{aligned} \cos A \cos B &= \frac{\cos(A+B) + \cos(A-B)}{2}, \\ \cos A \sin B &= \frac{\sin(A+B) - \sin(A-B)}{2}, \\ \sin A \sin B &= \frac{\cos(A-B) - \cos(A+B)}{2}, \end{aligned}$$

we obtain an equation of the form

$$f(x_1, y_1, \ldots, x_m, y_m) = \sum_{|\mathbf{r}|+|\mathbf{s}|\leq n} d_{\mathbf{r},\mathbf{s}} \cos(\mathbf{r}\cdot\boldsymbol{\alpha} + \mathbf{s}\cdot\boldsymbol{\beta}) + e_{\mathbf{r},\mathbf{s}} \sin(\mathbf{r}\cdot\boldsymbol{\alpha} + \mathbf{s}\cdot\boldsymbol{\beta}).$$

Finally, for each i and j, choose $f_{\mathbf{r},\mathbf{s}}$ and $\gamma_{\mathbf{r},\mathbf{s}}$ such that, for all θ,

$$f_{\mathbf{r},\mathbf{s}} \cos(\theta + \gamma_{\mathbf{r},\mathbf{s}}) = d_{\mathbf{r},\mathbf{s}} \cos(\theta) + e_{\mathbf{r},\mathbf{s}} \sin(\theta).$$

Then, inside A,

$$f(x_1, y_1, \ldots, x_m, y_m) = \sum_{|\mathbf{r}|+|\mathbf{s}| \leq n} f_{\mathbf{r},\mathbf{s}} \cos(\mathbf{r} \cdot \boldsymbol{\alpha} + \mathbf{s} \cdot \boldsymbol{\beta} + \gamma_{\mathbf{r},\mathbf{s}}) \tag{4.5}$$

as desired. ■

4.2 Constructing the Angles

We now show how to construct vertices $v_{\mathbf{r},\mathbf{s}}$ whose x-coordinates are the terms $f_{\mathbf{r},\mathbf{s}} \cos(\mathbf{r} \cdot \boldsymbol{\alpha} + \mathbf{s} \cdot \boldsymbol{\beta} + \gamma_{\mathbf{r},\mathbf{s}})$.

Lemma 4.5. *For any $R_1 > R_2 > 0$, and polynomial $f \in \mathbb{R}[x_1, y_1, \ldots, x_m, y_m]$ of total degree n, define $f_{\mathbf{r},\mathbf{s}}, \gamma_{\mathbf{r},\mathbf{s}}$ as in Lemma 4.4. Then*

$$\begin{aligned} S = \{O = (0,0), \{M_k\}_{k=1}^m, \{N_k\}_{k=1}^m, \{u_k\}_{k=1}^m, \{v_{\mathbf{r},\mathbf{s}}\}_{|\mathbf{r}|+|\mathbf{s}|\leq n} :\ & \textit{there exist } \alpha_k, \beta_k \in \mathbb{R}^m \textit{ satisfying} \\ M_k = (R_1 \cos\alpha_k, R_1 \sin\alpha_k), N_k = (R_2 \cos\beta_k, R_2 \sin\beta_k), u_k = M_k + N_k & \qquad (4.6) \\ v_{\mathbf{r},\mathbf{s}} = (f_{\mathbf{r},\mathbf{s}} \cos\theta_{\mathbf{r},\mathbf{s}}, f_{\mathbf{r},\mathbf{s}} \sin\theta_{\mathbf{r},\mathbf{s}}),\ \textit{where } \theta_{\mathbf{r},\mathbf{s}} = \mathbf{r}\cdot\boldsymbol{\alpha} + \mathbf{s}\cdot\boldsymbol{\beta} + \gamma_{\mathbf{r},\mathbf{s}}\} & \end{aligned}$$

is nicely constructible using $O\left(\binom{n+2m}{2m}\right)$ bars.

Proof. Construct the linkage $\mathcal{L}$ as follows. Start with a (braced) parallelogram $OM_ku_kN_k$ for each output vertex u_k, which by Corollary 3.2 nicely constructs vertices (O, M_k, N_k, u_k) satisfying equations (4.6). Let α_k be the angle from the x-axis to $\overrightarrow{OM_k}$, and β_k be the angle from the x-axis to $\overrightarrow{ON_k}$. Glue these parallelograms together at the common vertex O.

Use $O\left(\binom{n}{m}\right)$ additors (detailed in Proposition 3.7) and reversors (detailed in Proposition 3.6) to iteratively construct points $A_{\mathbf{r}}$ with $|OA_{\mathbf{r}}| = 1$ at angles from the x-axis of $\mathbf{r} \cdot \boldsymbol{\alpha}$ for all $\mathbf{r} \in \mathbb{Z}^m$ satisfying $|\mathbf{r}| \leq n$. Similarly, construct points $B_{\mathbf{s}}$ with $|OB_{\mathbf{s}}| = 1$ at angles from the x-axis of $\mathbf{s} \cdot \boldsymbol{\beta}$ for all $\mathbf{s} \in \mathbb{Z}^m$ satisfying $|\mathbf{s}| \leq n$. Now use $O\left(\binom{n+2m}{2m}\right)$ additors adding the angles of each pair $A_{\mathbf{r}}$, $B_{\mathbf{s}}$, to construct points $Q_{\mathbf{r},\mathbf{s}}$ with $|OQ_{\mathbf{r},\mathbf{s}}| = 1$ at angles $\mathbf{r} \cdot \boldsymbol{\alpha} + \mathbf{s} \cdot \boldsymbol{\beta}$ for all $|\mathbf{r}| + |\mathbf{s}| \leq n$. For each pair $(\mathbf{r}, \mathbf{s})$, include a fixed point $P_{\mathbf{r},\mathbf{s}}$ such that $OP_{\mathbf{r},\mathbf{s}}$ has angle $\gamma_{\mathbf{r},\mathbf{s}}$ from the x-axis. Finally, use $O\left(\binom{n+2m}{2m}\right)$ additors to construct bars of length $f_{\mathbf{r},\mathbf{s}}$ at angles $\mathbf{r} \cdot \boldsymbol{\alpha} + \mathbf{s} \cdot \boldsymbol{\beta} + \gamma_{\mathbf{r},\mathbf{s}}$ for all $\mathbf{r}, \mathbf{s}$ satisfying $|\mathbf{r}| + |\mathbf{s}| \leq n$. Since additors and reversors take $O(1)$ bars each, we have used a total of $O\left(\binom{n+2m}{2m}\right)$ bars.

This construction is a mass gluing of nicely constructible components, followed by a projection that forgets about the $A_{\mathbf{r}}$'s, $B_{\mathbf{s}}$'s, $P_{\mathbf{r},\mathbf{s}}$'s, and $Q_{\mathbf{r},\mathbf{s}}$'s. All the vertices being forgotten in this projection are continuous functions of the α_k's and β_k's, which are in turn continuous functions of the coordinates of the M_k's and N_k's, so this projection is a homeomorphism. Thus S is nicely constructible using $O\left(\binom{n+2m}{2m}\right)$ bars, as desired. ∎

4.3 Proving Kempe's Theorem

We are now ready to prove Theorem 4.2. Let $\mathcal{L}'$ be the linkage of Lemma 4.5.

Order the set of points $v_{\mathbf{r},\mathbf{s}}$ into a list $P_1, P_2, \ldots, P_N$. Write $P_i = (u_i, w_i)$. Let $W_0 = (a, b)$ be a fixed point further from the origin than the sum of the lengths of all bars used in constructing $\mathcal{L}'$. For each $1 \leq i \leq N$, connect a translator (detailed in Proposition 3.8) enforcing for the pair $(\mathbf{r}, \mathbf{s})$ corresponding to i that $|OP_i| = f_{\mathbf{r},\mathbf{s}}$ and $\overrightarrow{OP_i} + \overrightarrow{OW_{i-1}} = \overrightarrow{OW_i}$. Then the x-coordinate of W_N is

$$a + \sum_{i=1}^{N} u_i = a + \sum_{|\mathbf{r}|+|\mathbf{s}|\leq n} f_{\mathbf{r},\mathbf{s}} \cos(\mathbf{r} \cdot \boldsymbol{\alpha} + \mathbf{s} \cdot \boldsymbol{\beta} + \gamma_{\mathbf{r},\mathbf{s}}) = a + f(x_1, y_1, \ldots, x_m, y_m).$$

by Lemma 4.4.

Now finish the construction of $\mathcal{L}_S$ by attaching to W_N a sufficiently large modified Peaucellier linkage (detailed in Proposition 3.9) that constrains it to the half-plane $x \geq a$. By "sufficiently large" we mean that its U should contain the disk centered at $W_0 = (a, b)$ of radius equal to the sum of the lengths of all bars used so far. The output vertices $u_k = (x_k, y_k)$ are then constrained by $f(x_1, y_1, \ldots, x_m, y_m) \geq 0$, as desired. Moreover, for any configuration of the u_k's inside A satisfying $f(x_1, y_1, \ldots, x_m, y_m) = 0$, there is at least one corresponding configuration of $\mathcal{L}_S$. By projecting from $\mathrm{Conf}(\mathcal{L}_S)$ down to just the u_k's, we see that $\mathcal{L}_S$

constructs the desired set $A^m \cap \{ (x_1, y_1, \ldots, x_m, y_m) \in \mathbb{R}^2 : f(x_1, y_1, \ldots, x_m, y_m) \geq 0 \}$ using $O\left(\binom{n+2m}{2m}\right)$ bars.

The construction before adding the Peaucellier linkage is nicely constructible, because for each i, $|\overrightarrow{OP_i}|$ is bounded away from 0, and by Proposition 3.8, this is the property needed for gluing a translator to another set to produce a nicely constructible set. Attaching the modified Peaucellier linkage leaves us with a continuously and rigidly constructible set.

The W_i's are continuous functions of the α_k's and β_k's, which are in turn a continuous function of the locations of the M_k's and N_k's, so the projection that forgets about all the W_i's is a homeomorphism.

Finally, because O is fixed and, for each k, there are at most two choices for which side of $\overrightarrow{Ou_k}$ vertices M_k and N_k are on in the initial parallelogram, projecting down to just the trace of u_k is a covering space map to a closed set. Applying Proposition 2.14, we find that S is continuously and rigidly constructible using $O\left(\binom{n+2m}{2m}\right)$ bars, as desired.

Linkage $\mathcal{L}_T$ is the same as $\mathcal{L}_S$, except we use a standard Peaucellier inversor instead of a modified Peaucellier inversor. Because the standard Peaucellier inversor nicely constructs the line, the argument for $\mathcal{L}_S$ shows that $\mathcal{L}_T$ nicely constructs T using $O\left(\binom{n+2m}{2m}\right)$ bars. ■

4.4 Computational Issues

In this section, we give an algorithm for computing a complete configuration of a linkage that is necessary for proving the coNP-hardness of rigidity testing.

Theorem 4.6. *Let $f \in \mathbb{R}[x_1, y_1, \ldots, x_m, y_m]$ be a polynomial of total degree n, and let A be a closed annulus in the plane with integral radii. Let $S = A^m \cap \{ (x_1, y_1, \ldots, x_m, y_m) \in (\mathbb{R}^2)^m : f(x_1, y_1, \ldots, x_m, y_m) = 0 \}$. Then there is an algorithm running in time polynomial in $\binom{n+2m}{2m}$ that given a point $P \in S$ such that the vectors of angles $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ defined by Equation 4.6 have the property that $\cos(\frac{\alpha_k}{2}), \cos(\frac{\beta_k}{2}), \sin(\frac{\alpha_k}{2}), \sin(\frac{\beta_k}{2})$ are all rational, computes a linkage $\mathcal{L}$ that continuously and rigidly constructing S, and a configuration $C \in \mathrm{Conf}(\mathcal{L})$ such that $p(C) = P$.*

Proof. We start from the linkage $\mathcal{L}'$ defined in Theorem 4.2, and then show how to adjust it so that all vertices have rational coordinates and so that we can compute a configuration C in time polynomial in $\binom{n+2m}{n}$.

By the trigonometric sum-to-product formulas, a polynomial in $\sin\alpha_k, \sin\beta_k, \cos\alpha_k, \cos\beta_k$ can be efficiently converted into a polynomial in $\cos\frac{\alpha_k}{2}, \cos\frac{\beta_k}{2}, \sin\frac{\alpha_k}{2}, \sin\frac{\beta_k}{2}$ of twice the original degree. Thus, all of the vertices in the constructions of Lemma 4.5 and Theorem 4.2 are a polynomial function of the variables $\cos\frac{\alpha_k}{2}, \cos\frac{\beta_k}{2}, \sin\frac{\alpha_k}{2}, \sin\frac{\beta_k}{2}$ of degree polynomial in $\binom{n+2m}{n}$.

Given the point $P = \{x_1, y_1, \ldots, x_m, y_m\}$, set $u_k = (x_k, y_k)$. Then we can compute a set of coordinates for each point M_k defined in Equation 4.6 by solving the quadratic equation defined by the intersection of the circles of radius R_1 about O and R_2 about u_k. Because each M_k must have rational coefficients, both solutions have rational coordinates. Now, given the point M_k, we can compute $N_k = u_k - M_k$.

Now, the $A_{\mathbf{r}}$'s, $B_{\mathbf{s}}$'s, $Q_{\mathbf{r},\mathbf{s}}$'s, and $P_{\mathbf{r},\mathbf{s}}$'s have coordinates

$$
\begin{aligned}
A_{\mathbf{r}} &= (\cos(\mathbf{r}\cdot\boldsymbol{\alpha}), \sin(\mathbf{r}\cdot\boldsymbol{\alpha})) \\
B_{\mathbf{s}} &= (\cos(\mathbf{s}\cdot\boldsymbol{\beta}), \sin(\mathbf{s}\cdot\boldsymbol{\beta})) \\
Q_{\mathbf{r},\mathbf{s}} &= (\cos(\mathbf{r}\cdot\boldsymbol{\alpha}+\mathbf{s}\cdot\boldsymbol{\beta}), \sin(\mathbf{r}\cdot\boldsymbol{\alpha}+\mathbf{s}\cdot\boldsymbol{\beta})) \\
P_{\mathbf{r},\mathbf{s}} &= (f_{\mathbf{r},\mathbf{s}}\cos(\mathbf{r}\cdot\boldsymbol{\alpha}+\mathbf{s}\cdot\boldsymbol{\beta}+\gamma_{\mathbf{r},\mathbf{s}}), f_{\mathbf{r},\mathbf{s}}\sin(\mathbf{r}\cdot\boldsymbol{\alpha}+\mathbf{s}\cdot\boldsymbol{\beta}+\gamma_{\mathbf{r},\mathbf{s}})) \\
&= (d_{\mathbf{r},\mathbf{s}}\cos(\mathbf{r}\cdot\boldsymbol{\alpha}+\mathbf{s}\cdot\boldsymbol{\beta}) + e_{\mathbf{r},\mathbf{s}}\sin(\mathbf{r}\cdot\boldsymbol{\alpha}+\mathbf{s}\cdot\boldsymbol{\beta}), \\
&\qquad d_{\mathbf{r},\mathbf{s}}\sin(\mathbf{r}\cdot\boldsymbol{\alpha}+\mathbf{s}\cdot\boldsymbol{\beta}) - e_{\mathbf{r},\mathbf{s}}\cos(\mathbf{r}\cdot\boldsymbol{\alpha}+\mathbf{s}\cdot\boldsymbol{\beta})) \\
W_i &= W_{i-1} + P_i, \text{ for } i > 0;\ W_0 = (a, b)
\end{aligned}
$$

where the $d_{\mathbf{r},\mathbf{s}}$'s and $e_{\mathbf{r},\mathbf{s}}$'s are as defined in Lemma 4.4.

We now show how to compute the $d_{\mathbf{r},\mathbf{s}}$'s and $e_{\mathbf{r},\mathbf{s}}$'s. These are a function of only the coefficients of f. Then by applying the Binomial Theorem to Equation 4.4, we obtain

$$\begin{aligned} f(\alpha,\beta) &= \sum_{|\mathbf{i}|+|\mathbf{j}|\le n} c_{\mathbf{i},\mathbf{j}} \prod_{k=1}^{m} (R_1\cos\alpha_k + R_2\cos\beta_k)^{i_k}(R_1\sin\alpha_k + R_2\sin\beta_k)^{j_k} \\ &= \sum_{|\mathbf{a}|+|\mathbf{b}|+|\mathbf{c}|+|\mathbf{d}|\le n} c_{\mathbf{a}+\mathbf{b},\mathbf{c}+\mathbf{d}} \binom{a_k+b_k}{a_k}\binom{c_k+d_k}{c_k} \times \\ &\qquad\qquad \prod_{k=1}^{m}(R_1\cos\alpha_k)^{a_k}(R_2\cos\beta_k)^{b_k}(R_1\sin\alpha_k)^{c_k}(R_2\sin\beta_k)^{d_k} \end{aligned}$$

where the vectors $\mathbf{i},\mathbf{j},\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$ are all vectors in $\mathbb{Z}^m_{\ge 0}$. Let B_1 be the basis of terms the form $\prod_{k=1}^{m}(\cos\alpha_k)^{a_k}(\cos\beta_k)^{b_k}(\sin\alpha_k)^{c_k}(\sin\beta_k)^{d_k}$ such that $|\mathbf{b}|+|\mathbf{c}|+|\mathbf{d}|+|\mathbf{a}| \le n$. This expansion allows us to efficiently compute the coefficients of f in the basis B_1. Basis B_1 contains $\binom{n+4m}{n}$ basis vectors because it is the space of all polynomials in $4m$ variables of total degree at most n.

To compute the $d_{\mathbf{r},\mathbf{s}}$'s and $e_{\mathbf{r},\mathbf{s}}$'s, we need to express f in the basis B_2 generated by $\cos(\mathbf{r}\cdot\boldsymbol{\alpha}+\mathbf{s}\cdot\boldsymbol{\beta})$ and $\sin(\mathbf{r}\cdot\boldsymbol{\alpha}+\mathbf{s}\cdot\boldsymbol{\beta})$, where $\mathbf{r}$ and $\mathbf{s}$ vary, as usual, over integer vectors satisfying $\mathbf{r},\mathbf{s}$ such that $|\mathbf{r}|+|\mathbf{s}| \le n$. Basis B_2 contains one basis vector for each $2m$-dimensional vector over $\mathbb{Z}$ with L_1 norm at most n. Because this is at most the number of $4m$-dimensional vectors over $\mathbb{Z}_{\ge 0}$ with L_1 norm at most n, basis B_2 has at most $O\left(\binom{n+4m}{n}\right)$ basis vectors.

To do this, we compute the matrix M defining the transformation from B_2 to B_1. We can then invert the matrix, and multiply it with the coefficients of f in the basis B_1 to obtain an expansion of f in the basis B_2.

We can efficiently compute the expansions of each term $\cos(\boldsymbol{\alpha}\cdot\mathbf{r}+\boldsymbol{\beta}\cdot\mathbf{s})$ (or $\sin(\boldsymbol{\alpha}\cdot\mathbf{r}+\boldsymbol{\beta}\cdot\mathbf{s})$) using dynamic programming. We first compute the coefficients of each term where $|\mathbf{r}|+|\mathbf{s}| \le t-1$, and then from that compute the coefficients of each term where $|\mathbf{r}|+|\mathbf{s}| = t$, using a single application of the trigonometric product-to-sum formulas. The total time consumed for each element at level t is the cost of adding two coefficients at level $t-1$. Thus, computing M requires performing

$$O\left(\sum_{t=0}^{n}\binom{t+4m}{n}\right) = O\left(n\binom{n+4m}{n}\right).$$

rational addition operations.

Since M is a matrix with each dimension at most $O\left(\binom{n+4m}{n}\right)$, we can solve an equation of the form $Mx = b$ in time $O\left(\binom{n+4m}{n}^3\right)$ using Gaussian elimination. By construction, M has integer entries, and one can also verify using the trigonometric product-to-sum formulas that each $d_{\mathbf{r},\mathbf{s}}$ and $e_{\mathbf{r},\mathbf{s}}$ is an integer multiple of $\frac{1}{2^n R_1^n R_2^n}$. Remember that R_1 and R_2 are integers, so that these have rational denominators.

Thus, we can compute all the coefficients $d_{\mathbf{r},\mathbf{s}}$ and $e_{\mathbf{r},\mathbf{s}}$ needed to compute the coordinates of the $P_{\mathbf{r},\mathbf{s}}$'s in time polynomial in $O\left(\binom{n+4m}{n}\right)$. Further, these coefficients are rational numbers, with denominators containing only polynomially many bits.

We can also use the matrix M to compute the coordinates of the points $A_{\mathbf{r}}$, $B_{\mathbf{s}}$, and $Q_{\mathbf{r},\mathbf{s}}$. To evaluate $\cos(\boldsymbol{\alpha} \cdot \mathbf{r} + \boldsymbol{\beta} \cdot \mathbf{s})$, we convert it to the basis B_1 by extracting the corresponding column c of M. Since each element of the basis B_1 is a product of at most n terms of the form $\cos \alpha_k, \cos \beta_k, \sin \alpha_k, \sin \beta_k$, and we know all these rational numbers in our target configuration C, we can simply evaluate each element of B_1 and sum them up to obtain $\cos(\boldsymbol{\alpha} \cdot \mathbf{r} + \boldsymbol{\beta} \cdot \mathbf{s})$. A similar technique can be used for computing those of the form $\sin(\boldsymbol{\alpha} \cdot \mathbf{r} + \boldsymbol{\beta} \cdot \mathbf{s})$. Each of these computations takes time polynomial in $\binom{n+4m}{n}$, and there are $\binom{n+4m}{n}$ basis elements, so in total computing the coordinates of these points consumes time polynomial in $\binom{n+4m}{n}$, as desired.

We can now compute the coordinates of the $P_{\mathbf{r},\mathbf{s}}$'s from the coordinates of the $Q_{\mathbf{r},\mathbf{s}}$'s, along with the values $d_{\mathbf{r},\mathbf{s}}$ and $e_{\mathbf{r},\mathbf{s}}$ that we have already obtained.

Given the coordinates of the $P_{\mathbf{r},\mathbf{s}}$'s, one can add up the partial sums to compute the coordinates of all the W_i's. We have now shown how to compute coordinates for all the vertices projected away in the proofs of Lemma 4.5 and Theorem 4.2.

Thus, it now suffices to, for each of the additors, reversors, translators, and peaucellier linkages used in our construction (and the reversors, parallelograms, and contraparallelograms used in building them), we can efficiently compute some set of rational coordinates for all of the vertices projected away in those constructions. This process may involve modifying the linkage $\mathcal{L}'$ in order to ensure the rationality of the relevant coordinates, so long as we do not change the structure of S in the process.

1. For each translator enforcing $\overrightarrow{OP_i} + \overrightarrow{OW_{i-1}} = \overrightarrow{OW_i}$, we compute a location for point N

by computing the perpendicular bisector of OY, and picking a point N with rational coordinates along it such that $|ON| > R$; then we compute $M = N + \overrightarrow{OX}$. Note that this transformation does not necessarily preserve the size of the translator, but that the new translator with a slightly different radius greater than R constructs the same set as the original translator. The new translator has the advantage that all its points have rational coordinates in the configuration that are constructing.

2. For the Peaucellier linkage, we choose an origin O for our new Peaucellier linkage that has the same x-coordinate as the point $A = W_N$. Then pick B to be any point between O and A, and $X = O + \frac{\overrightarrow{OB}}{2}$. Finally, choose M and N sufficiently far from the origin using the same trick used to compute points M and N for the translator, so that we draw a sufficiently large line segment, as desired.

3. For the additors adding two angles, the only vertex of the construction we forget is the angle bisector M. This point is on the unit circle at an angle that is a half-integer linear combination of the α_k's and β_k's. It follows that it can be expressed as polynomial with rational coefficients in the $\cos\frac{\alpha_k}{2}$, $\sin\frac{\alpha_k}{2}$, $\cos\frac{\beta_k}{2}$, and $\sin\frac{\beta_k}{2}$, and thus it has rational coordinates. We can efficiently compute those coordinates using the trigonometric half-angle formulas.

4. For the reversors, the vertices W and P that are forgotten are the reflection of vertex O across the perpendicular bisectors of ZY and YX, respectively. These forgotten vertices are easily computed, and clearly have rational coordinates if the coordinates of points O,X,Y and Z are rational.

5. For the parallelograms, we simply need to compute coordinates M and N, which is trivial since they are the at the midpoints of existing bars. These coordinates are rational if the coordinates of the vertices are rational.

6. For the contraparallelograms, we must find a location for vertices K, L, M, N, X. Vertices K, L, M, N are simply the midpoints of existing bars. For vertex X, we compute the perpendicular bisector of DB, and pick X with rational coordinates sufficiently far along this line to satisfy our radii constraints.

Given the coordinates of all the vertices of a configuration, we can represent the lengths of all the bars in the configuration in terms of the distances between the relevant pairs of vertices in the configuration we have constructed.

Thus we have shown how to compute a linkage $\mathcal{L}$ such that $p(\mathrm{Conf}(\mathcal{L})) \cap U^N = p(\mathrm{Conf}(\mathcal{L}')) \cap U^N = S \cap U^N$, along with a configuration $C \in \mathrm{Conf}(\mathcal{L})$ such that $p(C) = P$.

This computation runs in time polynomial in $O\left(\binom{n+2m}{n}\right)$ because

$$\frac{\binom{n+4m}{n}}{\binom{n+2m}{n}^2} = \prod_{i=1}^{n} \frac{i+4m}{i} \frac{i^2}{(i+2m)^2} = \prod_{i=1}^{n} \frac{i(4m+i)}{(2m+i)^2} \leq 1$$

so that $\binom{n+4m}{n} = O\left(\binom{n+2m}{n}^2\right)$. ∎

Chapter 5

Rigidity$_2$ is coNP-hard

Definition 5.1. A configuration C of a linkage $\mathcal{L}$ is *rigid* if C is an isolated point of $\text{Conf}(\mathcal{L})$, modulo rigid motions of $\mathcal{L}$.

Definition 5.2. RIGIDITY$_d$ is the problem of deciding whether a given configuration of a linkage in d-dimensional space is rigid.

The universality theorems of Kempe and Kapovich and Millson relate the configuration spaces of linkages to real algebraic varieties. Combined with the NP$_{\mathbb{R}}$-hardness of the real Nullstellensatz [BCSS], one might conjecture that RIGIDITY$_d$ is computationally intractable. In this chapter, we prove that this intuition is correct. There are two challenges in this approach. One is that the Nullstellensatz is fundamentally a question about the existence of a solution, while rigidity is a question about (local) uniqueness of a solution. We solve this problem by using a certain NP-hardness result of Koiran for testing whether a point on an algebraic variety defined by homogeneous equations has a nontrivial point.

The other challenge is that one needs to efficiently compute a configuration of a continuously and rigidly constructible linkage given by one of the universality theorems. This challenge is somewhat subtle, because we would like to avoid having to use the real algebraic computation model [BCSS]. Theorem 4.6 provides the necessary algorithm.

In rigidity theory, fixed vertices are typically not allowed, whereas in the setting of Kempe's Universality Theorem, they are. The following proposition shows that this distinction is not important.

Proposition 5.3. *Given a configuration C of a linkage $\mathcal{L}$ with fixed vertices, one can efficiently construct a configuration C' of a linkage $\mathcal{L}'$ with no fixed vertices that is rigid if and only if $\mathcal{L}$ is rigid.*

PROOF. We construct $\mathcal{L}'$ from $\mathcal{L}$ as follows. Place a nondegenerate d-dimensional simplex linkage with vertices $Y_1Y_2 \dots Y_{d+1}$ anywhere. For any vertex Q that was fixed in $\mathcal{L}$, add bars Y_iQ, and replace Q with an unpinned vertex. Because distances from $d+1$ points not lying in a $d-1$-dimensional hyperplane determine a point in d-dimensional space, none of the previously pinned vertices Q can move relative to the simplex Y_i. Thus the configurations of linkage $\mathcal{L}'$ differs from those of linkage $\mathcal{L}$ only in the rigid motions of the simplex $Y_1Y_2 \dots Y_{d+1}$. It follows that configurations of $\mathcal{L}'$ are rigid if and only if the corresponding configurations of $\mathcal{L}'$ is rigid. This construction takes at most $(k+d)(d+1)$ extra bars, where k is the number of fixed points in $\mathcal{L}$, and runs in time linear in the number of bars of $\mathcal{L}'$. ■

To show that RIGIDITY$_2$ is coNP-hard, we reduce from ISO($\mathbb{R}$), a problem closely related to the problem $H_2N(\mathbb{R})$ [Koiran].

Definition 5.4. ISO($\mathbb{R}$) is the problem of deciding whether a system of s homogeneous polynomials of total degree 2 in m variables with coefficients in $\mathbb{Z}$ (given the in dense representation), have an isolated point over $\mathbb{R}$. $H_2N(\mathbb{R})$ is the problem of deciding whether a a system of s homogeneous polynomials of total degree 2 in m variables with coefficients in $\mathbb{Z}$ (given in the dense representation) has a nontrivial solution.

We use the following result from [Koiran].

Theorem 5.5 (Koiran). $H_2N(\mathbb{R})$ *is NP-hard.*

Corollary 5.6. ISO($\mathbb{R}$) *is coNP-hard.*

PROOF. A system of homogeneous polynomials is in ISO($\mathbb{R}$) if and only if it is not in $H_2N(\mathbb{R})$. If $\mathbf{0}$ is not isolated, then there must be a nontrivial solution. Conversely, if there is a nontrivial solution $\mathbf{x}$, the line between $\mathbf{0}$ and $\mathbf{x}$ is in V by homogeneity, so $\mathbf{0}$ is not isolated. Thus ISO($\mathbb{R}$) is coNP-hard. ■

Theorem 5.7. RIGIDITY$_2$ *is coNP-hard.*

For polynomials $f_1, \ldots, f_s$, let $\mathcal{Z}(f_1, \ldots, f_j)$ be the variety of common zeroes of the f_js.

PROOF. Without loss of generality, we may assume that instances of ISO($\mathbb{R}$) have an even number of variables, because $x_m = 0$ is a homogeneous equation.

Let $\{f_j(x_1, y_1, \ldots, x_m, y_m)\}_{j=1,\ldots,s}$ be an instance of ISO($\mathbb{R}$). Pick U any neighborhood of $\mathbf{0}$. For each f_j, compute a linkage $\mathcal{L}_j$ as follows. Set $R_1 = 1$ and $R_2 = 2$. Let θ be any angle with rational sine and cosine (e.g. an acute angle of a 3-4-5 right triangle). Set $\frac{\alpha_k}{2} = \frac{\beta_k}{2} = \theta$ for all k. Let $(x_0, y_0) = (3\cos(2\theta), 3\sin(2\theta))$ be the point defined by these choices of $\boldsymbol{\alpha}, \boldsymbol{\beta}$, R_1, and R_2. Let $\mathcal{L}'_j$ be the linkage obtained by running the algorithm of Theorem 4.6 on the polynomial $f'_j = f_j(x_1 - x_0, y_1 - y_0, \ldots, x_m - x_0, y_m - y_0)$ and the point (x_0, y_0) with the above choices. Let $\mathcal{L}_j$ be the result of translating $\mathcal{L}'_j$ by $(-x_0, -y_0, \ldots, -x_0, -y_0)$. It is easy to check that $\mathcal{L}_j$ continuously and rigidly constructs $\mathcal{Z}(f_j)$.

We construct the linkage $\mathcal{L}$ by gluing together the output vertices of the $\mathcal{L}_j$s. Each $\mathcal{L}_j$ continuously and rigidly constructs $A^m \cap \mathcal{Z}(f_j)$, so $\mathcal{L}$ continuously and rigidly constructs $S = A^m \cap \bigcap_{j=1}^s \mathcal{Z}(f_j) = A^m \cap \mathcal{Z}(f_1, \ldots, f_j)$ (and thus the projection $p : \mathrm{Conf}(\mathcal{L}) \cap p^{-1}(U^r) \to S \cap U^r$ has the path-lifting property and finite fibres). By Theorem 4.6, we can compute a configuration C of $\mathcal{L}$ satisfying $p(C) = \mathbf{0}$ in $O\left(\binom{n+2m}{n}^k\right) = O(m^{2k})$ time, since $n = 2$. We claim C is rigid if and only if $\mathbf{0}$ is an isolated point of $\mathcal{Z}(f_1, \ldots, f_j)$.

First suppose $\mathbf{0}$ is not an isolated point of $\mathcal{Z}(f_1, \ldots, f_j)$. Then there is a path in $\mathcal{Z}(f_1, \ldots, f_j)$ starting at $\mathbf{0}$. As $\mathbf{0}$ is in the interior of U, there is a path in $\mathcal{Z}(f_1, \ldots, f_j) \cap U^m$ starting at $\mathbf{0}$. By the path lifting property of p, there is a path in $\mathrm{Conf}(\mathcal{L}) \cap p^{-1}(U^r)$ about $C \in p^{-1}(\mathbf{0})$ as well. Then C is not rigid.

Conversely, suppose that C is not rigid. Then there is a nontrivial path in $\mathrm{Conf}(\mathcal{L})$ starting at C. Because C is in the interior of $p^{-1}(U^m)$, this implies $\mathrm{Conf}(\mathcal{L}) \cap p^{-1}(U^m)$ is infinite. Then $\mathrm{Conf}(\mathcal{L}) \cap p^{-1}(U^r)$ is infinite. Because p has finite fibers, $p(\mathrm{Conf}(\mathcal{L})) \cap U^r \subset \mathcal{Z}(f_1, \ldots, f_j)$ is also infinite. But then $\mathcal{Z}(f_1, \ldots, f_j)$ has a point P other than $\mathbf{0}$. The f_j's are homogeneous, so the line from $\mathbf{0}$ to P is in $\mathcal{Z}(f_1, \ldots, f_j)$. Thus $\mathbf{0}$ is not an isolated point on $\mathcal{Z}(f_1, \ldots, f_j)$.

Thus given an instance of ISO($\mathbb{R}$), we can efficiently construct a configuration C of a linkage $\mathcal{L}$ that is rigid if and only if $\mathcal{Z}(f_1, \ldots, f_s)$ has an isolated point at $\mathbf{x} = \mathbf{0}$. It follows that RIGIDITY$_2$ is coNP-hard. ■

We are now ready to prove Theorem 1.2.

Theorem 1.2. $\textsc{Rigidity}_d$ is coNP-hard for all $d \geq 2$.

Proof. The case $d = 2$ was Theorem 5.7. For $d > 2$, we reduce from $\textsc{Rigidity}_{d-1}$. Let C be a configuration of a (connected) $d-1$-dimensional linkage $\mathcal{L}$. Create a new linkage $\mathcal{L}'$ by replacing each edge in $\mathcal{L}$ with a rectangular degenerate tetrahedron with height 1 and width equal to the length of the edge (replacing each vertex v with two vertices v_1 and v_2), as shown in Figure 5-1. Construct a configuration C' of $\mathcal{L}'$ as follows. If vertex v was at $(x_1, \ldots, x_{d-1})$ in C, in C' we place the v_1 at $(x_1, \ldots, x_{d-1}, 0)$ and v_2 at $(x_1, \ldots, x_{d-1}, 1)$. This transformation runs in linear time. We claim that that C' is rigid if and only if C is.

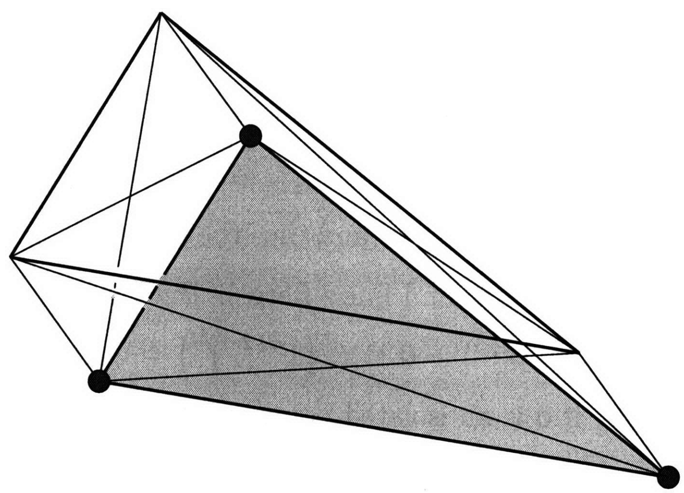


Figure 5-1: Reduction from $\textsc{Rigidity}_2$ to $\textsc{Rigidity}_3$.

If C is not rigid, then C' is not either, because any path in $\mathrm{Conf}(\mathcal{L})$ can be lifted to a path in $\mathrm{Conf}(\mathcal{L}')$ by maintaining $v_2 - v_1 = (0, \ldots, 0, 1)$. To show the converse, we use the following lemma.

Lemma 5.8. *If $\mathcal{L}$ is connected, then in every configuration of $\mathcal{L}'$, $\overrightarrow{v_1v_2}$ is the same for all vertices v of $\mathcal{L}$.*

Proof. Consider an edge (u, v) in $\mathcal{L}$. Since the rectangular degenerate tetrahedron $u_1u_2v_2v_1$ is rigid (simplices are rigid in any number of dimensions) and opposite edges of a rectangle are parallel, $\overrightarrow{u_1u_2} = \overrightarrow{v_1v_2}$ in any configuration of $\mathcal{L}'$. Since $\mathcal{L}$ is connected, the result follows by transitivity. ■

From Lemma 5.8, it follows that configurations of $\mathcal{L}'$ are always two copies of a $(d-1)$-dimensional configuration of $\mathcal{L}$, spaced by 1 unit in some direction that is perpendicular to all the edges of $\mathcal{L}$. If C is not contained entirely in a $(d-2)$-dimensional hyperplane, then the only direction perpendicular to all the edges is the normal direction. If C is contained entirely in a $(d-2)$-dimensional hyperplane, then the additional motions available to C' of varying the direction $\overrightarrow{v_1 v_2}$ are rotations, hence rigid motions. In either case, if C is a rigid configuration of $\mathrm{Conf}(\mathcal{L})$, then C' must be a rigid configuration of $\mathrm{Conf}(\mathcal{L}')$. Thus RIGIDITY$_d$ is coNP-hard. ∎

RIGIDITY$_1$ is trivial because configuration spaces of 1-dimensional linkages are discrete modulo rigid motions, and thus any connected linkage in 1-dimensional space is rigid.

Chapter 6

Higher Dimensions

A natural question is whether the results of Chapter 4 generalize to d dimensions. Namely, consider the problem of drawing a portion of the surface $f(x_1, \ldots, x_d) = 0$ where f is a polynomial in d variables.

In this chapter, we extend Kempe's Universality Theorem to linkages in d dimensions for any $d \geq 2$ (at the end of this chapter, we treat the special case $d = 1$). We first construct a d-dimensional analogue of the Peaucellier linkage that constrains a vertex to a $(d-1)$-dimensional hyperplane. We may then restrict a vertex to a plane or a line by attaching multiple such linkages. This allows us to re-use most of the proof of the two-dimensional Kempe's theorem for the general case.

In constructing the d-dimensional Peaucellier linkages, we need to deal with spheres of various dimensions in various ambient spaces. We set the following convention.

Definition 6.1. A *d-sphere* of radius $r \geq 0$ is the set of points in $\mathbb{R}^{d+1}$ at distance r from the origin, or any isometric subset of $\mathbb{R}^m$ for $m \geq d+1$. In particular, a 0-sphere is a one- or two-point set, and a (-1)-sphere is a zero- or one-point set.

6.1 d-dimensional Peaucellier linkages

We need some technical lemmas in order to construct linkages that avoid degeneracies.

Lemma 6.2. *Let $d \geq 2$ and let S be a unit $(d-2)$-sphere. There exist reals $0 < \alpha < \beta$ such that there exist d points on S with any two points at distance at least β, but there do not exist d points on S in a $(d-2)$-dimensional hyperplane with any two points at distance at least α.*

PROOF. First consider the d points $v_1, \ldots, v_d$ on S that form the vertices of a regular $(d-1)$-dimensional simplex. Then $||v_i - v_j|| = \sqrt{2 - 2v_i \cdot v_j}$ is constant for all $i \neq j$ so $v_i \cdot v_j = c$ for all $i \neq j$ for some constant c. By symmetry $\sum_{i=1}^{d} v_i = 0$ so

$$0 = \Big(\sum_{i=1}^{d} v_i\Big) \cdot \Big(\sum_{i=1}^{d} v_i\Big) = \sum_{i=1}^{d} v_i \cdot v_i + \sum_{i \neq j} v_i \cdot v_j = d + d(d-1)c.$$

Hence $c = -1/(d-1)$ and thus $||v_i - v_j|| = \sqrt{2 + 2/(d-1)}$. Now suppose $\{v_1, \ldots, v_d\}$ is any set of d points on S such that for all $i \neq j$, $||v_i - v_j|| \geq \sqrt{2 + 2/(d-1)}$, so $v_i \cdot v_j \leq -1/(d-1)$. Now

$$0 \leq \Big(\sum_{i=1}^{d} v_i\Big) \cdot \Big(\sum_{i=1}^{d} v_i\Big) = \sum_{i=1}^{d} v_i \cdot v_i + \sum_{i \neq j} v_i \cdot v_j \leq d + d(d-1)\frac{-1}{d-1} = 0.$$

Thus equality must hold and $v_i \cdot v_j = -1/(d-1)$. Hence the distances between v_i and v_j are all $\sqrt{2 + 2/(d-1)}$ so the v_i are the vertices of a regular $(d-1)$-dimensional simplex. Thus we have shown that the minimum distance between any pair of d points on S is at most $\sqrt{2 + 2/(d-1)}$, with equality if and only if the d points are the vertices of a regular $(d-1)$-dimensional simplex.

Now the set of d-tuples of points on S that lie in a $(d-2)$-dimensional hyperplane is a closed set, because it is given by the condition that the volume of the simplex they define is 0. It is also bounded so the minimum distance between two of these points takes on a maximum value α' on this set. By the above result, this maximum value is less than $\sqrt{2 + 2/(d-1)}$. Set $\beta = \sqrt{2 + 2/(d-1)}$ and choose $\alpha' < \alpha < \beta$. ∎

Lemma 6.3. *Let $d \geq 2$. Suppose $w_1, \ldots, w_d$ are d points that do not lie in a common $(d-2)$-hyperplane. Suppose v_1, v_2, v_3 are three points such that the distances $d(v_i, w_j)$ are equal for all i and j. Then at least two of the points v_1, v_2, v_3 coincide.*

PROOF. Suppose not; then v_1, v_2, v_3 form a triangle. Consider the projection of $\mathbb{R}^d$ onto the plane of this triangle. The vertices w_i are equidistant from v_1, v_2, v_3, so they must all project onto the circumcenter of this triangle. But by assumption they do not all lie on a common $(d-2)$-hyperplane, a contradiction. ∎

Proposition 6.4. *For any $d \geq 2$ and any $0 < r < R$ there exists a $C > 0$ and a linkage $\mathcal{L}$ containing vertices v_1, v_2 and $w_1, \ldots, w_d$ (and others) such that*

(i) v_i is connected by a bar of length C to w_j for $i = 1, 2$, $j = 1, \ldots, d$;

(ii) there exists a configuration of $\mathcal{L}$ with $d(v_1, v_2) = d$ if and only if $r \leq d \leq R$;

(iii) in any configuration of $\mathcal{L}$, the vertices $w_1, \ldots, w_d$ do not lie in any $(d-2)$-dimensional hyperplane.

PROOF. Join v_1 to v_2 with a hook with minimum length r and maximum length R. Let α and β be as in the result of Lemma 6.2. Since

$$\lim_{C\to\infty} \frac{\sqrt{C^2 - r^2}}{\sqrt{C^2 - R^2}} = 1 \quad \text{we can choose } C > 0 \text{ such that} \quad \frac{\sqrt{C^2 - r^2}}{\sqrt{C^2 - R^2}} < \frac{\beta}{\alpha}.$$

Join vertices $w_1, \ldots, w_d$ to v_1 and v_2 by bars of length C, and join w_i, w_j by a hook constraining their distance to be at least $\alpha\sqrt{C^2 - r^2}$ for all i and j (and at most $2C$). This is our linkage $\mathcal{L}$.

For any $r \leq D \leq R$, we can place v_1 and v_2 to lie at distance D; then the $(d-1)$-spheres of radius C centered at v_1 and v_2 intersect in a $(d-2)$-sphere of radius $\sqrt{C^2 - D^2} \geq \sqrt{C^2 - R^2}$. By Lemma 6.2 we can find locations for $w_1, \ldots, w_d$ on this $(d-2)$-sphere of radius $\sqrt{C^2 - D^2}$ with any two at distance at least

$$\beta\sqrt{C^2 - D^2} \geq \beta\sqrt{C^2 - R^2} > \alpha\sqrt{C^2 - r^2}.$$

Conversely, for any configuration of this linkage, the vertices v_1 and v_2 lie at least r apart, so $w_1, \ldots, w_d$ lie on a $(d-2)$-sphere of radius at most $\sqrt{C^2 - r^2}$. Since their mutual distances are all at least $\alpha\sqrt{C^2 - r^2}$, we conclude from Lemma 6.2 that the vertices $w_1, \ldots, w_d$ cannot lie in a $(d-2)$-dimensional hyperplane. ∎

Theorem 6.5. *Let $d \geq 2$. There exists a linkage $\mathcal{L}$ with vertices O, v_1, v_2 (among others) such that*

(i) O is fixed at the origin;

(ii) vertex v_1 draws a set with nonempty interior, not containing O; and

(iii) in any configuration of $\mathcal{L}$, v_1 and v_2 lie on a common ray through the origin and $Ov_1 \cdot Ov_2$ is a constant.

PROOF. Let $\mathcal{L}'$ be the linkage from Proposition 6.4 with $r = 1$, $R = 2$. Construct $\mathcal{L}$ from $\mathcal{L}'$ by fixing a vertex O at the origin and join O to $w_1, \ldots, w_d$ by bars of length D for some $D > C$. It is easy to see that this linkage has configurations with v_1 any point between two distinct $(d-1)$-spheres centered at O (i.e. any point in a shell centered at O). Thus it suffices to check condition (iii). In any configuration of $\mathcal{L}$ the points $w_1, \ldots,$ w_d lie in the intersection of the spheres of radius C centered at v_1 and v_2, which lies in a $(d-1)$-dimensional hyperplane, but they do not lie in a $(d-2)$-dimensional hyperplane, so they lie in a unique $(d-1)$-dimensional hyperplane. Consider v_1', the reflection of v_1 in this hyperplane; it also lies at distance C from each of the w_i, so by Lemma 6.3, $v_1' = v_2$. There exist two distinct points O', O'' on the line through v_1 and v_2 and not between v_1 and v_2 that are each distance D from each of the w_i. By Lemma 6.3 again O coincides with O' or O'', hence v_1 and v_2 lie on a common ray through O. It remains to show that $Ov_1 \cdot Ov_2$ is a constant. Consider the plane through O, v_1, v_2, and w_1. Let v be the midpoint of v_1v_2, that is also the foot of the perpendicular from w_1 to the line Ov_1v_2, because $|v_1w_1| = C = |v_2w_1|$. Let $h = |vw_1|$. Assume without loss of generality that v_1 lies between O and v_2. Now

$$|Ov_1| \cdot |Ov_2| = (|Ov| - |v_1v|)(|Ov| + |vv_2|) = |Ov|^2 - |v_1v|^2 = (D^2 - h^2) - (C^2 - h^2) = D^2 - C^2.$$

This is a constant, so we are done. ∎

Corollary 6.6. *Let $d \geq 2$, and let P be a $(d-1)$-dimensional hyperplane in $\mathbb{R}^d$. Then P is continuously constructible using $O(d^3)$ bars. If H is a d-dimensional half-space in $\mathbb{R}^d$, then H is continuously constructible.*

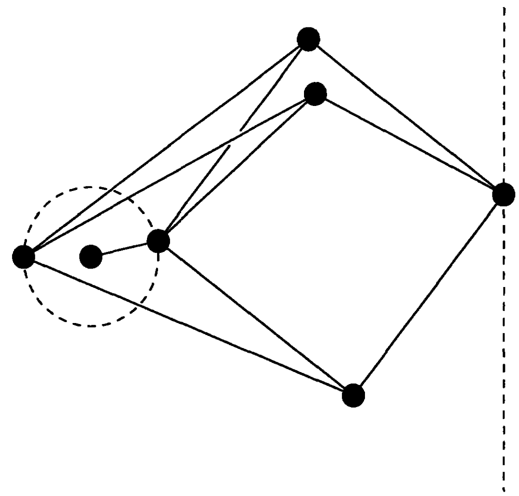


Figure 6-1: A 3-dimensional Peaucellier linkage.

PROOF. We prove the result for some specific P and U; the result follows by translating, rotating, and scaling the linkage. Take the linkage $\mathcal{L}$ of Theorem 6.5 and let U' be an open ball contained in the trace of v_1. Let v be the center of this ball and let b be the distance from O to v. Fix a vertex v' at the midpoint of Ov and add a bar of length $b/2$ connecting v' to v_1. Then v_1 is constrained to a sphere S through O so by properties of inversion v_2 is constrained to a plane P. Moreover, the trace of v_1 contains the intersection of S with the open ball U', so the trace of v_2 contains the intersection of P with an open ball. Let U be this open ball and let $x = v_2$. Then because v_1 and v_2 are related by geometric inversion (Theorem 6.5(iii)), x is constrained to the plane P. Conversely, by the construction of U, any point of U lifts to a configuration of $\mathcal{L}$. Thus P is constructible.

That P is continuously constructible follows from the fact that $\mathcal{L}$ is constructed from a number of hook linkages with nonzero inner radii, and by Proposition 2.10, such hooks continuously constructs their traces.

To construct H, we modify linkage the linkage for P by replacing the bar $v'v_1$ of length $b/2$ with a hook of inner radius 0 and outer radius $b/2$. By geometric inversion, this constrains $x = v_2$ to H, with configurations within an open subset of that half-space containing a point of the boundary. Again by scaling, translating, and rotating, we extend the result to all U. Because the configurations of the Peaucellier linkage are such that $|v' - v_1|$ is bounded from below, the relevant hook is continuously constructible (see the nicely constructible part of the proof of Proposition 3.8 for an argument); it follows that H is continuously constructible.

Both constructions require $O(d^2)$ bars, with the cost dominated by the hooks connecting each pair of w_is. ■

By choosing the set U sufficiently large, we may effectively constrain a vertex v to a fixed hyperplane of any desired dimension by attaching up to d of these linkages to v.

Corollary 6.7. *Any vector subspace H of $\mathbb{R}^d$ is continuously constructible using $O(d^3)$ bars.*

6.2 Bracing the Translators

Lemma 6.8. *For any $a, b > 0$, the parallelogram configuration space embedded in d dimensions*

$$S = \left\{(A,B,C,D) \in (\mathbb{R}^d)^4 : |AB| = |CD| = a, |BC| = |AD| = b, AB \parallel CD, BC \parallel AD\right\}$$

is nicely constructible using $O(1)$ bars.

PROOF. It suffices to show that every configuration of the braced parallelogram of Proposition 3.1 is planar in any number of dimensions; the result then follows by Proposition 3.1.

The degenerate triangles AMB and DNC are simplices, and thus are rigid in any number of dimensions. Thus $ABCD$ is planar if $\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$. This follows from $|AD| = |MN| = |BC| = b$ via a simple coordinates analysis. ■

Proposition 6.9. *Let $a > 0$. Then the translation by a bar OX,*

$$S = \left\{(O,X,Y,Z) \in (\mathbb{R}^d)^4 : |OX| = |YZ| = a, \overrightarrow{OX} + \overrightarrow{OY} = \overrightarrow{OZ}\right\}$$

is constructible using $O(1)$ bars. If $T \subset (\mathbb{R}^d)^k$ is a continuously constructible set containing vertices O, X, and Y such that inside T, $|C(O) - C(Y)|$ is bounded away from zero, then the set obtained by gluing S and T along vertices O, X, and Y is continuously constructible.

PROOF. We use the construction of 3.8. The proof in Proposition 3.8 that S is constructible for the case $d = 2$ relied only on the fact that opposite edges of a parallelogram are parallel, which is true here by Lemma 6.8. Thus, S is constructible.

If additionally S is glued along vertices O, X, and Y to a set T nicely constructed by a linkage $\mathcal{L}_T$ where, within T, $d(O,Y)$ is bounded from below, then N is the vertex projected away from hook ONY which is glued to a set on which $2R > |OY| > 0$. By Proposition 2.15, the projection forgetting N has the path-lifting property. Given N, $M = N + \overrightarrow{OX}$. It follows that the set obtained by gluing S and T is continuously constructible. ∎

6.3 Kempe's Theorem

For $j = 1, \ldots, m$, let $\mathbf{x_j} \in \mathbb{R}^d$, so that we have $O(dm)$ variables $x_{j,i}$ over $\mathbb{R}$.

Theorem 6.10. *Let $d \geq 2$. Let $f \in \mathbb{R}[\{x_{j,i}\}]$ be a polynomial with real coefficients in dm variables of total degree n, and let A be a closed shell in $\mathbb{R}^d$. Then there exists a linkage over $\mathbb{R}^d$ that draws and continuously constructs the set $A^m \cap \mathcal{Z}(f)$. Moreover, this linkage contains $O\left(d^3\binom{n+dm}{n}\right)$ bars.*

Theorems 1.4 and 1.6 follow from Theorem 6.10 using an argument similar to that of Lemma 4.3 (we treat d as a constant in this asymptotic analysis). We now prove Theorem 6.10.

Proof. Assume without loss of generality that B is centered at the origin. Let R be the radius of the ball. For each vertex j and each dimension x_i, construct a vertex $w_{j,i}$ that is constrained to lie in the intersection of the x_i axis with the ball B, using $d-1$ d-dimensional Peaucellier linkages. Now, construct the output vertex u_j, by using translators to add together the vectors for these $w_{j,i}$, so that $w_{j,i}$ is the projection of u_j onto the x_i axis. Constrain u_j to lie in the ball B using a pair of bars of length $\frac{R}{2}$. Now, it suffices to implement the constraint $f(x_{1,1}, \ldots, x_{m,d}) = 0$. We do this by first moving the lengths $w_{j,i}$ onto the x_1 axis, and then using the 2-dimensional Kempe construction (with all vertices constrained to lie in that plane with $d-2$ d-dimensional Peaucellier linkages) on the resultant md angles to construct a point with x_1 coordinate $f(x_{1,1}, \ldots, x_{m,d})$, that we then set to zero using a Peaucellier linkage. The construction so far has used $O(dm)$ d-dimensional translators, for a total of $O(dm)$ bars.

First, we show how to move the length $w_{j,i}$ onto the x_1-axis, giving vertices $z_{j,i} = (w_{j,i}, 0)$

in the x_1x_2-plane. If $i = 1$ then there is nothing to do, so assume $i > 1$. Consider for a moment the x_1x_i-plane. The vertex with coordinates in the x_1x_i plane $(0, w_{j,i})$ lies between $-R$ and R on the x_i-axis, so we can build a braced rhombus with bars of length $R/2$ with one vertex at the origin and opposite vertex at $(0, w_{j,i})$ and with its other vertices constrained to lie in the x_1x_i-plane. Fix a bar along the negative x_i-axis, so it makes a $-\pi/2$ angle with the x_1-axis. Using Kempe's additor (still constraining vertices to the x_1x_i-plane) we can add $-\pi/2$ to the angles of the bars of this rhombus and construct a braced rhombus with two of its edges along these rotated edges. Then its fourth vertex lies at distance $w_{j,i}$ along the x_1-axis, as desired. This construction uses $O(dm)$ additors, each constrained by $d-2$ d-dimensional Peaucellier linkages, for a total of $O(md^4)$ bars.

Now we restrict our attention, and our vertices, to the x_1x_2-plane. We can construct bars of length $R/2$ forming a braced rhombus with opposite endpoints at the origin O and at $z_{j,i}$, so that the angles $\theta_{j,i}$ from the x-axis formed by the bars out of the origin would have cosine $\frac{w_{j,i}}{R}$. (Note that we might get either the positive or the negative form of the angle, but this is fine, because cosine is an even function). The trigonometric algebra from Lemma 4.4 generalizes to a polynomial of total degree n in dm variables in a straightforward fashion, giving

$$f(x_{1,1}, \ldots, x_{m,d}) = \sum_{|\boldsymbol{\delta}| \le n} \sum_{j=1}^{m} f_{j,\boldsymbol{\delta}} \cos\left(\sum_{i=1}^{d} \delta_{j,i}\theta_{j,i} + \gamma_{j,\boldsymbol{\delta}}\right)$$

for some constants $f_{j,\boldsymbol{\delta}}$, $\gamma_{j,\boldsymbol{\delta}}$, where the sum is over all vectors $\boldsymbol{\delta} \in \mathbb{Z}^{dm}$ with L_1 norm at most n. There are at most $O\left(\binom{n+md}{n}\right)$ terms in this sum. Following Lemma 4.4, we construct from the angles $\theta_{j,i}$ all the angles $\sum_{i=1}^{d} \delta_{j,i}\theta_{j,i} + \gamma_{j,i}$ for $|\boldsymbol{\delta}| \le n$ using $O\left(\binom{n+dm}{n}\right)$ additors and reversors. Following the proof of Theorem 4.2, we use $O\left(\binom{n+dm}{n}\right)$ planar translators to create a linkage with a vertex w such that the x_1-coordinate of w is $f(x_{1,1}, \ldots, x_{m,d})$. As in Theorem 4.2, we then use a sufficiently large 2-dimensional modified Peaucellier linkage to force the x_1-coordinate of w to be nonnegative. Thus, in any configuration of this linkage, we have that $f(x_{1,1}, \ldots, x_{m,d}) \ge 0$. Conversely, for any $(x_{1,1}, \ldots, x_{m,d}) \in A$ with $f(x_{1,1}, \ldots, x_{m,d}) \ge 0$ there is a configuration of the linkage where u_j is located at $(x_{j,1}, \ldots, x_{j,d})$. We need $d-2$ d-dimensional Peaucellier linkages on each vertex to keep them in the plane, and we need $O\left(\binom{n+dm}{n}\right)$ bars in the plane, so the total cost of this construction is $O\left(d^3\binom{n+dm}{n}\right)$ bars.

Because $n \geq 1$, this constribution dominates the number bars used earlier in this argument.

This is continuously constructible because the individual gadgets are continuously constructible, along with an argument analogous to the continuity argument in Theorem 4.2, replacing the phrase "nicely constructible" with "continuously constructible" (the individual arguments for the various classes of points are quite similar, so we do not repeat them). ■

Open Question 1. *Which drawable sets in d dimensions are rigidly constructible?*

Both the work of King [King3] and our d-dimensional Peaucellier inversor are not rigid constructions. It is not obvious whether either set of gadgets can be modified to be rigidly constructible.

6.4 Lower Dimensions

Lemma 6.11. *Let $S \subset \mathbb{R}^1$ be a finite set and let $r > 0$. Then there is a one-dimensional linkage $\mathcal{L}$ that draws a finite set $S' \subset \mathbb{R}^1$ such that S' contains S and every element of S' not in S is at distance more than r from any point of S.*

Proof. Write $S = \{s_1, s_2, \ldots, s_n\}$, and let y be a large real number to be determined later, but in particular greater than $-s_i$ for every i. Let $\mathcal{L}$ be a linkage consisting of a vertex v_0 fixed at a point x_0 (also to be determined later) and a chain of bars v_0v_1, v_1v_2, $\ldots$, $v_{n-1}v_n$, with the length of bar $v_{i-1}v_i$ equal to $\frac{y+s_i}{2}$. Then the trace of vertex v_n is the set S' of values of the form

$$x_0 \pm \frac{y+s_1}{2} \pm \frac{y+s_2}{2} \pm \cdots \pm \frac{y+s_n}{2}.$$

Let m be the minimum value attained by such an expression, that is, the value of the expression when all $\pm$ signs are replaced by $-$. Then $m + y + s_i$ is in the trace of $\mathcal{L}$; choose x_0 so that $m + y = 0$ and thus S is contained in the trace of $\mathcal{L}$. Every element of S' not of the form $m + y + s_i$ either is m or has at least two $\pm$ signs replaced by $+$, and thus differs from every element of S by at least $y - 2\max\{|s_1|, \ldots, |s_n|\}$. By choosing y sufficiently large we can ensure that all of these differences are greater than r. ■

Lemma 6.12. *Let $S \subset \mathbb{R}^1$ be a finite subset. Then there is a linkage $\mathcal{L}$ that draws S.*

PROOF. We use Lemma 6.11 twice. First, let $r_1 > 0$ be arbitrary and apply Lemma 6.11 to construct a linkage $\mathcal{L}_1$ with a vertex v_1 whose trace is a set S_1 containing S. Then let r_2 be larger than the maximum distance between a point of S and a point of S_1, and apply Lemma 6.11 again to construct a linkage $\mathcal{L}_2$ with a vertex v_2 whose trace is a set S_2, also containing S, and such that every point of S_2 not in S is at distance at least r_2 from S. Then $S_1 \cap S_2 = S$, so letting $\mathcal{L}$ be the linkage formed from $\mathcal{L}_1$ and $\mathcal{L}_2$ by gluing v_1 to v_2, the trace of the vertex $v_1 = v_2$ of $\mathcal{L}$ is exactly S. ∎

Theorem 6.13. *Let $S \subset \mathbb{R}^1$. Then S is drawable if and only if S is either finite or $S = \mathbb{R}^1$.*

PROOF. Suppose the trace of vertex v of $\mathcal{L}$ is S. If the connected component of $\mathcal{L}$ containing v does not contain a fixed vertex, then either $S = \emptyset$ or $S = \mathbb{R}^1$. Otherwise, the location of v in any configuration of $\mathcal{L}$ is determined by the directions of the n bars in that component. Because there are only 2^n choices, the trace of v must be finite in this case. The interesting case of the converse is Lemma 6.12. ∎

Chapter 7

Characterization of Drawable Sets

In Section 6.4, we characterized the drawable sets in $\mathbb{R}^1$. Thus, let $d \geq 2$.

We have shown that there exist linkages drawing any set defined by a single polynomial equation or inequality. A natural question is what other sets can be drawn by linkages. The main result of this section is that the drawable sets in $\mathbb{R}^d$ are precisely the compact semi-algebraic sets in $\mathbb{R}^d$, along with $\mathbb{R}^d$ itself. This theorem was previously shown by Henry C. King in [King2] and [King3]. We obtain a slightly stronger result than King, proving that the compact semi-algebraic sets in $\mathbb{R}^d$ are continuously constructible as well.

Theorem 7.1. *Let S_1 and S_2 be drawable sets in $\mathbb{R}^d$. Then $S_1 \cap S_2$ is also drawable. If S_1 and S_2 are also continuously constructible, so is $S_1 \cap S_2$.*

PROOF. Let $\mathcal{L}_1$ be a linkage with a vertex v_1 whose trace is S_1 and let $\mathcal{L}_2$ be a linkage with a vertex v_2 whose trace is S_2. Let $\mathcal{L}$ be the union of $\mathcal{L}_1$ and $\mathcal{L}_2$, identifying vertices v_1 and v_2. Then it is easy to see that this vertex of $\mathcal{L}$ draws $S_1 \cap S_2$. ∎

Theorem 7.2. *Let S_1 and S_2 be drawable sets in $\mathbb{R}^d$. Then $S_1 \cup S_2$ is also drawable. If S_1 and S_2 are also continuously constructible, so is $S_1 \cup S_2$.*

PROOF. If $x, y, z \in \mathbb{R}^d$, define $f(x, y, z)$ to be

$$f(x, y, z) = \left(\sum_{i=1}^{n} (z_i - x_i)^2 \right) \left(\sum_{i=1}^{n} (z_i - y_i)^2 \right).$$

Now, f is a polynomial in $3d$ variables of total degree 2, and thus by Theorem 6.10, we can construct a multi-input continuously constructible Kempe linkage $\mathcal{L}$ for the set $T = \{x, y, z | f(x, y, z) = 0\}$. Let $\mathcal{L}_1$ be the linkage drawing S_1 and $\mathcal{L}_2$ be the linkage drawing S_2. Construct a new linkage $\mathcal{L}'$ from $\mathcal{L}, \mathcal{L}_1$, and $\mathcal{L}_2$ by gluing the output vertex of $\mathcal{L}_1$ to x from $\mathcal{L}$, and the output vertex of $\mathcal{L}_2$ to y from $\mathcal{L}$. We now project to the locations of vertex z. By inspecting f, we see that in every configuration of $\mathcal{L}$, either $z = x$ or $z = y$, and thus in every configuration of $\mathcal{L}'$, $z = x \in S_1$ or $z = y \in S_2$. Conversely, for any $u \in S_1$, there is a configuration of $\mathcal{L}$ with $u = z = x$ and y an arbitrary element of S_2 (and similarly for $u \in S_2$). Thus $\mathcal{L}'$ draws $S_1 \cup S_2$.

Suppose S_1 and S_2 are continuously constructible. Let $g : [0,1] \to S_1 \cup S_2$ be a path in $S_1 \cup S_2$. We construct a continuous map $g' : [0,1] \to \mathrm{Conf}(\mathcal{L})$ lifting g as follows. If $g(t) \in S_i$, then lift $g(t)$ to $\mathcal{L}_i$ continuously using the lift guaranteed to exist by the fact that S_i is continuously constructible. Set S_i is closed, so if $g(t) \notin S_i$, there is a neighborhood U of $S_1 \cup S_2$ such that $g(t) \in U$ and $U \cap S_i = \emptyset$. Pick any closed subset $[t_0, t_1]$ of the open interval $g^{-1}(U)$. In the interval $[t_0, t_1]$, rearrange $\mathcal{L}_i$ to a configuration consistent with the next entry of path g into S_i. At all points t that we have not defined the life of g to $\mathcal{L}_i$ for, lift $g(t)$ into $\mathcal{L}_i$ with to a locally constant function. The resulting lifted function is piecewise continuous, and is continuous at the transition points, hence it is continuous. ∎

Note that this construction does not preserve the property of being rigidly constructible.

Corollary 7.3. *Any finite union of bounded sets of the form*

$$\{\, (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : f_1(x_1, x_2, \ldots, x_d) \geq 0, \ldots, f_s(x_1, x_2, \ldots, x_d) \geq 0 \,\}$$

is drawable and continuously constructible.

Proof. Let S be a bounded set of the form

$$S = \{\, (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : f_1(x_1, x_2, \ldots, x_d) \geq 0, \ldots, f_s(x_1, x_2, \ldots, x_d) \geq 0 \,\}.$$

Let B be a closed ball containing S. Then the sets $B \cap \{\, (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : f_i(x_1, x_2, \ldots, x_d) \geq 0 \,\}$ are drawable and continuously constructible by Theorem 1.4 and their intersection

$B \cap S = S$ is drawable and continuously constructible by Theorem 7.1. A finite union of such sets is then drawable and continuously constructible by Theorem 7.2. ∎

It turns out that the above sets, together with $\mathbb{R}^d$, are all of the drawable sets. To prove this, we need some results from real algebraic geometry.

Definition 7.4. An *algebraic subset* of $\mathbb{R}^n$ is one of the form $\{ x \in \mathbb{R}^n : f(x) = 0 \}$ for some polynomial f.

Definition 7.5. A *semi-algebraic subset* of $\mathbb{R}^n$ is one formed from algebraic sets by the operations of intersection, union, and complement.

Proposition 7.6. *The intersection of two algebraic subsets of* $\mathbb{R}^n$ *is algebraic.*

PROOF. If V_1 is defined by $f_1 = 0$ and V_2 is defined by $f_2 = 0$ then $V_1 \cap V_2$ is defined by $f_1^2 + f_2^2 = 0$. ∎

We are now ready to prove Theorem 1.5:

Theorem 1.5. A set $S \subset \mathbb{R}^d$ is drawable if and only if S is compact and semi-algebraic or $S = \mathbb{R}^d$.

PROOF. By Theorem 2.7.2 of [BCR], any compact semi-algebraic set is a finite union of bounded sets defined by a finite number of non-strict polynomial inequalities, hence drawable by Corollary 7.3. Conversely, suppose S is a set drawn by a vertex v of some linkage $\mathcal{L}$. Assume that $S \neq \mathbb{R}^d$; then we must show that S is compact and semi-algebraic. Further assume that S is nonempty; then S is also drawn by the connected component of $\mathcal{L}$ containing v, hence we may assume $\mathcal{L}$ is connected. Linkage $\mathcal{L}$ has at least one configuration, but $S \neq \mathbb{R}^d$, so by the proof of Proposition 2.5, it must have at least one fixed vertex. Then in any configuration of $\mathcal{L}$ each vertex lies in the closed ball of radius R about the location of this fixed vertex, where R is the sum of the lengths of all bars of $\mathcal{L}$. Hence $\mathrm{Conf}(\mathcal{L})$ is bounded. Moreover $\mathrm{Conf}(\mathcal{L})$ is defined by the polynomial equations $|C(v) - C(w)|^2 = \ell(v, w)^2$ for each edge (v, w) in $\mathcal{L}$, so $\mathrm{Conf}(\mathcal{L})$ is an algebraic subset of $\mathrm{Map}(V(\mathcal{L}), \mathbb{R}^d)$. In particular, $\mathrm{Conf}(\mathcal{L})$ is closed, hence compact. Now S is the projection of $\mathrm{Conf}(\mathcal{L})$ to $\mathbb{R}^d$, so S is compact and semi-algebraic by Theorem 2.2.1 of [BCR]. ∎

Chapter 8

Optimality

In this chapter, we address the question of how many bars are needed to build linkages constructing desired sets in $\mathbb{R}^d$ for a fixed dimension $d \geq 2$. We first show that the bound on the number of bars used in Theorem 1.4 to draw the zero set of a polynomial cannot be reduced. Later, we consider the problem of drawing sets of n points in $\mathbb{R}^d$; we show that the minimum number of bars needed to draw a given n-point set lies between $\Theta(\log n)$ and $\Theta(n)$, and that both these bounds are asymptotically optimal.

The arguments in this chapter use more technical tools from real algebraic geometry than the previous chapters. Refer to Appendix A for the statements of the various results from [BCR] that we use throughout this chapter.

8.1 Varieties Defined by a Single Equation

Fix $d \geq 2$. In this section, we show that our construction of a linkage constructing the zero set in $\mathbb{R}^d$ of a polynomial of total degree n using $O(n^d)$ bars is asymptotically optimal: there exist polynomials of degree n whose zero sets cannot be drawn with fewer than $\Omega(n^d)$ bars. Our argument is a dimension count; roughly speaking, there are $\Theta(n^d)$ different zero sets of degree n polynomials, and so we need a space of linkages of dimension $\Omega(n^d)$ to be able to draw all of them. To make this precise, we need some tools from real algebraic geometry.

Definition 8.1. An ideal of $\mathbb{R}[x_1, \ldots, x_d]$ is *real* if for any sequence $a_1, \ldots, a_r \in \mathbb{R}[x_1, \ldots, x_d]$,

$$a_1^2 + \cdots + a_r^2 \in I \implies a_i \in I \text{ for each } i.$$

Lemma 8.2. *Identify the set of polynomials in* $\mathbb{R}[x_1, \ldots, x_d]$ *of total degree* n *and constant term 1 with* $\mathbb{R}^{\binom{n+d}{d}-1}$. *Let* Y_n *be the subset of* $\mathbb{R}^{\binom{n+d}{d}-1}$ *consisting of irreducible polynomials* f *such that* $f(1, 0, \ldots, 0) < 0$. *Then* Y_n *is a semi-algebraic set that is Zariski dense in* $\mathbb{R}^{\binom{n+d}{d}-1}$. *In particular,* $\dim Y_n = \binom{n+d}{d} - 1$.

Proof. Let Z_n be the subset of $\mathbb{R}^{\binom{n+d}{d}-1}$ consisting of reducible polynomials. A polynomial $f \in Z_n$ can be factored into two polynomials of total degree summing to n and each with constant term 1, so there is a surjective algebraic map

$$\coprod_{1 \leq k \leq n-1} \mathbb{R}^{\binom{k+d}{d}-1} \times \mathbb{R}^{\binom{n-k+d}{d}-1} \to Z_n.$$

Therefore Z_n is a semi-algebraic subset of $\mathbb{R}^{\binom{n+d}{d}-1}$ of dimension at most

$$\max_{1 \leq k \leq n-1} \binom{k+d}{d} - 1 + \binom{n-k+d}{d} - 1 = \binom{d+1}{d} - 1 + \binom{n-1+d}{d} - 1 < \binom{n+d}{d} - 1.$$

because $\binom{k+d}{d} + \binom{(n-k)+d}{d}$ is a convex function of k, and $n \geq 2$.

Its closure then has dimension less than $\binom{n+d}{d} - 1$, by Theorem 2.8.2 of [BCR]. Thus the complement of its closure is an open dense set in the norm topology, by Theorem 2.8.5(i) of [BCR]. This set is clearly nonempty, so $\dim(\{f \in \mathbb{R}^{\binom{n+d}{d}-1} : f(1, 0, \ldots, 0) < 0\}) = \binom{n+d}{d} - 1$, by Theorem 2.8.4 of [BCR]. Hence by Theorem 2.8.5(i) of [BCR], the intersection of these open dense sets,

$$Y_n = \left\{ f \in \mathbb{R}^{\binom{n+d}{d}-1} : f \notin Z_n, f(1, 0, \ldots, 0) < 0 \right\}$$

has dimension $\binom{n+d}{d} - 1$. ■

We next describe how to reconstruct an element of Y_n from the intersection of its zero set with the closed unit ball.

Lemma 8.3. *Let $f \neq g \in Y_n$. Then the intersections of the zero sets of f and g with the closed unit ball are distinct.*

PROOF. Let $f \in Y_n$. Denote its zero set by $\mathcal{Z}(f)$ and the closed unit ball by B. It suffices to show that f is the unique nonzero polynomial of minimum degree and constant term 1 that vanishes on $\mathcal{Z}(f) \cap B$.

As $f(0,0,\dots,0) = 1$ and $f(1,0,\dots,0) < 0$, the sets $U_1 = B \cap \{x \in \mathbb{R}^d : f(x) < 0\}$ and $U_2 = B \cap \{x \in \mathbb{R}^d : f(x) > 0\}$ are disjoint, nonempty, open semi-algebraic subsets of B. Thus by Theorem 4.5.2 of [BCR], $\dim(B \setminus U_1 \setminus U_2) = \dim(\mathcal{Z}(f) \cap B) = d - 1$. It follows by Theorems 2.8.2 and 2.8.3(i) of [BCR] that the Zariski closure of $\mathcal{Z}(f) \cap B$ is all of $\mathcal{Z}(f)$. As the ideal (f) is real, we conclude by the Real Nullstellensatz that the ideal of polynomials vanishing on $\mathcal{Z}(f) \cap B$ is (f). The unique nonzero polynomial in (f) of minimum degree and constant term 1 is f itself. ∎

Theorem 8.4. *For every n, there exists a polynomial $f(x_1, \dots, x_d)$ of total degree n such that any linkage that has a vertex whose trace is the intersection of the zero set of f with the closed unit ball in $\mathbb{R}^d$ contains at least $\Omega(n^d)$ bars.*

PROOF. Without loss of generality, we may consider only connected linkages. Let k be such that for any polynomial f of total degree n, there is a connected linkage with a vertex whose trace is the intersection of the zero-set of f with the closed unit ball in $\mathbb{R}^d$ using at most k bars. We now show that $k \geq \Omega(n^d)$.

Let $\mathcal{P}_k$ be the "parameter space" of connected marked linkages with at most k bars. Informally, a point of $\mathcal{P}_k$ describes a choice of a graph G, a marked vertex $v \in G$, a set of fixed vertices $W \subset V(G)$, a choice of locations $f : W \to \mathbb{R}^d$, and a choice of bar lengths $\ell : \mathcal{E}(G) \to \mathbb{R}_{\geq 0}$. We are interested in the trace of the vertex v in the linkage defined by these data. There are only a finite number N of choices of the triple (G, v, W) up to isomorphism, and for each such choice a space of dimension

$$d|W| + |E| \leq d|V| + |E| \leq d(|E| + 1) + |E| \leq (2d + 1)k,$$

as G is a connected graph with at most k edges. So we may view $\mathcal{P}_k$ as a semi-algebraic set of dimension at most $(2d+1)k$, living inside $\{1, 2, \dots, N\} \times \mathbb{R}^{(2d+1)k}$. We identify a connected marked linkage $\mathcal{L}$ with at most k bars with the point of $\mathcal{P}_k$ that represents it.

There is a logical formula in $\mathcal{P}_k \times \mathbb{R}^d$ that describes whether the trace of the marked vertex in a linkage $\mathcal{L} \in \mathcal{P}_k$ contains a point $x \in \mathbb{R}^d$. Write $\mathcal{L}$ in the form

$$(g, f(v_{i_1}), \dots, f(v_{i_w}), \ell(e_1), \dots, \ell(e_j)),$$

where g encodes the choice of (G, v, W), the vertices of G are labeled $v_1, \dots, u_k$, the edges of G are labeled $e_1, \dots, e_j$, where e_i connects s_i and t_i, the indices of the vertices in W are $i_1, \dots, i_w$, and the index of the marked vertex v is m. Then this formula has the form

$$\begin{aligned}(g = 1 \wedge \exists x_1, \dots, x_k \in \mathbb{R}^d : x_m = x \wedge x_{i_1} = f(v_{i_1}) \wedge \cdots \wedge ||x_{s_1} - x_{t_1}||^2 = \ell(e_1)^2 \wedge \cdots)& \\ \vee (g = 2 \wedge \cdots) \vee \cdots .& \quad (8.1)\end{aligned}$$

Denote this formula by $D(\mathcal{L}, x)$. We can use it to define an incidence graph

$$Q = \{ (\mathcal{L}, f) : \mathcal{L} \in \mathcal{P}_k,\ f \in Y_n, \text{ and } \mathcal{L} \text{ draws } \mathcal{Z}(f) \cap B \}$$

by a logical formula:

$$Q = \left\{ (\mathcal{L}, f) \mid \mathcal{L} \in \mathcal{P}_k, f \in Y_n, \forall x \in \mathbb{R}^d : D(\mathcal{L}, x) \iff (||x|| \leq 1 \wedge f(x) = 0) \right\}.$$

Here of course $\mathcal{Z}(f)$ denotes the zero-set of f and B the closed unit ball as in the proof of Lemma 8.3. Thus Q is a semi-algebraic set by Tarski's theorem (Theorem 2.2.4 in [BCR]). The projections $Q \to \mathcal{P}_k$ and $Q \to Y_n$ are semi-algebraic maps; $Q \to \mathcal{P}_k$ is injective by Lemma 8.3, as a linkage only draws one set, but $Q \to Y_n$ is surjective by the definition of k. Then by Theorem 2.8.8 of [BCR],

$$\dim Y_n \leq \dim Q \leq \dim \mathcal{P}_k \leq (2d+1)k.$$

As $\dim Y_n = \binom{n+d}{d} - 1$, we conclude that $k \geq \frac{1}{2d+1}(\binom{n+d}{d} - 1)$, or $k \geq \Omega(n^d)$. ∎

Theorem 1.7. Drawing the zero-set of a polynomial function of total degree n in d variables requires $\Omega(n^d)$ bars in the worse case.

PROOF. Follows from Theorems 1.4 and 8.4. ■

8.2 Finite sets

For a finite subset S of $\mathbb{R}^d$, define $\beta(S)$ to be the minimum number of bars of a linkage that has a vertex whose trace is S. We study the range of β, showing that for a set S of n points, $\Theta(\log n) \leq \beta(S) \leq \Theta(n)$, and that both of these bounds are asymptotically tight.

The upper bound can be handled by arguments similar to those from the previous section.

Theorem 8.5. $\max\{\,\beta(S) : S \subset \mathbb{R}^d, |S| = n\,\} = \Theta(n)$.

PROOF. First we must show that every set S of size n can be drawn by a linkage with at most $\Theta(n)$ bars. Let $S = \{P_1, \dots, P_n\}$, and let $\mathcal{L}_i$ be the linkage with a single vertex fixed at P_i for $i = 1, \dots, n$. Then $\mathcal{L}_i$ draws $\{P_i\}$, so applying Theorem 7.2 $n-1$ times, we can find a linkage $\mathcal{L}$ that draws S. Moreover, the construction of Theorem 7.2 requires $O(1)$ bars in addition to the bars used to draw the two sets, so $\mathcal{L}$ contains $\Theta(n)$ bars in total.

To prove that some n-point set requires $\Theta(n)$ bars we imitate the proof of Theorem 8.4, replacing Y_n by the space Y'_n of all n-tuples of points $(y_1, y_2, \dots, y_m)$, $y_i \in \mathbb{R}^d$, such that $\pi_1(y_1) < \pi_1(y_2) < \dots < \pi_1(y_n)$ where π_1 denotes the value of the first coordinate. Then Y'_n is semi-algebraic of dimension dn. Replace Q by the set of pairs $(\mathcal{L}, y)$ such that the trace of the marked vertex of $\mathcal{L}$ is exactly the set of points of y. Then again each linkage in $\mathcal{P}_k$ corresponds to at most one point of Q, so if every finite set of n points can be drawn by a linkage with at most k bars, $\dim \mathcal{P}_k \geq \dim Q \geq \dim Y'_n = dn$. As $\dim \mathcal{P}_k \leq (2d+1)k$, this entails $k \geq \Omega(n)$. ■

We now treat the lower bound.

Theorem 8.6. *For every n, there exists a linkage with $\Theta(\log n)$ bars that draws a finite set of size n.*

PROOF. We use Kempe's multiplicator and additor (with braced contraparallelograms) to construct a linkage $\mathcal{L}$ that multiplies an angle in the x_1x_2-plane by n. With the "double and optionally add 1" algorithm, we can achieve this construction with $\Theta(\log n)$ bars. Fix the output vertex of $\mathcal{L}$; then there are n possible locations for the input vertex. In other words, the trace of the input vertex is an n-point set. ■

Theorem 8.7. *A linkage with a vertex that draws an n-point set in $\mathbb{R}^d$ must contain at least $\Theta(\log n)$ bars.*

PROOF. Suppose $\mathcal{L}$ is such a linkage with V vertices and E edges. We may assume without loss of generality that $\mathcal{L}$ is connected. Because $\mathcal{L}$ has a vertex that draws an n-point set, the configuration space of $\mathcal{L}$ must have at least n path-connected components. But $\mathrm{Conf}(\mathcal{L})$ is a real algebraic subset of $\mathbb{R}^{dV}$ defined by equations of degree at most 2. Thus by Theorem 11.5.3 of [BCR], the sum of the Betti numbers of $\mathrm{Conf}(\mathcal{L})$ (in singular homology) is at most $2 \cdot 3^{dV-1}$. The sum of these Betti numbers is at least the number of path-connected components of $\mathrm{Conf}(\mathcal{L})$, hence $2 \cdot 3^{dV-1} \geq n$ so $V \geq \Theta(\log n)$. Because $\mathcal{L}$ is connected, $E \geq V - 1 = \Theta(\log n)$. ■

Corollary 8.8. *For a set S of n points in the plane, $\Theta(\log n) \leq \beta(S) \leq \Theta(n)$, and both of these bounds are asymptotically tight.*

PROOF. Theorems 8.6 and 8.7 show that $\min\{\beta(S) : S \subset \mathbb{R}^d, |S| = n\} = \Theta(\log n)$. Together with Theorem 8.5, this implies the result. ■

Open Question 2. *Can one exhibit an explicit family of sets S_n, $|S_n| = n$, such that $\beta(S_n) = \Theta(n)$?*

One might expect for instance that a set $S = \{x_1, \ldots, x_n\}$ with $d(x_i, x_j) \approx 2^{2^j}$ for $i < j$ satisfies $\beta(S) = \Theta(n)$, but we have no proof of this claim.

Appendix A

Results from Real Algebraic Geometry

The following results are proved in [BCR]. We have compiled them here for convenience. We have preserved the numbering from that book.

Theorem ([BCR], 2.2.1). *Let S be a semi-algebraic subset of $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ and let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the projection onto the first factor. Then $\pi(S)$ is a semi-algebraic subset of $\mathbb{R}^k$.*

Theorem ([BCR], 2.2.4, Tarski's Theorem). *Let $\phi(x_1, \ldots, x_n)$ be a first-order formula of the language of ordered fields, with parameters in $\mathbb{R}$, with free variables $x_1, \ldots, x_n$. Then $\{x \in \mathbb{R}^n : \phi(x)\}$ is a semi-algebraic set.*

Theorem ([BCR], 2.2.6 (i)). *The composition $g \circ f$ of semi-algebraic mappings $f : A \to B$ and $g : B \to C$ is semi-algebraic.*

Theorem ([BCR], 2.4.5). *A semi-algebraic subset A of $\mathbb{R}^n$ is semi-algebraically connected if and only if it is connected. Every semi-algebraic set (and in particular, every algebraic subset of $\mathbb{R}^n$) has a finite number of connected components, which are semi-algebraic.*

Theorem ([BCR], 2.7.2). *Let $V \subset \mathbb{R}^n$ be a closed semi-algebraic set. Then V is a finite union of sets of the form $\{x \in \mathbb{R}^n : f_1(x) \geq 0, \ldots, f_s(x) \geq 0\}$ where $f_1, \ldots, f_s \in \mathbb{R}[X_1, \ldots, X_n]$.*

Theorem ([BCR], 2.8.2). *Let $A \subset \mathbb{R}^n$ be a semi-algebraic set. Then*

$$\dim(A) = \dim(\mathrm{clos}(A)) = \dim(\mathrm{clos}_{\mathrm{Zar}}(A)),$$

where $\mathrm{clos}_{\mathrm{Zar}}(A) = \mathcal{Z}(\mathcal{I}(A))$ *is the Zariski closure of* A.

Theorem ([BCR], 2.8.3(i)). *An algebraic set $V \subset \mathbb{R}^n$ is said to be* irreducible, *if, whenever $V = F_1 \cup F_2$, where F_1 and F_2 are algebraic sets, then $V = F_1$ or $V = F_2$. Every algebraic set V is the union – in a unique way – of a finite number of irreducible algebraic sets $V_1, \ldots, V_p$, such that $V_i \not\subset \bigcup_{j \neq i} V_j$ for $i = 1, \ldots, p$, are the* irreducible components *of V. We have* $\dim(V) = \max(\dim(V_1), \ldots, \dim(V_p))$.

Theorem ([BCR], 2.8.4). *Let U by a nonempty open semi-algebraic subset of $\mathbb{R}^n$. Then* $\dim(U) = n$.

Theorem ([BCR], 2.8.5 (i)). *Let $A = \bigcup_{i=1}^{p} A_i$ be a finite union of semi-algebraic sets. Then*

$$\dim(A) = \max(\dim(A_1), \ldots, \dim(A_p)).$$

Theorem ([BCR], 2.8.8). *Let A be a semi-algebraic set and $f : A \to \mathbb{R}^p$ a semi-algebraic mapping. Then* $\dim(A) \geq \dim(f(A))$. *If f is a bijection from A onto $f(A)$, then* $\dim(A) = \dim(f(A))$.

Theorem ([BCR], 4.1.4, Real Nullstellensatz). *Let I be an ideal of $\mathbb{R}[X_1, \ldots, X_n]$. Then $I = \mathcal{I}(\mathcal{Z}(I))$ if and only if I is real.*

Theorem ([BCR], 4.5.1). *Let f be an irreducible polynomial in $\mathbb{R}[X_1, \ldots, X_n]$. Then the following properties are equivalent:*

(i) The ideal (f) is real.

(ii) $(f) = \mathcal{I}(\mathcal{Z}(f))$.

(iii) The polynomial f has a nonsingular zero in $\mathbb{R}^n$ (i.e. there is an $x \in \mathbb{R}^n$ such that $f(x) = 0$ and $\frac{\partial f}{\partial X_i}(x) \neq 0$ for some $i \in \{1, \ldots, n\}$).

(iv) The sign of the polynomial f changes on $\mathbb{R}^n$ (i.e. $f(x)f(y) < 0$ for some x, y in $\mathbb{R}^n$).

(v) $\dim(\mathcal{Z}(f)) = n - 1$.

Theorem ([BCR], 4.5.2). *Let B be an open ball of $\mathbb{R}^n$ (or $B = \mathbb{R}^n$) and U_1 and U_2 two disjoint nonempty semi-algebraic open subsets of B. Then*

$$dim(B \setminus (U_1 \cup U_2)) \geq n - 1.$$

Theorem ([BCR], 11.5.3). *Let $V \subset \mathbb{R}^n$ be an algebraic set defined by equations of degree less than or equal to d. Then the sum of the Betti numbers of V is less than or equal to $d(2d-1)^{n-1}$.*

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