Polytopes, generating functions, and new statistics related to descents and inversions in permutations

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Abstract

We study new statistics on permutations that are variations on the descent and the inversion statistics. In particular, we consider the *alternating descent set* of a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ defined as the set of indices i such that either i is odd and $\sigma_i > \sigma_{i+1}$, or i is even and $\sigma_i < \sigma_{i+1}$. We show that this statistic is equidistributed with the 3-*descent set* statistic on permutations $\tilde{\sigma} = \sigma_1\sigma_2\cdots\sigma_{n+1}$ with $\sigma_1 = 1$, defined to be the set of indices i such that the triple $\sigma_i\sigma_{i+1}\sigma_{i+2}$ forms an odd permutation of size 3. We then introduce Mahonian inversion statistics corresponding to the two new variations of descents and show that the joint distributions of the resulting descent-inversion pairs are the same. We examine the generating functions involving *alternating Eulerian polynomials*, defined by analogy with the classical Eulerian polynomials $\sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}(\sigma)+1}$ using alternating descents. By looking at the number of alternating inversions in alternating (down-up) permutations, we obtain a new q-analog of the Euler number E_n and show how it emerges in a q-analog of an identity expressing E_n as a weighted sum of Dyck paths.

Other parts of this thesis are devoted to polytopes relevant to the descent statistic. One such polytope is a "signed" version of the Pitman-Stanley parking function polytope, which can be viewed as a generalization of the chain polytope of the zigzag poset. We also discuss the family of descent polytopes, also known as order polytopes of ribbon posets, giving ways to compute their f-vectors and looking further into their combinatorial structure.

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Contents

Chapter 1

Introduction

1.1 Background

Descents and inversions are classical and very well studied statistics on permutations. The work in this thesis revolves around these and more refined statistics, the generating functions that encode their distribution on the set of permutations of a given size, and the connections they have with other combinatorial objects such as symmetric functions and poset polytopes.

1.1.1 Descents and inversions

A permutation has a *descent* in position i if the element in position i is greater than the element in position $i+1$. The number of descents of a permutation σ, denoted $d(\sigma)$ or $\mathrm{des}(\sigma)$ in the literature, is a fundamental statistic whose distribution over the set $\mathfrak{S}_n$ of permutations of size n gives rise to *Eulerian polynomials* $A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{d(\sigma)+1}$. For example, for $n = 3$ the six permutations 123, 132, 213, 231, 312, and 321, have 0, 1, 1, 1, 1, and 2 descents, respectively, and hence $A_3(t) = t + 4t^2 + t^3$. The Eulerian polynomials have many properties of classical combinatorial polynomials: they are symmetric, unimodal, and have only real and non-positive zeroes. They have been widely generalized and refined. For instance, the basic identity

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{m \geq 1} m^n t^m \tag{1.1}$$

is a special case of a theorem true for all partially ordered sets, in which $A_n(t)$ is the polynomial obtained by adding $t^{d(\sigma)+1}$ over all linear extensions of the poset, and instead of m^n one has the number of order-preserving maps from the poset to the set $[m] = \{1, 2, \ldots, m\}$. (A *partially ordered set*, or *poset*, is a set of elements in which some, but not necessarily all, pairs of distinct elements satisfy the relation $x < y$, and the natural transitivity relation holds: if $x < y$ and $y < z$, then $x < z$. A *linear extension* of a poset is a way to write a permutation of the elements so that if $x < y$, then x appears before y.) The above identity corresponds to the case when the poset is the n-element antichain, that is, no two elements are comparable.

The number of *inversions* in a permutation σ, denoted $\mathrm{inv}(\sigma)$, is the number of pairs of elements of σ such that the larger element appears before the smaller one. As with the descent statistic, one can encode the distribution of the number of inversions over $\mathfrak{S}_n$ by a polynomial, which is commonly written as $\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)}$. The most notable fact about this polynomial is that it is the *q-factorial* $[n]_q!$, defined as the product $[1]_q[2]_q\cdots[n]_q$, where $[i]_q = 1+q+q^2+\cdots+q^{i-1}$ is the q-analog of the positive integer i. This property makes the number of inversions a *Mahonian* statistic on permutations, a common term for permutation statistics that are equidistributed with the *major index*, denoted $\imath(\sigma)$ or sometimes $\mathrm{maj}(\sigma)$, and defined as the sum of positions where σ has descents. (For instance, we have $\imath(3142) = 1+3 = 4$, as 3142 has descents in the first and the third positions.)

There are many results, classical and recent, related to the generating functions for the distributions of the above statistics, including the joint distribution of two statistics, one of which is Eulerian, i.e. equidistributed with the number of descents, and the other is Mahonian. One notable bivariate identity is due to Stanley [18]:

$$1+\sum_{n\geq 1}\sum_{\sigma\in\mathfrak{S}_n} t^{d(\sigma)}q^{\mathrm{inv}(\sigma)}\cdot\frac{u^n}{[n]_q!} = \frac{1-t}{\mathrm{Exp}_q\big(u(t-1)\big)-t}, \tag{1.2}$$

where $\mathrm{Exp}_q(x) = \sum_{n\geq 0} q^{\binom{n}{2}}x^n/[n]_q!$. The $q=1$ specialization reduces $\mathrm{Exp}_q(x)$ to e^x, and the above identity to the exponential generating function for the Eulerian polynomials $A_n(t)$. A good reference featuring discussion of this and other ways of combining classical Eulerian and Mahonian permutation statistics is a recent paper [17] of Shareshian and Wachs.

1.1.2 The descent set of a permutation

A refined permutation statistic arises if one looks at not just the number of descents, but the set of positions at which descents occur. The *descent set* of a permutation σ is denoted $D(\sigma)$, and the number of permutations of size n with descent set S is denoted $\beta_n(S)$. There are several ways to express $\beta_n(S)$ — for example, as an alternating sum of multinomial coefficients or as a determinantal formula (see [19, p. 69]). The former involves the closely related numbers $\alpha_n(S)$, defined as the number of permutations of size n whose descent set is *contained* in S. From the inclusion-exclusion principle it follows that $\beta_n(S) = \sum_{T\subseteq S}(-1)^{|S-T|}\alpha_n(T)$.

One can ask for a generating function for $\alpha_n(S)$ or $\beta_n(S)$, and such a function would need to encode the set $D(\sigma)$ in some way. A subset $S\subseteq[n-1]$ can be turned into an algebraic object via the *monomial quasisymmetric function* $M_\gamma(x_1,x_2,\ldots)$, where $\gamma=(\gamma_1,\gamma_2,\ldots)$ is a *composition* of n, that is, a sequence of positive integers that add up to n. The function M_γ is a polynomial in infinitely many variables x_1, x_2, ..., defined as the sum of all monomials of the form $x_{i_1}^{\gamma_1}x_{i_2}^{\gamma_2}\cdots$, where $i_1<i_2<\cdots$. The standard correspondence between a set $S=\{s_1<s_2<\cdots<s_k\}\subseteq[n-1]$ and the composition $\mathrm{co}(S) = (s_1, s_2-s_1, s_3-s_2, \ldots, s_k-s_{k-1}, n-s_k)$ of n allows to write a generating function for the numbers $\alpha_n(S)$ in terms of monomial quasisymmetric

functions:

$$\sum_{S\subseteq[n-1]} \alpha_n(S) M_{\mathrm{co}(S)} = (x_1 + x_2 + \cdots)^n.$$

This identity is equivalent to saying that $\alpha_n(S)$ is the multinomial coefficient $\binom{n}{\mathrm{co}(S)} = \frac{n!}{s_1!(s_2-s_1)!(s_3-s_2)!\cdots}$.

Another natural approach is to associate a subset S of $[n-1]$ with a monimial in two non-commuting variables $\mathbf{a}$ and $\mathbf{b}$ as follows: given S, define $\mathbf{u}_S$ to be the monomial of degree $n-1$, or the $(\mathbf{a},\mathbf{b})$-word of length $n-1$, obtained by writing $\mathbf{a}$ in position i if $i \notin S$, or $\mathbf{b}$ if $i \in S$. Thus for $n = 4$ and $S = \{2,3\}$, one would obtain $\mathbf{u}_S = \mathbf{a}\,\mathbf{b}\,\mathbf{b} = \mathbf{a}\,\mathbf{b}^2$. Then the desired generating functions are $\Upsilon_n(\mathbf{a},\mathbf{b}) = \sum_{S\subseteq[n-1]} \alpha_n(S)\,\mathbf{u}_S$ and $\Psi_n(\mathbf{a},\mathbf{b}) = \sum_{S\subseteq[n-1]} \beta_n(S)\,\mathbf{u}_S$, and the aforementioned inclusion-exclusion principle yields $\Psi_n(\mathbf{a},\mathbf{b}) = \Upsilon_n(\mathbf{a}-\mathbf{b},\ \mathbf{b})$.

1.1.3 Enumerating chains of a poset and the *cd*-index of a polytope

The polynomial $\Psi_n(\mathbf{a},\mathbf{b})$ is known as the *ab-index* of the *Boolean algebra* B_n, which is the poset of subsets of an n-element set ordered by inclusion. The *ab*-index can be defined for any finite ranked poset.

A *chain* of length k, or a *k-chain*, in a poset is a sequence of pairwise comparable elements, commonly written in the increasing order: $x_1 < x_2 < \cdots < x_{k+1}$. A chain is called *saturated* if there is no element y in the poset such that $x_i < y < x_{i+1}$ for some i. A poset is called *ranked* if for every element x, all saturated chains whose largest (top) element is x have the same length, and this length is the *rank* of x, denoted $r(x)$. The rank of the whole poset is the largest rank among all its elements.

Given a ranked poset P, the chain enumerating quasisymmetric function

$$F_P(x_1, x_2, \ldots) := \sum_{x_0<x_1<\cdots<x_k} M_{(r(x_1)-r(x_0),\ r(x_2)-r(x_1),\ \ldots,\ r(x_k)-r(x_{k-1}))},$$

where x_0 and x_k are a minimal and a maximal element of P, has many interesting properties. For example, $F_{P\times Q} = F_P \cdot F_Q$, where $P \times Q$ is the Cartesian product of posets; see [3]. Equivalently, one can write down the chain enumerating polynomial

$$\Upsilon_P(\mathbf{a},\mathbf{b}) := \sum_{x_0<x_1<\cdots<x_k} \mathbf{u}_{\{r(x_1),\ r(x_2),\ \ldots,\ r(x_{k-1})\}}.$$

Then the *ab*-index of P is defined by $\Psi_P(\mathbf{a},\mathbf{b}) := \Upsilon_P(\mathbf{a}-\mathbf{b},\ \mathbf{b})$.

For a large class of posets, the *ab*-index can be reduced to a more compact polynomial called the *cd-index*. This class of posets includes *face lattices* of polytopes, whose elements are the faces of the polytope and the empty set, ordered by inclusion. The rank of a face is its dimension plus one, and the rank of the empty set is 0. Thus the empty set is the unique minimal element, often denoted $\hat{0}$, and the polytope itself is the unique maximal element whose rank is the dimension of the

polytope. The *ab*-index of the face lattice of a polytope can be expressed in terms of variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{a}\,\mathbf{b} + \mathbf{b}\,\mathbf{a}$, and the resulting polynomial $\Phi_P(\mathbf{c}, \mathbf{d})$ is called the *cd*-index of the polytope (or of the face lattice). Remarkably, the coefficients of the *cd*-index of a polytope are positive integers, yielding a variety of combinatorial results related to these coefficients in the cases of common polytopes. In [22] Stanley gives a detailed introduction to the theory of the *cd*-index and discusses the case of the Boolean algebra which is intimately related to the enumeration of permutations by their descent sets.

1.1.4 Alternating permutations

An important class of permutations defined in terms of the descent set statistic are *alternating* permutations. Depending on the situation, they are defined to be the *up-down* permutations, which are permutations $\sigma = \sigma_1\sigma_2\cdots$ satisfying $\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \cdots$, or equivalently $D(\sigma) = \{2, 4, 6, \ldots\}$, or the *down-up* permutations, which satisfy $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \cdots$ or $D(\sigma) = \{1, 3, 5, \ldots\}$. The map $\sigma_1\sigma_2\cdots\sigma_n \leftrightarrow \sigma'_1\sigma'_2\cdots\sigma'_n$, where $\sigma'_i = n+1-\sigma_i$, is a natural bijection between up-down and down-up permutations, hence with either definition the number of alternating permutations of size n is the same and equals the *n-th Euler number* E_n.

Perhaps the one most significant fact about Euler numbers is the celebrated exponential generating function

$$\sum_{n\geq 0} E_n \cdot \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{5x^4}{4!} + \frac{16x^5}{5!} + \frac{61x^6}{6!} + \cdots = \tan x + \sec x.$$

Since $\tan x$ is an odd function and $\sec x$ is even, the sequence of odd-indexed Euler numbers 1, 2, 16, 272, ... is known as the *tangent numbers*, and the sequence of even-indexed Euler numbers 1, 1, 5, 61, 1385, ... — as the *secant numbers*. One way to derive this generating function is to turn the recurrence relation

$$\sum_{i=0}^{n-1} \binom{n-1}{i} \cdot E_i \cdot E_{n-1-i} = 2E_n$$

into the differential equation

$$(E(x))^2 = 2E'(x) - 1,$$

where $E(x) = \sum_{n\geq 0} E_n x^n/n!$. Solving with the initial condition $E(0) = 1$ one obtains $E(x) = \tan x + \sec x$.

More results on enumeration of alternating permutations, including several combinatorial interpretations of E_n, appear in the paper [10] by Kuznetsov, Pak, and Postnikov.

A now classical generalization of Euler numbers are the *q-Euler numbers*, defined combinatorially by enumerating alternating permutations according to the number of

inversions:

$$E_n(q) := \sum_{\sigma \in \mathrm{Alt}'_n} q^{\mathrm{inv}(\sigma)},$$

where Alt'_n is the set of up-down permutations of size n. The polynomials $E_n(q)$ satisfy

$$\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} q^{2i} \begin{bmatrix} n-1 \\ 2i \end{bmatrix}_q \cdot E_{2i}(q) \cdot E_{n-1-2i}(q) = E_n(q),$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! \cdot [n-k]_q!}$ is the *q-binomial coefficient.* The q-Euler numbers give rise to the q-analogs of tangent and secant given by

$$\tan_q x + \sec_q x = \sum_{n \geq 0} E_n(q) \cdot \frac{x^n}{[n]_q!}.$$

In his paper [2] Désarménien studies these q-analogs and their connections to symmetric functions.

1.1.5 Polytopes related to the descent statistic

To put the combinatorics of permutations in geometric context one often introduces polytopes whose crucial characteristics, such as volume or number of integer (lattice) points, carry a combinatorial significance of being equal to the number of permutations with a certain property.

A great opening example is a celebrated result about slicing an n-dimensional hypercube with parallel hyperplanes so that the volumes of the resulting pieces are, up to a common scalar, the coefficients of the Eulerian polynomial $A_n(t)$. More specifically, if the unit hypercube C_n consisting of all points $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ such that $0 \leq x_i \leq 1$ is split into n regions by the hyperplanes $\sum x_i = k$ for $k \subseteq [n-1]$, then the relative volumes of these regions are $A(n,1)$, $A(n,2)$, $\ldots$, $A(n,n)$, where $A(n,k)$ is the number of permutations of size n with $k-1$ descents. In his famous short paper [21] Stanley gives a combinatorial proof of this fact.

A more straightforward way to partition the unit hypercube C_n into regions corresponding to permutations is to define the polytope R_σ, where $\sigma \in \mathfrak{S}_n$, to be the closure of the set of points $(x_1, x_2, \ldots, x_n) \in C_n$ such that the relative order of the coordinates x_1, x_2, $\ldots$, x_n is the same as that of the elements of σ. For instance, if $\sigma = 312 \in \mathfrak{S}_3$, then R_σ is the set of points in $\mathbb{R}^3$ satisfying $0 \leq x_2 \leq x_3 \leq x_1 \leq 1$. Permutation of coordinates provides a volume-preserving map transforming R_σ into the simplex $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1$, hence the volume of R_σ is $1/n!$ for every $\sigma \in \mathfrak{S}_n$.

In the study of finite posets, a frequently arising class of permutations is the set of linear extensions of a poset called the *Jordan-Hölder set.* To make linear extensions of an n-element poset permutations of $[n]$, one can choose a labeling of the elements of the poset with numbers in $[n]$. Then a linear extension, written down as a list of labels of elements, becomes a permutation in $\mathfrak{S}_n$. The union of R_σ taken over

all linear extensions σ is the *order polytope* of the poset, normally defined as the set of points in C_n satisfying $x_i \leq x_j$ whenever the element of the poset labeled i is smaller than the element labeled j. A special case directly relevant to the descent set statistic is the class of n-element ribbon posets Z_S, where S is a subset of $[n-1]$, whose elements $z_1, z_2, \ldots, z_n$ are partially ordered by the relations $z_i < z_{i+1}$ for $i \notin S$ and $z_i > z_{i+1}$ for $i \in S$. The volume of the order polytope of Z_S is $1/n!$ times the number of permutations in $\mathfrak{S}_n$ with descent set S. Another very closely related poset polytope is the *chain polytope*, also a polytope lying inside C_n and defined by the inequalities $x_{i_1} + x_{i_2} + \cdots + x_{i_k} \leq 1$ whenever the elements labeled $i_1, i_2, \ldots, i_k$ form a chain in the poset.

A notable connection between the order polytope $\mathcal{O}(P)$ and the chain polytope $\mathcal{C}(P)$ of the same poset P is that they have a common *Ehrhart polynomial* $i(\mathcal{O}(P), k) = i(\mathcal{C}(P), k)$. For a convex polytope $\mathcal{P}$ whose vertices are integer points (that is, every coordinate of every vertex is an integer) and a non-negative integer k, the value of $i(\mathcal{P}, k)$ is defined to be the number of integer points inside or on the boundary of $\mathcal{P}$ dilated by a factor of k, which can be thought of as the convex hull of the set of points obtained by multiplying the coordinates of every vertex of $\mathcal{P}$ by k. It turns out that $i(\mathcal{P}, k)$ is a polynomial in k of degree equal to the dimension of $\mathcal{P}$ and that it encodes many properties of $\mathcal{P}$: for instance, the volume of $\mathcal{P}$ is the leading coefficient of this polynomial. In [23] Stanley proved the following result relating the Ehrhart polynomial $i(\mathcal{O}(P), k) = i(\mathcal{C}(P), k)$ to the *order polynomial* $\Omega(P, k)$ of P, defined to be the number of order preserving maps $f : P \to [k]$ (that is, $f(x) \leq f(y)$ whenever $x < y$):

$$i(\mathcal{O}(P), k) = i(\mathcal{C}(P), k) = \Omega(P, k+1).$$

In particular, for $k = 1$ this result states that the number of vertices of $\mathcal{O}(P)$ and $\mathcal{C}(P)$ is equal to the number of subsets $I \subseteq P$ such that if $y \in I$ and $x < y$, then $x \in I$; such subsets are called the *order ideals* of P.

1.2 Thesis overview

In the second part of the introduction we present a summary of the two main chapters of this work. Chapter 4 features an open-ended discussion of topics for future research inspired by our results.

1.2.1 Variations on the descent and the inversion statistics

Specifying the descent set of a permutation can be thought of as giving information on how the elements are ordered locally, namely, which pairs of consecutive elements are ordered properly and which are not, the latter constituting the descents. The original idea that became the starting point of this research was to generalize descent sets to indicators of relative orders of k-tuples of consecutive elements, the next simplest case being $k = 3$. In this case there are 6 possible relative orders, and thus the analog of the descent set enumerator $\Psi_n(\mathbf{a}, \mathbf{b})$ introduced in Section 1.1.3 would involve 6 non-commuting variables. In order to defer overcomplication, to keep the number

of variables at 2, and to stay close to classical permutation statistics, we can divide triples of consecutive elements into merely "proper" or "improper", defined as having the relative order of an even or an odd permutation of size 3, respectively. We call the improper triples 3-*descents*, and denote the set of positions at which 3-descents occur in a permutation σ by $D_3(\sigma)$.

Computing the number of permutations with a given 3-descent set S yields a few immediate observations. For example, the number of permutations $\sigma \in \mathfrak{S}_n$ with $D_3(\sigma)$ equal to a fixed subset $S \subseteq [n-2]$ is divisible by n. This fact becomes clear upon the realization that $D_3(\sigma)$ is preserved when the elements of σ are cyclically shifted, so that 1 becomes 2, 2 becomes 3, and so on. As a result, it makes sense to focus on the set $\tilde{\mathfrak{S}}_n$ of permutations of $[n]$ with the first element equal to 1. A second, less trivial observation arising from early calculations is that the number of permutations in $\tilde{\mathfrak{S}}_n$ whose 3-descent set is empty is the Euler number E_{n-1}.

This second observation follows from the equidistribution of the statistic D_3 on the set $\tilde{\mathfrak{S}}_{n+1}$ with another variation on the descent set statistic, this time on $\mathfrak{S}_n$, which we call the *alternating descent set* (Theorem 2.1.3). It is defined as the set of positions i at which the permutation has an *alternating descent*, which is a regular descent if i is odd or an ascent if i is even. Thus the alternating descent set $\hat{D}(\sigma)$ of a permutation σ is the set of places where σ deviates from the alternating pattern.

Many of the results in this thesis that were originally motivated by the generalized descent statistic $d_3(\sigma) = |D_3(\sigma)|$ are actually given in terms of the alternating descent statistic $\hat{d}(\sigma) = |\hat{D}(\sigma)|$. We show that the alternating Eulerian polynomials, defined as $\hat{A}_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\hat{d}(\sigma)+1}$ by analogy with the classical Eulerian polynomials, have the generating function

$$\sum_{n \geq 1} \hat{A}_n(t) \cdot \frac{u^n}{n!} = \frac{t\big(1 - h\big(u(t-1)\big)\big)}{h\big(u(t-1)\big) - t}$$

where $h(x) = \tan x + \sec x$, so that the difference with the classical formula (1.2) (specialized at $q = 1$) is only in that the exponential function is replaced by tangent plus secant (Theorem 2.3.2).

A similar parallel becomes apparent in our consideration of the analog of the identity (1.1) for $\hat{A}_n(t)$. Given a formal power series $f(x) = 1 + \sum_{n \geq 1} a_n x^n/n!$, we define the symmetric function

$$g_{f,n} := \sum_{\gamma \models n} \binom{n}{\gamma} \cdot a_{\gamma_1} a_{\gamma_2} \cdots \cdot M_\gamma,$$

where γ runs over all compositions of n, and $\binom{n}{\gamma}$ and M_γ are as defined in Section 1.1.2. Then (1.1) can be written as

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{m \geq 1} g_{\exp,n}(1^m) \cdot t^m,$$

and we have

$$\frac{\hat{A}_n(t)}{(1-t)^{n+1}} = \sum_{m\geq 1} g_{\tan+\sec,n}(1^m)\cdot t^m,$$

where 1^m denotes setting the variables $x_1, x_2, \ldots, x_m$ to 1 and the remaining variables to 0 (Proposition 2.4.2).

In Section 2.6 we discuss the generating function $\hat{\Psi}(\mathbf{a},\mathbf{b})$ for the number of permutations in $\mathfrak{S}_n$ with a given alternating descent set $S \subseteq [n-1]$, denoted $\hat{\beta}_n(S)$, which is analogous to the polynomial $\Psi_n(\mathbf{a},\mathbf{b})$ introduced in Section 1.1.2. Recall that $\Psi_n(\mathbf{a},\mathbf{b})$ can be expressed as the *cd*-index $\Phi_n(\mathbf{c},\mathbf{d})$ of the Boolean algebra B_n, where $\mathbf{c} = \mathbf{a}+\mathbf{b}$ and $\mathbf{d} = \mathbf{a}\,\mathbf{b}+\mathbf{b}\,\mathbf{a}$. We show that $\hat{\Psi}_n$ can also be written in terms of $\mathbf{c}$ and $\mathbf{d}$ as $\hat{\Phi}_n(\mathbf{c},\mathbf{d}) = \Phi_n(\mathbf{c},\ \mathbf{c}^2-\mathbf{d})$ (Proposition 2.6.2), and that the sum of absolute values of the coefficients of this $(\mathbf{c},\mathbf{d})$-polynomial, which is the evaluation $\Phi_n(1,2)$, is the n-th term of a notable combinatorial sequence counting permutations in $\mathfrak{S}_n$ with no consecutive descents and no descent at the end (Theorem 2.6.6). This sequence has properties relevant to this work; in particular, the logarithm of the corresponding exponential generating function is an odd function, which is a crucial property of both e^x and $\tan x + \sec x$ that emerges repeatedly in the derivations of the results mentioned above. We discuss the similarities with Euler numbers and alternating permutations in Section 4.2.

It is natural to wonder if the variations of descents introduced thus far can be accompanied by corresponding variations of inversions. For alternating descents it seems reasonable to consider *alternating inversions* defined in a similar manner as pairs of indices $i<j$ such that either i is odd and the elements in positions i and j form a regular inversion, or else i is even and these two elements do *not* form a regular inversion. As for 3-descents, we define the accompanying 3-*inversion statistic*, where a 3-inversion is defined as the number of pairs of indices (i,j) such that $i+1<j$ and the elements in positions i, $i+1$, and j, taken in this order, constitute an odd permutation of size 3. Let $\hat{\imath}(\sigma)$ and $i_3(\sigma)$ be the number of alternating inversions and 3-inversions of a permutation σ, respectively. We find that the joint distribution of the pair $(\hat{d},\hat{\imath})$ of statistics on the set $\mathfrak{S}_n$ is identical to the distribution of the pair (d_3,i_3) of statistics on the set $\widetilde{\mathfrak{S}}_{n+1}$ (Theorem 2.2.7).

It would be nice to produce an analog of the generating function (1.2) for these descent-inversion pairs, but this task appears to be challenging, and it is not even clear what form such a generating function should have, as the q-factorials in the denominators of (1.2) are strongly connected to q-binomial coefficients, which have a combinatorial interpretation of the number of inversions in a permutation obtained by concatenating two increasing runs of fixed size. Nevertheless the bivariate polynomial $\hat{A}_n(t,q) := \sum_{\sigma\in\mathfrak{S}_n} t^{\hat{d}(\sigma)}q^{\hat{\imath}(\sigma)}$ seems to be of interest, and in Section 2.7 we direct our attention to the q-polynomials that result if we set $t=0$. This special case concerns up-down permutations and, more precisely, their distribution according to the number of alternating inversions. For down-up permutations this distribution is essentially the same, the only difference being the order of the coefficients in the q-polynomial, and for our purposes it turns out to be more convenient to work with down-up permutations, so we use the distribution of $\hat{\imath}$ on them to define a q-analog $\hat{E}_n(q)$ of Euler numbers.

The formal definition we give is

$$\hat{E}_n(q) := q^{-\lfloor n^2/4 \rfloor} \sum_{\sigma \in \mathrm{Alt}_n} q^{\hat{\imath}(\sigma)},$$

where Alt_n is the set of down-up permutations of $[n]$. The polynomial $\hat{E}_n(q)$ is monic with constant term equal to the Catalan number $c_{\lfloor n/2 \rfloor}$ (Proposition 2.7.2), which hints at the possibility to express $\hat{E}_n(q)$ as the sum of $c_{\lfloor n/2 \rfloor}$ "nice" polynomials with constant term 1. We discover such an expression in the form of a q-analog of a beautiful identity that represents E_n as the sum of weighted Dyck paths of length $2\lfloor n/2 \rfloor$. In this identity we imagine Dyck paths as starting at $(0,0)$ and ending at $(2\lfloor n/2 \rfloor, 0)$. We set the weight of an up-step to be the level at which that step is situated (the steps that touch the "ground" are at level 1, the steps above them at level 2, and so on) and the weight of a down-step to be either the level of the step (for even n) or one plus the level of the step (for odd n). We set the weight of the path to be the product of the weights of all its steps. The sum of the weights taken over all $c_{\lfloor n/2 \rfloor}$ paths then equals E_n, and if we replace the weight of a step with the q-analog of the respective integer, we obtain $\hat{E}_n(q)$ (Theorem 2.7.5).

The original $q = 1$ version of the above identity provides a curious connection between Catalan and Euler numbers. A notable difference between these numbers is in the generating functions: one traditionally considers the ordinary generating function for the former and the exponential one for the latter. An interesting and hopefully solvable problem is to find a generating function interpolating between the two, and a potential solution could be to use the above q-analog $\hat{E}_n(q)$ of Euler numbers to write

$$H(q,x) := \sum_{n \geq 0} \hat{E}_n(q) \cdot \frac{x^n}{[n]_q!},$$

so that $H(1,x) = \tan x + \sec x$ and

$$H(0,x) = \sum_{n \geq 0} c_{\lfloor n/2 \rfloor} x^n = \frac{(1+x)\left(1 - \sqrt{1-4x^2}\right)}{2x^2}.$$

1.2.2 Descent polytopes

For a positive integer n and a subset $S \subseteq [n-1]$, we define the *descent polytope* DP_S to be the order polytope of the ribbon poset Z_S as described in Section 1.1.5. Descent polytopes occur as a subdivision of the n-cube in the recent work [4] of Ehrenborg, Kitaev, and Perry.

Our efforts in this part of the thesis are aimed at calculating the *f-vector* of the n-dimensional polytope DP_S for arbitrary n and S. We represent this f-vector as the polynomial $F_S(t) = f_0 + f_1 t + f_2 t^2 + \cdots + f_{n-1} t^{n-1} + t^n$, where f_i is the number of i-dimensional faces of DP_S, and the last term t^n represents the polytope itself viewed as an n-dimensional face.

To obtain a closed-form result we once again invoke the technique of encoding the

subset $S \subseteq [n-1]$ by a word in two non-commuting variables, this time $\mathbf{x}$ and $\mathbf{y}$ to avoid confusion with $\mathbf{a}$ and $\mathbf{b}$ that arise in the discussion of the *ab*-index of DP_S. Let $\mathbf{v}_S$ be the $(\mathbf{x},\mathbf{y})$-word constructed in the same way as $\mathbf{u}_S$ in Section 1.1.2, only using letters $\mathbf{x}$ and $\mathbf{y}$ instead of $\mathbf{a}$ and $\mathbf{b}$. Then we have the following formula (Theorem 3.3.2):

$$\sum_{n\geq 1}\sum_{S\subseteq[n-1]} F_S(t)\,\mathbf{v}_S = \left(1+\frac{t+1}{1-(t+1)\Big((1-\mathbf{y})^{-1}\mathbf{x}+(1-\mathbf{x})^{-1}\mathbf{y}\Big)}\right)\cdot\frac{1}{1-\mathbf{x}-\mathbf{y}}.$$

The above expression is not very helpful if one wants to actually compute $F_S(t)$ for some particular S. We give two sets of recurrence relations that allow to carry out such a calculation. The first one (Lemma 3.3.1) is used to derive the main generating function, and the second one (Lemmas 3.4.1 and 3.4.2) is advantageous if the composition $\mathrm{c}(S) = (\gamma_1, \gamma_2, \ldots)$, defined by $\mathbf{v}_S = \mathbf{x}^{\gamma_1}\mathbf{y}^{\gamma_2}\mathbf{x}^{\gamma_3}\mathbf{y}^{\gamma_4}\cdots$, does not have too many parts. We exclude technical details here to avoid complicating this introductory outline.

Of course, it would be interesting to extend the theory of descent polytopes by finding ways of computing the *flag f-vector* and, more preferrably, the *cd*-index of DP_S. In Section 3.5 we give a description of the face lattice of DP_S, and in Section A.2 we list the *cd*-indices of small descent polytopes.

Chapter 2

Alternating descents and inversions and related statistics

2.1 Variations on the descent statistic

In this chapter, we investigate permutation statistics that are similar to the descent and the inversion statistics.

Let $\mathfrak{S}_n$ be the set of permutations of $[n] = \{1, \ldots, n\}$, and let $\tilde{\mathfrak{S}}_n$ be the set of permutations $\sigma_1\sigma_2\cdots\sigma_n$ of $[n]$ such that $\sigma_1 = 1$. For a permutation $\sigma = \sigma_1\cdots\sigma_n$, define the *descent set* $D(\sigma)$ of σ by $D(\sigma) = \{i \mid \sigma_i > \sigma_j\} \subseteq [n-1]$, and set $d(\sigma) = |D(\sigma)|$.

We say that a permutation σ has a 3-*descent* at position i if the permutation $\sigma_i\sigma_{i+1}\sigma_{i+2}$, viewed as an element of $\mathfrak{S}_3$, is odd. Let $D_3(\sigma)$ be the set of positions at which a permutation σ has a 3-descent, and set $d_3(\sigma) = |D_3(\sigma)|$. An important property of the 3-descent statistic is the following.

Lemma 2.1.1 *Let ω_n^c be the cyclic permutation $(2\ 3\ \ldots\ n\ 1)$, and let $\sigma \in \mathfrak{S}_n$. Then $D_3(\sigma) = D_3(\sigma\omega_n^c)$.*

Proof. Multiplying σ on the right by ω_n^c replaces each $\sigma_i < n$ by $\sigma_i + 1$, and the element of σ equal to n by 1. Thus the elements of the triples $\sigma_i\sigma_{i+1}\sigma_{i+2}$ that do not include n maintain their relative order under this operation, and in the triples that include n, the relative order of exactly two pairs of elements is altered. Thus the 3-descent set of σ is preserved. □

Corollary 2.1.2 *For all $i, j, k, \ell \in [n]$ and $B \subseteq [n-2]$, the number of permutations $\sigma \in \mathfrak{S}_n$ with $D_3(\sigma) = B$ and $\sigma_i = j$ is the same as the number of permutations with $D_3(\sigma) = B$ and $\sigma_k = \ell$.*

Proof. The set $\mathfrak{S}_n$ splits into orbits of the form $\{\sigma, \sigma\omega_n^c, \sigma(\omega_n^c)^2, \ldots, \sigma(\omega_n^c)^{n-1}\}$, and each such subset contains exactly one permutation with a j in the i-th position for all $i, j \in [n]$. □

Next, we define another variation on the descent statistic. We say that a permutation $\sigma = \sigma_1 \cdots \sigma_n$ has an *alternating descent* at position i if either $\sigma_i > \sigma_{i+1}$ and i is odd, or else if $\sigma_i < \sigma_{i+1}$ and i is even. Let $\hat{D}(\sigma)$ be the set of positions at which σ has an alternating descent, and set $\hat{d}(\sigma) = |\hat{D}(\sigma)|$.

Our first result relates the last two statistics by asserting that the 3-descent sets of permutations in $\tilde{\mathfrak{S}}_{n+1}$ are equidistributed with the alternating descent sets of permutations in $\mathfrak{S}_n$.

Theorem 2.1.3 *Let $B \subseteq [n-1]$. The number of permutations $\sigma \in \tilde{\mathfrak{S}}_{n+1}$ with $D_3(\sigma) = B$ is equal to the number of permutations $\omega \in \mathfrak{S}_n$ with $\hat{D}(\omega) = B$.*

Proof (by Pavlo Pylyavskyy). We construct a bijection between $\tilde{\mathfrak{S}}_{n+1}$ and $\mathfrak{S}_n$ mapping permutations with 3-descent set B to permutations with alternating descent set B.

Start with a permutation in $\sigma \in \tilde{\mathfrak{S}}_n$. We construct the corresponding permutation ω in $\mathfrak{S}_n$ by the following procedure. Consider $n+1$ points on a circle, and label them with numbers from 1 to $n+1$ in the clockwise direction. For convenience, we refer to these points by their labels. For $1 \leq i \leq n$, draw a line segment connecting σ_i and σ_{i+1}. The segment $\sigma_i\sigma_{i+1}$ divides the circle into two arcs. Define the sequence $C_1, \ldots, C_n$, where C_i is one of the two arcs between σ_i and σ_{i+1}, according to the following rule. Choose C_1 to be the arc between σ_1 and σ_2 corresponding to going from σ_1 to σ_2 in the clockwise direction. For $i > 1$, given the choice of C_{i-1}, let C_i be the arc between σ_i and σ_{i+1} that either *contains* or *is contained in* C_{i-1}. The choice of such an arc is always possible and unique. Let $\ell(i)$ denote how many of the i points $\sigma_1, \ldots, \sigma_i$, including σ_i, are contained in C_i.

Now, construct the sequence of permutations $\omega^{(i)} = \omega_1^{(i)} \ldots \omega_i^{(i)} \in \mathfrak{S}_i$, $1 \leq i \leq n$, as follows. Let $\omega^{(1)} = \ell(1)$. Given $\omega^{(i-1)}$, set $\omega_i^{(i)} = \ell(i)$, and let $\omega_1^{(i)} \ldots \omega_{i-1}^{(i)}$ be the permutation obtained from $\omega^{(i-1)}$ by adding 1 to all elements which are greater than or equal to ℓ_i. Finally, set $\omega = \omega^{(n)}$.

Next, we argue that the map $\sigma \mapsto \omega$ is a bijection. Indeed, from the subword $\omega_1 \ldots \omega_i$ of ω one can recover $\ell(i)$ since ω_i is the $\ell(i)$-th smallest element of the set $\{\omega_1, \ldots, \omega_i\}$. Then one can reconstruct one by one the arcs C_i and the segments connecting σ_i and σ_{i+1} as follows. If $\ell(i) > \ell(i-1)$ then C_i contains C_{i-1}, and if $\ell(i) \leq \ell(i-1)$ then C_i is contained in C_{i-1}. Using this observation and the number $\ell(i)$ of the points $\sigma_1, \ldots, \sigma_i$ contained in C_i, one can determine the position of the point σ_{i+1} relative to the points $\sigma_1, \ldots, \sigma_i$.

It remains to check that $D_3(\sigma) = \hat{D}(\omega)$. Observe that σ has a 3-descent in position i if and only if the triple of points $\sigma_i, \sigma_{i+1}, \sigma_{i+2}$ on the circle is oriented counterclockwise. Also, observe that $\omega_i > \omega_{i-1}$ if and only if $C_{i-1} \subset C_i$. Finally, note that $C_{i-1} \subset C_i \supset C_{i+1}$ or $C_{i-1} \supset C_i \subset C_{i+1}$ if and only if triples $\sigma_{i-1}, \sigma_i, \sigma_{i+1}$ and $\sigma_i, \sigma_{i+1}, \sigma_{i+2}$ have the same orientation. We now show by induction on i that $i \in D_3(\sigma)$ if and only if $i \in \hat{D}(\omega)$. From the choice of C_1 and C_2, it follows that $C_1 \subset C_2$ if and only if $\sigma_3 > \sigma_2$, and hence ω has an (alternating) descent at position 1 if and only if $\sigma_1\sigma_2\sigma_3 = 1\sigma_2\sigma_3$ is an odd permutation. Suppose the claim holds for $i-1$. By the above observations, we have $\omega_{i-1} < \omega_i > \omega_{i+1}$ or $\omega_{i-1} > \omega_i < \omega_{i+1}$ if and only if the permutations $\sigma_{i-1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_{i+1}\sigma_{i+2}$ have the same sign. In other

words, $i-1$ and i are either both contained or both not contained in $\hat{D}(\omega)$ if and only if they are either both contained or both not contained in $D_3(\sigma)$. It follows that $i \in D_3(\sigma)$ if and only if $i \in \hat{D}(\omega)$. $\square$

An important special case of Theorem 2.1.3 is $B = \varnothing$. A permutation $\sigma \in \mathfrak{S}_n$ has $\hat{D}(\sigma) = \varnothing$ if and only if it is an *alternating (up-down)* permutation, i.e. $\sigma_1 < \sigma_2 > \sigma_3 < \cdots$. The number of such permutations of size n is the *Euler number* E_n. Thus we get the following corollary:

Corollary 2.1.4 *(a) The number of permutations in $\tilde{\mathfrak{S}}_{n+1}$ with no 3-descents is E_n. (b) The number of permutations in $\mathfrak{S}_{n+1}$ with no 3-descents is $(n+1)E_n$.*

Proof. Part (b) follows from Corollary 2.1.2: for each $j \in [n+1]$, there are E_n permutations in $\mathfrak{S}_{n+1}$ beginning with j. $\square$

Permutations with no 3-descents can be equivalently described as simultaneously avoiding *generalized* patterns 132, 213, and 321 (meaning, in this case, triples of *consecutive* elements with one of these relative orders), and Corollary 2.1.4(b) appears in the Ph.D. thesis [8] of Kitaev. Thus the above construction yields a bijective proof of Kitaev's result.

2.2 Variations on the inversion statistic

In this section we introduce analogs of the inversion statistic on permutations corresponding to the 3-descent and the alternating descent statistics introduced in Section 2.1. First, let us recall the standard inversion statistic. For $\sigma \in \mathfrak{S}_n$, let a_i be the number of indices $j > i$ such that $\sigma_i > \sigma_j$, and set $\mathrm{code}(\sigma) = (a_1, \ldots, a_{n-1})$ and $\mathrm{inv}(\sigma) = a_1 + \cdots + a_{n-1}$.

For a permutation $\sigma \in \mathfrak{S}_n$ and $i \in [n-2]$, let $c_i^3(\sigma)$ be the number of indices $j > i+1$ such that $\sigma_i\sigma_{i+1}\sigma_j$ is an odd permutation, and set $\mathrm{code}_3(\sigma) = (c_1^3(\sigma), c_2^3(\sigma), \ldots, c_{n-2}^3(\sigma))$. Let C_k be the set of k-tuples $(a_1, \ldots, a_k)$ of non-negative integers such that $a_i \leq k+1-i$. Clearly, $\mathrm{code}_3(\sigma) \in C_{n-2}$.

Lemma 2.2.1 *Let ω_n^c be the cyclic permutation $(2\ 3\ \ldots\ n\ 1)$, and let $\sigma \in \mathfrak{S}_n$. Then $\mathrm{code}_3(\sigma) = \mathrm{code}_3(\sigma\omega_n^c)$.*

Proof. The proof is analogous to that of Lemma 2.1.1. $\square$

Proposition 2.2.2 *The restriction $\mathrm{code}_3 : \tilde{\mathfrak{S}}_n \to C_{n-2}$ is a bijection.*

Proof. Since $|\tilde{\mathfrak{S}}_n| = |C_{n-2}| = (n-1)!$, it suffices to show that the restriction of code_3 to $\tilde{\mathfrak{S}}_n$ is surjective. We proceed by induction on n. The claim is trivial for $n = 3$. Suppose it is true for $n-1$, and let $(a_1, \ldots, a_{n-2}) \in C_{n-2}$. Let τ be the unique permutation in $\tilde{\mathfrak{S}}_{n-1}$ such that $\mathrm{code}_3(\tau) = (a_2, \ldots, a_{n-2})$. For $1 \leq \ell \leq n$, let $\ell * \tau$

be the permutation in $\mathfrak{S}_n$ beginning with ℓ such that the relative order of last $n-1$ elements of $\ell * \tau$ is the same as that of the elements of τ. Setting $\ell = n - a_1$ we obtain $\mathrm{code}_3(\ell * \tau) = (a_1, \ldots, a_{n-2})$ since ℓ 1 m is an odd permutation if and only if $\ell < m$, and there are exactly a_1 elements of $\ell * \tau$ that are greater than ℓ. Finally, by Lemma 2.2.1, the permutation $\sigma = (\ell * \tau)(\omega_n^c)^{1-a_1} \in \tilde{\mathfrak{S}}_n$ satisfies $\mathrm{code}_3(\sigma) = (a_1, \ldots, a_{n-2})$. □

Let $i_3(\sigma) = c_1^3(\sigma) + c_2^3(\sigma) + \cdots + c_{n-2}^3(\sigma)$. An immediate consequence of Proposition 2.2.2 is that $i_3(1 * \sigma)$ is a *Mahonian statistic* on permutations $\sigma \in \mathfrak{S}_n$:

Corollary 2.2.3 *We have*

$$\sum_{\sigma \in \mathfrak{S}_n} q^{i_3(1*\sigma)} = (1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1}).$$

For a permutation $\sigma \in \mathfrak{S}_n$ and $i \in [n-1]$, define $\hat{c}_i(\sigma)$ to be the number of indices $j > i$ such that $\sigma_i > \sigma_j$ if i is odd, or the number of indices $j > i$ such that $\sigma_i < \sigma_j$ if i is even. Set $\hat{\mathrm{code}}(\sigma) = (\hat{c}_1(\sigma), \ldots, \hat{c}_{n-1}(\sigma)) \in C_{n-1}$ and $\hat{\imath}(\sigma) = \hat{c}_1(\sigma) + \cdots + \hat{c}_{n-1}(\sigma)$.

Proposition 2.2.4 *The map* $\hat{\mathrm{code}} : \mathfrak{S}_n \to C_{n-1}$ *is a bijection.*

Proof. The proposition follows easily from the fact that if $\mathrm{code}(\sigma) = (a_1, \ldots, a_{n-1})$ is the standard inversion code of σ, then $\hat{\mathrm{code}}(\sigma) = (a_1, n-2-a_2, a_3, n-4-a_4, \ldots)$. Since the standard inversion code is a bijection between $\mathfrak{S}_n$ and C_{n-1}, so is $\hat{\mathrm{code}}$. □

Corollary 2.2.5 *We have*

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\hat{\imath}(\sigma)} = (1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1}).$$

Another way to deduce Corollary 2.2.5 is via the bijection $\sigma \leftrightarrow \sigma^\vee$, where

$$\sigma^\vee = \sigma_1 \sigma_3 \sigma_5 \cdots \sigma_6 \sigma_4 \sigma_2.$$

Proposition 2.2.6 *We have* $\hat{\imath}(\sigma) = \mathrm{inv}(\sigma^\vee)$.

Proof. It is easy to verify that a pair (σ_i, σ_j), $i < j$, contributes to $\hat{\imath}(\sigma)$ if and only if it contributes to $\mathrm{inv}(\sigma^\vee)$. □

Next, we prove a fundamental relation between the variants of the descent and the inversion statistics introduced thus far.

Theorem 2.2.7 *We have*

$$\sum_{\sigma \in \tilde{\mathfrak{S}}_{n+1}} t^{d_3(\sigma)} q^{i_3(\sigma)} = \sum_{\omega \in \mathfrak{S}_n} t^{\hat{d}(\omega)} q^{\hat{\imath}(\omega)}.$$

Proof. The theorem is a direct consequence of the following proposition.

Proposition 2.2.8 *If* $\mathrm{code}_3(\sigma) = \hat{\mathrm{code}}(\omega)$ *for some* $\sigma \in \tilde{\mathfrak{S}}_{n+1}$ *and* $\omega \in \mathfrak{S}_n$, *then* $D_3(\sigma) = \hat{D}(\omega)$.

Proof. The alternating descent set of ω can be obtained from $\hat{\mathrm{code}}(\omega)$ as follows:

Lemma 2.2.9 *For* $\omega \in \mathfrak{S}_n$, *write* $(a_1, \ldots, a_{n-1}) = \hat{code}(\omega)$, *and set* $a_n = 0$. *Then* $\hat{D}(\omega) = \{i \in [n-1] \mid a_i + a_{i+1} \geq n - i\}$.

Proof. Suppose i is odd; then if $\omega_i > \omega_{i+1}$, i.e. $i \in \hat{D}(\omega)$, then for each $j > i$ we have $\omega_i > \omega_j$ or $\omega_{i+1} < \omega_j$ or both, so $a_i + a_{i+1}$ is not smaller than $n - i$, which is the number of elements of ω to the right of ω_i; if on the other hand $\omega_i < \omega_{i+1}$, i.e. $i \notin \hat{D}(\omega)$, then for each $j > i$, at most one of the inequalities $\omega_i > \omega_j$ and $\omega_{i+1} < \omega_j$ holds, and neither inequality holds for $j = i + 1$, so $a_i + a_{i+1} \leq n - i - 1$, which is the number of elements of ω to the right of ω_{i+1}. The case of even i is analogous. □

We now show that the 3-descent set of σ can be obtained from $(a_1, \ldots, a_{n-1})$ in the same way.

Lemma 2.2.10 *For* $\sigma \in \tilde{\mathfrak{S}}_{n+1}$, *write* $(a_1, \ldots, a_{n-1}) = \mathrm{code}_3(\sigma)$, *and set* $a_n = 0$. *Then* $D_3(\sigma) = \{i \in [n-1] \mid a_i + a_{i+1} \geq n - i\}$.

Proof. Let $B = D_3(\sigma)$, and let $\sigma' = \sigma(\omega_{n+1}^c)^{1-\sigma_i} \in \mathfrak{S}_{n+1}$. Then $\sigma'_i = 1$, and by Lemmas 2.1.1 and 2.2.1, we have $D_3(\sigma') = D_3(\sigma) = B$ and $\mathrm{code}_3(\sigma') = \mathrm{code}_3(\sigma)$.

Suppose that $1 = \sigma'_i < \sigma'_{i+1} < \sigma'_{i+2}$. Then $i \notin B$, and for each $j > i + 2$, at most one of the permutations $\sigma'_i\sigma'_{i+1}\sigma'_j = 1\sigma'_{i+1}\sigma'_j$ and $\sigma'_{i+1}\sigma'_{i+2}\sigma'_j$ is odd, because $1\sigma'_{i+1}\sigma'_j$ is odd if and only if $\sigma'_{i+1} > \sigma'_j$, and $\sigma'_{i+1}\sigma'_{i+2}\sigma'_j$ is odd if and only if $\sigma'_{i+1} < \sigma'_j < \sigma'_{i+2}$. Hence $a_i + a_{i+1}$ is at most $n - 1 - i$, which is the number of indices $j \in [n+1]$ such that $j > i + 2$.

Now suppose that $1 = \sigma'_i < \sigma'_{i+1} > \sigma'_{i+2}$. Then $i \in B$, and for each $j > i + 2$, at least one of the permutations $\sigma'_i\sigma'_{i+1}\sigma'_j = 1\sigma'_{i+1}\sigma'_j$ and $\sigma'_{i+1}\sigma'_{i+2}\sigma'_j$ is odd, because $\sigma'_{i+1} > \sigma'_j$ makes $1\sigma'_{i+1}\sigma'_j$ odd, and $\sigma'_{i+1} < \sigma_j$ makes $\sigma'_{i+1}\sigma'_{i+2}\sigma'_j$ odd. Thus each index $j > i + 1$ contributes to at least one of a_i and a_{i+1}, so $a_i + a_{i+1} \geq n - i$, which is the number of indices $j \in [n+1]$ such that $j > i + 1$. □

Proposition 2.2.8 follows from Lemmas 2.2.9 and 2.2.10. □

Combining the results of the above discussion, we conclude that both polynomials of Theorem 2.2.7 are equal to

$$\sum_{(a_1,\ldots,a_{n-1}) \in C_{n-1}} t^{|\hat{D}(a_1,\ldots,a_{n-1})|}\, q^{a_1+\cdots+a_{n-1}},$$

where $\hat{D}(a_1, \ldots, a_{n-1}) = \{i \in [n-1] \mid a_i + a_{i+1} \geq n - i\}$. □

Note that the bijective correspondence

$$\sigma \in \mathfrak{S}_n \xrightarrow{\widehat{\text{code}}} c \in C_{n-1} \xrightarrow{(\text{code}_3)^{-1}} \omega \in \tilde{\mathfrak{S}}_{n+1}$$

satisfying $\hat{D}(\sigma) = D_3(\omega)$ yields another bijective proof of Theorem 2.1.3.

Besides the inversion statistic, the most famous Mahonian statistic on permutations is the *major index*. For $\sigma \in \mathfrak{S}_n$, define the major index of σ by

$$\text{maj}(\sigma) = \sum_{i \in D(\sigma)} i.$$

Our next result reveals a close relation between the major index and the 3-inversion statistic i_3.

Proposition 2.2.11 *For $\sigma \in \mathfrak{S}_n$, write $\sigma^r = \sigma'_n \cdots \sigma'_2 \sigma'_1$, where $\sigma'_i = n + 1 - \sigma_i$. Then*

$$i_3(1 * \sigma) = \text{maj}(\sigma^r).$$

Proof. Let $\sigma = 1 * \omega \in \tilde{\mathfrak{S}}_{n+1}$. Let $D(\sigma) = \{b_1 < \cdots < b_d\}$. Write $\sigma = \tau^{(1)}\tau^{(2)} \cdots \tau^{(d+1)}$, where $\tau^{(k)} = \sigma_{b_{k-1}+1}\sigma_{b_{k-1}+2} \cdots \sigma_{b_k}$ and $b_0 = 0$ and $b_{d+1} = n$. In other words, we split σ into ascending runs between consecutive descents. Fix an element σ_j of σ, and suppose $\sigma_j \in \tau^{(k)}$. We claim that there are exactly $k-1$ indices $i < j-1$ such that $\sigma_i\sigma_{i+1}\sigma_j$ is an odd permutation. For each ascending run $\tau^{(\ell)}$, $\ell < k$, there is at most one element $\sigma_i \in \tau^{(\ell)}$ such that $\sigma_i < \sigma_j < \sigma_{i+1}$, in which case $\sigma_i\sigma_{i+1}\sigma_j$ is odd. There is no such element in $\tau^{(\ell)}$ if and only if the first element $\sigma_{b_{\ell-1}+1}$ of $\tau^{(\ell)}$ is greater than σ_j, or the last element σ_{b_ℓ} of $\tau^{(\ell)}$ is smaller than σ_j. In the former case we have $\sigma_{b_\ell - 1} > \sigma_{b_\ell} > \sigma_j$, so $\sigma_{b_\ell-1}\sigma_{b_\ell}\sigma_j$ is odd, and in the latter case, $\sigma_j > \sigma_{b_\ell} > \sigma_{b_\ell+1}$, so $\sigma_{b_\ell}\sigma_{b_\ell+1}\sigma_j$ is odd. Thus we obtain a one-to-one correspondence between the $k-1$ ascending runs $\tau^{(1)}, \ldots, \tau^{(k-1)}$ and elements σ_i such that $\sigma_i\sigma_{i+1}\sigma_j$ is an odd permutation.

We conclude that for each $\tau^{(k)}$, there are $(k-1)\cdot(b_k - b_{k-1})$ odd triples $\sigma_i\sigma_{i+1}\sigma_j$ with $\sigma_j \in \tau^{(k)}$, and hence

$$i_3(\sigma) = \sum_{k=1}^{d+1} (k-1)\cdot(b_k - b_{k-1}) =$$

$$= (b_{d+1} - b_d) + (b_{d+1} - b_d + b_d - b_{d-1}) + (b_{d+1} - b_d + b_d - b_{d-1} + b_{d-1} - b_{d-2}) + \cdots =$$

$$= \sum_{m=1}^{d} (n - b_m).$$

We have $D(\omega) = \{b_1 - 1, b_2 - 1, \ldots, b_d - 1\}$, from where it is not hard to see that $D(\omega^r) = \{n - b_d, n - b_{d-1}, \ldots, n - b_1\}$. The proposition follows. □

Observe that for a permutation π with $\pi^r = \pi'_m \cdots \pi'_1$, the triple $\pi_i\pi_{i+1}\pi_{i+2}$ is odd if and only if the triple $\pi_{i+2}\pi_{i+1}\pi_i$ is even, which in turn is the case if and only if the

triple $\pi'_{i+2}\pi'_{i+1}\pi'_i$ of consecutive elements of π^r is odd. Thus $d_3(\pi) = d_3(\pi^r)$, and we obtain the following corollary.

Corollary 2.2.12 *We have*

$$\sum_{\sigma\in\tilde{\mathfrak{S}}_{n+1}} t^{d_3(\sigma)}q^{i_3(\sigma)} = \sum_{\omega\in\mathfrak{S}_n} t^{d_3(\omega\circ(n+1))}q^{\mathrm{maj}(\omega)},$$

where $\omega \circ (n+1)$ *is the permutation obtained by appending* $(n+1)$ *to* ω.

Proof. To deduce the identity from Proposition 2.2.11, write $\sigma = 1*\pi$ and set $\omega = \pi^r$, so that $\omega \circ (n+1) = \sigma^r$. □

In the language of permutation patterns, the statistic $i_3(\sigma)$ can be defined as the total number of occurrences of generalized patterns 13-2, 21-3, and 32-1 in σ. (An occurrence of a generalized pattern 13-2 in a permutation $\sigma = \sigma_1\sigma_2\cdots$ is a pair of indices (i,j) such that $i+1 < j$ and σ_i, σ_{i+1}, and σ_j have the same relative order as 1, 3, and 2, that is, $\sigma_i < \sigma_j < \sigma_{i+1}$, and the other two patterns are defined analogously.) In [1] Babson and Steingrímsson mention the Mahonian statistic $\mathrm{STAT}(\sigma)$, which is defined as $i_3(\sigma)$ (treated in terms of the aforementioned patterns) plus $d(\sigma)$. In the permutation $\sigma \circ (n+1)$, where $\sigma \in \mathfrak{S}_n$, the descents of σ and the last element $n+1$ constitute all occurrences of the pattern 21-3 involving $n+1$, and hence $i_3\big(\sigma\circ(n+1)\big) = \mathrm{STAT}(\sigma)$.

2.3 Variations on Eulerian polynomials

Having introduced two new descent statistics, it is natural to look at the analog of the Eulerian polynomials representing their common distribution on $\mathfrak{S}_n$. First, recall the definition of the classical n-th Eulerian polynomial:

$$A_n(t) := \sum_{\sigma\in\mathfrak{S}_n} t^{d(\sigma)+1} = \sum_{k=1}^{n} A(n,k)\cdot t^k,$$

where $A(n,k)$ is the number of permutations in $\mathfrak{S}_n$ with $k-1$ descents. There is a well-known formula for the exponential generating function for Eulerian polynomials:

$$E(t,u) = \sum_{n\geq 1} A_n(t)\cdot\frac{u^n}{n!} = \frac{t(1-e^{u(t-1)})}{e^{u(t-1)}-t}. \tag{2.1}$$

In this section we consider analogs of Eulerian numbers and polynomials for our variations of the descent statistic. Define the *alternating Eulerian polynomials* $\hat{A}_n(t)$ by

$$\hat{A}_n(t) := \sum_{\sigma\in\mathfrak{S}_n} t^{\hat{d}(\sigma)+1} = \sum_{k=1}^{n} \hat{A}(n,k)\cdot t^k,$$

where $\hat{A}(n,k)$ is the number of permutations in $\mathfrak{S}_n$ with $k-1$ alternating descents. Our next goal is to find an expression for the exponential generating function

$$F(t,u) := \sum_{n\geq 1} \hat{A}_n(t) \cdot \frac{u^n}{n!}.$$

We begin by deducing a formula for the number of permutations in $\mathfrak{S}_n$ with a given alternating descent set. For $S \subseteq [n-1]$, let $\hat{\beta}_n(S)$ be the number of permutations $\sigma \in \mathfrak{S}_n$ with $\hat{D}(\sigma) = S$, and let $\hat{\alpha}_n(S) = \sum_{T\subseteq S} \hat{\beta}_n(T)$ be the number of permutations $\sigma \in \mathfrak{S}_n$ with $\hat{D}(\sigma) \subseteq S$. For $S = \{s_1 < \cdots < s_k\} \subseteq [n-1]$, let $\mathrm{co}(S)$ be the composition $(s_1, s_2 - s_1, s_3 - s_2, \ldots, s_k - s_{k-1}, n - s_k)$ of n, and for a composition $\gamma = (\gamma_1, \ldots, \gamma_\ell)$ of n, let S_γ be the subset $\{\gamma_1, \gamma_1+\gamma_2, \ldots, \gamma_1+\cdots+\gamma_{\ell-1}\}$ of $[n-1]$. Also, define

$$\binom{n}{\gamma} := \binom{n}{\gamma_1, \ldots, \gamma_\ell} = \frac{n!}{\gamma_1! \cdots \gamma_\ell!}$$

and

$$\binom{n}{\gamma}_E := \binom{n}{\gamma_1, \ldots, \gamma_\ell} \cdot E_{\gamma_1} \cdots E_{\gamma_\ell}.$$

Lemma 2.3.1 *We have*

$$\hat{\alpha}_n(S) = \binom{n}{\mathrm{co}(S)}_E$$

and

$$\hat{\beta}_n(S) = \sum_{T\subseteq S} (-1)^{|S-T|} \binom{n}{\mathrm{co}(T)}_E.$$

Proof. Let $S = \{s_1 < \cdots < s_k\} \subseteq [n-1]$. Set $s_0 = 0$ and $s_{k+1} = n$ for convenience. The alternating descent set of a permutation $\sigma \in \mathfrak{S}_n$ is contained in S if and only if for all $1 \leq i \leq k+1$, the subword $\tau_i = \sigma_{s_{i-1}+1}\sigma_{s_{i-1}+2}\cdots\sigma_{s_i}$ forms either an up-down (if s_{i-1} is even) or a down-up (if s_{i-1} is odd) permutation. Thus to construct a permutation σ with $\hat{D}(\sigma) \subseteq S$, one must choose one of the $\binom{n}{s_1-s_0, s_2-s_1, \ldots, s_{k+1}-s_k} = \binom{n}{\mathrm{co}(S)}$ ways to distribute the elements of $[n]$ among the subwords $\tau_1, \ldots, \tau_{k+1}$, and then for each $i \in [k+1]$, choose one of the $E_{s_i - s_{i-1}}$ ways of ordering the elements within the subword τ_i. The first equation of the lemma follows. The second equation is obtained from the first via the inclusion-exclusion principle. □

Now consider the sum

$$\sum_{S\subseteq[n-1]} \binom{n}{\mathrm{co}(S)}_E x^{|S|} = \sum_{S\subseteq[n-1]} \hat{\alpha}_n(S) \cdot x^{|S|} = \sum_{\sigma\in\mathfrak{S}_n} \left(\sum_{T\supseteq \hat{D}(\sigma)} x^{|T|} \right) \tag{2.2}$$

(a permutation σ contributes to $\hat{\alpha}_n(T)$ whenever $T \supseteq \hat{D}(\sigma)$). The right hand side of

(2.2) is equal to

$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}_n}\sum_{T\supseteq\hat{D}(\sigma)} x^{\hat{d}(\sigma)+|T-\hat{D}(\sigma)|} &= \sum_{\sigma\in\mathfrak{S}_n} x^{\hat{d}(\sigma)} \sum_{i=0}^{n-1-\hat{d}(\sigma)} \binom{n-1-\hat{d}(\sigma)}{i} x^i \\ &= \sum_{\sigma\in\mathfrak{S}_n} x^{\hat{d}(\sigma)}(1+x)^{n-1-\hat{d}(\sigma)}, \end{aligned} \tag{2.3}$$

as there are $\binom{n-1-\hat{d}(\sigma)}{i}$ subsets of $[n-1]$ containing $\hat{D}(\sigma)$. Continuing with the right hand side of (2.3), we get

$$\frac{(1+x)^n}{x}\cdot\sum_{\sigma\in\mathfrak{S}_n}\left(\frac{x}{1+x}\right)^{\hat{d}(\sigma)+1} = \frac{(1+x)^n}{x}\cdot\hat{A}_n\left(\frac{x}{1+x}\right). \tag{2.4}$$

Combining equations (2.2)–(2.4), we obtain

$$\sum_{n\geq 1}\left(\sum_{S\subseteq[n-1]}\binom{n}{\mathrm{co}(S)}_E x^{|S|}\right)\cdot\frac{y^n}{n!} = \frac{1}{x}\cdot\sum_{n\geq 1}\hat{A}_n\left(\frac{x}{1+x}\right)\cdot\frac{y^n(1+x)^n}{n!}. \tag{2.5}$$

Since $S\mapsto \mathrm{co}(S)$ is a bijection between $[n-1]$ and the set of compositions of n, the left hand side of (2.5) is

$$\sum_{n\geq 1}\left(\sum_{\gamma}\frac{E_{\gamma_1}\cdots E_{\gamma_\ell}}{\gamma_1!\cdots\gamma_\ell!}\cdot x^{\ell-1}\right)\cdot y^n = \frac{1}{x}\cdot\sum_{\ell\geq 1}x^\ell\cdot\left(\sum_{i\geq 1}\frac{E_i y^i}{i!}\right)^\ell, \tag{2.6}$$

where the inside summation in the left hand side is over all compositions $\gamma = (\gamma_1,\ldots,\gamma_\ell)$ of n. Applying the well-known formula $\sum_{j\geq 0}E_j y^j/j! = \tan y + \sec y$, the right hand side of (2.6) becomes

$$\frac{1}{x}\cdot\sum_{\ell\geq 1}x^\ell(\tan y+\sec y-1)^\ell = \frac{1}{x}\cdot\left(\frac{1}{1-x(\tan y+\sec y-1)}-1\right). \tag{2.7}$$

Now set $t=\frac{x}{1+x}$ and $u=y(1+x)$. Equating the right hand sides of (2.5) and (2.7), we obtain

$$F(t,u)=\sum_{n\geq 1}\hat{A}_n(t)\cdot\frac{u^n}{n!} = \left(\frac{1}{1-x(\tan y+\sec y-1)}-1\right). \tag{2.8}$$

Finally, applying the inverse substitution $x=\frac{t}{1-t}$ and $y=u(1-t)$ and simplifying

yields an expression for $F(t,u)$:

$$\begin{aligned} F(t,u) &= \frac{x(\tan y+\sec y-1)}{1-x(\tan y+\sec y-1)} \\ &= \frac{t}{1-t}\cdot\left(\frac{\tan y+\sec y-1}{1-\frac{t}{1-t}\cdot(\tan y+\sec y-1)}\right) \\ &= \frac{t\cdot\big(\tan(u(1-t))+\sec(u(1-t))-1\big)}{1-t\cdot\big(\tan(u(1-t))+\sec(u(1-t))\big)}. \end{aligned} \tag{2.9}$$

Using the property $(\tan z+\sec z)(\tan(-z)+\sec(-z))=1$, we can rewrite the above expression for $F(t,u)$ as follows:

Theorem 2.3.2 *We have*

$$F(t,u)=\frac{t\cdot\big(1-\tan(u(t-1))-\sec(u(t-1))\big)}{\tan(u(t-1))+\sec(u(t-1))-t}.$$

Thus $F(t,u)$ can be expressed by replacing the exponential function in the formula (2.1) for $E(t,u)$ by tangent plus secant. In fact, omitting the Euler numbers and working with standard multinomial coefficients gives a proof of (2.1).

A basic result on Eulerian polynomials is the identity

$$\frac{A_n(t)}{(1-t)^{n+1}}=\sum_{m\geq 1}m^n t^m. \tag{2.10}$$

Our next result is a similar identity involving alternating Eulerian polynomials. For a partition λ of n with r_i parts equal to i, define

$$z_\lambda := 1^{r_1}\cdot r_1!\cdot 2^{r_2}\cdot r_2!\cdot\cdots.$$

Theorem 2.3.3 *Let*

$$\hat{f}_n(m)=\sum_\lambda\frac{n!}{z_\lambda}\cdot\frac{E_{\lambda_1-1}E_{\lambda_2-1}\cdots}{(\lambda_1-1)!(\lambda_2-1)!\cdots}\cdot m^{\ell(\lambda)},$$

where the sum is over all partitions $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_{\ell(\lambda)})$ of n into odd parts. Then

$$\frac{\hat{A}_n(t)}{(1-t)^{n+1}}=\sum_{m\geq 1}\hat{f}_n(m)t^m.$$

Proof. Let us consider the generating function

$$G(t,u):=\sum_{n\geq 1}\frac{\hat{A}(t)}{(1-t)^{n+1}}\cdot\frac{u^n}{n!}.$$

Then, by (2.9), we have

$$G(t,u) = \frac{1}{1-t} \cdot F\left(t,\ \frac{u}{1-t}\right) = \frac{t \cdot (\tan u + \sec u - 1)}{(1-t)\big(1 - t \cdot (\tan u + \sec u)\big)}. \tag{2.11}$$

Define

$$H(m,u) := \sum_{n \geq 1} \hat{f}_n(m) \cdot \frac{u^n}{n!}.$$

This series can be rewritten as follows:

$$H(m,u) = \sum_{n \geq 1} \frac{\hat{f}_n(m)}{n!} \cdot u^n = -1 + \prod_{i \geq 0} \left(\sum_{j \geq 0} \frac{\left(\frac{E_{2i} m u^{2i+1}}{(2i+1)!}\right)^j}{j!} \right). \tag{2.12}$$

Indeed, for each i, the index j in the summation is the number of parts equal to $2i+1$ in a partition of n into odd parts, and it is not hard to check that the contribution of j parts equal to $2i+1$ to the appropriate terms of $\hat{f}_n(m)/n!$ is given by the expression inside the summation on the right. We subtract 1 to cancel out the empty partition of 0 counted by the product on the right but not by $H(m,u)$. Continuing with the right hand side of (2.12), we get

$$\begin{aligned} H(m,u) + 1 &= \prod_{i \geq 0} \exp\left(\frac{E_{2i} m u^{2i+1}}{(2i+1)!}\right) \\ &= \exp\left(m \sum_{i \geq 0} \left(\frac{E_{2i} u^{2i+1}}{(2i+1)!}\right)\right). \end{aligned} \tag{2.13}$$

The sum appearing in the right hand side of (2.13) is the antiderivative of $\sec u = \sum_{i \geq 0} E_{2i} u^{2i}/(2i)!$ that vanishes at $u = 0$; this antiderivative is $\ln(\tan u + \sec u)$. Therefore

$$H(m,u) + 1 = (\tan u + \sec u)^m.$$

Hence we have

$$\sum_{m \geq 1} H(m,u) \cdot t^m = \frac{(\tan u + \sec u) \cdot t}{1 - (\tan u + \sec u) \cdot t} - \frac{1}{1-t}. \tag{2.14}$$

It is straightforward to verify that the right hand sides of (2.11) and (2.14) agree, and thus

$$\sum_{n \geq 1} \frac{\hat{A}_n(t)}{(1-t)^{n+1}} \cdot \frac{u^n}{n!} = G(t,u) = \sum_{m \geq 1} H(m,u) t^m = \sum_{m,n \geq 1} \hat{f}_n(m) t^m \cdot \frac{u^n}{n!}. \tag{2.15}$$

Equating the coefficients of $u^n/n!$ on both sides of (2.15) completes the proof of the theorem. □

In the terminology of [19, Sec. 4.5], Theorem 2.3.3 states that the polynomials $\hat{A}_n(t)$ are the $\hat{f}_n$-*Eulerian polynomials*.

2.4 Eulerian polynomials and symmetric functions

The results of the previous section can be tied to the theory of symmetric functions. Let us recall some basics. For a composition $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$, the *monomial quasisymmetric function* $M_\gamma(x_1, x_2, \ldots)$ is defined by

$$M_\gamma := \sum_{1 \le i_1 < \cdots < i_k} x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_k}^{\gamma_k}.$$

Let $\pi(\gamma)$ denote the partition obtained by rearranging the parts of γ in non-increasing order. Then for a partition λ, the *monomial symmetric function* $m_\lambda(x_1, x_2, \ldots)$ is defined as

$$m_\lambda := \sum_{\gamma \,:\, \pi(\gamma) = \lambda} M_\gamma.$$

Let $f(x)$ be a function given by the formal power series

$$f(x) = 1 + \sum_{n \ge 1} \frac{a_n x^n}{n!}.$$

Define the symmetric function $g_{f,n}(x_1, x_2, \ldots)$ by

$$g_{f,n} := \sum_{\gamma \models n} \binom{n}{\gamma} \cdot a_{\gamma_1} a_{\gamma_2} \cdots \cdot M_\gamma = \sum_{\lambda \vdash n} \binom{n}{\lambda} \cdot a_{\lambda_1} a_{\lambda_2} \cdots \cdot m_\lambda,$$

where by $\gamma \models n$ and $\lambda \vdash n$ we mean that γ and λ are a composition and a partition of n, respectively. This function can be thought of as the generating function for numbers like $\alpha_n(S)$ or $\hat{\alpha}_n(S)$ (the number of permutations $\sigma \in \mathfrak{S}_n$ with $D(\sigma) \subseteq S$ or $\hat{D}(\sigma) \subseteq S$, respectively). Our first step is to express $g_{f,n}$ in terms of the *power sum symmetric functions* $p_k(x_1, x_2, \ldots) = \sum x_i^k$.

Consider the generating function

$$G_f(x_1, x_2, \ldots; u) := \sum_{n \ge 0} g_{f,n} \cdot \frac{u^n}{n!}. \tag{2.16}$$

Then we have

$$G_f = \sum_{n \ge 0} \sum_{\gamma \models n} \frac{a_{\gamma_1} a_{\gamma_2} \cdots}{\gamma_1! \gamma_2! \cdots} \cdot M_\gamma u^n = \prod_{i \ge 1} f(x_i u). \tag{2.17}$$

Now let us write

$$\ln(f(x)) = \sum_{n \ge 1} \frac{b_n x^n}{n!}. \tag{2.18}$$

Then from (2.17) we have

$$\ln G_f = \sum_{i\geq 1} \ln(f(x_i u)) = \sum_{n\geq 1} b_n p_n(x_1, x_2, \ldots) \cdot \frac{u^n}{n!}. \tag{2.19}$$

Since the power sum symmetric functions $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$, with λ ranging over all partitions of positive integers, form a basis for the ring of symmetric functions, the transformation $p_n \mapsto b_n p_n u^n/(n-1)!$, where u is regarded as a scalar, extends to a homomorphism of this ring. Applying this homomorphism to the well-known identity

$$\exp \sum_{n\geq 1} \frac{1}{n} \cdot p_n = \sum_{\lambda} z_\lambda^{-1} p_\lambda,$$

where λ ranges over all partitions of positive integers, we obtain from (2.19) that

$$\begin{aligned} G_f &= \exp \sum_{n\geq 1} \frac{1}{n} \cdot \left(\frac{b_n p_n u^n}{(n-1)!} \right) \\ &= \sum_{\lambda} z_\lambda^{-1} \cdot \frac{b_{\lambda_1} b_{\lambda_2} \cdots}{(\lambda_1 - 1)!(\lambda_2 - 1)! \cdots} \cdot p_\lambda u^{|\lambda|}. \end{aligned} \tag{2.20}$$

Comparing the coefficients of u^n in (2.16) and (2.20), we conclude the following:

Proposition 2.4.1 *For a function $f(x)$ with $f(0) = 1$ and $\ln(f(x)) = \sum_{n\geq 1} b_n x^n/n!$ we have*

$$g_{f,n} = \sum_{\lambda \vdash n} \frac{n!}{z_\lambda} \cdot \frac{b_{\lambda_1} b_{\lambda_2} \cdots}{(\lambda_1 - 1)!(\lambda_2 - 1)! \cdots} \cdot p_\lambda.$$

Two special cases related to earlier discussion are $f(x) = e^x$ and $f(x) = \tan x + \sec x$. For $f(x) = e^x$, we have $b_1 = 1$, $b_2 = b_3 = \cdots = 0$, and hence $g_{f,n} = p_1^n$. In the case of $f(x) = \tan x + \sec x$, we have

$$b_i = \begin{cases} E_{i-1} & \text{if } i \text{ is odd,} \\ 0 & \text{if } i \text{ is even,} \end{cases}$$

thus the coefficient at p_λ in the expression of Proposition 2.4.1 coincides with the coefficient in the term for λ in the definition of the polynomial $\hat{f}_n(m)$ of Theorem 2.3.3. These observations lead to the following restatements of the classical identity (2.10) and Theorem 2.3.3.

Proposition 2.4.2 *Let $g(1^m)$ denote the evaluation of $g(x_1, x_2, \ldots)$ at $x_1 = x_2 = \cdots = x_m = 1$, $x_{m+1} = x_{m+2} = \cdots = 0$. Then*

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{m\geq 1} g_{\exp,n}(1^m) \cdot t^m$$

and

$$\frac{\hat{A}_n(t)}{(1-t)^{n+1}} = \sum_{m\geq 1} g_{\tan+\sec,n}(1^m)\cdot t^m.$$

Proof. We have $p_i(1^m) = m$, and hence $p_\lambda(1^m) = m^{\ell(\lambda)}$. □

It is an interesting problem to prove Proposition 2.4.2 without referring to the results of Section 2.3. Observe that for $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k) \models n$, we have $M_\gamma(1^m) = \binom{m}{k}$, the number of monomials $x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_k}^{\gamma_k}$ where $1 \leq i_1 < \cdots < i_k \leq m$, which are the monomials in the definition of M_γ that evaluate to 1.

It would also be of interest to relate the observations of this section to Schur functions. One possibility is to consider the following generalization of the *complete homogeneous symmetric function*. Let φ_f be the homomorphism of the ring of symmetric functions defined by $p_n \mapsto b_n p_n/(n-1)!$, where the b_i's are as in equation (2.18). Let

$$h_{f,n} := \sum_{\lambda\vdash n} z_\lambda^{-1} \varphi_f(p_\lambda).$$

For $f(x) = (1-x)^{-1}$, the homomorphism φ is identity, and $h_{f,n}$ is the standard complete homogeneous symmetric function h_n, defined to be the sum of all monomials in $x_1, x_2, \ldots$, of degree n. Then (2.20) becomes

$$G_f = \sum_{n\geq 1} h_{f,n} u^n$$

(we do not really need u here because of homegeneity). We can define the generalized Schur function $s_{f,\lambda}$, where $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$, by the *Jacoby-Trudy identity*

$$s_{f,\lambda} := \det\left[\, h_{f,\ \lambda_i - i + j} \,\right]_{1\leq i,j\leq n},$$

where $h_{f,0} = 1$ and $h_{f,k} = 0$ for $k < 0$ (see [20, Sec. 7.16]). What happens for $f(x) = e^x$ and $f(x) = \tan x + \sec x$?

2.5 The alternating Eulerian numbers

In this section we give a recurrence relation that allows to construct a triangle of alternating Eulerian numbers $\hat{A}(n,k)$ introduced in Section 2.3. (Recall that $\hat{A}(n,k)$ denotes the number of permutations in $\mathfrak{S}_n$ with $k-1$ alternating descents.) The first few rows of this triangle are given in Table 2.1.

The following lemma provides a way to compute alternating Eulerian numbers given the initial condition $\hat{A}(n,1) = E_n$.

						1								
					1		1							
				2		2		2						
			5		7		7		5					
		16		26		36		26		16				
	61		117		182		182		117		61			
272		594		1056		1196		1056		594		272		

Table 2.1: Triangle of alternating Eulerian numbers

Lemma 2.5.1 *For $n \geq k \geq 0$ we have*

$$\sum_{i=0}^{n}\sum_{j=0}^{k}\binom{n}{i}\cdot\hat{A}(i,j+1)\cdot\hat{A}(n-i,k-j+1)$$
$$=(n+1-k)\hat{A}(n,k+1)+(k+1)\hat{A}(n,k+2). \tag{2.21}$$

Proof. First, suppose that k is even. The left hand side of the equation counts the number of ways to split the elements of $[n]$ into two groups of sizes i and $n-i$, arrange the elements in the first and the second group so that the resulting permutations have j and $k-j$ alternating descents, respectively, and writing down the second permutation after the first to form a permtutation of $[n]$. This permutation has either k or $k+1$ alternating descents, depending on whether an alternating descent is produced at position i. For a permutation $\sigma \in \mathfrak{S}_n$ with $\hat{\imath}(\sigma) = k$, there are exactly $n+1-k$ ways to produce σ by means of the above procedure, one for every choice of $i \in \hat{D}(\sigma) \cup \{0, n\}$. Similarly, for $\sigma \in \mathfrak{S}_n$ such that $\hat{\imath}(\sigma) = k+1$, there are exactly $k+1$ ways to produce σ, one for every choice of $i \in \hat{D}(\sigma)$. The identity follows.

As for odd k, the same argument is valid, except that the quantity $\hat{A}(n-i,k-j+1)$ in the left hand side should be interpreted as the number of ways to arrange the elements of the second group to form a permutation with $k-j$ alternating *ascents*, which become alternating descents when the two permutations are concatenated. □

Recall the generating function

$$F(t,u)=\sum_{n,k\geq 1}\hat{A}(n,k)\cdot\frac{t^k u^n}{n!}$$

introduced in Section 2.3. An alternative way to express $F(t,u)$ and obtain the result of Theorem 2.3.2 is by solving a partial differential equation arising from the recurrence of Lemma 2.5.1.

Proposition 2.5.2 *The function $F(t,u)$ is the solution of the partial differential equation*

$$F^2 - F = u \cdot \frac{\partial F}{\partial u} + (1-t) \cdot \frac{\partial F}{\partial t} \tag{2.22}$$

with the initial condition $F(0,u) = \tan u + \sec u$.

Proof. Since $\hat{A}(n,0) = 0$ for all n, the left hand side of (2.21) is $n!$ times the coefficient of $t^k u^n$ in $\big(F(t,u)\big)^2$, which we denote by $[t^k u^n]F^2$. The right hand side of (2.21) is

$$\begin{aligned}
& n! \cdot \left(\frac{\hat{A}(n,k+1)}{(n-1)!} + \frac{\hat{A}(n,k+1)}{n!} - \frac{k\hat{A}(n,k+1)}{n!} + \frac{(k+1)\hat{A}(n,k+2)}{n!} \right) \\
&= n! \cdot \Big([t^k u^n]F_u + [t^k u^n]F - [t^{k-1}u^n]F_t + [t^k u^n]F_t \Big) \\
&= n! \cdot [t^k u^n] \left(uF_u + F - tF_t + F_t \right),
\end{aligned}$$

where F_t and F_u denote partial derivatives of F with respect to t and u. Equating the above with $n! \cdot [t^k u^n]F^2$ proves (2.22). □

Upon reparameterization $u = we^z$ and $t = 1 + e^{-z}$, we obtain from (2.22) that

$$\begin{aligned}
F_z &= we^z F_u - e^{-z} F_t \\
&= uF_u + (1-t)F_t \\
&= F^2 - F,
\end{aligned} \tag{2.23}$$

where F_u, F_t, and F_z are partial derivatives. Hold w fixed and define $f(z) := F(we^z, 1 + e^{-z})$. Then (2.23) turns into an ordinary differential equation:

$$\frac{df}{dz} = f^2 - f,$$

which yields

$$\frac{f-1}{f} = ce^z,$$

where c depends only on $w = u(1-t)$. Hence

$$\frac{F-1}{F} = ug(w) \tag{2.24}$$

for some function $g(w)$. To find this function, set $t = 0$ in (2.24), so that w becomes u:

$$g(u) = \frac{\tan u + \sec u - 1}{u(\tan u + \sec u)} = \frac{1 + \tan - \sec u}{u}.$$

Finally, the formula of Theorem 2.3.2 can be obtained from the expression

$$F(t,u) = \frac{1}{1 - ug\big(u(1-t)\big)}.$$

2.6 The generating function for the alternating descent set statistic

Besides the generating polynomials for the alternating descent statistic, another natural generating function to consider is one counting permutations by their alternating descent set. We begin by stating some well-known facts about the analogous generating function for the classical descent set statistic.

Fix a positive integer n. For a subset $S \subseteq [n-1]$, define the monomial u_S in two non-commuting variables $\mathbf{a}$ and $\mathbf{b}$ by $u_S = u_1 u_2 \cdots u_{n-1}$, where

$$u_i = \begin{cases} \mathbf{a} & \text{if } i \notin S, \\ \mathbf{b} & \text{if } i \in S. \end{cases}$$

Consider the generating function

$$\Psi_n(\mathbf{a}, \mathbf{b}) := \sum_{S \subseteq [n-1]} \beta_n(S) u_S,$$

where $\beta_n(S)$ is the number of permutations in $\mathfrak{S}_n$ with descent set S. The polynomial $\Psi_n(\mathbf{a}, \mathbf{b})$ is known as the *ab-index of the Boolean algebra* B_n. A remarkable property of $\Psi_n(\mathbf{a}, \mathbf{b})$ (and also of *ab*-indices of a wide class of posets, including face lattices of polytopes) is that it can be expressed in terms of the variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{a}\,\mathbf{b} + \mathbf{b}\,\mathbf{a}$. The polynomial $\Phi_n(\mathbf{c}, \mathbf{d})$ defined by $\Psi_n(\mathbf{a}, \mathbf{b}) = \Phi_n(\mathbf{a}+\mathbf{b},\ \mathbf{a}\,\mathbf{b}+\mathbf{b}\,\mathbf{a})$ is called the *cd-index of* B_n.

The polynomial $\Phi_n(\mathbf{c}, \mathbf{d})$ has positive integer coefficients, for which several combinatorial interpretations have been found. Here we give one that will help establish a connection with the alternating descent set statistic. We proceed with a definition.

Definition 2.6.1 *A permutation is* simsun *if, for all* $k \geq 0$*, removing* k *largest elements from it results in a permutation with no consecutive descents.*

Let SS_n be the set of simsun permutations in $\mathfrak{S}_n$ whose last element is n. (Thus SS_n is essentially the set of simsun permutations of $[n-1]$ with an n attached at the end.) It is known that $|\,\mathrm{SS}_n\,| = E_n$.

For a permutation $\sigma \in \mathrm{SS}_n$, define the $(\mathbf{c}, \mathbf{d})$-monomial $\mathrm{cd}(\sigma)$ as follows: write out the descent set of σ as a string of pluses and minuses denoting ascents and descents, respectively, and then replace each occurrence of " $-\,+$ " by $\mathbf{d}$, and each remaining plus by $\mathbf{c}$. This definition is valid because a simsun permutation has no consecutive descents. For example, consider the permutation $423516 \in \mathrm{SS}_6$. Its descent set in the above notation is " $-\,+\,+\,-\,+$ ", and thus $\mathrm{cd}(423516) = \mathbf{d}\,\mathbf{c}\,\mathbf{d}$.

The simsun permutations provide a combinatorial expression for the *cd*-index of B_n:

$$\Phi_n(\mathbf{c}, \mathbf{d}) = \sum_{\sigma \in \mathrm{SS}_n} \mathrm{cd}(\sigma). \tag{2.25}$$

Now let us define the analog of $\Psi_n(\mathbf{a},\mathbf{b})$ for the alternating descent set statistic:

$$\hat{\Psi}_n(\mathbf{a},\mathbf{b}) := \sum_{S\subseteq[n-1]} \hat{\beta}_n(S)u_S.$$

Proposition 2.6.2 *There exists a polynomial* $\hat{\Phi}_n(\mathbf{c},\mathbf{d})$ *such that*

$$\hat{\Phi}_n(\mathbf{a}+\mathbf{b},\ \mathbf{a}\,\mathbf{b}+\mathbf{b}\,\mathbf{a}) = \hat{\Psi}_n(\mathbf{a},\mathbf{b}),$$

namely, $\hat{\Phi}_n(\mathbf{c},\mathbf{d}) = \Phi_n(\mathbf{c},\ \mathbf{c}^2-\mathbf{d})$.

Proof. Note that $\hat{\Psi}_n(\mathbf{a},\mathbf{b})$ is the polynomial obtained from $\Psi(\mathbf{a},\mathbf{b})$ by switching the letters at even positions in all the $(\mathbf{a},\mathbf{b})$-monomials. For example, we have $\Psi_3(\mathbf{a},\mathbf{b}) = \mathbf{a}\,\mathbf{a}+2\,\mathbf{a}\,\mathbf{b}+2\,\mathbf{b}\,\mathbf{a}+\mathbf{b}\,\mathbf{b}$, so $\hat{\Psi}_3(\mathbf{a},\mathbf{b}) = \mathbf{a}\,\mathbf{b}+2\,\mathbf{a}\,\mathbf{a}+2\,\mathbf{b}\,\mathbf{b}+\mathbf{b}\,\mathbf{a}$. In terms of the variables $\mathbf{c}$ and $\mathbf{d}$, this operation corresponds to replacing $\mathbf{d} = \mathbf{a}\,\mathbf{b}+\mathbf{b}\,\mathbf{a}$ with $\mathbf{a}\,\mathbf{a}+\mathbf{b}\,\mathbf{b} = \mathbf{c}^2-\mathbf{d}$, and $\mathbf{c} = \mathbf{a}+\mathbf{b}$ with either $\mathbf{a}+\mathbf{b}$ or $\mathbf{b}+\mathbf{a}$, which in any case is still equal to $\mathbf{c}$. □

The polynomial $\hat{\Phi}_n(\mathbf{c},\mathbf{d})$ has both positive and negative coefficients, but the polynomial $\hat{\Phi}_n(\mathbf{c},\ -\mathbf{d}) = \Phi_n(\mathbf{c},\ \mathbf{c}^2+\mathbf{d})$ has only positive coefficients. It would be nice to give a combinatorial interpretation for these coefficients similar to that of the coefficients of $\Phi_n(\mathbf{c},\mathbf{d})$, so that the coefficients of $\hat{\Phi}_n(\mathbf{c},\ -\mathbf{d})$ enumerate permutations of a certain kind according to some statistic. In what follows we show that the sum of the coefficients of $\hat{\Phi}_n(\mathbf{c},\ -\mathbf{d})$ is equal to the number of permutations containing no consecutive descents and not ending with a descent. Let $\mathcal{R}_n$ denote the set of such permutations of $[n]$.

In working with the different kinds of permutations that have emerged thus far we use the approach of min-tree representation of permutations introduced by Hetyei and Reiner [7]. To a word w whose letters are distinct elements of $[n]$, associate a labeled rooted planar binary tree according to the following recursive rule. Let m be the smallest letter of w, and write $w = w_1 \circ m \circ w_2$, where $\circ$ denotes concatenation. Then form the tree $T(w)$ by labeling the root with m and setting the left and the right subtrees of the root to be $T(w_1)$ and $T(w_2)$, respectively. To the empty word we associate the empty tree. Thus $T(w)$ is an increasing rooted planar binary tree, i.e. the distinction between left and right children is being made. For example, $T(423516)$ is the tree shown in Figure 2-1.

To get the word w back from the tree $T(w)$, simply read the labels of the nodes of $T(w)$ in topological order.

Next, we formulate some of the permutation properties from the above discussion in terms of the min-tree representation.

Lemma 2.6.3 *A permutation* σ *has no consecutive descents if and only if the tree* $T(\sigma)$ *has no node whose only child is a left child, except maybe for the rightmost node in topological order.*

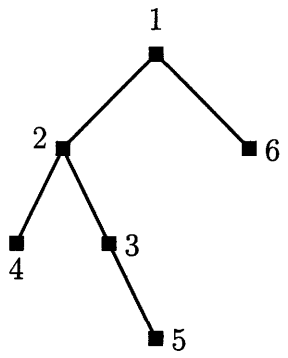


Figure 2-1: The tree $T(423516)$

Proof. Write $\sigma = s_1 s_2 \cdots s_n$ and $T = T(\sigma)$. For convenience, we refer to the nodes of T by their labels. We have $s_i > s_{i+1}$ if and only if s_{i+1} is an ancestor of s_i in T. Since s_i and s_{i+1} are consecutive nodes in the topological reading of T, it follows that s_{i+1} is an ancestor of s_i if and only if s_i has no right child. Thus we have $s_i > s_{i+1} > s_{i+2}$ if and only if s_{i+1} has no right child and s_i is a descendant of s_{i+1}, i.e. s_{i+1} has a lone left child. The proposition follows. □

Proposition 2.6.4 *A permutation σ is in $\mathcal{R}_n$ if and only if the tree $T(\sigma)$ has no node whose only child is a left child.*

Proof. We have $s_{n-1} > s_n$ if and only if the rightmost node s_n has a (lone) left child. The proposition now follows from Lemma 2.6.3. □

Proposition 2.6.5 *A permutation σ is in SS_n if and only if the rightmost node of $T(\sigma)$ is labeled n, no node has a lone left child, and for every node s not on the rightmost path (the path from the root to the rightmost node) that has both a left child t and a right child u, the inequality $t > u$ holds.*

Proof. If $T(\sigma)$ has a node s not on the rightmost path whose left child t is smaller than its right child u, then removing the elements of σ that are greater than or equal to u results in a permutation σ' such that in $T(\sigma')$, the node s has a lone left child t and is not the rightmost node, meaning that σ' contains a pair of consecutve descents, by Lemma 2.6.3. If on the other hand $T(\sigma)$ has no such node s, the removing k largest elements of σ does not create any nodes with a lone left child except maybe for the rightmost node. □

One can see that for $\sigma = 423516$, the tree $T(\sigma)$ shown in Figure 2-1 satisfies all conditions of Proposition 2.6.5, and hence $423516 \in \mathrm{SS}_6$. Next, we consider the sum of coefficients of $\hat{\Phi}_n(\mathbf{c}, -\mathbf{d})$.

Theorem 2.6.6 *The sum of coefficients of $\hat{\Phi}_n(\mathbf{c}, -\mathbf{d})$ is $|\mathcal{R}_n|$.*

Proof. The sum of coefficients of $\hat{\Phi}_n(\mathbf{c}, -\mathbf{d})$ is $\hat{\Phi}_n(1,-1) = \Phi_n(1,2)$, which equals

$$\sum_{\sigma \in \mathrm{SS}_n} 2^{d(\sigma)},$$

where $d(\sigma)$ is the number of $\mathbf{d}$'s in $\mathrm{cd}(\sigma)$, or, equivalently, the number of descents of σ. Since the descents of σ correspond to nodes of $T(\sigma)$ that have no right child (except for the rightmost node, which corresponds to the last element of σ), it follows from Proposition 2.6.4 that the descents of a permutation $\sigma \in \mathcal{R}_n$ correspond to the leaves of $T(\sigma)$ minus the rightmost node. Thus for $\sigma \in \mathcal{R}_n$ we have that $2^{d(\sigma)}$ is the number of leaves in $T(\sigma)$ minus one, which equals the number of of nodes of $T(\sigma)$ with two children. (The latter can be proved easily by induction.)

For a min-tree T and a node s of T with two children, let $F_s(T)$ be the tree obtained by switching the left and the right subtrees of T. (This operation is called the *Foata-Strehl action* on the permutation encoded by T; see [7].) For example, if T is the tree $T(423516)$ shown above, then $F_2(T)$ is the tree shown in Figure 2-2.

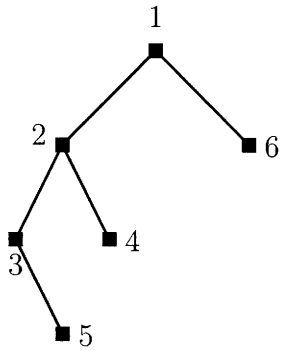


Figure 2-2: The tree $F_2(T(423516))$

Note that the action of F_s preserves the set of nodes with two children and does not create any nodes with a lone left child if the original tree contained no such nodes. Hence the set $T(\mathcal{R}_n)$ is invariant under this action. Observe also that the operators F_s commute and satisfy $F_s^2 = 1$. Thus these operators, viewed as operators on permutations corresponding to trees, split the set $\mathcal{R}_n$ into orbits of size $2^{d(\sigma)}$, where σ is any member of the orbit. It remains to show that each orbit contains exactly one permutation in SS_n.

Given $\sigma \in \mathcal{R}_n$, there is a unique, up to order, sequence of operators F_s, where s is on the rightmost path, that, when applied to $T(\sigma)$, makes n the rightmost node of the resulting tree. An example is shown in Figure 2-3. (One needs to find the closest ancestor of n on the rightmost path and then apply the corresponding operator to bring the node n closer to the rightmost path.) Once n is the rightmost node, apply the operator F_s to all nodes s with two children for which the condition of Proposition 2.6.5 is violated. We obtain a tree corresponding to a permutation in SS_n in the orbit of σ. To see that each orbit contains only one member of SS_n, observe

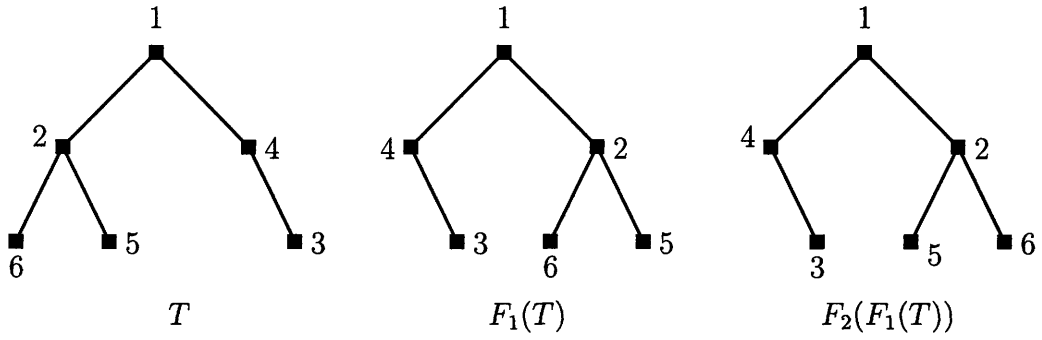


Figure 2-3: The action of F_1 and F_2 on a min-tree

that the action of F_s preserves the sequence of elements on the path from 1 to k for each k, and given the sequence of ancestors for each $k \in [n]$, there is a unique way of arranging the elements of $[n]$ to form a min-tree satisfying the conditions of Proposition 2.6.5: first, set the path from 1 to n to be the rightmost path, and then set all lone children to be right children, and for all nodes with two children, set the greater element to be the left child.

The proof is now complete. □

Table 2.2 lists the polynomials $\hat{\Phi}_n(\mathbf{c}, \mathbf{d})$ for $n \leq 6$. Section A.1 contains values of these polynomials for $n \leq 9$.

n	$\hat{\Phi}_n(\mathbf{c}, \mathbf{d})$
1	1
2	$\mathbf{c}$
3	$2\,\mathbf{c}^2 - \mathbf{d}$
4	$5\,\mathbf{c}^3 - 2(\mathbf{c}\,\mathbf{d} + \mathbf{d}\,\mathbf{c})$
5	$16\,\mathbf{c}^4 - 7(\mathbf{c}^2\,\mathbf{d} + \mathbf{d}\,\mathbf{c}^2) - 5\,\mathbf{c}\,\mathbf{d}\,\mathbf{c} + 4\,\mathbf{d}^2$
6	$61\,\mathbf{c}^5 - 26(\mathbf{c}^3\,\mathbf{d} + \mathbf{d}\,\mathbf{c}^3) - 21(\mathbf{c}\,\mathbf{d}\,\mathbf{c}^2 + \mathbf{c}^2\,\mathbf{d}\,\mathbf{c}) + 10\,\mathbf{d}\,\mathbf{c}\,\mathbf{d} + 12(\mathbf{c}\,\mathbf{d}^2 + \mathbf{d}^2\,\mathbf{c})$

Table 2.2: The polynomials $\hat{\Phi}_n(\mathbf{c}, \mathbf{d})$

2.7 A q-analog of Euler numbers

Let $\hat{A}_n(t, q)$ denote the bivariate polynomial of Theorem 2.2.7:

$$\hat{A}_n(t, q) := \sum_{\sigma \in \mathfrak{S}_n} t^{\hat{d}(\sigma)} q^{\hat{i}(\sigma)}.$$

Then the alternating Eulerian polynomial $\hat{A}_n(t)$ is just the specialization $t\hat{A}_n(t,1)$. We also noted earlier (Corollary 2.2.5) that

$$\hat{A}_n(1,q) = [n]_q!,$$

the classical q-analog of the factorial defined by $[n]_q! := [1]_q[2]_q \cdots [n]_q$, where $[i]_q := 1 + q + q^2 + \cdots + q^{i-1}$. One can ask about other specializations of $\hat{A}_n(t,q)$, such as the ones with t or q set to 0. Clearly, we have $\hat{A}_n(t,0) = 1$ because the only permutation $\sigma \in \mathfrak{S}_n$ for which $\hat{\imath}(\sigma) = 0$ also satisfies $\hat{d}(\sigma) = 0$. The case of $t = 0$ is more curious and is the subject of this section.

We have $\hat{d}(\sigma) = 0$ if and only if σ is an up-down permutation. Thus $\hat{A}_n(0,1) = E_n$, and the specialization $\hat{A}_n(0,q)$ gives a q-analog of the Euler number E_n with coefficients encoding the distribution of the number of alternating inversions among up-down permutations. The following lemma is key in understanding this q-analog.

Lemma 2.7.1 *For a permutation $\sigma \in \mathfrak{S}_n$, let $\hat{\mathrm{code}}(\sigma) = (\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_{n-1})$. Then σ is up-down (resp., down-up) if and only if $\hat{c}_i + \hat{c}_{i+1} \leq n-1-i$ (resp., $\hat{c}_i + \hat{c}_{i+1} \geq n-i$) for all i.*

Proof. This fact is just a special case of Lemma 2.2.9. □

For various reasons it is more convenient to study the distribution of $\hat{\imath}$ on down-up, rather than up-down, permutations. The q-analog obtained this way from down-up permutations is essentially equivalent to $\hat{A}_n(0,q)$, the difference being the reverse order of coefficients and a power of q factor. It follows from Lemma 2.7.1 that for a down-up permutation $\sigma \in \mathfrak{S}_n$, we have

$$\hat{\imath}(\sigma) \geq (n-1) + (n-3) + (n-5) + \cdots = \left\lfloor \frac{n^2}{4} \right\rfloor. \tag{2.26}$$

Therefore let Alt_n be the set of down-up permutations in $\mathfrak{S}_n$, and define

$$\hat{E}_n(q) := q^{-\lfloor n^2/4 \rfloor} \sum_{\sigma \in \mathrm{Alt}_n} q^{\hat{\imath}(\sigma)}.$$

The values of $\hat{E}_n(q)$ for small n are given in Table 2.3. We have the following facts about $\hat{E}_n(q)$.

Proposition 2.7.2 *(a) The polynomial $\hat{E}_n(q)$ is monic and has degree $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor$.*

(b) $\hat{A}_n(0,q) = q^{\lfloor (n-1)^2/4 \rfloor} \cdot \hat{E}_n(q^{-1})$.

(c) $\hat{E}_n(0) = c_{\lfloor n/2 \rfloor}$, the $\lfloor n/2 \rfloor$-th Catalan number.

Proof. (a) By Proposition 2.2.4, the unique permutation $\sigma \in \mathfrak{S}_n$ with the maximum possible number of alternating inversions is the one for which $\hat{\mathrm{code}}(\sigma) = (n-1, n-2, \ldots, 1)$. By Lemma 2.7.1, or by simply realizing that $\sigma = n \circ 1 \circ (n-1) \circ 2 \circ \cdots$,

n	$\hat{E}_n(q)$
0, 1, 2	1
3	$1+q$
4	$2+2q+q^2$
5	$2+5q+5q^2+3q^3+q^4$
6	$5+12q+16q^2+14q^3+9q^4+4q^5+q^6$
7	$5+21q+42q^2+56q^3+56q^4+44q^5+28q^6+14q^7+5q^8+q^9$

Table 2.3: The polynomials $\hat{E}_n(q)$ for $n \leq 7$

one can see that $\sigma \in \mathrm{Alt}_n$. We have $\hat{\imath}(\sigma) = n(n-1)/2$, and thus the degree of $\hat{E}_n(q)$ is $n(n-1)/2 - \lfloor n^2/4 \rfloor = \lfloor (n-1)^2/4 \rfloor$.

(b) This identity is an algebraic restatement of an earlier observation.

(c) The constant term $\hat{E}_n(0)$ of $\hat{E}_n(q)$ is the number of permutations $\sigma \in \mathrm{Alt}_n$ with exactly $\lfloor n^2/4 \rfloor$ alternating inversions. By (2.26), these are precisely the permutations in Alt_n satisfying $\hat{c}_i + \hat{c}_{i+1} = n - i$ for odd i. Let $\sigma \in \mathrm{Alt}_n$ be a permutation with this property.

For $j \geq 1$, we have $\hat{c}_{2j} \geq n - 2j - \hat{c}_{2j+1} = \hat{c}_{2j+2} - 1$. Thus $\hat{c}_2, \hat{c}_4, \ldots, \hat{c}_{2\lfloor n/2 \rfloor}$ is a strictly decreasing sequence of non-negative integers satisfying $\hat{c}_{2j} \leq n - 2j$ (for convenience, let $\hat{c}_n = 0$). Reversing the sequence and reducing the k-th term by $k-1$ for all k yields a bijective correspondence with sequences of $\lfloor n/2 \rfloor$ non-negative integers whose k-th term does not exceed $k-1$, and it is well known that there are $c_{\lfloor n/2 \rfloor}$ such sequences. Since $\hat{c}_{2j-1}$ is uniquely determined by $\hat{c}_{2j}$, it follows that there are $c_{\lfloor n/2 \rfloor}$ permutations $\sigma \in \mathrm{Alt}_n$ with $\lfloor n^2/4 \rfloor$ alternating inversions. □

It is curious to note that the permutations in Alt_n with $\lfloor n^2/4 \rfloor$ alternating inversions can be characterized in terms of pattern avoidance, so that Proposition 2.7.2(c) follows from a result of Mansour [11] stating that the number of 312-avoiding down-up permutations of size n is $c_{\lfloor n/2 \rfloor}$.

Proposition 2.7.3 *A permutation $\sigma \in \mathrm{Alt}_n$ has $\hat{\imath}(\sigma) = \lfloor n^2/4 \rfloor$ if and only if σ is 312-avoiding.*

The following lemma implies the above proposition and is useful in the later discussion as well.

Lemma 2.7.4 *For a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathrm{Alt}_n$, the number $\hat{\imath}(\sigma)$ is equal to $\lfloor n^2/4 \rfloor$ plus the number of occurrences of the generalized pattern 31-2 (that is, the number of pairs of indices $i < j$ such that $\sigma_{i+1} < \sigma_j < \sigma_i$).*

Proof. For $i \in [n-1]$, define

$$S_i := \begin{cases} \{j \mid j > i \text{ and } \sigma_i > \sigma_j\} & \text{if } i \text{ is odd;} \\ \{j \mid j > i \text{ and } \sigma_i < \sigma_j\} & \text{if } i \text{ is even.} \end{cases}$$

Thus $\hat{c}_i = |S_i|$. Let i be *odd*. Then $\sigma_i > \sigma_{i+1}$, so $i+1 \in S_i$ and for every $j > i+1$, either $\sigma_j < \sigma_i$ or $\sigma_j > \sigma_{i+1}$, or both. Hence $\{i+1, i+2, \ldots, n-1\} \subseteq S_i \cup S_{i+1}$ and $\hat{c}_i + \hat{c}_{i+1} = n - 1 - i + |S_i \cap S_{i+1}|$. But $S_i \cap S_{i+1}$ is the set of indices $j > i+1$ such that $\sigma_{i+1} < \sigma_j < \sigma_i$, i.e. the number of occurrences of the pattern 31-2 beginning at position i. Therefore the total number of alternating inversions is $\sum_{i \text{ odd}} (n-1-i) = \lfloor n^2/4 \rfloor$ plus the total number of occurrences of 31-2. $\square$

Proof of Proposition 2.7.3. Suppose that a permutation $\sigma \in \mathrm{Alt}_n$ has exactly $\lfloor n^2/4 \rfloor$ alternating inversions but is not 312-avoiding. Choose a triple $i < k < j$ such that $\sigma_k < \sigma_j < \sigma_i$ and the difference $k - i$ is as small as possible. Suppose that $k - i \geq 2$. If $\sigma_{k-1} < \sigma_j$, then we have $\sigma_{k-1} < \sigma_j < \sigma_i$, contradicting the choice of i, k, and j. If $\sigma_{k-1} > \sigma_j$, then we have $\sigma_k < \sigma_j < \sigma_{k-1}$, also contradicting the choice of i, k, and j. Hence $k = i+1$, and we obtain a contradiction by Lemma 2.7.4. $\square$

In view of Lemma 2.7.4, we can write $\hat{E}_n(q)$ as

$$\hat{E}_n(q) = \sum_{\sigma \in \mathrm{Alt}_n} q^{\mathrm{occ}_{31\text{-}2}(\sigma)}$$

where $\mathrm{occ}_{31\text{-}2}(\sigma)$ is the number of occurrences of 31-2 in σ. In what follows, we use this expression to show how a q-analog of a combinatorial identity representing the Euler number E_n as a weighted sum of Dyck paths yields a refined identity of $\hat{E}_n(q)$.

First, we need to introduce Dyck paths, which are perhaps the most famous combinatorial objects counted by Catalan numbers. A *Dyck path* of length $2m$ is a continuous path consisting of line segments, or *steps*, each of which connects an integer point (x, y) with either $(x+1, y-1)$ or $(x+1, y+1)$, such that the path starts at $(0, 0)$, ends at $(2m, 0)$, and never goes below the x-axis, that is, contains no point with a negative y-coordinate. The identity we are about to describe involves associating a certain weight with every step of a Dyck path, defining the weight of the entire path to be the product of the weights of the individual steps, and adding the weights of all Dyck paths of length $2m$ to obtain E_{2m} or E_{2m+1}, or, in the case of the refined identity, $\hat{E}_{2m}(q)$ or $\hat{E}_{2m+1}(q)$.

For a step in a Dyck path, define the *level* of that step to be the y-coordinate of the highest point of the corresponding segment of the path. Given a Dyck path $\mathcal{D}$ of length $2m$, let $\ell(i)$ be the level of the i-th step of $\mathcal{D}$. Define

$$w^e_{\mathcal{D},i}(q) := [\ell(i)]_q$$

and

$$w^o_{\mathcal{D},i}(q) := \begin{cases} [\ell(i)]_q, & \text{if the } i\text{-th step is an up-step;} \\ [\ell(i)+1]_q, & \text{if the } i\text{-th step is a down-step.} \end{cases}$$

As mentioned above, we set the weight of the entire path to be the product of step

weights:

$$w^e_{\mathcal{D}}(q) = \prod_{i=1}^{2m} w^e_{\mathcal{D},i}(q);$$
$$w^o_{\mathcal{D}}(q) = \prod_{i=1}^{2m} w^o_{\mathcal{D},i}(q).$$

Theorem 2.7.5 *We have*

$$\sum_{\mathcal{D}} w^e_{\mathcal{D}}(q) = \hat{E}_{2m}(q)$$

and

$$\sum_{\mathcal{D}} w^o_{\mathcal{D}}(q) = \hat{E}_{2m+1}(q),$$

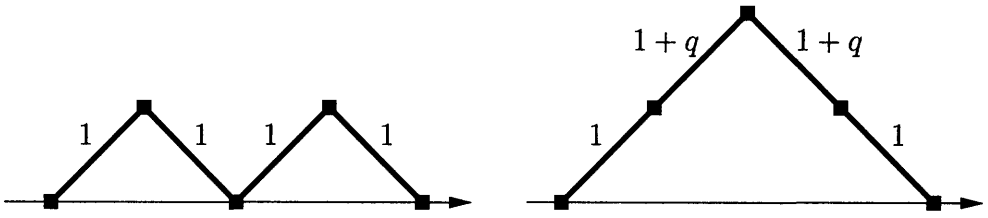
where both sums are taken over all Dyck paths of length $2m$.

For example, for $m = 2$ there are two Dyck paths, shown in Figures 2-4 and 2-5 with step weights given by $w^e_{\mathcal{D},i}(q)$ and $w^o_{\mathcal{D},i}(q)$. From these weighted paths, we get

$$1 + (1+q)^2 = 2 + 2q + q^2 = \hat{E}_4(q)$$

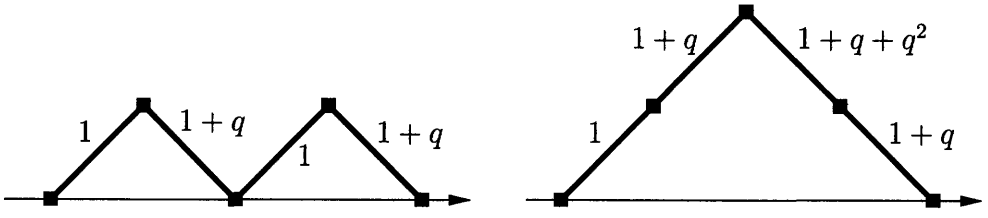
and

$$(1+q)^2 + (1+q)^2(1+q+q^2) = 2 + 5q + 5q^2 + 3q^3 + q^4 = \hat{E}_5(q).$$



$w^e = 1$ $\qquad$ $w^e = (1+q)^2$

Figure 2-4: Weighted Dyck paths adding up to $\hat{E}_4(q)$



$w^o = (1+q)^2$ $\qquad$ $w^o = (1+q)^2(1+q+q^2)$

Figure 2-5: Weighted Dyck paths adding up to $\hat{E}_5(q)$

In the classical case $q = 1$, the identities of Theorem 2.7.5 are due to Françon and Viennot [5], and are discussed in a broader context in the book [6, Sec. 5.2] by Goulden and Jackson. The proof of our identities is a refinement of the original argument.

Proof of Theorem 2.7.5. Fix a positive integer $n > 1$, and let $m = \lfloor n/2 \rfloor$. Recall that in Section 2.6 we associated to a permutation $\sigma \in \mathfrak{S}_n$ an increasing planar binary tree $T(\sigma)$ with vertex set $[n]$. Extending the argument in the proof of Lemma 2.6.3, we conclude that σ is in Alt_n if and only if the tree $T(\sigma)$ has no vertices with a lone child, except for the rightmost vertex in the case of even n, which has a lone left child. For $\sigma \in \mathrm{Alt}_n$, define the corresponding Dyck path $\mathcal{D}(\sigma)$ of length $2m$ as follows: set the i-th step of the path to be an up-step if vertex i of $T(\sigma)$ has at least one child, and set the i-th step to be a down-step if vertex i is a leaf of $T(\sigma)$. We leave it as an exercise for the reader to check that $\mathcal{D}(\sigma)$ is a valid Dyck path.

Fix a Dyck path $\mathcal{D}$ of length $2m$. We claim that

$$\sum_{\sigma \in \mathrm{Alt}_n \,:\, \mathcal{D}(\sigma) = \mathcal{D}} q^{\mathrm{occ}_{31\text{-}2}(\sigma)} = \begin{cases} w^e_{\mathcal{D}}(q), & \text{if } n \text{ is even;} \\ w^o_{\mathcal{D}}(q), & \text{if } n \text{ is odd.} \end{cases} \tag{2.27}$$

To prove the claim, consider for every i the subtree $T_i(\sigma)$ obtained from $T(\sigma)$ by removing all vertices labeled with numbers greater than i. For the sake of clarity, one should imagine the "incomplete" tree $T_i(\sigma)$ together with "loose" edges indicating those edges with parent vertices in $T_i(\sigma)$ that appear when $T_i(\sigma)$ is completed to $T(\sigma)$. For even n one should also think of a loose edge directed to the right coming out of the rightmost vertex of every tree $T_i(\sigma)$ including $T_n(\sigma) = T(\sigma)$ — this way the number of edges coming out of a vertex of $T_i(\sigma)$ is always 0 or 2.

Observe that for $1 \le i \le 2m$, the number of loose edges of $T_i(\sigma)$ is equal to $y_{\mathcal{D}}(i) + 1$, where $y_{\mathcal{D}}(i)$ is the y-coordinate of the point of $\mathcal{D}$ whose x-coordinate is i. Indeed, $T_1(\sigma)$ has two loose edges, and $T_{i+1}(\sigma)$ is obtained from $T_i(\sigma)$ by attaching a non-leaf to a loose edge, thus increasing the number of loose edges by one, if the i-th step of $\mathcal{D}$ is an up-step, or by attaching a leaf to a loose edge, thus reducing the number of loose edges by one, if the i-th step is a down-step. Hence we can count the number of permutations $\sigma \in \mathrm{Alt}_n$ with $\mathcal{D}(\sigma) = \mathcal{D}$ by multiplying together the number of possibilities to attach a vertex labeled $i + 1$ to $T_i(\sigma)$ to form $T_{i+1}(\sigma)$ for all $1 \le i \le n - 1$. The number of valid places to attach vertex $i + 1$ is equal to the number of loose edges in $T_i(\sigma)$ unless $i + 1$ is a leaf of $T(\sigma)$ and n is even, in which case we have one fewer possibilities, because we are not allowed to make the rightmost vertex a leaf. Note that the level $\ell(i)$ of the i-th step of $\mathcal{D}$ is equal to $y_{\mathcal{D}}(i)$ if it is an up-step, or $y_{\mathcal{D}}(i) + 1$ if it is a down-step. Comparing with the choice of step weights, we conclude that the number of possibilities to attach vertex $i + 1$ is $w^e_{\mathcal{D},i+1}(1)$ if n is even, or $w^o_{\mathcal{D},i+1}(1)$ if n is odd. (For odd n and $i = n - 1$ the latter assertion makes no sense as $\mathcal{D}$ does not have an n-th step; however, there is just one way to attach the last vertex, so the counting argument is not affected.)

The above computation proves the $q = 1$ case of (2.27). To prove the general claim, we need to show that if there are p possibilities to attach vertex $i + 1$ to a loose

edge of $T_i(\sigma)$, then the number of occurrences of the 31-2 pattern "induced" by the attachment is 0 for one of the possibilities, 1 for another possibility, 2 for another, and so on, up to $p-1$. Then choosing a place to attach vertex $i+1$ would correspond to choosing a term from $1+q+q^2+\cdots+q^{p-1} = [p]_q$, the weight of the i-th step of $\mathcal{D}$, which is a factor in the total weight of $\mathcal{D}$, and (2.27) would follow.

It remains to specify which occurrences of 31-2 in σ are induced by which vertex of $T(\sigma)$. Suppose there are p possible places to attach vertex $i+1$. Order these places according to the topological order of tree traversal, and suppose we choose to put vertex $i+1$ in the k-th place in this order. Let $r_1, r_2, \ldots, r_{k-1}$ be the numbers of the vertices immediately following the first $k-1$ places in the topological order, and let a_j denote the label of the rightmost vertex of the eventual subtree of $T(\sigma)$ rooted at what is currently the j-th of these $k-1$ places. Although a_j is not determined at the time vertex $i+1$ is attached, it is certain that $r_j < i+1 < a_j$ and that a_j and r_j will be consecutive elements of σ, with $i+1$ located somewhere to the right, resulting in an occurrence of 31-2. Thus the choice to put vertex $i+1$ in the k-th available place induces $k-1$ occurrences of 31-2, one for each $1 \le j \le k-1$. It is not hard to check that each occurrence of 31-2 is induced by some vertex of $T(\sigma)$, namely, the vertex corresponding to the rightmost element forming the pattern, in the way described above.

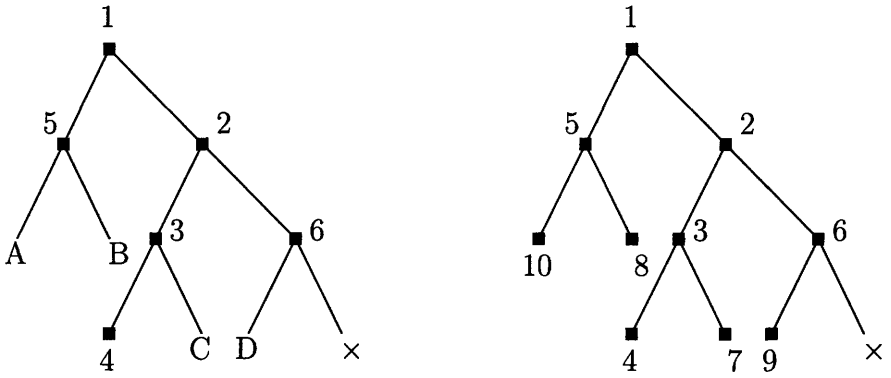


Figure 2-6: An intermediate tree $T_6(\sigma)$ and its completion $T(\sigma)$

Let us illustrate the argument with an example. The left side of Figure 2-6 shows the tree $T_6(\sigma)$ for some $\sigma \in \mathrm{Alt}_{10}$, with the four potential places for vertex 7 marked A, B, C, and D. If vertex 7 is put in position A, then it induces no occurrences of 31-2. If it is put in position B, it induces one occurrence of 31-2 as the triple a5-7 is created, where a stands for the number of the rightmost vertex in the subtree rooted at A in the eventual tree. If vertex 7 is put in position C, then in addition to the triple a5-7, one obtains a second 31-2 triple b1-7. Finally, putting vertex 7 in position D results in a third 31-2 triple c2-7. (Here b and c are defined by analogy with a.) On the right side of Figure 2-6 we have a possible completion of the tree on the left, which corresponds to the permutation $\sigma = 10\ 5\ 8\ 1\ 4\ 3\ 7\ 2\ 9\ 6$.

The theorem now follows by taking the sum of (2.27) over all Dyck paths $\mathcal{D}$ of length $2m$. $\square$

Chapter 3

The f-vector of the descent polytope

3.1 Preliminaries

For a set $S \subseteq [n-1]$, define the *descent polytope* DP_S to be the set of points $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ such that $0 \leq x_i \leq 1$, and

$$\begin{cases} x_i \geq x_{i+1} & \text{if } i \in S, \\ x_i \leq x_{i+1} & \text{if } i \notin S. \end{cases}$$

Thus DP_S is the order polytope of the ribbon poset $Z_S = \{z_1, z_2, \ldots, z_n\}$ defined by the cover relations $z_i \cdot > z_{i+1}$ if $i \in S$ and $z_i <\cdot z_{i+1}$ if $i \notin S$ (cf. Section 1.1.5). Therefore the volume of DP_S is equal to the number of linear extensions of Z_S or, equivalently, the number of permutations in $\mathfrak{S}_n$ with descent set S, times $1/n!$.

In this chapter our primary goal is to compute the *f-vector* of the descent polytope DP_S. Recall that for an n-dimensional polytope, the f-vector is the integer vector $(f_0, f_1, \ldots, f_{n-1})$, where f_i is the number of i-dimensional faces in the polytope. For $S \subseteq [n-1]$, define the polynomial

$$F_S(t) := \sum_{i=0}^{n} f_i t^i,$$

where $(f_0, f_1, \ldots, f_{n-1})$ is the f-vector of DP_S, and $f_n = 1$ by convention. To simplify notation, we will often write F_S instead of $F_S(t)$. As we show in Section 3.2, F_S can be expressed as a sum of polynomials taken over all subsets of S. To obtain a "closed form" result, in Section 3.3 we compute a generating function for F_S as a formal power series in two non-commuting variables. We describe the general setup here.

Let $\mathbf{x}$ and $\mathbf{y}$ be two non-commuting variables. For $S \subseteq [m]$, define $\mathbf{v}_S = \mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_m$ where

$$\mathbf{v}_i = \begin{cases} \mathbf{x} & \text{if } i \notin S, \\ \mathbf{y} & \text{if } i \in S. \end{cases}$$

For a given n, define $\Phi_n \in \mathbb{Z}[t]\langle \mathbf{x}, \mathbf{y} \rangle$ by

$$\Phi_n(\mathbf{x}, \mathbf{y}) := \sum_{S \subseteq [n-1]} F_S \, \mathbf{v}_S \, .$$

Now define $\Phi(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}[t]\langle\langle \mathbf{x}, \mathbf{y} \rangle\rangle$ by

$$\Phi(\mathbf{x}, \mathbf{y}) = \sum_{n \geq 1} \Phi_n(\mathbf{x}, \mathbf{y}),$$

so that $\Phi_n(\mathbf{x}, \mathbf{y})$ is the homogeneous component of $\Phi(\mathbf{x}, \mathbf{y})$ of degree $n-1$. Note that for $S^c = [n-1] - S$, we have $F_{S^c} = F_S$, and $\mathbf{v}_{S^c}$ is obtained from $\mathbf{v}_S$ by switching $\mathbf{x}$ and $\mathbf{y}$. It follows that $\Phi_n(\mathbf{x}, \mathbf{y})$, and hence $\Phi(\mathbf{x}, \mathbf{y})$, is symmetric with respect to $\mathbf{x}$ and $\mathbf{y}$, that is,

$$\Phi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{y}, \mathbf{x}).$$

3.2 An expression for F_S

The following theorem provides a way to compute F_S.

Theorem 3.2.1 *Let S be a subset of $[n-1]$ and let $\mathbf{v}_S = \mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_{n-1}$, where $\mathbf{v}_i \in \{\mathbf{x}, \mathbf{y}\}$. For $T = \{j_1 < j_2 < \cdots < j_k\} \subseteq [n-1]$, let $\mathbf{v}_S^T = \mathbf{v}_{j_1} \mathbf{v}_{j_2} \cdots \mathbf{v}_{j_k}$. Then F_S is given by*

$$F_S = 1 + \sum_{T \subseteq [n-1]} \left(\frac{t+1}{t} \right)^{\kappa(\mathbf{v}_S^T)} t^{|T|+1},$$

where $\kappa(\mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_m) = 2 + |\{i \ : \ \mathbf{v}_i \neq \mathbf{v}_{i+1}\}|$ for $m > 0$, and $\kappa(1) = 1$.

Proof. For a face $\mathcal{F}$ of a polytope, let $\mathcal{F}^I$ denote the relative interior of $\mathcal{F}$. Then the polytope is the disjoint union of $\mathcal{F}^I$ taken over all faces $\mathcal{F}$, including the polytope itself.

As we stated before, the polytope DP_S consists of all points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ belonging simultaneously to the halfspaces $x_i \geq 0$, $x_i \leq 1$ $(1 \leq i \leq n)$, $x_i \leq x_{i+1}$ $(i \notin S)$, and $x_i \geq x_{i+1}$ $(i \in S)$. A face $\mathcal{F}$ of DP_S can be uniquely identified by specifying which of these halfspaces contain $\mathcal{F}$ on their boundary hyperplanes, as long as the intersection of the whole polytope and the specified boundary hyperplanes is non-empty. Forming the specification just for the halfspaces of the form $x_i \leq x_{i+1}$ or $x_i \geq x_{i+1}$ restricts the location of $\mathcal{F}^I$ in $\mathbb{R}^n$ to the region defined by the relations

$$x_1 = x_2 = \cdots = x_{j_1} \lneqq x_{j_1+1} = x_{j_1+2} = \cdots = x_{j_2} \lneqq \cdots \lneqq x_{j_k+1} = x_{j_k+2} = \cdots = x_n \tag{3.1}$$

for some $T = \{j_1 < j_2 < \cdots < j_k\} \subseteq [n-1]$, where the symbol $\lneqq$ denotes strict inequality: $x_{j_i} < x_{j_i+1}$ if $j_i \notin S$, or $x_{j_i} > x_{j_i+1}$ if $j_i \in S$. Then T is the set of indices j for which $\mathcal{F}$ does *not* lie entirely on the boundary hyperplane $x_j = x_{j+1}$ and thus the relative interior $\mathcal{F}^I$ is contained in the interior of the corresponding halfspace. Let

$\mathcal{R}(T)$ denote the intersection of the region defined by (3.1) and the hypercube $[0,1]^n$. Each point $(x_1,\dots,x_n)$ of DP_S belongs to exactly one such region $\mathcal{R}(T)$, namely, the one for $T = \{j \mid x_j \neq x_{j+1}\}$. Thus we have the disjoint union

$$\mathrm{DP}_S = \bigsqcup_{T\subseteq[n-1]} \mathcal{R}(T).$$

Let us show that the term corresponding to $T \neq \varnothing$ in the expression in the statement of the theorem is the contribution to F_S of the faces $\mathcal{F}$ of DP_S for which $\mathcal{F}^I$ is contained in the region $\mathcal{R}(T)$. In other words, we claim that for $T \neq \varnothing$ we have

$$\sum_{\mathcal{F}\,:\,\mathcal{F}^I\subseteq\mathcal{R}(T)} t^{\dim\mathcal{F}} = \left(\frac{t+1}{t}\right)^{\kappa(\mathbf{v}_S^T)} t^{|T|+1}. \tag{3.2}$$

Fix $\varnothing \neq T \subseteq [n-1]$. To select a particular face $\mathcal{F}$ from the set of all faces with the property $\mathcal{F}^I \subseteq \mathcal{R}(T)$, we need to complete the specification started above, that is, we must specify which of the hyperplanes $x_i = 0, 1$ contain $\mathcal{F}$, and we must make sure that the intersection of the set of the specified hyperplanes and $\mathcal{R}(T)$ is non-empty. In terms of defining relations (3.1), this task is equivalent to setting the common value of some of the "blocks" of coordinates $(x_1,\dots,x_{j_1})$, $(x_{j_1+1},\dots,x_{j_2})$, $\dots$, $(x_{j_k+1},\dots,x_n)$ to 0 or 1. Since the relations must remain satisfiable by at least one point in $[0,1]^n$, only the blocks preceded in (3.1) by $>$ (or nothing) and succeeded by $<$ (or nothing) can be set to 0. Similarly, only the blocks preceded by $<$ (or nothing) and succeeded by $>$ (or nothing) can be set to 1. Thus each block can be set to at most one of 0 and 1. The letters of the $(\mathbf{x},\mathbf{y})$-word $\mathbf{v}_S^T = \mathbf{v}_{j_1}\cdots\mathbf{v}_{j_k}$ encode the inequality signs in (3.1) ($\mathbf{x}$ stands for $<$, and $\mathbf{y}$ stands for $>$), so the number of blocks that can be set to 0 or 1 is the total number of occurrences of $\mathbf{x}$ followed by $\mathbf{y}$, or $\mathbf{y}$ followed by $\mathbf{x}$, in $\mathbf{v}_S^T$, plus 2, as we also need to count the first and the last blocks. In other words, the number of such blocks is $\kappa(\mathbf{v}_S^T)$.

Observe that the dimension of the face of DP_S obtained by this specification procedure equals the number of blocks that have not been set to 0 or 1: the common values of the coordinates in those blocks form the "degrees of freedom" that constitute the dimension. Let us call such blocks *free*. The number of faces $\mathcal{F}$ with $\mathcal{F}^I \subseteq \mathcal{R}(T)$ for which the specification procedure results in m free blocks is

$$\binom{\kappa(\mathbf{v}_S^T)}{|T|+1-m},$$

the number of ways to choose $|T|+1-m$ blocks that are *not* free out of $\kappa(\mathbf{v}_S^T)$ possibilities. Hence we have

$$\sum_{\mathcal{F}\,:\,\mathcal{F}^I\subseteq\mathcal{R}(T)} t^{\dim\mathcal{F}} = \sum_{m=|T|+1-\kappa(\mathbf{v}_S^T)}^{|T|+1} \binom{\kappa(\mathbf{v}_S^T)}{|T|+1-m}\cdot t^m =$$

$$= t^{|T|+1-\kappa(\mathbf{v}_S^T)} \sum_{\ell=0}^{\kappa(\mathbf{v}_S^T)} \binom{\kappa(\mathbf{v}_S^T)}{\ell} \cdot t^\ell = t^{|T|+1-\kappa(\mathbf{v}_S^T)} \cdot (t+1)^{\kappa(\mathbf{v}_S^T)},$$

proving (3.2).

Finally, for $T = \varnothing$, we have $\mathcal{R}(T) = \{0 \leq x_1 = \cdots = x_n \leq 1\}$, which is just the line segment joining the two vertices $(0, \ldots, 0)$ and $(1, \ldots, 1)$ of DP_S. Thus the contribution of $\mathcal{R}(T)$ to F_S is

$$t + 2 = 1 + \left(\frac{t+1}{t}\right)^{\kappa(\mathbf{v}_S^\varnothing)} \cdot t.$$

Adding this equation to the sum of (3.2) taken over the non-empty T proves the theorem. □

Theorem 3.2.1 yields a combinatorial interpretation of the number of vertices of the polytope DP_S. Call an $(\mathbf{x}, \mathbf{y})$-word $\mathbf{v} = \mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_k$, where $\mathbf{v}_i \in \{\mathbf{x}, \mathbf{y}\}$, *alternating* if $\mathbf{v}_i \neq \mathbf{v}_{i+1}$ for all $1 \leq i \leq k-1$. Then we have the following corollary.

Corollary 3.2.2 *For $S \subseteq [n-1]$, the number of vertices of DP_S is one greater than the number of subsets $T \subseteq [n-1]$ for which the word $\mathbf{v}_S^T$ is alternating.*

Proof. The number of vertices of DP_S is the constant term of F_S. For the summand corresponding to a subset $T \subseteq [n-1]$ in the formula of Theorem 3.2.1, the constant term is either 0 or 1, the latter being the case if and only if $|T| + 1 - \kappa(\mathbf{v}_S^T) = 0$. This condition is equivalent to $\mathbf{v}_S^T$ being alternating. The corollary follows. □

3.3 The power series $\Phi(\mathbf{x}, \mathbf{y})$

We now discuss a more efficient way to compute $F_S(t)$ than the expression of Theorem 3.2.1. In this section we denote F_S by $F_{\mathbf{v}_S}$; such notation has an advantage, as $\mathbf{v}_S$ encodes not only $S \subseteq [n-1]$ but also the dimension n. Since pairs $(n, S \subseteq [n-1])$ are in bijective correspondence with $(\mathbf{x}, \mathbf{y})$-words via $S \mapsto \mathbf{v}_S$, it makes sense to parameterize the f-polynomials of descent polytopes by $(\mathbf{x}, \mathbf{y})$-words and write $F_{\mathbf{v}}$, where $\mathbf{v} = \mathbf{v}_S$ for some $S \subseteq [|\,\mathbf{v}\,|]$. (Here $|\,\mathbf{v}\,|$ denotes the length of the word $\mathbf{v}$.)

Let $\mathbf{v} = \mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_{n-1}$, where $\mathbf{v}_i \in \{\mathbf{x}, \mathbf{y}\}$. Consider the following polynomials:

$$K_{\mathbf{v}}(t) := \sum_{T \subseteq [n-1] \,:\, \mathbf{v}_{j_1} = \mathbf{x}} \left(\frac{t+1}{t}\right)^{\kappa(\mathbf{v}^T)} t^{|T|+1};$$

$$L_{\mathbf{v}}(t) := \sum_{T \subseteq [n-1] \,:\, \mathbf{v}_{j_1} = \mathbf{y}} \left(\frac{t+1}{t}\right)^{\kappa(\mathbf{v}^T)} t^{|T|+1},$$

where $\mathbf{v}_{j_1}$ denotes the first letter of the word $\mathbf{v}^T = \mathbf{v}_{j_1}\mathbf{v}_{j_2}\cdots\mathbf{v}_{j_k}$, as in the notation of Theorem 3.2.1. Since $\mathbf{v}^T$ begins with either $\mathbf{x}$ or $\mathbf{y}$ unless $T = \varnothing$, we have

$$F_{\mathbf{v}} = 1 + \left(\frac{t+1}{t}\right)^{\kappa(\mathbf{v}^{\varnothing})} \cdot t + K_{\mathbf{v}} + L_{\mathbf{v}} = K_{\mathbf{v}} + L_{\mathbf{v}} + t + 2. \tag{3.3}$$

We continue with a lemma that relates the two polynomials $K_{\mathbf{v}}$ and $L_{\mathbf{v}}$.

Lemma 3.3.1 *For an $(\mathbf{x},\mathbf{y})$-word $\mathbf{v} = \mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_{n-1}$, where $\mathbf{v}_i \in \{\mathbf{x},\mathbf{y}\}$, the following equalities hold:*

$$\begin{aligned} K_{\mathbf{y}\mathbf{v}} &= K_{\mathbf{v}}; \\ L_{\mathbf{x}\mathbf{v}} &= L_{\mathbf{v}}; \\ K_{\mathbf{x}\mathbf{v}} &= L_{\mathbf{y}\mathbf{v}} = (t+1)(K_{\mathbf{v}} + L_{\mathbf{v}} + t + 1). \end{aligned}$$

Proof. For an integer i and a set $U \subseteq \mathbb{Z}$, let $U + i$ denote the set obtained by adding i to each element of U.

Clearly, $(\mathbf{y}\,\mathbf{v})^T$ begins with $\mathbf{x}$ if and only if $1 \notin T$ and $\mathbf{v}^{T-1}$ begins with $\mathbf{x}$, in which case $(\mathbf{y}\,\mathbf{v})^T = \mathbf{v}^{T-1}$. Hence $K_{\mathbf{y}\mathbf{v}} = K_{\mathbf{v}}$.

Now, $(\mathbf{x}\,\mathbf{v})^T$ begins with $\mathbf{x}$ if and only if either $1 \in T$, or else $1 \notin T$ and $\mathbf{v}^{T-1}$ begins with $\mathbf{x}$. In the former case, we have $T = \{1 < j_1 + 1 < j_2 + 1 < \cdots < j_k + 1\}$, and $(\mathbf{x}\,\mathbf{v})^T = \mathbf{x}\,\mathbf{v}_{j_1}\mathbf{v}_{j_2}\cdots\mathbf{v}_{j_k}$. Set $U = T\backslash\{1\} - 1 = \{j_1 < \cdots < j_k\}$. Then $\kappa\big((\mathbf{x}\,\mathbf{v})^T\big) = \kappa\big(\mathbf{v}^U\big)$ if $\mathbf{v}_{j_1} = \mathbf{x}$, and $\kappa\big((\mathbf{x}\,\mathbf{v})^T\big) = \kappa\big(\mathbf{v}^U\big) + 1$ if $\mathbf{v}_{j_1} = \mathbf{y}$. Hence

$$\sum_{1\in T\subseteq[n]} \left(\frac{t+1}{t}\right)^{\kappa\left((\mathbf{x}\,\mathbf{v})^T\right)} t^{|T|+1} = \tag{3.4}$$

$$= (t+1)^2 + t\cdot \sum_{U\,:\,\mathbf{v}_{j_1}=\mathbf{x}} \left(\frac{t+1}{t}\right)^{\kappa\left(\mathbf{v}^U\right)} t^{|U|+1} + (t+1)\cdot \sum_{U\,:\,\mathbf{v}_{j_1}=\mathbf{y}} \left(\frac{t+1}{t}\right)^{\kappa\left(\mathbf{v}^U\right)} t^{|U|+1} =$$

$$= (t+1)^2 + tK_{\mathbf{v}} + (t+1)L_{\mathbf{v}},$$

where the leading term $(t+1)^2$ corresponds to $T = \{1\}$ and $U = \varnothing$. In the case where $1 \notin T$ and $\mathbf{v}^{T-1}$ begins with $\mathbf{x}$ we have, as before, $(\mathbf{x}\,\mathbf{v})^T = \mathbf{v}_{j_1}\mathbf{v}_{j_2}\cdots\mathbf{v}_{j_k} = \mathbf{v}^{T-1}$, and hence

$$\sum_{T\,:\,\mathbf{v}_{j_1}=\mathbf{x}} \left(\frac{t+1}{t}\right)^{\kappa\left((\mathbf{x}\,\mathbf{v})^T\right)} t^{|T|+1} = \sum_{\mathbf{v}_{j_1}=\mathbf{x}} \left(\frac{t+1}{t}\right)^{\kappa\left(\mathbf{v}^{T-1}\right)} t^{|T-1|+1} = K_{\mathbf{v}}. \tag{3.5}$$

Adding (3.4) and (3.5) yields

$$K_{\mathbf{x}\mathbf{v}} = (t+1)(K_{\mathbf{v}} + L_{\mathbf{v}} + t + 1).$$

The relations for $L_{\mathbf{x}\mathbf{v}}$ and $L_{\mathbf{y}\mathbf{v}}$ follow from symmetry that arises from exchanging the variables $\mathbf{x}$ and $\mathbf{y}$. □

Starting with $K_1 = L_1 = 0$, one can use Lemma 3.3.1 to compute $K_{\mathbf{v}}$ and $L_{\mathbf{v}}$, and hence $F_{\mathbf{v}}$, from (3.3). Recall the generating power series

$$\Phi(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{v}} F_{\mathbf{v}}\,\mathbf{v},$$

where the sum is over all $(\mathbf{x},\mathbf{y})$-words, including the empty word $\mathbf{v} = \mathbf{v}_\varnothing = 1$. Define the two generating power series

$$\mathrm{K}(\mathbf{x},\mathbf{y}) := \sum_{\mathbf{v}} K_{\mathbf{v}}\,\mathbf{v};$$

$$\Lambda(\mathbf{x},\mathbf{y}) := \sum_{\mathbf{v}} L_{\mathbf{v}}\,\mathbf{v}\,.$$

It is not hard to see from the definition of $K_{\mathbf{v}}$ and $L_{\mathbf{v}}$ that $K_{\mathbf{v}} = L_{\mathbf{v}(\mathbf{y},\mathbf{x})}$, where $\mathbf{v}(\mathbf{y},\mathbf{x})$ denotes the word obtained from $\mathbf{v}$ by switching $\mathbf{x}$ and $\mathbf{y}$. It follows that

$$\mathrm{K}(\mathbf{x},\mathbf{y}) = \Lambda(\mathbf{y},\mathbf{x}).$$

Then, by (3.3), we have

$$\begin{aligned}
\Phi(\mathbf{x},\mathbf{y}) &= \mathrm{K}(\mathbf{x},\mathbf{y}) + \Lambda(\mathbf{x},\mathbf{y}) + (t+2)\sum_{\mathbf{v}} \mathbf{v} \\
&= \mathrm{K}(\mathbf{x},\mathbf{y}) + \mathrm{K}(\mathbf{y},\mathbf{x}) + (t+2)\sum_{r\geq 0}(\mathbf{x}+\mathbf{y})^r \\
&= \mathrm{K}(\mathbf{x},\mathbf{y}) + \mathrm{K}(\mathbf{y},\mathbf{x}) + (t+2)\cdot\frac{1}{1-\mathbf{x}-\mathbf{y}}. \qquad (3.6)
\end{aligned}$$

From the equations of Lemma 3.3.1 we obtain

$$\begin{aligned}
\sum_{\mathbf{v}} K_{\mathbf{x}\,\mathbf{v}}\,\mathbf{x}\,\mathbf{v} &= (t+1)\,\mathbf{x}\sum_{\mathbf{v}}(K_{\mathbf{v}} + L_{\mathbf{v}} + t + 1)\,\mathbf{v}; \\
\sum_{\mathbf{v}} K_{\mathbf{y}\,\mathbf{v}}\,\mathbf{y}\,\mathbf{v} &= \mathbf{y}\sum_{\mathbf{v}} K_{\mathbf{v}}\,\mathbf{v}\,.
\end{aligned}$$

We add these two identities and recall that $K_1 = 0$ to get

$$\begin{aligned}
\mathrm{K}(\mathbf{x},\mathbf{y}) &= \sum_{\mathbf{v}\neq 1} K_{\mathbf{v}}\,\mathbf{v} \\
&= (t+1)\,\mathbf{x}\sum_{\mathbf{v}}(K_{\mathbf{v}} + L_{\mathbf{v}} + t + 1)\,\mathbf{v} + \mathbf{y}\sum_{\mathbf{v}} K_{\mathbf{v}}\,\mathbf{v} \\
&= (t+1)\,\mathbf{x}\left(\mathrm{K}(\mathbf{x},\mathbf{y}) + \Lambda(\mathbf{x},\mathbf{y}) + (t+1)\cdot\frac{1}{1-\mathbf{x}-\mathbf{y}}\right) + \mathbf{y}\,\mathrm{K}(\mathbf{x},\mathbf{y}) \\
&= (t+1)\,\mathbf{x}\left(\Phi(\mathbf{x},\mathbf{y}) - \frac{1}{1-\mathbf{x}-\mathbf{y}}\right) + \mathbf{y}\,\mathrm{K}(\mathbf{x},\mathbf{y}),
\end{aligned}$$

or, after rearranging terms,

$$\mathrm{K}(\mathbf{x},\mathbf{y}) = (t+1)(1-\mathbf{y})^{-1}\mathbf{x}\left(\Phi(\mathbf{x},\mathbf{y}) - \frac{1}{1-\mathbf{x}-\mathbf{y}}\right).$$

Adding this equation and its symmetric version obtained by switching $\mathbf{x}$ and $\mathbf{y}$ gives, via (3.6),

$$\Phi(\mathbf{x},\mathbf{y}) - (t+2)\cdot\frac{1}{1-\mathbf{x}-\mathbf{y}} = \mathrm{K}(\mathbf{x},\mathbf{y}) + \mathrm{K}(\mathbf{y},\mathbf{x}) =$$

$$= (t+1)\Big((1-\mathbf{y})^{-1}\mathbf{x} + (1-\mathbf{x})^{-1}\mathbf{y}\Big)\left(\Phi(\mathbf{x},\mathbf{y}) - \frac{1}{1-\mathbf{x}-\mathbf{y}}\right)$$

(recall that $\Phi(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{y},\mathbf{x})$). Solving for $\Phi(\mathbf{x},\mathbf{y})$, we arrive at the following theorem.

Theorem 3.3.2 *The generating power series $\Phi(\mathbf{x},\mathbf{y})$ is given by*

$$\Phi(\mathbf{x},\mathbf{y}) = \left(1 + \frac{t+1}{1-(t+1)\Big((1-\mathbf{y})^{-1}\mathbf{x} + (1-\mathbf{x})^{-1}\mathbf{y}\Big)}\right)\cdot\frac{1}{1-\mathbf{x}-\mathbf{y}}.$$

3.4 More recurrence relations

In this section we derive a different set of recurrences determining $F_S(t)$ than the ones we used to obtain Theorem 3.3.2. Here it will be more convenient to associate integer sets with compositions. Let $\mathrm{Comp}'(m)$ denote the set of integer compositions $(\gamma_1,\gamma_2,\ldots)$ of m with $\gamma_1 \geq 0$ and $\gamma_2,\gamma_3,\ldots > 0$. For $\gamma = (\gamma_1,\gamma_2,\ldots) \in \mathrm{Comp}'(m)$, define $\mathbf{v}_\gamma = \mathbf{x}^{\gamma_1}\mathbf{y}^{\gamma_2}\mathbf{x}^{\gamma_3}\mathbf{y}^{\gamma_4}\cdots$. There is a bijective correspondence $\mathrm{c} : [m] \to \mathrm{Comp}(m)$ arising from the defining relation $\mathbf{v}_S = \mathbf{v}_{\mathrm{c}(S)}$. For example, if we have $S = \{1,3,4\} \subseteq [6]$, then $\mathbf{v}_S = \mathbf{yxyyxx} = \mathbf{x}^0\mathbf{y}^1\mathbf{x}^1\mathbf{y}^2\mathbf{x}^2$ and thus $\mathrm{c}(S) = (0,1,1,2,2)$.

Write $F_S(t) = F_{\mathrm{c}(S)}(t)$. In the notation of Theorem 3.2.1, define

$$G_\gamma(t) := t+1+\sum_{T\,:\,x_1>\gamma_1}\left(\frac{t+1}{t}\right)^{\kappa(\mathbf{v}_\gamma^T)} t^{|T|+1};$$

$$H_\gamma(t) := \sum_{T\,:\,x_1\leq\gamma_1}\left(\frac{t+1}{t}\right)^{\kappa(\mathbf{v}_\gamma^T)} t^{|T|+1},$$

where $T = \{x_1 < x_2 < \cdots\} \subseteq [n-1]$. Thus $F_\gamma = 1 + G_\gamma + H_\gamma$. The extra $t+1$ in G_γ corresponds to the term for $T = \varnothing$. Note that G_γ does not depend on the value of γ_1, and that $\gamma_1 = 0$ implies $H_\gamma = 0$. For $\gamma = (\gamma_1,\gamma_2,\ldots)$, write $\gamma^{(\ell)} = (\gamma_{\ell+1},\gamma_{\ell+2},\ldots)$. Also, for a nonnegative integer r, define

$$p_r = p_r(t) := \frac{(t+1)^r - 1}{t} = [r]_{q=t+1},$$

where $[r]_q = \frac{q^r-1}{q-1}$ is the classical q-analog of r. Breaking up the summation formula for F_γ allows to obtain the following recurrence relations.

Lemma 3.4.1 *For $r \geq 0$, we have $G_{(r)} = t+1$ and $H_{(r)} = p_r(t+1)^2$. If $\gamma = (\gamma_1, \gamma_2, \ldots)$ has at least two parts, then*

$$G_\gamma = G_{\gamma^{(1)}} + H_{\gamma^{(1)}};$$

$$H_\gamma = p_{\gamma_1} \cdot \left((t+1)^2 + t(H_{\gamma^{(1)}} + H_{\gamma^{(2)}} + \cdots) + H_{\gamma^{(1)}} + H_{\gamma^{(3)}} + H_{\gamma^{(5)}} + \cdots\right).$$

Proof. Comparing with Theorem 3.2.1, observe that $G_\gamma = F_{\gamma^{(1)}} - 1 = G_{\gamma^{(1)}} + H_{\gamma^{(1)}}$.

Now consider the terms in the definition of H_γ. These terms correspond to $T \subseteq [n-1]$ such that $T \cap [\gamma_1] \neq \varnothing$. Write $T = \{x_1 < \cdots < x_\ell < y_1 < \cdots < y_{k-\ell}\}$, where $x_\ell \leq \gamma_1$ and $y_1 > \gamma_1$. Let $X = \{x_1 < \cdots < x_\ell\}$ and $Y = \{y_1 < \cdots < y_{k-\ell}\}$, so that $\mathbf{v}_\gamma^T = \mathbf{v}_\gamma^X \mathbf{v}_\gamma^Y$. Since $\mathbf{v}_\gamma^X$ is a positive power of $\mathbf{x}$, we have $\kappa(\mathbf{v}_\gamma^T) = \kappa(\mathbf{v}_\gamma^Y)$ if $\mathbf{v}_\gamma^Y$ begins with an $\mathbf{x}$, and $\kappa(\mathbf{v}_\gamma^T) = \kappa(\mathbf{v}_\gamma^Y) + 1$ if $\mathbf{v}_\gamma^Y$ begins with a $\mathbf{y}$ or if $Y = \varnothing$. Hence we have

$$H_\gamma = \left(\sum_{\varnothing \neq X \subseteq [\gamma_1]} t^{|X|}\right) \cdot \left(\sum_{Y \subseteq [n-1] \setminus [\gamma_1]} \left(\frac{t+1}{t}\right)^{\kappa(\mathbf{v}_\gamma^Y) + \epsilon_\gamma^Y} t^{|Y|+1}\right),$$

where ϵ_γ^Y is 0 if $\mathbf{v}_\gamma^Y$ begins with an $\mathbf{x}$, and 1 otherwise. The sum on the left is $(t+1)^{\gamma_1} - 1 = tp_{\gamma_1}$. Therefore

$$H_\gamma = tp_{\gamma_1} \cdot \left(\sum_Y \left(\frac{t+1}{t}\right)^{\kappa(\mathbf{v}_\gamma^Y)} t^{|Y|+1} + \frac{1}{t} \sum_{Y \,:\, \epsilon_\gamma^Y = 1} \left(\frac{t+1}{t}\right)^{\kappa(\mathbf{v}_\gamma^Y)} t^{|Y|+1}\right) \tag{3.7}$$

$$= tp_{\gamma_1} \cdot \left(H_{\gamma^{(1)}} + H_{\gamma^{(2)}} + \cdots + t + 1 + \frac{1}{t}\left(H_{\gamma^{(1)}} + H_{\gamma^{(3)}} + H_{\gamma^{(5)}} + \cdots + t + 1\right)\right),$$

because $H_{\gamma^{(i)}}$ is the sum of $\left((t+1)/t\right)^{\kappa(\mathbf{v}_\gamma^Y)} t^{|Y|+1}$ taken over Y with $\gamma_1 + \cdots + \gamma_i < y_1 \leq \gamma_1 + \cdots + \gamma_{i+1}$, and $\epsilon_\gamma^Y = 1$ if and only if this condition on y_1 holds for odd i (or if $Y = \varnothing$, which is accounted for by the two $t+1$ terms). $\square$

The next lemma provides a more concise recurrence relation for H_γ.

Lemma 3.4.2 *For a composition $\gamma = (\gamma_1, \gamma_2, \ldots)$ with at least two parts, the following equality holds:*

$$H_\gamma + H_{(\gamma_1, \gamma_3, \gamma_4, \ldots)} = p_{\gamma_1} \cdot \left(tG_{\gamma^{(1)}} + (t+1)(1 + G_\gamma)\right).$$

Proof. Observe that, by Lemma 3.4.1,

$$G_\gamma = G_{\gamma^{(1)}} + H_{\gamma^{(1)}} = G_{\gamma^{(2)}} + H_{\gamma^{(1)}} + H_{\gamma^{(2)}} = \cdots = t + 1 + \left(H_{\gamma^{(1)}} + H_{\gamma^{(2)}} + \cdots\right)$$

since $G_{(0)} = t+1$. Apply the relation (3.7) to H_γ and $H_{(\gamma_1,\gamma_3,\gamma_4,\ldots)}$, and then add the two resulting equations:

$$\begin{aligned} H_\gamma + H_{(\gamma_1,\gamma_3,\gamma_4,\ldots)} &= tp_{\gamma_1} \cdot \left(G_\gamma + \frac{1}{t}\left(H_{\gamma^{(1)}} + H_{\gamma^{(3)}} + H_{\gamma^{(5)}} + \cdots + t + 1 \right) \right) \\ &\quad + tp_{\gamma_1} \cdot \left(G_{\gamma^{(1)}} + \frac{1}{t}\left(H_{\gamma^{(2)}} + H_{\gamma^{(4)}} + H_{\gamma^{(6)}} + \cdots + t + 1 \right) \right) \\ &= tp_{\gamma_1} \cdot \left(G_\gamma + G_{\gamma^{(1)}} + \frac{1}{t}\left(G_\gamma + t + 1 \right) \right). \end{aligned}$$

The lemma follows. □

A useful consequence of Lemma 3.4.1 is that in working with G_γ and H_γ we can concentrate on just the compositions with the first part equal to 1. Specifically, we have the following corollary.

Corollary 3.4.3 *The polynomials G and H satisfy*

$$G_{(\gamma_1,\gamma_2,\ldots)} = G_{(1,\gamma_2,\gamma_3,\ldots)}$$

and

$$H_{(\gamma_1,\gamma_2,\ldots)} = p_{\gamma_1} H_{(1,\gamma_2,\gamma_3,\ldots)}.$$

Proof. The first identity follows from an earlier observation that G_γ is independent of the first part of γ, and the second one follows from Lemma 3.4.1 since $p_1 = 1$. □

Thus for a composition γ with k parts, we can compute F_γ by applying the recurrence relations of Lemmas 3.4.1 and 3.4.2 k times. For instance, to compute $F_{(2,4,3)}$ we proceed as follows:

$$\begin{aligned} G_{(1)} = G_{(3)} &= t+1; \\ H_{(1)} &= (t+1)^2; \\ G_{(1,3)} = G_{(4,3)} &= G_{(3)} + H_{(3)} = G_{(1)} + p_3 H_{(1)}; \\ H_{(1,3)} &= -H_{(1)} + \left(tG_{(3)} + (t+1)\left(1 + G_{(1,3)}\right)\right); \\ G_{(1,4,3)} = G_{(2,4,3)} &= G_{(4,3)} + H_{(4,3)} = G_{(1,3)} + p_4 H_{(1,3)}; \\ H_{(1,4,3)} &= -H_{(1,3)} + \left(tG_{(4,3)} + (t+1)\left(1 + G_{(1,4,3)}\right)\right), \end{aligned}$$

and finally

$$F_{(2,4,3)} = 1 + G_{(2,4,3)} + H_{(2,4,3)} = 1 + G_{(1,4,3)} + p_2 H_{(1,4,3)}.$$

The special case of the alternating pattern, corresponding to $S = \{2,4,6,\ldots\} \cap [n-1]$ or $\mathbf{v} = \mathbf{x}\mathbf{y}\mathbf{x}\mathbf{y}\cdots$ or $\gamma = 1^{n-1} = (1,1,\ldots,1) \models n-1$, is connected to Fibonacci numbers F_n as in this case the descent polytope DP_S has F_{n+1} vertices. We can obtain the generating function for $F_{1^{n-1}}(t)$ from the recurrences of this section.

From Lemmas 3.4.1 and 3.4.2 we get the relations

$$\begin{aligned} G_{1^{n-1}} &= G_{1^{n-2}} + H_{1^{n-2}}; \\ H_{1^{n-1}} + H_{1^{n-2}} &= tG_{1^{n-2}} + (t+1)\,(1 + G_{1^{n-1}}) \end{aligned}$$

for $n \geq 2$, where we put $G_{1^0} = 0$ and $H_{1^0} = t + 1$ for convenience (it can be easily seen that the relations are valid for $n = 2$). Then we multiply the above equations by x^n and sum over all $n \geq 2$ to obtain the system of equations

$$\begin{cases} G &= x(G + H); \\ H + xH &= txG + (t+1)\big(x^2(1-x)^{-1} + G\big) + (t+1)x, \end{cases}$$

where $G = G(t,x) := \sum_{n\geq 1} G_{1^{n-1}}(t)\ x^n$ and $H = H(t,x) := \sum_{n\geq 1} H_{1^{n-1}}(t)\ x^n$. Solving this system for G and H, we get

$$1 + \sum_{n\geq 1} F_{1^{n-1}}(t)\ x^n = \frac{1}{1-x} + G + H = \frac{1-(t+1)x^2}{1-(2+t)x+(1+t)x^3}. \tag{3.8}$$

We add 1 in the left hand side so that (3.8) refines the generating function

$$\sum_{n\geq 0} F_{n+1}\ x^n = \frac{1+x}{1-x-x^2}$$

for Fibonacci numbers.

Setting $t = 1$ in the polynomial $F_{1^{n-1}}(t)$ we obtain the number of faces of the descent polytope $\mathrm{DP}_{1^{n-1}}$. The sequence $\{F_{1^{n-1}}(1)\}_{n=1,2,\ldots} = 3, 7, 19, 51, \ldots$ appears to have combinatorial significance, as it matches the sequence A052948 in the Online Encyclopedia of Integer Sequences [12] defined as the number of paths from $(0,0)$ to $(n+1,0)$ with allowed steps $(1,1)$, $(1,0)$, and $(1,-1)$ contained within the region $-2 \leq y \leq 2$. The generating function

$$\frac{1-2x^2}{1-3x+2x^3}$$

given in [12] indeed results if $t = 1$ is substituted into (3.8). Is there a bijective proof?

3.5 Face lattice of the descent polytope

In this section we describe the face lattice of the polytope DP_S based on the description of its faces given in the proof of Theorem 3.2.1.

We use the following notation to identify faces of the descent polytope. We associate a face of DP_S with the pair $(T, \mathbf{w})$, where T is a subset of S, and $\mathbf{w}$ is what we call a *starred word* obtained from the word $\mathbf{v}_S^T$ by inserting stars of one of two kinds ($\underline{\star}$ and $\overline{\star}$) in some of the following places: before the first letter, between consecutive *distinct* letters, and after the last letter, – according to the rule that a star before

an $\mathbf{x}$ or after a $\mathbf{y}$ must be a $\underline{\star}$, and a star before a $\mathbf{y}$ or after a $\mathbf{x}$ must be a $\overline{\star}$. The reasoning behind this notation is this: removing certain letters of $\mathbf{v}_S$ corresponds to replacing some of the inequalities between coordinates with equalities, and placing stars $\underline{\star}$ and $\overline{\star}$ corresponds to setting certain blocks of coordinates to be equal to 0 or to 1, respectively.

The covering relations in the face lattice can be described as follows. Given a face of DP_S specified in the above way, one can go down one step in the lattice by either

(1) putting a new star in a valid position, or

(2) picking a letter not surrounded by two stars and removing that letter (together with the corresponding element of T), keeping the star if there was one next to that letter, provided that this operation results in a legitimate starred word, or

(3) picking an occurrence of the substring $\underline{\star}\mathbf{x}\mathbf{y}\underline{\star}$ or $\overline{\star}\mathbf{y}\mathbf{x}\overline{\star}$ in $\mathbf{w}$ and replacing all four characters with a single star $\underline{\star}$ or $\overline{\star}$, respectively (again, removing the two corresponding elements of T).

We must also add the minimal element to the face lattice, which should not be confused with the elements having $T = \varnothing$. The elements having $T = \varnothing$ are $\underline{\star}$, $\overline{\star}$, and 1 (empty word, no stars), corresponding to vertices $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$, and the edge connecting these two vertices.

Figure 3-1 shows the face lattice of the 3-dimensional polytope $\mathrm{DP}_{\mathbf{xy}}$, where $\mathbf{xy}$ stands for the subset $S = \{2\}$ of $[2]$. Observe that $\mathrm{DP}_{\mathbf{xy}}$ is the square based pyramid defined by the inequalities $x_1 \leq x_2 \geq x_3$ and $0 \leq x_1, x_2, x_3 \leq 1$; its base is the square $[0, 1] \times \{1\} \times [0, 1]$, and its apex is the point $(0, 0, 0)$.

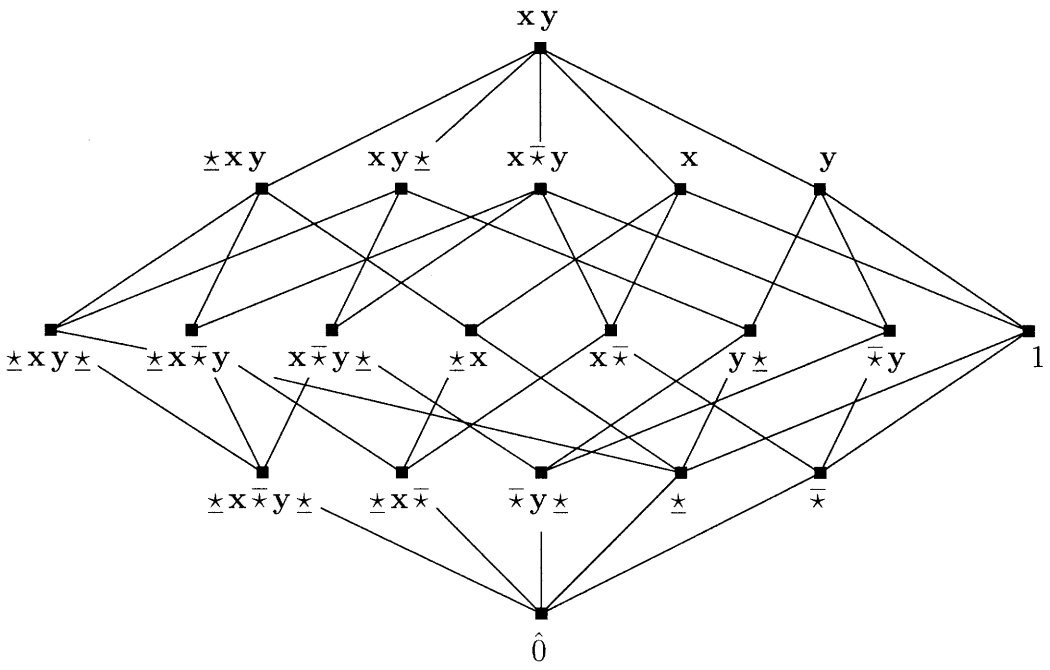


Figure 3-1: Face lattice of the descent polytope $\mathrm{DP}_{\mathbf{xy}}$

Chapter 4

Diversions and open questions

We now describe some open problems and ideas for further research on the topics treated in the other parts of this thesis. Note that some questions are mentioned in Sections 2.4 and 3.4.

4.1 Generalized chain polytope of the zigzag poset

The characterization of alternating (up-down) permutations σ in terms of inequalities satisfied by the components of $\widehat{\text{code}}(\sigma)$ given in Lemma 2.7.1 motivates the study of the following polytope.

Let $b_1, b_2, \ldots, b_n$ be positive real numbers, and let $\mathcal{Z}_n(b_1, b_2, \ldots, b_n)$ be the polytope in $\mathbb{R}^n$ defined by the equations $x_i \geq 0$ for $i \in [n]$, $x_i + x_{i+1} \leq b_i$ for $i \in [n-1]$, and $x_n \leq b_n$. Then setting $b_i = n - i$ yields a polytope of dimension $n-1$ whose integer points correspond to alternating inversion codes of up-down permutations of size $n+1$, and hence we have

$$|\mathcal{Z}_n(n-1, n-2, \ldots, 1, 0)| = E_{n+1}$$

(the number of integer points in a polytope $\mathcal{P}$ is commonly denoted by $|\mathcal{P}|$). Another important special case is that of $b_1 = b_2 = \cdots = b_n = 1$, which makes $\mathcal{Z}_n$ into the chain polytope of the n-element zigzag poset $Z_{\{2,4,6,\ldots\}}$ whose elements obey the relations $z_1 < z_2 > z_3 < z_4 < \cdots$. As mentioned in Section 1.1.5, the volume of this polytope is $1/n!$ times the number of linear extensions of the zigzag poset, that is, the number E_n of alternating permutations of size n. The integer points of $\mathcal{Z}_n(1, 1, \ldots, 1)$ are the $(0,1)$-sequences of length n with no consecutive 1's, and the number of such sequences is well known to be the $(n+1)$-th Fibonacci number F_{n+1}. The polytope $\mathcal{Z}_n(b_1, b_2, \ldots, b_n)$ is a fairly natural generalization of the aforementioned chain polytope, thus it seems reasonable to call it the *generalized chain polytope* of the zigzag poset.

In the case $b_1 \geq b_2 \geq \cdots \geq b_n$ the volume of $\mathcal{Z}_n(b_1, b_2, \ldots, b_n)$ has a nice combinatorial expression. Let K_n denote the set of *weak* compositions $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ of n (that is, parts equal to 0 are allowed) satisfying $\gamma_1 + \gamma_2 + \cdots + \gamma_i \leq i$.

Theorem 4.1.1 *Suppose $b_1 \geq b_2 \geq \cdots \geq b_n$. Then the volume of $\mathcal{Z}_n(b_1, b_2, \ldots, b_n)$ is given by*

$$\frac{1}{n!} \cdot \sum_{\gamma \in \mathrm{K}_n} (-1)^{\gamma_2+\gamma_4+\gamma_6+\cdots} \cdot \binom{n}{\gamma} \cdot b_1^{\gamma_1} b_2^{\gamma_2} \cdots b_n^{\gamma_n},$$

where $\binom{n}{\gamma} = \frac{n!}{\gamma_1!\gamma_2!\cdots\gamma_n!}$.

Proof. Let

$$V_n(b_1, b_2, \ldots, b_n) := \frac{1}{n!} \cdot \sum_{\gamma \in \mathrm{K}_n} (-1)^{\gamma_2+\gamma_4+\gamma_6+\cdots} \cdot \binom{n}{\gamma} \cdot b_1^{\gamma_1} b_2^{\gamma_2} \cdots b_n^{\gamma_n}. \tag{4.1}$$

The volume of $\mathcal{Z}_n(b_1, b_2, \ldots, b_n)$ is given by

$$\mathrm{Vol}(\mathcal{Z}_n(b_1, b_2, \ldots, b_n)) = \int_0^{b_n} \int_0^{b_{n-1}-x_n} \int_0^{b_{n-2}-x_{n-1}} \cdots \int_0^{b_1-x_2} dx_1\, dx_2\, \cdots\, dx_n.$$

The assumption that $b_1 \geq b_2 \geq \cdots \geq b_n$ ensures that the upper limits of the integrals are non-negative. Let $J_k(b_1, b_2, \ldots, b_k; x_{k+1})$ be the evaluation of the k inside integrals in the above equation. Set $x_{n+1} = 0$ for convenience. We claim that for $1 \leq k \leq n$, we have

$$J_k(b_1, b_2, \ldots, b_k; x_{k+1}) =$$
$$= \frac{(-1)^{\lfloor k/2 \rfloor}}{k!} \cdot \sum_{\substack{\gamma_1+\cdots+\gamma_{k+1}=k \\ \gamma_1+\cdots+\gamma_i \leq i}} (-1)^{\gamma_2+\gamma_4+\gamma_6+\cdots} \cdot \binom{k}{\gamma} \cdot b_1^{\gamma_1} b_2^{\gamma_2} \cdots b_k^{\gamma_k} x_{k+1}^{\gamma_{k+1}} \tag{4.2}$$

We prove (4.2) by induction on k. In the base case $k = 1$, we have $J_1 = b_1 - x_2$, and the terms in the defining expression (4.1) for $V_2(b_1, x_2)$ corresponding to the two members $(1, 1)$ and $(0, 2)$ of K_2 are $b_1 x_2$ and $-x_2^2$. Suppose the claim is true for $k-1$. Then

$$J_k(b_1, b_2, \ldots, b_k; x_{k+1}) = \int_0^{b_k - x_{k+1}} J_{k-1}(b_1, b_2, \ldots, b_{k-1}; x_k)\, dx_k =$$
$$= (-1)^{\lfloor \frac{k-1}{2} \rfloor} \cdot \sum_{\substack{\gamma_1+\cdots+\gamma_k=k-1 \\ \gamma_1+\cdots+\gamma_i \leq i}} (-1)^{\gamma_2+\gamma_4+\cdots} \cdot \frac{b_1^{\gamma_1} b_2^{\gamma_2} \cdots b_{k-1}^{\gamma_{k-1}}}{\gamma_1!\gamma_2! \cdots \gamma_{k-1}!} \cdot \frac{(b_k - x_{k+1})^{\gamma_k+1}}{(\gamma_k+1)!}.$$

Expanding $(b_k - x_{k+1})^{\gamma_k+1}$ in the right hand side by the binomial theorem, we get that $J_k(b_1, b_2, \ldots, b_k; x_{k+1})$ equals

$$(-1)^{\lfloor \frac{k-1}{2} \rfloor} \cdot \sum_{\substack{\gamma_1+\cdots+\gamma_{k-1}+r+s=k \\ \gamma_1+\cdots+\gamma_i \leq i}} (-1)^{\gamma_2+\gamma_4+\cdots} \cdot \frac{b_1^{\gamma_1} b_2^{\gamma_2} \cdots b_{k-1}^{\gamma_{k-1}}}{\gamma_1!\gamma_2! \cdots \gamma_{k-1}!} \cdot \frac{(-1)^s b_k^r x_{k+1}^s}{r!\, s!}.$$

To complete the inductive step it remains to check that the signs work out. For odd k, the factor $(-1)^s$ inside the sum enters $(-1)^{\gamma_2+\gamma_4+\cdots}$ because s is the $(k+1)$-th part of the newly obtained composition of k, and outside the sum, the sign factor stays the

same as $\lfloor (k-1)/2 \rfloor = \lfloor k/2 \rfloor$. For even k, the factor $(-1)^{\gamma_2+\gamma_4+\cdots}$ includes $(-1)^{\gamma_k} = (-1)^{r+s-1}$, which, combined with the other $(-1)^s$, gives $(-1)^{r-1}$. Entering $(-1)^r$ in the factor $(-1)^{\gamma_2+\gamma_4+\cdots}$, we are left with an extra -1 to make the $(-1)^{\lfloor (k-1)/2 \rfloor}$ on the outside into $(-1)^{\lfloor k/2 \rfloor}$.

Now set $k = n$ so that $x_{k+1} = 0$; then the non-vanishing terms of (4.2) are the ones corresponding to compositions with $\gamma_{k+1} = 0$. The theorem follows. □

With the exception of the sign factor and the direction of the inequality in the definition of K_n, the above formula is exactly the formula for the volume of the *parking function polytope* introduced by Pitman and Stanley [14]. This polytope, which we denote $\Pi_n(b_1, b_2, \ldots, b_n)$, is defined by the relations $x_i \geq 0$ and $x_1 + x_2 + \cdots + x_i \leq b_1 + b_2 + \cdots + b_i$ for $i \in [n]$. The volume-preserving change of coordinates $y_i = b_n + b_{n-1} + \cdots + b_{n+1-i} - (x_1 + x_2 + \cdots + x_i)$ transforms the defining relations into $y_i \geq 0$ for $i \in [n]$, $y_1 \leq b_1$, and $y_i - y_{i+1} \geq b_i$ for $i \in [n-1]$, and these new relations look much like the ones defining $\mathcal{Z}_n(b_1, b_2, \ldots, b_n)$: in essence we have here a difference instead of a sum. This similarity somewhat explains the close resemblance of the volume formulas for the two polytopes.

As we mentioned above, the volume of $\Pi_n(b_1, b_2, \ldots, b_n)$ can be written as

$$\mathrm{Vol}(\Pi_n(b_1, b_2, \ldots, b_n)) = V_n(b_n, -b_{n-1}, b_{n-2}, -b_{n-3}, \ldots, (-1)^{n-1} b_1).$$

The name of the polytope $\Pi_n(b_1, b_2, \ldots, b_n)$ comes from the fact that the above volume formula, when viewed as a polynomial in the indeterminates b_1, b_2, ..., b_n, is a generating function enumerating parking functions of size n by content. Let us give the relevant definitions. A *parking function* of size n is a sequence $(a_1, a_2, \ldots, a_n)$ of positive integers such that for each $i \in [n]$, the number of elements of the sequence not exceeding i is at least i. The *content* of a parking function is the sequence $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ of non-negative integers where β_i is the number of elements of the parking function equal to i. Thus the content sequence satisfies $\beta_1 + \beta_2 + \cdots + \beta_i \geq i$ for $i \in [n]$, which is equivalent to $\beta_n + \beta_{n-1} + \cdots + \beta_{n+1-i} \leq i$ for $i \in [n]$, that is, to $(\beta_n, \beta_{n-1}, \ldots, \beta_1) \in \mathrm{K}_n$. The number of parking functions of size n with content β is known to be $\binom{n}{\beta}$. The total number of parking functions of size n is even better known to be $(n+1)^{n-1}$, and thus setting $b_i = 1$ for all i yields the identity

$$\sum_{\beta \in \mathrm{K}_n} \binom{n}{\beta} = (n+1)^{n+1}$$

(reversing the order of the elements of β does not affect $\binom{n}{\beta}$).

A more general specialization $\mathrm{Vol}(\Pi_n(1, q, q^2, \ldots, q^{n-1}))$ is closely related to a notable combinatorial polynomial $I_n(q)$ known as the *inversion enumerator* of labeled rooted trees. Let $\mathcal{T}_n$ denote the set of all trees on vertices labeled 0, 1, 2, ..., n, where vertex 0 is considered to be the root, thus defining ancestor-descendant relations between some pairs of vertices. (Here we make no distinction on how the children of a particular vertex are ordered.) We say that two vertices i and j of such a tree form an *invesion* if $i > j$ and j is a descendant of i, that is, i lies on the unique path in the

tree from j to the root 0. Let $\mathrm{inv}(T)$ be the number of inversions in a tree T. Then the inversion enumerator is defined by

$$I_n(q) := \sum_{T \in \mathcal{T}_n} q^{\mathrm{inv}(T)}.$$

Cayley's formula states that $|\mathcal{T}_n| = (n+1)^{n-1}$, so that trees on $n+1$ labeled vertices are equinumerous with parking functions of size n. A classical bijection by Kreweras [9] establishes a correspondence between trees in $\mathcal{T}_n$ with $\binom{n}{2} - k$ inversions and parking functions of size n whose components add up to $k+n$, and hence we have

$$\begin{aligned} q^{\binom{n}{2}} \cdot I_n(q^{-1}) &= \sum_{T \in \mathcal{T}_n} q^{\binom{n}{2} - \mathrm{inv}(T)} \\ &= \sum_{\gamma^r \in \mathrm{K}_n} \binom{n}{\gamma} \cdot q^{\gamma_1 + 2\gamma_2 + \cdots + n\gamma_n - n} \\ &= \sum_{\gamma^r \in \mathrm{K}_n} \binom{n}{\gamma} \cdot q^{\gamma_2 + 2\gamma_3 + \cdots + (n-1)\gamma_n} \\ &= n! \cdot \mathrm{Vol}(\Pi_n(1, q, q^2, \ldots, q^{n-1})) \\ &= n! \cdot V_n(q^{n-1}, -q^{n-2}, q^{n-3}, -q^{n-4}, \ldots, (-1)^{n-1}), \end{aligned} \tag{4.3}$$

where γ^r denotes the composition γ with the order of the parts reversed.

Setting $b_i = 1$ for all i in the formula for the volume of $\mathcal{Z}_n(b_1, b_2, \ldots, b_n)$, we get the normalized volume of the chain polytope of the zigzag poset:

$$\sum_{\gamma \in \mathrm{K}_n} (-1)^{\gamma_2 + \gamma_4 + \gamma_6 + \cdots} \binom{n}{\gamma} = E_n.$$

Up to sign, we get the same value by plugging in $q = -1$ into (4.3). After careful analysis of signs, we arrive at a remarkable formula

$$I_n(-1) = E_n.$$

This formula can be proved analytically from generating functions (see [6, Exercise 3.3.49(d)] or [9]), and a combinatorial proof is advertised in [13].

In addition to an elegant volume formula, the Pitman-Stanley parking function polytope has an equally elegant expression for its Ehrhart polynomial. (Recall that the Ehrhart polynomial evaluated at a non-negative integer k equals the number of integer points in the dilation of the polytope by a factor of k.) Let us assume that b_1, b_2, ..., b_n are positive integers. Since the dilation of $\Pi_n(b_1, b_2, \ldots, b_n)$ by a factor of k is another parking function polytope $\Pi_n(kb_1, kb_2, \ldots, kb_n)$, the Ehrhart polynomial of $\Pi_n(b_1, b_2, \ldots, b_n)$ is obtained automatically from an expression for the number of integer points in $\Pi_n(b_1, b_2, \ldots, b_n)$. In [14] Pitman and Stanley give such

an expression:

$$|\Pi_n(b_1, b_2, \ldots, b_n)| = \sum_{\gamma^r \in \mathrm{K}_n} \left(\!\!\binom{b_1+1}{\gamma_1}\!\!\right) \prod_{i=2}^{n} \left(\!\!\binom{b_i}{\gamma_i}\!\!\right)$$

where $\left(\!\!\binom{x}{\ell}\!\!\right) = \frac{x(x+1)(x+2)\cdots(x+\ell-1)}{\ell!}$.

The above formula can be thought of as the formula

$$\mathrm{Vol}(\Pi_n(b_1, b_2, \ldots, b_n)) = \frac{1}{n!} \cdot \sum_{\gamma^r \in \mathrm{K}_n} \binom{n}{\gamma} \cdot b_1^{\gamma_1} b_2^{\gamma_2} \cdots b_n^{\gamma_n}$$

with $b_i^{\gamma_i}$ replaced with the "raising power" $b_i(b_i+1)\cdots(b_i+\gamma_i-1)$ (or, for $i=1$, with $(b_1+1)(b_1+2)\cdots(b_1+\gamma_1)$). One could ask if the number of integer points in $\mathcal{Z}_n(b_1, b_2, \ldots, b_n)$, and hence the Ehrhart polynomial, could be obtained from the volume formula of Theorem 4.1.1 in a similar way.

Another natural direction for further work is to investigate generalized chain polytopes of other posets. After the zigzag poset, a logical class of posets to consider next are the ribbon posets Z_S. Defining $\mathcal{Z}_S(b_1, b_2, \ldots)$ for Z_S in the same way $\mathcal{Z}_n$ is defined for $Z_{\{2,4,6,\ldots\}}$, we get a polynomial in b_1, b_2,..., whose value at $b_1 = b_2 = \cdots = 1$ is $1/n!$ times the descent number $\beta_n(S)$, resulting in an interesting combinatorial identity. Is there a way to relate this polytope to objects similar to parking functions or labeled trees?

4.2 Shapiro-Woan-Getu permutations

In this section we take a closer look at the class of permutations which we denoted by $\mathcal{R}_n$ in Section 2.6. Recall that $\mathcal{R}_n$ is the set of permutations with no consecutive (double) descents and no descent at the end. They appear in the paper [16] by Shapiro, Woan, and Getu, hence the section title, who call them *reduced* permutations. The paper studies enumeration of permutations by the number of runs or slides, and in [15, Sec. 11.1] Postnikov, Reiner, and Williams put these results in the context of structural properties of permutohedra: for instance, the polynomial encoding the distribution of permutations in $\mathcal{R}_n$ by the number of descents is the γ-polynomial of the classical permutohedron.

In Section 2.6, we found the number R_n of SWG permutations of size n to be the sum of absolute values of coefficients of a $(\mathbf{c}, \mathbf{d})$-polynomial that, when expanded in terms of $\mathbf{a}$ and $\mathbf{b}$, gave the generating function for the alternating descent set statistic. Shapiro, Woan, and Getu provide a generating function for R_n:

$$R(x) := \sum_{n \geq 0} R_n \cdot \frac{x^n}{n!} = 1 + \frac{2\tan(x\sqrt{3}/2)}{\sqrt{3} - \tan(x\sqrt{3}/2)}$$

(we put $R_0 = 1$). Observe that $R(x)R(-x) = 1$, a property that $R(x)$ shares with e^x and $\tan x + \sec x$, which are the two fundamental generating functions in the analisys

done in Chapter 2. There is a further resemblance with the Euler numbers E_n if one looks at the logarithm of $R(x)$:

$$\ln(R(x)) = -x + 2\sum_{n\geq 0} R_{2n} \cdot \frac{x^{2n+1}}{(2n+1)!}. \tag{4.4}$$

Comparing with

$$\ln(\tan x + \sec x) = \sum_{n\geq 0} E_{2n} \cdot \frac{x^{2n+1}}{(2n+1)!}, \tag{4.5}$$

we see that taking the logarithm has a similar effect on both $R(x)$ and $\tan x + \sec x$ of taking the even part and integrating, except that for $R(x)$ all coefficients excluding that of x are doubled.

The fact that

$$\int \sec x \; dx = \ln(\tan x + \sec x) \tag{4.6}$$

(omitting the arbitrary constant of integration) has been used in the proof of Theorem 2.3.3. This textbook integral formula can be proved combinatorially using the exponential formula for generating functions (see [20, Sec. 5.1]). Given an up-down permutation σ, divide σ into blocks by the following procedure. Put the subword of σ starting at the beginning of σ and ending at the element equal to 1 in the first block, and remove this block from σ. In the resulting word, find the *maximum* element m_2 and put the subword consisting of initial elements of the word up to, and including, m_2 in the second block, and remove the second block. In the remaining word, find the *minimum* element m_3, and repeat until there is nothing left, alternating between cutting at the minimum and at the maximum element of the current word. For example, for $\sigma = 593418672$, the blocks would be 59341, 8, and 672. Note that given the blocks one can uniquely recover the order in which they must be concatenated to form the original permutation σ. Indeed, the first block is the one containing 1, the second block contains the largest element not in the the first block, the third block contains the smallest element not in the first two blocks, and so on. Thus to construct an up-down permutation of size n we need to divide the elements of $[n]$ into blocks of *odd* size, then determine the order of concatenation using the above principle, and then arrange the elements of odd numbered blocks in up-down order and those of even numbered blocks in down-up order. There are E_{k-1} ways to arrange the elements in a block of size k for odd k, and 0 ways for even k since we do not allow blocks of even size. Thus (4.5), which is equivalent to (4.6), follows from the exponential formula. This argument "combinatorializes" the proof of Theorem 2.3.3. It would be nice to give a similar argument for reduced permutations $\mathcal{R}_n$.

Problem 4.2.1 *Find a combinatorial proof of the formula (4.4) for* $\ln(R(x))$.

Another problem emerging from the results of Section 2.6 is the following.

Problem 4.2.2 *Give a combinatorial interpretation of the coefficients of the polynomial* $\hat{\Phi}_n(\mathbf{c}, -\mathbf{d})$ *by partitioning the set* $\mathcal{R}_n$ *into classes corresponding to the* F_{n-1} *monomials.*

It is worth pointing out here that even though one can split $\mathcal{R}_n$ into F_{n-1} classes corresponding to $(\mathbf{c}, \mathbf{d})$-monomials by descent set, like it was done for simsun permutations in Section 2.6, the resulting polynomial is different from $\hat{\Phi}_n(\mathbf{c}, -\mathbf{d})$. There are a few hints on what the correct way to refine permutations in $\mathcal{R}_n$ could be. The coefficient of $\mathbf{c}^{n-1}$ in $\hat{\Phi}_n(\mathbf{c}, -\mathbf{d})$ is the Euler number E_n, and the set $\mathcal{R}_n$ includes at least three kinds of permutations mentioned in this thesis that are counted by E_n: alternating permutations ending with an ascent, simsun permutations, and permutations $\sigma \in \mathfrak{S}_n$ such that $\sigma \circ (n+1)$ has no 3-descents. Values of $\hat{\Phi}_n$ listed in Section A.1 present evidence that the common coefficient of $\mathbf{c}^{n-3}\,\mathbf{d}$ and $\mathbf{d}\,\mathbf{c}^{n-3}$ is $\hat{A}(n-1, 2)$ (the number of permutations of size $n-1$ with exactly one alternating descent).

4.3 Circular posets

In this section we convey some thoughts on why 3-descents as a generalization of ordinary descents is a potentially fruitful subject involving quite natural combinatorial concepts.

In the proof of Theorem 2.1.3 the 3-descent set of a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ is given the following interpretation: if the elements $\sigma_1, \sigma_2, \ldots,$ are written in clockwise order on a circle, then the set $D_3(\sigma)$ consists of all those indices $1 \le i \le n-2$ such that the triple $\sigma_i, \sigma_{i+1}, \sigma_{i+2}$ is oriented counterclockwise. If we write the elements of σ^{-1} on the circle instead of those of σ, we obtain a correspondence between permutations in $\sigma \in \tilde{\mathfrak{S}}_n$ with $D_3(\sigma) = S$ and arrangements of the numbers $1, 2, \ldots, n$ on a circle (with no distinction of the starting position) such that the triple $i, i+1, i+2$ is oriented clockwise if and only if $i \notin S$. Thus we can single out such arrangements from the total of $(n-1)!$ arrangements by prescribing the orientation (clockwise or counterclockwise) of every triple $i, i+1, i+2$ for $i \in [n-2]$, leaving the orientation of every other triple unspecified.

This viewpoint invokes a parallel with partially ordered sets, in which "orientation" is prescribed for pairs, rather than triples, of elements. To give a formal definition of this poset-like concept, note that a *finite* poset in the traditional sense can be defined as a collection of ordered pairs $x < y$ of elements of a base set such that there exists a linear extension – a way to *linearly* order the elements so that if $x < y$ is in the collection, then x appears before y. Now consider the following concept suggested by Pavlo Pylyavskyy (private communication).

Definition 4.3.1 *A* circular poset Q *on an underlying set* X *is a collection* $R(Q)$ *of ordered triples of distinct elements of* X *such that:*

(I) Q *has a* circular extension *– a way to arrange the elements of* X *on a circle so that if a triple* (x, y, z) *is in* $R(Q)$*, then it is oriented clockwise on the circle;*

(II) if an ordered triple (x, y, z) *is oriented clockwise in every circular extension of* Q*, then* $(x, y, z) \in R(Q)$*.*

For the rest of the section, we assume the base sets of ordinary and circular posets in question to be finite unless stated otherwise.

Let us call $R(Q)$ the *set of relations* of Q. The following properties are apparent from the definition.

Lemma 4.3.2 *Let Q be a circular poset.*

(a) If $(x,y,z) \in R(Q)$, then (y,z,x) and (z,x,y) are in $R(Q)$, whereas (x,z,y), (z,y,x), and (y,x,z) are not.

(b) If $(x,y,z) \in R(Q)$ and $(z,t,x) \in R(Q)$, then (y,z,t) and (t,x,y) are in $R(Q)$.

In view of part (a) of Lemma 4.3.2, the cyclic shifts (x,y,z), (y,z,x), and (z,x,y) are the same triple for the purposes of $R(Q)$. To shorten notation and to make the statements of the lemma more apprehensible, let $((x,y,z))$ denote the "cyclic" triple meaning any of these three cyclic shifts. We can extend the notion of a cyclic triple to a cyclic k-tuple, and write $((x_1,x_2,\ldots,x_k)) \in R(Q)$ to mean that every cyclic subtriple $((x_i,x_j,x_\ell))$, where $i<j<\ell$, is in $R(Q)$. (That is, in a circular extension of Q, the entire k-tuple $((x_1,x_2,\ldots,x_k))$ appears in the given order in the clockwise direction.) Then Lemma 4.3.2 can be rewritten as follows.

Lemma 4.3.3 *Let Q be a circular poset.*

(A) If $((x,y,z)) \in R(Q)$, then $((x,z,y)) \notin R(Q)$.

(B) If $((x,y,z)) \in R(Q)$ and $((z,t,x)) \in R(Q)$, then $((x,y,z,t)) \in R(Q)$.

If one defines ordinary (finite) posets by analogy with Definition 4.3.1, then the existence of a linear extension implies the usual poset axioms stating that $x \not< x$, that $x<y$ implies $y \not< x$, and that $x<y$ and $y<z$ together imply $x<z$. It is easy to see that if these axioms hold, then there exists a linear extension. Moreover, for every pair (x,y) of elements that are not forced to be comparable by the axioms, there exists a linear extension in which x appears before y and one in which x appears after y. It is a very interesting question whether the same is true for circular posets.

Question 4.3.4 *Do the conditions of Lemma 4.3.3 characterize circular posets completely? In other words, if R is a collection of cyclic triples of an underlying set X satisfying the conditions of Lemma 4.3.3,*

(1) does R always have a valid circular extension?

(2) does R always include every cyclic triple that is oriented clockwise in every valid circular extension?

Even if the above dilemma with the proper set of axioms is not resolved, it still makes sense to think about examples of circular posets. There are trivial examples, such as an "antichain", for which $R(Q)=\varnothing$. There is also a "total order" in the case when the underlying set X is countable, which can be defined by presenting a map $\alpha : X \to \mathbb{Q}$ and prescribing a triple $((x,y,z))$ to be in $R(Q)$ whenever $\alpha(x)<\alpha(y)<\alpha(z)$. (In the case of finite X we can define the total order simply by $((x_1,x_2,\ldots,x_n)) \in R(Q)$, where $\{x_1,x_2,\ldots,x_n\}=X$.)

Note that an ordinary poset P can be made into a circular poset P° by adding a new element O to the underlying set of P and defining $R(P^\circ)$ to be the collection of triples

$$\Big\{(\!(\mathrm{O}, x, y)\!) \mid x, y \in P \text{ and } x < y\Big\}.$$

It is not hard to see that P° is indeed a circular poset, for which a circular extension is a linear extension of P with O acting as "the point of infinity" that makes the line into a circle.

When describing a circular poset it is probably unnecessary to prove that the presented set of relations satisfies condition (II) of Definition 4.3.1. Instead, it suffices to show that these relations allow a circular extension (condition (I)), and then let $R(Q)$ be the closure of the presented set of relations R under the implications forced by condition (II). Let us write $R(Q) \leftarrow R$ to denote that $R(Q)$ was obtained from R in this way.

The discussion at the beginning of the section essentially provides a circular analog of the ribbon poset Z_S. Let the underlying set be $[n]$, fix a subset $S \subseteq [n-2]$, and put

$$R_S = \Big\{(\!(i, i+1, i+2)\!) \mid i \notin S\Big\} \cup \Big\{(\!(i, i+2, i+1)\!) \mid i \in S\Big\},$$

where $1 \leq i \leq n-2$. Then a permutation $\sigma \in \tilde{\mathfrak{S}}_n$ is a circular extension of R_S if and only if $D_3(\sigma^{-1}) = S$. Hence R_S satisfies condition (I) of Definition 4.3.1, and we can define the circular poset O_S by $R(O_S) \leftarrow R_S$. It is an interesting problem to answer Question 4.3.4 for $R = R(O_S)$.

Many questions emerge from attempts to further develop the theory of circular posets along the lines of the theory of ordinary posets. One such question is how to properly define a circular counterpart of a chain. On one hand, a chain in an ordinary poset is a totally ordered subset, which prompts to adopt the same definition for circular posets. On the other hand, if we regard a chain in an ordinary poset as a sequence $x_1, x_2, \ldots$ of elements satisfying $x_i < x_{i+1}$ for all i, then the natural circular analog would be a sequence $x_1, x_2, \ldots, x_k$ such that $(\!(x_i, x_{i+1}, x_{i+2})\!) \in R(Q)$ for all $1 \leq i \leq k-2$, which in general is different from a totally ordered subset. To differentiate between the two definitions, let us call the latter version a *weak chain* and a totally ordered subset of a circular poset a *strong chain*.

By Corollary 2.1.4, the number of circular extensions of a weak chain with k elements is E_{k-1}, if there are no extra relations besides $(\!(x_i, x_{i+1}, x_{i+2})\!)$. This interpretation of the corollary suggests that there could be a connection between circular posets and the results of Chapter 2, where there were E_n ways of arranging n objects in "proper" order instead of a trivial 1. Another occurrence of Euler numbers replacing the sequence of all 1's is in the polynomials $\hat{\Phi}_n$, where the coefficient of $\mathbf{c}^{n-1}$ is E_n, excluding the possibility of $\hat{\Phi}_n(\mathbf{c}, -\mathbf{d})$ being the *cd*-index of a polytope or a poset. Is there a way to relate $\hat{\Phi}_n(\mathbf{c}, -\mathbf{d})$ to a circular poset instead?

Another class of circular posets that should not be too hard to analyze is a natural generalization of *permutation posets* P_σ. For a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, define the poset P_σ on the set $[n]$ by the relations $i <_P j$ whenever $i < j$ and i appears before j in σ. By analogy, for a permutation $\sigma \in \mathfrak{S}_n$, define the circular poset O_σ

on $[n]$ from the set of relations containing the triple $((i,j,k))$ whenever $i<j<k$ and i, j, and k appear in clockwise order when all the elements σ are written in clockwise order on a circle. As is the case with P_σ, (strong) chains of O_σ become antichains and vice versa if the permutation σ is replaced with $\sigma^r = \sigma_n \cdots \sigma_2 \sigma_1$.

For any finite poset P, the Greene-Kleitman theorem asserts that there exists a partition $\lambda(P) = (\lambda_1 \geq \lambda_2 \geq \cdots)$ such that for all i, the largest number of elements in a union of i chains of P is $\lambda_1 + \lambda_2 + \cdots + \lambda_i$, and that the partition $\mu(P)$ defined in the same way for antichains is in fact the conjugate partition $\lambda'(P)$. In the case of permutation posets, $\lambda(P_\sigma)$ is the common shape of the two standard Young tableaux produced by the RSK algorithm (see [20, Sec. 7.11]). It would be very nice to have a similar theory for circular posets, or at least the ones of the form O_σ. For example, it is well known that for a poset P on n elements whose longest chain and largest antichain have $\ell(P)$ and $a(P)$ elements, respectively, the inequality

$$\ell(P)a(P) \geq n$$

holds. Defining $\ell(O_\sigma)$ and $a(O_\sigma)$ for $\sigma \in \mathfrak{S}_n$ analogously, the above result implies

$$\big(\ell(O_\sigma)-1\big)\big(a(O_\sigma)-1\big) \geq n-1,$$

and it can be shown that this inequality is sharp. As for the partitions $\lambda(O_\sigma)$ and $\mu(O_\sigma)$ defined in terms of unions of strong chains and unions of antichains, computer experiments show that $\lambda(O_\sigma)$ and $\mu(O_\sigma)$ exist but do not determine each other uniquely. To prove that these partitions exist, that is, that $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\mu_1 \geq \mu_2 \geq \cdots$ in the case of circular permutation posets seems to be a non-trivial problem. And, of course, there is a much less trivial problem of generalizing to arbitrary circular posets.

Appendix A

Computational results

A.1 The *cd*-index of the Boolean algebra and the polynomials $\hat{\Phi}_n(\mathbf{c}, \mathbf{d})$

The following tables show the coefficients of the *cd*-index $\Phi_n(\mathbf{c}, \mathbf{d})$ of the Boolean algebra B_n and the polynomials $\hat{\Phi}_n(\mathbf{c}, \mathbf{d}) = \Phi_n(\mathbf{c},\ \mathbf{c}^2 - \mathbf{d})$ for $n \leq 9$. The notation $[w]P$ is used to denote the coefficient of w in P. Note that a word w and the word w^r obtained by reading w backwards have the same coefficient in the polynomials in question, thus many entries in the columns labeled "w" actually read "$w + w^r$".

n	w	$[w]\Phi_n$	$[w]\hat{\Phi}_n$
1	1	1	1
2	$\mathbf{c}$	1	1
3	$\mathbf{c}^2$	1	2
	$\mathbf{d}$	1	-1
4	$\mathbf{c}^3$	1	5
	$\mathbf{c}\,\mathbf{d} + \mathbf{d}\,\mathbf{c}$	2	-2
5	$\mathbf{c}^4$	1	16
	$\mathbf{c}^2\,\mathbf{d} + \mathbf{d}\,\mathbf{c}^2$	3	-7
	$\mathbf{c}\,\mathbf{d}\,\mathbf{c}$	5	-5
	$\mathbf{d}^2$	4	4

Table A.1: Coefficients of the polynomials $\Phi_n(\mathbf{c}, \mathbf{d})$ and $\hat{\Phi}_n(\mathbf{c}, \mathbf{d})$ for $n \leq 5$.

n	w	$[w]\Phi_n$	$[w]\hat{\Phi}_n$
6	$\mathbf{c}^5$	1	61
	$\mathbf{c}^3\mathbf{d}+\mathbf{d}\mathbf{c}^3$	4	-26
	$\mathbf{c}^2\mathbf{d}\mathbf{c}+\mathbf{c}\mathbf{d}\mathbf{c}^2$	9	-21
	$\mathbf{c}\mathbf{d}^2+\mathbf{d}^2\mathbf{c}$	12	12
	$\mathbf{d}\mathbf{c}\mathbf{d}$	10	10
7	$\mathbf{c}^6$	1	272
	$\mathbf{c}^4\mathbf{d}+\mathbf{d}\mathbf{c}^4$	5	-117
	$\mathbf{c}^3\mathbf{d}\mathbf{c}+\mathbf{c}\mathbf{d}\mathbf{c}^3$	14	-91
	$\mathbf{c}^2\mathbf{d}\mathbf{c}^2$	19	-103
	$\mathbf{c}^2\mathbf{d}^2+\mathbf{d}^2\mathbf{c}^2$	25	59
	$\mathbf{c}\mathbf{d}\mathbf{c}\mathbf{d}+\mathbf{d}\mathbf{c}\mathbf{d}\mathbf{c}$	35	35
	$\mathbf{c}\mathbf{d}^2\mathbf{c}$	42	42
	$\mathbf{d}\mathbf{c}^2\mathbf{d}$	18	52
	$\mathbf{d}^3$	34	-34
8	$\mathbf{c}^7$	1	1385
	$\mathbf{c}^5\mathbf{d}+\mathbf{d}\mathbf{c}^5$	6	-594
	$\mathbf{c}^4\mathbf{d}\mathbf{c}+\mathbf{c}\mathbf{d}\mathbf{c}^4$	20	-468
	$\mathbf{c}^3\mathbf{d}\mathbf{c}^2+\mathbf{c}^2\mathbf{d}\mathbf{c}^3$	34	-510
	$\mathbf{c}^3\mathbf{d}^2+\mathbf{d}^2\mathbf{c}^3$	44	292
	$\mathbf{c}^2\mathbf{d}\mathbf{c}\mathbf{d}+\mathbf{d}\mathbf{c}\mathbf{d}\mathbf{c}^2$	84	196
	$\mathbf{c}^2\mathbf{d}^2\mathbf{c}+\mathbf{c}\mathbf{d}^2\mathbf{c}^2$	100	236
	$\mathbf{c}\mathbf{d}\mathbf{c}^2\mathbf{d}+\mathbf{d}\mathbf{c}^2\mathbf{d}\mathbf{c}$	72	208
	$\mathbf{c}\mathbf{d}\mathbf{c}\mathbf{d}\mathbf{c}$	140	140
	$\mathbf{d}\mathbf{c}^3\mathbf{d}$	28	252
	$\mathbf{c}\mathbf{d}^3+\mathbf{d}^3\mathbf{c}$	136	-136
	$\mathbf{d}\mathbf{c}\mathbf{d}^2+\mathbf{d}^2\mathbf{c}\mathbf{d}$	112	-112

Table A.2: Coefficients of the polynomials $\Phi_n(\mathbf{c},\mathbf{d})$ and $\hat{\Phi}_n(\mathbf{c},\mathbf{d})$ for $n = 6, 7, 8$.

n	w	$[w]\Phi_n$	$[w]\hat{\Phi}_n$
9	$\mathbf{c}^8$	1	7936
	$\mathbf{c}^6\,\mathbf{d}+\mathbf{d}\,\mathbf{c}^6$	7	−3407
	$\mathbf{c}^5\,\mathbf{d}\,\mathbf{c}+\mathbf{c}\,\mathbf{d}\,\mathbf{c}^5$	27	−2673
	$\mathbf{c}^4\,\mathbf{d}\,\mathbf{c}^2+\mathbf{c}^2\,\mathbf{d}\,\mathbf{c}^4$	55	−2951
	$\mathbf{c}^3\,\mathbf{d}\,\mathbf{c}^3$	69	−2841
	$\mathbf{c}^4\,\mathbf{d}^2+\mathbf{d}^2\,\mathbf{c}^4$	70	1690
	$\mathbf{c}^3\,\mathbf{d}\,\mathbf{c}\,\mathbf{d}+\mathbf{d}\,\mathbf{c}\,\mathbf{d}\,\mathbf{c}^3$	168	1092
	$\mathbf{c}^3\,\mathbf{d}^2\,\mathbf{c}+\mathbf{c}\,\mathbf{d}^2\,\mathbf{c}^3$	198	1314
	$\mathbf{c}^2\,\mathbf{d}\,\mathbf{c}^2\,\mathbf{d}+\mathbf{d}\,\mathbf{c}^2\,\mathbf{d}\,\mathbf{c}^2$	196	1312
	$\mathbf{c}^2\,\mathbf{d}\,\mathbf{c}\,\mathbf{d}\,\mathbf{c}+\mathbf{c}\,\mathbf{d}\,\mathbf{c}\,\mathbf{d}\,\mathbf{c}^2$	378	882
	$\mathbf{c}^2\,\mathbf{d}^2\,\mathbf{c}^2$	268	1492
	$\mathbf{c}\,\mathbf{d}\,\mathbf{c}^2\,\mathbf{d}\,\mathbf{c}$	324	936
	$\mathbf{c}\,\mathbf{d}\,\mathbf{c}^3\,\mathbf{d}+\mathbf{d}\,\mathbf{c}^3\,\mathbf{d}\,\mathbf{c}$	126	1134
	$\mathbf{d}\,\mathbf{c}^4\,\mathbf{d}$	40	1468
	$\mathbf{c}^2\,\mathbf{d}^3+\mathbf{d}^3\,\mathbf{c}^2$	364	−860
	$\mathbf{c}\,\mathbf{d}\,\mathbf{c}\,\mathbf{d}^2+\mathbf{d}^2\,\mathbf{c}\,\mathbf{d}\,\mathbf{c}$	504	−504
	$\mathbf{c}\,\mathbf{d}^2\,\mathbf{c}\,\mathbf{d}+\mathbf{d}\,\mathbf{c}\,\mathbf{d}^2\,\mathbf{c}$	504	−504
	$\mathbf{d}\,\mathbf{c}^2\,\mathbf{d}^2+\mathbf{d}^2\,\mathbf{c}^2\,\mathbf{d}$	256	−752
	$\mathbf{d}\,\mathbf{c}\,\mathbf{d}\,\mathbf{c}\,\mathbf{d}$	420	−420
	$\mathbf{c}\,\mathbf{d}^3\,\mathbf{c}$	612	−612
	$\mathbf{d}^4$	496	496

Table A.3: Coefficients of the polynomials $\Phi_n(\mathbf{c},\mathbf{d})$ and $\hat{\Phi}_n(\mathbf{c},\mathbf{d})$ for $n = 9$.

A.2 The *cd*-indices of descent polytopes

In this section we present the coefficients of the *cd*-indices of descent polytopes. In the following tables, rows are labeled by $(\mathbf{c},\mathbf{d})$-words of degree n, and columns are labeled by $(\mathbf{x},\mathbf{y})$-words of length $n-1$, where $2 \leq n \leq 6$. The coefficient of a word w in the *cd*-index of DP_S is written in the intersection of the row corresponding to w and the column corresponding to $\mathbf{v}_S = \mathbf{v}_1\,\mathbf{v}_2 \cdots \mathbf{v}_{n-1}$, where $\mathbf{v}_i = \mathbf{x}$ if $i \notin S$ and $\mathbf{v}_i = \mathbf{y}$ if $i \in S$. We include only $(\mathbf{x},\mathbf{y})$-words that are essentially different for our purposes, as reversing an $(\mathbf{x},\mathbf{y})$-word or changing $\mathbf{x}$'s to $\mathbf{y}$'s and vice versa clearly does not affect the corresponding *cd*-index.

$n=2$	$\mathbf{x}$
$\mathbf{c}^2$	1
$\mathbf{d}$	1

$n=3$	$\mathbf{x}^2$	$\mathbf{xy}$
$\mathbf{c}^3$	1	1
$\mathbf{cd}$	2	3
$\mathbf{dc}$	2	3

$n=4$	$\mathbf{x}^3$	$\mathbf{xxy}$	$\mathbf{xyx}$
$\mathbf{c}^4$	1	1	1
$\mathbf{c}^2\mathbf{d}$	3	5	6
$\mathbf{cdc}$	5	8	10
$\mathbf{dc}^2$	3	4	5
$\mathbf{d}^2$	4	7	10

$n=5$	$\mathbf{x}^4$	$\mathbf{x}^3\mathbf{y}$	$\mathbf{x}^2\mathbf{yx}$	$\mathbf{x}^2\mathbf{y}^2$	$\mathbf{xy}^2\mathbf{x}$	$\mathbf{xyxy}$
$\mathbf{c}^5$	1	1	1	1	1	1
$\mathbf{c}^3\mathbf{d}$	4	7	9	8	10	11
$\mathbf{c}^2\mathbf{dc}$	9	15	20	17	22	25
$\mathbf{cdc}^2$	9	13	17	14	18	21
$\mathbf{cd}^2$	12	22	33	26	36	44
$\mathbf{dc}^3$	4	5	6	5	6	7
$\mathbf{dcd}$	10	18	26	21	28	35
$\mathbf{d}^2\mathbf{c}$	12	20	28	23	30	37

$n=6$	$\mathbf{x}^5$	$\mathbf{x}^4\mathbf{y}$	$\mathbf{x}^3\mathbf{y}\mathbf{x}$	$\mathbf{x}^2\mathbf{y}\mathbf{x}^2$	$\mathbf{x}^3\mathbf{y}^2$	$\mathbf{x}^2\mathbf{y}\mathbf{x}\mathbf{y}$	$\mathbf{x}^2\mathbf{y}^2\mathbf{x}$	$\mathbf{x}\mathbf{y}\mathbf{x}\mathbf{y}\mathbf{x}$	$\mathbf{x}\mathbf{y}\mathbf{x}^2\mathbf{y}$	$\mathbf{x}\mathbf{y}^3\mathbf{x}$
$\mathbf{c}^6$	1	1	1	1	1	1	1	1	1	1
$\mathbf{c}^4\mathbf{d}$	5	9	12	13	11	16	15	19	17	14
$\mathbf{c}^3\mathbf{d}\mathbf{c}$	14	24	33	36	29	45	41	55	48	38
$\mathbf{c}^2\mathbf{d}\mathbf{c}^2$	19	29	39	42	33	52	46	64	55	43
$\mathbf{c}^2\mathbf{d}^2$	25	47	72	81	60	106	92	136	113	82
$\mathbf{c}\mathbf{d}\mathbf{c}^3$	14	19	24	25	20	30	26	36	31	25
$\mathbf{c}\mathbf{d}\mathbf{c}\mathbf{d}$	35	65	98	109	82	145	123	188	153	110
$\mathbf{c}\mathbf{d}^2\mathbf{c}$	42	72	104	114	88	149	128	190	157	116
$\mathbf{d}\mathbf{c}^4$	5	6	7	7	6	8	7	9	8	7
$\mathbf{d}\mathbf{c}^2\mathbf{d}$	18	33	48	52	41	69	59	88	72	54
$\mathbf{d}\mathbf{c}\mathbf{d}\mathbf{c}$	35	60	86	93	73	122	105	156	128	96
$\mathbf{d}^2\mathbf{c}^2$	25	38	52	55	43	70	60	88	73	56
$\mathbf{d}^3$	34	64	100	112	82	151	127	200	159	112

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