The Theory of Commuting Boolean Algebras

by

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Abstract

In this thesis we give a definition of commutativity of Boolean subalgebras which generalizes the notion of commutativity of equivalence relations, and characterize the commutativity of complete Boolean subalgebras by a structure theorem.

We study the lattice of commuting Boolean subalgebras of a complete Boolean algebra. We characterize this class of lattices, and more generally, a similar class of lattices in a complete Heyting algebra. We develop a proof theory for this class of lattices which extends Haiman's proof theory for lattices of commuting equivalence relations.

We study the representation theory of commuting Boolean subalgebras. We associate to every complete subalgebra a normal, closed, completely additive operator, and prove that the commutativity of Boolean algebras is equivalent to the commutativity of the associated completely additive operators under composition. We then represent subalgebras of a Boolean algebra in terms of partitions of the Boolean space of that Boolean algebra. We obtain the following representation theorem: two Complete Boolean subalgebras commute if and only if they commute as partitions on the Boolean space.

We conclude with applications in Probability. We propose a notion of stochastic commutativity, which is a generalization of stochastic independence. We obtain a structure theorem for pairwise stochastically commuting σ-algebras and give some applications in the lattices of stochastically commuting σ-algebras.

Thesis Supervisor: Gian-Carlo Rota

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Contents

Introduction

The classical theory of lattices, as it evolved out of the nineteenth century through the work of Boole, Charles Saunders Peirce, and Schröder, and later in the work of Dedekind, Ore, Birkhoff, Von Neumann, Bilworth, and others, can today be viewed as essentially the study of two classes of lattices, together with their variants and their implications for their naturally occurring models. These are the classes of distributive lattices, whose natural models, which they capture exactly, are systems of sets or, from another point of view, of logical propositions; and modular lattices, whose natural but by no means only models are quotient structures of algebraic entities such as groups, rings, modules, and vector spaces.

In actuality, the lattices of normal subgroups of a group, ideals of a ring, or subspaces of a vector spaces are more than modular; as Birkhoff and Dubreil-Jacotin were first to observe, they are lattices of equivalence relations which commute relative to the operation of composition of relations. The combinatorial properties of lattices of commuting equivalence relations are not mere consequences of their modularity, but rather the opposite; the consequences of the modular law derived since Dedekind, who originally formulated it, have mainly been guessed on the basis of examples which were lattices of commuting equivalence relations.

The lattices of commuting equivalence relations were named **linear lattices**, a term suggested by G-C Rota for its evocation of the archetypal example of projective geometry. It is predicated on the supposition that in the linear lattice case, there is hope of carrying out the dream of Birkhoff and Von Neumann, to understand modular lattices through a "modular" extension of classical logic, just as distributive lattices had been so effectively understood through the constellation of ideas connecting classical propositional logic, the theory of sets, and visualization via the device of Venn diagrams.

It is not known whether linear lattices may be characterized by identities. Nevertheless, they can be characterized by a simple, elegant proof theory (Haiman). Such a proof theory is in several ways analogous to the classical Gentzen system of natural deduction for the predicate calculus. It is the deepest results to date on linear lattices. It provided a way to visualize statements pertaining to linear lattices with the aid of series-parallel network (extensively studied in combinatorics and circuit theory).

Haiman's proof theory for linear lattices is an iterative algorithm performed on a lattice inequality that splits the inequality into sub-inequalities by a tree-like procedure and eventually establishes that the inequality is true in all linear lattices, or else it automatically provides a counterexample. A proof theoretic algorithm is at least as significant as a decision procedure, since a decision procedure is merely an assurance that the proof theoretic algorithm will eventually stop.

Haiman's proof theory for linear lattices brings to fruition the program that was set forth in the celebrated paper "The logic of quantum mechanics", by Birkhoff and Von Neumann. This paper argues that modular lattices provides a new logic suited to quantum mechanics. The authors did not know the modular lattices of quantum mechanics are linear lattices. In the light of Haiman's proof theory, we may now confidently assert that Birkhoff and Von Neumann's logic of quantum mechanics is indeed the long awaited new "logic" where meet and join are endowed with a logical meaning that is a direct descendant of "and " and "or" of propositional logic.

Our objective is to develop a similar proof theory to lattices of subalgebras of a Boolean algebra. It is known that the lattice of Boolean subalgebras of a finite Boolean algebra is anti-isomorphic to a lattice of equivalence relations on a finite set. This anti-isomorphism leads us to a definition of **independence** of two Boolean subalgebras. Two Boolean subalgebras B and C are said to be independent if for all nonzero elements $b \in B$ and $c \in C$, we have $b \wedge c \neq 0$. Note that this definition does not require the Boolean algebra to be atomic. However, there was no analogous definition of commutativity of Boolean algebras which generizes the commutativity of equivalence relations.

In the first half of this thesis, we propose a definition of commutativity for Boolean algebras, and we study lattices of commuting Boolean algebras. We will characterize such a class of lattices, or more generally, the class of lattices defined similarly on Heyting algebras, by developing a proof theory which extends Haiman's proof theory for linear lattices. Indeed, commuting equivalence relations can be understood as commuting complete subalgebras of a complete atomic Boolean algebra. Hence the class of linear lattices is contained in the class of commuting Boolean algebras.

The commutativity of Boolean subalgebras arises naturally from the algebraic and topological structures of Boolean algebra. In Chapter 4, we give two equivalent definitions of commutativity of Boolean subalgebras. One is guided by the beautiful work of Jonsson and Tarski on Boolean Algebras with Operators. We associate to every complete subalgebra a completely additive operator and prove that the commutativity of two Boolean algebras is equivalent to the commutativity of their associated completely additive operators. We then represent subalgebras of a Boolean algebra A in terms of partitions on the Boolean space of

A, and obtain: two Boolean subalgebras commute if and only if they commute as partitions on the Boolean space.

In the second half of the thesis, we relate our work to probability and study the stochastic analog of commuting equivalence relations. Classical probability is a game of two lattices defined on a sample space: the Boolean σ-algebra of events, and the lattice of Boolean σ-subalgebras.

A σ-subalgebra of a sample space is a generalized equivalence relation on the sample points. In a sample space, the Boolean σ-algebra of events and the lattice of σ-subalgebras are dual notions, but whereas the Boolean σ-algebra of events has a simple structure, the same cannot be said of the lattice of σ-subalgebras. For example, we understand fairly well measures on a Boolean σ-algebra, but the analogous notion for the lattice of σ-subalgebras, namely, entropy, is poorly understood.

Stochastic independence of two Boolean σ-subalgebras is a strengthening of the notion of independence of equivalence relations. Commuting equivalence relations also have a stochastic analog, which is best expressed in terms of random variables. We say that two σ-subalgebras Σ_1 and Σ_2 commute when any two random variables X_1 and X_2 defining the σ-subalgebras Σ_1 and Σ_2 are conditionally independent. We studied the probabilistic analog of a lattice of commuting equivalence relations, namely, lattices of non-atomic σ-subalgebras any two of which are stochastically commuting. Subspaces of a real vector space with Lebesgue measure is a natural example of a lattice of stochastically commuting σ-algebras in our sense.

We have obtained a set of deduction rules from which we expect to develop a proof theory for the lattices of stochastically commuting non-atomic σ-algebras. We believe this line of work is new on probability. It is also a vindication of Dorothy Maharam's pioneering work in the classification of Boolean σ-algebras.

We stress the value of these results as a practical method of guessing and verifying lattice identities. A major application of our method is to finding and proving theorems of projective geometry relating to incidence of subspaces, independent of dimensions. For example, use the deduction rules we are able to prove that the lattice of stochastically commuting σ-algebras satisfy most of the classical theorems of projective geometry, such as various generalizations of Desargues' theorem.

We wish to emphasize also the relevance of the present work to the invariant theory of linear varieties, approached along the lines initiated by Gel'fand and Ponomarev in their influential papers on representations of free modular lattices and later further developed by Herrmann, Huhn, Wille, and others. We expect that the remarkable structural features found by Gel'fand and Ponomarev in their linear (in the sense of linear algebra) quotients of free modular lattices will manifest themselves already in the lattices of commuting Boolean algebras. If so, the techniques and results we have established here may contribute substantial insights and simplifications to this line of work.

This thesis may be read as an argument for the contention that much of the combinatorial subtlety of synthetic projective geometry (typically, the Von Staudt/Von Neumann coordinatization theorem) resides in the combinatorics of commuting Boolean algebras; and further that commutativity of Boolean subalgebras can be understood by a parallel reasoning to the classical logical ideas that explain distributivity. It is our belief that the theory of lattices of commuting Boolean algebras, because of its combinatorial elegance, its intuitively appealing proof theory, and its broad range of potential applications, may finally come to exert on algebra, combinatorics, probability, and geometry the unifying influence that modular lattices, despite their great historical significance, failed to achieve.

Chapter 1

Commutativity for Boolean Algebras

We begin by reviewing the theory of commuting equivalence relations. We propose a definition of commutativity for Boolean algebras which generalizes the commutativity of equivalence relations. We characterize the commutativity of complete Boolean algebras by a structure theorem. We study lattices of commuting Boolean algebras and characterize such a class of lattices, or more generally, the class of lattices defined similarly on Heyting algebras, by developing a proof theory analogous to Haiman's proof theory for linear lattices. We will further develop a similar theory for Boolean σ-algebras, and apply our results to Probability theory, and Logic in later chapters.

1.1 Commuting Equivalence Relations

Given a set S, a **relation** on S is a subset R of $S \times S$. On the set $\mathcal{R}$ of all relations on S, all Boolean operations among sets are defined. For example, $\cup$ and $\cap$ are the usual union and intersection; similarly, one defines the complement of a relation. The identity relation is the relation $I = \{(x, x) \mid x \in S\}$. In addition, composition of relations is defined, which is the analog for relations of functional composition. If R and T are relations, set:

$$R \circ T = \{(x, y) \in S \times S \mid \text{There exists } z \in S \text{ s. t. } (x, z) \in R, (z, y) \in T\}.$$

Given a relation R, one defines the **inverse relation** as follows:

$$R^{-1} = \{ (x, y) \mid (y, x) \in R\}.$$

Recall that a relation R is an **equivalence relation** if it is reflexive, symmetric and transitive. In the notation introduced above, these three properties are expressed by $I \subseteq R$, $R^{-1} = R$ and $R \circ R \subseteq R$.

The notion of equivalence relations on a set S and the partitions of this set are mathematically identical. Given an equivalence relation R on a set S, the equivalence classes form a partition of S. Conversely, every partition π of S defines a unique equivalence relation whose equivalence classes are blocks of π. We denote by R_π the equivalence relation associated to the partition π. The lattice of equivalence relations on a set S is identical to the partition lattice of S.

Two equivalence relations R_π and R_σ (or equivalently, two partitions π and σ) are said to be **independent** when, for any blocks $A \in \pi, B \in \sigma$, we have $A \cap B \neq \emptyset$. Two equivalence relations R_π and R_σ are said to be **commuting** if $R_\pi \circ R_\sigma = R_\sigma \circ R_\pi$. We sometimes say that partitions π and σ commute if R_π and R_σ commute.

The following theorems about commuting equivalence relations were proved in (Finberg, Mainetti, Rota, [5]).

Theorem 1.1 *Equivalence relations R_π and R_σ commute if and only if $R(\pi \vee \sigma) = R_\pi \circ R_\sigma$.*

Theorem 1.2 *Two equivalence relations R and T commute if and only if $R \circ T$ is an equivalence relation.*

Theorem 1.3 *If equivalence relations R_π and R_σ commute, and $\pi \vee \sigma = \hat{1}$, then R_π and R_σ are independent. Here $\hat{1}$ is the unique maximal element in the partition lattice, namely, the partition has only one block.*

Theorem 1.4 (Dubreil-Jacotin) *Two equivalence relations R_π and R_σ associated with partitions π and σ commute if and only if for every block C of the partition $\pi \vee \sigma$, the restrictions $\pi|_C$, $\sigma|_C$ are independent partitions.*

1.2 Definition of Commutativity for Boolean Algebras

The lattice of Boolean subalgebras of a finite Boolean algebra is dually-isomorphic to the lattice of equivalence relations on a finite set. This dual-isomorphism leads us to define a notion of independence of two Boolean subalgebras. Recall in Sec. 1.1, two equivalence relations R_π and R_σ are independent if for any blocks $A \in \pi$ and $B \in \sigma$, we have $A \cap B \neq \emptyset$. Extended to Boolean algebras, two Boolean subalgebras B and C are said to be independent if for all $0 \neq b \in B$ and $0 \neq c \in C$, we have $b \wedge c \neq 0$.

The notion of independence of Boolean algebras has long been known. However, no analogous definition of commutativity for Boolean algebras has been given to date. Here we propose a notion of commutativity of Boolean subalgebras which generalizes the notion of commutativity of equivalence relations.

Let A be a fixed Boolean algebra. Given a subalgebra B, we define a relation $h(B)$ on A as follows: $(x,y) \in h(B)$ whenever $ann(x) \cap B = ann(y) \cap B$ where $ann(x) = \{t | t \wedge x = 0\}$. It is easy to check that $h(B)$ is an equivalence relation on A since it is reflexive, transitive, and symmetric.

Definition 1.1 *Two Boolean subalgebra B and C of Boolean algebra A are said to* **commute** *if for any pair of elements x, y in A,*

$$ann(x) \cap B \cap C = ann(y) \cap B \cap C$$

implies that there exists $z \in A$ such that

$$ann(x) \cap B = ann(z) \cap B, \qquad ann(z) \cap C = ann(y) \cap C.$$

In another word, B and C commute if for any pair of elements x and y, $(x,y) \in h(B \cap C)$ implies there exists an elements z such that $(x,z) \in h(B)$ and $(z,y) \in h(C)$.

Obviously, if $B \subseteq C$, then B and C commute.

Next we show that the definition of commutativity is a generalization of the classical results about finite Boolean algebras.

Suppose that S is a finite set and let A be its power set $P(S)$. Let $\pi^*(S)$ be the set of all Boolean subalgebras of $P(S)$ ordered by inclusion. Then $\pi^*(S)$ is a lattice where $B \wedge C = B \cap C$ and $B \vee C$ is the Boolean subalgebra generated by $B \cup C$.

The lattice $\pi^*(S)$ is dually-isomorphic to the lattice of all partitions of S. Let us denote by γ this dual-isomorphism: $\pi^*(S) \longrightarrow Par(S)$.

Theorem 1.5 *Let S be a finite set and $A = P(S)$. Let $\pi^*(S)$ be the lattice of Boolean subalgebras of A. Then for two elements $B, C \in \pi^*(S)$, B and C commute if and only if $\gamma(B)$ and $\gamma(C)$ commute as equivalence relations on S.*

Proof. We will use the following trivial fact: if $P \in \pi^*(S)$ and p is an atomic element of P, i.e., p as a subset of S is a block of $\gamma(P)$, then for all $x \in A = P(S)$,

$$ann(x) \cap P = ann(p) \cap P \qquad \text{iff} \qquad x \subseteq p.$$

Assume that B and C commute as Boolean subalgebras. Since γ is an anti-isomorphism, we have $\gamma(B \wedge C) = \gamma(B) \vee \gamma(C)$. Take a block of $\gamma(B) \vee \gamma(C)$, say t, and pick arbitrarily a block b of $\gamma(B)$ and a block c of $\gamma(C)$ contained in t. By the lemma,

$$ann(b) \cap (B \cap C) = ann(c) \cap (B \cap C) = ann(t) \cap (B \cap C).$$

Thus there exists $z \in A$ such that

$$ann(b) \cap B = ann(z) \cap B, \qquad ann(z) \cap C = ann(c) \cap C.$$

It is clear $z \neq 0$ in A. But since b is an block of $\gamma(B)$, by the lemma, $z \subseteq b$. Similarly, $z \subseteq c$. So $b \cap c \neq \emptyset$. That is, $\gamma(B)|_t$ and $\gamma(C)|_t$ are independent. By Dubreil-Jacotin Theorem, $\gamma(B)$ commutes with $\gamma(C)$.

Conversely, assume that equivalence relations $\gamma(B)$ and $\gamma(C)$ commute. Given a pair of elements $x, y \in A$, $x \neq y$, if

$$ann(x) \cap B \cap C = ann(y) \cap B \cap C,$$

then for any atomic element t in $B \wedge C$, set

$$B_t = \{\ b \mid b \text{ is atomic in } B, b \wedge x \neq 0, b \leq t\},$$
$$C_t = \{\ c \mid c \text{ is atomic in } C, c \wedge y \neq 0, c \leq t\}.$$

Clearly $B_t \neq \emptyset$ if and only if $C_t \neq \emptyset$. And $S = \{\ t \mid B_t \neq \emptyset\}$ is not empty if $x \neq 0$.

Note that for any $t \in S$, if $b \in B_t$ and $c \in C_t$, then $b \cap c \neq 0$ since $\gamma(B)$ and $\gamma(C)$ are independent when restricted to t. Now let

$$z = \vee_t(\vee_{b\in B_t, c\in C_t} b \wedge c),$$

then to any atomic element $b \in B$, $b \wedge z \neq 0$ if and only if $b \wedge x \neq 0$. Hence $ann(x) \cap B = ann(z) \cap B$. Similarly, $ann(y) \cap C = ann(z) \cap C$. This proves that B and C commute as Boolean subalgebras. □

REMARK. Theorem 1.5 remains true when S is an infinite set, but here the lattice of subalgebras should be replaced by the lattice of all closed subalgebras. In other words, in a complete atomic Boolean algebra $P(S)$– the power set of S, two closed Boolean subalgebras commute if and only their associated partitions of S commute.

In a Boolean algebra, $ann(x)$ is the principal ideal generated by x^c, the complement of x. So we can state the commutativity as follows. Two subalgebra B and C of a Boolean algebra A are said to commute whenever for any pair of principal ideals I and J of A, if

$$I \cap B \cap C = J \cap B \cap C,$$

then there is a principal ideal K such that

$$I \cap B = K \cap B, \quad J \cap C = K \cap C.$$

Corollary 1.6 *Let B and C be two subalgebras of a Boolean algebra A where B and C commute, and $t \in B \cap C$, $I(t)$ is the principal ideal generated by t. For $x, y \in I(t)$, if*

$$ann(x) \cap B \cap C \cap I(t) = ann(y) \cap B \cap C \cap I(t),$$

then there exists $z \in I(t)$ such that

$$ann(x) \cap B \cap I(t) = ann(z) \cap B \cap I(t), \quad ann(z) \cap C \cap I(t) = ann(y) \cap C \cap I(t).$$

Proof.
We prove the corollary by the following lemma.

Lemma: $ann(s) \cap I(t) = ann(s \wedge t) \cap I(t)$ for all $s \in A$.

It is obvious that the left hand-side belongs to the right hand side since $ann(s) \subseteq ann(s \wedge t)$.

Conversely, assume that $a \leq t$ and $a \wedge (s \wedge t) = 0$. Then $(a \wedge s) \wedge t = 0$. But $a \wedge s \leq a \leq t$, so $0 = (a \wedge s) \wedge t = a \wedge s$. That is, $a \in ann(s) \cap I(t)$.

Now given $x, y \in I(t)$ such that

$$ann(x) \cap B \cap C \cap I(t) = ann(y) \cap B \cap C \cap I(t),$$

note that $ann(x)$ in A equal to $(ann(x) \cap I(t)) \vee I(t^c)$, so in A,

$$ann(x) \cap B \cap C = ann(y) \cap B \cap C.$$

Since B and C commute as subalgebras of A, there exists $z \in A$ such that

$$ann(x) \cap B = ann(z) \cap B, \qquad ann(y) \cap C = ann(z) \cap C.$$

Consider $z \wedge t$, we have

$$ann(x) \cap B \cap I(t) = ann(z) \cap B \cap I(t) = ann(z \wedge t) \cap B \cap I(t),$$

$$ann(y) \cap C \cap I(t) = ann(z) \cap C \cap I(t) = ann(z \wedge t) \cap C \cap I(t).$$

That finishes our proof. □

1.3 Equivalence Relations Induced by Boolean Subalgebras

As in the previous section, let A be a Boolean algebra and B be a Boolean subalgebra. Recall that we define an equivalence relation $h(B)$ on A as following: $(x, y) \in h(B)$ whenever $ann(x) \cap B = ann(y) \cap B$. Sometimes we write $(x, y) \in h(B)$ as $x \sim y$ $(h(B))$. Then h is a map from the lattice of Boolean subalgebras to the lattice of equivalence relations on A.

Lemma 1.7 *If B and C are two Boolean subalgebras of A and $B \subseteq C$, then $h(B) \supseteq h(C)$.*

Proof. Assume $B \subseteq C$ and $x \sim y$ $(h(C))$, then $ann(x) \cap C = ann(y) \cap C$, hence

$$ann(x) \cap C \cap B = ann(y) \cap C \cap B.$$

Since $C \cap B = B$, we have $ann(x) \cap B = ann(y) \cap B$. That is, $x \sim y$ $(h(B))$. □

We expect that h is an one-to-one map. But in general, it is not true.

Example 1 Let A be the power set of $[0, \infty)$, B be the least Boolean algebra generated by all the open set on $[0, \infty)$ (so B is not complete). Obviously B is a proper subalgebra of A. For any $x \in A$, $ann(x) = I(x^c)$ =power set of $[0, \infty) \setminus x$. It is easy to see that both A and B induce the same equivalence relation, the identity relation on A. But $B \subset A$.

The next lemma says that if $h(B) = h(C)$, then B and C are closely related.

Lemma 1.8 *Let A be a complete Boolean algebra, where B and C are two subalgebras and $h(B) = h(C)$. Let $\overline{C}$ be the minimal complete subalgebra containing C, then $B \subseteq \overline{C}$.*

Proof. Take an arbitrary element $b \in B \setminus C$, let $\bar{c} = \wedge\{c | c \in C, c \geq b\}$. Then $b \leq \bar{c}$ and $ann(b) \supseteq ann(\bar{c})$.

For any $t \in C, t \wedge b = 0$, we have $b \leq t^c$, so $\bar{c} \leq t^c$, hence $\bar{c} \wedge t = 0$. That is, $ann(b) \cap C = ann(\bar{c}) \cap C$.

By assumption, $h(B) = h(C)$, so $ann(b) \cap B = ann(\bar{c}) \cap B$. But $b^c \in ann(b) \cap B$, so $b^c \wedge \bar{c} = 0$, that implies $\bar{c} \leq b$. So we have $b = \bar{c} \in \overline{C}$, this is, $B \subseteq \overline{C}$. □

This lemma suggests that we should restrict ourselves to complete subalgebras of a complete Boolean algebra. More precisely, we have the following corollary.

Corollary 1.9 *Let A be a complete Boolean algebra and B, C are complete Boolean subalgebras. Then $h(B) = h(C)$ implies $B = C$.*

Corollary 1.10 *Let A, B and C be the same as in the previous corollary, then $h(B) \subseteq h(C)$ implies $C \subseteq B$. Hence h is an one-to one map from complete subalgebras into equivalence relations which reverses the order.*

In the rest of this paper, we always assume that A is a complete Boolean algebra and all Boolean subalgebras we talk about are complete subalgebras in the sense that for a set S of elements in a subalgebra B, $\vee\{x|x \in S\}$ and $\wedge\{x|x \in S\}$ exist in B, and they are equal to $\vee\{x|x \in S\}, \wedge\{x|x \in S\}$ in A, respectively.

Proposition 1.11 *If B is a complete Boolean subalgebra of A, then 0 forms a single equivalence class of $h(B)$.*

Proof. It is obvious because $x = 0$ if and only if $1 \in ann(x)$. □

Lemma 1.12 *If $x_\alpha \in A$ where α belongs to some index set I, then*

$$ann(\vee x_\alpha) = \cap_\alpha ann(x_\alpha),$$

Proof. Given $t \in ann(\vee x_\alpha)$, then $t \wedge x_\alpha = 0$ for all $\alpha \in I$, so $t \in \cap_\alpha ann(x_\alpha)$.

Conversely, if $t \in ann(x_\alpha)$ for all α, then $t \wedge x_\alpha = 0$. This implies $x_\alpha \leq t^c$ for all α. So $\vee x_\alpha \leq t^c$, that means, $t \wedge (\vee x_\alpha) = 0$. □

From this lemma, it is easy to see the following.

Proposition 1.13 *$h(B)$ preserves arbitrary joins; i.e., if $x_\alpha \sim y_\alpha$ $(h(B))$ for $\alpha \in$ some index set I, then $\vee x_\alpha \sim \vee y_\alpha$ $(h(B))$.*

Proposition 1.14 *$h(B)$ is hereditary; i.e., it commutes with the partial order of A. In other words, if $x \sim y$ $(h(B))$ and $a \in A, a \leq x$, then there exists $b \in A$ such that $b \leq y$ and $a \sim b$ $(h(B))$.*

Proof. First, we need a lemma: Let T be a subset of A, and $s_\alpha \wedge t = 0$ for all $t \in T$. Let $s = \vee s_\alpha$, then $s \wedge t = 0$ for all $t \in T$.

The proof of the lemma is easy. Take any $t \in T$, $t \wedge s_\alpha = 0$ implies $s_\alpha \leq t^c$ for all α. Hence $s = \vee s_\alpha \leq t^c$. Therefore $s \wedge t = 0$. For simplicity, we write $s \cdot T = 0$.

Assume now $x \sim y$ $(H(B))$ and $a \leq x$. Let

$$y_1 = \bigvee \{y' \mid y' \leq y,\ y' \cdot (ann(a) \cap B) = 0\}.$$

Obviously $y_1 \leq y$. By the above lemma, $y_1 \cdot (ann(a) \cap B) = 0$. So $ann(a) \cap B \subseteq ann(y_1) \cap B$.

Claim: $ann(y_1) \cap B = ann(a) \cap B$.

Once it is proved, y_1 is the element which is less than of equal to y and equivalent to a.

Proof of the claim.

Step 1 Suppose the claim fails, then there exists $t \in B$ such that $t \wedge y_1 = 0$ and $t \wedge a \neq 0$. Let $b' = \wedge\{b \in B | b \geq t \wedge a\}$. We have $t \wedge a \leq b' \leq t$, hence

$$(b' \wedge x) \geq (t \wedge a \wedge x) = (t \wedge a) \neq 0.$$

Because $x \sim y$ $(h(B))$, we have $b' \wedge y \neq 0$.

step 2 A corollary of step 1 is $(b' \wedge y) \vee y_1 > y_1$. Otherwise $b' \wedge y \leq y_1$, which leads to $b' \wedge y = t \wedge b' \wedge y \leq t \wedge y_1 = 0$. It is a contradiction to the step 1.

Step 3 We have $b' \cdot (ann(a) \cap B) = 0$. It can be proved by contradiction. Suppose not, then there exists $b \in ann(a) \cap B$, $b' \wedge b \neq 0$. Thus $(t \wedge a) \wedge b \leq a \wedge b = 0$, therefore $t \wedge a \leq b^c$. Hence $t \wedge a \leq b' \wedge b^c < b'$, which contradicts the fact that $b' = \wedge\{b \in B \mid b \geq t \wedge a\}$.

Step 4 Let us compute $y_1 \vee (b' \wedge y) \cdot (ann(a) \cap B)$. By distributive law and step 3, it is 0. But from step 2, $(b' \wedge y) \vee y_1) > y_1$. It contradicts the maximality of y_1. This proves the Claim. □

Theorem 1.15 (Characterization) *Given a complete Boolean algebra A, there is an one-to-one correspondence between complete Boolean subalgebras and the equivalence relation R on A satisfying the following conditions:*

C1 *0 forms a single equivalence class;*

C2 *R preserves arbitrary joins;*

C3 *R commutes with the partial order of A.*

Denote by C-relation the equivalence relations satisfying $C1, C2, C3$. Then from the preceding three propositions, h gives the injection from complete Boolean subalgebras to the set of C-relations. In fact, h is also surjective. This is a consequence of Theorem 1.16 – Theorem 1.19 below.

Theorem 1.16 *Conditions $C1, C2, C3$ on a complete Boolean algebra A are equivalent to the conditions $C1$, $C2'$, $C3$, and $C4$, where*

C2' *The equivalence relation R is join preserving, i.e., $x_1 \sim y_1$ (R) and $x_2 \sim y_2$ (R) imply $(x_1 \vee x_2) \sim (y_1 \vee y_2)$ (R).*

C4 *Every equivalence class is closed under infinite join operations. In particular, every equivalence class has a maximal element.*

Proof. Assume $C2$ holds for an equivalence relation R. Take an equivalence class $[a]$, if $\forall \alpha$ belongs to some index set I, $b_\alpha \in [a]$, then $b_\alpha \sim a$ (R), hence $\vee b_\alpha \sim a$ (R). Therefore the equivalence class is closed under infinite join. In particular, every equivalence class has a maximal element $\vee_{b \in [a]} b$.

Conversely, if $C2'$ and $C4$ hold for R, and $x_\alpha \sim y_\alpha$ (R) for which α belongs to some index set I, let $x = \vee x_\alpha$, $y = \vee y_\alpha$, and $x_\alpha \vee y_\alpha = z_\alpha$, then $\vee z_\alpha = x \vee y$. By $C2'$, $x_\alpha \sim z_\alpha$ (R), hence $x \vee z_\alpha \sim x \vee x_\alpha = x$ (R). By $C4$, $x \sim \vee(x \vee z_\alpha)$ (R), That is, $x \sim (x \vee y)$ (R). Similarly, $y \sim (x \vee y)$ (R), this proves $x \sim y$ (R). □

Theorem 1.17 *Every equivalence relation on a complete Boolean algebra A satisfying conditions $C1, C2$ and $C3$ defines a complete Boolean subalgebra.*

Proof. Given a C-relation R on A, by $C4$, every equivalence class contains a maximal element. Denote $\bar{x}$ the maximal element of the equivalence class containing x, (in particular, $x \sim y$ (R) implies $\bar{x} = \bar{y}$). Let

$$g(R) = \{\bar{x} \mid x \in A\},$$

we shall prove that $g(R)$ is a complete Boolean subalgebra.

Step 1. Lemma. If $x \leq y$, then $\bar{x} \leq \bar{y}$.

Because $x \sim \bar{x}$ (R), then $(\bar{x} \vee y) \sim (x \vee y) = y \sim \bar{y}$ (R), hence $(\bar{x} \vee y) \leq \bar{y}$, $\bar{x} \leq \bar{y}$.

Step 2. It is obvious that $1 \in g(R)$. And $0 \in g(R)$ by the condition $C1$.

Step 3. If $x_\alpha \in g(R)$, then $x = \wedge_\alpha x_\alpha \in g(R)$. In particular, $g(R)$ is closed under meet operation.

It suffice to show that if $t \sim x$ (R), then $t \leq x$. But note $x \leq x_\alpha$ for all α, so $(t \vee x_\alpha) \sim (x \vee x_\alpha) = x_\alpha$. $x_\alpha \in g(R)$ means that x_α is maximal in its equivalence class, so $t \leq x_\alpha$ for all α. Thus $t \leq x$.

Step 4. If $x_\alpha \in g(R)$, then $x = \vee_\alpha x_\alpha \in g(R)$. In particular, $g(R)$ is closed under join operation.

Suffice to show that if $t \sim x$ (R), then $t \wedge x^c = 0$. Now suppose $t_1 = t \wedge x^c \neq 0$, by $C1$ and $C3$. there exists $z \neq 0$ such that $z \leq x$ and $z \sim t_1$ (R). Note $z \leq x = \vee x_\alpha$, so $z \wedge x_\alpha \neq 0$ for some α. Again by $C1$ and $C3$, there exists $t_2 \neq 0$ such that $t_2 \leq t_1$, and $t_2 \sim (z \wedge x_\alpha)$. But $t_2 \leq \bar{t_2} = \overline{z \wedge x_\alpha} \leq \bar{x}_\alpha = x_\alpha$, so $t_2 \leq (t_1 \wedge x_\alpha) \leq (t_1 \wedge x) = 0$, which is a contradiction.

Step 5. If $x \in g(R)$, then $x^c \in g(R)$.

Similar to the previous two steps, only need to show that is $t \sim x^c$, then $t \wedge x = 0$. Suppose not, then there exists $y \neq 0$ such that $y \leq x^c$ and $y \sim (t \wedge x)$. But now $y \leq \bar{y} = \overline{t \wedge x} \leq \bar{x} = x$ which contradicts the fact $x \wedge x^c = 0$.

Conclusion: $g(R)$ is a complete Boolean subalgebra. □

Let h and g be defined as above. Then g is a map from C-relation to complete Boolean subalgebras.

Theorem 1.18 *$h \circ g = Id$, i.e., given any C-relation R, $h(g(R)) = R$.*

Proof. Assume $x \sim y$ (R), then $\bar{x} = \bar{y}$. To show that $ann(x) \cap g(R) = ann(y) \cap g(R)$, it is sufficient to show that $ann(x) \cap g(R) = ann(\bar{x}) \cap g(R)$. The "$\supseteq$" part is trivial since $x \leq \bar{x}$. To show the other side, note if for some $b \in g(R), x \wedge b = 0$ but $\bar{x} \wedge b \neq 0$, then $x \leq (\bar{x} \wedge b^c) < \bar{x}$. Since $x \sim \bar{x}$ (R), it implies that both $\bar{x}$ and $\bar{x} \wedge b^c$ belong to the same equivalence class of R, which is impossible. Hence we have $x \sim \bar{x}$ $(h(g(R))$. This proves $R \subseteq h(g(R))$.

Conversely, if $x \sim y$ $(h(g(R))$, then by the above argument, $\bar{x} \sim \bar{y}$ $(h(g(R))$, (where $\bar{x}$ is the maximal element of R-equivalence class). But since both $\bar{x}$ and $\bar{y}$ belong to $g(R)$, $ann(\bar{x}) \cap g(R) = ann(\bar{y}) \cap g(R)$ implies $\bar{x} = \bar{y}$. That means $x \sim y$ (R). So $h(g(R)) \subseteq R$. □

Theorem 1.19 *$g \circ h = Id$, i.e., given a complete Boolean subalgebra B, $g(h(B)) = B$.*

Proof. Denote $g(h(B))$ by B'. Given $b \in B$, then $b^c \in ann(b) \cap B$. So if $x \sim b$ $(h(B))$, i.e., $ann(x) \cap B = ann(b) \cap B$, then $x \wedge b^c = 0$, $x \leq b$. Thus b is the maximal element in $h(B)$-equivalence class, $b \in g(h(B))$.

Conversely, given $b' \in B' = g(h(B))$, then b' is maximal in its equivalence class of $h(B)$. Consider $ann(b') \cap B$, this set has a maximal element $t \in B$ since B is complete. Hence $t \wedge b' = 0$, and $b' \leq t^c$. Since $ann(b') \cap B = Ideal(t) \cap B = ann(t^c) \cap B$, we have $b' \sim t^c$ $(H(B))$. Therefore b' is maximal in its equivalence class which implies $b' = t^c \in B$. □

Theorem 1.16 – Theorem 1.19 show that h and g are inverse to each other. They induce an one-to-one correspondence between complete Boolean subalgebras and C-relations on A.

Corollary 1.20 *Let $\mathcal{C}$ be the set of all C-relations on a complete Boolean algebra A, then h induces an anti-isomorphism between the p.o.set of complete Boolean subalgebras of A and $\mathcal{C}$.*

REMARK.

1. $\mathcal{C}$ has a lattice structure induced via the map h. Hence we can define meet($\underline{\wedge}$) and join ($\underline{\vee}$) on $\mathcal{C}$ and

$$h(B \wedge C) = h(B)\underline{\vee}h(C),$$
$$h(B \vee C) = h(B)\underline{\wedge}h(C).$$

But generally, $\mathcal{C}$ is not a sublattice of the lattice of equivalence relation on A.

2. When $h(B)$ and $h(C)$ commute, then

$$h(B)\underline{\vee}h(C) = h(B) \circ h(C) = h(C) \circ h(B),$$

In this case $h(B \wedge C) = h(B) \circ h(C) = h(C) \circ h(B)$. This explains why our definition of commutativity of Boolean subalgebras is reasonable.

1.4 Commutativity of Complete Boolean Subalgebras

In this section we assume that A is a complete Boolean algebra, and all the subalgebras are complete Boolean subalgebras.

Given a Boolean subalgebra T of A, for any $a \in A$, denote by $\nu_T(a)$ the maximal element in T which belongs to $ann(a) \cap T$, i.e.,

$$\nu_T(a) = \vee\{x \mid x \in ann(a) \cap T\}.$$

Proposition 1.21 *If two Boolean subalgebras B and C commute, then $\nu_B(c) \in B \cap C$ for any $c \in C$.*

Proof. Let $\bar{b} = \nu_B(c)$. Obviously $\bar{b} \in B$, and $\bar{b} = \nu_B(c) \geq \nu_{B \cap C}(c) = \bar{d}$. Assume that $\bar{b} \notin C$, then $\bar{b} > \bar{d}$.

Let $x = \bar{b} - \bar{d} = \bar{b} \wedge (\bar{d})^c$, and let t be the minimal element of $B \cap C$ such that $\bar{b} \leq t$. I.e., $t = \wedge\{x|x \in B \cap C, x \geq \bar{b}\}$. It follows that $t^c = \nu_{B \wedge C}(\bar{b})$. Let $y = t \wedge c$.

Claim 1. $\nu_{B\cap C}(x) = \bar{d} \vee t^c$.

Proof. Note that

$$x \wedge \bar{d} = \bar{b} \wedge (\bar{d})^c \wedge \bar{d} = 0,$$

$$x \wedge t^c \leq \bar{b} \wedge t^c \leq t \wedge t^c = 0,$$

so $\bar{d} \vee t^c \leq \nu_{B\cap C}(x)$.

Conversely, if $z \in B \cap C$ and $z \wedge x = 0$, then

$$z = z \wedge (\bar{d} \vee \bar{d}^c) = (z \wedge \bar{d}) \vee (z \wedge \bar{d}^c).$$

So $z \wedge x = 0$ implies

$$(z \wedge \bar{d}^c) \wedge \bar{b} = z \wedge (\bar{d}^c \wedge \bar{b}) = z \wedge x = 0.$$

Thus $z \wedge \bar{d}^c \leq t^c$. Hence

$$z = (z \wedge \bar{d}) \vee (z \wedge \bar{d}^c) \leq \bar{d} \vee t^c.$$

This shows $\bar{d} \vee t^c = \nu_{B\cap C}(x)$.

Claim 2. $\nu_{B\cap C}(y) = \bar{d} \vee t^c$.

Proof. Note that

$$y \wedge t_c = t \wedge c \wedge t^c = 0,$$

$$y \wedge \bar{d} = t \wedge c \wedge \nu_{B\cap C}(c) = 0.$$

So $\bar{d} \vee t^c \leq \nu_{B\cap C}(y)$.

Conversely, if $z \in B \cap C$, and $z \wedge y = 0$, then $z = (z \wedge t) \vee (z \wedge t^c)$. But

$$(z \wedge t) \wedge y = 0 \Longrightarrow (z \wedge t) \wedge c = 0,$$

hence $z \wedge t \leq \nu_{B\cap C}(c) = \bar{d}$. And $z = (z \wedge t) \vee (z \wedge t^c) \leq \bar{d} \vee t^c$.

Now we conclude that $\nu_{B\cap C}(x) = \nu_{B\cap C}(y)$, that is,

$$ann(x) \cap B \cap C = ann(y) \cap B \cap C = \text{Ideal generated by } \nu_{B\cap C}(x) \text{ in } B \cap C.$$

Since B and C commute, there exists $s \in A$ such that

$$ann(x) \cap B = ann(s) \cap B, \qquad ann(s) \cap C = ann(y) \cap C.$$

But $0 \neq x = \bar{b} - \bar{d} \in B$, $y = t \wedge c \in C$, so $0 \neq s \leq x$ and $s \leq y$. Thus $s \leq x \wedge y \leq \bar{b} \wedge c = 0$, it is a contradiction. □

Similar argument shows that if B and C commute, then $\nu_C(b) \in B \cap C$ for any $b \in B$.

Corollary 1.22 *If B and C commute, $0 \neq b \in B, 0 \neq c \in C$ and $\nu_{B\cap C}(b) = \nu_{B\cap C}(c)$, then $b \wedge c \neq 0$.*

Proof. Since B and C commute, then $\nu_C(b) = \nu_{B\cap C}(b)$. If $b \wedge c = 0$, then

$$c \leq \nu_C(b) = \nu_{B\cap C}(b) = \nu_{B\cap C}(c),$$

So $c \wedge \nu_{B\cap C}(c) \geq c \neq 0$, a contradiction. □

The above proposition shows that if two complete Boolean subalgebras B C commute, then $\nu_B(c) \in B \cap C$, and $\nu_C(b) \in B \cap C$ for any $b \in B, c \in C$. The following proposition and lemmas (1.23–1.26) show that it is also sufficient.

Proposition 1.23 *Let B and C be two complete Boolean subalgebras of A. if $\nu_B(c) \in B\cap C$ for any $c \in C$, then $\nu_C(b) \in B \cap C$ for any $b \in B$.*

Proof. Given $b \in B$, if $c \in C$, and $b \wedge c = 0$, then $b \leq \nu_B(c) \in B \cap C$, hence $b \wedge (\nu_B(c))^c = 0$. Note that $c \wedge \nu_B(c) = 0$, so $c \leq (\nu_B(c))^c$. Therefore for any $c \in ann(b) \cap C$, there is an element $d = (\nu_B(c))^c \in ann(b) \cap B \cap C$ such that $d \geq c$, hence

$$\begin{aligned}\nu_C(b) &= \max\{x \in C \mid x \wedge b = 0\} \\ &\leq \max\{x \in B \cap C \mid x \wedge b = 0\} \\ &= \nu_{B\cap C}(b).\end{aligned}$$

This forces $\nu_C(b) = \nu_{B\cap C}(b) \in B \cap C$. □

Definition 1.2 *Let S and T be two subset of A. We say that S* **covers** *T if for any element $t \in T$, there is an element $s \in S$ such that $s \geq t$.*

Proposition 1.23 is equivalent to say that for two Boolean subalgebras B and C, $ann(b) \cap B \cap C$ covers $ann(b) \cap C$ for any $b \in B$ if and only if $ann(c) \cap B \cap C$ covers $ann(c) \cap B$ for any $c \in C$.

Lemma 1.24 *Let B and C be two complete Boolean subalgebras of A, and $\nu_B(c) \in B \cap C$ for all $c \in C$. Assume that for non-zero elements $b \in B$, $c \in C$,*

$$\nu_{B\cap C}(b) = \nu_{B\cap C}(c),$$

then $b \wedge c \neq 0$.

Proof. By the previous proposition, $\nu_B(c) = \nu_{B\cap C}(c)$, and $\nu_C(b) = \nu_{B\cap C}(b)$. So if $b \wedge c = 0$, then

$$c \leq \nu_C(b) = \nu_{B\cap C}(b) = \nu_{B\cap C}(c),$$

which is a contraction. □

Lemma 1.25 *Let B and C be two complete Boolean subalgebras of A, and $\nu_B(c) \in B \cap C$ for all $c \in C$. Assume that for non-zero elements $b \in B$, $c \in C$,*

$$\nu_{B\cap C}(b) = \nu_{B\cap C}(c),$$

Then

$$\nu_B(b \wedge c) = \nu_B(b) = b^c,$$

$$\nu_C(b \wedge c) = \nu_C(c) = c^c.$$

Proof. We only need to show the first one, the other one is similar.

It is obvious that $\nu_B(b\wedge c) \geq b^c$. Now for any $t \in B$ and $t \wedge (b \wedge c) = 0$, we have $(t \wedge b) \wedge c = 0$. So

$$t \wedge b \leq \nu_B(c) = \nu_{B\cap C}(c) = \nu_{B\cap C}(b),$$

thus $(t \wedge b) \wedge b = 0$, that means $t \wedge b = 0$, $t \leq b^c$. So

$$\nu_B(b \wedge c) = b^c.$$

□

Theorem 1.26 *let B and C be two complete Boolean subalgebras of A. If $\nu_B(c) \in B \cap C$ for all $c \in C$, then B, C commute.*

Proof. Given $x, y \in A$ such that

$$ann(x) \cap B \cap C = ann(y) \cap B \cap C,$$

Let $b = (\nu_B(x))^c$, $c = (\nu_C(y))^c$, then:

$$ann(x) \cap B = ann(b) \cap B = \text{Ideal generated by } b^c \text{ in } B,$$

$$ann(y) \cap C = ann(c) \cap C = \text{Ideal generated by } c^c \text{ in } C.$$

Consider b and c, we have:

$$\begin{aligned} ann(b) \cap B \cap C &= ann(x) \cap B \cap C \\ &= ann(y) \cap B \cap C \\ &= ann(c) \cap B \cap C. \end{aligned}$$

Hence

$$\nu_{B \cap C}(b) = \nu_{B \cap C}(c).$$

By Lemma 1.24 and Lemma 1.25, $b \wedge c \neq 0$ and

$$ann(b) \cap B = ann(b \wedge c) \cap B,$$

$$ann(c) \cap C = ann(b \wedge c) \cap C.$$

That is, B and C commute. □

Conclusion. *Two complete Boolean subalgebras B and C of a complete Boolean algebra A commute if and only if one of the followings holds:*

1. $\nu_B(c) \in B \cap C$, *for all* $c \in C$;
2. $\nu_C(b) \in B \cap C$ *for all* $b \in B$.

There are various statements that are equivalent to the above conclusion. They are listed in the next theorem. Any of them can be used as a criteria for commutativity of complete Boolean subalgebras.

Theorem 1.27 *Two complete Boolean subalgebras B and C commute if and only if one of the following five statements is true:*

1. *$\nu_B(c) \in B \cap C$ for all $c \in C$ or $\nu_C(b) \in B \cap C$ for all $b \in B$.*

2. *$ann(b) \cap B \cap C$ covers $ann(b) \cap C$ for all $b \in B$ or $ann(c) \cap B \cap C$ covers $ann(c) \cap B$ for all $c \in C$.*

3. *For any $b \in B$, if $c \in C$ and $b \wedge c = 0$, then there exists $t \in B \cap C$, such that $t \wedge b = 0$, and $t \geq c$. Or the same statement if we exchange B and C.*

4. *For any $b \in B$, if $C \in C$ and $b \wedge c = 0$, then there exists $t \in B \cap C$ such that $t \wedge b = 0$, $t \wedge c \neq 0$. Or the same statement if we exchange B and C.*

5. *For all $b \in B$ and $c \in C$, let $cl_{B\cap C}(x)$ be the complement of $\nu_{B\cap C}(x)$, i.e., $cl_{B\cap C}(x)$ is the minimal element in $B \cap C$ which is bigger than or equal to x. Then $b \wedge c = 0$ if and only if $cl_{B\cap C}(b) \wedge cl_{B\cap C}(c) = 0$.*

Proof.

(1) was just proved. And (2) is another form or (1).

To show that (2) implies (3), note that given $b \in B$, $c \in C$ and $b \wedge c = 0$, then $c \in ann(b) \cap C$. Since $ann(b) \cap B \cap C$ covers $ann(b) \cap C$, then there exists $t \in ann(b) \cap B \cap C$ such that $t \geq c$.

(3) implies (2) follows from the definition of **cover**.

(3) implies (4) is obvious. Conversely, if condition (4) holds, for any $b \in B$, if $c \in C$, and $b \wedge c = 0$, let

$$\bar{t} = \vee\{t \mid t \wedge b = 0, t \wedge c \neq 0, t \in B \cap C\},$$

then $\bar{t} \wedge b = 0$ and $\bar{t} \geq c$.

At last we show that (3) is equivalent to (5). First note that $cl_{B\cap C}(x) \geq x$, then

$$cl_{B\cap C}(b) \wedge cl_{B\cap C}(c) = 0 \Longrightarrow b \wedge c = 0.$$

Now assuming (5), then for $b \in B, c \in C$, and $b \wedge c = 0$, we choose t to be $cl_{B\cap C}(b)$. So

$$cl_{B\cap C}(b) \wedge c \leq cl_{B\cap C}(b) \wedge cl_{B\cap C}(c) = 0.$$

Conversely, assume (3) holds, and $b \wedge c = 0$, then there exists $t \in B \cap C$, such that $t \wedge b = 0$, $t \geq c$. So $t \geq cl_{B\cap C}(c)$, and $t^c \geq cl_{B\cap C}(b)$, hence $cl_{B\cap C}(b) \wedge cl_{B\cap C}(c) = 0$.□

Corollary 1.28 *Let B and C be two complete subalgebras of a complete Boolean algebra A, and let Σ be the smallest complete subalgebra of A which contains both B and C. Then B and C commute as subalgebras of A if and only if they commute as subalgebras of Σ.*

REMARK. We say that a subset T of A **separates** subsets P and Q if for any element $p \in P, q \in Q$, if $p \wedge q = 0$, then there exists element $t \in T$ such that $t \wedge p = 0$, $t \geq q$. In this sense, two complete Boolean subalgebras commute if and only if their intersection separates them.

Chapter 2

Proof Theory of CH-lattices

2.1 C-relations on Heyting Algebras

Definition 2.1 The pseudo-complement $a * b$ of an element a relative to an element b in a lattice L is an element c such that $a \wedge x \leq b$ if and only if $x \leq c$. A lattice in which $a * b$ exists, for all a, b is called **relatively pseudo-complemented** lattice, or **Heyting algebra**. The element $a * 0$ is called the pseudo-complement of a, and denoted by a^*.

Proposition 2.1 *Any Heyting algebra is distributive.*

Proof. Let H be a Heyting algebra, and $a, b, c \in H$. Let $y = (a \wedge b) \vee (a \wedge c)$, so $a \wedge b \leq y$, which implies $b \leq a * y$; similarly $c \leq a * y$. Hence $b \vee c \leq a * y$ in H. Therefore $a \wedge (b \vee c) \leq y = (a \wedge b) \vee (a \wedge c)$. This finishes the proof. □

Theorem 2.2 *Let F be a distributive lattice. Denote by $I(F)$ the lattice of all ideals of F with the order of inclusion. Then $I(F)$ is a complete Heyting algebra.*

Proof. Let $I_i, i \in \Lambda$ be a set of ideals of F. Then

$$\begin{aligned} I &= \vee(I_i \mid i \in \Lambda) \\ &= \{x \mid x \leq t_1 \vee t_2 \vee \ldots \vee t_n, \text{ for some } n, t_i \in I_i, i \in \Lambda\} \end{aligned} \tag{2.1}$$

is an ideal of F. Thus $I(F)$ is complete.

Now given two ideals $I, J \in I(F)$, let

$$T = \{K \in I(F) \mid K \wedge I = K \cap I \leq J\}$$

Then $T \neq \emptyset$ since $J \in T$. Let

$$\bar{K} = \vee_{K \in T} K$$

Claim: $I \wedge \bar{K} \leq J$.

Indeed, given $i \in I \wedge \bar{K} = I \cap \bar{K}$, by Eq.2.1 there exist $k_1, k_2, \ldots, k_n$ for some n such that $i \leq k_1 \vee k_2 \vee \ldots \vee k_n$, where $k_n \in K_n \in T$. So $i = \vee_{l=1}^{n}(i \wedge k_l)$ and $i \wedge k_l \in I \cap K_l = I \wedge K_l \subseteq J$, which implies $i \in J$. Hence $\bar{K} \wedge I \leq J$.

Given K' such that $K' \wedge I \leq J$, by the definition of T, $K' \in T$, so $K' \leq \vee_{K \in T} K = \bar{K}$. That is, $\bar{K}$ is the pseudo-complement of I relative to J. Since I and J are arbitrary elements in $I(F)$, we conclude that $I(F)$ is a complete Heyting algebra. □

Proposition 2.3 (Birkhoff) *A complete lattice H is a Heyting algebra if and only if it satisfies*

$$x \wedge (\bigvee_B y_\beta) = \bigvee_B (x \wedge y_\beta).$$

Now assume H is a complete Heyting algebra. The completeness of H guarantees the existence of the minimal and the maximal elements $\hat{0}$ and $\hat{1}$ in H. Consider an equivalence relation R on H such that:

C1 $\hat{0}$ forms a single equivalence class;

C2 R preserves arbitrary joins;

C3 R commutes with the partial order of H.

Such an equivalence relation on H is called **C-relation** on H.

Similar to Boolean algebra, every complete Heyting subalgebra can define an equivalence C-relation. The proof of this fact is identical to that of complete Boolean algebras, except that we have to replace t^c –the complement of an element t by the pseudo-complement t^*.

Conversely, given a C-relation R on a complete Heyting algebra H, we have the following properties.

1. Every equivalence class has a maximal element. Denote $\tilde{a}$ the maximal element in the equivalence class containing a, and let $T = \{\tilde{a} | a \in H\}$, we have
2. $\hat{0} \in T$,
3. if $x, y \in T$, then $x \wedge y \in T$,
4. if $x_\alpha \in T$, then $\wedge\{x_\alpha |$ for all $\alpha\} \in T$,
5. if $t \in T$, then $t^* \in T$.

The proof is the same as the corresponding results of complete Boolean algebras. Then the set T as defined above is a subset of H closed under infinite meet operation, and contains the pseudo-complements of its elements. Moreover, if in addition the equivalence C-relation R has the property that t is equivalent to t^{**} for every $t \in H$, then T is a complete Heyting subalgebra. In fact, it is a complete Boolean algebra contained in H.

2.2 Natural Deduction for CH-lattices

A lattice is a **CH-lattice** if it is isomorphic to a sublattice of the lattice of equivalence relations on a complete Heyting algebra H, with the properties that any equivalence relation in the lattice is a C-relation, and any two equivalence relations in the lattice commute, in the sence of composition of relations. In particular, a CH-lattice is a linear lattice.

In the next we shall describe a system of natural deduction whose intended models are CH-lattices.

- **Variables.**
  Variables will be of two sorts. Variables of the first sort will range over a countable Roman alphabet (not capitalized) $A = \{a, b, c, \ldots\}$. Variables of the second sort will range over a countable Greek alphabet (not capitalized) $B = \{\alpha, \beta, \gamma, \ldots\}$, where B represents the free distributive lattice generated by countably many elements with $\mathbf{0}$ and $\mathbf{1}$. (I.e., take the distributive lattice freely generated by a countable set of elements, add $\mathbf{0}$ and $\mathbf{1}$ to it, and denote its elements by Greek letters $\{\alpha, \beta, \gamma, \ldots\}$.)

  We denote by Free(A) the free lattice generated by the set A. An element of Free(A) will be called a **lattice polynomial** in the variables $a, b, c, \ldots$, and denoted by $P(a, b, c, \ldots)$.

- **Connectives.**
  There are three connectives: lattice join $\vee$ and meet $\wedge$, which are binary connectives used in lattice polynomials, and a unary connective R.

- **Formation rules.**
  The formation rules for lattice polynomials are understood. We define an **equation** to be an expression of the form
  $$\alpha R(P)\beta,$$
  where α and β are any Greek letters, P is any lattice polynomial. We define an **atomic equation** to be an expression of the form
  $$\alpha R(a)\beta,$$
  where a is any Roman letter.

- **Well formed formulas.**
  Any equation is a well formed formula.

  $\alpha \leq \beta$ and $\alpha \geq \beta$ are well formed formulas for any $\alpha, \beta \in B$.

  We denoted by Γ, Δ, etc. sets of well formed formulas.

- **Admissible pair.**
  A pair (Γ, Δ) of sets of well formed formulas is said to be admissible if all variables (Roman and Greek) occurring in Δ also occur in Γ, and Δ consists of equations.

- **Models.**
  A model $\{L, f, g\}$ is a CH-lattice L consisting of equivalence C-relations of a complete Heyting algebra H, together with lattice homomorphism $f : B \to H$ and function $g : A \to L$. It follows that a unique lattice homomorphism from Free(A) to L is defined. This homomorphism will also be denoted by g. An equation $\alpha R(P)\beta$ is said to hold in a given model, whenever $f(\alpha)R(g(P))f(\beta)$, that is, whenever the ordered pair $(f(\alpha), f(\beta))$ is an element of the C-relation $R(g(P))$ on the Heyting algebra H.

- **Validity.**
  An admissible pair (Γ, Δ) of sets of well formed formulas is said to be **valid** when every equation in Δ holds in every model in which every well formed formula in Γ holds.

- **Deduction rules.**
  A **proof** is a sequence of sets of well formed formulas $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ such that

$$\frac{\Gamma_i}{\Gamma_{i+1}}$$

is an instance of a linear deduction rule (v. below). In such circumstance, we write

$$\frac{\Gamma_1}{\Gamma_n},$$

to signify that the set of sentences Γ_n can be proved from the set of sentences Γ_1.

The set Γ_1 is the **premise** of the deduction rule, and the set Γ_n is the **conclusion**.

- **Provability.**
  If Γ_1 and Γ_n are sets of well formed formulas, we say that Γ_n is **provable** from Γ_1 if there exists a proof $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$.

- **Linear deduction rules for the theory of CH-lattices.**

1. **Reflexivity.**

$$\frac{\Gamma}{\Gamma,\ \alpha R(P)\alpha}$$

where P is any lattice polynomial, and where α is arbitrary Greek variable.

2. **Transitivity.**

$$\frac{\Gamma,\ \alpha R(P)\beta,\ \beta R(P)\gamma}{\Gamma,\ \alpha R(P)\beta,\ \beta R(P)\gamma,\ \alpha R(P)\gamma}$$

where P is any lattice polynomial, and where α, β and γ are arbitrary Greek variables.

3. **Splitting Meets.**

$$\frac{\Gamma,\ \alpha R(P \wedge Q)\beta}{\Gamma,\ \alpha R(P \wedge Q)\beta,\ \alpha R(P)\beta,\ \alpha R(Q)\beta}$$

where α and β are Greek variables, and where P and Q are arbitrary lattice polynomials.

4. **Combining Meets.**

$$\frac{\Gamma,\ \alpha R(P)\beta,\ \alpha R(Q)\beta}{\Gamma,\ \alpha R(P)\beta,\ \alpha R(Q)\beta,\ \alpha R(P\wedge Q)\beta}$$

   where α and β are Greek variables, and where P and Q are arbitrary lattice polynomials.

5. **Splitting Joins.**

$$\frac{\Gamma,\ \alpha R(P\vee Q)\beta}{\Gamma,\ \alpha R(P\vee Q)\beta,\ \alpha R(P)\gamma,\ \gamma R(Q)\beta}$$

   with the same provisos as in the preseeding rule for linear deduction, and with the additional proviso that γ is a new variable, that is, the Greek letter γ does not appear in Γ and is unequal to α,β.

6. **Combining Joins.**

$$\frac{\Gamma,\ \alpha R(P)\gamma,\ \gamma R(Q)\beta}{\Gamma,\ \alpha R(P)\gamma,\ \gamma R(Q)\beta,\ \alpha R(P\vee Q)\beta}$$

   where α, β and γ are Greek variables, and P and Q are arbitrary lattice polynomials.

7. **Commutativity.**

$$\frac{\Gamma,\ \alpha R(P)\gamma,\ \gamma R(Q)\beta}{\Gamma,\ \alpha R(P)\gamma,\ \gamma R(Q)\beta,\ \alpha R(Q)\delta,\ \delta R(P)\beta}$$

   where δ is again a new variable, it does not appear in Γ, and is unequal to α,β,γ.

8. **Hereditarity.**

$$\frac{\Gamma,\ \alpha\leq\beta,\ \beta R(P)\gamma}{\Gamma,\ \alpha\leq\beta,\ \beta R(P)\gamma,\ \alpha R(P)\delta,\ \delta\leq\gamma}$$

   where α, β are is Greek variables, and δ is a new variable which does not appear in Γ and is unequal to α,β,γ.

9. **Symmetry.**

$$\frac{\Gamma,\ \alpha R(P)\beta}{\Gamma,\ \alpha R(P)\beta,\ \beta R(P)\alpha}$$

   where P is a lattice polynomial, and α and β are arbitrary Greek variables.

10. **Preserving Joins.**

$$\frac{\Gamma,\ \alpha_i R(P)\beta_i}{\Gamma,\ (\vee_i\alpha_i)R(P)(\vee_i\beta_i)}$$

   where α_i and β_i are arbitrary Greek letters, and i belongs to some index set I.

Here whenever a new variable is introduced, unless otherwise stated, it is an element in B which is not comparable with any variables already present except the two distinguished elements $\mathbf{0}$ and $\mathbf{1}$.

Theorem 2.4 (Soundness) *If*

$$\frac{\Gamma}{\Delta},$$

that is, if (Γ, Δ) *is provable, then the set of well formed formulas* Δ *holds in every CH-lattice in which the set* Γ *holds.*

Note that every CH-lattices is a linear lattices, and the deduction rules of CH-lattices are contained in the linear deduction rules of linear lattice, so we have:

Corollary 2.5 *A pair* (Γ, Δ) *of sets of equations such that all variables occurring in* Δ *also occur in* Γ *is valid in any CH-lattices if it is valid in any linear lattices.*

In the following we give two examples of the use of linear deduction. First, we prove that every CH-lattice is a modular lattice. In fact this is true for every linear lattice, since in the proof we will only use those deduction rules which are valid in any linear lattices.

Recall that a lattice is said to be **modular** when it satisfies the inequality

$$a \wedge (b \vee (a \wedge c)) \leq (a \wedge b) \vee (a \wedge c) \text{ for any } a, b, c \in L.$$

Proposition 2.6 *Every CH-lattice is modular.*

Proof. In what follows, bear in mind the deduction rules 1-10 described above. Using **3**,

$$\frac{\alpha R(a \wedge (b \vee (a \wedge c)))\beta}{\alpha R(a \wedge (b \vee (a \wedge c)))\beta,\ \alpha R(a)\beta,\ \alpha R(b \vee (a \wedge c))\beta}. \tag{2.2}$$

Using **5**,

$$\frac{\alpha R(b \vee (a \wedge c))\beta}{\alpha R(b \vee (a \wedge c))\beta,\ \alpha R(b)\gamma,\ \gamma R(a \wedge c)\beta.} \tag{2.3}$$

Using **3**,

$$\frac{\gamma R(a \wedge c)\beta}{\gamma R(a \wedge c)\beta,\ \gamma R(a)\beta,\ \gamma R(c)\beta} \tag{2.4}$$

Using **9**,

$$\frac{\gamma R(a)\beta}{\gamma R(a)\beta,\ \beta R(a)\gamma} \tag{2.5}$$

From (2.2) and (2.5), Using **2**,

$$\frac{\alpha R(a)\beta,\ \beta R(a)\gamma}{\alpha R(a)\beta,\ \beta R(a)\gamma,\ \alpha R(a)\gamma} \tag{2.6}$$

From (2.3) and (2.6), using **4**

$$\frac{\alpha R(b)\gamma,\ \alpha R(a)\gamma}{\alpha R(b)\gamma,\ \alpha R(a)\gamma,\ \alpha R(a \wedge b)\gamma} \tag{2.7}$$

From (2.7) and (2.3), using **6**,

$$\frac{\alpha R(a \wedge b)\gamma,\ \gamma R(a \wedge c)\beta}{\alpha R(a \wedge b)\gamma,\ \gamma R(a \wedge c)\beta,\ \alpha R((a \wedge b) \vee (a \wedge c))\beta}. \tag{2.8}$$

Therefore, we have inferred

$$\frac{\alpha R(a \wedge (b \vee (a \wedge c)))\beta}{\alpha R((a \wedge b) \vee (a \wedge c))\beta}.$$

□

Example 1. Let $\Gamma = \{\alpha R(P)\beta,\ \alpha \leq \delta \leq \beta\}$, and $\Delta = \{\alpha R(P)\delta\}$. Then (Γ, Δ) is valid in any CH-lattice.

Proof. Using **1**

$$\frac{\Gamma}{\Gamma,\ \delta R(P)\delta}.$$

Using **10**,

$$\frac{\alpha R(P)\beta,\ \delta R(P)\delta}{\alpha R(P)\beta,\ \delta R(P)\delta,\ (\alpha \vee \delta)R(P)(\delta \vee \beta)}.$$

That is,

$$\frac{\alpha R(P)\beta,\ \delta R(P)\delta}{\alpha R(P)\beta,\ \delta R(P)\delta,\ \delta R(P)\beta}.$$

Using **2**,

$$\frac{\alpha R(P)\beta,\ \delta R(P)\beta}{\alpha R(P)\beta,\ \delta R(P)\beta,\ \alpha R(P)\delta}.$$

Therefore we proved

$$\frac{\alpha R(P)\beta,\ \alpha \leq \delta \leq \beta}{\alpha R(P)\delta}.$$

□

2.3 CH-lattices Generated by Equations

In this section, we introduce the notion of a CH-lattice generated by a countable set of well formed formulas. Thus, let Γ be a countable set of well formed formulas. The **graph** $Graph(\Gamma)$ of the set Γ is defined as follows.

1. If the equation $\alpha R(P)\beta$ belong to the set Γ, then the graph $Graph(\Gamma)$ has an edge labeled by the lattice polynomial P, whose adjacent vertices are the elements α and β;

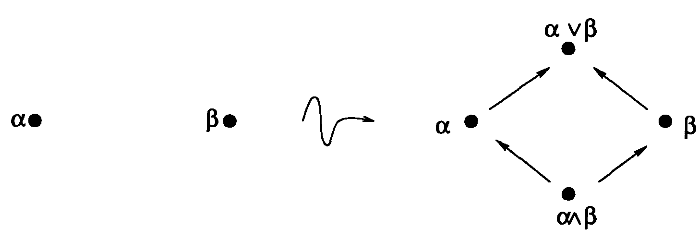
2. If two Greek variables α and β represent elements in B such that $\alpha \leq \beta$, then the graph $Graph(\Gamma)$ has a directed edge from α to β;

3. $Graph(\Gamma)$ has two distinguished vertices $\mathbf{0}$ and $\mathbf{1}$ which are the minimal and maximal elements of B, respectively.

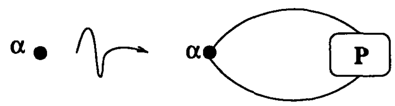
We define the **saturation** of the graph $Graph(\Gamma)$ in the following steps. We define an infinite sequence of graphs $G_0, G_1, \ldots,$ as follows. Set $G_0 = Graph(\Gamma)$. Having defined $G_n(\Gamma)$, we construct $G_{n+1}(\Gamma)$ by applying to $G_n(\Gamma)$ the following operations in the given order.

1. **Join and Meet**: To any two vertices α and β in G_n, add $\alpha \vee \beta$ and $\alpha \wedge \beta$ as new vertices. (If they are already in G_n, then do nothing.)

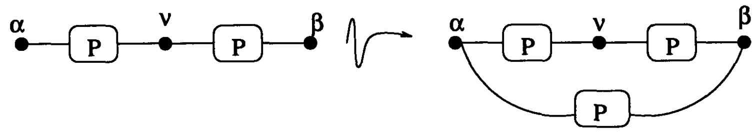
2. **Partial Ordering**: To any two vertices α and β with $\alpha \leq \beta$, connect α and β by a directed edge from α to β.



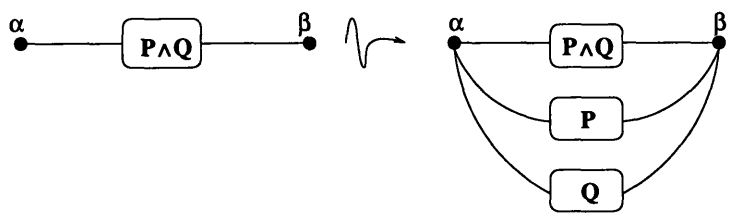
3. **Reflexive**: To any vertex α in G_n, add loops around α, one with each of the labels in any edges of G_n.



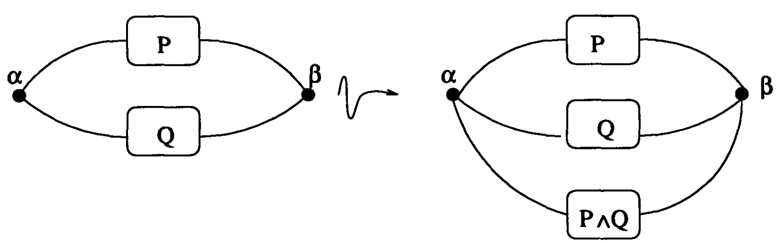
4. **Transitive**: If $\alpha R(P)\gamma$ and $\gamma R(P)\beta$ are edges of G_n, connect α and β by an edge labeled P.



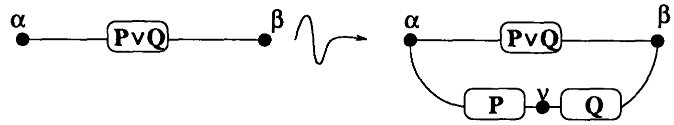
5. **Spiltting Meets**: For every edge haing vertices α β labeled by $P \wedge Q$, add two new edges with vertices α, β, labeled P and Q.



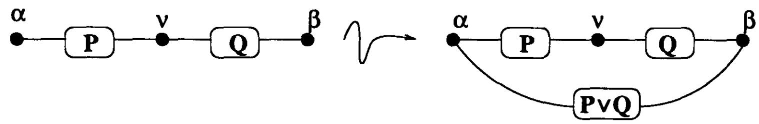
6. **Combining Meets**: If E is any multiset of cardinality at most n, of edges in G_n whose endpoints are vertices α and β and whose labels are lattice polynomials $P, Q, \ldots, R$, add a new edge labeled $P \wedge Q \wedge \ldots \wedge R$.



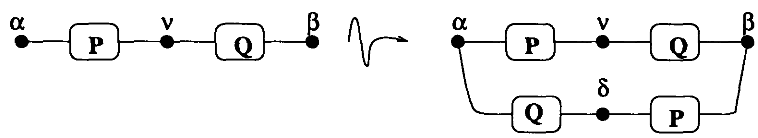
7. **Splitting Joins**: For evry edges with vertices α and β labeled by $P \vee Q$, add two new edges with endpoints α, γ and γ, β, labeled P and Q, respectively, where γ is a Greek letter not comparable with the vertices already present except **0** and **1**.



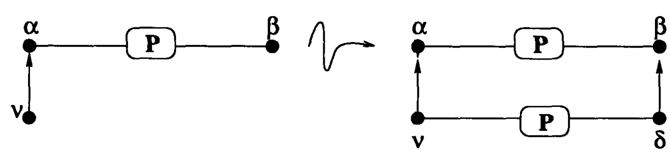
8. **Combining Joins**: Given an ordered sequence of edges of cardinality at most n, whose vertices are $\alpha, \gamma, \gamma, \delta, \ldots, \rho, \sigma, \sigma, \beta$, whose labels are the lattice polynomials $P, Q, \ldots, S, T$, add an edge whose endpoints are α, β, labeled by the polynomial $P \vee Q \vee \ldots \vee S \vee T$.



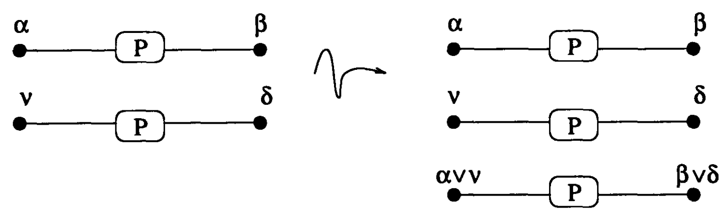
9. **Commutativity**: Given two edges whose vertices are α, γ and γ, β, and whose labels are polynomials P and Q, add new vertex δ which is not comparable with the vertices already present except **0** and **1**, together with edges having endpoints α, δ and δ, β, and labeled Q and P, respectively.



10. **Hereditarity**: Given an edge whose vertices are α and β and whose label is P, given $\gamma \leq \alpha$, add a new vertex, say δ, together with an edge having endpoints γ, δ, which is labeled P, and a directed edge from δ to β. Here δ is not appearing among the vertices already present, and is not comparable with vertices which are not belong to $F(\beta) \cap \{\mathbf{0}\}$, where $F(\beta) = \{\nu \mid \nu \geq \beta, \nu \text{ is a vertex in the graph }\}$.



11. **Preserving Joins**: Given two edges whose vertices are α, β and γ, δ, and whose labels are both P, add a new edge having endpoints $\alpha \vee \gamma, \beta \vee \delta$, and is labeled P.



The sequence of graphs thus constructed has the property that $G_0 \subset G_1 \subset G_2 \subset \ldots$. We denote by $Sat(\Gamma)$ the union of these graphs. The graph $Sat(\Gamma)$ is called the **saturation** of the set of equations Γ.

We note that the vertices of $Sat(\Gamma)$ is a sublattice of B, where the partial order is defined by setting $\alpha \leq \beta$ whenever there is a directed edge from α to β in $Sat(\Gamma)$. Furthermore, it is a distributive lattice with **0** and **1**. Denote it by F.

Proposition 2.7 *$Sat(\Gamma)$ contains countably many vertices if Γ is a countable set of well formed formulas. Moreover, if G_0 is a finite graph, then every G_n is also a finite graph.*

The preceding construction yields the following propositions.

Proposition 2.8 *On the lattice F, define an equivalence relation $R(P)$ for every lattice polynomial P by setting $\alpha R(P)\beta$ whenever the vertices α and β are connected by an edge labeled P. Then $R(P)$ has the following properties:*

1. ***0** forms a single equivalence class;*
2. *$R(P)$ preserves joins: if $\alpha R(P)\beta$ and $\gamma R(P)\delta$, then $(\alpha \vee \gamma)R(P)(\beta \vee \delta)$;*
3. *$R(P)$ is hereditary, i.e., it commutes with the partial order of F.*

Proposition 2.9 *To every two lattice polynomials P and Q appearing in $Sat(\Gamma)$, the equivalence relations $R(P)$ and $R(Q)$ commute.*

Corollary 2.10 *$\{R(P) \mid P$ is a lattice polynomial appearing in $Sat(\Gamma)\}$ is a linear lattice.*

Let $I(F)$ be the lattice of all ideals of F ordered by inclusion. Then by theorem 2.2, $I(F)$ is a complete Heyting algebra.

To each lattice polynomials P, associate a relation $\bar{P}$ on $I(F)$ by setting

$$I \sim J\ (\bar{P}), \text{ where } I, J \in I(F)$$

whenever for any element $\alpha \in I$ and $\beta \in J$, there exists $\gamma \in J$ and $\delta \in I$ such that $\alpha R(P)\gamma$ and $\beta R(P)\delta$. Obviously $\bar{P}$ is an equivalence relation on $I(F)$.

Proposition 2.11 *$\bar{P}$ is a C-relation on $I(F)$.*

Proof.

1. Since the minimal element $\hat{0}$ of $I(F)$ consists of a single element $\mathbf{0}$ of F, by property (1) in Proposition 2.8, $\hat{0}$ forms a single equivalence class of $\bar{P}$.

2. Recall that

$$I = \vee_{i \in \Lambda} I_i = \{x \mid x \le t_1 \vee t_2 \vee \ldots \vee t_n, \text{ for some } n, t_i \in I_i, i \in \Lambda\}$$

   Given $I_i \sim J_i\ (\bar{P})$, where $i \in \Lambda$, Let $I = \vee_i I_i$ and $J = \vee_i J_i$, pick an arbitrary element $\alpha \in I$, then there exist $\tau_1, \tau_2, \ldots, \tau_n$ for some n such that $\tau_i \in I_i$ and $\alpha \le \tau_1 \vee \tau_2 \vee \ldots \vee \tau_n$. Consider the pair $(\alpha \wedge \tau_i, \tau_i)$, Since $\tau_i \in I_i, I_i \sim J_i\ (\bar{P})$, there exists $\delta_i \in J_i$ such that $\delta_i R(P)\tau_i$. Now using the property (3) of $R(P)$, we see that there is $\xi_i \le \delta_i$ such that $\xi_i R(P)(\alpha \wedge \tau_i)$. Thus,

$$\alpha = \vee_{i=1}^{n}(\alpha \wedge \tau_i), \qquad (\vee_{i=1}^{n}(\alpha \wedge \tau_i))R(P)(\vee_{i=1}^{n}\xi_i),$$

   and $\vee_i \xi_i \in J$.

   Similarly, given $\beta \in J$, there exists an element $\gamma \in I$ such that $\beta R(P)\gamma$. That proves $I \sim J\ (\bar{P})$, i.e., $\bar{P}$ preserves arbitrary joins.

3. Given $I \sim J\ (\bar{P})$, and $A \subseteq I$ in $I(F)$, let

$$B = \{\beta \mid \beta \in J, \beta R(P)\alpha \text{ for some } \alpha \in A\}.$$

   and $I(B)$ be the ideal generated by B, i.e., $I(B) = \vee_{\beta \in B} I(\beta)$ where $I(\beta)$ is the principal ideal generated by β. Thus we have $I(B) \subseteq J$ and for any element $\alpha \in A$, there is $\beta \in I(B)$ such that $\alpha R(P)\beta$.

Conversely, given $\beta \in I(B)$, there exist $b_1, b_2, \ldots, b_n$ for some n such that $b_i \in B$ and $\beta \le b_1 \vee b_2 \vee \ldots \vee b_n$. Now apply the same argument as in step 2 to the pair $(\beta \wedge b_i, b_i)$, we can find elements $\xi_i \in A$ such that $(\beta \wedge b_i)R(P)\xi_i$. So $\beta R(P)(\vee_i \xi_i)$, and $\vee_i \xi_i \in A$. This proves $A \sim I(B)$ $(\bar{P})$. That is, $\bar{P}$ commutes with the partial order of $I(F)$. □

Proposition 2.12 *If lattice polynomials P and Q both appear in $Sat(\Gamma)$, then $\bar{P}$ and $\bar{Q}$ commute as equivalence relations on $I(F)$.*

Proof. We prove this proposition by showing that $\overline{P \vee Q} = \bar{P} \circ \bar{Q} = \bar{Q} \circ \bar{P}$. It is sufficient to show the first equality, the second one can be proved similarly.

First assume $I \sim K$ $(\bar{P} \circ \bar{Q})$, that is, there exists $J \in I(F)$ such taht

$$I \sim J \ (\bar{P}) \text{ and } J \sim K \ (\bar{Q}).$$

Thus for any element $\alpha \in I$, there is $\beta \in J$ such that $\alpha R(P)\beta$. To this β, there is $\gamma \in K$ such that $\beta R(Q)\gamma$. Hence $\alpha R(P \vee Q)\gamma$. Similarly, for any $\gamma \in K$, there is $\alpha \in I$ such that $\gamma R(P \vee Q)\alpha$. That proves $I \sim K$ $(\overline{P \vee Q})$.

Conversely, assume

$$I \sim K \ (\overline{P \vee Q})$$

Then for any $\alpha \in I$, there is $\beta_\alpha \in K$ such that $\alpha R(P \vee Q)\beta_\alpha$. Since $R(P)$ and $R(Q)$ commute, there exists $\gamma_\alpha \in F$ such that

$$\alpha R(P)\gamma_\alpha, \qquad \gamma_\alpha R(Q)\beta_\alpha.$$

Similarly for any $\beta \in K$ there is $\alpha_\beta \in I$ and $\gamma_\beta \in F$ such taht

$$\alpha_\beta R(P)\gamma_\beta, \qquad \gamma_\beta R(Q)\beta.$$

Let $J =$ Ideal generated by $\{\gamma_\alpha, \gamma_\beta |$ for all $\alpha \in I, \beta \in K\}$.

Claim: $I \sim J(\bar{P})$.

Proof: Given $\alpha \in I$, then we have $\gamma_\alpha \in J$ with the property $\alpha R(P)\gamma_\alpha$.

Given $j \in J$, by the definition of J, we have:

$$j \le \gamma_{\alpha_1} \vee \gamma_{\alpha_2} \vee \ldots \vee \gamma_{\alpha_n} \vee \gamma_{\beta_1} \vee \ldots \vee \gamma_{\beta_m}$$

for some m, n and where $\alpha_i \in I, \beta_i \in K$. Note that

$$\gamma_{\alpha_i} R(P) \alpha_i, \qquad \gamma_{\beta_i} R(P) \alpha_{\beta_i},$$

so by the property (3) of $R(P)$, there are elements ξ_i and τ_i belonging to I such that

$$(j \wedge \gamma_{\alpha_i}) R(P) \xi_i, \qquad (j \wedge \gamma_{\beta_i}) R(P) \tau_i.$$

Let $t = (\vee_i \xi_i) \vee (\vee_i \tau_i)$, then $jR(P)t$ and $t \in I$. That proves $I \sim J\ (\bar{P})$.

Similarly, $J \sim K(\bar{Q})$. So $I \sim K(\bar{P} \circ \bar{Q})$.

Conclusion: $\overline{P \vee Q} = \bar{P} \circ \bar{Q} = \bar{Q} \circ \bar{P}$. □

Theorem 2.13 *Let $CH(\Gamma) = \{\bar{P} \mid P$ is a lattice polynomial appearing in $Sat(\Gamma)\ \}$. Then $CH(\Gamma)$ is a CH-lattice on $I(F)$. We call such a CH-lattice the* **lattice generated by the set Γ of well formed formulas.**

Proof. By the preceding propositions, we only need to show that if $\bar{P}, \bar{Q} \in CH(\Gamma)$, then $\bar{P} \wedge \bar{Q} \in CH(\Gamma)$.

Claim: $\bar{P} \wedge \bar{Q} = \overline{P \wedge Q}$.

Proof. It is obvious that $\overline{P \wedge Q} \subseteq \bar{P} \wedge \bar{Q}$ as relations on $I(F)$.

Conversely, assume that $I \sim K(\bar{P} \wedge \bar{Q})$, then for any $\alpha \in I$ there are $\gamma_1, \gamma_2 \in K$ such that

$$\alpha R(P) \gamma_1, \qquad \alpha R(Q) \gamma_2.$$

Then

$$\begin{aligned} &(\alpha \vee \gamma_1 \vee \gamma_2) R(P)(\gamma_1 \vee \gamma_2), \\ &(\alpha \vee \gamma_1 \vee \gamma_2) R(Q)(\gamma_1 \vee \gamma_2), \\ \Longrightarrow\ &(\alpha \vee \gamma_1 \vee \gamma_2) R(P \wedge Q)(\gamma_1 \vee \gamma_2). \end{aligned}$$

Since $\alpha \leq (\alpha \vee \gamma_1 \vee \gamma_2)$, and $R(P \wedge Q)$ commutes with the partial order of F, then there exists $\gamma \leq \gamma_1 \vee \gamma_2$ such that $\alpha R(P \wedge Q) \gamma$.

Similarly, for any $\beta \in K$, there is $\delta \in I$ such that $\delta R(P \wedge Q) \beta$. So

$$I \sim K(\overline{P \wedge Q}).$$

From the above argument, we have:

$$\bar{P} \wedge \bar{Q} = \overline{P \wedge Q}.$$

□

Note that we can embed F into $I(F)$ by mapping α to the principal ideal $I(\alpha)$ generated by α. And $\alpha \leq \beta$ if and only if $I(\alpha) \leq I(\beta)$.

Proposition 2.14 *For two elements α and β in F,*

$$\alpha R(P)\beta \text{ if and only if } I(\alpha) \sim I(\beta)\ (\bar{P}).$$

Proof. The necessary part is followed from property (3) of $R(P)$.

To show the sufficient part, assume that $I(\alpha) \sim I(\beta)\ (\bar{P})$, by definition, there exists $\gamma \leq \beta$ such that $\alpha R(P)\gamma$, and there exists $\delta \leq \alpha$ such that $\delta R(P)\beta$. So $(\alpha \vee \delta)R(P)(\gamma \vee \beta)$, that is, $\alpha R(P)\beta$. □

We now ready to prove a completeness theorem for the natural deduction system described in the preceding section.

Theorem 2.15 (Completeness) *An admissible pair (Γ, Δ) of finite sets of well formed formulas is provable if and only if it is valid.*

Proof. Suppose that we are given a deduction of Δ from Γ. Each of the linear deduction rules holds for every CH-lattice, and thus the conclusion is valid.

Suppose now that the pair (Γ, Δ) is valid. We want to show that

$$\frac{\Gamma}{\Delta},$$

that is, there exists a sequence of deductions $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ where $\Gamma_1 = \Gamma$ and $\Gamma_n \supseteq \Delta$.

Since (Γ, Δ) holds in every CH-lattice, it holds in particular in the lattice $CH(\Gamma)$. Note in this model, α is represented by $I(\alpha)$. So every well formed formula in Γ holds in $CH(\Gamma)$. Thus, every equation in Δ holds in $CH(\Gamma)$ as well. By the previous proposition, it follows that in the sequence of graphs defining $Sat(\Gamma)$ there exists one graph, say G_n, which contains all equations in Δ. Let Γ_i be the set of well formed formulas corresponding to the edges of G_i. The construction of $Sat(\Gamma)$ shows that

$$\frac{\Gamma_i}{\Gamma_{i+1}},$$

since each of the operations by which Γ_{i+1} is constructed corresponds to the linear deduction bearing the same name. Thus, the sequence $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ provides the proof of a set of well formed formulas Γ_n of which Δ is a subset. □

Theorem 2.16 *Let P and Q be lattice polynomials. The lattice inequality $P \leq Q$ holds in all CH-lattice if and only if*

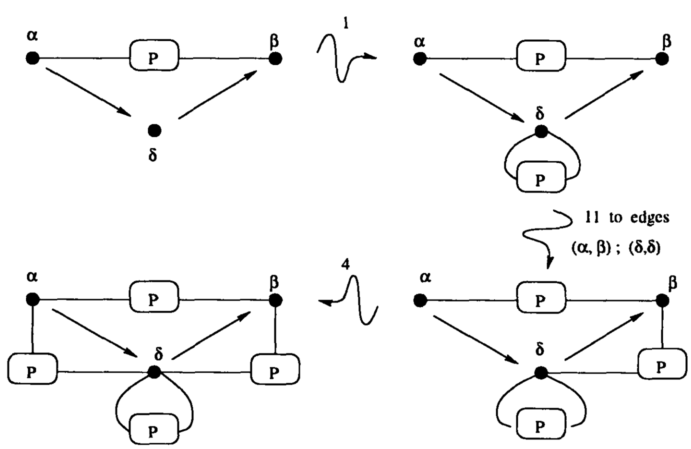
$$\frac{\{P\}}{\{Q\}};$$

in other words, the inequality $P \leq Q$ holds in all CH-lattices if and only if the equation $\alpha R(Q)\beta$ is provable from the equation $\alpha R(P)\beta$ for any α, β.

Here we show how to use the graph to get proof of Example 1.

Example 2 Let $\Gamma = \{\alpha R(P)\beta, \alpha \leq \delta \leq \beta\}$, and $\Delta = \{\alpha R(P)\delta\}$. Then (Γ, Δ) is valid in any CH-lattices.

Proof. The proof is shown in the figure. (Compare with Example 1 of Sec. 2.2.)



2.4 Inequalities and Horn Sentences

Actually, we can have a stronger version of the completeness theorem. In this section, we extend the completeness theorem to include Horn sentences.

Definition 2.2 *A* **Horn sentence** *is an expression of the form*

$$P_1 \leq Q_1, \ldots, P_n \leq Q_n \text{ imply } P \leq Q.$$

where P_i, Q_i, P, Q are lattice polynomials.

We say that a Horn sentence is **valid** in the theory of CH-lattices when every CH-lattice satisfying inequalities $P_i \leq Q_i$ for $i = 1, \ldots, n$ also satisfies the inequality $P \leq Q$. A single inequality is a special case of a Horn sentence —one with no assumptions.

In order to define a notion of proof for Horn sentences, an additional linear deduction rule has to be added to the list given before, to wit:

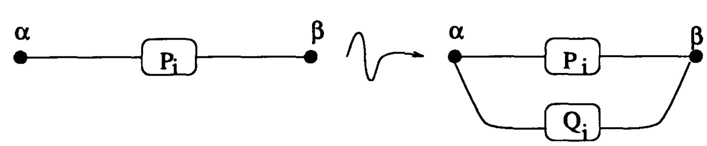
11. Conditional implication.

$$\frac{\Gamma, \alpha R(P_i)\beta}{\Gamma, \alpha R(P_i)\beta, \alpha R(Q_i)\beta}$$

where α, β are variables in B, and where P_i, Q_i are lattice polynomials.

Similarly, we extend the notion of saturation to include conditional implication, by adding an additional operation to the operations used in constructing the saturation of a set of sentences, to wit:

12. Conditional Implication: For every edge labeled P_i with vertices α and β, add a new edge labeled Q_i, with the same endpoints α and β.



Thus, just as before, G_n will be a finite graph for all n, and there is a one to one correspondence between deduction rules and operations in G_n. The union of this infinite increasing sequence of graphs gives a graph which may be called the **saturation** of the set of equations Γ **relative to the (finite) set of implications** $\Im = \{P_1 \leq Q_1, \ldots, P_n \leq Q_n\}$. Again, we denote this graph by $Sat(\Gamma; \Im)$. As in the previous completeness theorem, the graph $Sat(\Gamma; \Im)$ can induce a CH-lattice $CH(\Gamma; \Im)$.

The CH-lattice $CH(\Gamma; \Im)$ is the CH-lattice generated by the set of equations Γ and by the set of implications $\Im$.

Theorem 2.17 (Completeness 2) *The Horn sentence*

$$P_1 \leq Q_1, \ldots, P_n \leq Q_n \text{ imply } P \leq Q$$

is provable in the theory of CH-lattices by deduction rules 1– 12 if and only if it is valid in the theory of CH-lattices.

The proof is identical to that of the preceding completeness theorem since all the propositions and theorems about $Sat(\Gamma)$ remain true for $Sat(\Gamma; \Im)$.

We can even prove this completeness theorem without the assumption that $\Im$ is finite. Assume that $\Im$ is a countable set, i.e., $\Im = \{P_i \leq Q_i \mid i = 1, 2, \ldots\}$. Let $\Im(n) = \{P_i \leq Q_i \mid i = 1, 2, \ldots, n\}$. As before, let $G_0 = Graph(\Gamma)$. Having defined G_n, construct G_{n+1} by applying operations 1-11 and the operation of Conditional implication with implications $\Im(n)$. The sequence of graphs thus constructed has the properties that $G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots$, and every G_n is a finite graph if G_0 is. Now all the argument we used before can go through without difficulty. And we obtain a completeness theorem for Horn sentences with a countable set of implications.

As corollaries, we give the graph proof of some theorems in projective geometry. In fact, they are valid in all linear lattices. One can see from the proof that we only use the deduction rules which can be applied to all linear lattices. These proofs were first given by Haiman (c.f. [9]).

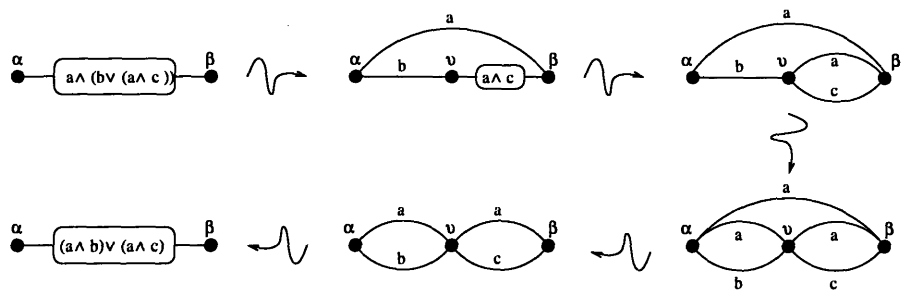
Example 2.1 *Every CH-lattice is modular.*

Proof. (compare with the proof of the proposition 2.6)
The modular inequality is

$$a \wedge (b \vee (a \wedge c)) \leq (a \wedge b) \vee (a \wedge c).$$

The opposite inequality is always true in any lattice.

The following graphs is the proof of modular inequality.



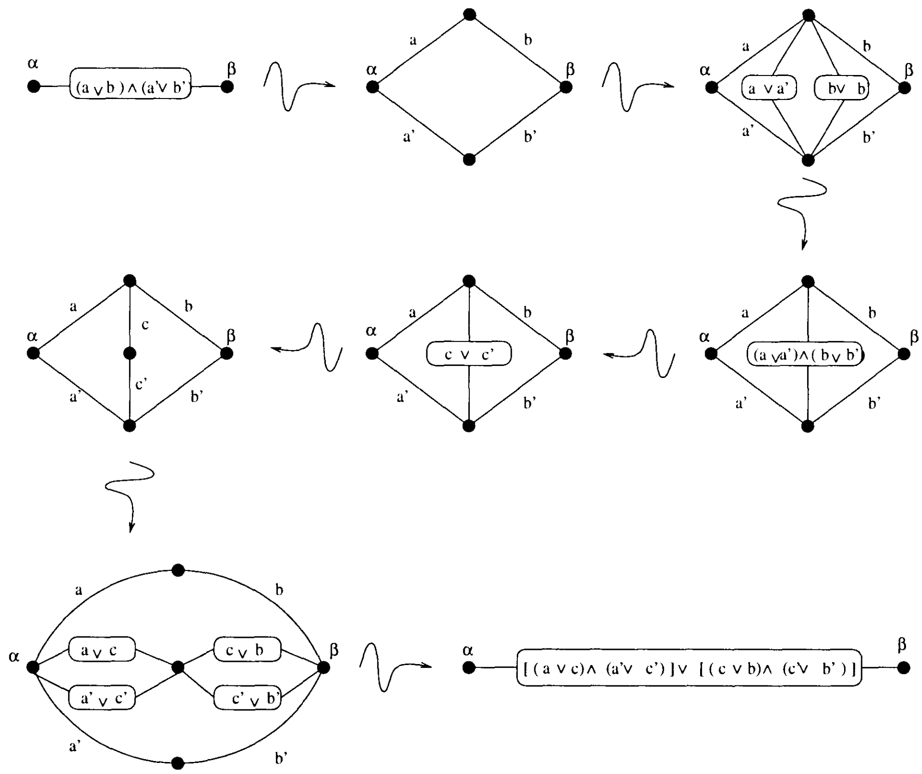
Example 2.2 (Desargues) *Given $a, b, c; a', b', c'$ in any CH-lattice, then*

$$(a \vee a') \wedge (b \vee b') \leq c \vee c',$$

implies

$$(a \vee b) \wedge (a' \vee b') \leq ((a \vee c) \wedge (a' \vee c')) \vee ((c \vee b) \wedge (c' \vee b')).$$

Proof. See the figure. □



α
(a ∨ b) ∧ (a'∨ b')
β
a
b
a'
b'
a ∨ a'
b∨ b
c
c'
c ∨ c'
(a ∨a')∧(b∨ b')
a ∨ c
a' ∨ c'
c ∨ b
c' ∨ b'
[(a ∨ c)∧ (a'∨ c')]∨ [(c ∨ b)∧ (c'∨ b')]

Chapter 3

Proof Theory for CB-lattices

3.1 Natural Deduction for CB-lattices

Definition 3.1 A lattice is called a **CB-lattice** if it is isomorphic to a sublattice of the lattice of equivalence relations on a complete Boolean algebra A, with the properties that any equivalence relation in the lattice is a C-relation, and two equivalence relations in the lattice commute, in the sense of composition of relations.

From the definition we see that:

$$\text{Class of CB-lattices} \subseteq \text{Class of CH-lattices} \subseteq \text{Class of linear lattices.}$$

The system of natural deduction whose intended models are CB-lattices is basicly as same as that of CH-lattices, except some small modifications.

- **Variables.**

  Variables will be of two sorts as in the case of CH-lattices. Variables of the first sort will range over a countable Roman alphabet (not capitalized) $A = \{a, b, c, \ldots\}$. Variables of the second sort will range over a countable Greek alphabet (not capitalized) $B = \{\alpha, \beta, \gamma, \ldots\}$, where B represents the free Boolean algebra generated by countably many elements.

  As before we denote by Free(A) the free lattice generated by the set A. An element of Free(A) will be called a **lattice polynomial** in the variables $a, b, c, \ldots$, and denoted by $P(a, b, c, \ldots)$.

- **Connectives.**
  In addition to the three connectives defined in the preceding section, we have another unary connective: lattice complement (^{c}), i.e., α^c is the complement of α.

- **Formation rules.**
  As before a lattice polynomial is an element of Free(A). An **equation** is an expression of the form

$$\alpha R(P)\beta,$$

  where α and $\beta \in B$, and P is any lattice polynomial.

- **Well formed formulas.**
  Any equation is a well formed formula.

  $\alpha \leq \beta$ and $\alpha \geq \beta$ are well formed formulas for any $\alpha, \beta \in B$.

  We denote by Γ, Δ, etc. sets of well formed formulas.

- **Admissible pair.**
  A pair (Γ, Δ) of sets of well formed formulas is said to be admissible if all variables (Roman and Greek) occurring in Δ also occur in Γ, and Δ consists of equations.

- **Models.**
  A model $\{L, f, g\}$ is a CB-lattice L consisting of equivalence C-relations of a complete Boolean algebra T, together with a homomorphism $f : B \to T$ and a function $g : A \to L$. It follows that a unique lattice homomorphism from Free(A) to L is defined. This homomorphism will also be denoted by g. An equation $\alpha R(P)\beta$ is said to hold in a given model, whenever $f(\alpha)R(g(P))f(\beta)$, that is, whenever the ordered pair $(f(\alpha, \beta))$ is an element of the C-relation $R(g(P))$ on the complete Boolean algebra T.

- **Validity.**
  A pair (Γ, Δ) of sets of well formed formulas is said to be **valid** when every equation in Δ holds in every model in which every well formed formula in Γ holds.

- **Deduction rules.**
  A **proof** is a sequence of sets of well formed formulas $\Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots, \Gamma_\omega$ such that

$$\frac{\Gamma_i}{\Gamma_{i+1}}$$

is an instance of a linear deduction rule, where ω is the countable cardinality. In such circumstance, we write

$$\frac{\Gamma_1}{\Gamma_\omega},$$

to signify the set of sentences Γ_ω can be proved from the set of sentences Γ_1.

- **Provability.**
  If Γ_1 and Δ are sets of well formed formulas, we say that Δ is **provable** from Γ_1 if there exists a proof $\Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots, \Gamma_\omega$ and $\Delta \subseteq \Gamma_\omega$.

- **Deduction rules for the theory of CB-Lattices.**
  Linear deduction rules for the theory of CB-lattices are same as those for CH-lattices defined in the previous section.

Theorem 3.1 (Soundness) *If*

$$\frac{\Gamma}{\Delta},$$

that is, if (Γ, Δ) *is provable, then the set of well formed formulas* Δ *holds in every CB-lattice in which the set* Γ *holds.*

As in the case of CH-lattices, we have the following corollary.

Corollary 3.2 *A pair* (Γ, Δ) *of sets of equations such that all variables occurring in* Δ *also occur in* Γ*, and Greek variables occurred are pairwise uncomparable, is valid in any CB-lattices if it is valid in any linear lattices.*

3.2 Proof Theory for CB-lattices

Given a countable set of well formed formulas Γ, we can define a graph $Graph(\Gamma)$ as we did for CH-lattices. In addition to the graph operations listed there, we add a new operation:

0. Complement For every vertex α in the graph, add the element α^c in B as a new vertex.

We define **saturation** of the graph $Graph(\Gamma)$ in the following steps. First let $G_0 = Graph(\Gamma)$. Having defined $G_n(\Gamma)$, construct $G_{n+1}(\Gamma)$ by applying to $G_n(\Gamma)$ the operations 0–11 in the given order.

The sequence of graphs thus constructed has the property that $G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots$. Again denote by $Sat(\Gamma)$ the union of these graphs. It is called the **saturation** of the set of equations Γ.

From the preceding definition and construction we see that the vertices of $Sat(\Gamma)$ form a subalgebra of the Boolean algebra B. Denote it by G.

Proposition 3.3 *$Sat(\Gamma)$ contains countably many vertices if Γ is a countable set of well formed formulas. Moreover, if G_0 is a finite graph, then every G_n is also a finite graph.*

Theorem 3.4 (Sufficient condition) *Given an admissible pair (Γ, Δ) of finite sets of well formed formulas, if $Sat(\Gamma)$ contains all equations in Δ, then (Γ, Δ) is valid.*

Proof. Since $Sat(\Gamma)$ contains every equation in Δ, and $Sat(\Gamma)$ is the union of all $G_n(\Gamma)$, it follows that there exists some n such that G_n contains all equations in Δ. Let Γ_i be the set of well formed formulas corresponding to the edges of G_i. The construction of $Sat(\Gamma)$ shows that

$$\frac{\Gamma_i}{\Gamma_{i+1}},$$

since each of the operations by which G_{i+1} is constructed corresponds to a linear deduction rule, thus, the sequence $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ provides the proof of a set of well formed formulas Γ_n of which Δ is a subset. And by the Soundness Theorem (Γ, Δ) is valid. □

The construction of $Sat(\Gamma)$ yields the following propositions.

Proposition 3.5 *On the lattice G, define an equivalence relation $R(P)$ for every lattice polynomial P by setting $\alpha R(P)\beta$ whenever the vertices α and β are connected by an edge labeled P. Then $R(P)$ has the following properties:*

1. *$\hat{0}$ forms a single equivalence class;*
2. *$R(P)$ preserves joins: if $\alpha R(P)\beta$ and $\gamma R(P)\delta$, then $(\alpha \vee \gamma)R(P)(\beta \vee \delta)$;*
3. *$R(P)$ is hereditary, i.e., it commutes with the partial order of G.*

Proposition 3.6 *To every two lattice polynomials P and Q appearing in $Sat(\Gamma)$, the equivalence relations $R(P)$ and $R(Q)$ commute.*

Corollary 3.7 *$\{R(P) \mid P$ is a lattice polynomial appearing in $Sat(\Gamma)\}$ is a linear lattice.*

For an equation $\alpha R(P)\beta$ in Δ, we say that $Sat(\Gamma)$ contains a **subgraph approaching** $\alpha\mathbf{R}(\mathbf{P})\beta$ if in the Boolean algebra G, there are sequences $\{\alpha_i\}, \{\beta_i\}$ such that $\alpha_i R(P)\beta_i$ for all i and $l.u.b.\{\alpha_i\} = \alpha, l.u.b.\{\beta_i\} = \beta$. If for every equation in Δ, $Sat(\Gamma)$ contains a subgraph approaching it, we say that $Sat(\Gamma)$ contains a **subgraph approaching** Δ.

Theorem 3.8 (Necessary condition) *Given an admissible pair (Γ, Δ) of countable sets of well formed formulas, if it is valid, then $Sat(\Gamma)$ contains a subgraph approaching Δ.*

This theorem is a consequence of the following propositions and theorems.

First, we need some propositions about the completion of a Boolean algebra.

Definition 3.2 *Let B be a Boolean algebra. A subset X of $B^+ = B \setminus \{0\}$ is* **dense** *in B if for every $b \in B^+$ there is some $x \in X$ such that $0 < x \leq b$. A subalgebra A of B is a* **dense subalbegra** *if A^+ is dense in B.*

Lemma 3.9 *The following are equivalent, for $X \subseteq B^+$:*

1. *X is dense in B;*
2. *for every $b \in B$, there is a pairwise disjoint family $M \subseteq X$ such that $b = \vee M$;*
3. *for every $b \in B$, there exists $M \subseteq X$ such that $b = \vee M$;*
4. *for every $b \in B$, $b = \vee\{x \in X | x \leq b\}$.*

Proof. Only the direction from 1 to 2 is non-trivial. Assume $b \in B$ and, by Zorn's lemma, let M be maximal with respect to the properties that $M \subseteq X \cap I(b)$, and M is a pairwise disjoint family. If $b \neq \vee M$, then there is an upper bound b' of M which is strictly smaller than b. By the deneseness of X, there exists $x \in X$ such that $0 < x \leq b \wedge b'^c$. Then the pairwise disjoint family $M \cup \{x\}$ contradicts maximality of M. □

Definition 3.3 *Let B be a Boolean algebra. A* **completion** *of B is a complete Boolean algebra C having B as a dense subalgebra. We write $C = \overline{B}$ if C is a completion of B.*

The following theorem is proved in Handbook of Boolean Algebras ([4]).

Theorem 3.10 *Every Boolean algebra has a unique completion, up to an isomorphism.*

Recall that G is the Boolean algebra consisting of vertices of $Sat(\Gamma)$, let $\overline{G}$ be a completion of G. We have:

1. G is dense in $\overline{G}$;

2. $\overline{G}$ is a complete Boolean algebra.

Definition 3.4 *A Boolean algebra B is said to satisfy the* **countable chain condition** *if every disjoint set of non-zero elements of B is countable.*

Proposition 3.11 *$\overline{G}$ satisfies the countable chain condition.*

Proof.
Suppose X is a disjoint family of $\overline{G}$, i.e., $0 \notin X$ and elements in X are pairwise disjoint. For any $x \in X$, by the denseness of G, there is a $y_x \in G$ such that $0 < y_x \leq x$. Let

$$Y = \{y_x \mid x \in X\},$$

then Y is a disjoint family of G and Card(X)=Card(Y). But from the construction of $Sat(\Gamma)$, G is a countable Boolean algebra, so Y, and hence X, are at most countable. □

Proposition 3.12 *For any $X \subseteq \overline{G}$, there is a countable set $M \subseteq X$ such that $\vee X = \vee M$.*

Proof. Assume that $\text{Card}(X) = \kappa$, we can give the set X a well ordering such that

$$\vee X = \vee_{\alpha<\kappa} x_\alpha.$$

Take

$$b_\alpha = x_\alpha - \vee_{\beta<\alpha} x_\beta,$$

then

$$\vee_{\alpha<\kappa} x_\alpha = \vee_{\alpha<\kappa} b_\alpha.$$

And elements in $\{b_\alpha\}$ are pairwise disjoint.

Consider the subset of $\{b_\alpha\}$ consisting of non-zero elements. Denote it by N. Thus N is a disjoint family in $\overline{G}$, then N is countable. Let $M = \{x_\alpha \mid b_\alpha \in N\}$, we have:

$$\vee_\alpha x_\alpha = \vee_\alpha b_\alpha = \vee N \leq \vee M \leq \vee_\alpha x_\alpha.$$

and Card(M)=Card(N) is countable. □

Proposition 3.13 *Let S be a non-empty subset of $\overline{G}$ which is closed under countable join. Then S has a unique maximal element.*

Proof. Suppose that T is an ascending chain of S, then by the previous proposition, $\vee T = \vee T'$, where $T' \subseteq T$ and T' is a countable set. By assumption, $\vee T' \in S$. That means every ascending chain in S has a maximal element. By Zorn's lemma, S has a maximal element.

The uniqueness of the maximal element is obvious. □

Key Lemma. *Suppose that $\alpha \in G, \beta_i \in G$ for all i and $\beta = \vee_i \beta_i$ exists in G. If $\alpha R(P)\beta$, then α can be written in the form*

$$\vee_{i=1}^{\infty} \vee_{j=1}^{\infty} \alpha_{i,j},$$

such that $\alpha_{i,j} \in G$ and $\alpha_{i,j} R(P)\beta_i$.

Proof. Since $\alpha R(P)\beta$ and the equivalence relation R(P) is hereditary, for every β_i, there is $\alpha_i \in G$ such that $\alpha_i \leq \alpha$ and $\alpha_i R(P)\beta_i$. Let

$$\alpha' = \vee_{i=1}^{\infty} \alpha_i \in \overline{G},$$

and let

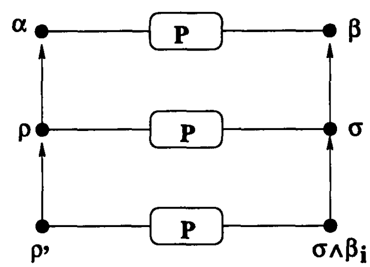
$$S = \{\delta \in \overline{G} \mid \delta \leq \alpha, \delta \text{ can be written in the form } \vee_{i,j=1}^{\infty} \delta_{i,j}, \text{ where } \delta_{i,j} \in G \text{ and } \delta_{i,j} R(P)\beta_i.\}.$$

Then S is non-empty since $\alpha' \in S$. And S is closed under countable join.

By proposition 3.13, S has a unique maximal element $\lambda \in \overline{G}$.

Claim: $\lambda = \alpha$.

Suppose not, then $\lambda < \alpha$ in $\overline{G}$. Let $\xi = \alpha - \lambda = \alpha \wedge \lambda^c$, then $0 \neq \xi \in \overline{G}$. By the denseness of G in $\overline{G}$, there exists $\rho \in G$ such that $0 < \rho \leq \xi$.



Consider the pair $(\rho < \alpha)$ in G, and since $\alpha R(P)\beta$, there exists $\sigma \in G$ with the properties $\sigma \leq \beta$ and $\rho R(P)\sigma$. It is obvious that $\sigma \wedge \beta_i > 0$ for some i. Use the hereditarity of R(P) to the pair $(\sigma \wedge \beta_i, \sigma)$, we obtain an element $\rho' \leq \rho$ in G such that $\rho' R(P)(\sigma \wedge \beta_i)$. Therefore

$$\rho' \vee \lambda > \lambda,$$

moreover,

$$(\rho' \vee \alpha_i')R(P)((\sigma \wedge \beta_i) \vee \beta_i) = \beta_i,$$

for any $\alpha_i' \in G$ with the property that $\alpha_i' R(P)\beta_i$, hence we have $\rho' \vee \lambda \in S$ which contradicts maximality of λ.

Conclusion: α is the maximal element in S. In particular, α can be written in the form

$$\alpha = \vee_{i=1}^{\infty} \vee_{j=1}^{\infty} \alpha_{i,j},$$

where $\alpha_{i,j} \in G$ and $\alpha_{i,j} R(P)\beta_i$. □

For each lattice polynomial P, one associate it an relation $\overline{P}$ on $\overline{G}$ defined as follows. Note that by Lemma 1 and countable chain condition, for any $\rho \in \overline{G}$, ρ can be written as $\rho = \vee\{\alpha \mid \alpha \in G, \alpha \leq \rho\} = \vee_i \alpha_i$ where $\alpha_i \in G$.

Set $\alpha \sim \beta$ $(\overline{P})$ for $\alpha, \beta \in \overline{G}$ whenever α, β can be written in the form

$$\alpha = \vee_{i=1}^{\infty} \alpha_i, \qquad \beta = \vee_{i=1}^{\infty} \beta_i,$$

where $\alpha_i, \beta_i \in G$ for all i and $\alpha_i R(P)\beta_i$.

Theorem 3.14 *$\overline{P}$ is an equivalence relation of $\overline{G}$.*

Proof. It is clear that $\overline{P}$ is reflexive and symmetric.

Now assume that $\alpha \sim \beta\ (\overline{P})$ and $\beta \sim \gamma\ (\overline{P})$. Then by definition,

$$\alpha = \vee_i \alpha_i, \quad \beta = \vee_i \beta_i = \vee_i \delta_i, \quad \gamma = \vee_i \gamma_i,$$

where $\alpha_i, \beta_i, \delta_i, \gamma_i \in G$ for all i and $\alpha_i R(P) \beta_i, \delta_i R(P) \gamma_i$.

Note that

$$\beta = \bigvee_{i,j} (\beta_i \wedge \delta_j),$$

and $\alpha_i R(P)(\vee_j (\beta_i \wedge \delta_j))$. By the Key lemma, α_i can be written in the form

$$\alpha_i = \bigvee_{j,k} \alpha_{i,j,k},$$

where $\alpha_{i,j,k} R(P)(\beta_i \wedge \delta_j)$.

Similarly, γ_j can be written in the form

$$\gamma_j = \bigvee_{k,l} \gamma_{j,k,l},$$

where $\gamma_{j,k,l} R(P)(\beta_k \wedge \delta_j)$.

So we have:

$$\alpha = \bigvee_{i,j,k} \alpha_{i,j,k}, \qquad \gamma = \bigvee_{j,i,l} \gamma_{j,i,l},$$

where $\alpha_{i,j,k} R(P) \gamma_{j,i,l}$ for all k and l. Rearranging the order of these terms, we get

$$\alpha \sim \gamma\ (\overline{P}).$$

□

Theorem 3.15 *$\overline{P}$ is a C-relation on $\overline{G}$.*

Proof.

1. It is clear that $\hat{0}$ forms a single equivalence class;

2. By proposition 3.11, we only need to show that $\overline{P}$ preserves countable join operation. Suppose in $\overline{G}$ there are two sequences $\{\alpha_i\}$ and $\{\beta_i\}$ such that $\alpha_i \sim \beta_i$ $(\overline{P})$. That is,

$$\alpha_i = \bigvee_i \alpha_{i,j}, \qquad \beta_i = \bigvee_j \beta_{i,j},$$

where $\alpha_{i,j}$ and $\beta_{i,j}$ belong to G, and $\alpha_{i,j} R(P) \beta_{i,j}$ for all i, j.

Note that $\omega \cdot \omega = \omega$ where ω stands for countable cardinality, so if

$$\alpha = \bigvee_i \alpha_i = \bigvee_{i,j} \alpha_{i,j}, \quad \beta = \bigvee_i \beta_i = \bigvee_{i,j} \beta_{i,j},$$

then $\alpha \sim \beta$ $(\overline{P})$.

3. Suppose $\alpha \sim \beta$ $(\overline{P})$ in $\overline{G}$, and γ is an element less than α in $\overline{G}$. We can write them as:

$$\alpha = \vee_i \alpha_i, \quad \beta = \vee_i \beta_i, \quad \gamma = \vee_i \gamma_i,$$

where $\alpha_i, \beta_i, \gamma_i \in G$ for all i and $\alpha_i R(P) \beta_i$.

Let $\gamma_i \wedge \alpha_j = \delta_{ij}$, then

$$\gamma_i = \bigvee_j \delta_{ij}, \text{ and } \gamma = \bigvee_{ij} \delta_{ij}.$$

For every pair $(\delta_{ij} \leq \alpha_j)$, by the heredity of equivalence relation $R(P)$, there exists a $\xi_{ij} \in G$ such that $\xi_{ij} \leq \beta_j$ and $\delta_{ij} R(P) \xi_{ij}$. Let

$$\xi = \bigvee_{i,j} \xi_{ij},$$

then

$$\xi \leq \beta, \text{ and } \xi \sim \gamma(\overline{P}).$$

□

Theorem 3.16 *For any two lattice polynomials P and Q,*

$$\overline{P \vee Q} = \overline{P} \circ \overline{Q} = \overline{Q} \circ \overline{P}.$$

Proof. It is sufficient to prove the first equation.

Suppose that $\alpha \sim \beta$ $(\overline{P \vee Q})$ in $\overline{G}$, then

$$\alpha = \vee_i \alpha_i, \qquad \beta = \vee_i \beta_i,$$

where $\alpha_i, \beta_i \in G$ and $\alpha_i R(P \vee Q)\beta_i$. By the commutativity of $R(P)$ and $R(Q)$ as equivalence relations on G, there exists $\gamma_i \in G$ such that

$$\alpha_i R(P)\gamma_i, \qquad \gamma_i R(Q)\beta_i.$$

Let $\gamma = \vee_i \gamma_i$, then $\alpha \sim \gamma$ $(\overline{P})$, $\gamma \sim \beta$ $(\overline{Q})$. This shows $\overline{P \vee Q} \subseteq \overline{P} \circ \overline{Q}$ as relations on $\overline{G}$.

Conversely, suppose that $\alpha \sim \gamma$ $(\overline{P})$ and $\gamma \sim \beta$ $(\overline{Q})$ in $\overline{G}$. Then

$$\alpha = \vee_i \alpha_i, \qquad \gamma = \vee_i \gamma_i = \vee_i \delta_i, \qquad \beta = \vee_i \beta_i,$$

where $\alpha_i, \gamma_i, \delta_i, \beta_i \in G$ for all i and

$$\alpha_i R(P)\gamma_i. \qquad \delta_i R(Q)\beta_i.$$

By the Key lemma,

$$\gamma = \bigvee_{i,j} (\gamma_i \wedge \delta_j),$$

and α_i, β_j can be written as

$$\alpha_i = \bigvee_{j,k} \alpha_{i,j,k}, \text{ where } \alpha_{i,j,k} R(P)(\gamma_i \wedge \delta_j),$$

$$\beta_j = \bigvee_{i,l} \beta_{j,i,l}, \text{ where } \beta_{j,i,l} R(Q)(\gamma_i \wedge \delta_j).$$

So $\alpha_{i,j,k} R(P \vee Q)\beta_{j,i,l}$ for all k and l. Since

$$\alpha = \bigvee_i \alpha_i = \bigvee_{i,j,k} \alpha_{i,j,k}, \qquad \beta = \bigvee_j \beta_j = \bigvee_{j,i,l} \beta_{j,i,l},$$

we have $\alpha \sim \beta$ $(\overline{P \vee Q})$. This finishes the proof. □

Theorem 3.17 *For any two lattice polynomials P and Q,*

$$\overline{P \wedge Q} = \overline{P} \wedge \overline{Q}.$$

Proof. Suppose $\alpha \sim \beta$ $(\overline{P \wedge Q})$ in $\overline{G}$. Then

$$\alpha = \vee_i \alpha_i, \qquad \beta = \vee_i \beta_i,$$

where $\alpha_i, \beta_i \in G$ for all i and $\alpha_i R(P \wedge Q)\beta_i$ which is

$$\alpha_i R(P)\beta_i, \text{ and } \alpha_i R(Q)\beta_i.$$

Hence $\alpha \sim \beta$ $(\overline{P})$ and $\alpha \sim \beta$ $(\overline{Q})$.

Conversely, suppose $\alpha, \beta \in \overline{G}$ such that $\alpha \sim \beta$ $(\overline{P} \wedge \overline{Q})$. Then

$$\alpha = \vee_i \alpha_i = \vee_i \gamma_i, \qquad \beta = \vee_i \beta_i = \vee_i \delta_i,$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i \in G$ for all i and

$$\alpha_i R(P)\beta_i, \qquad \gamma_i R(Q)\delta_i$$

By writing $\alpha = \vee_{i,j}(\alpha_i \wedge \gamma_j)$ and applying the Key lemma, we can assume that α has the same expression in these two formulas, which is,

$$\alpha = \vee_i \rho_i, \qquad \beta = \vee_i \eta_i = \vee_i \lambda_i,$$

where $\rho_i, \eta_i, \lambda_i \in G$ for all i and

$$\rho_i R(P)\eta_i, \qquad \rho_i R(Q)\lambda_i.$$

Hence

$$\begin{aligned}(\rho_i \vee \eta_i \vee \lambda_i) \quad R(P) \quad (\eta_i \vee \lambda_i),\\ (\rho_i \vee \eta_i \vee \lambda_i) \quad R(Q) \quad (\eta_i \vee \lambda_i).\end{aligned}$$

Thus $(\alpha \vee \beta) \sim \beta$ $(\overline{P \wedge Q})$.

By the same argument, $(\alpha \vee \beta) \sim \alpha$ $(\overline{P \wedge Q})$. Since $\overline{P \wedge Q}$ is an equivalence relation on $\overline{G}$, we have $\alpha \sim \beta$ $(\overline{P \wedge Q})$. □

Theorem 3.18 *Let $CB(\Gamma) = \{\overline{P} \mid P$ is a lattice polynomial appearing in $Sat(\Gamma)\}$. Then $CB(\Gamma)$ is a CB-lattice on the complete Boolean algebra $\overline{G}$. We call such a CB-lattice the* **lattice generated by the set Γ of well formed formulas**.

We are now ready to prove the Theorem of Necessary Condition.

Given an admissible pair (Γ, δ) of countable sets of well formed formulas, if (Γ, Δ) is valid, then particularly it is valid in the CB-lattice $CB(\Gamma)$. Note that G is included in $\overline{G}$ as a subalgebra, and if $\alpha R(P)\beta$ in G, then by the definition of $\overline{P}$ we have

$$\alpha \sim \beta \ (\overline{P}).$$

Thus $CB(\Gamma)$ is a model in which the well formed formulas in the set Γ hold. Consequently each of the equations in Δ holds in $CB(\Gamma)$, which means, for every equation $\alpha R(Q)\beta$ in Δ, there are sequences $\{\alpha_i\}$ and $\{\beta_i\}$ in G such that $\alpha = \vee_i \alpha_i, \beta = \vee_i \beta_i$ and $\alpha_i R(Q)\beta_i$. Therefore the graph $Sat(\Gamma)$ contains a subgraph approaching Δ. □

3.3 Implications and Horn Sentences

As in the proof theory of CH-lattices, we can extend all the theorem about CB-lattices to include Horn sentences by adding a deduction rule of Conditional implications and the corresponding operation on the graphs. All the arguments in this chapter remain true when one adds Conditional implications. In particular, the theorems of sufficient condition and necessary condition are true for Horn sentences. Explicitly, we have

Theorem 3.19 (Sufficient condition) *Given a Horn sentence*

$$P_1 \leq Q_2, \ldots, P_n \leq A_n \textit{ imply } P \leq Q.$$

Let $\Gamma = \{\alpha P \beta\}$, $\Delta = \{\alpha Q \beta\}$. *If* $Sat(\Gamma)$, *the saturated graph constructed by apply all operations 0-12 defined before, contains a subgraph of* Δ, *then the horn sentence is valid.*

Theorem 3.20 (Necessary condition) *Given a Horn sentence*

$$P_1 \leq Q_1, \ldots, P_n \leq Q_n \textit{ imply } P \leq Q.$$

Let $\Gamma = \{\alpha P \beta\}$, $\Delta = \{\alpha Q \beta\}$. *If this Horn sentence is valid, then the graph* $Sat(\Gamma)$ *contains a subgraph approaching* Δ.

From the sufficient theorem, we see that in CB-lattice, both modular law and Desargues theorem are true.

Chapter 4

Representation Theory of Commuting Boolean Algebras

In this chapter we study the representation theory of commuting Boolean algebras. We associate to every complete subalgebra of a complete Boolean algebra A a normal, closed, completely additive operator, and prove that the commutativity of Boolean subalgebras is equivalent to the commutativity of the associated completely additive operators under composition. We then represent subalgebras of a Boolean algebra A in terms of partitions of the Boolean space of that Boolean algebra. We obtain the following representation theorem: two complete Boolean subalgebras commute if and only if they commute as partitions on the Boolean space. Finally, we extend the definition of commutativity to subrings of an arbitrary commutative ring. The structure of pairwise commuting subrings will be a topic of its own interests.

4.1 Boolean Algebra with Operators

Let A be a complete Boolean algebra. Recall that every complete subalgebra B uniquely determines a C-relation $h(B)$, which can be characterized as an equivalence relation R on A satisfying the following conditions:

C1. 0 is an equivalence class;

C2. R preserves arbitrary joins;

C3. R is hereditary, i.e., R commutes with the partial order of the Boolean algebra A. In another word, for any elements a, x, $y \in A$ such that xRy and $a \leq x$, there exists an element $b \in A$ such that $b \leq y$ and aRb.

For every complete subalgebra B, we may define an operator $f_B : A \longrightarrow B$ by setting:

$$f_B(x) = \max\{y \mid y \in A \text{ and } yh(B)x\}.$$

Proposition 4.1 *The operator f_B has the following properties.*

1. *Range$(f_B) = B$ and f_B is the identity map when restricted to B;*
2. *f_B is normal, i.e., $f_B(x) = 0$ if and only if $x = 0$.*
3. *f_B is a closure operator, i.e.,*

$$\begin{aligned} x &\leq f_B(x), \\ f_B(f_B(x)) &= f_B(x). \end{aligned}$$

   for any $x \in A$.

4. *f_B is completely additive, i.e.,*

$$f_B(\bigvee_{\alpha \in I} x_\alpha) = \bigvee_{\alpha \in I} f_B(x_\alpha);$$

5. *f_B is an averaging operator, i.e., it satisfies the identity*

$$f_B(f_B(x) \wedge y) = f_B(x) \wedge f_B(y),$$

   for any $x, y \in A$.

Proof. Properties (1)–(4) are corollarys from the characterization of the C-relation $h(B)$. To prove Property (5), it is sufficient to show that

$$f_B(t \wedge y) = t \wedge f_B(y),$$

for any $t \in B$ and $y \in A$.

From Property (5), we have:

$$x \leq y \Longrightarrow f_B(x) \leq f_B(y).$$

Thus

$$\begin{cases} t \wedge y \leq t, \\ t \wedge y \leq y \end{cases} \Longrightarrow \begin{cases} f_B(t \wedge y) \leq f_B(t) = t, \\ f_B(t \wedge y) \leq f_B(y) \end{cases} \Longrightarrow f_B(t \wedge y) \leq t \wedge f_B(y).$$

Let $f_B(t \wedge y) = a$. If $a < t \wedge f_B(y)$, let $s = t \wedge f_B(y) - a$. Thus $s \in B$, and $a \wedge s = 0$. Hence we have

$$\begin{aligned} f_B(t \wedge y) \wedge s = 0 &\Longrightarrow (t \wedge y) \wedge s = 0 \Longrightarrow y \wedge (t \wedge s) = 0 \\ &\Longrightarrow y \leq (t \wedge s)^c \in B \Longrightarrow f_B(y) \leq (t \wedge s)^c \\ &\Longrightarrow f_B(y) \wedge t \wedge s = 0 \Longrightarrow s = 0. \end{aligned}$$

It contradicts the definition of s. Therefore $f_B(t \wedge y) = t \wedge f_B(y)$. □

From the above argument, for every complete subalgebra B, we can associate an operator f_B satisfying properties (1)–(5). Conversely, given such an operator f_B, we can recover the C-relation $h(B)$ (c.f Chap.1), therefore the complete subalgebra B.

Proposition 4.2 *Given an operator $f : A \longrightarrow A$ satisfying the properties (2)–(5), Let Range$(f) = B$, and define an equivalence relation R by setting*

$$x \sim y \text{ if and only if } f(x) = f(y),$$

then B is a complete subalgebra of A and R is the C-relation $h(B)$.

Proof. From Property (2) and (3), we know that 0 is an equivalence class of R. From property (4), we know that the equivalence relation R preserves arbitrary joins. Now if for some elements $a, x, y \in A$, we have $a \leq x$ and xRy, then $f(a) \leq f(x) = f(y)$. Let $b = f(a) \wedge y$, then $b \leq y$ and

$$f(b) = f(f(a) \wedge y) = f(a) \wedge f(y) = f(a) \wedge f(x) = f(a).$$

Hence R is hereditary. So R is a C-relation on A.

Since $Range(f) = B$, it is easy to see that B consists of all the maximal elements of the equivalence classes of R. Therefore $R = h(B)$ and B is complete. There is a one-to-one correspondence between complete subalgebras B and operators f_B which satisfies properties (2)–(5). □

REMARK.

1. It is a consequence of $x \leq f_B(x)$ and the averaging property that $f_B(1) = 1$ and $f_B(f_B(x)) = f_B(x)$.

2. f_B is self-conjugate, i.e., $f_B(x) \wedge y = 0$ if and only if $x \wedge f_B(y) = 0$. It is sufficient to prove only one direction:

$$f_B(x) \wedge y = 0 \Longrightarrow f_B(f_B(x) \wedge y) = 0 \Longrightarrow f_B(x) \wedge f_B(y) = 0 \Longrightarrow x \wedge f_B(y) = 0.$$

3. $f_B(x) = cl_B(x)$ for all $x \in A$.

We are ready to state the main theorem, which connects commuting Boolean algebras B and C via the completely additive operators f_B and f_C.

Theorem 4.3 *Two complete subalgebras B and C commute if and only if f_B and f_C commute under composition of functions, i.e., $f_B \circ f_C = f_C \circ f_B$.*

Proof. First let us assume $f_B \circ f_C = f_C \circ f_B$. For any elements $b \in B$, $c \in C$ with $b \wedge c = 0$, let $d = f_B \circ f_C(b) = f_C \circ f_B(b) \in B \wedge C$. Obviously $d \geq b$ and $d = f_C(f_B(b)) = f_C(b)$. Since $b \wedge c = 0$, we have $f_C(b \wedge c) = 0$, hence $f_C(b) \wedge c = 0$. Thus $d \in B \wedge C$, $d \geq b$ and $d \wedge c = 0$. Therefore B and C commute.

Conversely, suppose B and C are commuting subalgebras of A. For any $x \in A$, let t be $f_{B\wedge C}(x)$ which is equal to $cl_{B\wedge C}(x) = \min\{y \mid y \in B \wedge C, y \geq x\}$. Obviously $f_B \circ f_C(x) \leq t$.

Claim: $f_B \circ f_C(x) = t$.

Indeed, let $s = f_B \circ f_C(x)$ and $z = t - s \in B$. If $s < t$, then $z \wedge f_C(x) \leq z \wedge f_B(f_C(x)) = 0$. Sine B and C commute, there exists $d \in B \wedge C$ such that $d \geq f_C(x)$ and $d \wedge z = 0$. Hence $d \wedge t \in B \wedge C$ is bigger than $f_B(f_C(x))$, but less than t, which contradicts to the definition of t. Hence $z = 0$ and $s = t$. Similarly, we have: $f_B \circ f_C = f_C \circ f_B = f_{B\wedge C}$. □

Corollary 4.4 *Two completely additive operators f and g satisfying properties (2)-(5) commute if and only if the following equations are equivalent:*

$$f(x) \wedge g(y) = 0 \Longleftrightarrow f(y) \wedge g(x) = 0.$$

for any elements $x, y \in A$.

Proof. Assume f and g commute and their associated complete subalgebras are B and C respectively.

If for some elements x and $y \in A$, we have $f(x) \wedge g(y) = 0$, then from the commutativity of B and C, and the fact $f(x) \in B$ and $g(y) \in C$, we can find an element $d \in B \wedge C$ such that $d \geq f(x)$ and $d^c \geq g(y)$. Thus $g(f(x)) \leq d$ and $f(g(y)) \leq d^c$. Hence $g(x) \wedge f(y) \leq g(f(x)) \wedge f(g(y)) \leq d \wedge d^c = 0$.

Conversely, assume that for f and g, $f(x) \wedge g(y) = 0$ if and only if $f(y) \wedge g(y) = 0$. For any pair of elements $b \in B, c \in C$ and $b \wedge c = 0$, we derive from $b \wedge c = f(b) \wedge g(c)$ that $g(b) \wedge f(c) = 0$. Consider the sequence of pairs

$$(b, c) \longrightarrow (g(b), f(c)) \longrightarrow (f(g(b)), g(f(c))) \longrightarrow (g(f(g(b))),\ f(g(f(c)))\ ) \longrightarrow \cdots,$$

The meet of each pair in the sequence is 0. But from the definition of f and g,

$$\begin{aligned} g(b) \vee f(g(b)) \vee \cdots &= f_{B \wedge C}(b) \in B \wedge C, \\ f(c) \vee g(f(c)) \vee \cdots &= f_{B \wedge C}(c) \in B \wedge C. \end{aligned}$$

Hence elements b and c are separated by $B \wedge C$. Thus B and C, and consequently f and g, commute. □

Corollary 4.5 *Let f_B be the completely additive operator defined by the complete subalgebra B, then for every $x \in A$,*

$$f_B(x) = \bigwedge_{\{y | f_B(y) \wedge x = 0\}} y^c.$$

It is a consequence of the fact that f_B is self-conjugate.

For the representation theorem and the extension theorem of B. Jonsson and A. Tarski, we need some definitions and known results.

Theorem 4.6 (Representation theorem, Stone) *Every Boolean algebra is isomorphic to a set-field consisting of all open and closed sets in a totally-disconnected compact space.*

Before stating the corresponding form of the extension theorem, we shall introduce the notions of a **regular subalgebra** and a **perfect extension**. The results stated in the following text are substantially known, and the proofs will therefore be omitted.

Definition 4.1 Let

$$\mathcal{T} = < T, \vee, 0, \wedge, 1, ^c > \text{ and } \mathcal{A} = < A, \vee, 0, \wedge, 1, ^c >$$

be two Boolean algebras, where $\vee, \wedge$ and ^{c} are Boolean addition, Boolean multiplication and complement. We say that $\mathcal{A}$ is a **regular subalgebra** of $\mathcal{T}$ and $\mathcal{T}$ is a **perfect extension** of $\mathcal{A}$ if the following conditions are satisfied:

1. $\mathcal{T}$ is complete and atomistic, and $\mathcal{A}$ is a subalgebra of $\mathcal{T}$.
2. If I is an arbitrary set, and if the elements $x_i \in T$ with $i \in I$ satisfy

   $$\bigvee_{i \in I} x_i = 1,$$

   then there exists a finite subset J of I such that

   $$\bigvee_{i \in J} x_i = 1.$$

3. If u and v are distinct atoms of $\mathcal{T}$, then there exists an element $a \in A$ such that

   $$u \leq a, \qquad \text{and} \qquad v \wedge a = 0.$$

Definition 4.2 Let

$$\mathcal{T} = < T, \vee, 0, \wedge, 1, ^c >$$

be a complete atomistic Boolean algebra, and let

$$\mathcal{A} = < A, \vee, 0, \wedge, 1, ^c >$$

be a regular subalgebra of $\mathcal{T}$. An element $x \in T$ is said to be

1. **open** if

   $$x = \vee\{y | \; x \geq y \in A\};$$

2. **closed** if

$$x = \wedge\{y |\ x \leq y \in A\}.$$

Theorem 4.7 *Let*

$$\mathcal{T} =< T, \vee, 0, \wedge, 1,^c >$$

be a complete and atomistic Boolean algebra, and let

$$\mathcal{A} =< A, \vee, 0, \wedge, 1,^c >$$

be a regular subalgebra of $\mathcal{T}$. We have:

1. *For any $x \in T$, x is open if and only if x^c is closed.*
2. *For any $x \in T$, x is open and closed if and only if $x \in A$.*
3. *If $x \in T$ is closed, I is an arbitrary set, the elements $y_i \in T$ with $i \in I$ are open, and*

$$x \leq \bigvee_{i \in I} y_i,$$

   then there exists a finite subset J of I such that

$$x \leq \bigvee_{i \in J} y_i.$$

4. *If $x \in T$ is open, I is an arbitrary set, the elements $y_i \in T$ with $i \in I$ are closed, and*

$$x \geq \bigwedge_{i \in I} y_i,$$

   then there exists a finite subset J of I such that

$$x \geq \bigwedge_{i \in J} y_i.$$

5. *If u is an atom of $\mathcal{T}$, then u is closed.*
6. *If u is an atom of $\mathcal{A}$, then u is an atom of $\mathcal{T}$.*

The extension theorem for Boolean algebras may be stated as follows:

Theorem 4.8 (Extension theorem) *For any Boolean algebra $\mathcal{A}$, there exists a complete and atomistic Boolean algebra $\mathcal{T}$ which is a perfect extension of $\mathcal{A}$.*

The perfect extension $\mathcal{T}$ is essentially determined uniquely by the Boolean algebra $\mathcal{A}$.

In the following we shall consider a fixed complete Boolean algebra

$$\mathcal{A} =< A, \vee, 0, \wedge, 1,^c >$$

with a complete and atomistic perfect extension

$$\mathcal{T} =< T, \vee, 0, \wedge, 1,^c > .$$

The set of all closed elements of $\mathcal{T}$ will be denoted by $Cl(T)$. Let f be a completely additive operator on A.

Definition 4.3 For any function f from A to A, f^+ is the function from T to T defined by the formula

$$f^+(x) = \bigvee_{x \geq y \in Cl(T)} \bigwedge_{y \leq z \in A} f(z),$$

for any $x \in T$.

Here we list some of the propositions of the function f^+ proved by B. Jonsson and A. Tarski.

Proposition 4.9 *If f is an additive function from A to A, then f^+ is a completely additive function from T to T, and $f^+|_A = f$.*

Proposition 4.10 *If f is an additive function from A to A, and g is a completely additive function from T to T such that $g|_A \leq f$, then $g \leq f^+$.*

Proposition 4.11 *Let f and g be additive operators from A to A, then*

$$(f \circ g)^+ = f^+ \circ g^+.$$

As an immediate corollary, if both f and g are completely additive operators on A, then f and g commute if and only if f^+ and g^+ commute.

The work of Jonsson and Tarski also implies the following results.

Proposition 4.12 *Let f be a completely additive operator on A, and f^+ defined as in definition 3.*

1. *if f is self-conjugate, so is f^+.*

2. *Let g be another additive operator on A. Then $f \leq g$ implies $f^+ \leq g^+$.*

3. *$x \leq f(x)$ for all $x \in A$ implies $x \leq f^+(x)$ for all $x \in T$.*

4. *f^+ is normal if and only if f is normal.*

5. *If f is an averaging operator, i.e.,*

$$f(f(x) \wedge y) = f(x) \wedge f(y),$$

then so is f^+.

Hence if a complete additive operator $f = f_B$ is defined by a complete subalgebra B of A, then f^+ a normal and closed averaging operator which is completely additive. Furthermore, f_B^+ and f_C^+ commute if and only if f_B and f_C commute, where C is another complete subalgebra of A. It is also proven in [19] that for every equations only involving Boolean addition, multiplication and the additive functions, if it is satisfied in the Boolean algebra A, it is also satisfied in the complete and atomic Boolean algebra T.

A complete Boolean algebra A with a set $\{f_i\}_{i \in I}$ of pairwise commuting completely additive operators which are normal, closed averaging operators is called **generalized cylindric algebra** and denoted by $< A, \vee, \wedge, ^c, 0, 1, f_i >_{i \in I}$. This algebra is said to be atomic if the Boolean algebra A is. Applying the above theorems and propositions, we have:

Theorem 4.13 *A complete algebra $\mathcal{A} =< A, \vee, \wedge, ^c, 0, 1, f_i >_{i \in I}$ is a generalized cylindric algebra if and only if $\mathcal{A}$ is isomorphic into an algebraic system $\mathcal{T} =< T, \cup, \cap, ^c, \emptyset, 1, f_i^+ >_{i \in I}$ where $\mathcal{T}$ is a complete atomic generalized cylindric algebra with the usual set-theoretic operations, A is regular set-field and $f_i^+(x) \in A$ whenever $x \in A$ for any $i \in I$.*

Note that every completely additive operator on an complete and atomistic Boolean algebra is uniquely determined by its value on the set of atoms, which induces a partition of the set of atoms, hence we have:

Theorem 4.14 *Any sublattice of the lattice of complete subalgebras of a complete Boolean algebra, with the property that any two subalgebras commute is dually-isomorphic to a sub-system of a partition lattice R on some set U, where the sub-system is join-closed and any two partitions in this system commute.*

4.2 Partitions on the Stone Representation Space

Given a complete Boolean algebra A, let $\mathcal{U}$ be its Boolean space. Then we may consider A to be the field of closed and open subsets of $\mathcal{U}$. Let $\mathcal{F}$ be the family of partitions on $\mathcal{U}$ which divide $\mathcal{U}$ into two clopen sets. The collection of all intersects of members of $\mathcal{F}$ forms a complete lattice P of partitions of $\mathcal{U}$.

D. Sachs [34] showed that the lattice P is dually-isomorphic to the lattice of subalgebras of the Boolean algebra A. And P is a sub-system of the full partition lattice $Par(\mathcal{U})$ of $\mathcal{U}$ which is meet-closed.

For any subalgebra B of the Boolean algebra A, we associate a partition π of the Stone space $\mathcal{U}$ by the following rule: two points x and y lie in the same block of π if and only if for any elements $b \in B$, x and y both belong to the clopen set b or they both belong to the clopen set b^c.

Given two subalgebras B and C of A, let π and σ be the associated partition in P, we have

Theorem 4.15 *The partitions π and σ commute if and only if B and C satisfy the following condition:*
($\star$) If $b \wedge c = 0$ for some $b \in B$ and $c \in C$, then there exist $b_1 \in B$ and $c_1 \in C$ such that $b_1 \geq c$, $c_1 \geq b$ and $b_1 \wedge c_1 = 0$.

Proof. Assume the condition ($\star$) holds for subalgebras B and C. Let π_1 be a block of π and σ_1 be a block of σ, and $\pi_1 \cap \sigma_1 = \emptyset$. We show that π_1 and σ_1 must be in different blocks of $\pi \vee \sigma$.

Claim: There exist elements $b \in B$, and $c \in C$ such that in the Stone space $\mathcal{U}$, $\pi_1 \in b$, $\sigma_1 \in c$ and $b \wedge c = 0$.

Indeed, the Boolean space $\mathcal{U}$ is a totally disconnected, compact Hausdorff space. So it is normal. Moreover, for any two disjoint closed subsets on $\mathcal{U}$, there is a clopen set separating them. Using this fact and note that π_1 is a π-block which is disjoint from σ_1, we derive that for every point y in σ_1, there exists a clopen set b_y in B which contains y and disjoint from π_1. The family of set $\{b_y\}$ is a open cover of the closed set σ_1, from compactness, there is a finite subset $\{b_y(i)|i = 1, 2, \ldots, n\}$ whose union covers σ_1. Let $\tilde{b} = (\vee_i b_y(i))^c \in B$, then $\tilde{b} \supset \pi_1$ and $\tilde{b} \cap \sigma_1 = \emptyset$.

Similarly there is an element $\tilde{c} \in C$ such that $\pi_1 \cap c = \emptyset$ and $\sigma_1 \subset \tilde{c}$.

If $\tilde{b} \wedge \tilde{c} = 0$, then the claim is true.

Otherwise, let $d = \tilde{b} \wedge \tilde{c}$, which is a clopen set of $\mathcal{U}$, and disjoint from either π_1 or σ_1. Apply the above argument to the closed set π_1 and d, we obtain a clopen set $\hat{b} \in B$ such that $\pi_1 \in \hat{b}$ and $\hat{b} \cap d = \emptyset$. now let $b = \tilde{b} \wedge \hat{b}$ and $c = \tilde{c}$, we have $\pi_1 \in b$, $\sigma_1 \in c$, and $b \wedge c = 0$.

Back to the Proof of theorem. Assume that π_1 and σ_1 belong to the same block of $\pi \vee \sigma$. Then there exists a finite sequence

$$\pi_a, \sigma_b, \pi_c, \ldots, \pi_y, \sigma_z,$$

such that the adjacent blocks intersect each other and $\pi_1 = \pi_a, \sigma_1 = \sigma_z$.

We already have $\pi_1 \in b$, $\sigma_1 \in c$ and $b \wedge c = 0$. From the condition ($\star$), there exist $c_1 \geq b$, $b_1 \geq c$ and $b_1 \wedge c_1 = 0$. Since σ_b is a block of σ and intersects with π_1, it must be contained in c_1, i,e, $\sigma_b \subset c_1$, and $c_1 \wedge \sigma_1 = \emptyset$. Repeat this procedure, we have a sequence of elements which are disjoint with σ_1,

$$b, c_1, b_2, c_3, \ldots, b_k, c_l,$$

such that $\pi_1 \in b$, $\sigma_b \in c_2$, $\pi_c \in b_2$, $\ldots$, $\pi_y \in b_k$ and $\sigma_1 = \sigma_z \in c_l$. It is a contradiction.

Hence the partitions π and σ commute.

Conversely, assume that the partitions π and σ commute. We prove that B and C satisfy the condition ($\star$).

Let $b \in B$, $c \in C$ and $b \wedge c = 0$. Then the π-blocks contained in b are disjoint from the σ-blocks contained in c. Choose $\tilde{c} \in C$ and $\tilde{b} \in B$ such that $\tilde{c} \geq b$ and $\tilde{b} \geq c$. If $\tilde{b} \wedge \tilde{c} = 0$, then we are done. Otherwise, for any point $t \in \tilde{b} \wedge \tilde{c}$, consider the block π_t of π and σ_t of σ which contain the point t. Since the partitions π and σ commute, hence we have either $\sigma_t \cap b = \emptyset$ or $\pi_t \cap c = \emptyset$. If $\sigma_t \cap b = \emptyset$, then there is an element $c_t \in C$ such that $\sigma_t \subset c_t$ and $b \wedge c_t = 0$. Similarly, if $\pi_t \cap c = \emptyset$, then there is an element $b_t \in B$ such that $\pi_t \subset b_t$ and $b_t \wedge c = 0$.

That is, for any $t \in \tilde{b} \wedge \tilde{c}$, there is either a clopen subset $b_t \in B$ such that b_t separates t and c, or there is a clopen subset $c_t \in C$ such that c_t separates t and b.

The family of set $S = \{b_t \mid \pi_t \cap c = \emptyset\} \cup \{c_t \mid \sigma_t \cap b = \emptyset\}$ is a open cover of closed set $d = \tilde{b} \wedge \tilde{c}$. Hence there is a finite subset T of S whose elements cover d. Now let

$$\begin{aligned} b_1 &= \tilde{b} - \vee\{b_t \mid \pi_t \cap c = \emptyset, b_t \in T\}, \\ c_1 &= \tilde{c} - \vee\{c_t \mid \sigma_t \cap b = \emptyset, c_t \in T\}, \end{aligned}$$

then $b_1 \geq c$, $c_1 \geq b$ and $b_1 \wedge c_1 = 0$. □

Corollary 4.16 *If B and C are complete subalgebras of A, then B and C commute if and only if their associated partitions π and σ commute. In this case, the join of π and σ in P is the same as the join of π and σ in the full partition lattice $Par(S)$.*

Proof. To show the first part, it is enough to show that two complete subalgebras B and C commute if and only if B and C satisfy the condition $(\star)$.

One of the characterization of commutativity of complete subalgebras B and C is the disjoint elements of B and C are separated by $B \wedge C$. It follows immediately that if B and C commute, they satisfy the condition $(\star)$.

Conversely, if two complete subalgebras B and C satisfy the condition $(\star)$, let $b \in B$, $c \in C$ and $b \wedge c = 0$, then there exist $b_1 \in B$, $c_1 \in C$ such that $b_1 \geq c$, $c_1 \geq b$ and $b_1 \wedge c_1 = 0$. Repeat this procedure, we have a sequence of pairs

$$(b, c), (c_1, b_1), (b_2, c_2), \ldots$$

such that $b_i \in B$, $c_i \in C$ and $b_i \wedge c_i = 0$, and

$$\begin{aligned} b &\leq c_1 \leq b_2 \leq c_3 \leq \cdots \\ c &\leq b_1 \leq c_2 \leq b_3 \leq \cdots \end{aligned}$$

Let t be the limit of the monotonic sequence $b, c_1, b_2, c_3, \ldots$, then $t = \vee b_{2i} = \vee c_{2i+1} \in B \wedge C$ and $t \geq b$, $t \wedge c = 0$. That proves B and C commute.

To show the second part, let π_1 and σ_1 be disjoint blocks of π and σ. from the proof of the theorem 4.15, there are elements $b \in B$ and $c \in C$ such that $\pi_1 \subset b$, $\sigma_1 \in c$ and $b \wedge c = 0$.

By the commutativity of Boolean subalgebras, there exists an element $d \in B \wedge C$ such that $d \geq b$ and $d \wedge c = 0$. As a clopen set, $d \in B \wedge C$ separates π_1 and σ_1. This proves the partition associated to $B \wedge C$ is the join of π and σ as in the full partition lattice $\mathrm{Par}(\mathcal{U})$. □

Corollary 4.17 *Let L be a sublattice of the lattice of subalgebras of a Boolean algebra A, with the property that for any two elements B and C in this lattice L, disjoint elements of B and C are separated by $B \cap C$. Then the lattice L is a linear lattice.*

Proof. Given two subalgebras B and C in the lattice L, let π and σ be the associated partitions. From the proof of the theorem 4.15, with the assumption that disjoint elements of B and C are separated by $B \wedge C$, we conclude that π and σ commute. Then argue similarly to that of proof in Corollary 4.16, we derive disjoint blocks of π and σ are separated by a clopen set in $B \wedge C$. So the join of π and σ in the lattice L is same as the join of π and σ in the full partition lattice Par(U).

By the fact that P is a meet-closed sub-system of $\mathrm{Par}(\mathcal{U})$ and L is isomorphic to a sublattice of P, we conclude that L is a linear lattice. □

4.3 Commuting Subrings of Commutative Rings

Every Boolean algebra is a commutative ring with the product and addition defined as:

$$\begin{aligned} x \times y &= x \wedge y, \\ x + y &= (x \vee y) \wedge (x \wedge y)^c. \end{aligned}$$

In fact, a Boolean ring can be characterized as a commutative ring such that every element is idempotent, i.e., $x^2 = x$ for any x belongs to the ring.

A subset P of a Boolean algebra A is an **ideal** if

1. for any $x \in P$ and $y \leq x$, we have $y \in P$; and

2. if both x and y belong to P, then so is $x \vee y$.

An ideal P is a **prime ideal** if for any x and y in P, $x \wedge y \in P$ implies at least one of them is in P.

The notion of prime ideal of a Boolean algebra is the same as an ideal of the Boolean ring. P is a prime ideal of the Boolean algebra A if and only if it is a prime ideal of the Boolean ring A. Also B is a subalgebra of the Boolean algebra A if and only if it is a subring of the Boolean ring A.

Notice that the Stone representation space of the Boolean algebra A is isomorphic to $Spec(A)$, the set of all prime ideals of A. Consider the partition (equivalence relation) defined by a subalgebra in the Stone space as stated in previous section, one can describe it in terms of prime ideals. As before let π be the partition on the Stone space associated with subalgebra B, and its equivalence relation is R_π. Two prime ideals P and Q of a Boolean algebra A, when taken as two points on the Stone space, is equivalent under the relation R_π, if and only if for any clopen subset $b \in B$ on the Stone space, points P and Q both lie in b or both lie in b^c. In another word, element b of the Boolean algebra A belongs to both prime ideals P and Q, or it belongs to neither of them.

This suggests a way to define the commutativity of subrings of an arbitrary commutative ring $\mathcal{R}$.

Let $\mathcal{R}$ be a commutative ring. Let $Spec(\mathcal{R})$ be the set of all prime ideas of $\mathcal{R}$. Given two subrings B and C, we say that subrings B and C **commute** if for any two prime ideas $P, Q \in Spec(\mathcal{R})$,

$$P \cap B \cap C = Q \cap B \cap C,$$

implies there exists a prime idea $R \in Spec(\mathcal{R})$ such that

$$P \cap B = R \cap R \qquad \text{and} \qquad R \cap C = Q \cap C.$$

From the previous section we know that when $\mathcal{R}$ is a Boolean ring, this definition is equivalent to say that the partitions associated with subrings B and C on the Stone representation space commute. In that case, let P and Q be two prime ideals, if

$$P \cap B = Q \cap B, \qquad \text{and} \qquad P \cap C = Q \cap C,$$

then we have:

$$P \cap (B \vee C) = Q \cap (B \vee C).$$

Indeed, if $t \in P \cap (B \vee C)$, then $t \in P$ and

$$t = \sum_{i=1}^{n} b_i \wedge c_i,$$

for some $b_i \in B$ and $c_i \in C$, where $\{b_i\}$ is a disjoint family. Since P is an ideal, so $b_i \wedge c_i \in P$ for every i. From the fact that P is a prime ideal, we have either $b_i \in P$ or $c_i \in P$.

From $P \cap B = Q \cap B$ and $P \cap C = Q \cap C$, we have, either $b_i \in Q$ or $c_i \in Q$. Hence $t = \sum b_i \wedge c_i \in Q$. Therefore $P \cap (B \vee C) \subseteq Q \cap (B \vee C)$. Similarly, $Q \cap (B \wedge C) \subseteq P \cap (B \wedge C)$, thus they are equal.

Let $r(B)$ be the equivalence relation on $Spec(\mathcal{R})$ defined by:

$$(P, Q) \in r(B) \text{ whenever } P \cap B = Q \cap B.$$

From the above argument, for Boolean ring $\mathcal{R}$, we have

$$r(B) \cap r(C) = r(B \vee C),$$

and if B and C commute as subrings, then

$$r(B) \vee r(C) = r(B) \circ r(C) = r(C) \circ r(B) = r(B \wedge C).$$

Denote by $Er(Spec(\mathcal{R}))$ the set of all equivalence relations on $Spec(\mathcal{R})$, where $\mathcal{R}$ is a Boolean ring. Suppose $\mathcal{L}$ is a sublattice of the lattice of subrings of the Boolean ring $\mathcal{R}$, with the property that any two elements in $\mathcal{L}$ commute as subrings. Then $\gamma : \mathcal{L} \longrightarrow Er(Spec(\mathcal{R}))$ is an anti-isomorphism from $\mathcal{L}$ into the set of equivalence relations on $Spec(\mathcal{R})$, where any two equivalence relations in Image(γ) commute. Hence we have

Theorem 4.18 *A sublattice $\mathcal{L}$ of the lattice of subrings of a Boolean ring A, with the property that any two elements in $\mathcal{L}$ commute as subrings, is a linear lattice.*

REMARK. For general commutative rings, we don't have such a nice structure. Notice that in defining the commutativity of subrings, we do not require that the subrings to be complete. In general, even for a Boolean algebra A, the commutativity of B and C as subalgebras and subrings are not equivalent.

Chapter 5

Commutativity in Probability Theory

In this chapter we study the commutativity of complete Boolean σ-algebras. We shall restrict ourselves on probability spaces and their measure algebras. In the following, by qualitative commutativity we mean the commutativity defined in Chapter 1. Similar to the notions of qualitative independence and stochastic independence in probability theory, we propose a notion of stochastic commutativity, which is a generalization of stochastic independence. We show that for a σ-Boolean algebra with a normal and positive measure, qualitative commutativity is a necessary condition for stochastic commutativity. And with reasonable assumptions, it is also a sufficient condition for the existence of a probability measure with respect to which stochastic commutativity holds. We study the structure of pairwise stochastically commuting σ-algebras, and give some applications in the lattices of stochastically commuting σ-algebras.

5.1 Conditional Probability

Let $S = \{\Omega, \Sigma, P\}$ be a probability space, where Ω is the sample space, Σ is a σ-algebra of the subsets of Ω, and P is the probability measure.

Recall that two σ-subalgebras B and C of Σ are said to be stochastic independent if for every elements $b \in B$ and $c \in C$, we have

$$P(b)P(c) = P(b \cap c).$$

In order to define the stochastic commutativity, let us recall the notion of conditional probability.

1. *Conditional probability* given an event b.

   Given an event $b \in \Sigma$ with $P(b) > 0$, for an arbitrary event $a \in \Sigma$, the conditional probability of a given b is $P(a|b)$, which can be computed as

   $$P(a|b) = \frac{P(a \cap b)}{P(b)}.$$

2. *Conditional probability with respect to a partition of* Ω.

   Let $\{b_i, i \geq 1\}$ be a disjoint set of events which satisfy

   $$\bigcup_i b_i = \Omega, \qquad \text{and} \qquad P(b_i) > 0,$$

   for all $i = 1, 2, \ldots$, i.e., $\{b_i, i \geq 1\}$ forms a partition of the sample space Ω. Let B be the smallest σ-algebra containing all b_i. To every event $a \in \Sigma$, define a random variable $P_B(a)$ by letting

   $$P_B(a)(w) = P(a|b_i) \qquad \text{if } w \in b_i.$$

   The random variable $P_B(a)$ is called the conditional probability of event a with respect to the σ-algebra B.

3. In general, let B be an arbitrary σ-subalgebra of Σ. To define the conditional probability with respect to B, we shall need the Radon-Nikodym theorem.

   Theorem 5.1 (Radon-Nikodym) *Let $\mathcal{A}$ be a σ-algebra of the subset of a set Ω, let $\mu(a)$ be a σ-finite measure and $\nu(a)$ a σ-additive real set function on $\mathcal{A}$. Let further $\nu(a)$ be absolutely continuous with respect to $\mu(a)$, i.e., $\mu(a) = 0$ implies $\nu(b) = 0$ for every $b \in \mathcal{A}$, and $b \subset a$. Under these conditions there exists a function $f(w)$, measurable with respect to the σ-algebra $\mathcal{A}$, such that for every $b \in \mathcal{A}$, the relation*

   $$\nu(a) = \int_a f(w) d\mu$$

   holds. If ν is nonnegative, then $f(w) \geq 0$. The function $f(w)$ is unique up to a set of measure zero.

Now in the σ-algebra Σ, fix an event $a \in \Sigma$. Consider the measures $\mu(b) = P(b)$ and $\nu(b) = P(a \cap b)$ on the σ-algebra B, and apply the Radon-Nikodym theorem. Obviously ν is absolutely continuous with respect to μ. Hence there exists a function $f(w)$ which is B-measurable and

$$p(b \cap a) = \int_b f(w) dP_B,$$

where P_B is the restriction of P on B. $f(w)$ is determined up to a set of measure zero, and $0 \leq f(w) \leq 1$. The random variable $f(w)$ is called the *conditional probability* of the event a with respect to the σ-subalgebra B, and denoted by $P_B(a)$.

In short, $P_B(a)$ is a random variable which is unique up to a set of measure zero such that

(a) $P_B(a)$ is B-measurable;

(b) For any $b \in B$,

$$P(a \cap b) = \int_b P_B(a) dP_B.$$

In particular, when $b = \Omega$,

$$P(a) = \int_\Omega P_B(a) dP_B,$$

i.e., its expectation is $P(a)$.

Definition Let $S = (\Omega, \Sigma, P)$ be a probability space. Let B and C be σ-subalgebras of Σ, and $D = B \cap C$. The σ-subalgebras B and C are **stochastically commuting** whenever for every $b \in B$ and $c \in C$, we have

$$P_D(b)P_D(c) = P_D(b \cap c) \qquad (a.e.)$$

REMARK. When $D = B \cap C$ is atomic with atoms $d_1, d_2, \ldots$ such that $P(d_i) > 0$ for all i, then to say that B and C stochastically commute is equivalent to say that $B|d_i$ and $C|d_i$ are stochastic independent, for all i.

5.2 Conditional Expectation Operator

The notion of stochastical commutativity can also be expressed in terms of conditional expectation operators.

Again let $S = (\Omega, \Sigma, P)$ be a probability space, and B be an arbitrary σ-subalgebra of Σ. For every random variable f with finite expectation ($Ef < \infty$), by Radon-Nikodym theorem, there exists a B-measurable random variable $E_B(f)$, which is unique up to a set of measure zero, such that for every $b \in B$,

$$\int_b f dP = \int_b E_B(f) dP_B,$$

where P_B is the restriction of P on the σ-subalgebra B.

The random variable $E_B(f)$ is called the *conditional expectation* of random variable f with respect to the σ-subalgebra B, and the operator $E_B : \mathcal{L}^1(\Omega, \Sigma, P) \longmapsto \mathcal{L}^1(\Omega, B, P)$ is called the *conditional expectation operator*. It is a projection in the space of all integrable random variables of (Ω, Σ, P). Note that from the definition,

1. integrable random variables f and $E_B(f)$ have the same expectation:
$$E(f) = E(E_B(f));$$
2. if $B = \{\emptyset, \Omega\}$, then $E_B(f) = E(f) \in \mathbb{R}$;
3. if f is B-measurable, then $E_B(f) = f$ *(a.e.)*. In particular, $E_\Sigma(f) = f$ *(a.e.)*.

Conditional probability $P_B(a)$ with respect to B is a special case of conditional expectation. For any event $a \in \Sigma$, let χ_a be its indicator, i.e.,

$$\chi_a(w) = \begin{cases} 1 & \text{if } w \in a, \\ 0 & \text{otherwise.} \end{cases}$$

Then $P_b(a) = E_B(\chi_a)$ *(a.e.)*

Conditional expectation operator has been heavily studied in Functional Analysis and Linear Operator Theory. The following is a list of principal properties of conditional expectation operator.

Theorem 5.2 *Let B be a σ-subalgebra of $S = (\Omega, \Sigma, P)$. Let f and g be integrable random variables. The conditional expectation operator E_B has the following properties.*

1. *Linearity.*
$$E_b(af + bg) = aE_B(f) + bE_B(g),$$
where a and b are real numbers.

2. *Positivity.*

$$f \geq 0 \text{ (a.e.)} \quad \textit{implies} \quad E_B(f) \geq 0 \text{ (a.e.)};$$
$$f \geq g \text{ (a.e.)} \quad \textit{implies} \quad E_B(f) \geq E_B(g) \text{ (a.e.)}$$

3. *Monotonic convergence.* $f_n \to f$ *(a.e.) monotonically implies* $E_B(f_n) \to E_B(f)$ *(a.e.) monotonically.*

4. *Independence. Two* σ*-subalgebras* B *and* C *are stochastic independent if and only if* $E_B(f) = E(f)$ *for any* C*-measurable integrable random variable* f*.*

5. *Averaging identity.*

$$E_B(E_B(f) \cdot g) = E_B(f)E_B(g), \qquad \text{(a.e.)}$$

In particular if f *is a* B*-measurable random variable then*

$$E_B(fg) = f \cdot E_B(g). \qquad \text{(a.e.)}$$

6. *Triangle inequality.*

$$|E_B(f)| \leq E_B(|f|), \qquad \text{(a.e.)}$$

7. *Inclusion. If* $B \subset C$ *are* σ*-subalgebras, then*

$$E_B(E_C(f)) = E_C(E_B(f)) = E_B(f).$$

In general, a conditional expectation operator can be characterized as an averaging operator, (c.f. G-C Rota [33]). An averaging operator A in $\mathcal{L}^p(\Omega, \Sigma, \mu)$ (where p is a fixed real number, $1 \leq p \leq \infty$) is a linear operator in $\mathcal{L}^p(\Omega, \Sigma, \mu)$ with the following three properties:

1. A is a contraction operator:

$$\int_\Omega |(Af)(s)|^p \mu(ds) \leq \int_\Omega |f(s)|^p \mu(ds),$$

   for all $f \in \mathcal{L}^p(\Omega, \Sigma, \mu)$.

2. If f is of class $\mathcal{L}^p(\Omega, \Sigma, \mu)$ and g is an essentially bounded function on (Ω, Σ, μ), then the function $(Af)(s)(Ag)(s)$ is of class $\mathcal{L}^p(\Omega, \Sigma, \mu)$ and

$$A(gAf) = (Ag)(Af).$$

3. If $I(s)$ is the function identically equal to one on Ω, then $AI = I$.

Proposition 5.3 *Two σ-subalgebras B and C stochastically commute if and only if the following equation*

$$E_D(f)E_D(g) = E_D(fg) \qquad (a.e.) \tag{5.1}$$

holds for every B-measurable integrable random variables f and C-measurable integrable random variable g, where $D = B \cap C$.

Proof. If

$$E_D(f)E_D(g) = E_D(fg)$$

holds for all B-measurable random variable f and C-measurable random variable g, then let $f = \chi_b$ and $g = \chi_c$ for $b \in B$ and $c \in C$, we get that B and C stochastically commute.

Conversely, if B and C stochastically commute, then the Eq. 5.1 holds for all nonnegative simple random variables f and g, where f is B-measurable and g is C-measurable. Since every nonnegative random variable is a limit of monotonic nonnegative simple random variables, our result follows from the linearity and monotone convergence of the conditional expectation operators. □

Fix a σ-subalgebra Σ', two σ-subalgebras B and C are said to be conditional independent with respect to Σ' whenever for any B-measurable integrable random variable f and C-measurable integrable random variable g, we have

$$E_{\Sigma'}(f)E_{\Sigma'}(g) = E_{\Sigma'}(fg).$$

If two σ-subalgebras B and C stochastically commute, we also say that they are conditional independent with respect to their intersection $D = B \cap C$.

We have the following proposition about conditional independent σ-algebras, (c.f. Rao [30]).

Proposition 5.4 *Let B, C, Σ' be σ-subalgebras of (Ω, Σ, P). Let $A_1 = \sigma(B, \Sigma')$ be the σ-subalgebra generated by B and Σ', and $A_2 = \sigma(C, \Sigma')$ the σ-subalgebra generated by C and Σ'. Then the following are equivalent statements.*

1. *B and C are conditional independent with respect to Σ'.*

2. *For each* $b \in B$,

$$P_{A_2}(b) = P_{\Sigma'}(b) \qquad (a.e.)$$

3. *For each* $c \in C$,

$$P_{A_1}(c) = P_{\Sigma'}(c) \qquad (a.e.)$$

4. *For each* B*-measurable integrable random variable* $f : \Omega \to \mathbb{R}^+$,

$$E_{A_2}(f) = E_{\Sigma'}(f) \qquad (a.e.)$$

5. *For each* C*-measurable integrable random variable* $g : \Omega \to \mathbb{R}^+$,

$$E_{A_1}(g) = E_{\Sigma'}(g) \qquad (a.e.)$$

Proof. (1) $\Longrightarrow$ (2) Assume that B and C are conditional independent with respect to Σ'. We need to verify that $P_{A_2}(b)$ and $P_{\Sigma'}(b)$ satisfy the functional equation on A_2 for each $b \in B$. Since the σ-algebra A_2 is generated by the sets of the form $\{c \cap t : c \in C, t \in \Sigma'\}$, it suffices to verify the desired equation for the generators. We have

$$\begin{aligned}
\int_{c\cap t} P_{A_2}(b) dP_{A_2} &= \int_{c\cap t} E_{A_2}(\chi_b) dP_{A_2} \\
&= \int_t \chi_{b\cap c} dP \\
&= \int_t E_{\Sigma'}(\chi_{b\cap c}) dP_{\Sigma'} \\
&= \int_t P_{\Sigma'}(b) P_{\Sigma'}(c) dP_{\Sigma'} \\
&= \int_t E_{\Sigma'}(\chi_c P_{\Sigma'}(b) dP_{\Sigma'} \\
&= \int_{t\cap c} P_{\Sigma'}(b) dP_{A_2},
\end{aligned}$$

where the first three equations are by definition, the fourth one is the hypothesis, the fifth is the averaging property of conditional expectation operators, and the last one because of $\Sigma' \subseteq A_2$.

Both $P_{A_2}(b)$ and $P_{\Sigma'}(b)$ are A_2-measurable, and $c \cap t$ is an arbitrary generator of A_2, by uniqueness of conditional probability, $P_{A_2}(b) = P_{\Sigma'}(b)$ almost everywhere.

(2) $\Longrightarrow$ (1) To prove the conditional independence of B and C with respect to Σ', consider $b \in B$ and $c \in C$, with $A_2 = \sigma(C, \Sigma')$, then

$$\begin{aligned}
P_{\Sigma'}(b \cap c) &= E_{\Sigma'}(\chi_{b \cap c}) \\
&= E_{\Sigma'}(E_{A_2}(\chi_{b \cap c})), \\
&= E_{\Sigma'}(\chi_c E_{A_2}(\chi_b)) \\
&= E_{\Sigma'}(\chi_c P_{A_2}(b)) \\
&= E_{\Sigma'}(\chi_c P_{\Sigma'}(b)), \qquad a.e. \\
&= P_{\Sigma'}(b) P_{\Sigma'}(c), \qquad a.e.
\end{aligned}$$

where the first equation is by definition, the second is because $c \in A_2$, the third is by definition, the fourth is the hypothesis, and the last one is the averaging property.

(1) $\Longleftrightarrow$ (3). Similar to the above proof.

(2) $\Longleftrightarrow$ (4) By expressing (2) in terms of conditional expectations, we have

$$E_{A_2}(\chi_b) = E_{\Sigma'}(\chi_b) \qquad a.e. \tag{5.2}$$

for all $b \in B$. Hence by linearity of the conditional expectation operator, the equation holds for all B-measurable simple random variables. Since every nonnegative B-measurable random variable can be expressed as a limit of simple random variables, we get (4).

(4) $\Longrightarrow$ (2) is trivial.

(3) $\Longleftrightarrow$ (5) is similar to the above. □

Theorem 5.5 *let $S = (\Omega, \Sigma, P)$ be a probability space. Two σ-subalgebras B and C stochastically commute if and only if their corresponding conditional expectation operators commute, i.e.,*

$$E_B \circ E_C = E_C \circ E_B = E_D, \tag{5.3}$$

where $D = B \cap C$. Indeed, the equality of any two operators of equation (5.3) implies B and C stochastically commute.

Proof. Assume B and C stochastically commute, and let $D = B \cap C$. For any nonnegative integrable random variable f, $g = E_B(f)$ is B-measurable. Apply Proposition 5.4 (4),

$E_C(g) = E_D(g)$, i.e., $E_C(E_B(f)) = E_D(E_B(f))$, the latter one is equal to $E_D(f)$ since $D \subset B$. Similarly, $E_B \circ E_C = E_D$.

Conversely, if $E_B \circ E_c = E_D$, then $E_B(E_C(f)) = E_D(f) = E_D(E_C(f))$, by proposition (1) and (5), B and C are conditional independent with respect to D. Similarly, $E_C \circ E_B = E_D$ implies B and C are conditional independent with respect to D.

If $E_B \circ E_C = E_C \circ E_B$, Let $T = E_B \circ E_C$. For arbitrary B and C, we have

$$E_D = \lim T^n.$$

Since conditional expectation operators are projections and $T = E_B \circ E_C = E_C \circ E_B$, we get $T^n = T$, so $E_D = T = E_B \circ E_C$, then by the above argument, B and C stochastically commute. □

Example

1. Inclusion. Assume σ-subalgebras $B \subseteq C$. Then $D = B \cap C = B$. The commutativity follows from the property (7) of conditional expectation operator. Also we may prove it as follows. For any integrable random variable f, $E_B(f)$ is B-measurable, hence C-measurable. So we have

   $$E_C(E_B(f)) = E_B(f) = E_D(f).$$

   By Theorem 5.5, B and C stochastically commute.

2. Independence. Assume that B and C are stochastically independent. Then to any event $a \in \Sigma$,

   $$P_D(a, w) = P(a) \qquad (a.e.)$$

   Thus the independence relation $P(b)P(c) = P(b \cap c)$ implies the commutativity of σ-subalgebras B and C.

3. Let $\Omega = X \times Y \times Z$ where $X = Y = Z = [0, 1]$. Let B be the algebra of measurable sets independent of z, and C the algebra of measurable sets independent of x. For any integrable random variable $f(x, y, z)$, it is easy to see

   $$\begin{aligned} E_B(f) &= \int f(x, y, z)dz; \\ E_C(f) &= \int f(x, y, z)dx. \end{aligned}$$

   The commutativity of operators E_B and E_C follows from the Fubini's theorem.

5.3 Stochastically Commuting and Qualitatively Commuting σ-algebras

In this section we discuss the relation between qualitatively commuting and stochastically commuting. We shall show that they are equivalent.

Theorem 5.6 *Let $S = (\Omega, \Sigma, P)$ be a probability space where P is positive, i.e., $P(x) = 0$ if and only if $x = 0$. Assume two σ-subalgebras B and C stochastically commute. Then B and C qualitatively commute as Boolean subalgebras.*

Proof. As before let $D = B \cap C$. For any $b \in B$ and $c \in C$, since $P_D(b)P_D(c) = P_D(b \cap c)$, we have

$$P(b \cap c) = \int_\Omega P_D(b \cap c) dP_D = \int_\Omega P_D(b) P_D(c) dP_D.$$

Now assume $b \cap c = 0$, then $P(b \cap c) = 0$. Let d_1 be the minimal closure of b in D, thus $P_D(b) = 0$ (a.e.) on d_1^c. Therefore

$$\int_{d_1} P_D(b) P_D(c) dP_D = 0,$$

moreover, since on d_1, $P_D(b) > 0$ (*a.e.*), we have

$$P_D(c) = 0 \qquad (a.e.)$$

on d_1. Hence $P(c \cap d_1) = 0$. Thus $c \cap d_1 = 0$. Therefore we prove that disjoint elements of B and C are separated by elements in D. Hence B and C qualitatively commute. □

REMARK If we remove the condition that P is positive, we will get the following result: Let σ-subalgebras B and C stochastically commute. If $b \wedge c = 0$ for some $b \in B$ and $c \in C$, then there exists an element $d \in B \cap C$ such that $d \geq b$ and $P(d \wedge c) = 0$.

Conversely, we have

Theorem 5.7 *Let (Ω, B, P_1) and (Ω, C, P_2) be two probability spaces with the same basic set Ω. Suppose that the σ-algebras B and C are qualitatively commuting, i.e., if $b \wedge c = 0$ for some $b \in B$ and $c \in C$, then there exists $d \in B \cap C$ such that $d \geq b$ and $d \wedge c = 0$. And the probability P_1 coincides with P_2 on $B \cap C$. Let Σ denote the least σ-algebra containing both B and C. Then there exists a probability measure P on Σ which is a common extension of P_1 and P_2, and for which the σ-algebras B and C stochastically commute.*

Special case: When $D = B \cap C$ is an atomic Boolean σ-algebra with atoms $\{d_i, i \geq 1\}$ where $P_1(d_i) = P_2(d_i) > 0$, then B and C qualitatively commute implies they are qualitative independent when restricted on the atoms of D. In this case the theorem become a direct corollary of the following theory of Renyi.

Theorem 5.8 (Renyi) *Let $(\Omega, \mathcal{A}_1, P_1)$ and $(\Omega, \mathcal{A}_2, P_2)$ be two probability spaces with the same basic set Ω. Suppose that the σ-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are qualitatively independent, i.e., if $a_i \in \mathcal{A}_i$, $a_i \neq \emptyset$ $(i = 1, 2)$ then $a_1 \cap a_2 \neq \emptyset$. Let $\mathcal{A}$ be the least σ-algebra containing both $\mathcal{A}_1$ and $\mathcal{A}_2$. Then there exists a probability measure P on $\mathcal{A}$ which is a common extension of both P_1 and P_2, and for which the σ-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are stochastic independent.*

We will outline the proof of theorem 5.7 in general case below. The details will be given in the appendix.

Sketch of the proof of theorem 5.7.

Let A_0 be the minimal Boolean subalgebra containing both B and C. Then $\Sigma = \sigma(A_0)$. Elements in A_0 are of form

$$a = \sum_{k=1}^{n} b_k \cap c_k,$$

where b_k forms a partition of Ω.

For such an element in A_0, define a probability P by

$$P(a) = \sum_{k=1}^{n} \int_{\Omega} P_{1,D}(b_k) P_{2,D}(c_k) dP_D.$$

Then we show that P is well-defined on A_0 and finitely additive. Thus it can be uniquely extended to Σ. We can prove that as a probability on Σ, P is σ-additive, and under which B and C stochastically commute.

5.4 Structure Theorem

As before let $S = (\Omega, \Sigma, P)$ be a probability space, and let $(\mathcal{A}, P)$ be the associated measure algebra of measurable sets Σ module sets of measure zero. The term "measure algebra" $(\mathcal{A}, P)$ means a Boolean σ-algebra $\mathcal{A}$ on which P is normal ($P(e) = 1$ where e is the maximal element) and positive ($P(x) = 0$ if and only if $x = 0$).

Conversely, if $(\mathcal{A}, P)$ is the measure algebra associated with a measure space (Ω, P), we call (Ω, P) a *realization* of $(\mathcal{A}, P)$. Every measure algebra $(\mathcal{A}, P)$ has at least one realization: the Stone representation space X, where the measurable set form the field generated by the clopen subsets of X.

For any Boolean σ-algebra M, let $dim(M)$ be the least cardinal number of a σ-basis of M. M is called **homogeneous** if for every element $m \in M$ and $m \neq 0$, we have $dim(I(m)) = dim(M)$, where $I(m)$ is the principal ideal generated by m. An example of homogeneous measure algebra is the Boolean algebra $I^{\mathfrak{m}}$ of all measurable sets (module null sets) of an infinite product space $\Omega_{\mathfrak{m}} = \prod_\alpha I_\alpha$ of intervals $I_\alpha : 0 \leq x_\alpha \leq 1$, where α and $\mathfrak{m}$ are ordinal numbers, $0 \leq \alpha \leq \mathfrak{m}$, and the measure on $\Omega_{\mathfrak{m}}$ is defined as the product measure of ordinary Lebesgue measure on each interval.

D. Maharam has the following classification theorem of homogeneous measure algebras. (c.f. [25])

Theorem 5.9 (Maharam) *Every homogeneous measure algebra M is isomorphic to I^{γ_0}, where $\gamma_0 = dim(M)$.*

In a measure algebra $(\mathcal{A}, P)$ with a σ-subalgebra B, given an element $x \in \mathcal{A}$, consider all sets $Q \subseteq \mathcal{A}$ such that $I(x) \subseteq \{x \wedge t \mid t \in \sigma(B \cup Q)\}$, where $I(x)$ is the principal ideal of $\mathcal{A}$ generated by x. The smallest cardinal $\mathfrak{m}$ of such a Q is called the **order** of x over B. In particular, an element x is **of order 0** over B if the principal ideal $I(x)$ of $\mathcal{A}$ coincides with $xB = \{x \cap b | b \in B\}$.

Lemma 5.10 *If x is a nonzero element of finite order over B, then there exists a nonzero element $y \leq x$ of order 0 over B.*

Proof. Assume x is of order n, and $I(x) = \{x \wedge t \mid t \in \sigma(B, a_1, a_2, \ldots, a_n)\}$. Consider elements of form $e_1 \wedge e_2 \wedge \cdots, \wedge e_n$ where $e_i = a_i$ or $e_i = a_i^c$. It is easy to see that there is a s_1 of such a form and $x \wedge s_1 \neq 0$. Let $y = x \wedge s_1$. If $z \leq y$, then $z \leq x$, thus $z \in I(x) = \{x \wedge t \mid t \in \sigma(B, a_1, a_2, \ldots, a_n)\}$. Hence z can be expressed as $\vee(b_j \wedge s_j)$, where $b_j \in B$ and s_j ranges over all 2^n elements of form $e_1 \wedge e_2 \wedge \ldots \wedge e_n$. Since $z \leq s_1$ and $s_i \wedge s_j = 0$ if $i \neq j$, we have $z = z \wedge s_1 = b_1 \wedge s_1$. Also note that $z \leq x$, hence $z = z \wedge x = b_1 \wedge s_1 \wedge x = b_1 \wedge y \in \{y \wedge t \mid t \in B\}$. Thus y is of order 0 over B. □

An element x is called **homogeneous** of order $\mathfrak{m}$ over B if $x \neq 0$ and every y such that $0 < y \leq x$ has order $\mathfrak{m}$ over B.

From the lemma, it follows that the order of an homogeneous element x over B is either 0 or an infinite cardinal $\mathfrak{m}$.

Theorem 5.11 *Given a σ-subalgebra B of a measure algebras $(\mathcal{A}, P)$, where all elements in $\mathcal{A}$ have the same order over B. Then there exists a measure algebra (J, μ) such that $\mathcal{A}$ is isomorphic to the direct product $(J, \mu) \times (B, P)$.*

Proof. We will use a result proved by Maharam in her paper Decompositions of Measure Algebras and Spaces ([25]), and The Representation of Abstract Measure Functions ([24]).

> Theorem. Given a σ-subalgebra B of a measure algebra (A, P), there exist disjoint elements $a_1, a_2, \ldots$ of $\mathcal{A}$ and $b_1, b_2 \ldots$ of B such that
>
> 1. $\vee a_i = e$, where e is the maximal element in $\mathcal{A}$.
> 2. a_i is the maximal element which is homogeneous of order $\mathfrak{m}_n$ over B;
> 3. $(\mathcal{A}, P)$ is the direct sum of principal ideals $(I(a_i), P)$, where
>
> $$(I(a_i), P) \cong \begin{cases} \{0,1\} \times (I_B(b_i), P) & \text{if the order of } a_i \text{ is } 0, \\ I \times (I_B(b_i), P) & \text{if the order of } a_i \text{ is } \aleph_0, \\ I^{m_i} \times (I_B(b_i), P) & \text{if the order of } a_i \text{ is } \mathfrak{m}_n. \end{cases}$$
>
> Here $I_B(b)$ is the ideal generated by b in B.

By the theorem, if all elements in $\mathcal{A}$ have the same infinite order $\mathfrak{m}$ over B, then only one summand survives and $(\mathcal{A}, P)$ is isomorphic to $I \times (B, P)$ if $\mathfrak{m} = \aleph_0$ or to $I^{\mathfrak{m}} \times (B, P)$ if $\mathfrak{m} > \aleph_0$.

If all the elements of $\mathcal{A}$ have order 0 over B, then $\mathcal{A}$ is isomorphic to B, and this case is trivial. □

Remark. If the measure algebra $(\mathcal{A}, P)$ and its σ-algebra B are homogeneous, and $dim(\mathcal{A}) > dim(B)$, then from the above decomposition theorem, $\mathcal{A}$ is isomorphic to a direct product of $(J, \mu) \times (B, p)$.

As a corollary, if the measure algebra A has a countable σ-basis, and it has no element of order 0 over B, then from the theorem 5.11, we have $(\mathcal{A}, P) \cong I \times (B, P)$. In fact, in this case, instead of the preceding algebraic decomposition theorem, we may have an analogous result for measure spaces. The following theorems are due to Maharam and Rohlin.

A probability space $S = (\Omega, \Sigma, P)$ is **strictly separable** if the σ-algebra Σ has a countable σ-basis. It is **complete** if for any $w_1 \neq w_2$ in Ω, there is a measurable set a such that $w_1 \in a$ and $w_2 \notin a$. Given a σ-subalgebra of Σ, let Ω_B denote the quotient space $\Omega/\sim$, where $w_1 \sim w_2$ if and only if for all measurable set $b \in B$, w_1 and w_2 both belong to b or both belong to b^c.

Lemma 5.12 (Rohlin, Doob) *If (Ω, Σ, P) is complete and strictly separable, then for each $a \in \Sigma$, there exists a function $f_B(a, w)$ on Ω, with the properties (a) $f_B(a, w) = P_B(a, w)$ (a.e.), and (b) for almost all $w \in \Omega$, fix w, $f_B(a, w)$ is a measure on Σ.*

In ([35], Rohlin], the functions $f_B(a, w)$ are called **canonical system of measures.**

Given a probability space (Ω, Σ, P), there can be at most countably many events with positive measure. List them in a sequence $a_1, a_2, a_3, \ldots,$ such that $P(a_i) \geq P(a_{i+1})$. The sequence $(P(a_1), P(a_2), P(a_3), \ldots)$ is called the **type** of the measure P, and denoted by $\tau(P)$. The type of P is invariant under isomorphism (module null sets).

If $\tau(P) = (0, 0, \ldots,)$, then we call the measure P **continuous**. A measure is continuous if and only if there is no point of positive measure.

Lemma 5.13 *A complete and strictly separable probability space (Ω, Σ, P) with continuous measure P is isomorphic (mod 0) to $(I, \mathcal{B}, \mu)$ where I is the unit interval, $\mathcal{B}$ is the Borel sets and μ is the ordinary Lebesgue measure.*

Theorem 5.14 (Rohlin, Maharam) *Let (Ω, Σ, P) be a probability space which is complete and strictly separable, and B is a σ-subalgebra of the measurable subsets. Let $f_B(a, w)$ be the canonical system of measures. If all the measures $f_B(a, w)$ are of the same type (a.e.), then there exists another σ-subalgebra C of Σ such that*

$$(\Omega, \Sigma, P) = (\Omega_B, B, P) \times (\Omega_C, C, P). \qquad (mod\ 0)$$

In particular, when the measure P is continuous and there is no element of order 0 over B, (in this case all (mod 0) measure $f_B(a, w)$ are continuous), then (Ω, Σ, P) is isomorphic to the algebra of Borel sets on the unit square, and B is isomorphic to the algebra of Borel sets independent of one side.

Since the notion of conditional probability, random variable, and conditional expectation operator can be defined abstractly for measure algebras, (c.f. D. Kappos), we can extend the notion of stochastic commutativity to abstract measure algebras. Now we are ready to state the Structure Theorem.

Theorem 5.15 (Structure) *Let (Σ, P) be a measure algebra, and B and C be two σ-subalgebras. Let $D = B \cap C$, and $A = \sigma(B, C)$ the minimal σ-algebra containing both B and C. Assume that elements in B (C) have the same order over D. If B and C stochastically commute. Then there are measure algebras X, Y and Z such that*

$$A \cong X \times Y \times Z,$$

and when the isomorphism is restricted to B, C, and D, we have

$$\begin{aligned} D &\cong X, \\ B &\cong X \times Y, \\ C &\cong X \times Z. \end{aligned}$$

Proof. Let $X = D$. By theorem 5.11, there exist measure algebra X and Y such that $B \cong X \times Y$, $C \cong X \times Z$. To show that the isomorphism can be extended to A such that $A \cong X \times Y \times Z$, it suffices to show that Y is independent of C. For any $y \in Y$ and $c \in C$,

$$P(y \cap c) = \int_\Omega P_D(y \cap c) dP_D = \int_\Omega P_D(y) P_D(c) dP_D,$$

where Ω is the representation space of A.

Since X and Y are independent, $P_D(y) = P(y)$ is a constant. Hence

$$\int_\Omega P_D(y) P_D(c) dP_d = P(y) \cdot \int_\Omega P_D(c) dP_D = P(y) \cdot P(c),$$

That is, $P(y \cap c) = P(y) \cdot P(c)$, which finishes the proof. □

REMARK. Theorem 5.15 remains true if we remove the condition that all elements in B (C) have the same order over D, and assume that B, C and D are homogeneous, with $dim(B) > dim(D)$, $dim(C) > dim(D)$.

In the case that $dim(\Sigma) = \aleph_0$, we have the following version:

Theorem 5.16 (Structure 2) *Let B and C be two σ-subalgebras of a complete probability space (Ω, Σ, P) such that B and C stochastically commute. Let $D = B \cap C$, and $A = \sigma(B, C)$. Assume A, B, and C are strictly separable. If the conditional probability $g_D(s, w)$ is continuous for almost all $w \in \Omega$. then there exist probability algebras (Δ_i, X_i, μ_i), $i = 1, 2, 3$ such that the following isomorphisms hold (mod 0):*

$$\begin{aligned} (\Omega_D, D, P) &\cong (\Delta_1, X_1, \mu_1) \\ (\Omega_B, B, P) &\cong (\Delta_1, X_1, \mu_1) \times (\Delta_2, X_2, \mu_2) \\ (\Omega_C, C, P) &\cong (\Delta_1, X_1, \mu_1) \times (\Delta_3, X_3, \mu_3), \\ \textit{and} \quad (\Omega_A, A, P) &\cong (\Delta_1, X_1, \mu_1) \times (\Delta_2, X_2, \mu_2) \times (\Delta_3, X_3, \mu_3). \end{aligned}$$

Proof. From the construction, all probability spaces (Ω_K, K, P) are complete and strictly separable, where $K \in \{A, B, C\}$. Let $(\Delta_1, X_1, \mu_1) = (\Omega, D, P)$, then from Theorem 5.14, there exist σ-algebra X_2 of B and X_3 of C such that $(\Delta_2, X_2, \mu_2) = (\Omega_{X_2}, X_2, P)$, $(\Delta_3, X_3, \mu_3) = (\Omega_{X_3}, X_3, P)$ and

$$\begin{aligned} (\Omega_B, B, P) &= (\Delta_1, X_1, \mu_1) \times (\Delta_2, X_2, \mu_2); \qquad (mod\ 0) \\ (\Omega_C, C, P) &= (\Delta_1, X_1, \mu_1) \times (\Delta_3, X_3, \mu_3). \qquad (mod\ 0). \end{aligned}$$

The independence of X_2 and C follows from the proof of Structure Theorem 5.15. Since $A = \sigma(X_2, C)$, and both X_2 and C are strictly separable, there exists a σ-basis T of A consisting elements in X_2 and C such that for any $w_1 \neq w_2$ in Ω_A, there exists an element $t \in T$ such that $w_1 \in t$ and $w_2 \notin t$. Hence X_2 and C defines a pair of mutually complementary decompositions on Ω. Then from the result of Rohlin, (c.f. [35], Sect. 3, No 4) we have that the space Ω_A is the direct product of the factor spaces Ω_{X_2} and Ω_C. This finishes the proof. □

Corollary 5.17 (Join) *Let B and C be two σ-algebras which stochastically commute. Let $D = B \cap C$. If there exist subalgebras Y and Z such that $B = D \times Y$ and $C = D \times Z$, then $B \vee C = \sigma(B, C) = D \times X \times Y$.*

The dual of this corollary is trivial.

Corollary 5.18 (Meet) *If for some measure algebras X, Y and Z, we have $B \vee C \cong X \times Y \times Z$, under the isomorphism $B \cong X \times Y$ and $C \cong X \times Z$, then B and C stochastically commute and $D = B \cap C \cong X$.*

5.5 Lattices of Stochastically Commuting σ-algebras

Let $(\mathcal{A}, P)$ be a Boolean σ-algebra with a non-negative measure P, and B be a σ-subalgebra. If there exists a σ-subalgebra J such that $\mathcal{A} \cong B \times J$, then we say the pair $(\mathcal{A}, B)$ satisfies the **continuous assumption**. Let $\mathcal{L}(\sigma)$ be a lattice of pairwise stochastically commuting σ-algebras. Furthermore, assume that all pairs $(\mathcal{A}, B)$ satisfy the continuous assumption whenever $B \leq \mathcal{A}$.

Example 5.1 *Let B and C stochastically commute, and $B_1 \leq B$. Suppose the continuous assumption holds. Let $C' = C \vee B_1$, then C' and B stochastically commute.*

Proof. Let $B \wedge C = X$. By the Structure Theorem, we have $B = X \times Y$, $C = X \times Z$ and $B \vee C = X \times Y \times Z$. Since $B_1 \leq B$, $B = (X \vee B_1) \times T$ for some $T \leq B$.

Note that $C' = C \vee B_1 = Z \times (X \vee B_1)$. And we have

$$B \vee C' = B \vee C = X \times Y \times Z = (X \vee B_1) \times T \times Z.$$

Thus from corollary 5.18, $B \wedge C' = X \vee B_1$, and B stochastically commutes with C'. □

Use the Structure Theorem and the two corollaries, we can prove the modular law.

Theorem 5.19 *Let A, B, C be three elements in the lattice $\mathcal{L}(\sigma)$ of stochastically commuting σ-algebras with $A \geq C$. Then we have*

$$A \wedge (B \vee C) = (A \wedge B) \vee C.$$

Proof. Let $A \wedge B \wedge C = B \wedge C = X$. By the Structure Theorem, we have $C = X \times Y$, $A \wedge B = X \times Z$, and $(A \wedge B) \vee C = X \times Y \times Z$ for some measure algebras Y and Z.

Since $B \geq (A \wedge B)$, so $B = X \times Z \times T$ for some measure algebra T, apply the Structure Theorem to the pair B, C, we have: $B \vee C = X \times Y \times Z \times T$.

Since $A \geq (A \wedge B) \vee C$, we have $A = X \times Y \times Z \times S$ for some measure algebra S. Apply the Structure Theorem to the pair A, B we have: $A \vee B = X \times Y \times Z \times T \times S$. Hence $A \wedge (B \vee C) = X \times Y \times Z$, which is equal to $(A \wedge B) \vee C$. □

From the fact that the lattice of commuting σ-algebras is modular, we can easily construct the free lattice of commuting σ-algebras with three generators. It turns out that this free lattice has 28 elements, hence it is a free modular lattice. However, we can derive this fact directly from the Structure Theorem without assuming modular law.

Theorem 5.20 *The free lattice of commuting σ-algebras with three generators is isomorphic to the free modular lattice with three generators.*

Proof. Let x, y, z be the three generators. Let $u = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ and $v = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$. Obviously we have $u \leq v$.

Step 1. The sublattice generated by $x \wedge y, y \wedge z$, and $z \wedge x$ is a Boolean algebra.

Let $x \wedge y \wedge z = P$. It is the minimal element in the lattice. So we may assume $x \wedge y = P \times A$, $x \wedge z = P \times B$ and $z \wedge y = P \times C$. Apply the Structure Theorem, we have:

$$\begin{aligned} (x \wedge y) \vee (x \wedge z) &= P \times A \times B, \\ (x \wedge z) \vee (y \wedge z) &= P \times B \times C, \\ (y \wedge z) \vee (x \wedge z) &= P \times C \times A. \end{aligned}$$

Consider elements $(x \wedge y) \vee (x \wedge z)$ and $y \wedge z$, since

$$((x \wedge y) \vee (x \wedge z)) \leq x \wedge (y \wedge z),$$

and note that $x \wedge y \wedge z$ is the minimal element in the lattice, the equation holds. Apply the Structure Theorem, we have

$$u = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) = P \times A \times B \times C.$$

Hence $x \wedge y, y \wedge z$, and $z \wedge x$ form a Boolean algebra with 8 elements.

Step 2. Since

$$v \geq u > P \times A \otimes B,$$

$$x \geq (x \wedge y) \vee (x \wedge z) = P \otimes A \times B,$$

we have

$$x \wedge v \geq P \times A \times B.$$

Assume that

$$\begin{aligned} x \wedge v = x \wedge (y \vee z) &= P \times A \times B \times D, \\ x &= P \times A \times B \times D \times E. \end{aligned}$$

Consider elements x and $y \wedge z$, since $x = P \times A \times B \times D \times E$, $y \wedge z = P \times C$ and $x \wedge (y \wedge z) = P$, we have

$$x \vee (y \wedge z) = P \times A \times B \times C \times D \times E.$$

Note $x \vee u = x \vee (y \wedge z)$, so $x \vee u = P \times A \times B \times C \times D \times E$. Apply the structurerule for meet to pair (x, u), we have $x \wedge u = P \times A \times B = (x \wedge z) \vee (x \wedge y)$. Thus

$$\begin{aligned}(x \wedge v) \wedge u &= x \wedge u = P \times A \times B,\\ (x \wedge v) \vee u &= P \times A \times B \times C \times D.\end{aligned}$$

Step 3. Consider element z. Similar to x, we may assume

$$\begin{aligned}z \wedge v &= P \times B \times C \times F,\\ z &= P \times B \times C \times F \times G.\end{aligned}$$

Hence

$$\begin{aligned}(z \vee v) \wedge u &= P \times A \times B \times C \times F,\\ z \vee u &= P \times A \times B \times C \times F \times G.\end{aligned}$$

Consider the pair (x, z),

$$x \wedge z = P \times B, \qquad x = P \times A \times B \times D \times E, \qquad z = P \times B \times C \times F \times G,$$

$$\Longrightarrow x \vee z = P \times A \times B \times C \times D \times E \times F \times G.$$

Consider the pair $(x \wedge v, z \wedge v)$, we have

$$v \geq (x \wedge v) \vee (z \wedge v) = P \times A \times B \times C \times D \times F. \tag{5.4}$$

Now apply the Structure Theorem to pair $(z \wedge v, x)$,

$$\begin{aligned}z \wedge v \wedge x &= z \wedge (x \vee y) \wedge x = z \wedge x = P \times B,\\ z \wedge v &= P \times B \times C \times F,\\ x &= P \times A \times B \times D \times E,\\ \Longrightarrow (z \wedge (x \vee y)) \vee x &= P \times A \times B \times C \times D \times E \times F.\end{aligned}$$

Note that $x \vee y \geq z \wedge (x \vee y)$, hence we may assume that

$$x \vee y = p \times A \times B \times C \times D \times E \times F \times T.$$

Consider the pair $(x \vee y, z)$ with $(x \vee y) \wedge z = z \wedge v = P \times B \times C \times F$, we have

$$x \vee y \vee z = P \times A \times B \times C \times D \times E \times F \times G \times T.$$

Hence

$$(x \vee y) \wedge (x \vee z) = P \times A \times B \times C \times D \times E \times F.$$

Compare with v, we get

$$v \leq (x \vee y) \wedge (x \vee z) = P \times A \times B \times C \times D \times E \times F.$$

Similarly,

$$v \leq (z \vee y) \wedge (z \vee x) = P \times A \times B \times C \times D \times F \times G.$$

So

$$\begin{aligned} V &\leq ((x \vee y) \wedge (x \vee z)) \wedge ((z \vee y) \wedge (z \vee x)) \\ &= P \times A \times B \times C \times D \times F. \end{aligned} \tag{5.5}$$

Compare Eq(5.4) and Eq(5.5), we obtain $v = P \times A \times B \times C \times D \times F$.

Step 4. Similarly by introducing y to the lattice, we have

$$y \wedge v = P \times A \times C \times I,$$
$$y = P \times A \times C \times I \times J,$$

and

$$v = P \times A \times B \times C \times D \times I = P \times A \times B \times C \times F \times I.$$

Look at the sublattice generated by $x \vee y, y \vee z$, and $z \vee x$, we have

$$v = (x \vee y) \wedge (y \vee z) \wedge (z \vee x) = P \times A \times B \times C \times D \times F,$$

$$\begin{aligned} (x \vee y) \wedge (x \vee z) &= P \times A \times B \times C \times D \times E \times F, \\ (x \vee y) \wedge (y \vee z) &= P \times A \times B \times C \times F \times I \times J, \\ (x \vee z) \wedge (y \vee z) &= P \times A \times B \times C \times D \times F \times G, \\ x \vee y &= P \times A \times B \times C \times D \times E \times F \times J, \\ x \vee z &= P \times A \times B \times C \times D \times E \times F \times G, \\ y \vee z &= P \times A \times B \times C \times D \times F \times G \times J \end{aligned}$$

where $D \times F = D \times I = I \times F$.

Apply the Structure Theorem to the pair $(x \vee y, (x \vee z) \wedge (y \vee z))$, we have

$$\begin{aligned} x \vee y \vee z &\leq (x \vee y) \vee ((y \vee z) \wedge (x \vee z)) && (5.6) \\ &= P \times A \times B \times C \times D \times E \times F \times G \times J. && (5.7) \end{aligned}$$

Since $x \vee y \vee z$ is the maximal element in this lattice, in the formula (5.6), the equation holds. Now we have

$$x \vee y \vee z = P \times A \times B \times C \times D \times E \times F \times G \times J.$$

It is easy to check that all the joins and meets are closed and this lattice has exctly 28 elements. Compare with the free modular lattice with three generators, we see that they are isomorphic. □

Definition 5.1 *A triple (A, B, C) is called a* **weakly modular triple** *if $A \vee B = A \vee C = B \vee C$. It is a* **modular triple** *if A, B, C are pairwise independent and $A \times B = A \times C = B \times C$.*

Corollary 5.21 *For any three elements x, y, z in a lattice of commuting σ-algebras, we have the following decomposition:*

$$\begin{aligned} x &= P \times A \times B \times 0 \times D \times 0 \times 0 \times L, \\ y &= P \times A \times 0 \times C \times 0 \times E \times 0 \times M, \\ z &= P \times 0 \times B \times C \times 0 \times 0 \times F \times N, \end{aligned}$$

where (L, M, N) is a modular triple, and

$$x \vee y \vee z = P \times A \times B \times C \times D \times E \times F \times L \times M.$$

As a conclusion, we list some properties of the lattice $\mathcal{L}(\sigma)$ of stochastically commuting σ-algebras, and we use them to show that Desargues Theorem holds in $\mathcal{L}(\sigma)$. We will use these properties to define a set of deduction rules, and therefore develop a proof theory for the class of lattices of stochastically commuting σ-algebras in a coming paper.

Theorem 5.22 *Properties for the lattice of stochastically commuting σ-algebras:*

1. **Continuous assumption:**

   For any pair of element $B \leq A$, there exists a σ-subalgebra J of A such that $A = B \times J$.

2. **Lattice operations:**

   $A \leq B \wedge C$ if and only if $A \leq B$ and $A \leq C$; $A \geq b \vee C$ if and only if $A \geq B$ and $A \geq C$.

3. **Structure rule for join:**

$$\frac{B = P \otimes X,\ c = P \otimes Y,\ B \wedge C = P}{B \vee C = P \otimes X \otimes Y}.$$

4. **Structure rule for meet:**

$$\frac{B \vee C = P \otimes X \otimes Y,\ B = P \otimes X,\ C = P \otimes Y}{B \wedge C = P}.$$

5. **Independence:**

   B and C are independent if and only if $B \vee C = B \otimes C$.

   B and C are independent if and only if $B \wedge C = 0$, where 0 is the Boolean σ-algebra with two elements.

6. If A and B are independent, and $C \leq A$, then B and C are independent.

7. If $A = P \otimes X,\ B = P$, and $B \geq A$, then $X = 0$.

8. **Splitting join:** if $P \leq Q \vee R$,then $P = P_1 \otimes P_2 \otimes P_3$, where $P_1 \leq Q$, $P_2 \leq R$, $P_3 \leq Q \vee R$ and P_3 is independent with both Q and R.

   This is a corollary of the structure of the free lattice generated by three elements.

9. If $P \leq Q \vee R$, where P is independent with both Q and R, then There exists $Q_1 \leq Q$ and $R_1 \leq R$ such that (P, Q_1, R_1) is a modular triple.

   It also follows from the structure of the free lattice generated by three elements.

10. **Modular triple decomposition:** if (A, B, C) is a modular triple, and $A = A_1 \otimes A_2$, then we can write $B = B_1 \otimes B_2$ and $C = C_1 \otimes C_2$ such that both (A_1, B_1, C_1) and (A_2, B_2, C_2) are modular triples.

Proof. Let

$$B_1 = B \wedge (A_1 \vee C), \qquad B_2 = B \wedge (A_2 \vee C),$$
$$C_1 = C \wedge (A_1 \vee B), \qquad C_2 = C \wedge (A_2 \vee B).$$

Then by modular law,

$$\begin{aligned} B_1 \vee C_1 &= (A_1 \vee B) \wedge (A_1 \vee C) \wedge (B \vee C), \\ A_1 \vee B_1 &= (A_1 \vee B) \wedge (A_1 \vee C), \\ A_1 \vee C_1 &= (A_1 \vee B) \wedge (A_1 \vee C), \end{aligned}$$

Hence (A_1, B_1, C_1) is a modular triple. Similarly, (A_2, B_2, C_2) is a modular triple as well.

Also note that $B_1 \wedge B_2 = B \wedge (A_1 \vee C) \wedge (A_2 \vee C) = B \wedge C = 0$, and

$$\begin{aligned} B_1 \vee B_2 &= (B \wedge (A_1 \vee C)) \vee (B \wedge (A_2 \vee C)) \\ &= B \wedge (A_1 \vee C \vee (B \wedge (A_2 \vee C))) \\ &= B \wedge (A_1 \vee ((A_2 \vee C) \wedge (B \vee C))) \\ &= B \wedge (A_1 \vee A_2 \vee C) \\ &= B. \end{aligned}$$

Hence $B_1 \times B_2 = B$. Similarly, $C_1 \times C_2 = C$. That finishes the proof.

In general, if (A, B, C) is a weakly modular triple, and $A = A_1 \vee A_2$, then we can write $B = B_1 \vee B_2$ and $C = C_1 \vee C_2$ such that both (A_1, B_1, C_1) and (A_2, B_2, C_2) are weakly modular triples.

11. If $(A \vee A') \wedge (B \vee B') = T$, $(A \vee B) \wedge (A' \vee B') = P$, where (A, A', T) and (B, B', T) are modular triples, then (A, B, P) and (A', B', P) are modular triples.

    Proof. From $(A \vee A') \wedge (B \vee B') = T$, applying the Structure Theorem we have:

    $$\begin{aligned} A \vee B \vee A' \vee B' &= A \times A' \times B = A \times A' \times B' \\ &= A \times B \times B' = A' \times B \times B'. \end{aligned}$$

    If $(A \vee B) \wedge (A' \vee B') = P$, then $P \leq A \vee B$, and $P \wedge A \leq A \wedge (A' \vee B') = 0$. So $P \wedge A = P \wedge B = P \wedge A' = P \wedge B' = 0$.

From the structure of free lattice with three generators, we have $A = A_1 \times Q$, $B = B_1 \times R$, where (P, Q, R) is a modular triple. Similarly, $A' = A_1' \times Q'$, $B' = B_1' \times R'$ and (P, Q', R') is a modular triple. From $(A \vee B) \wedge (A' \vee B') = P$, we also have

$$A \vee B \vee A' \vee B' = A_1 \times B_1 \times Q \times R \times A_1' \times B_1' \times Q'.$$

But $A \vee B \vee A' \vee B' = A \times B \times A' = A_1 \times B_1 \times Q \times R \times A_1' \times Q'$. Compare these two formulas, we have $B_1' = 0$. Similarly, $A_1 = B_1 = A_1' = 0$. Hence both (P, A, B) and (P, A', B') are modular triples.

For simplicity, we shall write AB instead of $A \vee B$ when there is no confusion.

Theorem 5.23 (Desargues) *If*

$$(AA') \wedge (BB') = (AA') \wedge (CC') = (CC') \wedge (BB'),$$

then

$$(AB \wedge A'B') \vee (AC \wedge A'C') = (AB \wedge A'B') \vee (BC \wedge B'C') = (BC \wedge B'C') \vee (AC \wedge A'C').$$

Proof. By symmetry, it is enough to show that if $T = AA' \wedge BB' = BB' \wedge CC' = AA' \wedge CC'$, then $AB \wedge A'B' \leq (AC \wedge A'C') \vee (BC \wedge B'C')$.

Let $AB \wedge A'B' = P, AC \wedge A'C' = Q$, and $BC \wedge B'C' = R$. To show $P \leq Q \vee R$, it suffices to show that for some decomposition $P = P_1 \times P_2 \times \cdots \times P_n$, we have $P_i \leq Q \vee R$ for $i = 1, 2, \ldots, n$.

From $P \leq AB$, apply Property (8), we have $P = P_1 \times P_2 \times P_3$ where $P_1 \leq A$, $P_2 \leq B$, $P_3 \leq AB$ and P_3 is independent with A and B.

Case 1. $P_1 \leq A \wedge (A'B')$.

From $P_1 \leq A'B'$, we have

$$P_1 = U_1 \times U_2 \times U_3,$$

where $U_1 \leq A'$, $U_2 \leq B'$, $U_3 \leq A'B'$, and U_3 is independent with both A' and B'.

1. $U_1 \leq Q \vee R$.

   Because $U_1 \leq P_1 \leq A$ and $U_1 \leq A'$, then $U_1 \leq AC \wedge A'C' = Q$.

2. $U_2 \leq Q \vee R$.

   Indeed, $U_2 \leq P_1 \leq A$ and $U_2 \leq B'$. Thus

$$T = AB \wedge A'B' \geq U_2.$$

   From $T \leq CC'$, we derive that $U_2 \leq CC'$. Hence

$$\begin{aligned} Q \vee R &\geq (AC \wedge C') \vee (C \wedge B'C') \\ &= AC \wedge (C' \vee (C \wedge B'C')) \\ &= AC \wedge (B'C' \wedge CC') \\ &\geq U_2 \end{aligned}$$

   where the equations are obtained by applying modular law.

3. $U_3 \leq Q \vee R$.

   Indeed, $U_3 \leq A'B'$ and U_3 is independent with both A' and B', then there exist $A'_1 \leq A', B'_1 \leq B'$ such that (U_3, A'_1, B'_1) is a modular triple. Note that $T = AA' \wedge BB' \geq U_3A'_1 \wedge B'_1 = B'_1$, so we have $B'_1 \leq CC'$. Use the modular law, we obtain

$$\begin{aligned} Q \vee R &= (AC \wedge A'C') \vee (BC \wedge B'C') \\ &\geq (U_3C \wedge A'_1C') \vee (C \wedge B'_1C') \\ &= U_3C \wedge (A'_1C' \vee (C \wedge B'_1C')) \\ &= U_3C \wedge (A'_1 \vee C' \vee (C \wedge B'_1C')) \\ &= U_3C \wedge (A'_1 \vee (B'_1C' \wedge CC')) \\ &= U_3C \wedge (A'_1 \vee B'_1C') \\ &\geq U_3C \wedge A'_1B'_1 \\ &\geq U_3. \end{aligned}$$

Hence $P_1 = U_1 \times U_2 \times U_3 \leq Q \vee R$.

Case 2. $P_2 \leq B \wedge A'B'$. By the symmetry of A and B, we have $P_2 \leq Q \vee R$.

Case 3. $P_3 \leq AB$ and P_3 is independent with both A and B.

From $P_3 \leq A'B'$, we can write P_3 as $S_1 \times S_2 \times S$ where $S_1 \leq A'$, $S_2 \leq B'$ $S \leq A'B'$ and S is independent with A' and B'. By using the symmetry of (A, B) and (A', B') and repeating the previous argument, we have $S_1 \leq Q \vee R$ and $S_2 \leq Q \vee R$.

From the fact that $S \leq AB$, $S \leq A'B'$ and S is independent with A, B, A', B', alos applying Property (9), there exist $A_1 \leq A, B_1 \leq B, A'_1 \leq A'$, and $B'_1 \leq B'$ such that both $(S, A_1.B_1)$ and (S, A'_1, B'_1) are modular triples. Without loss of generality, we may assume that (S, A, B) and (S, A', B') are modular triples.

By the Structure Theorem, and note that $AB \wedge A'B' = S$, we have

$$AA'BB' = AB \vee A'B' = A \times B \times A' = A \times B \times B' = A \times A' \times B' = B \times A' \times B'.$$

Apply the Rule(10), and use the fact that $AA' \wedge BB' = T$, we derive that both (A, A', T) and (B, B', T) are modular triples. Moreover, from $T \leq CC'$, without loss of generality, we may assume that C, C' are independent. Let $C_1 = C \wedge (C'T)$ and $C'_1 = C' \wedge (CT)$, we have (T, C_1, C'_1) is a weakly modular triple. We may assume that $C = C_1$ and $C' = C'_1$ without loss of generality.

Note that $A \wedge CC' \leq AA' \wedge CC' = T$ and $A \wedge T = 0$, so A is independent with CC'. Similar to that of $AA'BB'$, we can get

$$AA'CC' = AA' \vee CC' = A \times C \times C' = A' \times C \times C',$$
$$BB'CC' = BB' \vee CC' = B \times C \times C' = B' \times C \times C',$$

and

$$AC = A \times C = C \times Q, \qquad A'C' = A' \times C' = C' \times Q,$$
$$BC = B \times C = C \times R, \qquad B'C' = B' \times C' = C' \times R.$$

We have derived that both A and B are independent with CC'. Now add the non-degenerating assumption: AB is independent with CC'. We have

$$\begin{aligned} ABC &= A \times B \times C = Q \times C \times B = Q \times R \times C, \\ A'B'C' &= A' \vee B' \vee C' = Q \vee R \vee C'. \end{aligned}$$

Note that

$$\begin{aligned} ABC \vee A'B'C' &= AA'BB'CC' \\ &= AA'CC' \vee BB'CC' \\ &= A \times C \times C' \vee B \times C \times C' \\ &= A \times B \times C \times C'. \end{aligned}$$

The last equation holds because AB is independent with CC'.

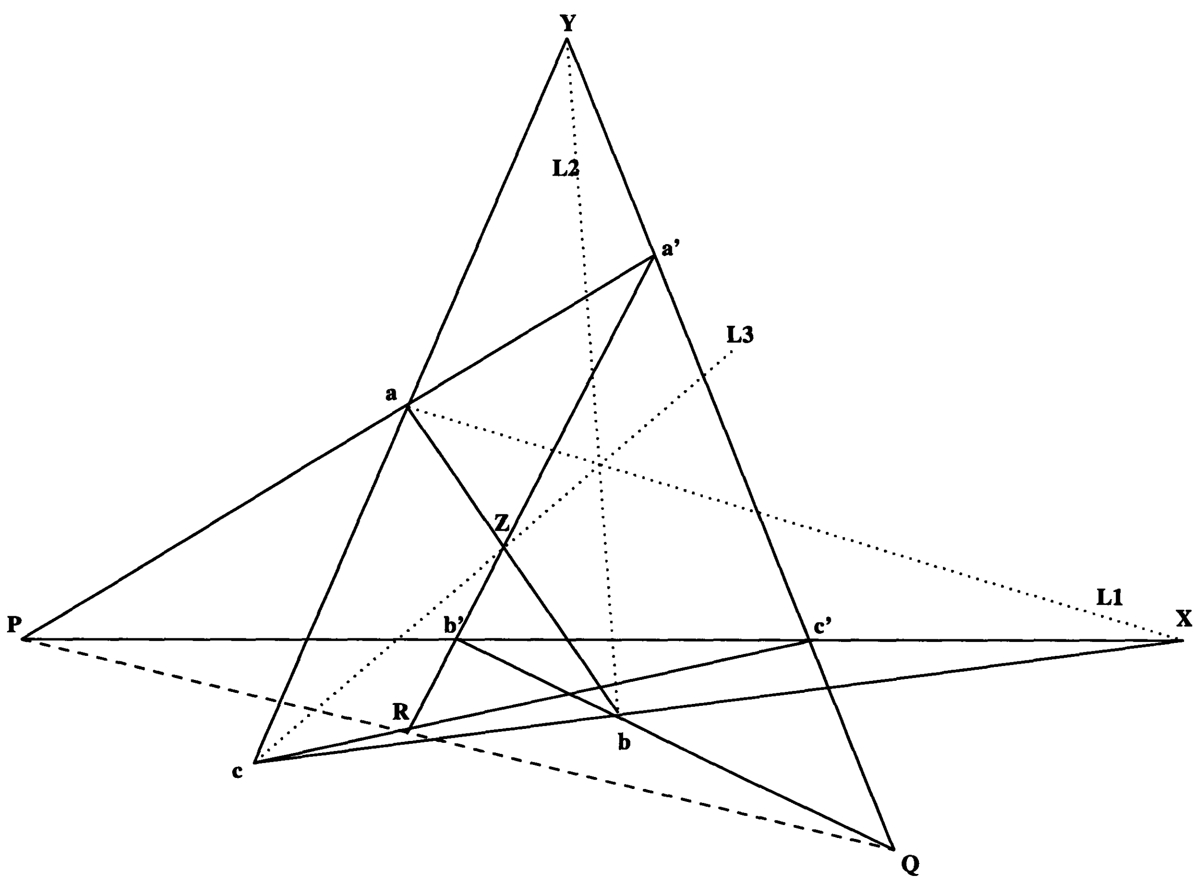
Moreover, $A \times B \times C \times C' = Q \times R \times C \times C'$, together with $ABC = Q \times R \times C$ and $A'B'C' = Q \times R \times C'$, we obtain

$$ABC \wedge A'B'C' = Q \vee R.$$

Finally note $S \leq ABC \wedge A'B'C'$, so $S \leq Q \vee R$ as desired. This finishes the proof. □

The properties of the lattices of stochastically commmute σ-algebras are similar to those of projective geometry. However they are not equivalent. As an example, we construct a lattice of stochastically commuting σ-algebras in which Bricard Theorem fails.

It is known that the Bricard Theorem holds in projective geometry (c.f. [15]). The geometric Bricard Theorem is the following.



Theorem 5.24 (Bricard) *Let a, b, c and a', b', c' be two triangles in the projective plane. Form the lines aa', bb', and cc' by joining respective vertices. Then these lines intersect the opposite edges $b'c'$, $a'c'$, and $a'b'$ in colinear points if and only if the join of the points $bc \cap b'c'$, $ac \cap a'c'$ and $ab \cap a'b'$ to the opposite vertices a, b, and c form three concurrent lines.*

In the language of Lattice Theory, Bricard Theorem can be stated as follows.

Bricard. Let a, b, c and a', b', c' be elements in a lattice L. Define

$$\begin{aligned} p &= aa' \wedge b'c', & x &= bc \wedge b'c', \\ q &= bb' \wedge a'c', & y &= ac \wedge a'c', \\ r &= cc' \wedge a'b', & z &= ab \wedge a'b'. \end{aligned}$$

Let $L_1 = a \vee x$, $L_2 = b \vee y$, and $L_3 = c \vee z$. Then $p \vee q = q \vee r = r \vee p$ if and only if $L_1 \wedge L_2 = L_2 \wedge L_3 = L_3 \wedge L_1$.

Counterexample. Let (α, β, γ) be a modular triple with $\Sigma = \alpha \times \beta$. Denote by 0 the minimal subalgebra of Σ which has only two elements. Define a, b, c and a', b', c' as follows.

$$\begin{aligned} a &= 0, & b &= \gamma, & c &= \alpha, \\ a' &= \gamma, & b' &= 0, & c' &= \beta. \end{aligned}$$

Then it is easy to see that they form a lattice of stochastically sommuting σ-algebras. Moreover, we have $p = 0$, $q = \gamma$ and $r = \gamma$. Hence

$$p \vee q = q \vee r = r \vee p.$$

We also have $x = \beta$, $y = \alpha$ and $z = \gamma$. Hence $L_1 = \beta$, $L_2 = L_3 = \Sigma$, and $L_1 \wedge L_2 \neq L_2 \wedge L_3$. Therefore in this lattice, Bricard Theorem fails. □

REMARK. From this counterexample we may construct a linear lattice in which the Bricard Theorem fails. For example, take a 2-dimensional vector space with basis $\{e_1, e_2\}$, and let $a = 0$, $b = < e_1 + e_2 >$, $c = < e_1 >$ and $a' = < e_1 + e_2 >$, $b' = 0$, $c' = < e_2 >$. Then it forms a linear lattice which is a counterexample of Bricard Theorem.

Appendix A

Proof of Theorem 5.7

To prove the theorem in general case, we need the following lemma.

Lemma A.1 *In a probability space $S = (\Omega, \Sigma, P)$, let B and C be two algebras of sets which are subalgebras of Σ. Let $\sigma(B)$ and $\sigma(C)$ denote the least σ-algebra containing them, and $\Sigma' = \sigma(B) \cap \sigma(C)$.*

If for each $b \in B$ and $c \in C$,

$$P(b \cap c) = \int_\Omega P_{\Sigma'}(b) P_{\Sigma'}(c) dP_{\Sigma'},$$

then the σ-algebras $\sigma(B)$ and $\sigma(C)$ stochastically commute.

Proof. First we show that for any element $b \in \sigma(B)$ and $c \in \sigma(C)$, we have

$$P(b \cap c) = \int_\Omega P_{\Sigma'}(b) P_{\Sigma'}(c) dP_{\Sigma'}.$$

Let c be a fixed element in C. The measures $\mu_1(b) = P(c \cap b)$ and

$$\mu_2(b) = \int_\Omega P_{\Sigma'}(b) P_{\Sigma'}(c) dP_{\Sigma'}$$

agrees on all the elements of B. Hence they must agree on the minimal σ-algebra containing B, namely $\sigma(B)$. Thus for every $b \in \sigma(B)$ and $c \in C$ we have:

$$P(b \cap c) = \int_\Omega P_{\Sigma'}(b) P_{\Sigma'}(c) dP_{\Sigma'}, \tag{A.1}$$

Let fix $b \in \sigma(B)$. As the two measures $\nu_1(c) = P(b \cap c)$ and

$$\nu_2(c) = \int_\Omega P_{\Sigma'}(b)P_{\Sigma'}(c)dP_{\Sigma'}$$

coincide for $c \in C$, they coincide on the minimal σ-algebra containing C, namely $\sigma(C)$. Thus for every $b \in \sigma(B)$ and $c \in \sigma(C)$, we have

$$P(b \cap c) = \int_\Omega P_{\Sigma'}(b)P_{\Sigma'}(c)dP_{\Sigma'}.$$

Next we show $\sigma(B)$ and $\sigma(C)$ stochastically commute. From the equation (A.1), for any element $d \in \Sigma'$, we have

$$\begin{aligned} P((b \cap d) \cap (c \cap d)) &= \int_\Omega P_{\Sigma'}(b \cap d)P_{\Sigma'}(c \cap d)dP_{\Sigma'}, \\ &= \int_d P_{\Sigma'}(b)P_{\Sigma'}(c)dP_{\Sigma'}, \end{aligned}$$

Hence $P((b \cap c) \cap d) = \int_d P_{\Sigma'}(b)P_{\Sigma'}(c)dP_{\Sigma'}$ for all $d \in \Sigma'$. By the definition of conditional probability, we have $P_{\Sigma'}(b \cap c) = P_{\Sigma'}(b)P_{\Sigma'}(c)$ almost everywhere. That finishes the proof. □

Proof of Theorem 5.7.

Let A_0 be the minimal Boolean subalgebra containing both B and C. Then $\Sigma = \sigma(A_0)$. Elements in A_0 are of form

$$a = \sum_{k=1}^{n} b_k \cap c_k,$$

where b_k forms a partition of Ω.

For such an element $a \in A_0$, define a probability P by letting

$$P(a) = \sum_{k=1}^{n} \int_\Omega P_{1,D}(b_k)P_{2,D}(c_k)dP, \tag{A.2}$$

We show that P is well-defined on A_0, and it is σ-additive. Then it can be uniquely extended to Σ. By the previous lemma and the definition of P, we have B and C stochastically commute.

For any element $t \in A_0$, let $b(t)$, $c(t)$, and $d(t)$ be its closure in B, C, and D, respectively.

Step 1 The probability P is well-defined on A_0 and is finite additive.

Lemma A.2 *Let $b \cap c = t$, then*

$$\int_\Omega P_{1,D}(b)P_{2,D}(c)dP = \int_\Omega P_{1,D}(b(t))P_{2,D}(c(t))dP.$$

Proof of the lemma. We have:

$$t \le b(t) \cap c(t) \le b \cap c = t,$$

hence $t = b(t) \cap c(t)$.

Let $b' = b - b(t)$, $C' = C - C(t)$, then $b' \cap c = 0$. Since B and C qualitatively commute, there is an element $d \in D$ such that $d \ge b'$ and $d \cap c = 0$. So $\int P_{1,D}(b')P_{2,D}(c)dP_D = 0$. Similarly $\int P_{1,D}(b)P_{2,D}(c')dP_D = 0$. Thus

$$\begin{aligned} \int P_{1,D}(b)P_{2,D}(c)dP &= \int P_{1,D}(b(t))P_{2,D}(c)dP \\ &= \int P_{1,D}(b(t))P_{2,D}(c(t))dP, \end{aligned}$$

which proves the lemma.

If an element $a \in A_0$ has two expressions

$$a = \sum_{k=1}^{n} b_k \cap c_k = \sum_{j=1}^{m} b'_j \cap c'_j,$$

we will show that $P(a)$ is well-defined. Note that a can be written as

$$a = \sum_{k=1}^{n} \sum_{j=1}^{m} (b_k \cap b'_j) \cap (c_k \cap c'_j),$$

it suffices to show that for some $b \in B$ and $c \in C$, if

$$b \cap c = \sum_{k=1}^{n} b_k \cap c_k,$$

where b_k forms a partition of b and $c_k \le c$, then

$$\int_\Omega P_{1,D}(b)P_{2,D}(c)dP = \sum_{k=1}^{n} \int_\Omega P_{1,D}(b_k)P_{2,D}(c_k)dP.$$

Let $b_k \cap c_k = t_k$. Since b_k are disjoint, then t_k are disjoint. Moreover $(b \cap c) \cap b_k \leq b_k \cap c_k$, so $b_k \cap c = b_k \cap c_k = t_k$. By the lemma,

$$\int P_{1,D}(b_k)P_{2,D}(c)dP = \int P_{1,D}(b_k)P_{2,D}(c_k)dP = \int P_{1,D}(b(t_k))P_{2,D}(c(t_k))dP.$$

Hence

$$\begin{aligned} \int_\Omega P_{1,D}(b)P_{2,D}(c)dP &= \sum_{k=1}^{n} \int_\Omega P_{1,D}(b_k)P_{2,D}(c)dP \\ &= \sum_{k=1}^{n} \int_\Omega P_{1,D}(b_k)P_{2,D}(c_k)dP. \end{aligned}$$

Given $a = \sum_{k=1}^{n} b_k \cap c_k$ and $a' = \sum_{j=1}^{m} b'_j \cap c'_j$, where a and a' disjoint, then

$$\begin{aligned} a \cup a' &= \left(\sum_{k=1}^{n}\sum_{j=1}^{m} b_k \cap b'_j \cap c_k\right) \cup \left(\sum_{k=1}^{n}\sum_{j=1}^{m} b'_j \cap b_k \cap c'_j\right), \\ &= \sum_{k=1}^{n}\sum_{j=1}^{m} (b_k \cap b'_j) \cap (c_k \cup c'_j). \end{aligned}$$

Since a and a' are disjoint, $b_k \cap b'_j \cap c_k$ is disjoint with $b_k \cap b'_j \cap c'_j$. Let $c(k,j) = c_k \cap c'_j$, then $(b_k \cap b'_j) \cap c(k,j) = 0$. Since B and C qualitatively commute, there exists an element $d \in D = B \cap C$ such that $d \geq c(k,j)$ and $d \cap (b_k \cap b'_j) = 0$. Hence

$$\begin{aligned} \int_\Omega P_{1,D}(b_k \cap b'_j)P_{2,D}(c_k \cap c'_j) &= \int_{d^c} P_{1,D}(b_k \cap b'_j)P_{2,D}(c_k \cap c'_j)dP \\ &= \int_{d^c} P_{1,D}(b_k \cap b'_j)(P_{2,D}(c_k) + P_{2,D}(c'_j))dP \\ &= \int_\Omega P_{1,D}(b_k \cap b'_j)(P_{2,D}(c_k) + P_{2,D}(c'_j))dP. \end{aligned}$$

Thus $P(a \cap a') = P(a) + P(a')$. Hence P is finitely additive.

The finitely additive measure P can be uniquely extended to the minimal σ-algebra containing A_0.

Step 2 P is σ-additive on Σ. we will need the following theorem.

Theorem A.3 *Let (Σ, μ) be a Boolean σ-algebra with a finitely additive measure μ. Then the σ-additivity of μ is equivalent to the continuity of μ on Σ, i.e., for every monotonically* *decreasing sequence $a_1 \geq a_2 \geq \ldots$ of events $a_n \in \Sigma$ with $\cap_{k=1}^{\infty} a_k = \emptyset$, we have*

$$\lim_{k\to\infty} \mu(a_k) = 0.$$

In order to show that the measure P defined above is σ-additive, it suffices to show that if $a_n \in A_0$, $a_n \geq a_{n+1}$, and $\cap_n a_n = 0$, then $\lim_{n\to\infty} P(a_n) = 0$. We prove it by contradiction.

Substep 1 Suppose that there exists a sequence $\{a_n\}$, $a_n \in A_0$ such that $a_n \geq a_{n+1}$, $\cap_n a_n = 0$, and $\lim_n P(a_n) = \epsilon > 0$. By the argument in the first part, we can assume

$$a_n = \sum_{k=1}^{i_n} b_k^{(n)} \cap c_k^{(n)},$$

where $b_k^{(n)} \cap c_k^{(n)} = t_k^{(n)}$ and $b_k^{(n)} = b(t_k^{(n)})$, and $c_k^{(n)} = c(t_k^{(n)})$. Furthermore, $b_k^{(n)}$ is disjoint from each other. Obviously $\sum_{k=1}^{n} b_k^{(n)} = b(a_n)$. From the fact that $a_{n+1} \leq a_n$, we have

$$\sum_{k=1}^{i_{n+1}} b_k^{(n+1)} \leq \sum_{k=1}^{i_n} b_k^{(n)}.$$

Substep 2 For any element $t_k^{(n)}$, let $d_k^{(n)} = d(t_k^{(n)})$.

Lemma A.4 *If $b_j^{(n+1)} \cap b_i^{(n)} \neq 0$, then $b_j^{(n+1)} \cap b_i^{(n)} \cap c_j^{(n+1)} \neq 0$.*

Proof of the lemma. If $b_j^{(n+1)} \cap b_i^{(n)} \cap c_j^{(n+1)} = 0$, let $\bar{b} = b_j^{(n+1)} \cap b_i^{(n)}$, $\bar{c} = c_j^{(n+1)}$, then $\bar{b} \cap \bar{c} = 0$, hence there is an element $d \in D$ such that $d \geq \bar{c}$ and $d \cap \bar{b} = 0$. The inequality $d \geq \bar{c}$ implies that $d \geq t_j^{(n+1)}$. It contradicts the fact that $b_j^{(n+1)} = b(t_j^{(n+1)})$.

Let

$$E_n = \{k | \text{ there exists } k, \text{ s. t. } d \leq d_k^{(n)}, P_{2,D}(c_k^{(n)}) \geq \frac{\epsilon}{2} \text{ on } d.\}$$

and for each $k \in E_n$, let $(d')_k^{(n)}$ be the maximal d described in above.

For $k \in E_n$, let

$$\begin{aligned} x_k^{(n)} &= b_k^{(n)} \cap (d')_k^{(n)} \\ y_k^{(n)} &= b_k^{(n)} \cap (d_k^{(n)} - (d')_k^{(n)}), \\ u_k^{(n)} &= c_k^{(n)} \cap (d')_k^{(n)}, \\ v_k^{(n)} &= c_k^{(n)} \cap (d_k^{(n)} - (d')_k^{(n)}), \end{aligned}$$

then

$$a_n = \sum_{k\in E_n} x_k^{(n)} \cap u_k^{(n)} + \sum_{k\in E_n} y_k^{(n)} \cap v_k^{(n)} + \sum_{k\notin E_n} b_k^{(n)} \cap c_k^{(n)},$$

where $\{x_k^{(n)}|k \in E_n\}$, $\{y_k^{(n)}|k \in E_n\}$ and $\{b_k^{(n)}|k \notin E_n\}$ forms a disjoint family of B.

Let $e_n = \sum_{k\in E_n} x_k^{(n)}$, then $e \in B$ and

$$\begin{aligned} \epsilon &\leq P(a_n) \\ &\leq \sum_{k\in E_n} \int_\Omega P_{1,D}(x_k^{(n)})dP + \frac{\epsilon}{2}\left(\sum_{k\in E_n} P_{1,D}(y_k^{(n)})dP + \sum_{k\notin E_n} P_{1,D}(b_k^{(n)})dP\right), \\ &\leq P(e_n) + \frac{\epsilon}{2}(1-P(e_n)), \end{aligned}$$

Hence $P(e_n) \geq \epsilon/(2-\epsilon) > 0$.

Substep 3 Claim: $e_{n+1} \leq e_n$.

By substep 1,

$$\sum_{k\in E_{n+1}} x_k^{(n+1)} + \sum_{k\in E_{n+1}} y_k^{(n+1)} + \sum_{k\notin E_{n+1}} b_k^{(n+1)} \leq \sum_{k\in E_n} x_k^{(n)} + \sum_{k\in E_n} y_k^{(n)} + \sum_{k\notin E_n} b_k^{(n)}.$$

One notices that $x_k^{(n+1)} \cap y_k^{(n)} = 0$. The reason is, if it is not zero, then by the lemma A.4, we have $x_k^{(n+1)} \cap y_k^{(n)} \cap u_k^{(n+1)} \neq 0$. But this element is contained in $y_k^{(n)} \cap v_k^{(n)}$. Exam the definition of E_n, it is impossible.

Similarly, $x_k^{(n+1)} \cap b_k^{(n)} = 0$, hence $e_{n+1} = \sum_{k\in E_{n+1}} x_k^{(n+1)} \leq \sum_{k\in E_n} x_k^{(n)} = e_n$.

Substep 4 Let $e = \cap e_n$, then $P_1(e) > 0$. Let $T = d(e)$, on every D-point t in T, choose an element $w(t) \in e$. Then on the D-point t, there exists one and only one h_n such that $w(t) \in b_{h_n}^{(n)}$. Let $b^* = \vee_t(\wedge_n b_{h_n}^{(n)})$, where t ranges over all D-points in T, so $b^* \neq 0$. On t, B and C are qualitative independent, hence we have

$$c_{h_{n+1}}^{(n+1)} \cap t \leq c_{h_n}^{(n)} \cap t.$$

Moreover on t, $P_{2,D}(c_{h_n}^{(n)}) \geq \frac{\epsilon}{2}$.

Let $c^* = \vee_t(\wedge_n(c_{h_n}^{(n)} \wedge t))$, where t ranges over all D-point in T, then $P_{2,D}(c^*) > 0$, hence $c^* \neq 0$. And for both b^* and c^*, their minimal closure in D is T.

We have $b^* \cap c^* \leq a_n$ for all n, hence $b^* \cap c^* = 0$. It is impossible since B and C qualitatively commute, but the minimal closure of b^* and c^* in D are both T, hence b^* and c^* can not be separated by any elements in D.

This finishes the proof. □

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