International Journal of Analysis and Applications
ISSN 2291-8639
Volume 15, Number 2 (2017), 211-221
DOI: 10.28924/2291-8639-15-2017-211

NEW BOUNDS OF OSTROWSKI–GRÜSS TYPE INEQUALITY FOR $(k+1)$ POINTS ON TIME SCALES

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ABSTRACT. The aim of this paper is to present three new bounds of the Ostrowski–Grüss type inequality for points $x_0, x_1, x_2, \cdots, x_k$ on time scales. Our results generalize result of Ngô and Liu, and extend results of Ujević to time scales with $(k+1)$ points. We apply our results to the continuous, discrete, and quantum calculus to obtain many new interesting inequalities. An example is also considered. The estimates obtained in this paper will be very useful in numerical integration especially for the continuous case.

1. INTRODUCTION

In 1997, Dragomir and Wang [6] proved that if $f : [a, b] \to \mathbb{R}$ is a differentiable function such that there exist constants $\gamma, \Gamma \in \mathbb{R}$ with $\gamma \le f'(x) \le \Gamma$ for all $x \in [a, b]$, then we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(s)ds - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \le \frac{1}{4}(b-a)(\Gamma - \gamma) \tag{1.1}$$

for all $x \in [a, b]$. The above inequality is known in the literature as the Ostrowski–Grüss type inequality. Under the same assumption, Cheng [5] obtained the following sharp version of (1.1).

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(s)ds - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \le \frac{1}{8}(b-a)(\Gamma - \gamma) \tag{1.2}$$

for all $x \in [a, b]$.

In 2003, Ujević [20] obtained another estimate of the left part of (1.2) as follows:

Theorem 1.1. *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a mapping differentiable in the interior* $\mathrm{Int}I$ *of I, and let $a, b \in \mathrm{Int}I,\ a < b$. If there exist constants $\gamma, \Gamma \in \mathbb{R}$ such that $\gamma \le f'(t) \le \Gamma$ for all $t \in [a, b]$ and $f' \in L_1(a, b)$, then, for all $x \in [a, b]$, we have*

$$\left| f(x) - \left(x - \frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\ dt\right| \le \frac{b-a}{2}(S - \gamma)$$

and

$$\left| f(x) - \left(x - \frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\ dt\right| \le \frac{b-a}{2}(\Gamma - S),$$

where $S = \frac{f(b)-f(a)}{b-a}$.

In 2012, Feng and Meng [7] generalized Inequality (1.1) to the case involving $(k+1)$ points $x_0, x_1, \cdots, x_k$. Their result is contained in the following theorem.

Received 18th August, 2017; accepted 20th October, 2017; published 1st November, 2017.
2010 *Mathematics Subject Classification.* 35A23, 26E70, 34N05.
Key words and phrases. Montgomery identity; Ostrowski-Grüss inequality; parameter; time scales.

Theorem 1.2. *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$, $a < b$, $f : I \to \mathbb{R}$ a differentiable function such that there exist constants $\gamma, \Gamma \in \mathbb{R}$ with $\gamma \leq f'(x) \leq \Gamma$ for all $x \in [a, b]$. Furthermore, suppose that $x_i \in [a, b]$, $i = 0, 1, 2, \cdots, k$, $I_k : a = x_0 < x_1 < \cdots < x_k = b$ is a division of the interval $[a, b]$ and $m_i \in [x_{i-1}, x_i]$, $i = 1, 2, \cdots, k$, $m_0 = a$, $m_{k+1} = b$. Then we have the following inequality*

$$\left|\frac{1}{b-a}\sum_{i=0}^{k}(m_{i+1}-m_i)f(x_i) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{(b-a)^2}\left[\frac{b^2-a^2}{2} - \sum_{i=0}^{k-1} m_{i+1}(x_{i+1}-x_i)\right]\right|$$
$$\leq \frac{1}{4}(b-a)(\Gamma-\gamma).$$

To unify the theory of continuous, discrete and quantum calculus, Stefan Hilger [8] in 1988 came up with the theory of time scales (see Section 2 for a brief introduction). Ever since, many classical integral inequalities have been extended to time scales; see, for example, the references [4, 9–12, 15–17, 19]. In [13], Liu and Ngô extended (1.1) to time scales. Following thereafter, the same authors in [14] obtained the following theorem which sharpens their earlier result.

Theorem 1.3. *Let $a, b, s, t \in \mathbb{T}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be differentiable. If f^Δ is rd-continuous and $\gamma \leq f^\Delta(t) \leq \Gamma$ for all $t \in [a, b]$, then we have*

$$\left|f(t) - \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s - \frac{f(b)-f(a)}{(b-a)^2}\left[h_2(t,a) - h_2(t,b)\right]\right|$$
$$\leq \frac{\Gamma-\gamma}{2(b-a)}\int_a^b \left|p(t,x) - \frac{h_2(t,a)-h_2(t,b)}{b-a}\right|\Delta x, \tag{1.3}$$

where $h_2(t, s)$ is given in Definition 2.8 and

$$p(t,x) = \begin{cases} x-a, & x \in [a,t), \\ x-b, & x \in [t,b]. \end{cases} \tag{1.4}$$

In this paper, we introduce a parameter $\lambda \in [0, 1]$ and then achieve the following goals, viz.,

(1) firstly, we extend Theorem 1.3 to $(k+1)$ points. Our first result provides another estimate for the left hand side of the inequality in Theorem 1.2 for the case when $\lambda = 0$ and the time scale is the set of real numbers (see Remark 4.1).
(2) Next, we generalize and extend Theorem 1.1 to time scales (see Remark 3.2).

This paper is made up of five sections. In Section 2, we lay a quick foundation of the theory of time scales. Our main results (Theorems 3.1 and 3.2) are then formulated and proved in Section 3. Thereafter, we then give some applications of our results in Section 4 to obtain many new inequalities. Finally, a brief conclusion follows in Section 5.

2. Time scale essentials

In this section, we collect basic time scale concepts that will aid in better understanding of this work. For more on this subject, we refer the interested reader to Hilger's Ph.D. thesis [8], the books [2, 3], and the survey [1].

Definition 2.1. *A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers.*

We assume throughout that $\mathbb{T}$ has the topology that is inherited from the standard topology on $\mathbb{R}$. It is also assumed throughout that in $\mathbb{T}$ the interval $[a, b]$ means the set $\{t \in \mathbb{T}\colon a \leq t \leq b\}$ for the points $a < b$ in $\mathbb{T}$. Since a time scale may not be connected, we need the following concept of jump operators.

Definition 2.2. *For each $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is definied by $\sigma(t) = \inf\{s \in \mathbb{T}\colon s > t\}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$.*

Definition 2.3. *If $\sigma(t) > t$, then we say that t is right-scattered, while if $\rho(t) < t$ then we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(t) = t$, then t is called right-dense, and if $\rho(t) = t$ then t is called left-dense. Points that are both right-dense and left-dense are called dense.*

Definition 2.4. *The mapping* $\mu : \mathbb{T} \to [0, \infty)$ *defined by* $\mu(t) = \sigma(t) - t$ *is called the graininess function. The set* $\mathbb{T}^k$ *is defined as follows: if* $\mathbb{T}$ *has a left-scattered maximum* m*, then* $\mathbb{T}^k = \mathbb{T} - \{m\}$ *; otherwise,* $\mathbb{T}^k = \mathbb{T}$.

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$, and when $\mathbb{T} = \mathbb{Z}$, we have $\mu(t) = 1$.

Definition 2.5. *Let* $f : \mathbb{T} \to \mathbb{R}$ *and* $t \in \mathbb{T}^k$. *Then we define* $f^{\Delta}(t)$ *to be the number (provided it exists) with the property that for any given* $\epsilon > 0$ *there exists a neighborhood* U *of* t *such that*

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t) [\sigma(t) - s] \right| \leq \epsilon \left| \sigma(t) - s \right|, \qquad \forall s \in U.$$

We call $f^{\Delta}(t)$ *the delta derivative of* f *at* t*. Moreover, we say that* f *is delta differentiable (or in short: differentiable) on* $\mathbb{T}^k$ *provided* $f^{\Delta}(t)$ *exists for all* $t \in \mathbb{T}^k$*. The function* $f^{\Delta} : \mathbb{T}^k \to \mathbb{R}$ *is then called the delta derivative of* f *on* $\mathbb{T}^k$.

In the case $\mathbb{T} = \mathbb{R}$, $f^{\Delta}(t) = \frac{df(t)}{dt}$. In the case $\mathbb{T} = \mathbb{Z}$, $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$, which is the usual forward difference operator.

Theorem 2.1. *If* $f, g : \mathbb{T} \to \mathbb{R}$ *are differentiable at* $t \in \mathbb{T}^k$*, then the product* $fg : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *and*

$$(fg)^{\Delta}(t) = f^{\Delta}(t) g(t) + f(\sigma(t)) g^{\Delta}(t).$$

Definition 2.6. *The function* $f : \mathbb{T} \to \mathbb{R}$ *is said to be rd-continuous on* $\mathbb{T}$ *provided it is continuous at all right-dense points* $t \in \mathbb{T}$ *and its left-sided limits exist at all left-dense points* $t \in \mathbb{T}$*. The set of all rd-continuous function* $f : \mathbb{T} \to \mathbb{R}$ *is denoted by* $C_{rd}(\mathbb{T},\mathbb{R})$*. Also, the set of functions* $f : \mathbb{T} \to \mathbb{R}$ *that are differentiable and whose derivative is rd-continuous is denoted by* $C^1_{rd}(\mathbb{T},\mathbb{R})$.

It follows from [2, Theorem 1.74] that every rd-continuous function has an anti-derivative.

Definition 2.7. *Let* $F : \mathbb{T} \to \mathbb{R}$ *be a function. Then* $F : \mathbb{T} \to \mathbb{R}$ *is called the anti-derivative of* f *on* $\mathbb{T}$ *if it satisfies* $F^{\Delta}(t) = f(t)$ *for any* $t \in \mathbb{T}^k$*. In this case, the Cauchy integral*

$$\int_a^b f(t)\Delta t = F(b) - F(a), \quad a, b \in \mathbb{T}.$$

Theorem 2.2. *Let* $f, g \in C_{rd}(\mathbb{T},\mathbb{R})$*,* $a, b, c \in \mathbb{T}$ *and* $\alpha, \beta \in \mathbb{R}$*. Then*

(1) $\int_a^b [\alpha f(t) + \beta g(t)] \Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t.$

(2) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t.$

(3) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t.$

(4) $\int_a^b f(t) g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t) g(\sigma(t))\Delta t.$

(5) *If* $|f(t)| \leq g(t)$ *on* $[a, b]$*, then*

$$\left| \int_a^b f(t)\Delta t \right| \leq \int_a^b g(t)\Delta t.$$

Definition 2.8. *Let* $h_k, g_k : \mathbb{T}^2 \to \mathbb{R}$ *,* $k \in \mathbb{N}_0$ *be defined by* $h_0(t,s) := g_0(t,s) := 1$*, for all* $s, t \in \mathbb{T}$ *and then recursively by* $g_{k+1}(t,s) = \int_s^t g_k(\sigma(\tau), s)\Delta\tau$*,* $h_{k+1}(t,s) = \int_s^t h_k(\tau, s)\Delta\tau$*, for all* $s, t \in \mathbb{T}$.

In view of the above definition, we make the following remarks that will come handy in the sequel.

– For $\mathbb{T} = \mathbb{R}$, $h_2(t,s) = \frac{(t-s)^2}{2}$.

– For $\mathbb{T} = \mathbb{Z}$, $h_2(t,s) = \frac{(t-s)(t-s-1)}{2}$.

3. Main results

In order to prove our results, we will need the following lemmas. The first lemma is given in [18,21] but with some typos. We present here the correct version.

Lemma 3.1 (Generalized Montgomery identity with a parameter). *Suppose that*

(1) $a, b \in \mathbb{T},\ \lambda \in [0,1],\ I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ *is a partition of the interval* $[a,b]$ *for* $x_0, x_1, \cdots, x_k \in \mathbb{T}$,
(2) $\alpha_i \in \mathbb{T}$ $(i = 0, 1, \cdots, k+1)$ *is* $k+2$ *points so that* $\alpha_0 = a,\ \alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \cdots, k)$ *and* $\alpha_{k+1} = b$,
(3) $f : [a,b] \to \mathbb{R}$ *is a differentiable function.*

Then we have the following equation

$$\begin{aligned} &\int_a^b K(t, I_k) f^{\Delta}(t)\Delta t + \int_a^b f^{\sigma}(t)\Delta t \\ &= (1-\lambda)\sum_{i=0}^{k}\Big(\alpha_{i+1} - \alpha_i\Big) f(x_i) + \lambda \sum_{i=0}^{k}\Big(\alpha_{i+1} - \alpha_i\Big)\frac{f(\alpha_i) + f(\alpha_{i+1})}{2}, \end{aligned} \tag{3.1}$$

where

$$K(t, I_k) = \begin{cases} t - \Big(\alpha_1 - \lambda\frac{\alpha_1 - a}{2}\Big),\ t \in [a, \alpha_1), \\ t - \Big(\alpha_1 + \lambda\frac{\alpha_2 - \alpha_1}{2}\Big),\ t \in [\alpha_1, x_1), \\ t - \Big(\alpha_2 - \lambda\frac{\alpha_2 - \alpha_1}{2}\Big),\ t \in [x_1, \alpha_2), \\ \vdots \\ t - \Big(\alpha_{k-1} + \lambda\frac{\alpha_k - \alpha_{k-1}}{2}\Big),\ t \in [\alpha_{k-1}, x_{k-1}), \\ t - \Big(\alpha_k - \lambda\frac{\alpha_k - \alpha_{k-1}}{2}\Big),\ t \in [x_{k-1}, \alpha_k), \\ t - \Big(\alpha_k + \lambda\frac{\alpha_{k+1} - \alpha_k}{2}\Big),\ t \in [\alpha_k, b], \end{cases} \tag{3.2}$$

provided for each $i \in \{0, 1, 2, \ldots, k-1\}$, $\alpha_{i+1} - \lambda\frac{\alpha_{i+1} - \alpha_i}{2}$ *and* $\alpha_{i+1} + \lambda\frac{\alpha_{i+2} - \alpha_{i+1}}{2}$ *belong to* $\mathbb{T}$.

Lemma 3.2 ([14]). *Let* $a, b, x \in \mathbb{T},\ f, g \in C_{rd}$ *and* $f, g : [a,b] \to \mathbb{R}$ *with* $\gamma \le g(x) \le \Gamma$ *for all* $x \in [a,b]$ *and for some* $\gamma, \Gamma \in \mathbb{R}$. *Then we have*

$$\begin{aligned} &\left|\int_a^b f(t)g(t)\Delta t - \frac{1}{b-a}\int_a^b f(t)\Delta t \int_a^b g(t)\Delta t\right| \\ &\qquad \le \frac{\Gamma - \gamma}{2}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\Delta s\right| \Delta t. \end{aligned} \tag{3.3}$$

Moreover, the inequality in (3.3) *is sharp.*

We now state and justify our first result.

Theorem 3.1. *Suppose f satisfies the conditions of Lemma 3.1. If, in addition, $f^{\Delta} \in C_{rd}$ with $\gamma \le f^{\Delta}(t) \le \Gamma$ for all $t \in [a,b]$ and some $\gamma, \Gamma \in \mathbb{R}$, then we have the inequality*

$$\begin{aligned}
&\Bigg|(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)+\lambda\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\frac{f(\alpha_i)+f(\alpha_{i+1})}{2}-\int_a^b f^{\sigma}(t)\Delta t\\
&-\frac{f(b)-f(a)}{b-a}\sum_{i=0}^{k-1}\Bigg[h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)-h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)\\
&+h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)-h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Bigg]\Bigg|\\
&\le\frac{\Gamma-\gamma}{2}\int_a^b\Bigg|K(t,I_k)-\frac{1}{b-a}\sum_{i=0}^{k-1}\Bigg[h_2\Big(\alpha_{i+1},\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)-h_2\Big(x_i,\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)\\
&+h_2\Big(x_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)-h_2\Big(\alpha_{i+1},\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Bigg]\Bigg|\Delta t, \qquad (3.4)
\end{aligned}$$

provided for each $i \in \{0,1,2,\dots,k-1\}$, $\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}$ and $\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}$ belong to $\mathbb{T}$. The inequality in (3.4) is sharp in the sense that the constant 1/2 cannot be replaced by a smaller one.

Proof. By applying Lemma 3.2 to the functions $f(t) := K(t,I_k)$ and $g(t) = f^{\Delta}(t)$, we have

$$\begin{aligned}
&\Big|\int_a^b K(t,I_k)f^{\Delta}(t)\Delta t-\frac{1}{b-a}\int_a^b K(t,I_k)\Delta t\int_a^b f^{\Delta}(t)\Delta t\Big|\\
&\qquad\le\frac{\Gamma-\gamma}{2}\int_a^b\Big|K(t,I_k)-\frac{1}{b-a}\int_a^b K(s,I_k)\Delta s\Big|\Delta t. \qquad (3.5)
\end{aligned}$$

Now, we observe that

$$\int_a^b f^{\Delta}(t)\Delta t = f(b)-f(a), \qquad (3.6)$$

and (by applying the items of Theorem 2.2 and Definition 2.8)

$$\begin{aligned}
\int_a^b K(t,I_k)\Delta t &= \sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}} K(t,I_k)\Delta t\\
&=\sum_{i=0}^{k-1}\Bigg[\int_{x_i}^{\alpha_{i+1}}\Big(t-\Big(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)\Big)\Delta t\\
&\quad+\int_{\alpha_{i+1}}^{x_{i+1}}\Big(t-\Big(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big)\Delta t\Bigg]\\
&=\sum_{i=0}^{k-1}\Bigg[\int_{x_i}^{\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}}\Big(t-\Big(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)\Big)\Delta t\\
&\quad+\int_{\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}}^{\alpha_{i+1}}\Big(t-\Big(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big)\Big)\Delta t\\
&\quad+\int_{\alpha_{i+1}}^{\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}}\Big(t-\Big(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big)\Delta t\\
&\quad+\int_{\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}}^{x_{i+1}}\Big(t-\Big(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Big)\Delta t\Bigg]
\end{aligned}$$

$$= \sum_{i=0}^{k-1} \Big[- \int_{\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}}^{x_i} \Big(t - \Big(\alpha_{i+1} - \lambda \frac{\alpha_{i+1}-\alpha_i}{2} \Big) \Big) \Delta t \tag{3.7}$$
$$+ \int_{\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}}^{\alpha_{i+1}} \Big(t - \Big(\alpha_{i+1} - \lambda \frac{\alpha_{i+1}-\alpha_i}{2} \Big) \Big) \Delta t$$
$$- \int_{\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}}^{\alpha_{i+1}} \Big(t - \Big(\alpha_{i+1} + \lambda \frac{\alpha_{i+2}-\alpha_{i+1}}{2} \Big) \Big) \Delta t$$
$$+ \int_{\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}}^{x_{i+1}} \Big(t - \Big(\alpha_{i+1} + \lambda \frac{\alpha_{i+2}-\alpha_{i+1}}{2} \Big) \Big) \Delta t \Big]$$
$$= \sum_{i=0}^{k-1} \Big[h_2\Big(\alpha_{i+1}, \alpha_{i+1} - \lambda \frac{\alpha_{i+1}-\alpha_i}{2}\Big) - h_2\Big(x_i, \alpha_{i+1} - \lambda \frac{\alpha_{i+1}-\alpha_i}{2}\Big)$$
$$+ h_2\Big(x_{i+1}, \alpha_{i+1} + \lambda \frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big) - h_2\Big(\alpha_{i+1}, \alpha_{i+1} + \lambda \frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big) \Big]. \tag{3.8}$$

Now using Lemma 3.1, we get

$$\int_a^b K(t, I_k) f^{\Delta}(t) \Delta t$$
$$= (1-\lambda) \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) + \lambda \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) \frac{f(\alpha_i) + f(\alpha_{i+1})}{2} - \int_a^b f^{\sigma}(t) \Delta t. \tag{3.9}$$

By substituting Equations (3.6), (3.7) and (3.9) into (3.5), we obtain

$$\Big| (1-\lambda) \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) + \lambda \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) \frac{f(\alpha_i) + f(\alpha_{i+1})}{2} - \int_a^b f^{\sigma}(t) \Delta t$$
$$- \frac{f(b)-f(a)}{b-a} \sum_{i=0}^{k-1} \Big[h_2\Big(\alpha_{i+1}, \alpha_{i+1} - \lambda \frac{\alpha_{i+1}-\alpha_i}{2}\Big) - h_2\Big(x_i, \alpha_{i+1} - \lambda \frac{\alpha_{i+1}-\alpha_i}{2}\Big)$$
$$+ h_2\Big(x_{i+1}, \alpha_{i+1} + \lambda \frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big) - h_2\Big(\alpha_{i+1}, \alpha_{i+1} + \lambda \frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big) \Big] \Big|$$
$$\le \frac{\Gamma - \gamma}{2} \int_a^b \Big| K(t, I_k) - \frac{1}{b-a} \sum_{i=0}^{k-1} \Big[h_2\Big(\alpha_{i+1}, \alpha_{i+1} - \lambda \frac{\alpha_{i+1}-\alpha_i}{2}\Big) - h_2\Big(x_i, \alpha_{i+1} - \lambda \frac{\alpha_{i+1}-\alpha_i}{2}\Big)$$
$$+ h_2\Big(x_{i+1}, \alpha_{i+1} + \lambda \frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big) - h_2\Big(\alpha_{i+1}, \alpha_{i+1} + \lambda \frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big) \Big] \Big| \Delta t. \tag{3.10}$$

Hence, the proof is complete. □

Remark 3.1. *If we take $\lambda = 0$, $k = 2$, and $x_1 = x$, $\alpha_0 = \alpha_1 = a$, $\alpha_2 = \alpha_3 = x_2 = b$ in Theorem 3.1, then we recapture Theorem 1.3.*

Next, we provide another bound for (3.4).

Theorem 3.2. *Under the conditions of Theorem 3.1, we obtain the following inequalities*

$$\Big| (1-\lambda) \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) + \lambda \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) \frac{f(\alpha_i) + f(\alpha_{i+1})}{2} - \int_a^b f^{\sigma}(t) \Delta t$$
$$- \frac{f(b)-f(a)}{b-a} \sum_{i=0}^{k-1} \Big[h_2\Big(\alpha_{i+1}, \alpha_{i+1} - \lambda \frac{\alpha_{i+1}-\alpha_i}{2}\Big) - h_2\Big(x_i, \alpha_{i+1} - \lambda \frac{\alpha_{i+1}-\alpha_i}{2}\Big)$$
$$+ h_2\Big(x_{i+1}, \alpha_{i+1} + \lambda \frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big) - h_2\Big(\alpha_{i+1}, \alpha_{i+1} + \lambda \frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big) \Big] \Big|$$
$$\le \begin{cases} M_k(b-a)(S-\gamma) \\ M_k(b-a)(\Gamma - S), \end{cases}$$

where $S = \frac{f(b)-f(a)}{b-a}$, *and* $M_k = \max_{t\in[a,b]} \left|K(t, I_k) - \frac{1}{b-a}\int_a^b k(s, I_k)\Delta s\right|$.

Proof. We start by observing that

$$\int_a^b \left[K(t, I_k) - \frac{1}{b-a}\int_a^b k(s, I_k)\Delta s\right]\Delta t = 0. \tag{3.11}$$

Using (3.11), we get that for any $C \in \mathbb{R}$,

$$\begin{aligned}&\int_a^b K(t, I_k)f^{\Delta}(t)\Delta t - \frac{1}{b-a}\int_a^b K(t, I_k)\Delta t \int_a^b f^{\Delta}(t)\Delta t\\&= \int_a^b \big(f^{\Delta}(t) - C\big)\left[K(t, I_k) - \frac{1}{b-a}\int_a^b k(s, I_k)\Delta s\right]\Delta t.\end{aligned} \tag{3.12}$$

For $C = \gamma$, and taking absolute values of both sides of (3.12), we have by using (3.6)

$$\begin{aligned}&\left|\int_a^b K(t, I_k)f^{\Delta}(t)\Delta t - \frac{1}{b-a}\int_a^b K(t, I_k)\Delta t \int_a^b f^{\Delta}(t)\Delta t\right|\\&\leq \int_a^b \left|f^{\Delta}(t) - \gamma\right|\left|K(t, I_k) - \frac{1}{b-a}\int_a^b k(s, I_k)\Delta s\right|\Delta t\\&\leq \max_{t\in[a,b]}\left|K(t, I_k) - \frac{1}{b-a}\int_a^b k(s, I_k)\Delta s\right|\int_a^b \left|f^{\Delta}(t) - \gamma\right|\Delta t\\&= M_k\int_a^b \big(f^{\Delta}(t) - \gamma\big)\Delta t\\&= M_k\Big[\frac{f(b) - f(a)}{b-a} - \gamma\Big](b-a).\end{aligned} \tag{3.13}$$

Similarly, for $C = \Gamma$, we get

$$\begin{aligned}&\left|\int_a^b K(t, I_k)f^{\Delta}(t)\Delta t - \frac{1}{b-a}\int_a^b K(t, I_k)\Delta t \int_a^b f^{\Delta}(t)\Delta t\right|\\&\leq M_k\Big[\Gamma - \frac{f(b) - f(a)}{b-a}\Big](b-a).\end{aligned} \tag{3.14}$$

The intended inequalities follow from Lemma 3.1 and Relations (3.13) and (3.14). □

Remark 3.2. *If we take* $\mathbb{T} = \mathbb{R}$, $\lambda = 0$, $k = 2$, *and* $x_1 = x$, $\alpha_0 = \alpha_1 = a$, $\alpha_2 = \alpha_3 = x_2 = b$ *in Theorem 3.2, then we get Theorem 1.1.*

4. Applications

In this section, we apply our theorems to the continuous, discrete, and quantum calculus to obtain the following results.

Corollary 4.1 (Continuous case)**.** *Let* $\mathbb{T} = \mathbb{R}$ *in Theorem 3.1. Then we have the inequality*

$$\begin{aligned}&\Bigg|(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1} - \alpha_i)f(x_i) + \lambda\sum_{i=0}^{k}(\alpha_{i+1} - \alpha_i)\frac{f(\alpha_i) + f(\alpha_{i+1})}{2} - \int_a^b f(t)dt\\&- \frac{f(b) - f(a)}{8(b-a)}\sum_{i=0}^{k-1}\Big[\lambda^2\Big(\alpha_{i+1} - \alpha_i\Big)^2 - \Big(2x_i - \lambda\alpha_i + (\lambda - 2)\alpha_{i+1}\Big)^2\\&+ \Big(2x_{i+1} - \lambda\alpha_{i+2} + (\lambda - 2)\alpha_{i+1}\Big)^2 - \lambda^2\Big(\alpha_{i+2} - \alpha_{i+1}\Big)^2\Big]\Bigg|\\&\leq \frac{\Gamma - \gamma}{2}\int_a^b \Bigg|K(t, I_k) - \frac{1}{8(b-a)}\sum_{i=0}^{k-1}\Big[\lambda^2\Big(\alpha_{i+1} - \alpha_i\Big)^2 - \Big(2x_i - \lambda\alpha_i + (\lambda - 2)\alpha_{i+1}\Big)^2\\&+ \Big(2x_{i+1} - \lambda\alpha_{i+2} + (\lambda - 2)\alpha_{i+1}\Big)^2 - \lambda^2\Big(\alpha_{i+2} - \alpha_{i+1}\Big)^2\Big]\Bigg|dt.\end{aligned} \tag{4.1}$$

Applying Corollary 4.1 to different values of λ and k, we obtain some novel inequalities. We present here some of these new results.

Remark 4.1. *If we take $\lambda = 0$ in Corollary 4.1, we get*

$$\begin{aligned}&\left|\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)-\int_a^b f(t)dt-\frac{f(b)-f(a)}{b-a}\left[\frac{b^2-a^2}{2}-\sum_{i=0}^{k-1}\alpha_{i+1}(x_{i+1}-x_i)\right]\right|\\&\leq\frac{\Gamma-\gamma}{2}\int_a^b\left|K(t,I_k)-\frac{1}{b-a}\left[\frac{b^2-a^2}{2}-\sum_{i=0}^{k-1}\alpha_{i+1}(x_{i+1}-x_i)\right]\right|dt,\end{aligned}\tag{4.2}$$

where

$$K(t,I_k)=\begin{cases}t-\alpha_1,\ t\in[a,x_1),\\ t-\alpha_2,\ t\in[x_1,x_2),\\ \vdots\\ t-\alpha_{k-1},\ t\in[x_{k-2},x_{k-1}),\\ t-\alpha_k,\ t\in[x_{k-1},b].\end{cases}\tag{4.3}$$

The above inequality is new and sharp. This gives a new estimate for the left hand side of the inequality in Theorem 1.2.

Furthermore, let $k = 2$ in Corollary 4.1. If in addition, one then sets $x_1 = x$, $\alpha_0 = \alpha_1 = a$, $\alpha_2 = \alpha_3 = x_2 = b$ in the resulting inequality, then one gets that for all $x \in [a, b]$ the following inequality holds:

$$\begin{aligned}&\Bigg|(1-\lambda)(b-a)f(x)+\lambda(b-a)\frac{f(a)+f(b)}{2}-\int_a^b f(t)dt\\&-\frac{f(b)-f(a)}{8(b-a)}\left[\Big(2x-\lambda b+(\lambda-2)a\Big)^2-\Big(2x-\lambda a+(\lambda-2)b\Big)^2\right]\Bigg|\\&\leq\frac{\Gamma-\gamma}{2}\int_a^b\left|K(t,x)-\frac{1}{8(b-a)}\left[\Big(2x-\lambda b+(\lambda-2)a\Big)^2-\Big(2x-\lambda a+(\lambda-2)b\Big)^2\right]\right|dt.\end{aligned}\tag{4.4}$$

Remark 4.2. *For $\lambda = 0$ in Inequality (4.4), we have the inequality*

$$\begin{aligned}&\left|(b-a)f(x)-\Big(f(b)-f(a)\Big)\left(x-\frac{a+b}{2}\right)-\int_a^b f(t)dt\right|\\&\leq\frac{\Gamma-\gamma}{2}\int_a^b\left|K(t,x)-\left(x-\frac{a+b}{2}\right)\right|dt,\end{aligned}\tag{4.5}$$

where

$$K(t,x)=\begin{cases}t-a,\ t\in[a,x)\\ t-b,\ t\in[x,b].\end{cases}$$

It is important to note here that Inequality (4.5) is sharper than (1.1) since

$$\max_{t\in[a,b]}\left|K(t,x)-\left(x-\frac{a+b}{2}\right)\right|=\frac{b-a}{2}.$$

Remark 4.3. *Next, we consider the case when $\lambda = 1$ in Inequality (4.4). For this, we obtain*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\ dt\right|\leq\frac{\Gamma-\gamma}{8(b-a)}\left[(b-a)^2+(2x-a-b)^2\right]\tag{4.6}$$

for all $x \in [a, b]$.

Applying the above inequality to the following example.

Example 4.1. *Consider the function $f : [0,1] \to \mathbb{R}_+$ defined by $f(x) = e^{x^2}$. We know that the integral of f cannot be achieved via an analytic method; but we can approximate it using numerical methods. For this function, we observe that $0 \le f'(x) \le 6$ for all $x \in [0,1]$. Choose $\gamma = 0$ and $\Gamma = 6$. Now, using* (4.6)*, on gets*

$$\Big| \int_0^1 e^{t^2}\ dt - \frac{e+1}{2} \Big| \le \frac{3}{2}(2x^2 - 2x + 1) \text{ for all } x \in [0,1].$$

In particular, for $x = 0$ or 1, we have

$$\frac{e}{2} - 1 \le \int_0^1 e^{t^2}\ dt \le \frac{e}{2} + 2.$$

Using MATLAB, one can verify that $\int_0^1 e^{t^2}\ dt \approx 1.46265$. This shows that the range given above is correct!

Corollary 4.2 (Discrete case)**.** *Let $\mathbb{T} = \mathbb{Z}$ in Theorem 3.1. Suppose $a = 0, b = n$ and*

1. *$\mathbb{I}_k := \{j_0, j_1, \cdots, j_k\} \subset \mathbb{Z}$, where $a = j_0 < j_1 < \cdots < j_k = b$, is a partition of the set $[0,n] \cap \mathbb{Z}$*
2. *$\{\alpha_0, \alpha_1, \cdots, \alpha_{k+1}\} \subset \mathbb{Z}$ is a set of $k+2$ points such that $\alpha_0 = 0, \alpha_i \in [j_{i-1}, j_i]$ for $i = 1, 2, \cdots, k$ and $\alpha_{k+1} = n$;*
3. *$f(k) = x_k$.*

We have the inequality,

$$\begin{aligned}
&\Bigg|(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)x_{j_i} + \lambda\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\frac{x_{\alpha_i}+x_{\alpha_{i+1}}}{2} - \sum_{j=1}^{n} x_j \\
&- \frac{x_n - x_0}{n}\sum_{i=0}^{k-1}\Bigg[h_2\Big(\alpha_{i+1}, \alpha_{i+1} - \lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) - h_2\Big(j_i, \alpha_{i+1} - \lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) \\
&+ h_2\Big(j_{i+1}, \alpha_{i+1} + \lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big) - h_2\Big(\alpha_{i+1}, \alpha_{i+1} + \lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Bigg]\Bigg| \\
&\le \frac{\Gamma-\gamma}{2}\sum_{j=0}^{n-1}\Bigg|K(j, I_k) - \frac{1}{n}\sum_{i=0}^{k-1}\Bigg[h_2\Big(\alpha_{i+1}, \alpha_{i+1} - \lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) - h_2\Big(j_i, \alpha_{i+1} - \lambda\frac{\alpha_{i+1}-\alpha_i}{2}\Big) \\
&+ h_2\Big(j_{i+1}, \alpha_{i+1} + \lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big) - h_2\Big(\alpha_{i+1}, \alpha_{i+1} + \lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\Big)\Bigg]\Bigg|, \qquad (4.7)
\end{aligned}$$

where $h_2(t,s) = \binom{t-s}{2} = \frac{(t-s)(t-s-1)}{2}$ for all $t, s \in \mathbb{Z}$.

Corollary 4.3 (Quantum case)**.** *Let $\mathbb{T} = q^{\mathbb{N}_0}, q > 1, a = q^m, b = q^n$ with $m, n \in \mathbb{N}$ and $m < n$. Suppose that*

1. *$\mathbb{I}_k : q^m = q^{j_0} < q^{j_1} < \cdots < q^{j_k} = q^n$, is a partition of the set $[q^m, q^n] \cap q^{\mathbb{N}_0}$ for $j_0, j_1, \cdots, j_k \in \mathbb{N}$;*
2. *$q^{\alpha_i} \in q^{\mathbb{N}_0} (i = 0, 1, \cdots, k+1)$ is a set of $k+2$ points such that $q^{\alpha_0} = q^m, q^{\alpha_i} \in [q^{j_{i-1}}, q^{j_i}] \cap q^{\mathbb{N}_0} (i = 1, 2, \cdots, k)$ and $q^{\alpha_{k+1}} = q^n$;*
3. *$f : [q^m, q^n] \to \mathbb{R}$ is differentiable.*

Then we have the inequality,

$$\Bigg|(1-\lambda)\sum_{i=0}^{k}(q^{\alpha_{i+1}}-q^{\alpha_i})f(q^{j_i})+\lambda\sum_{i=0}^{k}(q^{\alpha_{i+1}}-q^{\alpha_i})\frac{f(q^{\alpha_i})+f(q^{\alpha_{i+1}})}{2}-\int_{q^m}^{q^n}f(qt)d_qt$$
$$-\frac{f(q^n)-f(q^m)}{q^n-q^m}\sum_{i=0}^{k-1}\Bigg[h_2\Big(q^{\alpha_{i+1}},q^{\alpha_{i+1}}-\lambda\frac{q^{\alpha_{i+1}}-q^{\alpha_i}}{2}\Big)-h_2\Big(q^{j_i},q^{\alpha_{i+1}}-\lambda\frac{q^{\alpha_{i+1}}-q^{\alpha_i}}{2}\Big)$$
$$+h_2\Big(q^{j_{i+1}},q^{\alpha_{i+1}}+\lambda\frac{q^{\alpha_{i+2}}-q^{\alpha_{i+1}}}{2}\Big)-h_2\Big(q^{\alpha_{i+1}},q^{\alpha_{i+1}}+\lambda\frac{q^{\alpha_{i+2}}-q^{\alpha_{i+1}}}{2}\Big)\Bigg]\Bigg|$$
$$\leq\frac{\Gamma-\gamma}{2}\int_a^b\Bigg|K(t,I_k)-\frac{1}{q^n-q^m}\sum_{i=0}^{k-1}\Bigg[h_2\Big(q^{\alpha_{i+1}},q^{\alpha_{i+1}}-\lambda\frac{q^{\alpha_{i+1}}-q^{\alpha_i}}{2}\Big)-h_2\Big(q^{j_i},q^{\alpha_{i+1}}-\lambda\frac{q^{\alpha_{i+1}}-q^{\alpha_i}}{2}\Big)$$
$$+h_2\Big(q^{j_{i+1}},q^{\alpha_{i+1}}+\lambda\frac{q^{\alpha_{i+2}}-q^{\alpha_{i+1}}}{2}\Big)-h_2\Big(q^{\alpha_{i+1}},q^{\alpha_{i+1}}+\lambda\frac{q^{\alpha_{i+2}}-q^{\alpha_{i+1}}}{2}\Big)\Bigg]\Bigg|d_qt,$$

where $h_2(t,s)=\frac{(t-s)(t-qs)}{q+1}$ *for all* $t,s\in q^{\mathbb{N}_0}$.

We close this section by applying Theorem 3.2 to the continuous calculus.

Corollary 4.4 (Continuous case)**.** *Let* $\mathbb{T}=\mathbb{R}$. *Then we have the inequalities*

$$\Bigg|(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)+\lambda\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\frac{f(\alpha_i)+f(\alpha_{i+1})}{2}-\int_a^b f(t)dt$$
$$-\frac{f(b)-f(a)}{8(b-a)}\sum_{i=0}^{k-1}\Bigg[\lambda^2\Big(\alpha_{i+1}-\alpha_i\Big)^2-\Big(2x_i-\lambda\alpha_i+(\lambda-2)\alpha_{i+1}\Big)^2$$
$$+\Big(2x_{i+1}-\lambda\alpha_{i+2}+(\lambda-2)\alpha_{i+1}\Big)^2-\lambda^2\Big(\alpha_{i+2}-\alpha_{i+1}\Big)^2\Bigg]\Bigg|$$
$$\leq\begin{cases}M_k(b-a)(S-\gamma)\\ M_k(b-a)(\Gamma-S),\end{cases}$$

where $S=\frac{f(b)-f(a)}{b-a}$, *and* $M_k=\max\limits_{t\in[a,b]}\Big|K(t,I_k)-\frac{1}{b-a}\int_a^b k(s,I_k)ds\Big|$.

Remark 4.4. *By setting* $\lambda=0$ *in Corollary 4.4, we get a direct generalization of Theorem 1.1 to* $(k+1)$ *points* $x_0,x_1,\cdots,x_k$. *In fact, we obtain*

$$\Bigg|\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)-\int_a^b f(t)dt-\frac{f(b)-f(a)}{b-a}\Bigg[\frac{b^2-a^2}{2}-\sum_{i=0}^{k-1}\alpha_{i+1}(x_{i+1}-x_i)\Bigg]\Bigg|$$
$$\leq\begin{cases}M_k(b-a)(S-\gamma)\\ M_k(b-a)(\Gamma-S),\end{cases}\tag{4.8}$$

where $K(t,I_k)$ *is given by* (4.3).

5. Conclusion

We have established three new Ostrowski–Grüss type inequality with a parameter $\lambda\in[0,1]$. Loads of interesting results can be derived by choosing different values of $k\in\mathbb{N}$, and λ's. As an application, we considered the continuous, discrete, and quantum calculus from which many novel inequalities are obtained.

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