LORENTZ PROPERTIES OF AMPLITUDES

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This paper presents a brief summary of the various topics, connected with the invariant expansion of relativistic amplitudes. Most of the details of calculations can be find in three papers [1, 2, 6] where a complete list of references is also given.

1. Parametrization of 4-point amplitude

Instead of constructing amplitudes from one particle states (3), we treat the amplitude as a function of two variables. Choosing as independent variables the components of 4-velocity of one of the particles we define the amplitude as a function on the upper sheet of hyperboloid $u^2 = 1$. There are 5 possible parametrizations (coordinate systems): spherical (S) and hyperbolical (H) ones connected by the crossing transformation; two kinds of horispherical systems (O) and cylindrical system (C). Each of the systems corresponds to different reductions of the Lorentz group. The unitary transformation coefficients between different parametrizations appear to be essentially products of Clebsch — Gordon coefficients for complex values of angular momenta [7]. Another way for the parametrization of the amplitudes is connected with a complex sphere [4, 7] $z^2 = 1$. In this case the complex vector z is built from two 4-velocities $z = u + iv$. This method gives rise to the extremely simple form of matrix elements of the Lorentz group.

2. Expansion

The amplitude can be expanded into a series of orthonormal functions which give the representation of the group of motion: either Lorentz group or complex rotation one in three dimensions. In this way the Lorentz amplitudes or Wigner functions in a complex domain come into play. The most powerful method for expansion and normalization is the so-called horispherical method developed by Shapiro, Gelfand, Graev (see [1] for refs.). It starts with the expansion into a series of homogeneous functions (on the cone) which is followed by the transformation to the hyperboloid. In this way it is also possible to get the functions for nondegenerate representations of the Lorentz group. This is due to the fact that the helicity of a massless particle (connected with the light cone) is a Lorentz invariant quantity and can be identified with the second quantum number ν of the Lorentz group [5].

Some interesting features appear in the expansion into a series of functions defined on the complex sphere of a zero radius (a complex cone).

3. Crossing

Casimir operators of the Lorentz group generate new quantum numbers. In the spin zero case the expansion in S system produces the partial amplitude $a_l\,(\rho)$ which is a function of one discrete parameter l and the continious parameter ρ (which is also invariant with respect to crossing). In H system the partial amplitude depends upon two continious parameters $a\,(\rho,\ q)$. The second variable q is interpreted as a complex angular momentum. The amplitudes $a_l\,(\rho)$ and $a\,(\rho,\ q)$ are determined in an invariant way as integrals over the physical region in which the amplitudes are defined. Let us call them Lorentz amplitudes. In S system the coordinates are $\mathrm{ch}\alpha_S$ (energy in c. m. s. divided by the mass of the particle) and $\cos\alpha_S$ (α_S is a scattering angle). The transformation in t-shannel is given by $a_S + a_t = i\frac{\pi}{2}$ and $\beta_t = i\alpha_S$, where a_t and β_t are two hypergeometrical «angles» in H-system. The horispherical system is defined by constructing the isotropic (photon-like) vector $k_S = u_1e^A - u_2$, with $k_S^2 = 0$, or $k_H = u_1e^A - u_3$; $k_H^2 = 0$; the crossing transformation is defined in a similar way.

The Lorentz two-dimensional amplitudes define amplitudes in all physical parts of the Mandelstam plane provided the two conditions are fulfilled (usually they are not):

a) Integrals, which define the expansion coefficients, converge,

b) under the crossing transformation the boundary of the physical region in s-channel transforms into one in t- (or u-) channel.

4. Unequal masses

Condition b) is violated in the case of unequal masses. In this case the expression for coefficients in S-channel can not be used in t-channel because of the difference of the integration regions.

In order to improve the situation we consider instead of c. m. system the one in which two particles have equal (but opposite) 4-velocities. We define two

4-vectors of a unit length

$$n_0 = \frac{u_1 + u_2}{\sqrt{2(1 + u_1 u_2)}}, \; n_1 = \frac{u_1 - u_2}{\sqrt{2(1 - u_1 u_2)}}; \quad n_0^2 = n_1^2 = 1.$$

The first vector is time-like, the second is space-like. In order to complete the set we introduce a spatial vector which is orthogonal to the scattering plane (this vector is irrelevant) and one more space-like vector which is orthogonal to all the three. We call the latter n_2. Now consider the components of 4-velocity of one of the particles (say, 1) along these vectors; we denote them by x_0, x_1, x_2, ($x_3 = 0$). Then $x_0^2 - x_1^2 - x_2^2 = 1$, and x_2 appears to be proportional to the function Φ defining the boundary of the physical region [9]. Then it is not difficult to prove that in these coordinates the integral representations of Lorentz amplitudes are the same in all channels, since the physical boundary is now invariant with respect to cross-transformation. However the equal velocity frame is bad from the physical point of view; in physical applications we use the angular momentum partial wave expansion in c. m. system. I. e., we have to perform the Lorentz transformation in both channels. The transformation matrix for «boost» (the Wigner function) must be calculated, since only extremely unconvenient forms of this function (the double series of hypergeometric functions) are known in the literature. «Boost» functions will be discussed below.

5. Convergence

We have assumed that all the integrals converge, i. e. amplitudes vanish at infinity. The Regge behaviour results in the divergence of integrals.

In the invariant classification we start with the so-called Lorentz poles in the invariant variables ρ. The Lorentz poles induce Regge-poles of two sorts, depending on the parametrization of an amplitude. If we parametrize the amplitude by 4-velocity of particles the poles are essentially excited states. The true Regge poles appear as singularities in the two-particle parametrization (complex sphere). In both cases the Lorentz poles give rise to a family of poles and the connection between Lorentz poles and so-called daughter trajectories seems to be established.

6. Spins

So far we have discussed the spinless case only. For helicity the cross-transformation have a simple geometrical meaning: it is essentially the three-dimentional rotation. However in more complicated cases when we need all three components of spin, the analytical properties of the coordinate system become important. Here we have to build the coordinate system from 4-velocities and to pass on momenta only after performing the cross-transformation. This supplementary relativistic rotation gives rise to kinematical singularities which become factorized when we use the velocity frame as an intermediate coordinate system.

7. Matrix elements of Lorentz group

In order to make all the transformations mentioned above, we need the expression for matrix elements of the Lorentz transformation. The derivation of matrix elements starts from the observation that two sets of generators F, K built upon usual generators of the Lorentz group M and N

$$F_1 = \frac{1}{2}(iM_1 - N_1), \qquad F_2 = \frac{1}{2}(iM_2 - N_2), \qquad F_3 = \frac{1}{2}(M_3 + iN_3),$$

$$K_1 = \frac{1}{2}(-iM_1 - N_1), \; K_2 = \frac{1}{2}(-iM_2 - N_2), \; K_3 = \frac{1}{2}(M_3 - iN_3)$$

generate two (independent) algebras O (2, 1). Using C-system of coordinates, which corresponds to the diagonal form of F_3 and K_3 we can express the matrix element of the Lorentz group in terms of the Clebsch — Gordon coefficients for addition of two complex angular momenta into a real one. This result is verified by the explicit calculation using the expansion into a series of hypergeometrical functions ${}_3F_2$. The complex momenta to be added are $\frac{1}{2}(\sigma + \nu)$ and $\frac{1}{2}(\sigma - \nu)$, where in unitary case $\sigma = -1 + i\rho$, and ν is an integer.

$$d^{\nu\sigma}_{JmJ'} = N \int\limits_{-K-i\infty}^{-K+i\infty} dt \frac{e^{\pm i\pi t}}{\sin \pi t} \left(\frac{\sigma+\nu}{2}, t - \frac{\sigma-\nu}{2} + m; \frac{\sigma-\nu}{2}, \frac{\sigma-\nu}{2} - t \middle| Jm\right) \times$$

$$\times \left(\frac{\sigma+\nu}{2}, t - \frac{\sigma-\nu}{2} + m; \frac{\sigma-\nu}{2}, \frac{\sigma-\nu}{2} - t \middle| J'm\right) e^{-\theta(2t-\sigma+\nu+m)}$$

($\pm$ for Im $t \gtrless 0$, N is a normalization constant).

This opens the way for definition of complex angular momentum addition coefficients.

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