ANALYTIC PROPERTIES OF THE AMPLITUDES AND ASYMPTOTIC THEOREMS

Chairman	Yu. Novozhilov
Rapporteur	Nguyen van Hieu
Discussion leader	I. Todorov
Secretaries	G. Volkov
	V. Skarzhinsky
	V. Yatsun
	V. Ezela

ANALYTICAL PROPERTIES OF THE SCATTERING AMPLITUDE AND ASYMPTOTIC THEOREMS

Nguyen van Hieu

About 15 years ago N. N. Bogolubov proved rigorously the dispersion relation for the pion-nucleon scattering at fixed momentum transfers starting only from the general principles of the quantum field theory [1, 2]. Further investigations in this direction achieved significant successes. Different analytical properties of the scattering amplitude in one and two variables [3—8], the number of subtractions in the dispersion relations at fixed momentum transfers have found [9]. These analytical properties and the unitarity condition lead to a number of consequences which can be checked experimentally. Due to these investigations it is now possible to test the validity of the basic principles of the quantum field theory, and first of all, the microcausality principle.

Most of these experimental consequences refer to the high energy region. At present the study of the asymptotic behaviour of the amplitude at high energies is of great interest because of new possibilities of testing it experimentally up to 70 *GeV* at Serpukhov.

My talk is a review of the results of a series of theoretical works on studying the asymptotic behaviour of the amplitude and the form factor. In most of these works only the general properties of the amplitude: analyticity, unitarity and crossing symmetry are assumed. All the theoretical predictions from the references reviewed here are accessible to experimental test. We consider mainly the following problems:

1. The short range of strong interactions as a consequence of the analyticity in the momentum transfer, and bounds on the behaviour of cross sections.
2. Asymptotic theorems.

3. The experimental test of the dispersion relations.
4. Exact sum rules.
5. The behaviour of the form factor and bounds on the radii of elementary particles.

1. The Short Range of Strong Interactions and the Analyticity in Momentum Transfer. Bounds on the Behaviour of Cross Sections

In nuclear physics we know that one of the characteristic peculiarities of the nuclear forces is their short range character. These forces exist only in some region with a finite radius R, in contrast with the Coulomb force which decreases slowly as $1/r^2$. This is connected with the fact that the Coulomb forces between charged particles appear as a result of the exchange of photons — massless particles while the nuclear forces arise from the exchange of massive particles. For example, the effective potential of the interaction which is the result of the exchange of the meson with mass μ — Yukawa's potential — is of the form

$$V(r) = \frac{g}{r} e^{-\mu r}. \tag{1.1}$$

Its Fourier transform is

$$V(p) = \frac{g}{\mu^2 + p^2}. \tag{1.2}$$

The potential $V(r)$ is negligible at distances larger than R if

$$|ge^{-\mu R}| = 1.$$

The value

$$R = \frac{1}{\mu} \ln |g| \tag{1.3}$$

may be considered as the effective radius of the region in which the nuclear forces are not small.

In the works on the meson theory of nuclear forces these results have been obtained in the nonrelativistic approximation using the perturbation theory. A question naturally arises: is it possible to obtain these results in the framework of a relativistic theory without using the perturbation theory? The answer is positive. The connection between the short range character of strong interactions and the nonzero mass of exchange particles established in the nonrelativistic approximation using the perturbation theory has a significant physical content, and it should exist in any theory. The methods of the dispersion relation theory make it possible to establish this connection rigorously: without using the perturbation theory and taking into account the requirements of the relativistic invariance.

It is easy to understand the essence of this connection using the quasipotential approach proposed by Logunov and Tavkhelidze [10]. These authors showed that the elastic scattering amplitude $T(s, t)$ (where s and t are Mandelstam variables [11]) is the solution of a relativistic equation of the Lippmann — Schwinger type with a complex quasipotential $V(r, s)$ depending on. The following spectral representation for the quasipotential $V(r, s)$ results from the t-dispersion relation for the amplitude

$$V(r, s) = \int_{\mu_0}^{\infty} \frac{e^{-\mu r}}{r} \rho(\mu^2, s)\, d\mu^2, \tag{1.4}$$

or in the p-space,

$$V(p, s) = \int_{\mu_0}^{\infty} \frac{\rho(\mu^2, s)}{\mu^2 + p^2} d\mu^2 \quad (1.5)$$

where μ_0 is the minimal effective mass of the exchange particle systems (minimum total energy of these systems in their c. m. s.). The quasipotential $V(r, s)$ is the superposition of the Yukawa potentials (1.1). Comparing (1.5) with (1.1) and (1. 3) we see that the value R satisfying the condition

$$R \leqslant \max_{\mu \geqslant \mu_0} \frac{1}{\mu} \ln |\rho(\mu^2, s)| \quad (1.6)$$

may be used as the effective radius of strong interactions (12). At each finite value of s the radius R is finite. However, as s increases it may increase as ln s, so long as $\rho(\mu^2, s)$ in (1.6) is a polynomial in s.

Now let us start from the works in which it was shown that the short range character of strong interactions follows from the analytical properties of the elastic scattering amplitude in momentum transfer. For this purpose, following Logunov et al. [13], let us define the effective interaction radius in the frame of the relativistic theory.

Note that in the classical problem of the scattering on the potential with the finite radius R the partial amplitudes $f_l(s)$ are equal to zero if the orbital angular momentum $l > kR$, where k is the momentum of the incident particle. When passing to the relativistic problem of two particle scattering (with a spin, possibly), we use the total angular momentum J in the c. m. s. instead of the orbital momentum l. For any particle system the total angular momentum in the c. m. s. is a relativistic invariant quantity. It is the eigenvalue of one of the invariant operators — Casimir operators of the inhomogeneous Lorentz group. On the other hand, in the classical problem of the potential scattering of a spinless particle it reduces to the orbital angular momentum l. Using the total angular momentum J as a relativistic dynamical variable and studying the minimum value J_0 of the angular momentum J, beginning from which the partial waves practically do not contribute to the amplitude, we generalize the definition of the effective interaction radius to the relativistic case:

$$R = \frac{J_0}{k} \quad (1.7)$$

where k is the particle momentum in the c. m. s.

On the basis of the Mandelstam representation [11] for the elastic scattering amplitude $T(s, t)$, Froissart [14] showed that all partial waves with angular momenta J larger than some value proportional to ln s give a negligible contribution to the amplitude. Using the Greenberg-Low method [15], it is possible to obtain the results of Froissart on the basis of the rigorously proved analytical properties of the elastic scattering amplitude, the t-analyticity in the Martin ellipse [6—8] and the restrictions to the number of subtractions in the dispersion relations on s at all t in this ellipse [9]. In order to determine the minimum value of J, beginning from which the partial waves are practically negligible, it is necessary to indicate what magnitude of the contribution of the partial waves with $J \leqslant J_0$ we expect. It was shown in ref. [16] that if the total cross section of the elastic scattering σ_e decreases more slowly that $s^{-\rho}$

$$\sigma_{el} \geqslant \text{const}\, s^{-\rho}, \quad s \to \infty \quad (1.8)$$

then we have

$$J_0 \leqslant \left(1 + \frac{\rho}{2}\right) \sqrt{\frac{s}{t_0}} \ln \frac{s}{s_0} \quad (1.9)$$

where t_0 is the nearest to zero singularity in the t-plane: $t_0 = 4m_\pi^2$, s_0 is an unknown constant.

Thus, the theory of analytical functions turns out to be an adequate instrument for establishing the connection between the short range character of strong interactions and the nonvanishing of the masses of exchange particles. Note that the short range character of strong interactions as a consequence of analyticity in t has also been considered in refs. [17, 18].

The existence of a finite effective radius of strong interactions leads to a number of predictions. Consider the elastic scattering amplitude

$$T(s, t) = 8\pi \frac{\sqrt{s}}{k} \sum_{l=0}^{2l+1} (2l+1) f_l(s) P_l\left(1 + \frac{t}{2k^2}\right). \tag{1.10}$$

Let us divide the inifinite sum in this formula into two parts: a sum from $l = 0$ up to $l = J_0$ and a sum from $l = J_0 + 1$ up to infinity. In order to estimate the first sum, one may use the unitarity condition

$$|f_l(s)| \leqslant 1.$$

The contribution of these partial waves to the total cross section increases not faster than

$$R^2 \approx \frac{J_0^2}{k^2} \leqslant \text{const} \ln^2 \frac{s}{s_0}.$$

In comparison with this behaviour the contribution of all partial waves with $l = J_0$ is negligible even if we take $\rho = 0$ in (1.9). With this optimal choice of ρ we have

$$\sigma_{\text{tot}} \leqslant \frac{4\pi}{m_\pi^2} \ln^2 \frac{s}{s_0}. \tag{1.11}$$

Note that an unknown constant s_0 occurs in the last formula.

Different improvements of this result have been made. Lukaszuk and Martin [19] have shown that the value

$$J_0 \leqslant \frac{1}{2} \sqrt{\frac{s}{t_0}} \ln \frac{s}{s_0} \tag{1.12}$$

may be taken as the upper limit of the maximal angular momentum J_0 instead of the value (1.9). Thus, they improved the bound (1.11) by a factor of 1/4,

$$\sigma_{\text{tot}} \leqslant \frac{\pi}{m_\pi^2} \ln^2 \frac{s}{s_0}. \tag{1.13}$$

Another essential improvement has been made by Yndurian and Common [20—22]. In these papers the authors have obtained the bound (1.13) with some definite constant s_0. It is expressed through the D-wave scattering length in the t-channel. Thus, these authors have obtained absolute asymptotic upper bounds on the amplitude. Note that absolute upper limits were firstly found in the work bu Lukaszuk and Martin [19].

In the work of Blokhintsev, Barashenkov and Barbashov [23] it has been pointed out that the analysis of experimental data on the elastic scattering gives us some information about the interaction radius. The possibility of the experimental determination of the effective interaction radius of strong interactions has been discussed in many papers [13, 24, 25]. In particular, using the Buniakovski — Schwartz inequality, it is easy to show that the interaction radius may be estimated from experimental data on the elastic scattering cross section [13],

$$R^2 \geqslant \frac{4}{\sigma_{\text{el}}} \cdot \frac{d\sigma_{\text{el}}}{dt}\bigg|_{t=0}. \tag{1.14}$$

Note that according to (1.12)

$$R \leqslant \frac{1}{m_\pi} \ln \frac{s}{s_0}. \tag{1.15}$$

From (1.14) and (1.15) one gets

$$\frac{1}{\sigma_{el}} \cdot \frac{d\sigma_{el}}{dt}\bigg|_{t=0} \leqslant \frac{1}{4m_\pi^2} \ln^2 \frac{s}{s_0}. \tag{1.16}$$

This result was given in a paper by Singh and Roy [26]. A similar inequality with the right hand side

$$\frac{1}{m_\pi^2} \ln^2 \frac{s}{s_0}$$

was obtained earlier in ref. [16].

The introduced concept of the effective interaction radius may be generalized for inelastic processes as well as for backward elastic scattering. Following papers of Logunov, Mestvirishvili et al. [27] and Tiktopolous and Treiman [28], let us denote the differential cross section of the production of some particle «c» at an angle θ by an arbitrary system of two colliding particles «a» and «b», by

$$\frac{d\sigma_{inel}^c}{d\cos\theta}.$$

For the differential cross sections of the inelastic two-body processes we use the same notation.

Let σ_{inel}^c denotes the total cross section of the inelastic process which we are considering. Then for the effective radius R_{inel} of the given process one obtains the inequality

$$(R_{inel}^c)^2 \geqslant \frac{1}{2k^2} \cdot \frac{4}{\sigma_{inel}} \cdot \frac{d\sigma_{inel}^c}{d\cos\theta}\bigg|_{\theta=0}. \tag{1.17}$$

From the analyticity of the elastic scattering amplitude in t and the unitarity condition it is possible to obtain the same upper bound for R_{inel} as for the elastic scattering radius [16, 27, 28]

$$R_{inel}^c \leqslant \frac{1}{m_\pi} \ln \frac{s}{s_0}. \tag{1.18}$$

Note that the more significant statement is valid: the effective radius of any inelastic process does not exceed the effective radius of the corresponding elastic process. Owing to the unitarity condition, t-singularities of the inelastic amplitudes are automatically introduced into the elastic scattering amplitude, and therefore the singularity of the elastic scattering amplitude nearest to zero in the t-plane cannot lie to the right of the nearest singularity of the amplitude (cross section) of inelastic processes which contribute to the imaginary part of the elastic scattering amplitude. However, the upper limit in (1.18) for many inelastic processes may be too large.

The elastic backward scattering of two different particles may be also considered as an inelastic process at small angles. However, the total cross section of this particular inelastic scattering process coincides with the total cross section of the elastic scattering, and therefore the ratio (1.17) is partically of no interest. For the generalization of the concept of the effective radius in the case of backward scattering instead of the scattering in the whole angle interval $0 \leqslant \theta \leqslant \pi$ we shall consider only the scattering to the backward hemisphere $\theta \geqslant \frac{\pi}{2}$. Denote the total cross section of the elastic scattering to the backward hemisphere by σ_π. In this case the effective radius of backward scattering satisfies the condition

$$R_\pi^2 \geqslant \frac{4}{\sigma_\pi} \cdot \frac{d\sigma_{el}}{du}\bigg|_{u=0}, \tag{1.19}$$

where $u = 2(M^2 + m^2) - s - t$ is one of Mandelstam's variables, M and m are the particle masses. We may expect that for the scattering of a pion on a nucleon the backward scattering effective radius is much smaller than the forward scattering effective radius, since in the first case the process is a result of the exchange of heavy particles, and in the second case of light particles.

Logunov et al. [29, 30] noted that the ratio of the cross sections in the right hand sides of formulae (1.14), (1.17) and (1.19) may be considered as the definition of the inverse width of the diffraction peak of the corresponding process, so far as it characterizes the behaviour of the diffraction peak at high energies. Thus, we have

$$W = \frac{\sigma_{\mathrm{el}}}{\left.\dfrac{d\sigma_{\mathrm{el}}}{dt}\right|_{t=0}} \tag{1.20}$$

for the elastic scattering,

$$W^c_{\mathrm{inel}} = \frac{2k^2\sigma^c_{\mathrm{inel}}}{\left.\dfrac{d\sigma^c_{\mathrm{inel}}}{d\cos\theta}\right|_{\theta=0}} \tag{1.21}$$

for inelastic processes, and

$$W_\pi = \frac{\sigma_\pi}{\left.\dfrac{d\sigma_{\mathrm{el}}}{du}\right|_{u=0}} \tag{1.22}$$

for the backward scattering. Relations (1.14), (1.17) and (1.19) relate the effective radius of the interaction to the width of the diffraction peak

$$W \geqslant \frac{4}{R^2}, \tag{1.23}$$

$$W^c_{\mathrm{inel}} \geqslant \frac{4}{R^{c2}_{\mathrm{inel}}}, \tag{1.24}$$

$$W_\pi \geqslant \frac{4}{R^2_\pi}. \tag{1.25}$$

These inequalities show that if there is the shrinkage of the diffraction peak when the energy increases then the effective radius of the interaction should increase. This fact takes place, for example, if the amplitude has the Regge behaviour.

As the effective radius of the interaction cannot increase faster than $\ln\dfrac{s}{s_0}$ as $s \to \infty$, the width of the diffraction cone cannot decrease faster than $\ln^{-2}\dfrac{s}{s_0}$:

$$W \geqslant \frac{4m^2_\pi}{\ln^2\dfrac{s}{s_0}} \tag{1.26}$$

$$W^c_{\mathrm{inel}} \geqslant \frac{4m^2_\pi}{\ln^2\dfrac{s}{s_0}} \tag{1.27}$$

$$W_\pi \geqslant \frac{4m^2_\pi}{\ln^2\dfrac{s}{s_0}}. \tag{1.28}$$

Along with the definition (1.20) — (1.22) of the width of the diffraction peak there exists another definition according to which the inverse width of the diffraction peak is equal to the logarithmic derivative of the amplitude modulus at $t = 0$,

$$\frac{1}{W} = \left[\frac{d}{dt} \ln |T(s, t)| \right]_{t=0}. \tag{1.29}$$

Based on the analyticity of the amplitude in s, Bessis [31] proved that if the value of the cross section in the physical region does not exceed its value at $t = 0$,

$$|T(s, t)| \leqslant |T(s, 0)|, \; t \leqslant 0 \tag{1.30}$$

then the value of W has lower bound

$$W \geqslant \frac{1}{e} \left(\frac{\pi m \pi}{15} \right)^2 \ln^{-2} \frac{s}{s_0}. \tag{1.31}$$

To get this result Bessis used as the lower bound the elastic forward scattering amplitude obtained by Jin and Martin [32]:

$$|T(s, 0)| \geqslant \frac{\text{const}}{s^2}. \tag{1.32}$$

It is possible to improve the numerical factor in the right hand side of (1.31) by taking into account information on the behaviour of $T(s, 0)$ obtained in the experiment. Thus, for example, if the total cross sections tend to nonzero limits at $s \to \infty$, then instead of (1.32) we may take

$$|T(s, 0)| \geqslant \text{const} \cdot s. \tag{1.33}$$

In this case

$$W \geqslant \frac{1}{e} \left(\frac{4\pi m_\pi}{15} \right)^2 \ln^{-2} \frac{s}{s_0}. \tag{1.34}$$

The imaginary part of the amplitude, according to the unitarity condition deliberately satisfies the condition analogous to (1.30)

$$|A(s, t)| \leqslant A(s, 0), \quad t \leqslant 0. \tag{1.35}$$

Therefore, for the value

$$W_a = \left[\left(\frac{d}{dt} \ln |A(s, t)| \right)_{t=0} \right]^{-1} \tag{1.36}$$

we have the inequality of the type (1.34)

$$W_a \geqslant \frac{1}{e} \left(\frac{4\pi m_\pi}{15} \right)^2 \ln^{-2} \frac{s}{s_0} \tag{1.37}$$

if the total cross sections tend to finite limits at $s \to \infty$.

Note that in the case of the Regge asymptotic

$$T(s, t) \approx \text{const}\, s^{\alpha(t)}$$

we have

$$W = \frac{1}{\alpha'(0) \ln s/s_0}. \tag{1.38}$$

In a series of papers different bounds on the behaviour of the elastic scattering amplitude at $t \neq 0$ have been obtained. Kinoshita [33] showed that due to its analytical properties and the unitarity condition, the imaginary part $A(s, t)$ of the elastic scattering amplitude in the unphysical region $0 < t < 4m_\pi^2$ should have the Regge behaviour in the sense that

$$\left| \frac{d}{dt} \ln A(s, t) \right| \leqslant \beta(t) \ln \frac{s}{s_0} \tag{1.39}$$

at $s \to \infty$. Another result has been obtained in refs. [34, 35] where it was shown that in the physical region $t < 0$ as $s \to \infty$ the amplitude $T(s, t)$ cannot decrease faster than the Regge behaviour. Roughly speaking, we have

$$|T(s, t)| \geqslant \text{const}\, s^{\alpha(t)} \tag{1.40}$$

for some trajectory. This means that for the amplitude in the region $t < 0$ a lower bound having the Regge behaviour has been obtained.

In Kinoshita's report [36] presented at the conference the asymptotics of the elastic scattering amplitude has been studied at decreasing momentum transfers. On the basis of the analyticity of the amplitude in t and the unitarity condition, he showed that at the values of the momentum transfer decreasing as

$$t(s) \approx -\frac{c}{\ln^2 \frac{s}{s_0}} \tag{1.41}$$

for some rather small positive constant c the imaginary part $A(s, t(s))$ should have the same behaviour as its behaviour at $t = 0$,

$$\left|\frac{A(s, t(s))}{A(s, 0)}\right| \geqslant \text{const}, \quad s \to \infty. \tag{1.42}$$

It is likely that there exists an analogous property for the real part $D(s, t)$. The fact is that $D(s, t)$ may be written as

$$D(s, t) = D^+(s, t) - D^-(s, t) \tag{1.43}$$

where $D^+(s, t)$ and $D^-(s, t)$ are not negative at $t = 0$

$$D^{\pm}(s, t) \geqslant 0 \tag{1.44}$$

and at the values (1.41) of t they have the same behaviour as that at $t = 0$:

$$\left|\frac{D^{\pm}(s, t(s))}{D^{\pm}(s, 0)}\right| = \varepsilon^{\pm} \geqslant \text{const}, \quad s \to \infty. \tag{1.45}$$

As

$$D^+(s, 0) \neq D^-(s, 0),$$

it is likely that functions $D^+(s, t(s))$ and $D^-(s, t(s))$ cannot cancel at small c. Then we have

$$\left|\frac{D(s, t(s))}{D(s, 0)}\right| \geqslant \text{const}, \quad s \to \infty. \tag{1.46}$$

In ref. [37] it was shown rigorously that in any interval of momentum transfers of the form

$$-4k^2 + t_1(s) \leqslant t \leqslant -t_1(s) \tag{1.47}$$

where

$$t_1(s) = \frac{\alpha}{\ln^2 \frac{s}{s_0}} \tag{1.48}$$

with an arbitrary positive constant α, there exists at least one value of t at which the imaginary part or the modulus of the amplitude has the same behaviour as that of the corresponding quantity at $t \approx 0$:

$$\max_{t \leqslant -t_1(s)} \left|\frac{T(s, t)}{T(s, 0)}\right| \geqslant e^{-4\frac{\sqrt{\alpha}}{2m_\pi}}, \quad s \to \infty \tag{1.49}$$

$$\max_{t \leqslant -t_1(s)} \left|\frac{A(s, t)}{A(s, 0)}\right| \geqslant e^{-7\frac{\sqrt{\alpha}}{2m_\pi}}, \quad s \to \infty. \tag{1.50}$$

For this purpose, together with the t-analyticity of the amplitude the lower bound (1.32) for $|T(s, 0)|$ or the analogous bound for the imaginary part [32],

$$|A(s, 0)| \geqslant \frac{\text{const}}{s^5}, \quad s \to \infty \tag{1.51}$$

has been also used. If we assume that the total cross sections do not decrease at $s \to \infty$, then instead of (1.49) and (1.50) we have

$$\max_{t \leqslant -t_1(s)} \left| \frac{T(s, t)}{T(s, 0)} \right| \geqslant e^{-\frac{\sqrt{\alpha}}{2m_\pi}}, \tag{1.52}$$

$$\max_{t \leqslant -t_1(s)} \left| \frac{A(s, t)}{A(s, 0)} \right| \geqslant e^{-\frac{\sqrt{\alpha}}{2m_\pi}} \tag{1.53}$$

Similar results are valid for the real part if at $t = 0$ it has a lower bound s^{-n}.

2. Asymptotic Theorems

Recently, because of the new possibilities of the experimental investigation of elementary particle interactions at high energies in Serpukhov, a great interest has arisen in the asymptotic theorems, and particulary in the Pomeranchuk theorem [38] on the equality of the total cross sections of the particle and antiparticle interactions. It should be noted that without studying the behaviour of the real parts of the amplitudes we cannot conclude that the theorem is violated if in the experiment we shall observe that the particle and antiparticle interaction total cross sections tend to different nonzero limits.

Generally speaking, we can say that some theorem is violated only if all the conditions necessary for proving it are satisfied, but its conclusions do not take place. Remember that in order to prove the asymptotic equality of the particle and antiparticle interaction total cross sections we should make some assumption on the behaviour of the real parts of the amplitudes. Therefore before speaking about the violation of the Pomeranchuk theorem it is necessary to look at the behaviour of the real parts.

For convenience, instead of the particle and antiparticle elastic scattering amplitudes $T(s, t)$ and $\tilde{T}(s, t)$ at $t = 0$, we introduce new amplitudes

$$F(s) = \frac{T(s, 0)}{s - u}, \quad \tilde{F}(s) = \frac{\tilde{T}(s, 0)}{s - u} \tag{2.1}$$

where s and u are the Mandelstam variables. The total cross sections at high energies are proportional to the imaginary parts of these new amplitudes:

$$\sigma_{\text{tot}} \approx \operatorname{Im} F(s), \quad \tilde{\sigma}_{\text{tot}} \approx \operatorname{Im} \tilde{F}(s). \tag{2.2}$$

The forward differential cross-sections are

$$\left.\frac{d\sigma_{\text{el}}}{dt}\right|_{t=0} \approx |F(s)|^2, \quad \left.\frac{d\tilde{\sigma}_{\text{el}}}{dt}\right|_{t=0} \approx |\tilde{F}(s)|^2. \tag{2.3}$$

In terms of $F(s)$ and $\tilde{F}(s)$ the theorem states [38, 39]: if the total cross sections σ_{tot} and $\tilde{\sigma}_{\text{tot}}$ tend to finite limits at $s \to \infty$ and the amplitudes $F(s)$, $\tilde{F}(s)$ (i. e. their real parts) are bounded, then the following equality holds:

$$\sigma_{\text{tot}}(\infty) = \tilde{\sigma}_{\text{tot}}(\infty). \tag{2.4}$$

Martin [40] pointed out the following more general statement: if the difference of the total cross sections

$$\sigma_{tot} - \tilde{\sigma}_{tot}$$

tends to a limit at $s \to \infty$ and if the amplitudes $F(s)$ and $\tilde{F}(s)$ increase slower than $\ln s/s_0$,

$$\left|\frac{F(s)}{\ln \frac{s}{s_0}}\right| \to 0, \quad \left|\frac{\tilde{F}(s)}{\ln \frac{s}{s_0}}\right| \to 0 \tag{2.5}$$

then this limit of the difference of the cross sections is equal to zero.

It is evident that if the real parts of the amplitudes $F(s)$ and $\tilde{F}(s)$ increase logarithmically, then the equality (2.4) may not hold. If in the future we will observe in experiments, for example, that the limits $\sigma_{tot}(\infty)$ and $\tilde{\sigma}_{tot}(\infty)$ are not equal, and the cross sections $\left.\frac{d\sigma_{el}}{dt}\right|_{t=0}$ and $\left.\frac{d\tilde{\sigma}_{el}}{dt}\right|_{t=0}$ do not increase at high energies, then we shall be able to conclude that the Pomeranchuk theorem is violated. This would be an experimental indication on the invalidity of the principles of the local field theory.

By means of the dispersion relation it is easy to prove the inverse of the Pomeranchuk theorem: if the difference of the total cross sections $\sigma_{tot} - \tilde{\sigma}_{tot}$ tends to a nonzero limit, that is

$$\lim_{s\to\infty} [\operatorname{Im} F(s) - \operatorname{Im} \tilde{F}(s)] = \delta \neq 0, \tag{2.6}$$

then we have the following relation for the difference of the real parts

$$\lim_{s\to\infty} \frac{\operatorname{Re} F(s) - \operatorname{Re} \tilde{F}(s)}{\ln s/s_0} = -2\frac{\delta}{\pi}. \tag{2.7}$$

A very interesting report of Logunov, Mestvirishvili and Volkov has been presented at the conference [41]. In this work the authors investigated general conditions under which the asymptotic equality of the particle and antiparticle interaction total cross sections holds. Along with the analytical properties of the amplitude in s the analyticity in t and the unitarity conditions are also used. More exactly, when analysing the possible behaviours of the cross sections, inequalities (1.14) and (1.15), which mean the short-range character of the strong interactions, were effectively used.

It follows from the last inequalities that

$$\left.\frac{d\sigma_{el}}{dt}\right|_{t=0} \leqslant \text{const} \cdot \sigma_{tot} \ln^2 \frac{s}{s_0}, \quad \left.\frac{d\tilde{\sigma}_{el}}{dt}\right|_{t=0} \leqslant \text{const} \cdot \tilde{\sigma}_{tot} \ln^2 \frac{s}{s_0}. \tag{2.8}$$

On the basis of this relation between the differential cross sections of the forward elastic scattering and the total cross sections, we show now that if one of the total cross section, σ_{tot} for example, does not decrease, then the other one $\tilde{\sigma}_{tot}$ cannot also decrease [41]. Let us assume that $\tilde{\sigma}_{tot}$ decreases. From the dispersion relations it follows that the real parts of both amplitudes $F(s)$ and $\tilde{F}(s)$ increase faster than σ_{tot} by a logarithmic factor,

$$\overline{\lim_{s\to\infty}} \left|\frac{F(s)}{\sigma_{tot} \ln s/s_0}\right| > 0, \quad \overline{\lim_{s\to\infty}} \left|\frac{\tilde{F}(s)}{\tilde{\sigma}_{tot} \ln s/s_0}\right| > 0. \tag{2.9}$$

The last inequality can be rewritten in the following manner

$$\overline{\lim_{s\to\infty}}\, \frac{1}{\sigma_{tot}^2 \ln^2 s/s_0} \, \frac{d\tilde{\sigma}_{el}}{dt}\bigg|_{t=0} > 0. \tag{2.10}$$

In accordance with the assumption, σ_{tot} does not decrease and $\tilde{\sigma}_{tot}$ decreases, i. e.

$$\lim_{s\to\infty} \frac{\sigma_{tot}^2}{\tilde{\sigma}_{tot}} = \infty. \tag{2.11}$$

From (2.10) and (2.11) one obtains

$$\lim \frac{1}{\tilde{\sigma}_{tot} \ln^2 s/s_0} \cdot \frac{d\tilde{\sigma}_{el}}{dt}\bigg|_{t=0} = \infty \tag{2.12}$$

in contradiction with the condition (2.8). So, if σ_{tot} increases or tends to a nonzero limit, $\tilde{\sigma}_{tot}$ cannot decrease.

Let us prove another interesting statement: if both total cross sections σ_{tot} and $\tilde{\sigma}_{tot}$ increase and their ratio tends to a definite limit at $s \to \infty$, then this limit must be equal to unity,

$$\frac{\sigma_{tot}}{\tilde{\sigma}_{tot}} \to 1. \tag{2.13}$$

Let us assume that this limit is different from unity. Then from the dispersion relations it follows that

$$\overline{\lim_{s\to\infty}} \left| \frac{\operatorname{Re} F(s)}{\sigma_{tot} \ln \frac{s}{s_0}} \right| > 0, \quad \overline{\lim_{s\to\infty}} \left| \frac{\operatorname{Re} \tilde{F}(s)}{\tilde{\sigma}_{tot} \ln \frac{s}{s_0}} \right| > 0 \tag{2.14}$$

i. e.

$$\overline{\lim_{s\to\infty}}\, \frac{1}{\sigma_{tot}^2 \ln^2 \frac{s}{s_0}} \cdot \frac{d\sigma_{el}}{dt}\bigg|_{t=0} > 0, \quad \overline{\lim_{s\to\infty}}\, \frac{1}{\tilde{\sigma}_{tot}^2 \ln^2 \frac{s}{s_0}} \cdot \frac{d\tilde{\sigma}_{el}}{dt}\bigg|_{t=0} > 0. \tag{2.15}$$

As σ_{tot} and $\tilde{\sigma}_{tot}$ increase according to the assumption, then from here we obtain

$$\overline{\lim_{s\to\infty}}\, \frac{1}{\sigma_{tot} \ln^2 \frac{s}{s_0}} \cdot \frac{d\sigma_{el}}{dt}\bigg|_{t=0} = \infty, \quad \overline{\lim_{s\to\infty}}\, \frac{1}{\tilde{\sigma}_{tot} \ln^2 \frac{s}{s_0}} \cdot \frac{d\tilde{\sigma}_{el}}{dt}\bigg|_{t=0} = \infty \tag{2.16}$$

that contradicts the condition (2.8). This proves the statement. The last result has been obtained by Eden [42] and independently by Logunov, Mestvirishvili and Volkov [41].

Thus, although it is impossible to prove the asymptotic equality of the finite limits of the total cross sections (if they exist) only on the basis of the analytical properties of the scattering amplitudes without additional assumptions on the behaviour of the real parts, the following statements are valid:

I. If one of the total cross sections increases, then the other one also increases and the equality

$$\frac{\tilde{\sigma}_{tot}}{\sigma_{tot}} \to 1$$

must hold.

2. If one of the total cross sections decreases, then the other one also decreases.

From here it follows that if one of the total cross sections tends to a definite nonzero limit at $s \to \infty$, then the other one does not increase or decrease. In the last case there exists an upper bound for the difference of the total cross sections. In the refs. [41, 43, 44] it was shown that if the total cross sections tend to non-zero limits

$$\sigma_{tot} \to \sigma_{tot}(\infty), \quad \tilde{\sigma}_{tot} \to \tilde{\sigma}_{tot}(\infty) \tag{2.17}$$

then for the difference of these limits we have the inequality

$$|\sigma_{tot}(\infty) - \tilde{\sigma}_{tot}(\infty)| \leqslant \frac{\pi^{3/2}}{m_\pi} \min\left\{\sqrt{\sigma_{tot}(\infty)}, \quad \sqrt{\tilde{\sigma}_{tot}(\infty)}\right\}. \tag{2.18}$$

This means, roughly speaking, that if one of the total cross sections tends to a sufficiently small limit, the limit of the other can not be very big.

Expressing the difference of the $\pi^{\pm}p$-interaction total cross sections through the imaginary part of the charge exchange amplitude

$$\pi^- + p \to \pi^0 + n$$

by means of the isotopic relations and then using the above arguments, Singh and Roy [43] got the following interesting inequality:

$$\lim_{s\to\infty} |\sigma_{tot}^{\pi^+} - \sigma_{tot}^{\pi^-}| \leqslant \frac{\pi^{3/2}}{\sqrt{2} m_\pi} \lim \sqrt{\sigma_{ex}^{\pi^- p \to \pi^0 n}}, \tag{2.19}$$

where $\sigma_{ex}^{\pi^- p \to \pi^0 n}$ denotes the total cross section of the $\pi^- p$ charge exchange process.

In the above theorem the real parts are supposed to be bounded. Other conditions under which the asymptotic equality of the particle and antiparticle total cross sections holds were discussed also in the contribution of Logunov et al. [41].

Let us suppose that at $s \to \infty$ the total cross sections σ_{tot} and $\tilde{\sigma}_{tot}$ tend to different limits. Then the real parts of the amplitude increase logarithmically, and we have

$$\frac{1}{\sigma_{el}} \cdot \frac{d\sigma_{el}}{dt}\bigg|_{t=0} \geqslant \frac{1}{\sigma_{tot}} \cdot \frac{d\sigma_{el}}{dt}\bigg|_{t=0} \geqslant \text{const} \ln^2 \frac{s}{s_0},$$

$$\frac{1}{\tilde{\sigma}_{el}} \cdot \frac{d\tilde{\sigma}_{el}}{dt}\bigg|_{t=0} \geqslant \frac{1}{\tilde{\sigma}_{tot}} \cdot \frac{d\tilde{\sigma}_{el}}{dt}\bigg|_{t=0} \geqslant \text{const} \ln^2 \frac{s}{s_0}. \tag{2.20}$$

From (1.14) and the last inequalities it follows that

$$R \geqslant \text{const} \cdot \ln s/s_0. \tag{2.21}$$

This result together with the upper bound obtained as a consequence of the analyticity in momentum transfer shows that the radius R increases logarithmically,

$$R \approx \text{const} \cdot \ln s/s_0. \tag{2.22}$$

So, if σ_{tot} and $\tilde{\sigma}_{tot}$ tend to different constants, the effective radius of the interaction must increase as $\ln s$ as $s \to \infty$.

On the other hand according to inequalities (1.14) and (1.15)

$$\frac{1}{\sigma_{el}} \cdot \frac{d\sigma_{el}}{dt}\bigg|_{t=0} \leqslant \text{const} \cdot \ln^2 \frac{s}{s_0},$$

$$\frac{1}{\tilde{\sigma}_{el}} \cdot \frac{d\tilde{\sigma}_{el}}{dt}\bigg|_{t=0} \leqslant \text{const} \cdot \ln^2 \frac{s}{s_0}. \tag{2.23}$$

Since the real parts of the amplitudes increase as ln s if σ_{tot} and $\tilde{\sigma}_{\text{tot}}$ tend to different constants as $s \rightarrow \infty$, then in this case

$$\left.\frac{d\sigma_{\text{el}}}{dt}\right|_{t=0} \approx \left.\frac{d\tilde{\sigma}_{\text{el}}}{dt}\right|_{t=0} \approx \text{const} \cdot \ln^2 s/s_0 \tag{2.24}$$

and condition (2.23) gives

$$\sigma_{\text{el}} \geqslant \text{const}, \quad \tilde{\sigma}_{\text{el}} \geqslant \text{const} \tag{2.25}$$

i. e. if the limits of σ_{tot} and $\tilde{\sigma}_{\text{tot}}$ are different then the total cross sections of the elastic scattering cannot decrease.

Thus, in order to get the equality of the nonzero limits of the total cross sections

$$\sigma_{\text{tot}}(\infty) = \tilde{\sigma}_{\text{tot}}(\infty)$$

it is enough to have one of the following conditions: either the effective radius of the interaction does not increase or increases slower than ln s,

$$\lim_{s\to\infty} \frac{R}{\ln s} = 0, \tag{2.26}$$

or the total cross sections of the elastic scattering decrease as $s \rightarrow \infty$,

$$\lim_{s\to\infty} \sigma_{\text{el}} = 0, \quad \lim_{s\to\infty} \tilde{\sigma}_{\text{el}} = 0. \tag{2.27}$$

Along with the asymptotic equalities of the total cross sections, i. e. of the imaginary parts of the forward elastic scattering amplitudes of the particle and the antiparticle, the analytical properties and the crossing symmetry also lead to some relations for the real parts. For example, if the amplitudes $F(s)$ and $\tilde{F}(s)$ increase or decrease not faster than some power of ln s and if the ratio of the real parts to the imaginary ones does not tend to zero at $s \rightarrow \infty$, then the following asymptotic relation between the real parts must hold:

$$\operatorname{Re} F(s) \approx -\operatorname{Re} \tilde{F}(s) \tag{2.28}$$

as it was shown by Van Hove [45] and Logunov, Todorov, Khrustalev et al. [46]. From this follows the asymptotic equality of the differential crosssections at $t = 0$,

$$\left.\frac{d\sigma_{\text{el}}}{dt}\right|_{t=0} \approx \left.\frac{d\tilde{\sigma}_{\text{el}}}{dt}\right|_{t=0} \tag{2.29}$$

Now let us discuss in detail the general physical conditions under which the equality of the differential cross sections is valid. Logunov, Todorov et al. [47, 48] have proved that by using the unitarity conditions in the weak form

$$\operatorname{Im} F(s) \geqslant 0, \quad \operatorname{Im} \tilde{F}(s) \geqslant 0 \tag{2.30}$$

and applying the Phragmen — Lindelöff theorem in its most general formulation, one could prove equality (2.29) only on the basis of the assumption that the cross sections themselves in this formula have a smooth behaviour but do not oscillate as $s \rightarrow \infty$.

In particular, if the differential cross sections at $t = 0$ tend to some limits as $s \rightarrow \infty$, then these limits must be equal one another independently of the behaviours of the real and imaginary parts considered separately.

This result can be generalized to the case of the elastic scattering at nonzero t. Indeed, let us assume that at some values t_0 of the momentum transfer (probably depending on s) the imaginary parts of amplitude $F(s, t)$ and $\tilde{F}(s, t)$,

$$F(s, t) = \frac{T(s, t)}{s - u}, \quad \tilde{F}(s, t) = \frac{\tilde{T}(s, t)}{s - u} \tag{2.31}$$

are non-negative:

$$\operatorname{Im} F(s, t_0) \geqslant 0, \quad \operatorname{Im} \tilde{F}(s, t_0) \geqslant 0. \tag{2.32}$$

Then, according to the results of refs. [47, 48], we have

$$\left.\frac{d\sigma_{\text{el}}}{dt}\right|_{t=t_0} \approx \left.\frac{d\tilde{\sigma}_{\text{el}}}{dt}\right|_{t=t_0} \tag{2.33}$$

A question appears naturally: at what values t_0 is condition (2.32) satisfied? The study of the short-range character of the strong interactions will give the answer. Actually owing to this property of the strong interactions only the partial waves with $l \leqslant J_0$ contribute to the amplitude at high energies:

$$\operatorname{Im} F(s, t) \approx 8\pi \frac{\sqrt{s}}{k(s-u)} \sum_{l=0}^{J_0} (2l+1) \operatorname{Im} f_l(s) P_l\left(1 + \frac{t}{2k^2}\right),$$

$$\operatorname{Im} \tilde{F}(s, t) \approx 8\pi \frac{\sqrt{s}}{k(s-u)} \sum_{l=0}^{J_0} (2l+1) \operatorname{Im} \tilde{f}_l(s) P_l\left(1 + \frac{t}{2k^2}\right) \tag{2.34}$$

(see formula (1.10)). At $t = 0$ all the Legendre polynomials in (2.34) are equal to unity,

$$P_l(1) = 1. \tag{2.35}$$

Due to the unitarity conditions

$$\operatorname{Im} f_l(s) \geqslant 0, \quad \operatorname{Im} \tilde{f}_l(s) \geqslant 0 \tag{2.36}$$

we have the positivity conditions (2.30). Now we shall show that if the number of the partial waves in (2.3) is finite, then we have the similar result for $F(s, t)$, $\tilde{F}(s, t)$ at enough small t. Let us note firstly that from the property (2.35) and the continuity of $P_l\left(1 + \frac{t}{2k^2}\right)$ the positivity of these polynomials at small t follows. More exactly, if we denote the first zero of the Bessel function $J_0(x)$ by x_0,

$$x_0 = 2{,}4048, \tag{2.37}$$

then at all negative t, satisfying the condition,

$$|t| \leqslant \frac{k^2}{J_0^2} x_0^2 \approx \frac{x_0^2}{R^2} \tag{2.38}$$

we have

$$P_l\left(1 + \frac{t}{2k^2}\right) \geqslant 0, \ l \leqslant J_0. \tag{2.39}$$

From (2.36) and (2.39) we get immediately the positivity property (2.32) for all $t_0 < 0$ satisfying (2.38) and surely for all t_0 in the interval

$$-x_0^2 m_\pi^2 \ln^{-2} \frac{s}{s_0} \leqslant t_0 \leqslant 0. \tag{2.40}$$

Thus, for all t_0 in the interval (2.40) the asymptotic equality (2.33) is valid.

If we assume that the interaction radius does not increase with the increase of the energy, then it is seen from (2.38) that asymptotic equality (2.33) is valid at some fixed values of the momentum transfer. This result has been obtained by Logunov et al. [47, 48]. Slightly modifying their arguments we proved the equality of the differential cross sections in the interval (2.40). This equality has been proved in the report of Kinoshita submitted to the conference [36] in the case when the total cross-sections σ_{tot} and $\tilde{\sigma}_{tot}$ tend to different limits. For this aim he used the following property of the real part of the amplitude which has been reported earlier in sec. 11: If Re F $(s, 0)$ has a lower bound in the form

$$|\operatorname{Re} F(s, 0)| \geqslant \frac{\text{const}}{s^n}$$

then at the values t_0 in the interval

$$-\frac{c^2}{\ln^2 \frac{s}{s_0}} \leqslant t_0 \leqslant 0 \tag{2.41}$$

where c is some rather small constant, ReF (s, t_0) has the same behaviour as that of ReF $(s, 0)$. In ref. [36] the relation (2.33) has been proved for all these values of the constant c. By the method of the refs. [47, 48] we proved the equality (2.33), and at the same time we found some upper bound for the possible values of c in (2.41).

In sec. 11 we said that in any interval

$$t \leqslant -\frac{\alpha}{\ln^2 s/s_0} \tag{2.42}$$

with an arbitrary positive constant α, there must exist at least one point t_0 at which

$$\left|\frac{F(s, t_0)}{F(s, 0)}\right| \geqslant \text{const.} \tag{2.43}$$

At this t_0 the equality of the differential cross sections must hold if the nonzero limits of the total cross sections are different. Thus, we can conclude that if

$$\sigma_{tot}(\infty) \neq \tilde{\sigma}_{tot}(\infty)$$

then in any interval of the type (2.42) there must exist at least one point t_0 such that at $t = t_0$ we have (2.33).

The obtained results show that one can speak about the violation of the asymptotic theorems dealing only with the total interaction cross section in three following cases:

1) one of the total cross sections increases,

2) one of the total cross sections decreases, but the other one does not decrease,

3) one of the total cross sections tend to a rather small limit but the limit of the other one is extremely large. In other cases for experimentaly checking the asymptotic theorems it is necessary to measure both the total cross sections and the differential elastic scattering cross sections.

We note that in testing the asymptotic theorems for the elastic scattering we should compare the data obtained from different experiments. However there exists one process for which the experimental test of the asymptotic theorems needs to carry out only one experiment. This is the K_L^0-regeneration. As we know, the amplitude F_R (s) of the process

$$K_L^0 + p \rightarrow K_S^0 + p$$

is crossing-antisymmetrical. Logunov, Todorov, Khrustalev et al. [49] have shown that if $|F_R(s)|$ tends to a finite nonzero limit at $s \rightarrow \infty$, then $F_R(s)$ is purely real. More precisely

$$\overline{\lim_{s\rightarrow\infty}} \left|\frac{\operatorname{Re} F_R(s)}{\ln \frac{s}{s_0} \operatorname{Im} F_R(s)}\right| > 0. \tag{2.44}$$

The similar result is valid also when $|F_R(s)|$ increases or decreases as $(\ln s)^\beta$, for example. If the modulus $|F_R(s)|$ decreases as $s^{-\gamma}$, $\gamma > 0$, at large s, then we have [50]

$$\left|\frac{\operatorname{Im} F_R(s)}{\operatorname{Re} F_R(s)}\right| \approx \operatorname{tg}\frac{\gamma\pi}{2}. \qquad (2.45)$$

Remember that in the interval (2.40) of the momentum transfer we have also the asymptotic relation between the polarizations in the elastic scattering of the particle and the antiparticle. For example, the recoil baryon polarizations in these crossing processes at high energies should have equal magnitudes and different signs [51, 52]. This conclusion should also be checked experimentally.

In concluding this section I should like to underline once more that the violation of the equality of the total cross sections of particle and antiparticle interactions and the violation of the theorem on the equality of the total cross sections are different things, since when formulating the theorem some assumptions concerning other physical quantities have been made. The experiments in which we check both the conditions of the theorem and its conclusions would be of great importance for the theory. They provide an experimental test of the general principles of the local field theory. For example, in the future we shall be able to conclude surely that the experiment disproves the local field theory, if one of the following facts will be established experimentally:

1) the differential cross sections of the forward elastic scattering

$$\left.\frac{d\sigma_{el}}{dt}\right|_{t=0}, \quad \left.\frac{d\tilde{\sigma}_{el}}{dt}\right|_{t=0}$$

are bounded, and the total cross sections tend to different limits

$$\sigma_{tot}(\infty) \neq \tilde{\sigma}_{tot}(\infty)$$

2) one of the total cross sections increases, another does not increase,

3) one of the total cross sections decreases, another does not decrease,

4) both total cross sections increase and their ratio tends to a limit different from 1,

5) one of the total cross sections tends to a small limit and the limit of another is extremely high,

6) the total cross sections of the elastic scattering of the particle and the antiparticles, σ_{el} and $\tilde{\sigma}_{el}$, decrease and the total cross sections tend to different limits

$$\sigma_{el} \to 0, \quad \tilde{\sigma}_{el} \to 0,$$

$$\sigma_{tot}(\infty) \neq \tilde{\sigma}_{tot}(\infty),$$

7) the differential cross sections of the particle and antiparticle elastic scattering at $0 \gg t \gg -x_0^2 m_\pi^2 \ln^2 \frac{s}{s_0}$ have a smooth behaviour (in particular, they tend to finite nonzero limits or have a logarithmic behaviour) but their ratio does not tend to unity.

Different constraints on the behaviour of the elastic scattering amplitude following from the analyticity, the unitarity and the crossing symmetry were given also in a paper by Cornille [53]. He studied in detail all the possible simple behaviours. His main results are similar to some of those which I have reported above.

Finkelstein [54], Anselm et al. [55] studied some models of the amplitudes for which the total cross sections tend to different limits at high energies. Gribov et al. [44] have shown that for some rather general class of behaviours we have the following asymptotic relations between the real parts $D^{\pm}(s, t)$ and the imaginary parts $A^{\pm}(s, t)$ of the symmetrical and antisymmetrical amplitudes:

$$A^{-}(s, t) = -\frac{2}{\pi} \cdot \frac{\partial}{\partial \ln s} D^{-}(s, t), \tag{2.46}$$

$$D^{+}(s, t) = \frac{2}{\pi} \cdot \frac{\partial}{\partial \ln s} A^{+}(s, t). \tag{2.47}$$

This class includes the models in refs. [55, 56]. From the last relations it is easy to prove that if the nonzero limits of the total cross sections are not equal then at each rather small but fixed $t < 0$ the real part $D^{-}(s, t)$ grows with the increase of s in some interval, reaches it maximum at some value s such that

$$\ln s \approx |t|^{-1/2}$$

and then decreases. The imaginary part $A^{-}(s, t)$ should be equal to zero at this point. In considering the imaginary parts of the physical amplitudes of the particle and antiparticle elastic scattering we see that their angular dependence is quite different.

After this report was delivered I received a paper by Eden and Kaiser [56]. They showed that if the particle and antiparticle interaction total cross sections tend to different nonzero limits as $s \to \infty$ then the amplitude should have at least one zero at some point t with

$$|t| \approx \frac{\text{const}}{\ln^2 \frac{s}{s_0}}. \tag{2.48}$$

3. Dispersion Relations and the Fundamental Length

Let us consider some points associated with the experimental check of the dispersion relations for the elastic scattering of π-mesons on nucleon, which were proved rigorously by N. N. Bogolubov on the basis of the general principles of field theory. Note that all the quantities included in the dispersion relation at $t = 0$ are observable: the imaginary parts are proportional to the total cross sections, and from data on the differential cross sections the real parts can be determined.

Thus, dispersion relations can be used for the experimental check of the general principles of the field theory, and firstly the microcausality principle.

The comparison of the dispersion relations with experimental data plays an important role in studing the elementary particles. In particular, if experimental data are not in the agreement with the dispersion relations then we can conclude that the future theory of elementary particles should be a nonlocal one, in which, together with the known constants $\hbar$, c, there exists one more universal constant — the measure of the nonlocality of the fields, the fundamental length l. In the near future there will be good data up to the energy 70 *GeV* for testing the dispersion relations. Now is the time to discuss the possibility of getting the information about the fundamental length from the experimental data in the case when there is some discrepancy between these data and the dispersion relations.

Recall that in the local field theory the forward πN-scattering amplitude $T(\omega)$ averaged over spin states of nucleons, where ω is the energy of π-mesons in the lab. system of reference, is analytic in the complex ω plane with poles and cuts on the real axis, and as $\omega \to \infty$ it can increase only slower than any linear exponent of $|\omega|$ [57, 58]:

$$|T(\omega)| \leqslant \text{const}\, e^{\varepsilon |\omega|}, \quad \omega \to \infty \tag{3.1}$$

for any $\varepsilon > 0$. In the nonlocal field theory $T(\omega)$ can have complex singularities and at the same time can increase exponentially at $\omega \to \infty$. For example, in the nonlocal models of Blokhintsev and Kolerov [59], along with the real poles and cuts, there exist complex cuts going to the infinity in the upper and lower half-planes. However it is possible that in the nonlocal theory the amplitude $T(\omega)$ is analytic in the ω-plane with poles and cuts on the real axis, but increases exponentially. The experimental data on the imaginary and real parts of the amplitude surely provide some useful information on the singularities and on the growth of the amplitude that is necessary for estimating the fundamental length.

As a simple example of obtaining the information on the nonlocality from experimental data, we consider the case in which the amplitude in the nonlocal theory increases exponentially, but at finite points of the energy plane it has the same analytical properties as in the local theory. Besides, we make some assumptions on the growth of the amplitude on the real axis. These assumptions can be checked experimentally. More exactly, let us consider the elastic scattering of $\pi^{\pm}$-mesons on a proton and denote by $T^{\pm}(\omega)$ the amplitudes averaged over the spin states. We normalize these amplitudes such that the total cross sections are equal to

$$\sigma^{\pm}_{\text{tot}}(\omega) = \frac{4\pi}{\sqrt{\omega^2 - \mu^2}} \operatorname{Im} T^{\pm}(\omega). \tag{3.2}$$

Assume that:

1) $T^{\pm}(\omega)$ are analytic in the complex z-plane with the poles at $z = \pm\, \omega_0$,

$$\omega_0 = \frac{m_\pi^2}{2m_N}$$

and with the cuts along the real axis from $-\infty$ to $-m_\pi$ and from m_π to ∞;

2) $T^{\pm}(z)$ are the real functions in the sense that

$$T^{\pm}(z)^* = T^{\pm}(z^*) \tag{3.3}$$

and satisfy the crossing symmetry conditions

$$T^{+}(-z) = T^{-}(z) \tag{3.4}$$

for complex z, or

$$T^{+}(-\omega) = T^{-}(\omega)^* \tag{3.5}$$

for real ω;

3) $T^{\pm}(\omega)$ increase as $\omega \to \pm\infty$ not faster than $\omega^{2-\varepsilon}$:

$$|T^{\pm}(\omega)| \leqslant \text{const}\, \omega^{2-\varepsilon}, \quad \varepsilon > 0, \quad \omega \to \pm\infty; \tag{3.6}$$

4) $T^{\pm}(z)$ increase at $z \to \infty$ in the complex plane not faster than some linear exponential

$$|T^{\pm}(z)| \leqslant \text{const}\, e^{a|z|}, \quad a > 0, \quad |z| \to \infty. \tag{3.7}$$

Logunov et al. [60] have shown that in this case from the values of $T^{\pm}(\omega)$ on the real axis we can find some lower bound for the constant a in the condition (3.7). This constant can be considered as a measure of the nonlocality of the fields — the fundamental length.

Really, the amplitudes $T^{\pm}(z)$ possessing the properties 1) — 4) be represented in the form

$$T^{+}(z) = \Phi^{+}(z) + \frac{2f^2}{m_\pi^2} \cdot \frac{z^2 - m_\pi^2}{z - \omega_0} +$$

$$+\frac{z^2-m_\pi^2}{4\pi^2}\int\limits_{m_\pi}^{\infty}\frac{d\omega'}{\sqrt{\omega'^2-m_\pi^2}}\left[\frac{\sigma_{tot}^{+}(\omega')}{\omega'-z}+\frac{\sigma_{tot}^{-}(\omega')}{\omega'+z}\right],$$

$$T^-(z)=\Phi^-(z)-\frac{2f^2}{m_\pi^2}\cdot\frac{z^2-m_\pi^2}{z+\omega_0}+$$

$$+\frac{z^2-m_\pi^2}{4\pi^2}\int\limits_{m_\pi}^{\infty}\frac{d\omega'}{\sqrt{\omega'^2-m_\pi^2}}\left[\frac{\sigma_{tot}^{-}(\omega')}{\omega'-z}+\frac{\sigma_{tot}^{+}(\omega')}{\omega'+z}\right],\qquad(3.8)$$

where f is the πN coupling constant and $\Phi^\pm(z)$ are some entire functions of the exponential type satisfying the conditions

$$\Phi^\pm(z^*)=[\Phi^\pm(z)]^*,\qquad(3.9)$$

$$\Phi^+(-z)=\Phi^-(z).\qquad(3.10)$$

The dispersion integrals in (3.7) and (3.8) increase polynomially. Therefore, to study the growth of $T^\pm(z)$ we can consider only the entire functions $\Phi^\pm(z)$. From experimental data for $T^\pm(\omega)$ one can determine the values of $\Phi^\pm(\omega)$ for real ω, and this will give some information on the growth of $\Phi^\pm(z)$, i. e. on the growth of $T^\pm(z)$.

Assume that the functions $\Phi^\pm(\omega)$ are bounded at real ω. Then a lower bound for the constant a in the condition (3.7), i. e. for the fundamental length l, can be determined by using the formula

$$l^n\geqslant\frac{\max\limits_{-\infty<\omega<\infty}\left|\dfrac{d^n\Phi^\pm(\omega)}{d\omega^n}\right|}{\max\limits_{-\infty<\omega<\infty}|\Phi^\pm(\omega)|}.\qquad(3.11)$$

If $\Phi^\pm(\omega)$ are not bounded but increase slower than $\omega^{1-\varepsilon}$ or $\omega^{2-\varepsilon}$ for $\varepsilon>0$, then in formula (3.11) one can use the new functions

$$\frac{\Phi^\pm(\omega)-\Phi^\pm(0)}{\omega}\qquad(3.12)$$

in place of them, or

$$\frac{\Phi^\pm(\omega)-\Phi^\pm(0)-\omega\Phi^{\pm\prime}(0)}{\omega^2}.\qquad(3.13)$$

Let us assume that $|\Phi^\pm(\omega)|^2$ and $\left|\dfrac{d^n\Phi^\pm(\omega)}{d\omega^n}\right|^2$, $n\geqslant 1$ are integrable over the real axis. Then we have

$$l^n\geqslant\frac{\left\|\dfrac{d^n\Phi^\pm}{d\omega^n}\right\|}{\|\Phi^\pm\|},\qquad(3.14)$$

where

$$\|\Phi^\pm\|^2=\int\limits_{-\infty}^{\infty}|\Phi^\pm(\omega)|^2\,d\omega,\qquad(3.15)$$

$$\left\|\frac{d^n\Phi^\pm}{d\omega^n}\right\|^2=\int\limits_{-\infty}^{\infty}\left|\frac{d^n\Phi^\pm(\omega)}{d\omega^n}\right|^2d\omega.\qquad(3.16)$$

Thus, the presence of an exponential growth of the scattering amplitudes in the complex plane can be observed in studying their behaviour on the real axis.

From the values of the amplitudes and their derivatives we can in principle determine the lower bound of the constant a in formula (3.7). This constant a characterizes the growth of the amplitudes and is connected with the fundamental length. This, however, requires considerable experimental information.

In concluding this section we make one remark. The quantities included in the dispersion relations for πN scattering (the real and imaginary parts of the amplitudes) are measured in different experiments. However, for the check of the dispersion relation for the process $K_L^0 + p \to K_S^0 + p$ it is sufficient to carry out only one experiment: in one experiment we determine simultaneously both the real and imaginary parts of the amplitude. At high energies the contribution of the unphysical region can be expressed approximately by some polynomial of $\frac{1}{\omega}$ with any accuracy. Let us note that in the K_L^0-regeneration experiment we measure not the phase of the amplitude $T_R(\omega)$ but only the sum of the phase of this amplitude and some ω-independent phase η of the CP-violating K_L^0-decay amplitude, the latter being known with some error. Fortunately, for $e^{i\eta}T_R(\omega)$ the dispersion relation is also valid, if it is valid for $T_R(\omega)$, so that there is no question with the errors in the determination of η.

The K_L^0-regeneration experiments are being carried out now. They are surely of great interest and we hope that some work concerning the check of the dispersion relation for this very attractive process will be submitted to the next conference.

4. Exact Sum Rules

In the current algebra based on the equal-time commutation relations between vector and axial currents and the PCAC hypothesis, some sum rules for the imaginary parts of the amplitudes of different processes have been deduced. Bogolubov and Soloviev noted that most of these sum rules may be obtained on the basis of the analytical properties of the amplitudes if one assumes that at high energies they decrease not slower than $1/s^{1+\varepsilon}$ for some $\varepsilon > 0$ (see the review article [61]). Later, Logunov, Soloviev and Tavkhelidze [62] generalized this result for non-decreasing amplitudes. Assuming that at high energies the amplitudes have a definite behaviour, for example the Regge one, they have obtained the finite energy sum rules which allow us to relate the amplitudes in the low energy region to their asymptotics at high energies. Among the dispersion sum rules there exist those which can be proved without additional assumptions on the behaviour of the amplitude. They are obtained as the consequences of the analyticity, unitarity and crossing symmetry and are the subject for discussion in the present review.

Let us consider dispersion sum rules containing the differential cross sections. For simplicity let us consider the charge exchange process

$$\pi^- + p \to \pi^0 + n$$

at zero angle. Instead of s, introduce the crossing symmetrical variable

$$\omega = \frac{s-u}{4m_N}.$$

It is the pion energy in the lab. system. The differential cross section at zero angle includes only the following combination of the invariant amplitudes $A(s, t)$ and $B(s, t)$

$$T(\omega) = 2m_N [A(s, 0) - \omega B(s, 0)]. \qquad (4.1)$$

Denote by g the πN — coupling constant containing in the pole terms of $B(s, 0)$ in the following manner

$$B(s, 0) = \sqrt{2}\, g^2 \left[\frac{1}{s - m_N^2} + \frac{1}{u - m_N^2} \right] + \cdots \tag{4.2}$$

and define

$$f^2 = \frac{m_\pi}{2m_N} g^2. \tag{4.3}$$

Dao vong Duc et al. [63] showed that due to the analyticity and the crossing symmetry of the amplitude $T(\omega)$ the following dispersion sum rule holds:

$$\ln \left| \frac{f^2}{\sqrt{2}} \cdot \frac{m_\pi^2}{m_\pi^2 - \omega_0^2} \right| \leqslant \int_{m_\pi^2}^{\infty} \frac{1}{\omega^2 - \omega_0^2} \sqrt{\frac{m_\pi^2 - \omega_0^2}{\omega^2 - m_\pi^2}} \ln \left| T(\omega) \frac{m_\pi}{\omega} \right| d\omega^2. \tag{4.4}$$

Due to the unitarity condition the amplitude $T(\omega)$ cannot increase arbitrarily at $\omega \to \infty$ and the integral in the right hand side of (4.4) converges.

In Khalfin's [64] and Jenkovsky's [65] works another type of dispersion sum rules containing both the imaginary and real parts of amplitudes has been considered. For definiteness let us consider the elastic scattering of π^+ and π^--mesons on a proton at zero angle. Denote the sum and the difference of the corresponding amplitudes by $T^+(\omega)$ and $T^-(\omega)$. Let m_1 and m_2 be some constants satisfying the conditions $m_2 > m_1 > m_\pi$. Then we have

$$\int_0^{m_\pi} \frac{\operatorname{Im} T^+(\omega)\, d\omega}{\sqrt{(m_\pi^2 - \omega^2)(m_1^2 - \omega^2)(m_2^2 - \omega^2)}} + \int_{m_\pi}^{m_1} \frac{\operatorname{Re} T^-(\omega)\, d\omega}{\sqrt{(\omega^2 - m_\pi^2)(m_1^2 - \omega^2)(m_2^2 - \omega^2)}} -$$

$$- \int_{m_1}^{m_2} \frac{\operatorname{Im} T^+(\omega)\, d\omega}{\sqrt{(\omega^2 - m_\pi^2)(\omega^2 - m_1^2)(m_2^2 - \omega^2)}} = \int_{m_2}^{\infty} \frac{\operatorname{Re} T^-(\omega)\, d\omega}{\sqrt{(\omega^2 - m_\pi^2)(\omega^2 - m_1^2)(\omega^2 - m_2^2)}}. \tag{4.5}$$

The report of Vernov at the conference [66] includes an interesting result. If we denote the forward elastic scattering amplitudes for a particle and its antiparticle without pole terms by $T_0(\omega)$ and $\tilde{T}_0(\omega)$, then for the ratio of these amplitudes the inequality

$$-\pi^2 \leqslant \int_{\omega_1}^{\omega_2} \ln \left| \frac{\tilde{T}_0(\omega)}{T_0(\omega)} \right| \frac{d\omega}{\omega} \leqslant \pi^2 \tag{4.6}$$

holds at any ω_1 and ω_2. This result was generalized in the report of Baluni and Vernov [67] submitted to the conference. For the form factor $F(t)$ we have also some exact sum rules. One of them has been obtained by Geshkenbein and Ioffe [68]:

$$\int_{4m_\pi^2}^{\infty} \frac{\ln |F(t)|}{t \sqrt{t - 4m_\pi^2}} dt \geqslant 0. \tag{4.7}$$

Another sum rule including the values of the form factor modulus along the cut $t \geqslant 4m_\pi^2$ and in a part of the physical region of the scattering channel $t \leqslant -t_0$, $t_0 > 0$ is given in a paper by Nguyen thi Hong [69],

$$\int_{t_0}^{\infty} \frac{\ln |F(-t)|}{t \sqrt{(t + 4m_\pi^2)(t - t_0)}} dt + \int_{4m_\pi^2}^{\infty} \frac{\ln |F(t)|}{t \sqrt{(t - 4m_\pi^2)(t + t_0)}} dt \geqslant 0. \tag{4.8}$$

Generalizing the inequality (4.7), Truong nguyen Tran and Vinh Mau [70] showed that at any $t_0 < 4m_\pi^2$ we have

$$\ln |F(t_0)| \leqslant \frac{\sqrt{4m_\pi^2 - t_0}}{\pi} \int\limits_{4m_\pi^2}^{\infty} \frac{\ln |F(t)|}{(t_0 - t_0)\sqrt{t - 4m_\pi^2}} dt. \tag{4.9}$$

An interesting sum rule for the amplitude of the Compton scattering on a proton has been obtained in the work of Truong nguyen Tran [71]. Together with the analytical properties of the amplitude he used the known low energy theorem [72, 73], allowing us to get the precise expression for the amplitude of this electromagnetic process in the low energy limit. We write the amplitude in the form

$$f(\nu) = f_1(\nu)\,\vec{\varepsilon}_2 \cdot \vec{\varepsilon}_1 + f_2(\nu)\, i\vec{\sigma}\,[\vec{\varepsilon}_2, \vec{\varepsilon}_1], \tag{4.10}$$

where $\vec{\varepsilon}_1$ and $\vec{\varepsilon}_2$ are the polarization vectors of the photon, ν is the photon energy in the lab. system. The forward scattering differential cross section is equal to:

$$\left.\frac{d\sigma}{dt}\right|_{t=0} = \frac{\pi}{\nu^2}\{|f_1(\nu)|^2 + |f_2(\nu)|^2\}. \tag{4.11}$$

According to the low energy theorem

$$f_1(0) = -\frac{e^2}{m_N}. \tag{4.12}$$

Therefore, the dispersion relation for $f_1(\nu)$ does not include the arbitrary constant

$$\mathrm{Re}\, f_1(\nu) = -\frac{e^2}{m_N} + \frac{\nu^2}{\pi} P \int\limits_{\nu_0}^{\infty} \frac{\mathrm{Im}\, f_1(\nu')}{\nu'^2(\nu'^2 - \nu^2)} d\nu'^2. \tag{4.13}$$

Note that due to the optical theorem the imaginary part of the amplitude is expressed through the total cross section of the photon—proton interaction,

$$\mathrm{Im}\, f_1(\nu) = \frac{\nu}{4\pi}\sigma_{\mathrm{tot}}(\nu). \tag{4.14}$$

It is evident that $f_1(\nu)$ may be considered as a function of the complex variable $z = \nu^2$

$$\mathrm{Re}\, f_1^0(z) = -\frac{e^2}{m_N} + \frac{z}{\pi} P \int\limits_{z_0}^{\infty} \frac{\mathrm{Im}\, f_1^0(z')}{z'(z' - z)} dz'. \tag{4.15}$$

From the positivity of $\mathrm{Im}\, f_1(\nu)$ is follows that $f_1^0(z)$ cannot vanish if $\mathrm{Im}\, z > 0$. If $\mathrm{Re}\, f_1(\nu)$ is negative at the threshold of the pion production, then $f_1^0(z)$ cannot vanish also on the real axis. Then $1/f_1(\nu)$ satisfies the dispersion relation which may be written without subtractions, since it is likely that $f_1(\nu)$ increases with increasing ν:

$$\mathrm{Re}\frac{1}{f_1(\nu)} = \frac{\mathrm{Re}\, f_1(\nu)}{|f_1(\nu)|^2} = -\frac{1}{\pi} P \int\limits_{\nu_0}^{\infty} \frac{\mathrm{Im}\, f_1(\nu')}{|f_1(\nu')|^2(\nu'^2 - \nu^2)} d\nu'^2. \tag{4.16}$$

According to the low energy theorem (4.12)

$$\mathrm{Re}\frac{1}{f_1(0)} = -\frac{m_N}{e^2}.$$

From the dispersion relation (4.16) it follows that

$$\frac{e^2}{\pi m_N} \int\limits_{\nu_0}^{\infty} \frac{\mathrm{Im}\, f_1(\nu')}{|f_1(\nu')|^2} \cdot \frac{d\nu'^2}{\nu'^2} = 1. \tag{4.17}$$

5. The Behaviour of the Form Factors and the Bounds on the Radius of Elementary Particles

Analytical properties of the electromagnetic form factor and the form factors of the semileptonic weak interaction processes also lead to a number of experimental consequences. For simplicity we shall assume that the form factor $F(t)$ is analytic in the complex t-plane with the cut $t \geqslant 4m_\pi^2$ and increases at $t \to \infty$ slower than any linear exponential of $\sqrt{|t|}$

$$|F(t)| \leqslant \mathrm{const}\, e^{\varepsilon |t|^{1/2}}, \quad t \to \infty, \tag{5.1}$$

for any $\varepsilon > 0$. In the local field theory the last condition is always satisfied [57, 58].

The inequality (5.1) is some bound on the growth of the form factor. A question arises: is there some bound on the decrease of $F(t)$? The reply to this question is positive. Due to the condition (5.1) imposed on the growth of the form factor it cannot decrease too fast at $t \to \pm\infty$. Martin [74] and Jaffe [75] showed that at $t \to -\infty$ $F(t)$ cannot decrease faster than some negative linear exponential of $\sqrt{|t|}$. More exactly, at sufficiently large t_0 there exists some $a > 0$ such that

$$\max_{t \leqslant t_0} |F(t)| \geqslant \mathrm{const}\, e^{-a\sqrt{|t|}}, \quad t_0 \to +\infty. \tag{5.2}$$

Another restriction concerning the decrease of $F(t)$ on the positive semi-axis $t \geqslant 0$ has been obtained in ref. [76]. It was shown there that at $t \to +\infty$, $F(t)$ can decrease slower than any negative linear exponential in $\sqrt{t}$ only. In other words, for any $\varepsilon > 0$ there exists such a sequence $t_n \to +\infty$ that

$$|F(t_n)| \geqslant \mathrm{const}\, e^{-\varepsilon\sqrt{t_n}}. \tag{5.3}$$

The inequality (5.2) is a lower bound for $F(t)$ on the negative semi-axis. For establishing this condition we need no information on the behaviour of $F(t)$ on the positive semi-axis. However, the behaviour of $F(t)$ at $t \leqslant 0$ and its behaviour at $t \geqslant 4m_\pi^2$ are closely related to each other. Therefore, if we have some information on $F(t)$ in the region $t \geqslant 4m_\pi^2$ we may obtain stronger bounds on the behaviour of $F(t)$ at $t < 0$.

Assume that along the cut $t \geqslant 4m_\pi^2$ the form factor $F(t)$ is bounded:

$$|F(t)| \leqslant M, \quad t \geqslant 4m_\pi^2. \tag{5.4}$$

Then we have

$$\max_{t \leqslant -t_0} |F(t)| \geqslant \mathrm{const}\, e^{-\alpha\sqrt{\frac{t_0}{4m_\pi^2}}\ln M}, \quad t_0 \to +\infty \tag{5.5}$$

where α is some constant independent of M. In other words, the constant a in (5.2) may be chosen so that it is proportional to $\frac{\ln M}{2m_\pi}$ with some numerical coefficient α. In ref. [76] it was shown that this coefficient is not larger than 2:

$$\alpha \leqslant 2.$$

Later on the authors of refs. [77, 78] have obtained a more precise result:

$$\alpha \leqslant \frac{\pi}{2}$$

Thus, we have the following bound on the decrease of $F(t)$ at $t \to -\infty$;

$$\max_{t \leqslant -t_0} |F(t)| \geqslant \text{const}\, e^{-\frac{\pi}{2}\sqrt{\frac{t_0}{4m_\pi^2}}\ln M}, \quad t_0 \to +\infty. \tag{5.6}$$

If $F(t)$ decreases monotonically as $t \to -\infty$, the inequality (5.6) may be rewritten as

$$|F(t)| \geqslant \text{const}\, e^{-\frac{\pi}{2}\sqrt{\frac{|t|}{4m_\pi^2}}\ln M}, \quad t \to -\infty. \tag{5.7}$$

In order to see so far as this bound is close to the possible decrease, let us consider the following example:

$$F(t) = e^{1-\sqrt{1-t/4m_\pi^2}}.$$

For this form factor $M = e$, $\ln M = 1$, and the inequality (5.7) gives

$$|F(t)| \geqslant \text{const}\, e^{-\frac{\pi}{2}\sqrt{\frac{|t|}{4m_\pi^2}}}.$$

In fact

$$|F(t)| \approx \text{const}\, e^{-\sqrt{\frac{|t|}{4m_\pi^2}}}$$

Thus, it is impossible to improve the obtained result essentially.

The connection between the behaviour of the form factor as $t \to -\infty$ and its behaviour at $t \to +\infty$ has been considered earlier by Logunov, Todorov et al. [47]. Assuming that at $t \to +\infty$ the form factor does not oscillate but has a smooth asymptotic behaviour these authors showed that at large positive values of t the imaginary part of the form factor must tend to zero. Moreover, the asymptotic equality

$$\left|\frac{F(t)}{F(-t)}\right| \to 1, \quad t \to \infty \tag{5.8}$$

holds.

As the form factor is normalized by the condition

$$F(0) = 1,$$

the value of the maximum of $|F(t)|$ on the cut (condition (5.4)) determines the bound on the rate of change of $|F(t)|$. Therefore, if $F(t)$ satisfies the condition (5.4) the derivative $F'(0)$ cannot exceed some value depending on M. On the other hand, the mean-squared radius of the elementary particles is proportional to the derivative of its form factor at zero

$$\langle r^2 \rangle = 6F'(0).$$

Therefore we must have some upper bound for the particle radius.

An upper bound of the elementary radius when the modulus of the form factor on the cut has a given maximum was found in ref. [79]. In this work it was shown that

$$|F'(0)| \leqslant \begin{cases} \dfrac{\sqrt{M}\ln M}{2m_\pi^2} & \text{if } \ln M \leqslant 2 \\[2ex] \dfrac{e(\ln M)^2}{4m_\pi^2} & \text{if } \ln M \geqslant 2 \end{cases} \tag{5.9}$$

if $F(t)$ satisfies condition (5.4) and is smaller than 1 at $t \leqslant 0$.

Note that this result has been obtained without any assumption on the zeros of $F(t)$. Under definite assumptions on these zeros other results [80, 71] may be obtained.

In Geshkenbein's work [81] it was shown that if the number of zeros of $F(t)$ is finite, then the inequality

$$|F'(0)| \leqslant \frac{1}{16m_\pi^2} \cdot \frac{M^2-1}{M} \tag{5.10}$$

is satisfied.

An elegant proof of the last inequality without assumptions on the number of the zeros of $F(t)$ was given in the report by Nguyen thi Hong submitted to the conference [82].

On the right hand sides of (5.9) and (5.10) we have the expressions depending only on the maximum of $|F(t)|$ on the cut. If we change $F(t)$ but fix the maximum of its moduls on the cut these expressions do not change. On the other side, in the experiment we may determine not only the maximum of $|F(t)|$ but also the values of $|F(t)|$ at all $t \geqslant 4m_\pi^2$, if $F(t)$ is the pion form factor. Therefore, the use of formulae (5. 9) and (5.10) for estimating the upper bound on $F'(0)$ requires only a part of the information which we can get from experiment. In order to exploit all the information on the form factor $F(t)$ along the cut which we can get from the annihilation experiments, it is necessary to find the expression for the upper bound on $F'(0)$ which contains explicity the value of $|F(t)|$ along the whole cut. This has been done in the report by Dao vong Duc et al. [83] submitted to the conference. They have obtained the following result.

$$\frac{1}{16m_\pi^2}(J_1 - J_2) \leqslant F'(0) \leqslant \frac{1}{16m_\pi^2}(J_1 + J_2), \tag{5.11}$$

where

$$J_1 = \frac{2}{\pi}\int_1^\infty \frac{2-\nu}{\nu^2\sqrt{\nu-1}} \ln|F(4m_\pi^2\nu)|\,d\nu,$$

$$J_2 = 2\,\mathrm{sh}\left\{\frac{1}{\pi}\int_1^\infty \frac{1}{\nu\sqrt{\nu-1}} \ln|F(4m_\pi^2\nu)|\,d\nu\right\}. \tag{5.12}$$

Thus, together with the upper bound on $F'(0)$, a lower bound has been established. Moreover for the given values of $|F(t)|$ at $t \geqslant 4m_\pi^2$, there exist functions $F(t)$ for which $F'(0)$ reaches these upper and lower bounds. Therefore, the inequality (5.11) may be rewritten more precisely as follows:

$$\sup F'(0) = \frac{1}{16m_\pi^2}(J_1 + J_2),$$

$$\inf F'(0) = \frac{1}{16m_\pi^2}(J_1 - J_2). \tag{5.14}$$

The presented results for the pion concern only experimentally measurable quantities: the form factor $F(t)$ in the physical region of the scattering channel:

$$e^- + \pi \to e^- + \pi$$

and the moduls of $F(t)$ in the physical region $t \geqslant 4m_\pi^2$ of the annihilation channel

$$e^+ + e^- \to \pi^+ + \pi^-.$$

The comparison of these results with experimental data would mean an experimental test of general analytical properties of the form factor $F(t)$.

In a report submitted to the conference, Baluni [84] has generalized the results obtained in the work of Dao vong Duc [83]. He has solved the following more general problem: let the value of the integral

$$I = \int_{4m_\pi^2}^{\infty} \Phi(t) \, | F(t) | \, dt \tag{5.15}$$

be given for some function $\Phi(t)$ which decreases rather quickly at $t \to \infty$. Find the upper bound of possible values of $F'(0)$ for all form factors which are analytic in the cut plane and for which the integral I has the given value. This bound depends on the integral I.

* * *

I have briefly presented the results of a number of theoretical works on studying experimental consequences of the general analytical properties of scattering amplitude and the unitarity condition. Many of interesting works have not been reported because of the lack of time.

Among these works I would like to note, first of all, the work of Ezhela, Logunov and Mestvirishvili [85], in which the authors study the bounds on the differential cross sections of multiple production processes. On the basis of the analytic properties of the multiple production amplitudes in two variables,

$$z = \cos\theta \text{ and } w = e^{i\varphi},$$

where θ is the angle between the momentum of some particle and the relative momentum of colliding particles in the c. m. s., φ is some azimutal angle, these authors have obtained a very strong bound on the differential cross section at fixed θ and φ:

$$\left.\frac{d^2\sigma}{d\cos\theta d\varphi}\right|_{\theta\neq 0} \leqslant \text{const} \frac{(\ln s/s_0)^{1/2}}{s}.$$

I would like to say also some words about an interesting report by Ciulli and Cutkosky [86]. We know that if the dispersion relation is true for some amplitude $T(\omega)$, then it holds also for the functions $\Phi(\omega) T(\omega)$, where $\Phi(\omega)$ is any function analytical in the cut plane. We can choose $\Phi(\omega)$ in such a manner that it is very small in some intervals on the cuts. The contribution of these intervals to the modified dispersion integrals will be negligible.

In this case, however, a small error in the experimental determination of $T(\omega)$ in the intervals which give the main contributions can lead to a large error of the dispersion integrals. Ciulli and Cutkosky have found some optimal choice of $\Phi(\omega)$ such that this difficulty does not occur.

Atkinson's report [87] contains new results in his constructive programm. He found some $\pi\pi$-scattering amplitude satisfying the Mandelstam representation and the unitarity condition, for which the total cross section behaves like

$$\sigma_{\text{tot}} \approx \frac{1}{\left(\ln \frac{s}{s_0}\right)^{1+\varepsilon}}, \quad \varepsilon > 0.$$

* * *

In conclusion, I would like to say that by studying the analytical properties of the amplitudes we have obtained many consequences of the general principles of the field theory. For experimentaly checking the validity of these principles and, first of all, the microcausality principle, the analytic methods are seem to be powerful. We hope that in the near future many precise data will be accumulated for this purpose.

E f r e m o v:

Can somebody, probably Prof. Yndurain, say about the low limit of energy where do these unequalities work?

N g u e n:

Prof. Yndurain is here, I think he will answer this question.

Y n d u r a i n:

The bound derived by Common and myself is valid at all energies. You can compute its asymptotic form, i. e., the way it behaves at infinity, just to compare with the Froissart bound, but the actual bound is valid everywhere.

M a r t i n:

I want to indicate that unequalities of the type $\frac{1}{\sigma} \cdot \frac{d\sigma}{dt} < R^2$ have been obtained independently by Kinoshita and by Eden in 1966.

Concerning the radius of the pion form factor I would like to mention that this depends on assumptions in the behaviour of the form factor for large time like momenta. An above-general assumption $|F| < 5$ has been made by Baluni. May be Dr. Massam could indicate us the results of the Bologna group for $1\ GeV + 1\ GeV\ e^{+} + e^{-} \to \pi^{+} + \pi^{-}$ they get?

M a s s a m:

In the $e^{+}e^{-}$ colliding beam experiments at $s \approx 3\ (GeV)^2$ the pion form factor $|F_{\pi}| < 0{,}5$.

T a v k h e l i d z e:

When investigating deep non-elasic interactions of leptons with hadrons a principle of automodeling was put forward. Recently this principle is tried to be used for strong interactions. Is it possible to make any conclusion about the fulfilment of this principle in given strong interactions from the obtained bounds of Logunov A. A. et al.

N g u e n:

All predictions following from the principle of auto-modeling do not contradict to the results mentioned in my report. However, if, together with the principle of automodeling, we use these arguments to the deep nonelastic processes of the weak and electromagnetic interactions, new predictions are obtained.

J. G. T a y l o r:

I have a comment and then a question. First I would like to comment on non-localisable theories. An axiomatic framework has recently been given for them, and they are also of physical interest for strong interactions, through chiral Lagrangians. It is of interest to ask if the rigorous bounds and other properties discussed by the rapporteur are still valid in these non-localisable theories. This is not known, but obviously should be investigated. At least forward scattering dispersion relations can be proved, and very likely many of the other properties also. So the experimental testing of the various relations given by the rapporteur need not test nonlocality, but something else.

The question is that the rapporteur mentioned a fundamental length. Has the need for this been observed yet?

N g u e n:

Much more high energy data is needed to do that.

A. M a r t i n:

It seems to me that the introduction of fundamental lengths is very difficult. The experience we have with Glaser and Epstein was the following: we were working with the theory of local observables associated with a region of finite size a. We found all the standard results, polynomial bounds etc., without having to let the length a to zero. So the fundamental length must be introduced in a more sophisticated way (exponential decrease of commutators?)

J. G. T a y l o r:

This comment is additional to that of Prof. Martin. It is not possible to introduce a local algebra of observables for non-localisable theories, so it would seem even more difficult than in the localisable case to introduce a fundamental length in a consistent fashion.

B a l u n i:

I should like to comment the work mentioned by Prof. Martin. There is not up to now, at least I don't know, any rigorous consequence of the axiomatic field theory, which contradict to experimental data or coincide with it. That's why it is interesting that the result of the before mentioned work obtained from general principles and very reasonable assumption coincide with experimental data within experimental error.

A n s e l m:

I should like to note that if the equality of the total cross-sections is broken $\sigma_t \neq \bar{\sigma}_t$ such features of the appearing picture as the equality of differential cross-sections for the particles and antiparticles at $t \sim 1/\log^2 s$ can be displayed at incomparably large energies only. The last equality is based on the fact that at super-high energies the real part of the amplitude which differs only by sign for the particles and antiparticles contains the extra $\log s$ as compared with the imaginary part. On the other hand it contains a small factor $\Delta\sigma = \sigma_t - \bar{\sigma}_t$. Thus one should wait till $\log s$ compensates this smallness?

The comment mentioned belongs to L. Okun but since he keeps silence and this fact seems to me to be very important I decided to comment it.

G. V a t a g h i n:

It is well-known that with increasing of the energy in the frame of reference of two nucleons the number of possible «channels» for inelastic processes increases with the energy very rapidly. According to the non-local theory the «partial» cross-section for each of these inelastic channels should tend to zero if the energy E in the c. m. s. tends to infinity.

Was this problem investigated and what is known about the partial cross-sections from experiment?

I would like to add that the asymptotic behaviour of the cross-section of nucleon interactions (pp) and (pn) and the absorption cross-sections are known to physicists working in the field of cosmic rays since 1948 (See papers of G. T. Zatsepin and G. V. Vataghin for 1948). It is known that these cross-sections are constant in the wide range of the energy variation ($E_{\text{lab}} \approx 10^{11} \div 10^{20}\ eV$).

N g u e n:

I agree with you that the number of channels increases. However it is possible that the cross-sections for some channels do not decrease, but only those of the majority of the other processes do.

Yes, there are some upper bounds on the cross-sections of each process considered separately, but they are so high and therefore are not of interest.

L o m s a d z e:

Prof. Nguen V. H. in his talk pointed out the conditions under which in the quantum field theory the asymptotical relations and in particular the Pomeranchuk theorem must be fulfield. These, however, were the conditions for the physical energy ω' only. At the same time, the proofs of these asymptotical relations are based ultimately on the Generalized maximum principle of Phragmen, Lindelöf, Nevanlinna and consequently assume not only the validity of corresponding conditions for the real physical ω' but require also the fulfilment of Nevanlinna's limiting equality in the complex infinity. This is a very essential point.

The difficulty of the problem is that the averaged amplitude $\bar{T}(\omega') = (T(\varkappa), \varphi_D(\varkappa + \omega'))$ (being for the physical observable directly in experiment) in the complex region is not observable directly in experiment. And here one has no right to relay on any physical considerations or even on any physical intuition. For this reason the direct postulation of the Nevanlinna's limiting equality validity would not lesson the actuality or the attractivity of the problem of proving its validity proceeding only on physically justified postulates of the quantum field theory. Such a proof at the moment unfortunately is not at our disposal.

Quite recently we succeeded in the complete solving (with E. Sabad: preprints ITP — 70 — 17, ITP — 70 — 79, Kiev; with E. Sabad, A. Rowt: prepr. ITP — 70 — 80, Kiev) of the above problem for the averaged asymptotical amplitude $\bar{T}_\infty(\omega')$ relaying in fact only on an effective use of the Bogoljubov microcausality principle. This circumstance inspires the hope that the same will be valid for the averaged exact amplitude $\bar{T}(\omega')$ as well. But until obtaining a rigorous proof of this assertion or until confirming the asymptotical equivalence of $\bar{T}(\omega')$ and $\bar{T}_{\infty}(\omega')$ I would be careful to make the cathegorical assertion that a possible experimental nonfulfilment of some of the asymptotical relations (in the case of fulfilling all corresponding conditions for the physical ω') means inevitably breaking at least one of the physically comprehended postulates of the quantum field theory.

Nguen:

At our section we discuss only asymptotic theorems in the framework of commonly used Bogolubov — Wightman local theory. The localizable and nonlocalizable fields, the formulation of microcausality etc. will be discussed in more detail at the session «Fundamental theoretical problems».

REFERENCES

1. N. N. Bogolyubov, Report at the Seattle Conference, 1956.
2. N. N. Bogolyubov, B. V. Medvedev, M. K. Polivanov. Voprosy teorii dispersionnyh sootnosheniy. M., FM, 1958.
3. N. N. Bogolyubov, D. V. Shirkov. Vvedenie v teoriyu kvantovannyh poley, M., Gostechizdat, 1957.
4. H. Lehmann, Nuovo Cim., **10**, 579 (1958).
5. J. Bros, H. Epstein, V. Glaser, Nuovo Cim., 31, 1265 (1964).
6. A. Martin, Nuovo Cim., **42A**, 930 (1966); 1219 (1966).
7. G. Sommer, Nuovo Cim., **48A**, 92 (1967); **52A**, 373 (1967), 52A, 850 (1967); 52A, 866 (1967).
8. J. D. Bessis, V. Glaser, Nuovo Cim., **58**, 568 (1967).
9. Y. S. Jin, A. Martin, Phys. Rev., **135**, B1375 (1964).
10. A. A. Logunov, A. N. Tavkhelidze, Nuovo Cim., **29**, 380 (1969).
11. S. Mandelstam, Phys. Rev., **112**, 1344 (1958).
12. A. A. Logunov, A. N. Tavkhelidze, O. A. Khrustalev, Phys. Lett., **4**, 325 (1969).
13. A. A. Logunov, Nguyen van Hieu, TMF, **1**, 375 (1969).
14. M. Froissart, Phys. Rev., **123**, 1059 (1960).
15. O. W. Greenberg, F. Low, Phys. Rev., **124**, 2047 (1961).
16. Nguyen van Hieu, Nguyen ngoc Thuan, V. A. Suleymanov, Ann. Inst. Henri Poincaré, **9A**, 357 (1968).
17. R. Omnes, Phys. Rev., **146**, 1123 (1966).
18. M. Kugler and R. Roskies, Phys. Rev., **155**, 1685 (1967).
19. D. Lukaszuk, A. Martin, Nuovo Cim., **52A**, 122 (1967).
20. F. J. Yndurain, Phys. Lett., **31B**, 368 (1960).
21. A. K. Common, preprint, CERN, 1970.
22. A. K. Common, F. J. Yndurain, paper 18—2.
23. D. I. Blokhintsev, V. S. Barashenkov, B. M. Barbashov, Nuovo Cim., **12**, 602 (1959).
24. W. Rarita, P. Schwed, Phys. Rev., **112**, 271 (1958).
25. M. Levy, Phys. Rev. Lett., **5**, 380 (1960).
26. V. Singh, S. M. Roy, Ann. Phys., **57**, 461 (1970).
27. A. A. Logunov, M. A. Mestvirishvili, Nguyen van Hieu, Phys. Lett., **25B**, 611 (1967).
28. G. Tiktopolous, S. B. Treiman, Phys. Rev., **167**, 1408 (1968).
29. A. A. Logunov, Nguyen van Hieu, Lectures at the International School on Elementary Particle Physics, Popradske Pleso, Czechoslovakia, 1967.
30. A. A. Logunov, Nguyen van Hieu, Report at the Topical Conference on High Energy Collision of Hadrons, CERN, Geneva, 1968.
31. T. D. Bessis, Nuovo Cim., **45**, 974 (1966).
32. Y. S. Jin, A. Martin, Phys. Rev., **135**, B1369 (1964).
33. T. Kinoshita, Phys. Rev., **152**, 1266 (1966).
34. Nguyen van Hieu, JETP Letters, **7**, 391 (1968).
35. V. Baluni, Nguyen van Hieu, JETP Letters, **8**, 203 (1968).
36. T. Kinoshita, paper 18—11.
37. Nguyen van Hieu, preprint, Institute of Physics, Hanoi, 1970.
38. I. Ya. Pomeranchuk, JETP, **34**, 725 (1958).
39. N. N. Meiman, JETP, **43**, 4277 (1962).

40. A. Martin, CERN, preprint, 1968.
41. A. A. Logunov, M. A. Mestvirishvili, G. G. Volkov, paper 18—6.
42. R. J. Eden, Phys. Rev. Lett., **16**, 39 (1966).
43. V. Singh, S. M. Roy, paper 18—21.
44. V. N. Gribov et al., paper 18—8.
45. L. van Hove, Phys. Lett., **5**, 252 (1963).
46. A. A. Logunov, Nguyen van Hieu, I. T. Todorov, O. A. Khrustalev, Phys. Lett., **6**, 69 (1969).
47. A. A. Logunov, Nguyen van Hieu, I. T. Todorov. UFN, 88, 52 (1965).
48. A. A. Logunov, Nguyen van Hieu, I. T. Todorov, Ann. Phys., **31**, 203 (1965).
49. A. A. Logunov, Nguyen van Hieu, I. T. Todorov, O. A. Khrustalev, Phys. Lett., **7**, 7 (1963).
50. Nguyen van Hieu, TMF, to be published.
51. I. I. Levintov, JETP, **42**, 191 (1962).
52. S. S. Bilenkiy, Nguyen van Hieu, R. M. Ryndin. JETP, **46**, 1098 (1964).
53. H. Cornille, Report at this Conference.
54. I. Finkelstein, Report at this Conference.
55. A. A. Anselm, G. S. Danilov, I. T. Dyatlov, E. M. Levin, Report at this Conference.
56. R. J. Eden and G. D. Kaiser, preprint, 1970.
57. N. N. Meyman, JETP, **46**, 1502 (1964).
58. A. M. Jaffe, Phys. Rev., **158**, 1454 (1967).
59. D. I. Blokhintsev, G. I. Kolerov, Nuovo Cim., **34**, 163 (1964).
60. A. A. Logunov, Nguyen van Hieu, preprint JINR P—2873, Dubna 1966.
61. V. A. Matveev, L. D. Soloviev, B. V. Struminsky, A. N. Tavkhelidze, V. P. Shelest, preprint JINR, P2—3118, Dubna 1967.
62. A. A. Logunov, L. D. Soloviev, A. N. Tavkhelidze, Phys. Lett., **24B**, 181 (1967).
63. Dao vong Duc, Nguyen van Hieu (unpublished).
64. L. A. Khalfin. YaF, **6**, 651 (1967).
65. L. Jenkovsky, Nucl. Phys., **B5**, 119 (1968).
66. Yu. S. Vernov, Report at this Conference.
67. V. Baluni, Yu. S. Vernov, paper 18—13.
68. V. B. Geshkenbein, B. L. Ioffe, JETP, **46**, 902 (1964).
69. Nguyen thi Hong, Nucl. Phys., B11, 127 (1969).
70. Truong nguyen Tran, R. Vinh Mau, Phys. Rev., **177**, 2494 (1969).
71. Truong nguyen Tran, Phys. Lett., **31B**, 461 (1970).
72. M. Gell-Mann, M. Goldberger, Phys. Rev., **96**, 1533 (1954).
73. F. E. Low, Phys. Rev., **96**, 1428 (1954).
74. A. Martin, Nuovo Cim., **37**, 671 (1965).
75. A. M. Jaffe, Phys. Rev. Lett., **17**, 661 (1966).
76. Nguyen van Hieu, DAN SSSR, **182**, 1303 (1968).
77. V. Baluni, Nguyen van Hieu, V. A. Suleymanov, YaF, **9**, 635 (1969).
78. G. G. Volkov, V. V. Ezhela, M. A. Mestvirishvili, YaF, **9**, 857 (1969).
79. Nguyen van Hieu, YaF, **7**, 1111 (1968).
80. Truong nguyen Tran, R. Vinh Mau, Pham xuan Yem, Phys. Rev., **172**, 1645 (1968).
81. B. V. Geskenbein, YaF, **9**, 1232 (1969).
82. Nguyen thi Hong, paper 18—14.
83. Dao vong Duc, Nguyen van Hieu, Report at this Conference.
84. V. Baluni, paper 18—1.
85. V. V. Ezhela, A. A. Logunov, M. A. Mestvirishvili, paper 18—4.
86. S. Ciulli, R. Cutkosky, Report at this Conference.
87. D. Atkinson, paper 18—23.