SYMMETRY PROPERTIES IN ELEMENTARY PARTICLE THEORY

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REVIEW OF HADRON SYMMETRIES: MULTIPLETS, SUPERMULTIPLETS AND INFINITE MULTIPLETS

A. Barut

I. Introduction

This review concentrates on the classification schemes of hadron states and the problem of hadron structure — with special emphasis on high-lying «excited» states. Our present knowledge regarding these high-lying hadron states is rather incomplete. To show the status of our understanding in this area we discuss concrete examples and point to unsolved problems. We examine various theories, models, or points of view on hadrons and strong interactions with respect to their ability to make predictions about hadron symmetries and hadron levels. The limits of our knowledge can best be traced if we formulate specific questions with a possible yes-or-no answers. Some of these questions which we discuss are:

(I) Are there infinitely many excited states with the same internal quantum numbers?

(II) If there are infinitely many states, does the mass of the resonant states tend to infinity (as in the linear trajectories), or does it approach a saturation point (as in H-atom)?

(III) Is SU (3)-classification of the low-lying states also a good approximate symmetry for the high-lying states?

(IV) If so, do octets and decouplets repeat themselves (as in Regge recurrences, for example), or, are they higher dimensional SU (3)-multiplets for high-lying states (as in models with U (6, 6), for example)?

These questions are crucial for a more complete theory of strong interactions and we must examine and judge the existing hypotheses with respect to these questions. One cannot wait until all the (possibly infinitely many) hadrons states have been observed and identified experimentally, which itself is a difficult task and the identification process depends strongly on the theoretical models. We recall that atomic models have been established long before all the excited states have been seen.

There are more fundamental questions:

(I) What is the physical origin of the internal quantum numbers?

(II) Do hadrons consist of entirely new objects (quarks, magnetic charges)?

(III) If so, do the new constituents carry the internal quantum numbers, or are the internal quantum numbers the results of the composite structure of hadrons?

Keeping these questions in mind, we review in the next two Chapters the general hadron properties and then discuss the quark model, the relativistic infinite multiplets, the current algebra and other models in that order.

II. General views on hadron structure

The abstract classification of hadron states is not an end, but a means to discover the physical structure of hadrons and to understand the nature of strong interactions. When we speak of hadron symmetries we have this connection in the back of our minds. The time has come therefore to try to spell out the views that exist at present about the structure of hadrons, and the implications of these views to hadron states and symmetries. If we do this we are faced immediately with a large variety of models, hypotheses and philosophies, at first glance seemingly contradictory to each other, at least, seemingly at opposite ends of the scale.

To be specific, let me consider the best known member of all hadron, the proton. It is stable, and consequently it may be also the simplest of all baryon states. Furthermore, all baryons must «contain», in some sense, a proton, because they all eventually decay into a proton.

From the known properties of the proton, like form factors, it is now generally accepted that the physical proton is a **composite object.** The concept of an «elementary» proton as a Dirac particle may at best apply to a «bare» proton (in the field theory language), but even this is very doubtful, because for a composite object, like the H-atom, the «bare object» already is not a simple spin $\frac{1}{2}$ — Dirac particle but contains part of the internal interactions. I shall therefore consider the various views of the proton as a composite object. The predominant answers to the question «What is a proton?» are the following:

(1^0) Proton is made up of three quarks (naive quark model).

(2^0) Proton is a system with many degrees of freedom, but no constituents **per se.** (droplet model).

(3^0) Proton is a system with infinitely many constituents (partons, oscillators, ...).

(4^0) All hadrons are composed of each other, they have no constituents (bootstrap models).

(5^0) Proton is the ground state of **two** particles interacting via superstrong long-range forces, like an H-atom.

(6^0) Proton is composed of a core and a meson cloud around it (the old meson-theory picture).

(7^0) Inside the hadrons there is a new geometry of space-time due to the new energy-momentum tensor of the hadronic matter which couples to the gravitational field.

Undoubtedly, there are other views, or variations. I have listed those which have been, explicitly or implicitly, spelled out and investigated in some detail. These seemingly so different views of the proton structure may indeed all be valid; it depends at what level we begin. To illustrate this, note that the H-atom is not, strictly speaking, just an electron-proton system. It consists also of electron-proton plus $e^+ - e^-$-pairs, plus photons, etc., in fact, at higher and higher energies, infinitely many of them. The two-body description, thanks to the Schrödinger and Dirac equations, and to the smallness of the coupling constant, is a fairly good description, and more importantly, gives all the quantum numbers and levels of the system, if not their exact position. Thus, if hadrons consist of strongly interacting constituents, the importance of the virtual pairs is enhanced, and consequently other levels of descriptions are entirely justified. We shall come back to detailed discussion of these proton models.

III. Hadron quantum numbers, hadron properties and hadron multiplets

Some of the hadron models listed are constructed to give an immediate description of the scattering amplitude of hadrons. To this group belong the droplet model (and associated eikonal picture) and the parton model which seem to be rather convenient at high energies. But it is hard to see at the moment, how these models could give the discrete resonances and multiplets of hadron. On the other hand, other models (quark model, «atomic»-model, ...) can directly describe the discrete structure in hadrons, multiplets and excited states and individual hadron properties; but then the high energy scattering is a complicated many-body problem. The scattering properties are discussed in detail in other reports. I shall concentrate on the hadron quantum numbers and intrinsic hadron properties. The properties of hadrons to be described and understood are (1^0) the external quantum numbers of all hadron states (mass M, spin J, (relative) parity P), and (2^0) their internal quantum numbers (charge Q, baryon number N, isotopic spin I, hypercharge Y, as well as charge conjugation C, isospin parity G, whenever applicable), (3^0) the intrinsic properties of single hadron states: magnetic moment μ, form factors, $G_E(t)$ and $G_M(t)$, and (4^0) «excited states» of hadrons. We may use the «excited states» in two ways: (a) excitation of the spacetime quantum numbers, i. e. mass and spin spectrum with possible occurrence of principal quantum numbers n, but with the same $\mathbf{I}$, Y, Q, N; (b) «excitation» of internal quantum numbers $\mathbf{I}$, Y, with the same spin and parity.

There are two views about the internal quantum numbers $\mathbf{I}$, Y, N of strong interactions:

A) The internal quantum numbers are new abstract and intrinsic notions and are not explainable. This view is represented in the quark model. The quarks carry the same quantum numbers $\mathbf{I}$, Y, and they are the basic entities of hadron structure.

B) The internal quantum numbers are global or empirical quantum numbers of the composite structure, and they should be explainable in terms of the constituents. Examples can be found, e. g., in atomic and nuclear physics, in the quasi-spin formalism, or in seniority quantum number. Such a possibility for hadrons is given in Ch. VII.3.

We have so far listed the properties of the single particle hadron states. The scattering processes of hadrons are definitely much more complicated. In atomic physics, we arrived at the intrinsic structure of the atoms first (e. g. H-atom); subsequently one has begun to bescribe the scattering processes of atoms; these problems are complicated and not completely solved even at the present time; It would be perhaps almost hopeless to try to infer the structure of atoms from the collection of scattering data. In particle physics, on the other hand, the main trend has been, in the past decades, precisely the latter one: to try to infer the hadron structure by fitting all the complicated scattering experiments. The present report therefore deals mainly with attempts to start from the structure of single particle hadron states, which hopefully should be a simpler problem.

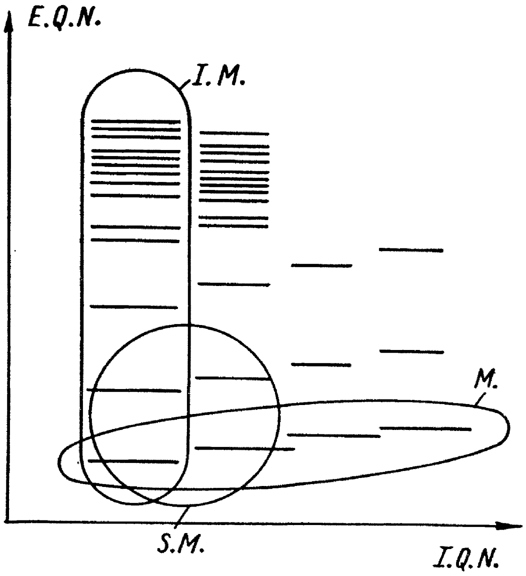


Fig. 1. Multiplets (M.), supermultiplets (S. M.) and infinite multiplets (I. M.) are shown on the plane «external quantum numbers» (E. Q. N.) — «internal quantum numbers» (I. Q. N.).

I now come to the definition of multiplets, supermultiplets and infinite multiplets in the subtitle of this report. Having divided the quantum numbers into external and internal we plot the hadron states schematically as shown in Fig. 1. A **multiplet** will denote states with the same spin J, parity P, and principal quantum numbers n (if any). Ideally, for a multiplet, the mass M should be the same for all members, but of course we are dealing with approximate symmetries of multiplets, not with exact symmetries of the S-matrix. I have not included the baryon number N into the definition of the multiplet, because of the general rule [1]

$$(-1)^{2J} = (-1)^{N} \tag{3.1}$$

for hadrons (which seems to be satisfied exactly), so that spin determines the baryon number N. Specifically, multiplets will mean irreducible representations of SU (3); singlets, octets, decouplets, ...

A **supermultiplet** consists of several multiplets of different spins taken together. Specifically, the representations of SU (6) [see Ch. IV; 1, 3], or SU (2) $\times$ $\times$ SU (2), or SU (3) $\times$ SU (3), or SU $(6)_W$, are supermultiplets. But one can introduce other or more general supermultiplets (see Ch. V. 2). In fact abstractly, there are quite a variety of ways to introduce supermultiplets, specially if we do not insist the members of the multiplets to have the same, or nearly the same mass. The spin **J** may be part of the supermultiplet quantum numbers, (e. g. SU (6)), or helicity alone and not the whole spin may be a supermultiplet quantum number. Similarly, one can consider supermultiplets with the same value of charge, **I** and Y, or with different values of these quantum numbers. We recall that even supermultiplets containing both mesons and baryons have been introduced.

The group SU (3) does not tell us how many multiplets we have and of what type. A larger group, like SU (6), gives a partial answer to this question: In an SU (6)-supermultiplet we have definite numbers and type of SU (3)-multiplets. But SU (6) does not tell us again how many supermultiplets we have and of what type. Does it make sense and is it useful to go to higher and higher multiplets with enormous mass differences? The answer is yes, if we do not interpret these multiplets as determined from a symmetry group (or an approximate symmetry group) in the usual sense (approximately commuting with the Hamiltonian), but from dynamical symmetry groups in a larger sense, in that the group determines all the states and quantum numbers of the system (spectrum generating groups). And the usefulness lies in the calculations of wave functions, therefore of form factors of the system. In this sense, an **infinite multiplet** consists of the set of states with the same internal quantum numbers **I**, Y, N, but with all spins, parities and principal quantum numbers n. An example is the set of excited states N^* of the proton $\left(I = \frac{1}{2},\ I_3 = +\frac{1}{2},\ Y = 1,\ N = 1\right)$, presumably an infinite number of them (e. g. a Regge trajectory). (A more familiar example from atomic physics is the set of all states of the H-atom — an infinite SO (4, 2)-multiplet.)

Again an infinite multiplet as defined here does not tell us what type and how many infinite multiplets we will have. The large values of the quantum numbers is really the region of «terra incognita». Specifically, will we have infinite repetitions of octets and decouplets, as in the idea of Regge recurrences of the multiplets, or will we have increasingly larger multiplets (27, 64, ...) with increasing spins, as given for example, by the larger groups SU (4,4) or SU (6,6)?

IV. The status on the «naive» quark model for high-lying states

The naive quark model * assumes the physical existence of spin $\frac{1}{2}$-particles having quantum numbers **I**, Y (hence charge Q), and pictures the hadrons as bound states of quarks, or quarks and antiquarks. The model came after the use of SU (3) and SU (6) groups. For the applicability of these groups it is of course not necessary at all that quarks or the quark model should exist.** Nor can one with absolute certainty say that quarks cannot exist. From the way they make up the hadrons it is seen that it would be highly unusual, if the quarks actually did exist. It is of course, by exact mathematical analogy, also highly unusual to say that every spin 1 particle is a bound state of two spin $\frac{1}{2}$ — «quarks», or that a three-dimensional harmonic oscillator is the bound state of three one-dimensional oscillators. At any rate, we wish to discuss now what the quark model has to say about the high-lying states. The detailed reviews of the quark model for lowlying states can be found in the previous conference reports [2], and elsewhere [3, 4].

1. MESON STATES

Mesons are quark-antiquark, $q\bar{q}$, bound states. Because $3 \times \bar{3} = 1 + 8$ in SU (3) **all mesons come in multiplets of singlets and octets.** The experimental identification of multiplets is not an easy matter because of

* The term «naive» quark model is used as distinct from the «quark field theory» in which three bare spin $\frac{1}{2}$ fields will be coupled with some interaction according to the laws, of quantum field theory. Unfortunately, no such field theory is soluble in detail so that we do not know if such a possibility can account for the observed hadron states.

** The two fundamental representations of the group SU (3) are of dimensions 3 (q and $\bar{q}$). Not all the representations of a group under which a Hamiltonian is invariant need to occur among the solutions: In the example of the non-relativistic Hamiltonian of spinless particles invariant under the quantum mechanical group SU (2) only integral values of spin occur. Half-integral values are eliminated by the condition of single-valuedness of the wave functions, or mathematically, by the restriction to the group $SU(2)/Z_2 \sim SO(3)$. Note that global groups SU (2) and $SU(2)/Z_2$ have the same Lie algebra. Similarly, the restriction of SU (3) to the group $SU(3)/Z_3$ would give the representations [8], [10], ..., but not [3] and [6]. Perhaps the best way to understand these restrictions is to notice that when SO (3) in imbedded in a larger dynamical group (see Ch. V) like SO (3, 1), then in any representations of the dynamical group, either only integer values or only half-integral values of spin occur.

the mixing problems specially for high-lying levels. Perhaps it is because of the quark model that the main experimental effort has gone into the identification of octets and singlets.

In SU (6), because $6 \times \overline{6} = 1 + 35$, all mesons come in supermultiplets [1] and [35]. This simplest non-trivial supermultiplet 35 is shown in Fig. 2, in our external-internal quantum numbers diagram.

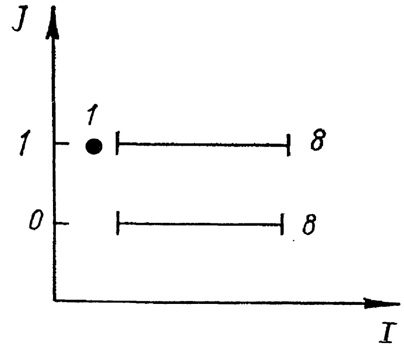


Fig. 2. Multiplets in the [35] — supermultiplet of SU (6).

What about the higher spin states? In the $q\overline{q}$-model, these must come via orbital and radial excitations, written symbolically as

$$[SU(6), L] \text{ and } [SU(6), L, n]. \tag{4.1}$$

Here the notation is that we have in addition to the SU (6)-quantum numbers a new orbital quantum number L ($L = 0, 1, 2, \ldots$) and a new principal quantum number n ($n = 1, 2, 3, \ldots$). Group-theoretically we may also write

$$[SU(6) \otimes O(3)_L] \text{ and } [SU(6) \otimes G]. \tag{4.1'}$$

where G will be specified below; it contains $O(3)_L$.

Because the $q\overline{q}$-system has total spins $S = 0$ and $S = 1$, we obtain for each L-value the total spins

$$J = L \text{ and } L-1,\ L,\ L+1, \tag{4.2}$$

or, in spectroscopic notation, for each L, the states

$$^1L_1 \text{ and } {}^3L_{L-1},\ {}^3L_{L},{}^3L_{L+1}. \tag{4.2'}$$

In the $q\overline{q}$-model, the parity P and charge conjugation C are completely determined: $P = (-1)^{L+1}$ for all the 4 states in (4. 2),

$$C = (-1)^L \text{ for } {}^1L_1\text{-state}$$

$$C = (-1)^{L+1} \text{ for } {}^3L\text{-states}. \tag{4.3}$$

The specifications (4.1) are incomplete and ambiguous, because they do not tell us, for example, how many L-values occur and each how many times. The simplest case is one in which each L occurs once. The resultant **infinite multiplet** is shown in Fig. 3, where a dot represents a single spin J-state, a dot surrounded by a circle two spin — J states. Thus, even in this simple case we have to deal with a large number of states. However, L does not occur alone. In all two-body problems we have also the principal quantum number n such that for each n there are several L-values. One proper way to settle these ambiguities is really to ask and answer the question: What are the forces between the «physical» quarks? Confronted with this dynamical question, the quark model faces numerous problems of formidable complexity, and difficulties, most of which must be pushed «under the rug» [5]. Among the complexities is the problem of the mixing of states which must be taken into account in their empirical identification. There will be spin-orbit type splitting of states (and splitting by tensor forces) and there are mass splitting within SU (3)-multiplets («SU (3)-breaking forces»); consequently mixing can occur among the

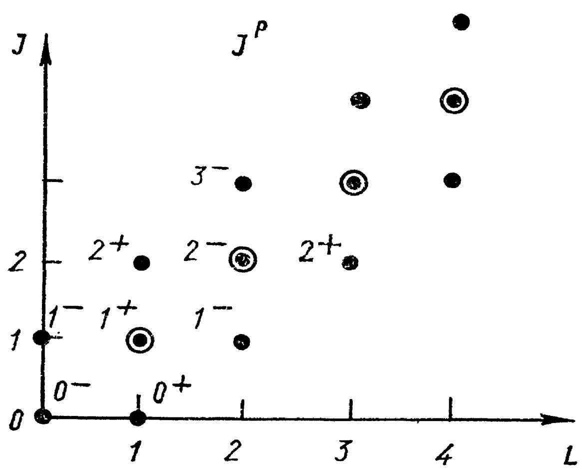


Fig. 3. Meson states $(q\overline{q})$ if each L occurs once.

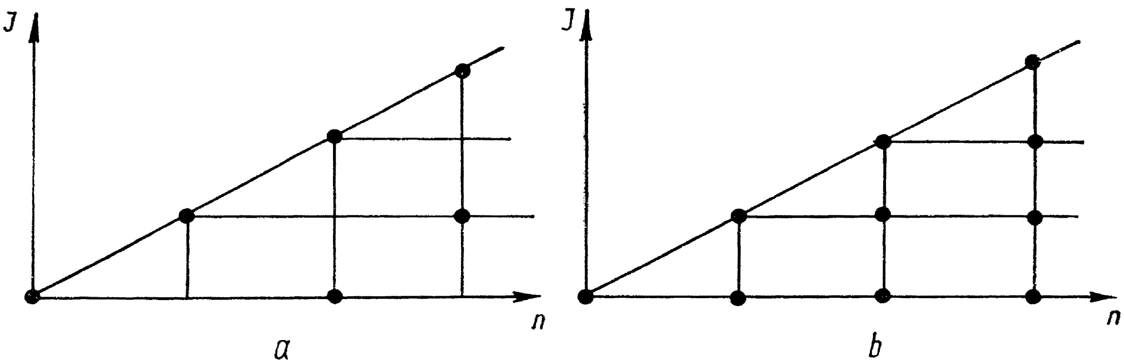


Fig. 4. Orbital and radial excitations (a) oscillator-type, (b) Kepler-type.

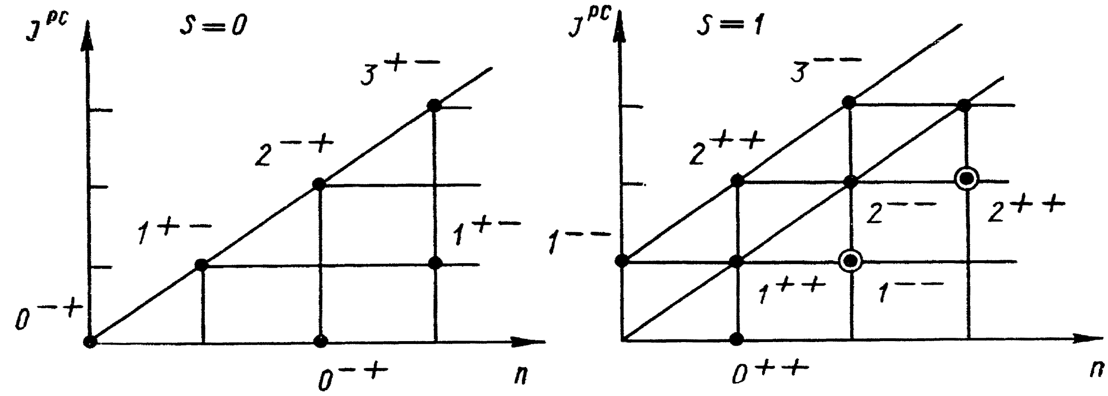


Fig. 5. Meson states $(q\bar{q})$ for «oscillator-type» excitations.

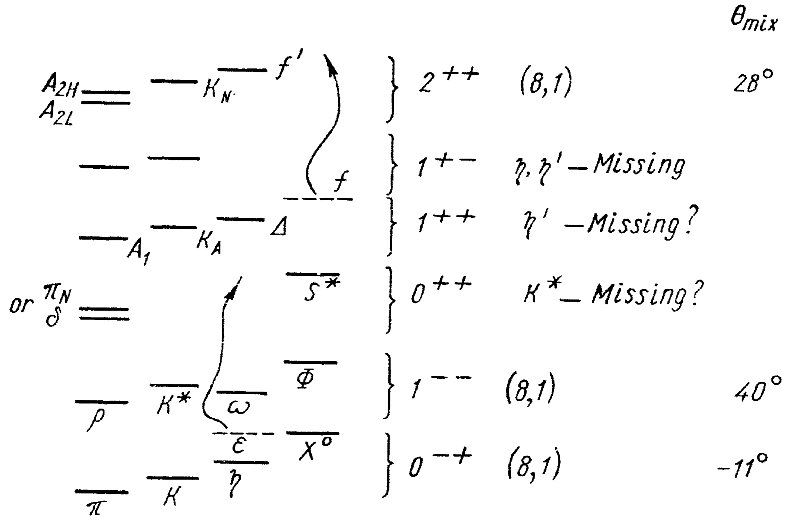


Fig. 6. «Identified» meson nonets.

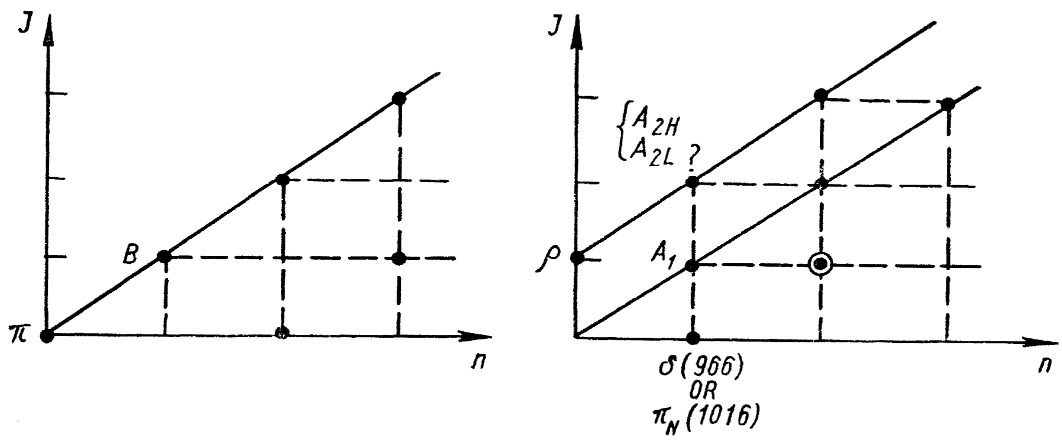


Fig. 7. «Identified» meson states in the «oscillator-type»-excitations.

states with the same I, Y and J (e. g. among the $I = Y = 0$ states in each unitary singlet and octet, among states with $S = 0$ and $S = 1$, or among states with $L = J - 1$ and $J + 1$ ($S = 1$)). Among the dynamical difficulties, I mention the saturation problem (why only $q\bar{q}$ and qqq — states?), the statistics obeyed by quarks [see also next Section, baryon states], the nature of the binding force, etc.

Nevertheless, let us see the possible simple values of the principal quantum number n. Fig. 4*a* and 4*b* show in the «J — n-diagram» all the states for an oscillator-type of potential and for a Coulomb-type of potential, respectively. The former leads to fewer states, for in this case we plot in Fig. 5 the resultant unitary singlets and octets in a J — n-diagram with their parity and charge conjugation assignments, for $S = 0$ and $S = 1$. The immediate important effect of the first radial excitations is the occurrence of new pseudo-scalar (0^{-+}) and vector meson (1^{--}) octets in the neighborhood of spin 2-states, and of course more of these for higher values of n.

2. EXPERIMENTAL STATUS OF MESON STATES

As mentioned previously, the efforts to systematize the meson states have proceeded in trying to identify singlet and octets and to determine the mixing angle between the two $I = Y = 0$ states such the Gell-Mann — Okubo mass formula and decay rates are approximately satisfied (see Sec. 4 below for a critique of this procedure). The six such octets are shown in Fig. 6. The identification is not complete as there are some missing states, and some extra states. A number of possible new resonances have been reported at this conference [6]. Fig. 7 shows the assignment of these observed multiplets in the J — n-diagram. The state E (1420) could be the $I = 0$, $Y = 0$ member of the new radially excited pseudo-scalar multiplets, and there are indications in photoproduction for the new radially excited vector mesons [7]. The extra states are one of the states δ (966) or π_N (1016), and more notable the A_2-splitting, for which two contradictory experiments (but not under exactly identical conditions) have been presented at this Conference [6]. A number of ideas have been expressed in the past concerning the A_2-splitting [2d—3b], and nothing new has been presented at this Conference. The most likely explanation seems to be still the occurrence of an accidental degeneracy, and therefore mixing of two 2^+-states, although it is unlikely that the other 2^+-state is of the $q\bar{q}$-type with an oscillator potential, because that would correspond to a state with $n = 3$ with a much higher mass. If the second 2^+-state is not the $q\bar{q}$-type and the accidental degeneracy is not an isolated case, then it must correspond to a different symmetry, as accidental degeneracies usually do. Related to these remarks is the idea [8] that we may have beside the SU (3) — multiplets, multiplets of another group G, say $SU(2)_{i_1} \otimes SU(2)_{i_2}$, $i_1 + i_2 = I$ = isospin (G not contained in SU (3)), in analogy to the use of two different overlapping schemes, in atomic physics: jj-coupling and LS-coupling.

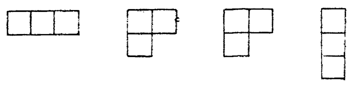
In conclusion, except for the A_2-splitting, there are no other inconsistencies with the quark model for low-lying states. For high-lying states the model predicts too many new states and we have ambiguities in the possible L and n-values in the absence of definite dynamical equations for the two-body problem.

3. THE BARYON STATES

Baryons are three quark systems in this model. Hence, because $3 \times 3 \times 3 = 1 + 8 + 10$ in SU (3), **baryons come only in singlets, octets and decouplets.** The range of quantum numbers in [1], [8] and [10] exclude the possible Z^*-resonances observed in K^+N-reactions with $Y = 2$ and with both $I = 0$ and $I = 1$: Z_0^* and Z_1^*. The experimental status of these Z^*-resonances is unresolved [9]. Again their existence, like the A_2-splitting in mesons, goes beyond the quark model, and might be accounted for, formally, by another approximate group not entirely contained in SU (3), e. g. $SU\,(2)_{i_1} \otimes SU\,(2)_{i_2}$, $I = i_1 + i_2$ [10].

In SU (6), $3q$-system gives the following **supermultiplets:**

$$6 \times 6 \times 6 = 56 + 70 + 70 + 20, \tag{4.4}$$



where we have also indicated the ***SU* (6) symmetry property** in each supermultiplet. The spin and multiplet content of each supermultiplet [i. e. reduction of SU (6) with respect to the subgroup $SU\,(3) \otimes SU\,(2)_J$] is given by

$$\begin{aligned}
[56] &\to \left(8, \tfrac{1}{2}\right) + (10, {}^3/_2) \\
[70] &\to \left(1, \tfrac{1}{2}\right) + \left(8, \tfrac{1}{2}\right) + \left(10, \tfrac{1}{2}\right) + 8, {}^3/_2) \\
[20] &\to \left(8, \tfrac{1}{2}\right) + (1, {}^3/_2)
\end{aligned} \tag{4.5}$$

where 1, 8, 10 represent, as usual, the dimension of the SU (3)-multiplets. The lowest supermultiplet observed agrees very well with [56]. The higher supermultiplets are described, as in the meson case, by orbital (total angular momentum in the center of mass of the three bodies) and radial excitations:

$$[SU\,(6),\ L,\ n]. \tag{4.6}$$

Because we have three basic supermultiplets, Eq. (4. 5), and a three-body problem, instead of two, the number of possibilities for the high-lying supermultiplets is enormous. Again, the nature and the choice of L and n values is ambiguous in the absence of a dynamical scheme, much more so than in the case of mesons.

In the so-called **«symmetric quark model»** [2d] the simplifying assumption is made that the total 3-body wave function is symmetric. This assumption contradicts the fact that quarks are fermions. There are a number of ways out of this difficulty [3], e. g. quarks obey parastatistic. No new thoughts have been presented on this problem. Because the symmetry of the total wave function is

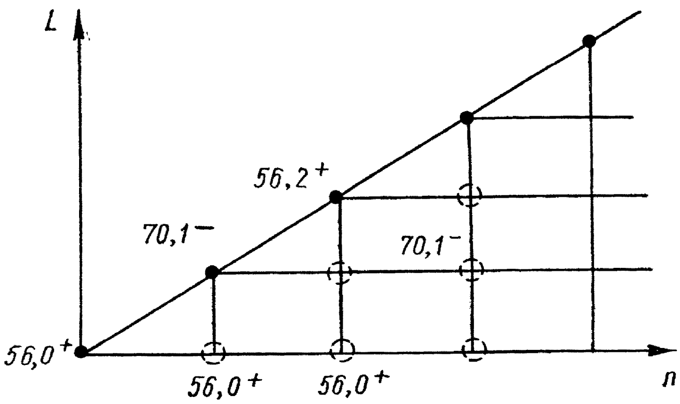


Fig. 8. Baryon SU (6) supermultiplets in the symmetric model. Black dots belong to the main sequence.

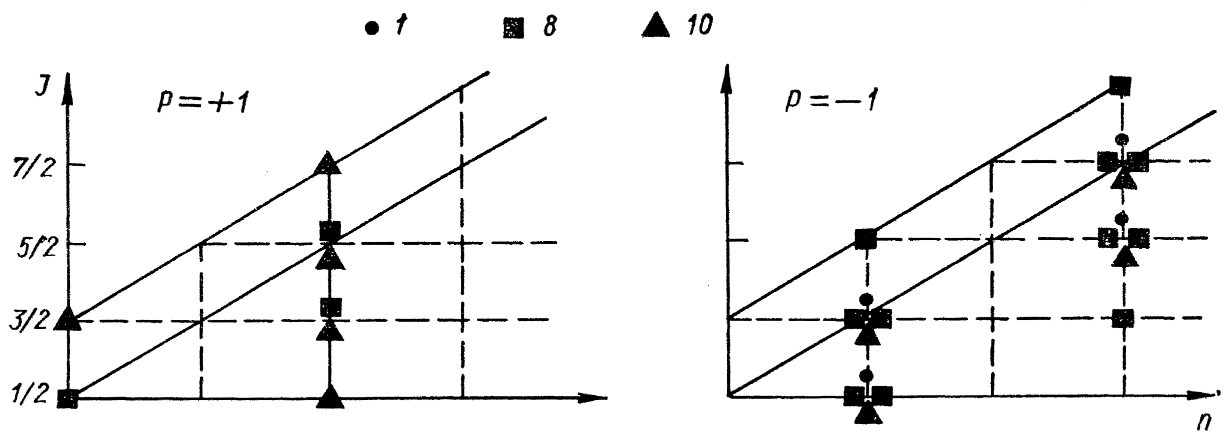


Fig. 9. Baryon SU (3)-multiplets corresponding to the MAIN SEQUENCE only.

determined by the symmetry properties of the quark exchange, and the exchange of space-coordinates we get the following possibilities:

$$\psi_{\text{tot}} = \psi_{SU(6)} \times \psi_{\text{space}}$$

$$\begin{array}{lllll} n = 0: & \square\square\square & \times \ \square\square\square & \rightarrow & (56,0^+),\ (56,2^+),\ (56,4^+)\ \ldots \\ n = 1: & \begin{smallmatrix}\square\square\\ \square\ \end{smallmatrix} & \times \ \begin{smallmatrix}\square\square\\ \square\ \end{smallmatrix} & \rightarrow & (70,\ 1^-),\ (70,\ 3^-),\ \ldots \\ n \geqq 2 & \begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix} & \times \ \begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix} & \rightarrow & \text{e. g. } (20,2^+),\ \ldots \end{array} \qquad (4.7)$$

For $n \gg 2$, there are many more possibilities [11]. The multiplets observed so far all seem to fit into the first two lines in (4.7), called the **main sequence** (the line $L = n$ in our $L - n$-diagram, Fig. 8). However, other symmetry combinations, with higher n-values besides the main sequence, have also been used in the harmonic oscillator model, apparently successively [12]. To show the large number of multiplets in this model we have plotted in Fig. 9 the $J - n$-diagram of the multiplets contained in **the main sequence only,** where positive and negative parity multiplets and singlets, octets and decouplets are separately indicated.

4. EXPERIMENTAL STATUS OF BARYON RESONANCES

Six multiplets seem to be established, and there are two more likely ones [9] (Table I). Again the identification is made mainly on the basis of Gell-Mann — Okubo mass formula. However, in the case of higher states,

Table I

Identified Baryon Multiplets

J^P	$1/2^+$	$3/2^+$	$5/2^+$	$1/2^-$	$3/2^-$	$5/2^-$	$7/2^-$	$7/2^+$
Multiplet	octet	decouplet	octet	nonet	nonet	octet	nonet	decouplet
	very good	very good	works well				?	?

pure reliance on mass formulas for classification purposes is rather dangerous on the basis of recent statistical studies which tend somehow against the existence of further octets [13]. On the other hand, due to the presence of large mixing

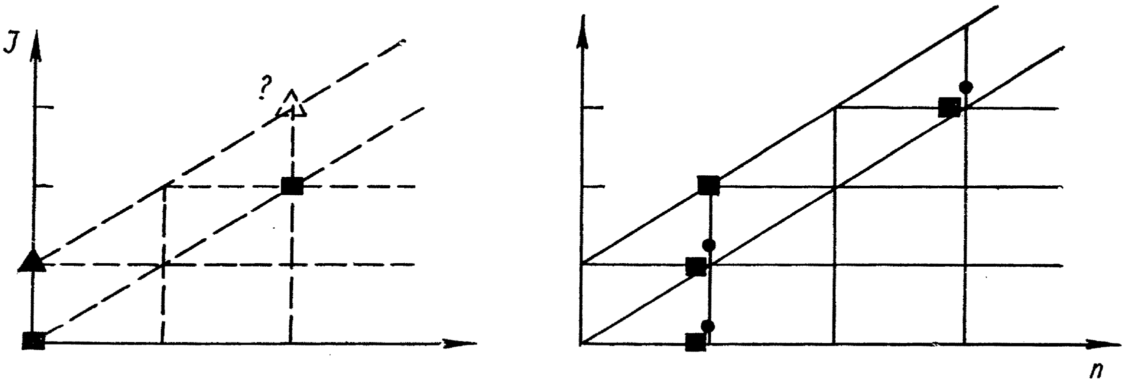


Fig. 10. «Identified» baryon multiplet assigned to the MAIN SEQUENCE.

effects, the octets may exist, but the mass formula is inadequate to identify the octets empirically [14]. Even octet-decouplet mixing is possible, but this possibility has not yet been considered in the experimental analysis. In either case, we are here clearly at the limit of applicability of purely abstract ideas, like *SU* (3), or the Regge recurrences, or the set of orbital excitations. We must have a solid theoretical belief, or a definite model to answer the riddle of higher states and to make predictions. Fig 10 shows the assignment of these multiplets to those contained in the main sequence (Fig. 9). There are certain regularities in this type of assignment:

(I) The square of the (average) masses of multiplets of the same type (same *SU* (3)-dimension and same parity) are approximately equal [15]. For example: $1/2^-$, $3/2^-$, $5/2^-$ octets. Referring to Figures 9 and 10, this means in our language of the J — n-diagram, that the masses depend essentially on the principal quantum number n, i. e. there is very little spin-orbit splitting.

(II) We can extract from Figs. 9 and 10, Regge trajectories connecting $3/2^+$—$7/2^+$ decouplets, $1/2^+$—$5/2^+$ octets, $3/2^-$—$7/2^-$ nonets, etc. [9, 15]. Unfortunately, these trajectories have only one or two definitely established points. The linear trajectories (J versus M^2) so obtained have a slope of about $.95\,(GeV)^{-2}$, the same as that of ρ-trajectory. Again referring to the J — n-diagrams of Fig. 9 and 10, this rule means that **M^2 depends linearly on the principal quantum number n** (at least for low-lying multiplets), and if we use the observation (I) , **M^2 depends very little on J for a given n** (see Ch. VIII).

5. COUPLINGS

The classification of states alone is not sufficient. We expect from hadron models prescriptions for evaluating couplings and interactions. The use of *SU* (3)-symmetry in the systematics of trilinear couplings is well known [4]. The quark model gives the general tensorial properties of the couplings, but this is not sufficient. Form factors must be introduced [17], which account for the dynamics. Again in the absence of a definite dynamical picture, these form factors in the quark model are phenomenological and unknown.

6. CONCLUSIONS

The quark model accounts well for the quantum numbers of the lowlying hadron states, except the A_2-splitting and the Z^*-resonances, which are however, unresolved. For high-lying states the status of the quark model is very ambiguous and almost hopelessly complicated. The orbital and radial excitation quantum numbers are introduced in an *ad hoc* manner and the form factors are unknown, in the absence of definite consistent ideas about the forces between the quarks, especially in the three-body problem. Even the most restrictive assumptions (subject to their consistency, e. g. spin-statistics) predict far too many states than the presently observed ones.

In Ch. VIII a more tractable and simpler model will be compared with the experiment.

V. Relativistic treatment of orbital and radial excitations. Passage to dynamical groups and infinite multiplets

The group structure of multiplets and supermultiplets is well known. What can we say about the group structure of the higher orbital and radial excitations? We observe that levels with the same internal quantum numbers, but varying mass and spin (i. e. different L and n-excitations) form in simple cases a discrete irreducible representation of a non-compact group G **containing the homogeneous Lorentz group** $\boldsymbol{SO}$ **(3, 1).** The last part of the sentence is the clue for a relativistic treatment. Table II shows this connection.

Table II

Groups Corresponding to Orbital and Radial Excitations

	States	Group G	Characterization of the Representation
L-excitations only	$\|l, m\rangle$	$O(3, 1)$	lowest spin j_0, j_1
L — n-excitations			
a) Oscillator type	$\|n, l, m\rangle$	$SU(3,1) \supset O(3,1)$	most degenerate, $n = 1, 2, \dots$; $l = 0,2,\dots (n-1)$ or 1 3, ... $(n-1)$
b) Kepler type	$\|n, l, m\rangle$	$SO(4, 1)$ or $SO(4, 2)$ $\supset O(3, 1)$	most degenerate (i. e. one non-vanishing second order Casimir operator) $n = 1, 2, \dots$; $l = 0,1,\dots n-1$

The states shown in Table II are examples of **infinite-multiplets.** The Group G whose representation gives all the state of an isolated composite system is called the **dynamical group** of the system [17], or the spectrum generating group [18], or also the non-invariance group [19]. Theories which describe particles with internal coordinates (bilocal theories, etc.) also lead to the notion of dynamical groups [20], as well as the concept of **«relativistic symmetries»** [21] when mass differences are neglected.

1. THE NATURE AND INTERPRETATION OF THE DYNAMICAL GROUP G

The groups G listed in the above Table are obviously not the symmetry groups in the usual sence that their generators **all** commute with the Hamiltonian. A subgroup G_0 of G may commute with H and corresponds to the degeneracy of the energy levels (G_0 may be called as the **group of degeneracy of** $\boldsymbol{H}$ — similarly there are groups of degeneracies of other quantities like spin,

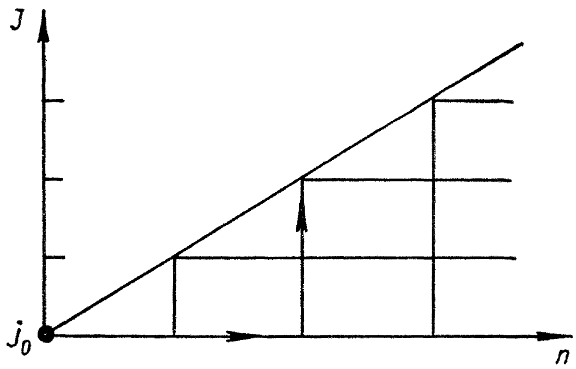


Fig. 11. *SO* (4,2)-weight diagram for a unitary representation with lowest spin j_0. The vertical and horizontal arrows correspond to compact and noncompact generators, respectively.

momentum, etc.). But G contains generators which connect states with different energies. These latter operations are physically realized if we let the system interact with certain external agents. The interacting system can have now a larger symmetry group: because there is energy exchange between the system and the external photon, for example, we can have larger multiplets whose members need not have the same energy [22]. In an explicit case, Malkin, Man'ko and Trifonov [23] have recently studied the time dependent Hamiltonian $H(t)$ representing an oscillator in external time-varying electromagnetic fields. They construct operators which commute with $\left(i\frac{\partial}{\partial t} - H(t)\right)$, and generate the dynamical group $SU(3,1)$ of the oscillator. The solutions of this time-dependent problem form a carrier space of the representations of the dynamical group G. Related to these considerations is the idea that the more general definition of a symmetry operator L is the equation (in the Heisenberg representation)

$$i\hbar \frac{dL}{dt} = [H, L] - i\hbar \frac{\partial L}{\partial t} = 0. \qquad (5.1)$$

The group properties of operators satisfying (5.1) have recently been investigated in classical and quantum mechanics by Dothan [24]. In any case, the important point is that G gives us **all the rest frame states of the system,** and includes the so-called non-compact generators which connect one energy level to another. The $J - n$-diagrams of the previous Chapters are now equivalent to the **multiplicity** patterns of the representation of G. For example, Fig. 11 shows the multiplicity pattern of a particular discrete representation of $O(4,2)$, characterized by the lowest spin j_0; the states are labeled by the quantum numbers $|njm\rangle$, which are the eigenvalues of $SO(4,2)$ operators Γ_0, J^2, J_z.

The group $SO(4, 2)$ has been found to have a rather universal place, in the sense that various systems (Dirac particle, H-atom, dyonium...) can be described by different representations of the same group. The $SO(4, 2)$-level scheme of the type shown in Fig. 11 is a rather general feature. This same $J - n$-diagram will also arise in models based on Regge trajectories, when the residue of the poles at $\alpha(s) = n$ is a polynomial z^n (rather than a $P_n(z)$-function). For then the projections

$$\int_{-1}^{+1} z^n P_j(z)\, dz \qquad (5.2)$$

gives for each n, j-values ranging from $j = 0, 1 \ldots n - 1$. This is the case, for example, in the Veneziano model [25], or, related models [26]. The same $SO(4, 2)$-pattern also arises naturally in $O(4)$-symmetric hadron models [27], and in the idea of the so-called Lorentz poles [28].

What distinguishes different system with the same $SO(4, 2)$ pattern is the actual form of the mass spectrum and the form of the wave functions. The pattern of Fig. 11 does not yet mean linear Regge trajectories (and daughters), unless the mass squares M^2 are linear functions of n (or j). (See remarks at the end of Ch. IV, § 4.)

2. RELATIVISTIC DESCRIPTION OF COMPOSITE SYSTEMS

According to the general philosophy of quantum theory, there **must exist a global relativistic theory of a (composite) system with internal degrees of freedom.** The system is characterized by its total momentum P_μ and angular momentum operators $J_{\mu\nu}$, and a set of other operators, among which we may choose a complete set of global quantum numbers: $\{n, \alpha\}$. In other words we are looking for a relativistic wave function for example of the form

$$\psi_{(n,\alpha)}(p) \tag{5.3}$$

where the linear momenta are assumed to be diagonalized. Moreover, the system can possess arbitrary mass-spectrum (range of $P_\mu P^\mu$) and/or spin spectrum.

There may or may not be an underlying quantum field theory in terms of interacting local fields describing the constituents, or an underlying particle theory in terms of the relative momenta and coordinates of the constituents. First of all we may not know what the constituents of the system are. Secondly, even if we knew the basic primitive constituents, the interactions will produce new virtual particles so that we have to go through the full machinery of perturbation theory to describe the system, and there seems to be no way at the present time to sum up **all** renormalized perturbation terms. (Note again that the H-atom is not just an $e - p$ system, but $e - p$ plus e^+e^--pairs, $e^+e^-\gamma$'s, etc.) For these reasons, we may introduce the global description of the system which is a final **non-perturbative** solution of an underlying dynamics. It is complementary to a possible, but untractable, field theory description. It may be argued, as one extreme point of view, that quantum theory of a given system must be formulated in terms of the observable quantum numbers of the system, and that it is meaningless to talk about unobservable relative coordinates. (The other extreme point being that every system must be described by local interacting fields, e. g. quark field theory.)

We will now review the concrete procedures in such global theories of quantum systems.

(2.1) Complete Algebraic Theory. In this approach the generators of the dynamical group G, defined in Sec. 1, will be enlarged to include the operators P_μ. The group G describes all the rest states of the system; with the inclusion of P_μ we will have the desired complete set of states of the form (5.3). Except for very simple cases, the new structure is not a finite Lie algebra containing the Poincaré group (note: G always contains the homogeneous Lorentz group), but an infinite Lie algebra, or an associative algebra which is generated by the successive commutators of finitely many generators [29].

For example, if P_μ and $L_{\mu\nu}$ are the generators of the Poincaré group, the new quantities B_μ defined by

$$B_\mu = P_\mu + \frac{1}{2}\lambda (P_\mu P^\mu)^{-\frac{1}{2}} \{P^\rho, L_{\rho\mu}\}$$

together with $L_{\mu\nu}$ generate a group $SO\,(4, 1)$. We can identify this $SO\,(4, 1)$ with our G. In an irreducible representation of $SO\,(4, 1)$ with fixed Casimir operators we can evaluate $P_\mu P^\mu$, the mass spectrum from $B_\mu B^\mu$. The result is an equation of the form [29]

$$m^2 = \lambda^2\alpha^2 - \frac{9}{4}\lambda^2 + \lambda^2 j\,(j + 1),$$

where α^2 is the value of the $SO\,(4, 1)$-Casimir operator.

(2.2) Sets of Finite-Dimensional Supermultiplet Wave Equations. Finite dimensional wave equations in general contain several mass and spin-states.

Therefore, one may group the set of hadrons into multiplets or supermultiplets. The mass spectrum in each supermultiplet is given by a wave equation. One can write, in the simplest case, a whole set of Lorentz-invariant equations [30—31], or, larger supermultiplet equations. A comprehensive case, recently studied [32] for hadrons, involves finite-dimensional supermultiplet wave equations on the group $SO\ (3,2) \otimes U\ (3,1)$. Each supermultiplet is characterized by the dimensions (N_0, N) of SO (3,2) and U (3,1) representations, respectively; e. g. (4,1), (5,6), (10, 10), ... The main problem with the set of, possibly infinitely many, finite dimensional equations is the question of **interactions.** The interactions will couple these equations, consequently we have again to quantize these fields. Then the question is what happens to the mass spectra that have been fitted in the construction of unquantized wave equations. The quantization and renormalization of higher spin wave equations are open questions also.

(2.3) Infinite Component Wave Equations. In contrast to the previous case it is possible to interpret the infinite component ware equations as non-perturbative c-number solutions, not to be quantized again. The masses and the form factors obtained from the wave equation are already the observed masses and form factors (see Sec. 2). This is indeed precisely what happens in the known cases like the relativistic H-atom, as we shall see. Although this is the view I have adopted, it has not always been the generally accepted view. Many authors investigated the (second) quantization of infinite-component wave equations [33]. It turns out that the general CPT, and spin and statistics property for the infinite-component wave equations are quite different [34]. Moreover, an infinite-component theory, which is local, must have a continuous mass spectrum [35].

One can understand why it would be desirable in principle to have a field theory with infinite component fields. If an infinite-component wave equation describes well the hadron states, then we could treat hadron-hadron interactions by coupling such infinite-component «fields». There may still be such a theory, but it will certainly have a different form than the **local** field theory. It is easy to see that the usual local perturbation theory diagrams do not lead to a correct counting of the interactions in our case. The example of hydrogen-hydrogen scattering with a local coupling of the wave functions may illustrate this remark. It is also clear that within our interpretation that the infinite-component equations describe composite structures, we cannot expect the vertex functions to be crossing symmetric. For example, the hydrogen-hydrogen-photon vertex in the crossed

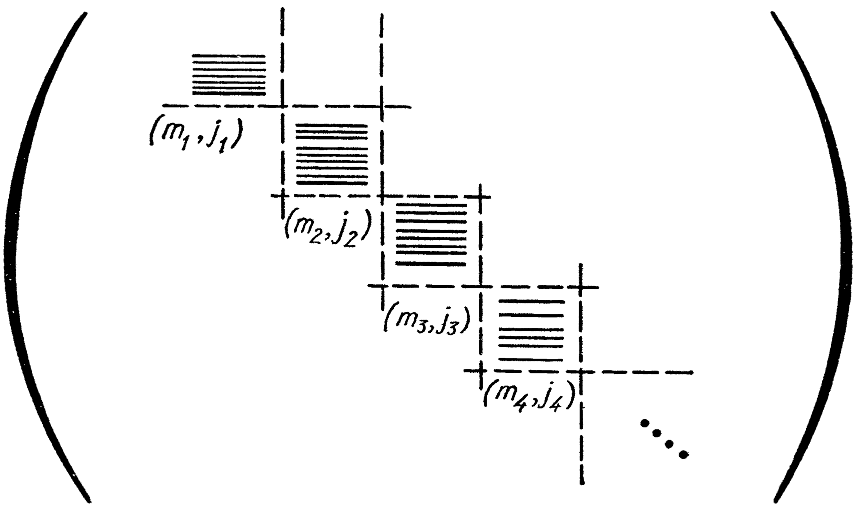


Fig. 12. Highly reducible representation of the Poincaré group (a many mass, many spin system). The collection of the ground states of each block form a representation of the dynamical group G.

channel means hydrogen-hydrogen annihilation into a photon which is an entirely different physical process. Similarly, the breakdown of *TCP*-and spin-statistic-theorems in some cases may be expected. Mathematically, these theorems rest on the possibility that covariant spinorial functions with respect to the real Lorentz group, are also covariant functions with respect to the complex Lorentz group which is not true in general for infinite-dimensional representations of the Lorentz group.

As long as we do not quantize the infinite-component wave equation, this approach is identical to the algebraic approach of Section (2.1).

In order to see the relation of these various cases, we show in Fig. 12 schematically the highly reducible Poincaré states of a multi-mass, multi-spin system. Each block is an irreducible representation of the Poincaré group characterized by (m, j). A set (m, j) may occur more than once. For each block we may write an irreducible wave equation, or for several blocks a (reducible) wave equation (supermultiplets). On the other hand the representation of the group G contains **one** ground state (i. e. the rest state) from each block. The fact that one state from each irreducible space (m, j) form a representation of a group G is the new remarkable fact which gives the theory its dynamical content.

3. WAVE FUNCTIONS AND FORM FACTORS

We start with the dynamical group G. Let $|nj\alpha\rangle$ be the basis in the representation space of a specific representation of G. Because G contains the Lorentz group, spin j will always occur among the labels of the basis states $|nj\alpha\rangle$. The remaining quantum numbers we define schematically by n and α. For the same reason, the generators of pure Lorentz transformations (boosts) $\mathbf{M}$ are among the generators of G. Consequently the Lorentz boosts are defined on the states $|nj\alpha\rangle$:

$$|nj\alpha;\ p\rangle = e^{i\xi\cdot M}\,|nj\alpha\rangle. \tag{5.4}$$

Here ξ are the parameters of pure Lorentz transformations: For time like states ξ is in the direction of the total momentum $\mathbf{p}$ of the system and

$$\cosh\xi = E/m,\ \sinh\xi = p/m. \tag{5.5}$$

The ket $|nj\alpha;\ p\rangle$ is the spinorial wave function of momentum P_μ, and will also be denoted occasionally as $\psi_{nj\alpha}(p)$.

Now according to the general probability interpretation of quantum theory and relativity, there exists a **conserved** (probability) **current operator** J_μ. A simple way to introduce formally a conserved current operator is to write (as in the free Dirac equation) a wave equation, «first order» in J_μ;

$$(J^\mu P_\mu + M)\,\psi_{nj\alpha}(p) = 0, \tag{5.6}$$

where, in general, M is a matrix on the space $|nj\alpha\rangle$.

From the wave equation (5.6) we obtain immediately two properties: (1^0) mass spectrum as a function of quantum numbers $(nj\alpha)$:

$$m^2 = m^2(nj\alpha). \tag{5.7}$$

(2^0) Form factors as matrix elements of the conserved current between two states (to lowest order in electromagnetic interactions):

$$F_\mu(t) = e\langle n'j'\alpha';\ p'\,|\,J_\mu\,|\,nj\alpha;\ p\rangle. \tag{5.8}$$

The last property assumes that the conserved electromagnetic current operator $J_\mu^{(e)}$, which causes transitions between the two states, is proportional to the probability current J_μ. In general, these two currents need not be the same.

Note that Eq. (5.6) has the form of a «free-particle» wave equation for the composite system as a whole. The internal dynamics (relative coordinates, etc.) are replaced by the set of quantum numbers $(nj\alpha)$.

Perhaps it is instructive to remark that the methodology of the use of the dynamical group G and the current operator J_μ (or the wave equation (5.6)) is precisely in the line of the basic philosophy of quantum mechanics as expressed by Heisenberg [36] that a system should be specified by (I) its possible frequencies (in our case the basis states of G), and (II) by certain quantities giving the line intensities (i. e. the matrix elements of the current J_μ).

4. EXAMPLES OF WAVE EQUATIONS

(4.1) THE DIRAC EQUATION

The dynamical group G of the Dirac equation

$$(\gamma^\mu P_\mu + m)\,\psi(p) = 0 \tag{5.9}$$

is again the group O (4,2), and the space of rest frame states $|njm\rangle$ is the 4-dimensional irreducible non-unitary representation of O (4,2) [37]. All finite-dimensional equations involve non-unitary representations of G. For the unitarity of the theory **it is not necessary that G is represented unitarily,** but the Poincaré group is represented unitarily and highly reducible via the wave equation. For Eq. (5.7) the weight diagram of the rest frame states is the simple picture shown in Fig. 13.

(4.2) THE INFINITE-DIMENSIONAL MAJORANA EQUATIONS

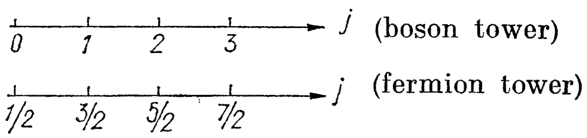
The first infinite-component wave equation was given by Majorana in 1932 [38]. It has the same form as the Dirac equation

$$(\Gamma^\mu P_\mu - \varkappa)\,\psi(p) = 0. \tag{5.10}$$

$$\varkappa = \text{constant}$$

but with the following properties:

(I) The dynamical group G is the Lorentz group O (3,1) or SL (2, C). The rest frame states belong to unitary infinite-dimensional irreducible representations of the type: *



exactly as the orbital excitations discussed in C. IV, and at the beginning of this Chapter.

(II) The mass spectrum is obtained first by going to the rest frame:

$$(\Gamma_0 m - \varkappa)\,\psi(0) = 0 \tag{5.11}$$

and then identifying the rest frame states with the basis vectors $|jm\rangle$ of the group representation. Because $\Gamma_0|jm\rangle = \left(j+\frac{1}{2}\right)|jm\rangle$, we find

$$m = \varkappa/\left(j+\frac{1}{2}\right). \tag{5.12}$$

* These two special representations of SO (3,1) are also irreducible representations of the larger group SO (3,2) generated by the Lorentz group generators **J**, **M** plus the vector operator Γ_μ.

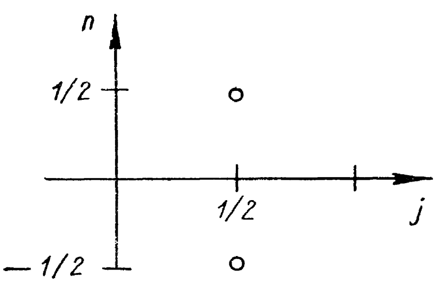
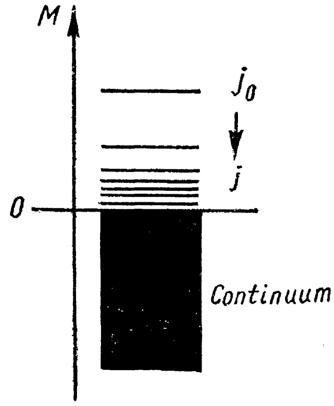


Fig. 13. Representation of O (4,2) for the Dirac-particle.

Fig. 14. Discrete and continuous (space-like) spectrum of the Majorana system.



(III) There are also space-like solutions of (5.10) [39], most easily obtained by going to the frame $p_\mu = (000p_3)$

$$(P^3\Gamma_3 - \varkappa)\,\psi(3) = 0, \tag{5.13}$$

and diagonalizing Γ_3 which has a continuous spectrum. The complete mass spectrum is shown in Fig. 14. In a (second) quantized field theory based on (5.10) both the discrete and the continuous spectrum must be used as a complete set, but the negative mass states cannot be **asymptotic states.** If only the positive mass states (Space I) are asymptotic states and the negative mass states (Space II) only occur as intermediate states, then the S-matrix is unitary only between states in Space I, as in indefinite metric theories. In fact space-like solutions have negative norm.

(IV) The electromagnetic form factors for the Majorana «particle» (under minimal coupling and to lowest order in electromagnetic interactions, i. e. $j_\mu \sim \Gamma_\mu$) have been evaluated exactly between any two states [40] $|j\rangle$ and $|j'\rangle$ (elastic or inelastic). For the ground state, for example,

$$G^E(t) = (1 - t/4m^2)^{-3/2}, \quad \text{scalar particle}$$
$$G^E(t) = \frac{1}{\mu} G^M(t) = (1 - t/4m^2)^{-3/2}, \quad \text{spin}\,\frac{1}{2}\text{-particle} \tag{5.14}$$

(V) Magnetic Moment: The magnetic moment of the spin $\frac{1}{2}$-ground state of a Majorana system is

$$\mu = -\frac{1}{2}\left(\frac{e\hbar}{mc}\right), \tag{5.15}$$

and has the wrong sign when applied to proton, for example. [In a more general representation of $SL(2, C)$, the magnetic moment is $\mu = -\frac{1}{2} - \frac{2}{3}\nu^2$ (ν is the second Casimir operator of the Lorentz group).]

(VI) Interpretation: The Majorana equation is a «free-particle» equation. It is possible to introduce new «dynamical» coordinates such that the equation is transformed into an ordinary Schrödinger equation in **two-dimensions** with a $1/r$-potential [41], or an oscillator potential [42]. This is in agreement with our views that infinite component wave equations describe composite systems as a whole, and in agreement with other examples to be discussed in the following Sections.

(4.3) GENERALIZATIONS OF THE MAJORANA EQUATIONS

Various generalizations of Eq. (5.10) are possible, in order to modify the unphysical mass spectrum (5.12) and the wrong sign of the magnetic moment (5.15). We can modify (a) the underlying representation of G, (b) the form of the equation keeping the representation fixed, or (c) by changing the dynamical group G itself.

(a) If one uses appropriate **reducible** representations of $SL\ (2,\ C)$, one can achieve an increasing spectrum as was shown by Komar and Slad [43]. The form factor and the value of the magnetic moment are not given in this paper.

(b) Using the same representation of $SL\ (2,\ C)$ as in the Majorana equation we can get a rising spectrum and the correct sign of the magnetic moment, if we generalize Eq. (5.10) to

$$[(\alpha_1\Gamma_\mu + \alpha_2 P_\mu)\, P^\mu - \varkappa]\, \psi(p) = 0. \tag{5.16}$$

For then, in the rest frame, we have

$$(\alpha_1\Gamma_0 M + \alpha_2 M^2 - \varkappa)\, \psi(p) = 0.$$

This equation together with the charge normalization condition leads to the mass spectrum

$$m = m_0\left(j + \frac{1}{2}\right), \tag{5.17}$$

and to a **positive** magnetic moment $\left(\mu = +\frac{1}{2}\right)$; magnetic form factor is unchanged [44].

(c) If we now change the group G itself, numerous new possibilities are open [45]. It is then necessary to have other physical criteria to select the applicable and appropriate equation.

A great deal has been learned from the treatment of the H-atom via the infinite-component wave equations regarding (I) the interpretation of the theory, (II) new relativistic treatment of the 2-body systems, and (III) for the purpose of generalizing the H-atom structure to the hadrons. Therefore, we discuss this example next.

(4.4) H-ATOM TREATED AS A SINGLE RELATIVISTIC PARTICLE

The group theoretical discussion of the non-relativistic atom is perhaps well known [45]. More relevant for our discussion is the relativistic case which we will discuss in detail. We neglect the spins of the particles, for the moment, and start from the equation of the motion in the total center of mass frame: $\left[-\frac{1}{2}\nabla^2 + \left(E - \frac{e^2}{r}\right)\right]\psi = 0$. This equation can be written in algebraic from simply as

$$(\alpha\Gamma_0 + \beta\Gamma_4 + \gamma)\, \psi(0) = 0\,, \tag{5.18}$$

$$\alpha = E - \frac{1}{2}, \quad \beta = -E - \frac{1}{2}, \quad \gamma = e^2 \qquad (m = c = \hbar = 1).$$

Here Γ_0, Γ_4 and $T = -i\ [\Gamma_0,\ \Gamma_4]$ generate an $O\ (2,1)$ — subgroup of a larger group $O\ (4,2)$ which includes in addition the generators $\mathbf{J}$ (angular momentum), $\mathbf{A}$ (Lenz vector), $\mathbf{M}$ (Lorentz boosters) and $\mathbf{\Gamma}$ (3-vector dipole operator). In Eq. (5. 18) the relative coordinates and the interparticle potential do not occur, they have been eliminated in favor of the global $O\ (4,\ 2)$-quantum numbers. Moreover Eq. (5. 18) contains both the bound — and the scattering states depending whether we diagonalize Γ_0 (which has only discrete spectrum), or Γ_4 (which has continuous spectrum).

From the rest frame equation (5.18) we can now pass to the frame where the system has a total momentum P. This can be done within the Galilei group [45], or within the Lorentz group. In the latter case, we use the so-called boost-equations

(5.4) and (5.5), and replace the non-relativistic probability density and probability current into a **four-vector current** such that the form factors have the correct non-relativistic limits. The resultant covariant equation is as follows [46]:

$$(J^{\mu}P_{\mu}+\beta\Gamma_4+\gamma)\psi(p)=0; \qquad J_{\mu}=\alpha_1\Gamma_{\mu}+\alpha_2P_{\mu}+\alpha_3P_{\mu}\Gamma_4. \tag{5.19}$$

Specifically for the H-atom the values of the parameters are:

$$\alpha_1=1,\quad \alpha_2=-\frac{\alpha}{2m_1},\quad \alpha_3=\frac{1}{2m_1},\quad \beta=\frac{m_1^2-m_2^2}{2m_1},\quad \gamma=\alpha\frac{m_1^2+m_2^2}{2m_1}. \tag{5.20}$$

Note: There is a second set of values with m_1 and m_2 interchanged in (5.20) with the same mass spectrum.

Equation (5.19) with (5.20) leads for bound states to the mass spectrum:

$$M^2=m_1^2+m_2^2+2m_1m_2\left[1-\frac{\alpha^2}{n^2+\alpha^2}\right]^{\frac{1}{2}}, \tag{5.21}$$

which is symmetric in m_1 and m_2.

Remark: A special case of Eq. (5.21) corresponding to $\alpha_2=0$ in (5.20); i. e.

$$M^2=m_1^2+m_2^2+2m_1m_2\left[1-\frac{\alpha^2}{n^2}\right]^{\frac{1}{2}}, \tag{5.22}$$

has in the meantime also been obtained within the framework of eikonal approximation [47], and of quasi-potential approach [48]. However, the general form in (5.21) is significant for the following reason. For small α, Eq. (5.21) leads to binding energies satisfying

$$\mu+B_n=\mu\left(1+\frac{\alpha^2}{n^2}\right)^{-\frac{1}{2}}-\frac{B_n^2}{2(m_1+m_2)};\quad \mu=\frac{m_1m_2}{m_1+m_2}, \tag{5.23}$$

i. e. Dirac spectrum plus recoil corrections$\left[-\frac{1}{8}\cdot\frac{m_2}{m_1}\left(\frac{\alpha}{4}\right)^4\right]$. For large α, Eq. (5. 22) does not make sense, but our Eq. (5.21) gives

$$M^2\cong m_1^2+m_2^2+2m_1m_2\frac{n}{\sqrt{n^2+\alpha^2}} \tag{5.24}$$

Large α-values are important in a model to be discussed in Ch. VIII.

Electron Spin: The effect of the electron spin and spin-orbit coupling is expressed by replacing in (5.19) Γ_{μ} and Γ_4 by Γ_{μ}' and Γ_4' which contain spin-terms. Γ_0 has the spectrum [49]

$$\Gamma_0: N=\sqrt{\left(j+\frac{1}{2}\right)^2-\alpha^2}+n' \tag{5.25}$$

$$n'=0,\ 1,\ 2,\ \ldots$$

Consequently, the result of the new equation is to replace everywhere n by N; in particular the mass spectrum is now

$$M^2=m_1^2+m_2^2+2m_1m_2\left[1-\frac{\alpha^2}{N^2+\alpha^2}\right]^{\frac{1}{2}} \tag{5.26}$$

Conclusions: The H-atom can advantageously be treated as a relativistic «particle» with internal degrees of freedom. It is coupled to the external electromagnetic field via a conserved current J_μ, Eq. (5.19), which contains convective terms, i. e. it is **not** minimal in the sense of the replacement $P_\mu \to P_m - \frac{e}{c} A_\mu$, but rather in the sense of the coupling $J_\mu A^\mu$. Recoil corrections are automatically included and relativistic form factors can be exactly evaluated. The covariant equations so far studied do not include yet (I) Lamb-shift terms, (II) nuclear spin (hyperfine) terms, and, (III) other new terms higher order in α. These terms may be essential in models with large α (see Ch. VIII).

CONCLUSIONS

A relativistic global treatment of a composite system with internal degrees of freedom is both possible and desirable. The unobserved relative coordinates and internal interactions are replaced by the observed set of quantum numbers of the system. The system is further specified by its conserved quantum mechanical current operator which can be specified algebraically, or by a wave equation. The mass spectrum, form factors and decay properties of the system are then directly derived from the relativistic wave function of the system. Various examples illustrate this point of view.

VI. Current algebra and infinite hadron states

In this Section we investigate the relation of the infinite component wave equations, discussed in the previous Section, to the postulates and results of current algebra.

1. BASIC PROPOSITIONS [50]

The method of current algebra describes the probing of the hadron structure by external electromagnetic and weak interactions. Let us assume that we have the complete Fock-space of all hadron states which are eigenstates of a strong interaction Hamiltoniag H_s; H_s and consequently the hadron states are unknown. The hadron states are assumed to be the nonperturbative solutions of H_s, i. e. to all order of strong interactions. Let now $j_\mu^\alpha(x)$, $j_{\mu s}^\alpha(x)$ be current operators defined on this Fock-space of hadron states. These operators are also unknown, because the space of states is unknown. Although the currents are unknown, we can **postulate** for them **equal-time commutation relations** (in the same way as the postulated Heisenberg commutation relations in ordinary quantum mechanics). The matrix elements of currents between the single particle states are interpreted as the experimental form factors of hadrons to lowest order in electromagnetic or weak interactions. The hadron states must now be obtained as the basic states of the carrier space of **representations of the current commutations relations** plus Lorentz invariance (i. e. the so-called angular conditions), because current commutators are only **equal time** commutation relations.

That these propositions are intuitive and reasonable can be seen from the discussion of the non-relativistic quantum mechanics in terms of current commutators [51].

The two approaches of current algebra and dynamical groups of the last Section are closely related as the comparison in Table III shows [52]. It must be emphasized, however, that the currents j_μ, $j_{\mu 5}$ in the current algebra framework are more general than the currents of the wave equations. The latter act by construction on a complete set of one-particle states, the former currents are field theory operators and act on the Fock-space (i. e. many particle states) as well. We should therefore compare the one-particle matrix elements in the current algebra with the solutions of the wave equation.

The relation of the wave equation and current algebra has been discussed in recent years by many authors [53], either starting from the wave equation, or

Table III

Comparison of methods based (a) on current algebra and (b) on wave equations of dynamical groups

	Current algebra	Models based on wave equations or dynamical groups
(A) Complete set of states of strong interactions	not specified (unknown)	specified by the solutions of the (postulated) wave equation
(B) Explicit form of currents	not specified (unknown)	conserved currents specified from the wave equation
(C) Weak and electromagnetic transition amplitudes	matrix elements of currents	matrix elements of currents
(D) Commutation relations of currents	specified (postulated), e. g., $\left[j_0^\alpha(x), j_\mu^\beta(y)\right] = if^{\alpha\beta\gamma} j_\mu^\gamma(x)\,\delta^3(x-y)$	to be evaluated from the explicit form of currents given above (B) (in general, different from current algebra)
(E) Comparison with experiments	via sum rules obtained from matrix elements of (D)	directly via the matrix elements in (C)

starting from the equal time commutation relations. Chang, Dashen and L. O'Raifeartaigh [54], in their detailed study of the commutation relations and angular conditions in the infinite momentum limit (and in the isospin-factorized case) conclude that the hadron states obtained are those given by infinite component wave equations (the infinite momentum frame is probably not essential to the results). The authors do not consider, however, this result to be completely satisfactory, because:

(a) the wave equations contain space-like solutions as part of the complete set of states, and in a field theory framework, there are transitions into the space-like solutions;

(b) there are also multiparticle states not contained in the wave equation, and it is not clear whether the infinite-momentum frame eliminates the multiparticle states. For a critical and pedagogical discussion we refer to a recent paper of Niederer and L. O'Raifeartaigh [55].

The space-like solutions of wave equations do not cause trouble, if the wave equation is interpreted, as we did in the previous Section, as the c-number equation providing non-perturbative solutions of the strong interactions; they should not be quantized again. In my opinion this is the only sensible interpretation. The space-like solution may then be used as intermediate states, not as asymptotic states [56].

A new study of the **current density** commutation relations has been initiated by Mendez and Ne'eman [57] which avoids the use of the complicated angular conditions. This would be important, because more information is contained in the commutation relations of the **densities**; the charge-density commutation relations are satisfied, from the point of view of wave equations, in a trivial sense [52].

2. ONE-PARTICLE MATRIX ELEMENTS

As we have remarked, the connection between the current algebra and the dynamical groups are in terms of the X-matrices [58]: one-particle matrix elements of currents. This is because the wave equations give only a complete set of one-particle states, with all possible quantum numbers, and the current operators of wave equations act on this space of one-particle states. To show this connection, we look at a general Baryon-Baryon-Meson vertex by the two methods (Fig. 15). The matrix element in question is

$$\langle N^* p', \pi q \,|\, S \,|\, Np \rangle \tag{6.1}$$

which is proportional to the invariant matrix element M. In the dynamical group approach the states $|Np\rangle$ and $|N^* p'\rangle$ are the known boosted (and tilted) group states, and one writes

$$M = \alpha \langle N^* p' \,|\, T \,|\, Np \rangle, \tag{6.2}$$

where T is an invariant pseudoscalar operator in our space of states (for pionic interactions). Once T is postulated, the invariant amplitude can be evaluated.

On the other hand, using the LSZ-reduction technique we have

$$\langle N^* p'; \pi q \,|\, S \,|\, Np \rangle = -i (2\pi)^{-3/2} (2p^0)^{-1/2} \int dx e^{-iqx} (\Box - m_\pi^2) \langle N^* p' \,|\, \varphi(x) \,|\, Np \rangle. \tag{6.3}$$

If one then uses either an effective Lagrangian $[(F_\pi)^{-1} A_\mu(x)\, \partial^\mu \varphi(x)]$, or the PCAC-hypothesis, one can express the pion field $\varphi(x)$ in terms of the matrix elements of the divergence of an axial vector current $A_\mu(x)$. In fact, using the translational invariance, one gets

$$M = \alpha (p' - p)^\mu \langle N^* p' \,|\, A_\mu(0) \,|\, Np \rangle. \tag{6.4}$$

Thus the invariant operator in (6.2) is just the divergence of an axial vector current which is correctly a pseudoscalar. The further evaluation of (6.2) or (6.4) proceeds in the same way: without loss of generalities we can go to the rest frame of N^* (for example), because M is an invariant quantity, and use the boosts given in Eq. (5.4):

$$M = a \langle N^* \,|\, T e^{i\xi \cdot \mathbf{M}} \,|\, N \rangle \tag{6.5}$$

The invariant amplitude is then essentially the matrix elements of finite group elements $e^{i\xi \cdot \mathbf{M}}$. How do we evaluate (6.4) in the current algebra approach? In a recent work, Noga and Katz [59] relate the matrix elements (6.4) to dynamical groups as follows: From (6.4) and (6.5):

$$M = \alpha (p' - p)^\mu \sum_n \langle N^* \,|\, A_\mu(0) \,|\, n \rangle \langle n \,|\, e^{i\xi \cdot M} \,|\, N \rangle \tag{6.6}$$

where $|n\rangle$ is a complete set of one-particle hadron states. [The assumption is made that we can saturate with one-particle states only (see also below).] The matrix elements $\langle n \,|\, e^{i\xi \cdot M} \,|\, N \rangle$ are in principle known. Let us consider the matrices

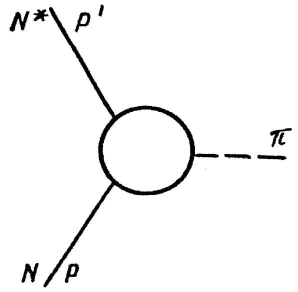


Fig. 15. $N^* \to N\pi$ process.

$$\langle k \,|\, A_\mu^\alpha(0) \,|\, n \rangle = (X_\mu^\alpha)_n (2\pi)^3, \tag{6.7}$$

where $|k\rangle$ and $|n\rangle$ are hadron states and α is an isospin-index which we had suppressed in the previous equations. (For collinear (helicity conserving) transitions of massless pions M is directly proportional to X.) Now if we assume the equal-time commutation relations

$$\left[\int d^3x A_0^\alpha(\mathbf{x}, t), \int d^3y A_0^\beta(\mathbf{y}, t)\right] = i\varepsilon^{\alpha\beta\gamma} I^\gamma, \tag{6.8}$$

where I^γ = isotopic spin generators, and sandwich it in between hadron states and use a «complete» set of one-particle hadron states as intermediate states and use translation invariance we find the matrix relation [58, 60]

$$[X_0^\alpha, X_0^\beta] = i\varepsilon^{\alpha\beta\gamma} I^\gamma. \tag{6.9}$$

Next, the problem is to combine (6.9) with the generators of the isospin group, I^α and of Lorentz group, $J_{\mu\nu}$. [We are in the rest frame, the momenta P_μ have been taken out.] The commutators $[I^\alpha, I^\beta]$, $[J_{\mu\nu}, J_{\sigma\rho}]$ are known; $[I^\alpha, J_{\mu\nu}] = 0$, and

$$[I^\alpha, X_\mu^\beta] = i\varepsilon^{\alpha\beta\gamma} X_\mu^\gamma,$$

$$[J_{\mu\nu}, X_\rho^\alpha] = i\,(g_{\nu\rho} X_\mu^\alpha - g_{\mu\rho} X_\nu^\alpha), \tag{6.10}$$

because X_μ^α is an isospin 3-vector and a Lorentz-four vector. It remains to evaluate $[X_\mu^\alpha, X_\nu^\beta]$. Here, one makes the usual assumption that $I = 2$ (exotic) mesons do not exist. Katz and Noga [59] find

$$[X_\mu^\alpha, X_\nu^\beta] = i g_{\mu\nu} \varepsilon^{\alpha\beta\gamma} I^\gamma - i\delta^{\alpha\beta} T_{\mu\nu}, \tag{6.11}$$

where

$$T_{\mu\nu} = J_{\mu\nu} - F_{\mu\nu},$$

and $F_{\mu\nu}$ commutes with I^α, $T_{\mu\nu}$ and X_μ^α and generates an $SO\,(3,1)$; $T_{\mu\nu}$, I^α, X_μ^α generate $SO\,(4, 3)$: L_{ab}, $(a,\ b = 1, \ldots, 7)$; $(L_{\alpha\mu} \equiv X_\mu^\alpha;\ L_{\mu\nu} \equiv T_{\mu\nu};\ L_{\alpha\rho} = -\,\varepsilon^{\alpha\beta\gamma} I^\gamma)$.

3. MASS RELATIONS

The operators X^α defined previously also satisfy the following relations with the mass-operator $M^2 = P_\mu P^\mu$, **for each value of helicity λ:**

$$[X_{(\lambda)}^\beta, [X_{(\lambda)}^\alpha, M^2]] = \frac{1}{3}\,\delta^{\beta\alpha}\,[X_{(\lambda)}^\gamma, [X_{(\lambda)}^\gamma, M^2]] \tag{6.12}$$

and [61]

$$[X_{(\lambda)}^\beta, [X_{(\lambda)}^\alpha, MJ_\pm]] = \frac{1}{3}\,\delta_{\alpha\beta}\,[X_{(\lambda)}^\gamma, [X_{(\lambda)}^\gamma, MJ_\pm]] \tag{6.13}$$

where J is the angular momentum matrix acting on helicity indices only. The relations (6.12) — (6.13) are again algebraic forms of the commutator properties of the axial charges with themselves and with the generators of the Poincaré group [62] (i.e., infinite-momentum frame saturation):

$$\begin{aligned} &[Q_A^\beta, Q_A^\alpha] = i\varepsilon^{\beta\alpha\gamma} V^\gamma,\ Q_A^\alpha = \int A_0^\alpha(\vec{x})\, d^3x \\ &[Q_A^\alpha, \boldsymbol{P}] = 0,\ [Q_A^\alpha, \boldsymbol{J}] = 0 \\ &[Q_A^\alpha, P_0] = i \int \partial^\mu A_\mu^\alpha(\boldsymbol{x})\, d^3x \\ &[Q_A^\mu, \boldsymbol{M}] = -\,i \int \boldsymbol{x}\partial^\mu A_\mu^\alpha(\boldsymbol{x})\, d\boldsymbol{x} \end{aligned} \tag{6.14}$$

from which one can deduce also

$$[Q_A^\alpha, W_0] = 0,\quad [Q_A^\alpha, \boldsymbol{W}] = i \int [\boldsymbol{J} - \boldsymbol{r} \times \boldsymbol{P}]\, D_A^\alpha(\boldsymbol{r})\, d^3x$$

where W_μ is the covariant spin operator:

$$W_\mu = \frac{1}{2}\,\varepsilon_{\mu\nu\lambda\rho} J^{\nu\lambda} P^\rho;\ \ W_0 = \boldsymbol{P}\cdot\boldsymbol{J},\ \ \boldsymbol{W} = P_0 \boldsymbol{J} + \boldsymbol{M} \times \boldsymbol{P}. \tag{6.15}$$

Equation (6.12) can also be obtained from the double commutator [63]

$$[\int d^3x A_0^\alpha, [M^2, \int d^3x A_0^\alpha]] = \delta^{\alpha\beta}\Lambda, \tag{6.16}$$

where Λ is an isoscalar, Lorentz-scalar operator. (Absence of $I = 2$ component assumed). The generalization of (6.12) to other components of $A^{\alpha}_{\mu}(x)$ has the form [64]

$$[X^{\alpha}_{\mu}, [M^2, X^{\beta}_{\nu}]] = \delta^{\alpha\beta} g_{\mu\nu} \frac{1}{12} g^{\sigma\rho} [X, [M^2, X^{\gamma}_{\rho}]] +$$

$$+ \varepsilon^{\alpha\beta\gamma} \frac{1}{2} \varepsilon^{\gamma\sigma\rho} [X^{\sigma}_{\mu}, [M^2, X^{\rho}_{\nu}]], \tag{6.17}$$

which can be used to derive approximate mass formulas. In one simple case, the dynamical group as well as the mass spectrum turns out to be identical to (5.17) derived from a wave equation.

Relations of the type (6.12) — (6.17) complement the dynamical group approach in the sense that they specify the nature of the transition operator (or current operator in the wave equation), in the present case, the nature of the axial current.

4. SATURATION PROBLEM AND DYNAMICAL GROUPS

There are other attempts to combine supermultiplets or infinite multiplets with the current algebra commutation relations (6.14). The ingredients are always (a) the single particle hadron states, (b) the form of the vector and axial vector current operators. Buccella et al. [65, 66] consider the infinite-multiplets $SU(6) \otimes O(3)$ or $SU(4) \otimes O(3)$. In $SU(6)$ it is natural to identify the vector and axial vector charges with the generators λ^i and $\sigma_z\lambda^i$, for then

$$[Q^i_A, Q^j_A] = [\sigma_z\lambda^i, \sigma_z\lambda^j] = if^{ijk}\lambda^k. \tag{6.18}$$

Under rotations the axial charge transforms like spin parity 1^+-tensor operator: e. g.

$$[J_x, \sigma_z\lambda^i] = [\sigma_x + L_x, \sigma_z\lambda^i] = i\varepsilon_{xzy}\sigma_y\lambda^i. \tag{6.19}$$

However, $\sigma_z\lambda^i$ acts on $SU(6)$-multiplets and does not connect states with different orbital and radial excitations in $SU(6) \otimes O(3)$, for example. The complicated mixing problems that arise can be described in a closed form by introducing [66] a tilting operator $e^{i\theta z}$:

$$Q^i_A = e^{i\theta z}\sigma_z\lambda^i e^{-i\theta z}, \tag{6.20}$$

where z is of the form $z = (\mathbf{W} \times \mathbf{K})_z$. Here W is a tensor in the spin 1, $SU(3)$ singlet of the 35 representation of $SU(6)$, and $\mathbf{K}$ is an ordinary three vector and $SU(6)$-scalar. The operator (6.20) connects the supermultiplets 56, $L = 0$ to 70, $L = 1$ (because z is a 35, $L = 1$ tensor), and 35, $L = 0$ to 35, $L = 1$.

CONCLUSIONS

The solution of the equal time commutation relations of currents within the framework of quantum field theory is still an open question, mainly because of the multi-particle states. The restricted algebraic relations on the one-particle states give results essentially equivalent to those of dynamical groups and infinite-component wave equations. The latter are interpreted as c-number equations whose solutions are the non-perturbative states of the strong interactions to all orders, and are not to be quantized again.

VII. Implications of other hadron models for hadron states

Some of the models of hadrons listed in Ch. II are specifically models of scattering processes. Let us briefly indicate the possible implications of these models regarding the discrete mass spectra and quantum numbers for hadrons.

(A) In the «liquid drop» model [67] particles of extended matter distributions at high energies go through each other. A coefficient of blackness is defined which is roughly equal to the product of the matter distributions of two particles that overlap and is a function of the impact parameter. The scattering matrix is then in the eikonal form. The model accounts for a number of general features of high energy elastic and inelastic scattering. Discrete hadron states will presumably imply discrete forms for the matter density distributions. Because the form of this distribution is an input into the model, or is taken from the form factor measurements, the model itself does not say anything about hadron symmetries and hadron states. There is some work on the relativistic treatment of liquid drops [68].

(B) Similarly, the parton model [69] visualizes that in the very high energy scattering hadrons behave as if composed of infinitely many freely moving constituents, the nature of which is not very well specified. Again the problem of discrete hadron states does not fall into the framework of this model.

(C) Related to this idea is the hadron picture that emerges from the dual resonance model [73], where hadrons appear to have an infinitely many (four-dimensional) oscillator-states (vibrational levels of many particle structures) and an exponentially increasing degeneracy of levels for a given mass and spin. This situation cannot be described by a simple group G of the type we have considered so far, although sets of representations of a non-compact group like $SO(2,1)$ or higher may be used [74].

(D) In the bootstrap models, the self-consistency conditions obtained from crossing and approximate use of unitarity and dispersion relations give relations between the coupling constants which may be compared to the structure constants of symmetry groups [72]. In that sense, there is an approximate relation to the compact approximate symmetry groups. In a latest investigation in this area, Capps [72] relates, using fixed angle dispersion relations and certain dynamical assumption, the trilinear baryon-baryon-meson coupling to matrix elements of group generators in the space of quarks. He finds $SU(n)$-invariant solutions, but an approximate $SU(6)$-invariant solution corresponds better to the experiment than any exact solution.

(E) The meson-theory picture of the proton (a core plus the meson cloud) did not produce any results on the hadron excited states and quantum numbers, because of difficulties in solving field theories (with large coupling constants) and because one does not know which basic set of fields to choose. There are new relativistic treatments of the two-body problem from field theory point of view, for example the quasi-potential method [74]. It remains to be seen if this method can make statements about hadron excited states and symmetries.

(F) Finally, I would like to mention an entirely different approach to hadron structure in terms of a curved space in the small. There are formal approaches of assumed curved spaces [75] which may be a geometrical interpretation of the group structure [17]. There is also a fundamental approach in which the energy-momentum tensor of the massive spin 2 f-mesons are coupled to the gravitational field [76]; hence there is a different geometry inside the proton than outside.

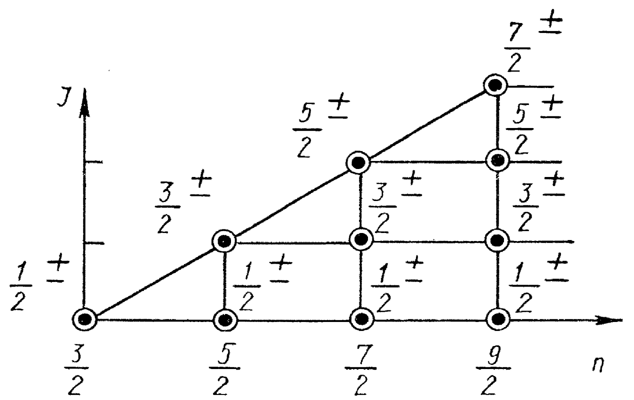


Fig. 16. States in the SO (4,2) fermion representation, $\mu = \frac{1}{2}$.

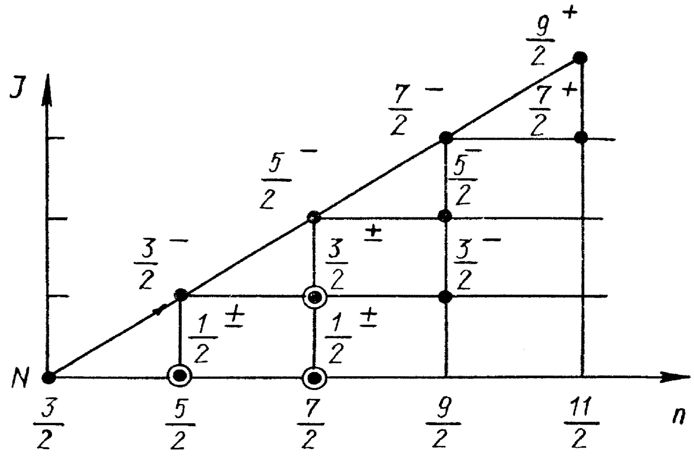


Fig. 17. Principal quantum number assignment of baryon levels.

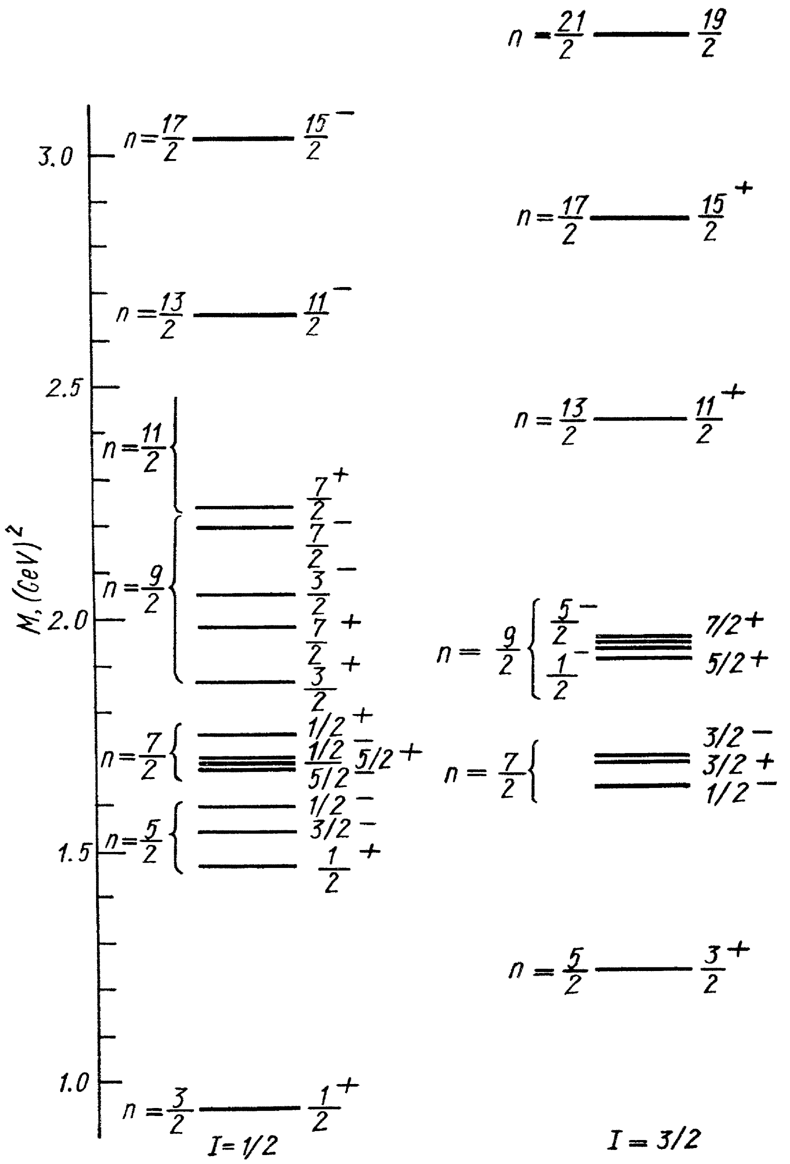


Fig. 18. Assignment of known $N^*\left(J = \frac{1}{2}, I = \frac{1}{2}\right)$ levels according to fermion representation.

VIII. A simple proton model

The predictions of the quark model and current algebra regarding the high-lying hadron states are at the moment incomplete, ambiguous and not very tractable. It is therefore of some interest, to present in this final Chapter, a tractable model, which makes precise statements about the high-lying states. It is in a sense, a completely «soluble» model, and the predictions are so far in agreement with experiment. It turns out that the group-theoretical formalism of the model has a physical interpretation in terms of bound states of magnetic charges. The final result is a unification of strong and electromagnetic interactions.

1. FORMALISM AND PREDICTIONS

ASSUMPTION 1

The first basic assumption of the model is that the excited states of the proton are assigned to the irreducible unitary **fermion** representation of the dynamical group $SO\,(4,2)$ (see Ch. IV), extended by parity. The quantum numbers are shown in Fig. 16. Figs. 17—18 show an assignment [77] of the $I = \frac{1}{2}$, $Y = 1$ baryon levels according to these quantum numbers. The latest $H_{9/2}$ state [78] are included in Fig. 17. The number of states in this model is much less than in the quark model.

The choice of the group G and the particular representation of it are dictated by the existence of a new principal quantum number n to distinguish, say, nucleon from N^*_{11} (1470) and N^*_{11} (1750), all $J^P = \frac{1^+}{2}$ — states, and by the nucleon form factors which is a characteristic of this representation of $SO\,(4,2)$ as we shall see.

All states are parity doublets including the ground-state, that is because the fermion representation is constructed out of two-representations with the invariants $\mu = +\frac{1}{2}$ and $\mu = -\frac{1}{2}$ (Appendix). The parity eigenstates are the superpositions $|\mu\rangle \pm |-\mu\rangle$.

ASSUMPTION 2

The second basic assumption is the form of the conserved current operator, J_μ, or that of the wave equation, as discussed in Ch. V, Eq. (5. 6). The conserved current is assumed to be the most general linear operator in the group generators and in the momenta:

$$J_\mu = \alpha_1\Gamma_\mu + \alpha_2 P_\mu + \alpha_3 P_\mu \Gamma_4 + i\alpha_4 L_{\mu\nu} q^\nu + i\alpha_5 \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} L_{\rho\sigma} Q q_\nu\,, \tag{8.1}$$

where the parameters α_i depend on internal quantum numbers (e. g. they are $SU\,(3)$-tensor operators to be specified) and the generators are defined in the Appendix. For a vertex $P_\mu = p'_\mu + p_\mu$ and $q_\mu = p_\mu - p_\mu$. The rest of the theory follows from the applications of equations (5.7) and (5.8).

RESULTS

First we state the general features of the results:

(A) **Form Factors:** Table IV shows a list of form factors evaluated in various groups and for various representations. The special position of the group $SO\,(4,2)$ is clear form this table. The special position of the fermion representation with $\mu = \frac{1}{2}$

Table IV

Ground State Form Factors (F. F.)

Group	Representation	Scalar F. F.	Electromagnetic F. F.
$SO(3,2)$	Majorana lowest spin $j_0=0$	$(1-t/4m^2)^{-\frac{1}{2}}$	$G_E=(1-t/4m^2)^{-\frac{3}{2}}$
$SO(3,2)$	lowest spin $j_0=\frac{1}{2}$	$(1-t/4m^2)^{-1}$	$G_E=G_M/\mu=(1-t/4m^2)^{-\frac{3}{2}}$
$SO(3,2)$	singleton $j_0=\frac{1}{2}$	$(1-t/4m^2)^{-1}$	$G_E=G_M/\mu=(1-t/4m^2)^{-\frac{3}{2}}$
$SO(4,2)$	most degenerate $j_0=0$	$(1-at)^{-1}$	$G_E=(1-at)^{-1}$
$SO(4,2)$	most degenerate $j_0=\frac{1}{2}$	$(1-t/4m^2)^{\frac{1}{2}}(1-at)^{-2}$	$G_E=G_M=(1-at)^{-2}$
$SO(N,2)$	most degenerate $j_0=0$	$(1-bt)^{-\frac{1}{2}(N+2)}$	$G_E=(1-at)^{-N/2}$
$SO(N,2)$	most degenerate $j_0=\frac{1}{2}$	$(1-t/4m^2)^{\frac{1}{2}}(1-at)^{N/2}$	$(1-at)^{-N/2}$

can be seen from the shape of the scalar form factor for arbitrary μ:

$$G(t)=\frac{\mathrm{Re}\,[\alpha^{2\,|\,\mu\,|}]}{[1+at]^{2\,|\,\mu\,|+1}}, \tag{8.2}$$

where

$$\alpha=\cosh\frac{\xi}{2}+i\sinh\frac{\xi}{2}\sinh\theta,\quad a=\cos^2 h\theta/4m^2;$$

θ is a parameter in the theory.

With the current (8.1) the ground state form factors and all the transition form factors can be evaluated

$$F_\mu=\langle N^*p'\,|\,J_\mu\,|\,Np\rangle. \tag{8.3}$$

(B) **The Mass Spectrum** can be derived from (8.1) by the method of current conservation, or by the wave equation method (as in eqs. (5.10—5.12)). The latter has the form

$$(J^\mu P_\mu+\beta\Gamma_4+\gamma)\,\tilde{\psi}(p)=0, \tag{8.4}$$

where J_μ is given in (8.1) and Γ_4 is the Lorentz-scalar generator of $SO(4,2)$. Because J_0 and Γ_4 do not commute, we define the so-called «untilted» states $\psi(p)$ [«group states» labeled by the **eigenvalue** n of Γ_0] by

$$\tilde{\psi}(p)=Ne^{i\theta L_{45}}\psi(p), \tag{8.5}$$

where the tilting parameters θ are determined in terms of the others. The physical states are the tilted states normalized by $\int\tilde{\psi}J_0\psi d^3x=1$, which gives

$$N^2=\alpha_1 n\cosh\theta+2M\alpha_3 n\sinh\theta+2M\alpha_2. \tag{8.6}$$

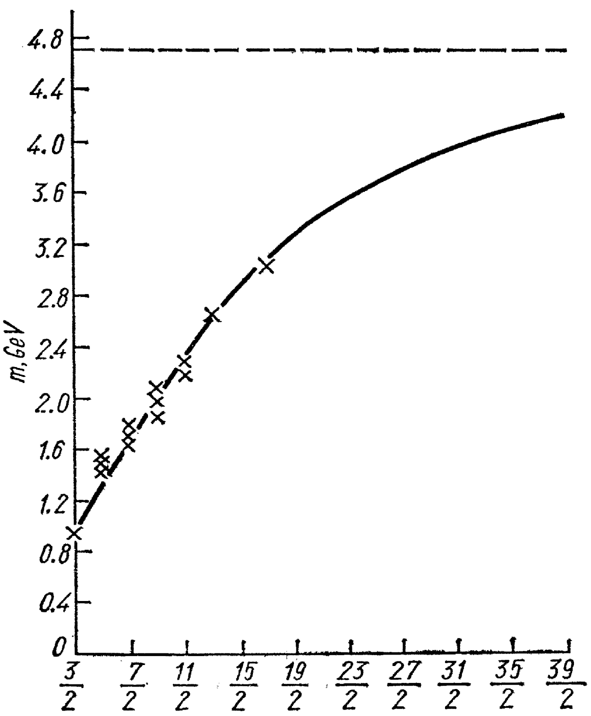


Fig. 19. N^*-octet mass spectrum.

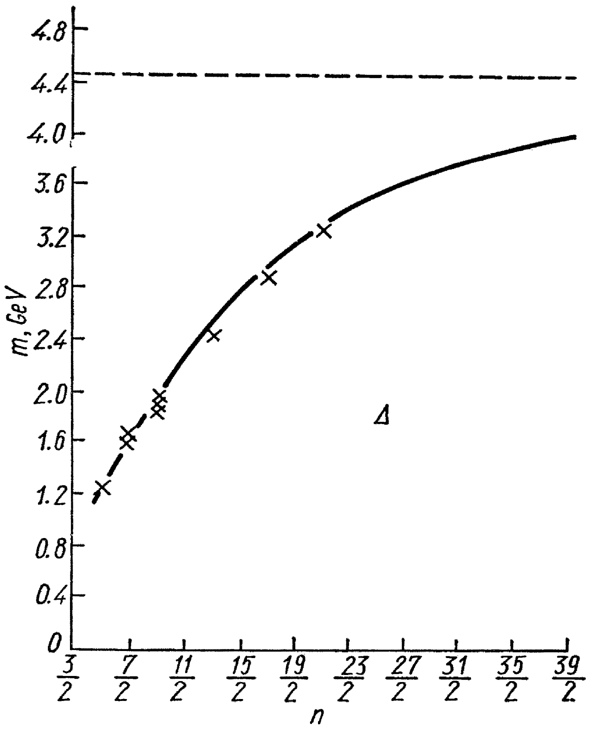


Fig. 20. Decimet mass spectrum.

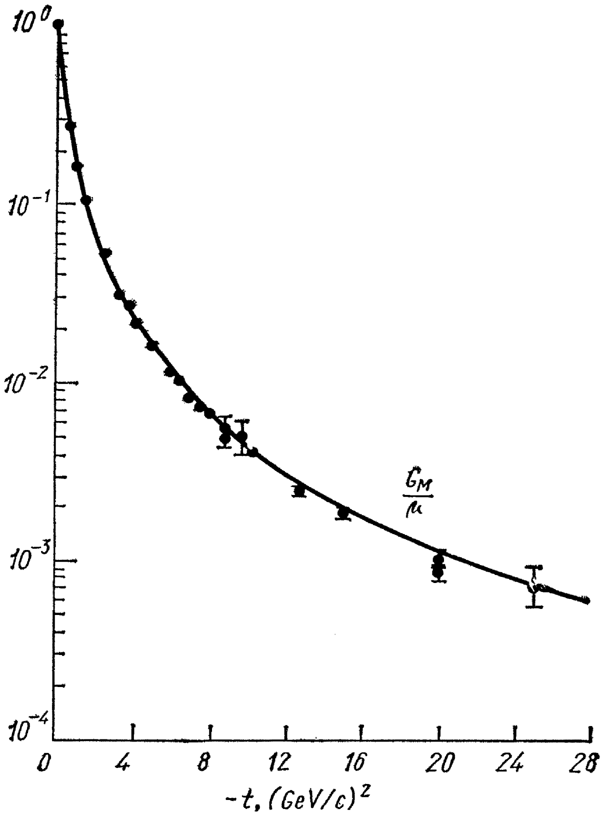


Fig. 21. Proton magnetic form factor fit.

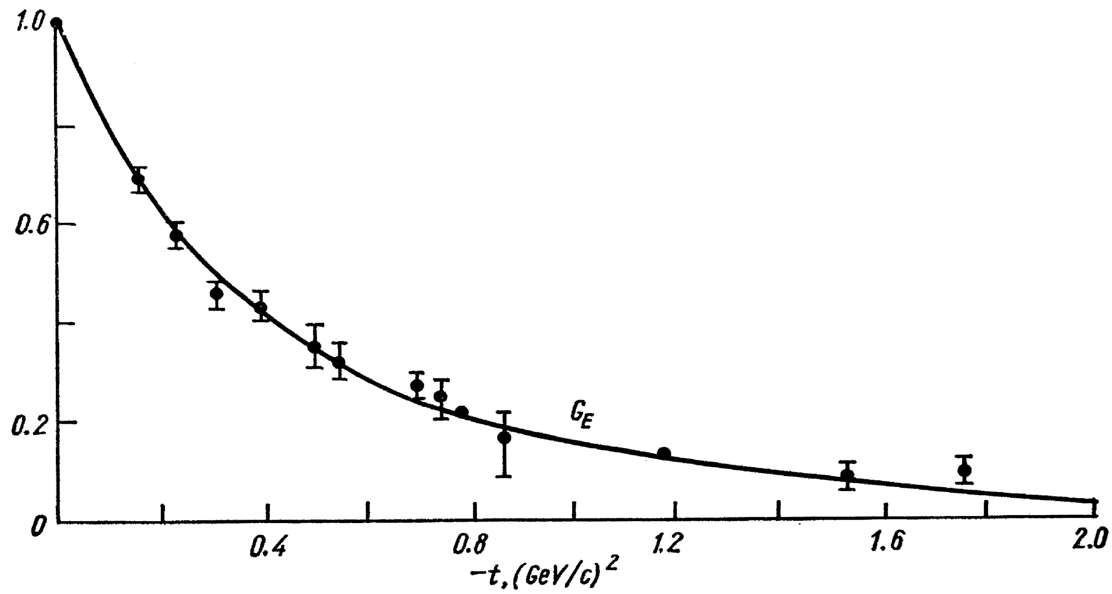


Fig. 22. Proton electric form factor fit.

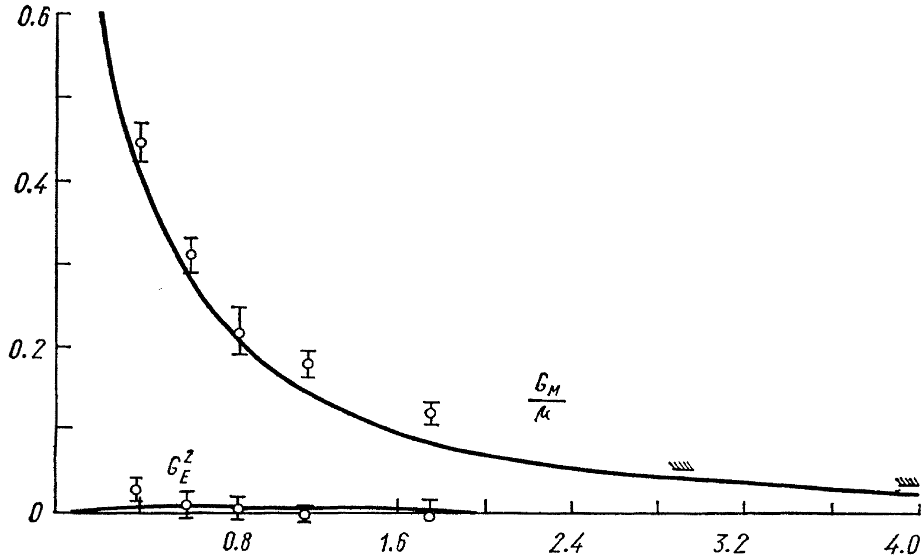


Fig. 23. Neutron magnetic and electric form factor fit.

The mass formula as a function of the principal quantum number n is the following

$$M_n^2 = (\alpha_3^2 + \alpha_2^2/n^2)^{-1} \left\{ \frac{1}{2}\alpha_1^2 + \beta\alpha_3 + \gamma\alpha_2/n^2 + \left[\left(\frac{1}{2}\alpha_1^2 + \beta\alpha_3 + \gamma\alpha_2/n^2 \right) - \left(\beta + \frac{\gamma^2}{n^2} \right) \left(\alpha_3^2 + \frac{\alpha_2^2}{n^2} \right) \right]^{\frac{1}{2}} \right\} \qquad (8.7)$$

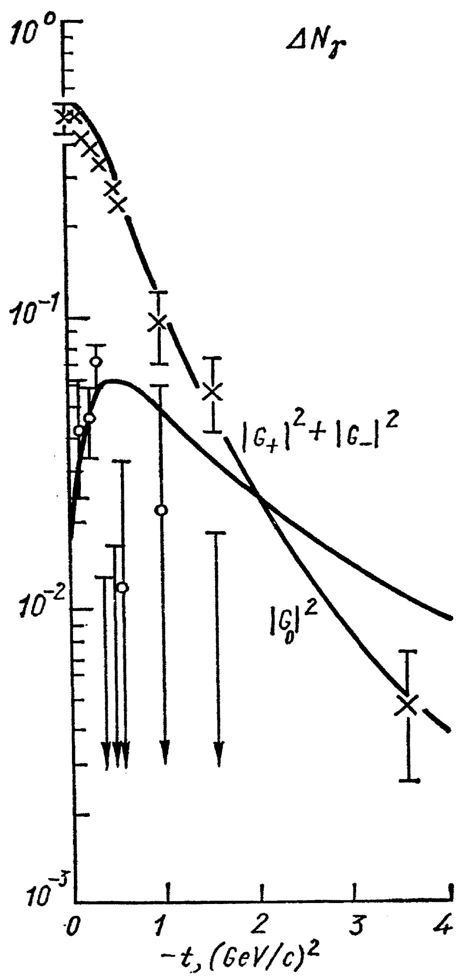


Fig. 24. The NN^* (1236) γ-transition form factors.

The spin-orbit type j-dependence of masses has been neglected but can be taken into account.

The parameters can be determined from the ground state (proton) mass, charge and magnetic moment and one point each on the mass spectrum and electric form factor curves and the slope of the form factor. Figs. 19 — 24 show the results [79 — 82].

A definite prediction is that **there are no nucleon-resonances beyond about 5 *GeV*,** the saturation point.

Table V

Magnetic Moments

	Pure SU (3)	O (4,2) $\times$ $\times$ SU (3)	Experimental
N_+	2.79	2.792	2.792763
N_0		—1.913	—1.913148
Ξ_0	—1.913	—2.027	
Ξ	—0.88	—0.707	
Λ	—0.957	—0.730	—0.73±0.16
Σ_0	0.957	0.865	
Σ_+	2.793	2.499	+2.5±0.5
Σ_-	—0.880	—0.629	

(C) **Inclusion of** SU **(3) — Quantum Numbers.** The simplest way is to assume the coefficients α_i, β, γ in the wave equation to be SU (3)-tensor operators, i. e. the rest frame group SO (4,2) $\otimes$ SU (3) $\otimes$ SU (3). From the transformation properties of currents one deduces that the parameters α_1, α_2 and α_3 are proportional to the charge operator Q, whereas α_4 and α_5 have a term proportional to charge Q, and another term proportional to Q' also transforming like Q (this is necessary to have non-vanishing form factors for neutral particles); β and γ are assumed to be the usual $Y = 0$, $I = 0$, $I_3 = 0$ tensor. It is then possible to

(I) account for the Gell-Mann — Okubo mass formula starting from (8.7), i. e. from the wave equation. In the wave equation, the current parameters α_i and β and γ are the fundamental quantities and mass is a derived quantity.

(II) account for the magnetic moments, mass splittings even within isomultiplets.

For further results concerning the partial strong decay rates of baryons and mesons, radiative decays of mesons, the pion form factor, etc. we refer to published literature [83—85].

2. REALIZATION IN TERMS OF MAGNETIC CHARGES

Is there a physical interpretation of the fermion representation of SO (4,2) with its characteristic dipole form factor? This is a vital question, because if the corresponding system is a composite system it is not an ordinary bound system of spin 0 and spin $\frac{1}{2}$-particles. The spin in our case belongs to the system as a whole and not to one of the constituents. The bound state of a spin 0 and spin $\frac{1}{2}$ particles would have a single-pole form factor, not a dipole [86]. We now show that the special fermion representation of SO (4,2) corresponds exactly to the dynamical group of **dyonium,** the bound state of two **spinless** particles having both electric and magnetic charges.

A particle carrying both an electric charge e, and a magnetic charge g, may be called a **dyon.** A spin $\frac{1}{2}$ dyon has also been identified with a quark [87]. We consider **spinless** dyons. The Maxwell — Dirac [88] electrodynamics is now specified by the equation

$$F'^{\nu}_{\mu\nu} = j^{(e)}_{\mu}, \quad \tilde{F}'^{\nu}_{\mu\nu} = j^{(m)}_{\mu}, \tag{8.8}$$

where we have introduced a **new magnetic axial-vector current** $j^{(m)}_{\mu}$. Both $j^{(e)}_{\mu}$ and $j^{(m)}_{\mu}$ are separately conserved. Equations (8.8) are invariant under the simultaneous chiral rotations in the two-dimensional space of electric and magnetic charges and fields, as shown in Fig. 25; only the relative angles α are observable.

The macroscopic matter is magnetically neutral. We can achieve this either (I) trivially if all magnetic charges are identically zero ($g_i = 0$), as was tacitly

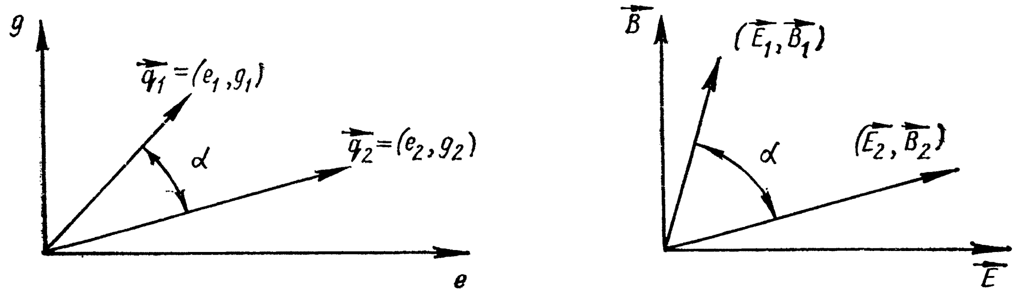


Fig. 25. Chiral invariance in the electric-magnetic plane.

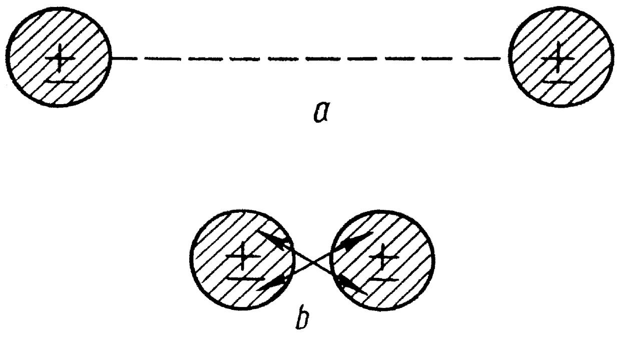


Fig. 26. Forces between two magnetically neutral dyoniums (a) Coulomb, (b) van der Waals.

assumed up to now, or (II) by assuming that the magnetic charges always occur in pairs $\pm g$, so that total $g = 0$, or (III) by assuming that, because g is an axial charge, we only have in strong interactions parity eigenstates

$$A_{\pm} = |e, g\rangle \pm |e, -g\rangle \tag{8.9}$$

in which the expectation value of the magnetic charge is zero. In the second case (II), with or without (III), we can say that, at large distances there is only the electric Coulomb force between the two particles, but at small distances there **must** be new, strong, short-ranged van der Waals type magnetic forces, as in the case of closed shell atoms (Fig. 26). Thus, had we started historically from the existence of magnetically charged pions we could predict the existence of strong interactions. The states (8.9) imply that the superselection rule on the magnetic charges would not hold [89]. The analogy with the neutral K-mesons is instructive, where in strong interactions the eigenstates of hypercharge Y are produced, K_0 and $\overline{K}_0$, and not K_1 and K_2 which are (approximate) eigenstates of CP.

The reason that magnetic charges have not been taken into account in the theory of strong interactions up to now is due to the fact that the following three points have been overlooked:

(1) The possibility of superpositions of the type (8.9) which would explain why magnetic charges are not seen readily.

(2) The magnetic charge g is an axial charge. In the Hamiltonian g should not be treated as a number, but as a discrete dichomic variable having two values $\pm g$. This resolves the difficulties of violation of parity P and time reversal T, because under P and T: $g \to -g$, and the Hamiltonians are invariant as we shall see. The use of g is very much like that of spin; we need a doubling of Hilbert space [90].

(3) The possibility of obtaining a spin $\frac{1}{2}$ — state out of two spinless particles with magnetic charges. This possibility follows from the so-called Dirac-quantization condition, which we explain in terms of the Hamiltonian of two interacting particles of charges $q_1 = (e_1, g_1)$ and $q_2 = (e_2, g_2)$. This Hamiltonian on the Klein — Gordon level is [91]

$$H^{(KG)} = [\pi^2 + m^2]^{\frac{1}{2}} - \alpha/r \tag{8.10}$$

where

$$\begin{aligned} \alpha &= -(e_1 e_2 + g_1 g_2), \\ \pi &= \mathbf{P} - \mu \mathbf{D}(\mathbf{r}), \\ \mu &= (e_1 g_2 - g_1 e_2), \\ \mathbf{D}(\mathbf{r}) &= \frac{\mathbf{r} \times \hat{\mathbf{n}} (\mathbf{r} \cdot \mathbf{n})}{r [r^2 - (\mathbf{r} \cdot \hat{\mathbf{n}})^2]} . \end{aligned} \tag{8.11}$$

The angular momentum that commutes with H and satisfies the commutation relations of the rotation group is

$$\mathbf{J} = \mathbf{r} \times \pi - \mu \hat{\mathbf{r}}. \tag{8.12}$$

Hence from the quantization of $\mathbf{J}$ in the direction of $\hat{r}$ we find that

$$\mu = 0, \ \pm\frac{1}{2}, \ \pm 1, \ \pm\frac{3}{2}, \ \dots \tag{8.13}$$

Each value of μ characterizes a new system and for each μ the possible values of angular momentum are

$$j = |\mu|, \ |\mu| + 1, \ |\mu| + 2, \ \ldots \tag{8.14}$$

For each μ the states are characterized by $|\mu; njm\rangle$, where n is the principal quantum number. Under parity

$$\mu \to -\mu$$

and parity eigenstates are $|\mu\rangle \pm |-\mu\rangle$. **For $\mu = \frac{1}{2}$ we thus have exactly the fermion representation of SO (4,2) used in the model of Sec. 1** [86, 91]. Note that the spin of the ground state $|\mu| = \frac{1}{2}$ does not belong to one of the constituents but to the system as a whole.

Consider now the following magnetically neutral dyoniums:

Case A: $\mu = 0$, $g_{\text{tot}} = g_1 + g_2 = 0$.

There are two solutions (1^0) $g_1 = g_2 = 0$: the system then corresponds to H-atom; (2^0) $g_1 = -g_2 \neq 0$; $e_1 = -e_2$. This is the dyon-antidyon system which has the quantum number of π^0-meson.

Case B: $\mu = \frac{1}{2}$, $g_{\text{tot}} = g_1 + g_2 = 0$.

There is only one solution: $g_1 = -g_2 \neq 0$; $e_1 + e_2 = e \neq 0$ hence $eg_2 = \frac{1}{2}$. This system clearly cannot be a dyon-antidyon; it has the quantum number of proton (or antiproton).

The interaction strength is then

$$\alpha = 137/4 \tag{8.15}$$

we thus have a **superstrong quantumelectrodynamics** and it is necessary to have a **relativistic** and **non-perturbative** treatment of the two-body problem. It is important to remark first that the proton according to this model does not have large electric and dipole magnetic moments, as might be expected at first sight [90]. This comes from the fact that the proton state is the positive parity state $|\mu\rangle + |-\mu\rangle$, hence the electric dipole moments cancel, and the fact that Bohr radius of the dyonium is $1/\alpha m \sim 10^{-16}$ whereas the classical radius of a single constituent is $\alpha/m \sim 10^{-13}$. This fact also explains why proton has the size 10^{-13} and not the classical radius of 10^{-16} (Fig. 27).

With a large value of α, Eq. (8.15), we expect strong pair production before ionization. A pair of emitted dyons are strongly bound and may be interpreted as a pion ($\mu = 0$). [Note that two representations of SO (4,2) with $|\mu| = \frac{1}{2}$ can give rise to $\mu = 0$, with $|\mu| = \frac{1}{2}$ and $|\mu| = 0$ can give rise to $|\mu| = \frac{1}{2}$ only, hence μ plays the role of baryon number as well, hence a physical explanation of Eq. (3.1).] We see that two dyons are as big as a single dyon, a pair emitted from a dyonium is as big as the emitter (Fig. 27), thus the size 10^{-13} (size of a dyon) occurs as an invariant magical size [67].

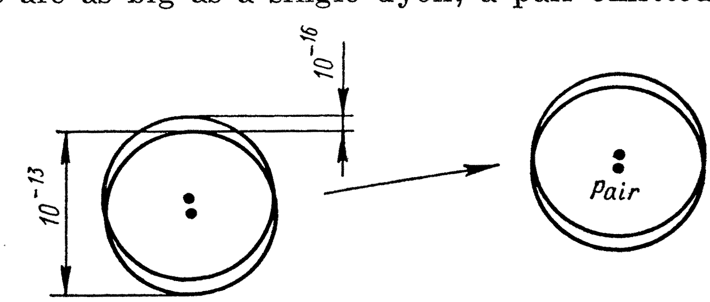


Fig. 27. Illustration of classical radius and Bohr radius of magnetically charged particles and size of the produced pair.

Because we do not have a tractable field theory with a large coupling constant 137/4, we treat the dyonium as a whole as a single relativistic particle according to our discussion in Ch. V—VI. We have seen that the relativistic mass

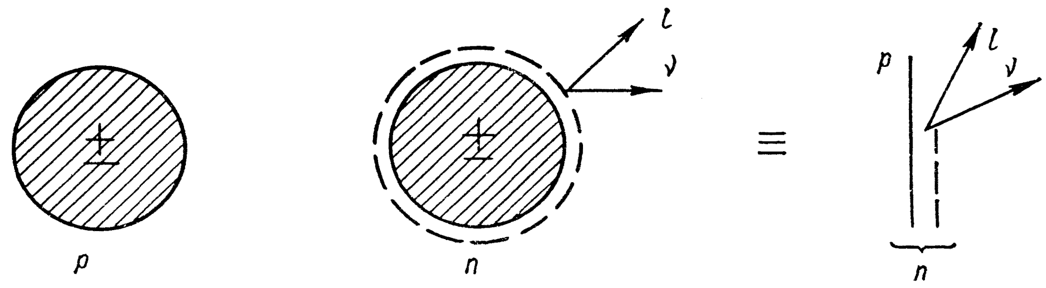


Fig. 28. Neutron model.

spectrum derived in Eq. (5.21) from infinite — component O (4,2) — equation (5.19) gives for large α a linear mass spectrum $M^2 \cong m_0^2 + \lambda N$, as a function of the principal quantum number N for not too large N. (See Eq. (5.26).) In these equations the Lamb-shift type effects were neglected, which may be very important for large α. For this reason, the more general equation with the same dynamical group O (4,2)-representation is Eq. (8.1) and (8.4). Hence we establish contact to the O (4,2)-model of hadrons. Unfortunately because of the new terms with coefficients α_4 and α_5 in (8.1)— which are missing in the H-atom equation — it is not possible at the moment to express the parameters α_i, β and γ of the wave equation in terms of the dyon masses and α, hence calculate the dyon masses precisely.

STRONG INTERACTION SYMMETRIES

Finally we discuss the problem of obtaining approximate symmetries of strong interaction, like isospin symmetry, from such a purely electromagnetic theory. The dyonium models so far discussed give us states identifiable with π^0, proton and antiproton. The neutron has the same strong interactions as the proton, but is otherwise quite a different particle. Consider the S-wave bound state of the dyonium («proton») with a spin 0 purely electric particle B^- (Fig. 28). We assume B^- to decay into a lepton and a neutrino (analogous to the hypothetical W-mesons of mediating the weak interactions, but spinless). At small distances the magnetic van der Waals-type of forces between proton and neutron or between two neutrons is now almost exactly the same as between two protons, up to electromagnetic B^--corrections. The electric form factor of the neutron is composed of two terms: the proton form factor minus that of the bound B^--particle which qualitatively accounts for the observed neutron-electric form factor [90]. The magnetic form factor of the neutron, due to the proton core, should be almost the same as that of proton. A consequence of this model of neutron is that the $n - p$ mass difference is not purely of electromagnetic origin, because of the B-meson. Furthermore, the lepton-neutrino pair is emitted at one point as the phenomenological V-A theory indicates.

IX. Some mathematical results

In this Chapter we report on some mathematical results pertaining to the symmetry problems.

A) Representations of Non-Compact Groups

All irreducible representations of **all** compact classical groups are in principle known. In contrast our knowledge of explicit representations of non-compact groups is in general fragmentary and rather incomplete. This area is mostly pursued by physicists rather than mathematicians; in fact the first representation of a non-compact group, namely **all** the representations of the Poincaré group, were found by Wigner [92]. The non-compact groups for which all representations are known are the type $SO(n, 1)$ or $SU(n, 1)$ [93] which include the Lorentz groups $O(2,1)$ [92] and $O(3,1)$ [95], and the De Sitter group $O(4,1)$ [96]. But even for the next simplest groups like $O(3,2)$ and the conformal group $SO(4,2) \sim SU(2,2)$ we do not have by far a complete list of all representations [97]. A. N. Leznov and M. V. Saveljev [98] claim to have now all representations of all non-compact group. It would be interesting to see their explicit results.

B) Mathematical Properties of the Internal Symmetry Breaking

Hadron internal symmetries are approximate, but the symmetry breaking follows well-defined rules: The (unknown) strong interaction Hamiltonian is a well defined tensor operator with respect to the internal symmetry group G and defines specific directions with respect to the group generators. Michel and Radicati [99] study in detail the mathematical characterization of these directions of symmetry breaking. They relate these directions to the extremal point of invariant functions by the theorem that «given a compact group G and a representation D, then invariant functions have extrema along special directions». For example, for $SO(2)$ acting on the 2-sphere $S_2 \subset E_3$, the two-poles are the special directions. For $SU(3)$ acting on $S_7 \subset E_8$, special directions are those of Q, Y and Cabibbo currents; the actual value of the Cabibbo angle is however undetermined — it is a dynamical question. In the case of $SU(3) \times SU(3) +$ + discrete symmetries and representation $(3,\bar{3}) \otimes (\bar{3},3)$ there are some new directions which might be associated with the direction of CP-violation.

C) Use of Non-Compact Groups in the Expansion of Scattering Amplitudes

The problem of various expansions of invariant scattering amplitude $A(s, t, \ldots)$ can be reduced mathematically to the problem of **harmonic analysis** on compact and non-compact group spaces, or on homogeneous spaces. Harmonic analysis deals with the expansion of functions over the group, $f(g)$, into a series (or integral) of irreducible representations of the group: $f(g) = \Sigma C_i D^i(g)$. For example, the ordinary partial wave expansion is a harmonic analysis on the surface of the sphere ($\mathbf{p}^2 =$ const) (i. e. homogeneous space $SO(3)/SO(2)$) when θ is identified with the scattering angle. Depending now how the variables of the amplitude $A(s, t, \ldots)$ are parametrized, we can perform a variety of harmonic analyses. For example, $p^2 = p_0^2 - \mathbf{p}^2$ is an equation of a hyperboloid on which the Lorentz group $SO(3,1)$ acts transitively, hence we can expand the amplitudes with respect to the representations of the homogeneous Lorentz group [100]. There are other possibilities [97]. The Lorentz group expansions can be used in the crossed as well as in the direct channels [101, 102]. An interesting result [102] is the expansion of the partial wave amplitudes in terms of the matrix elements of the Lorentz group (i. e. complete set of functions on a two-sheeted hyperboloid):

$$T_l(s) = \int_0^\infty A(s, \rho)\, \Phi_{00l}^{(\rho,0)}(-\xi_3)\, \Phi_{l00}^{(\rho,0)}(\xi_2)\, \rho^2 d\rho$$

where Φ's are known functions, ξ_i the Lorentz parameters. In this form the amplitudes have no kinematical singularities, correct threshold behavior factorized out, ($T_l(s) \to |\mathbf{p}'|^l |\mathbf{p}|^l$) and, in the l-plane $T_l(s) \underset{l\to\infty}{\to} B(s)\, e^{-l\frac{2m}{|\mathbf{p}|}}$.

1) *High-Lying Meson States.* A large number of high mass meson peaks are observed in the range $M \simeq 1700$ to $3600\ MeV$, the so-called $RSTU$ and various

X-peaks [6]. They all seem to be very narrow. The quantum numbers of these peaks are not established. If one plots M^2 versus a «peak number n» one obtains a linear curve up to X (2800).

Meson	ρ (767)	A_2 (1298)	R (1700)	S (1929)	T (2195)	U (2382)	X_7 (2620)	X_8 (2800)
«n»	1	2	3	4	5	6	7	8

Beyond this point, however, the linearity does not seem to hold any longer. In fact if we continue to assign the number n in integers for the following peaks

Meson	X_9 (2880)	X_{10} (3025)	X_{11} (3075)	X_{12} (3145)	X_{13} (3180)
«n»	9	10	11	12	13

we obtain a mass-spectrum that tends to a saturation, as in the model of Ch. VIII. The number «n» may indeed be identified with the principal quantum number.

2) Infinite Multiplets and Regge Behaviour. The connection between infinite-multiplets of hadron states and the Regge asymptotic behaviour of some hadron-hadron amplitudes mentioned in Ch. V, can be made precise by considering specific models.

L. van Hove considers the exchange of an infinite number of particles of spin $J = 0, 1, 2, \ldots$ and masses $M(J)$ between two spinless particles in the t-channel [103]. The amplitude is then of the form

$$A = \sum_J (2J+1)\, b(J, t)\, P_J(z_t) \frac{1}{2}(1 \pm e^{-i\pi J})\,[M^2(J) - t]^{-1}.$$

If $J = \alpha(t)$ is the solution of $M^2(J) = t$, then $[M^2(J) - t] \simeq (J - \alpha) \left.\frac{dM^2}{dJ}\right|_{J=\alpha}$ and performing a Sommerfeld — Watson transformation in the J-plane, we obtain from the leading term

$$A \sim -\pi b(\alpha, t)(\sin \pi\alpha)^{-1} P_\alpha(-z_t) \frac{d\alpha}{dt} \cdot \frac{1}{2}(1 \pm e^{-i\pi\alpha}) \xrightarrow[s\to\infty]{} (-s)^{\alpha(t)}.$$

The exchange of infinite multiplets has been studied further by a number of authors [104]. The van Hove model is incorporated in the dual models in the sense that the dual amplitude is a sum of infinitely many poles in the t-channel (and in the s-channel). The degeneracy of the 4-point Veneziano amplitude [25] is actually more than in the van Hove model, the former gives in fact a representation of $SO(4,2)$ (see Ch. V.1); the n-point dual amplitudes however have a much higher degeneracy [105] as we have noted.

Appendix

Rest frame states

$$|j_1 m_1 j_2 m_2\rangle = N a_1^{+\,j_1+m_1} a_2^{+\,j_2-m_1} b_1^{+\,j_2+m_2} b_2^{+\,j_2-m_2} |0\rangle$$

$$N^{-2} = (j_1+m_1)!\quad (j_1+m_1)!\quad (j_2-m_2)!$$

Operators **(A.1)**

$$L_{ij} = \frac{1}{2}(a^+\sigma_k a + b^+\sigma_k b) \equiv J_k$$

$$L_{i4} = -\frac{1}{2}(a^+\sigma_i a - b^+\sigma_i b) \equiv A_i$$

$$L_{i0} = -\frac{1}{2}(a^+\sigma_i Cb^+ - aC\sigma_i b) \equiv M_i$$

$$L_{i6} = \frac{1}{2i}(a^+\sigma_i Cb^+ + aC\sigma_i b) \equiv \Gamma_i$$

$$L_{46} = \frac{1}{2}(a^+Cb^+ + aCb) \equiv S \equiv \Gamma_4$$

$$L_{40} = \frac{1}{2i}(a^+Cb^+ - aCb) \equiv T$$

$$L_{06} = \frac{1}{2}(a^+a + b^+b + 2) \equiv \Gamma_0 \tag{A.2}$$

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

J = total spin; A = Lenz vector; M = generators of pure Lorentz transformations; Γ_μ = algebraic current operator; $S \equiv \Gamma_4$ = scalar generator, T = «tilter»; L_{ab}; $a, b = 1, 2, 3, 4, 0, 6$, generators of $O\,(4,2)$.

Representation

$$[Q, L_{ab}] = 0, \quad Q = \frac{1}{2}(a^+a - b^+b)$$

Eigenvalue of $Q = \nu = j_1 - j_2$ is an invariant.
L_{06} has eigenvalues $N = j_1 + j_2 + 1$
L_{12} has eigenvalues $m = m_1 + m_2$ (A.3)
L_{34} has eigenvalues $m_2 - m_1$

States can also be labeled by $|N\nu jm\rangle$
For a given N: $|\nu| \leqslant j \leqslant N - 1$.

Parity

$$P: a^+ \to b^+, \; b^+ \to -a^+.$$

$$P = e^{-\frac{\pi}{2}(a^+b - b^+a)} \tag{A.4}$$

Under P: $j_1 \to j_2$, hence $\nu \to -\nu$.

Eigenstates of parity

$$|N|\nu|jm\pm\rangle = \frac{1}{\sqrt{2}}[|N\nu jm\rangle \pm |N - \nu jm\rangle]. \tag{A.5}$$

DISCUSSION

B e g:

I am very confused by your identification of strong interactions with electromagnetism. If one accepts this identification, how can one understand the invariance of strong interactions under the isospin group?

B a r u t:

In this model isospin is not an intrinsic quantum number but an approximate result of the structure. As to how the isospin-symmetry of strong interactions might come about I refer to remarks at the end of Ch. VIII in my report.

Z w a n z i g e r:

I wish to emphasize that the validity of a relativistic quantum field theory of magnetic monopoles is independent of the consistency of nonrelativistic models involving monopoles. This is not the time or place to make specific criticism of such models.

Ogievetsky:

I would like to note that the pseudoscalar nonet should contain the E (1420) meson as its ninth member rather than the X (958) meson. This opinion is based on the mass rule of the algebraic realization of the unitary symmetry reported at this Conference and on the mass sum rule of the broken $SU\ (6)_w$. It should be stressed that the spin-parity of the X (958) has not been established firmly till now; it can be either 2^- or 0^- and 2^- seems to fit experimental data even better than 0^-.

J. G. Taylor:

I would like to comment that there are both parity and time reversal violations arising in the theory of magnetic monopoles. The parity problem can be removed by using an axial vector current as the source of the monopoles but the time reversal violation cannot. This makes me very suspicious of such a theory of matter; unless the problem is resolved such a theory is ruled out.

Barut:

In the dyonium model, under time reversal: $\mu \to -\mu$, and hence we have T-invariance.

Heusch:

Just a clarifying remark on our work mentioned by Dr. Barut.

It was not our intention to introduce amusing representations into the spectrum of observed states in the nonrelativistic quark model. Rather, starting from the observation that there are several nonstrange nucleonic states with the quantum numbers of the proton the question is followed up whether these might represent a radial sequence with $n = 0, 1, 2$. In this case, only $56, 0^+$ representations would be needed here. Instead, a simple calculation with a nonrelativistic Hamiltonian shows that the photoexcitation amplitude is much too small for the $n = 2\ 56,0^+$ to be reconciled with the experimental data. The only reasonable choice for the P_{11} states at 1460 and 1750 MeV is a linear combination of $n = 1$ radial excitation and two orbital excitations, populating the representations $56,0^+$ and $70,0^+$.

The conclusion is then in the framework of this very simple but very successful model: we do not see a radial excitation beyond $n = 1$, we do not see a candidate for a second daughter state in the form of the nucleon isobar P_{11} (1750).

REFERENCES

1. F. Lurcat and L. Michel, Nuovo Cimento **21**, 574 (1961).
2. (a) R. H. Dalitz, (I) Proc. Int. Conf. on Elem. Particles, Oxford (1965) (Rutherford Lab. Harwell 1966), p. 157; (II) Proc. XIIIth Int. Conf. on High Energy Phys., Berkeley (1966) (Univ. of Calif. Press, Berkeley 1967), p. 265.
(b) H. Harari, Proc. XIVth Int. Conf. on High. Energy Physics, Vienna (1968) (CERN, 1968), p. 195.
(c) G. Morpurgo, ibid, p. 225.
(d) O. W. Greenberg, Proc. Int. Conf. on Elementary Particles Lund (1969), p. 387.
3. (a) See the monograph by J. J. J. Kokkedee, The Quark Model W. A. Benjamin, Inc. Publ. (New York, 1969).
(b) R. H. Dalitz, Int. Conference on Symmetries and Quark Model (Wayne State Univ., (1969).
(c) A. N. Mitra, in Lectures in High-Energy Theoretical Physics, Ed. A. A. Aly (Gordon and Breach, London 1970).
4. For the background on $SU(3)$ see the books:
M. Gell-Mann and Y. Ne'eman, The Eightfold Way (W. A. Benjamin, N. Y. 1964); Y. Ne'eman, Algebraic Theory of Particle Physics, (W. A. Benjamin, N. Y., 1967); B. T. Feld, Models of Elementary Particles, (Blaisdell Publishing Co., 1969); J. Bernstein, Elementary Particles and Their Currents (W. H. Freeman, Co., 1968).
5. H. J. Lipkin, Proc. Coral Gables Conference on Fundamental Interactions, 1969 (Gordon and Breach, N. Y. 1969).
6. See A. Astier's Rapporteur talk in these Proceedings.
7. See K. Lubelsmeyer's Rapporteur talk in these Proceedings.
8. S. F. Tuan, Phys. Rev. Lett. **23**, 1198 (1969).
9. See N. P. Samios' Rapporteur's talk in these Proceedings.

10. S. F. Tuan (to be published).
11. See for example, G. Karl and E. Obryk, Nucl. Phys. **B8**, 609 (1968).
12. C. A. Heusch and F. Ravndal, Phys. Rev. Letters **25**, 253 (1970).
13. A. P. Roberts and M. Box, Phys. Rev. Letters **24**, 426 (1970).
14. B. T. Feld, Phys. Rev. **D2**, 952 (1970), and reply by A. P. Roberts and M. Box, Phys. Rev. D2, 953 (1970).
15. R. P. Feynman, S. Pakvasa and S. F. Tuan, Phys. Rev. D2, 1267 (1970).
16. A. N. Mitra, The Role of Form Factors in Hadron Resonances, Trieste preprint IC/70/58 and references therein.
17. A. O. Barut, Phys. Rev. **135**, B 839 (1964). A. O. Barut and A. Böhm, Phys. Rev. **139**, B 1107 (1965).
18. Y. Dothan, M. Gell-Mann, and Y. Ne'eman, Phys. Rev. Letters **17**, 148 (1965).
19. N. Mukunda, L. O'Raifeartaigh and E. C. G. Sudarshan, Phys. Rev. Letters **19**, 322 (1965).
20. T. Takabayashi, Prog. Theor. Phys. Suppl. (Kyoto), Extra number 339 (1965).
21. P. Budini and C. Fronsdal, Phys. Rev. Letters **14**, 968 (1965).
22. A. O. Barut, Phys. Rev. **156**, 1538 (1967). See also: The Formulation of Quantum Dynamics in Terms of Generalized Symmetries, Trieste Notes IC/68/104, 1968 (unpublished).
23. I. A. Malkin, V. I. Man'ko and D. A. Trifonov, Phys. Letters **30A**, 414 (1969); JETP **58**, N2 (1970), and Lebedev Inst. preprints (1970). See also V. S. Popov and A. M. Perelomov, Soviet Phys. JETP **29**, 738 (1969).
24. Y. Dothan, preprint (Institute for Advanced Study, Princeton, 1970).
25. G. Veneziano, Nuovo Cimento **57A**, 190 (1968).
26. A. O. Barut and J. W. Moffat, Phys. Rev. **D1**, 532 (1970).
27. G. Domokos, Phys. Rev. **159**, 1387 (1967). G. Domokos and P. Suranyi, Nuovo Cimento **56A**, 443 (1968). G. L. Tindle, Phys. Rev. **165**, 1906 (1968).
28. D. Z. Friedman and J. M. Wang, Phys. Rev. **153**, 1596 (1967). G. Cosenza, A. Sciarrino and M. Toller, Nuovo Cimento **57A**, 283 (1968). R. Delbourgo, A. Salam, and J. Strathdee, Phys. Rev. **172**, 1727 (1968).
29. A. Böhm, Phys. Rev. **145**, 1212 (1966). A. Böhm, in Lectures in Theoretical Physics, Vol. XB (Gordon and Breach, N. Y., 1968).
30. H. Abarbanel and Y. Frishman, Phys. Rev. **171**, 1442 (1968).
31. A. Chodos, Phys. Rev. **D1**, 2937 (1970); A. Chodos and R. W. Haymaker, Phys. Rev. **D2**, 793 (1970).
32. B. Kursunoǧlu, Phys. Rev. **D1**, 1115 (1970); **D2**, 117 (1970). See also the contributions of Kursunoǧlu and Perlmutter in Proc. Second Coral Gables Conference on Fundamental Interactions (Gordon and Breach, 1970).
33. E. Abers, I. T. Grodsky and R. E. Norton, Phys. Rev. **159**, 1222 (1967); C. Fronsdal, Phys. Rev. **156**, 1653 (1967).
34. G. Feldman and P. T. Mathews, Phys. Rev. **151**, 1176 (1966); **154**, 1241 (1967); Ann. of Physics 40, 19 (1966).
35. I. T. Grodsky and R. F. Streater, Phys. Rev. Letters **20**, 695 (1968). A. I. Oksak and I. T. Todorov, Comm. Math. Phys. **14**, 271 (1969) and Phys. Rev. **D1**, 3511 (1970).
36. W. Heisenberg, The Physical Principles of Quantum Theory (Dover Pub., 1930).
37. A. O. Barut, Phys. Rev. Letters **20**, 893 (1968). A. O. Barut, P. Cordero, and G.—C. Ghirardi, Phys. Rev. **D1**, 536 (1970).
38. E. Majorana, Nuovo Cimento **9**, 335 (1932). I. M. Gel'fand and Ya. M. Yaglom, Zh. Exper. i. Teoret. Fiz. **18**, 703 and 1096 (1948).
39. These were first pointed out by V. Bargmann, Math. Reviews **10**, 583, 584 (1949). See also W. Rühl, Comm. Math. Phys. **6**, 312 (1967).
40. A. O. Barut and H. Kleinert, Phys. Rev. **156**, 1546 (1967).
41. L. C. Biedenharn and A. Giovannini, Nuovo Cimento **51A**, 870 (1967). K. Morita, Nagoya Univ. preprints (1969).
42. A. O. Barut and I. H. Duru (to be published).
43. A. A. Komar and L. M. Slad, Journ. Theor. Math. Phys. USSR **1**, 50 (1969).
44. A. O. Barut, D. Corrigan and H. Kleinert, Phys. Rev. **167**, 1527 (1968). Other generalizations are given in this paper.
T. Takabayasi, Prog. Theor. Phys. **42**, 423 (1969).
Y. Nambu, Prog. Theor. Phys. Suppl. (Kyoto) **37** and **38**, 368 (1966), Phys. Rev. **160**, 1171 (1967).
45. References on the dynamical group treatment of the non-relativistic H-atom are:
C. Fronsdal, Phys. Rev. **156**, 1665 (1967).
A. O. Barut and H. Kleinert, Phys. Rev. **156**, 1541, 1180 (1967), **160**, 1149 (1967).
Y. Nambu, in Proc. Int. Conference on Particles and Fields (Interscience 1967).

See also the review: H. Kleinert, Lectures in Theoretical Physics Vol. XB (Gordon and Breach, 1968).
46. Relativistic equations of the type (5.19) were first used in connection with hadron models: A. O. Barut, D. Corrigan and H. Kleinert, Phys. Rev. Letters **20**, 167 (1968); Phys. Rev. **167**, 166 (1968).
For the application to relativistic H-atom see A. O. Barut and A. Baiquni, Phys. Rev. **184**, 1342 (1969); and Physics Letters **30A**, 352 (1969).
47. E. Brezin, C. Itzykson and J. Zinn-Justin, Phys. Rev. **D1**, 2349 (1970).
48. C. Itzykson, V. G. Kadyshevsky and I. T. Todorov, Phys. Rev. **D1**, 2823 (1970).
49. A. O. Barut and G. Bornzin, J. Math. Phys. (in press).
50. See also B. Zumino's Rapporteur talk in these Proceedings.
B. Renner, Current Algebra (Pergamon, 1969).
S. Adler and R. Dashen, Current Algebras (W. A. Benjamin, N. Y., 1968).
51. J. Grodnik and D. H. Sharp, Phys. Rev. **D1**, 1531 and 1546 (1970).
52. A. O. Barut and G. J. Komen, Phys. Rev. **D1**, 418 (1970).
53. H. Leutwyler, Phys. Rev. Letters **20**, 561 (1968).
H. Bebié, F. Ghielmetti, V. Gorgé and H. Leutwyler, Phys. Rev. **177**, 2133 and 2146 (1968).
B. Hamprecht and H. Kleinert, Phys. Rev. **180**, 1410 (1969).
H. Kleinert, D. Corrigan and B. Hamprecht, Springer Tracts in Modern Physics, Vol. 49.
G. Cocho, C. Fronsdal, R. White, Phys. Rev. **180**, 1547 (1969).
C. Fronsdal and A. M. Harun-Ar Rashid, Trieste preprint 1969.
54. S. J. Chang, R. Dashen and L. O' Raifeartaigh, Phys. Rev. **182**, 1805 and 1819 (1969). See also the review by L. O'Raifeartaigh, Proc. First Coral Gables Conference on Fundamental Interactions (Gordon and Breach, 1969).
55. U. H. Niederer, L. O' Raifeartaigh, Dublin preprint 1970.
56. A. O. Barut, Lett. Nuovo Cimento **1**, 601 (1969).
57. R. Mendez and Y. Ne'eman, Tel-Aviv preprint, TAUP—165—70.
See also A. Joseph, Tel-Aviv preprint, TAUP—120—20.
58. S. Weinberg, Phys. Rev. **177**, 2604 (1968).
59. M. Noga, in Lectures in Theoretical Physics, Vol. XIII (Gordon and Breach, N. Y., 1971).
J. Katz and M. Noga, Phys. Rev. **D2**, 1657 (1970).
60. G. Furlan, F. Lannoy, C. Rossetti and G. Segré, Nuovo Cimento **40A**, 597 (1965).
61. S. Weinberg, Phys. Rev. Letters **22**, 1023 (1969).
62. S. Fubini and G. Furlan, Lettere al Nuovo Cimento **3**, 168 (1970).
63. P. V. Collins and B. Renner, Nucl. Phys. **B23**, 436 (1970).
64. J. Noga, Purdue preprint 1970.
65. F. Buccella, E. Celeghini, E. Sorace, Nuovo Cimento Lett. **1**, 556 (1969); **2**, 571 (1969).
66. F. Buccella, E. Celeghini, H. Kleinert, C. Savoy and E. Sorace, Nuovo Cimento **69A**, 133 (1970).
67. T. T. Chow and C. N. Yang, Phys. Rev. **170**, 1591 (1968). See also the invited talk by C. N. Yang at this conference and the related theory by R. J. Glauber, Proc. Conference on High Energy Physics and Nuclear Structure (edited by G. Alexander) (North-Holland Publ. Co., Amsterdam, 1967), pp. 311—338.
68. S. Depaquit, Ph. Gueret and J. P. Vigier, Institut H. Poincaré preprint 1970.
69. R. P. Feynman, Proc. Intern. Conference on High Energy Physics, Stony Book, 1969. Phys. Rev. Letters **23**, 1415 (1969).
70. See the Rapporteur talk by G. Veneziano in these Proceedings.
71. L. Clarelli and P. Ramon, Phys. Rev. **D2**, 973 (1970).
72. R. Cutkosky, Phys. Rev. **131**, 1888 (1963).
73. R. H. Capps, Phys. Rev. **D2**, 780 (1970).
74. See A. Tavkhelidze's Rapporteur talk in these Proceedings.
75. M. B. Mensky, Yadernaya Fizika, Journ. of Nucl. Physics 9, 1293 (1969). A. Inomata, Proc. Conf. on De Sitter and Conformal Groups, Boulder, 1970, Lectures in Theoretical Physics Vol. XIII (Gordon and Breach, N. Y., 1971).
76. C. J. Isham, Abdus Salam and J. Strathdee, Trieste preprint IC/70/108. See also A. Salam's invited talk at this conference.
77. For this assignment see A. O. Barut, Physics Letters **26B**, 308 (1968).
78. See R. Plano's Rapporteur's talk in these Proceedings.
79. A. O. Barut, D. Corrigan and H. Kleinert, Phys. Rev. Letters **20**, 167 (1968).

80. D. Corrigan and B. Hamprecht, Lectures in Theoretical Physics, Vol. XIA, Part I (Gordon and Breach, N. Y., 1969), p. 299.
H. Kleinert, Springer Tracts of Modern Physics, Vol. **49**, 90 (1969).
81. D. Corrigan, B. Hamprecht and H. Kleinert, Nucl. Phys. **11B**, 1 (1969).
82. A. O. Barut and W. Monsma (to be published).
83. A. O. Barut and H. Kleinert, Phys. Rev. Letters **18**, 745 (1967).
84. A. O. Barut and K. C. Tripathy, Phys. Rev. Letters **19**, 918 and 1081 (1967), Nucl. Phys. **B7**, 125 (1968); Phys. Rev. **178**, 2278 (1969).
85. A. O. Barut, Nucl. Phys. **B4**, 455 (1968); A. O. Barut and S. Malin, Nucl. Phys. **B9**, 194 (1969).
86. See the discussion in A. O. Barut, Proceedings of the Second Coral Gables Conference on Fundamental Interactions at High Energy (Gordon and Breach, N. Y., 1970), p. 199—220, and Phys. Rev.
87. L. I. Schiff, Phys. Rev. Letters **17**, 714 (1967); Phys. Rev. **160**, 1257 (1967). J. Schwinger, Science **165**, 757 (1969). See also M. Y. Han and L. C. Biedenharn, Phys. Rev. Lett. **24**, 118 (1970).
88. P. A. M. Dirac, Proc. Roy. Soc. **A133**, 60 (1931); Phys. Rev. **74**, 817 (1948).
89. A. O. Barut, Superselection Rule on Magnetic Charges and the Search for Magnetic Monopoles, submitted to Phys. Rev.
90. A. O. Barut, Atoms with Magnetic Charges as Models of Hadron, in Topics in Modern Physics, E. U. Condon Volume (Univ. of Colorado Press, 1971).
91. A. O. Barut and G. Bornzin, J. Math. Phys. (May 1971). (References to many earlier papers can be found in 86—91.)
92. E. P. Wigner, Ann. Math. **40**, 19 (1939).
93. See for example U. Ottoson, Comm. Math. Phys. **8**, 228 (1968); **10**, 114 (1968).
94. V. Bargmann, Ann. of Math. **48**, 568 (1947). A. O. Barut and C. Fronsdal, Proc. Roy. Soc. (London) **A287**, 532 (1965).
95. M. A. Nai'mark, Linear Representations of the Lorentz Group (Pergamon, 1964).
96. J. Dixmier, Bull. Soc. Math. France **89**, 9 (1961).
97. For the latest results see Proceedings of the Conference on De Sitter and Conformal Groups, Boulder, 1970: Lectures in Theoretical Physics, Vol. XIII (Gordon and Breach, N. Y., (1971).
98. A. N. Leznov and M. V. Saveljev, TMP, **2**, 311 (1970).
99. L. Michel and L. A. Radicati, Annals of Physics (to be published); see also in Fifth Coral Gables Conference on Symmetry Principles at High Energy (W. A. Benjamin, N. Y., 1968), p. 19.
100. H. Joos, in Lectures in Theoretical Physics, Vol. VII (Univ. of Colo. Press, 1965); M. Toller, Nuovo Cimento **37**, 631 (1965). R. Delbourgo, A. Salam and J. Strathdee, Trieste preprint IC/67/21.
101. Ya. A. Smorodinski's invited talk at this conference; P. Winternitz, in Proc. Conference on De Sitter and Conformal Groups, Boulder 1970: Lectures in Theoretical Physics, Vol. XIII (Gordon and Breach, N. Y., 1971).
102. Yi. A. Verdiev, Dubna preprint 1970.
103. L. van Hove, Phys. Letters **24B**, 183 (1967).
R. Blanckenbecler, Proc. Conference on Fields and Particles, Rochester, 1967 (Interscience, 1967), and the remarks by R. P. Feynman in the discussion of this paper.
104. G. Cocho, Harun Ar-Rashid, Nuovo Cimento **47A**, 874 (1967).
C. Fronsdal, Phys. Rev. **168**, 1845 (1968).
K. Koller, Nuovo Cimento **54A**, 79 (1968).
R. Delbourgo, M. A. Rashid, Abdus Salam, J. Strathdee, Phys. Rev. **170**, 1477 (1968) and **172**, 1727 (1968).
Y. Matsumoto, Kyoto preprint 1970, KUNS —184.
105. G. Veneziano's Rapporteur's talk in these Proceedings.