Equivalence between Approximate Dynamic Inversion and Proportional-Integral Control

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***Abstract*—Approximate Dynamic Inversion (ADI) has been established as a method to control minimum-phase, nonaffine-in-control systems. Previous results have shown that for single-input nonaffine-in-control systems, every ADI controller admits a *linear* Proportional-Integral (PI) realization that is largely independent of the nonlinear function that defines the system. In this report, we first present an extension of the ADI method for single-input nonaffine-in-control systems that renders the closed-loop error dynamics independent of the reference model dynamics. The equivalent PI controller will be derived and both of these results are then extended to multi-input nonaffine-in-control systems.**

***Index Terms*—Dynamic inversion, feedback linearization, proportion-integral control, PI control.**

I. Introduction

DYNAMIC inversion (DI) or feedback linearization is a popular control design method that is well suited for minimum-phase nonlinear systems [1] [2, Chapter 13]. DI addresses the problem of designing a controller to transform a nonlinear system to a linear one by feedback. To overcome some limitations imposed by the requirements of exact linearization, approximate linearization has emerged as a viable alternative, where the problem is relaxed to enlarge the class of admissible controllers [3]. A notable departure from the approximate linearization literature is [4], where tracking control of *nonaffine-in-control* systems are considered.

In [4], an Approximate Dynamic Inversion (ADI) control law was proposed that drives a given minimum-phase nonaffine-in-control system towards a chosen *stable* reference model. The control signal was defined as a solution of "fast" dynamics, and Tikhonov's Theorem [2, Theorem 11.2, pp. 439 – 440] in singular perturbation theory was used to show that the control signal approaches the exact dynamic inversion solution and that the system states approach those of the reference model when the controller dynamics are made sufficiently fast.

In [5], the authors showed that for the single-input case, every ADI control law as formulated in [4] admits a *linear* Proportional-Integral (PI) *model reference* controller realization. The key characteristic of the equivalent PI controller is that it is largely independent of the system's nonlinearities, in contrast to the original ADI control law in [4]. However, when the controller has fast dynamics as required of the ADI method, the resulting PI controller is a high-gain controller with associated robustness problems [6].

Note that this equivalence holds only for the time response when applied to the *exact* system. The equivalence do not hold when applied to perturbed systems [7]. Even when restricted to *perfectly known* minimum-phase Linear Time-Invariant systems, the closed-loop systems differ in robustness properties [7].

This report extends the ADI method by decoupling the error dynamics specification from the reference model dynamics. This in essence decouples the "steady state" response specification from the transient response specification, when the reference model response is viewed as the "steady state" response. The derivation of the equivalent PI controller for this extension is very similar to [5]. A key result of this report is to extend the previous statements of PI and ADI equivalence to multi-input systems.

The report will proceed as follows. Section II presents the ADI extension and PI equivalent controller for single-input systems. The second (and last) section extends these results to multi-input systems.

II. Equivalence for Single-Input Systems

A. Nomenclature

In the sequel, italicized symbols (eg. x) denote scalars, boldface lowercase letters (eg. $\mathbf{x}$) denote column vectors, and boldface uppercase letters (eg. $\mathbf{A}$) denote matrices. Upright text subscripts (eg. $\mathbf{x}_\mathrm{r}$ with text subscript "r" to indicate state of reference model) are variable class indicators, and italicized subscript symbols (eg. $\mathbf{x}_\rho$ with subscript "ρ" to indicate the ρ-th element of the vector $\mathbf{x}$) are variables for numeric quantities.

B. Approximate Dynamic Inversion for Single Input Systems

Here, the ADI method [4] for single input systems is stated with a minor generalization, together with the main result. The proof in [4] applies with appropriate (trivial) substitutions, and will not be replicated here.

Consider an n-th order single-input nonaffine-in-control system of relative degree ρ, expressed in normal form

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{z}(t), u(t)), \quad \mathbf{x}(0) = \mathbf{x}_0,\\ \dot{\mathbf{z}}(t) &= \mathbf{g}(\mathbf{x}(t), \mathbf{z}(t), u(t)), \quad \mathbf{z}(0) = \mathbf{z}_0,\end{aligned} \tag{1a}$$

where

$$\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_\rho(t)]^\mathrm{T} \in \mathbb{R}^\rho,$$

$$\mathbf{f}(\mathbf{x}(t), \mathbf{z}(t), u(t)) = \begin{bmatrix} x_2(t)\\ \vdots \\ x_\rho(t) \\ f(\mathbf{x}(t), \mathbf{z}(t), u(t))\end{bmatrix} \in \mathbb{R}^\rho, \tag{1b}$$

for $(\mathbf{x}(t), \mathbf{z}(t), u(t)) \in D_\mathbf{x} \times D_\mathbf{z} \times D_u$, and the sets $D_\mathbf{x} \subset \mathbb{R}^\rho$, $D_\mathbf{z} \subset \mathbb{R}^{n-\rho}$ and $D_u \subset \mathbb{R}$ are domains containing the origins. Here, $[\mathbf{x}^\mathrm{T}(t), \mathbf{z}^\mathrm{T}(t)]^\mathrm{T}$ denotes the state vector of the system, $u(t)$ is the control input, and $f\colon D_\mathbf{x} \times D_\mathbf{z} \times D_u \mapsto \mathbb{R}$, $\mathbf{g}\colon D_\mathbf{x} \times D_\mathbf{z} \times D_u \mapsto \mathbb{R}^{n-\rho}$ are continuously differentiable functions of their arguments. Furthermore, assume that $\frac{\partial f}{\partial u}$ is bounded away from zero for $(\mathbf{x}(t), \mathbf{z}(t), u(t)) \in \Omega \subset D_\mathbf{x} \times D_\mathbf{z} \times D_u$, where Ω is a compact set. That is, there exists $b_0 > 0$ such that $|\frac{\partial f}{\partial u}| > b_0$ for all $(\mathbf{x}(t), \mathbf{z}(t), u(t)) \in \Omega$. Note that $|\frac{\partial f}{\partial u}| > b_0 > 0$ implies $\operatorname{sign}\left(\frac{\partial f}{\partial u}\right) \in \{-1, +1\}$ is a constant. In addition, assume that the function f cannot be inverted explicitly with respect to u.

It is desired for $\mathbf{x}(t)$ to track the states of a *stable* ρ-th order linear reference model described in the controllable canonical form

$$\dot{\mathbf{x}}_\mathrm{r}(t) = \mathbf{A}_\mathrm{r}\mathbf{x}_\mathrm{r}(t) + \mathbf{B}_\mathrm{r} r(t), \quad \mathbf{x}_\mathrm{r}(0) = \mathbf{x}_{\mathrm{r}0}, \tag{2a}$$

where

$$\mathbf{x}_\mathrm{r}(t) = [x_{\mathrm{r}1}(t), x_{\mathrm{r}2}(t), \ldots, x_{\mathrm{r}\rho}(t)]^\mathrm{T} \in \mathbb{R}^\rho, \tag{2b}$$

and the Hurwitz $\mathbf{A}_\mathrm{r}$ and column vector $\mathbf{B}_\mathrm{r}$ have the form

$$\mathbf{A}_\mathrm{r} = \begin{bmatrix} 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\\ -a_{\mathrm{r}0} & -a_{\mathrm{r}1} & \cdots & -a_{\mathrm{r}(\rho-1)}\end{bmatrix}, \quad \mathbf{B}_\mathrm{r} = \begin{bmatrix}0\\ \vdots\\ 0\\ b_\mathrm{r}\end{bmatrix}. \tag{2c}$$

Here, $r(t)$ is a continuously differentiable reference input signal, and $\mathbf{x}_\mathrm{r}(t)$ is the state of the reference model.

Let $\mathbf{e}(t) = \mathbf{x}(t) - \mathbf{x}_\mathrm{r}(t) \in \mathbb{R}^\rho$ be the tracking error signal, and let the desired *stable* error dynamics be specified by

$$\dot{\mathbf{e}}(t) = \mathbf{A}_\mathrm{e}\mathbf{e}(t), \tag{3}$$

where $\mathbf{A}_\mathrm{e}$ is Hurwitz and has identical structure as $\mathbf{A}_\mathrm{r}$, but with coefficients $a_{\mathrm{e}i}$ in place of $a_{\mathrm{r}i}$ for $i \in \{0, 1, \ldots, \rho-1\}$.

Observe that in [4], $\mathbf{A}_\mathrm{e}$ was set equal to $\mathbf{A}_\mathrm{r}$, while in the above, an independent Hurwitz matrix $\mathbf{A}_\mathrm{e}$ can be specified. In typical applications, $\mathbf{A}_\mathrm{r}$ and $\mathbf{B}_\mathrm{r}$ can be used to specify the desired *system* response to excitation $r(t)$, and $\mathbf{A}_\mathrm{e}$ can be used to independently specify the desired *error* dynamics. That is, how quickly the system response approaches that of the reference model. Thus the preceding is a slight generalization of the ADI as formulated in [4].

The open loop (time-varying) error dynamics are then given by the system

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \mathbf{f}(\mathbf{e}(t) + \mathbf{x}_\mathrm{r}(t), \mathbf{z}(t), u(t)) - \mathbf{A}_\mathrm{r}\mathbf{x}_\mathrm{r}(t) - \mathbf{B}_\mathrm{r} r(t),\\ \dot{\mathbf{z}}(t) &= \mathbf{g}(\mathbf{e}(t) + \mathbf{x}_\mathrm{r}(t), \mathbf{z}(t), u(t)),\end{aligned} \tag{4}$$

with initial conditions $\mathbf{e}(0) = \mathbf{e}_0$, $\mathbf{z}(0) = \mathbf{z}_0$. Define the selector vector

$$\mathbf{c} = [0, \ldots, 0, 1]^\mathrm{T} \in \mathbb{R}^\rho.$$

The ideal dynamic inversion control is then found by solving for $u(t)$ from the system of ρ equations

$$\mathbf{f}(\mathbf{x}(t), \mathbf{z}(t), u(t)) - \mathbf{A}_\mathrm{r}\mathbf{x}_\mathrm{r}(t) - \mathbf{B}_\mathrm{r} r(t) = \mathbf{A}_\mathrm{e}\mathbf{e}(t).$$

If this is a system of ρ arbitrary nonlinear equations, there are in general no solutions of $u(t)$ since there are more equations than available degrees of freedom. However, by the judicious choice of system, reference model and error dynamics representation, the first $(\rho-1)$ equations are automatically satisfied. Therefore, solving the above ρ equations reduces to solving the single equation

$$f(\mathbf{x}(t), \mathbf{z}(t), u(t)) - \mathbf{c}^\mathrm{T}\big(\mathbf{A}_\mathrm{r}\mathbf{x}_\mathrm{r}(t) + \mathbf{B}_\mathrm{r} r(t)\big) = \mathbf{c}^\mathrm{T}\mathbf{A}_\mathrm{e}\mathbf{e}(t), \tag{5}$$

resulting in the exponentially stable closed-loop tracking error dynamics (3). Since (5) cannot (in general) be solved explicitly for $u(t)$, the approximate dynamic inversion controller for the above formulation can be given in similar form to [4] as

$$\epsilon\dot{u}(t) = -\operatorname{sign}\left(\frac{\partial f}{\partial u}\right)\tilde{f}(t, \mathbf{e}(t), \mathbf{z}(t), u(t)), \tag{6a}$$

where

$$\begin{aligned}\tilde{f}(t,\mathbf{e}(t),\mathbf{z}(t),u(t)) &= f(\mathbf{e}(t)+\mathbf{x}_{\mathrm{r}}(t),\mathbf{z}(t),u(t)) \\ &\quad - \mathbf{c}^{\mathrm{T}}\big(\mathbf{A}_{\mathrm{r}}\mathbf{x}_{\mathrm{r}}(t)+\mathbf{B}_{\mathrm{r}}r(t)+\mathbf{A}_{\mathrm{e}}\mathbf{e}(t)\big),\end{aligned} \tag{6b}$$

for some initial control $u(0) = u_0$. Here, ϵ is a positive controller design parameter, chosen sufficiently small to achieve closed-loop stability and approximate dynamic inversion. Observe that (6) relaxes the requirement for exact dynamic inversion while increasing the control in a direction to reduce the discrepancy (5) so as to approach the exact dynamic inversion solution.

Let $u = h(t,\mathbf{e},\mathbf{z})$ be an isolated root of $\tilde{f}(t,\mathbf{e},\mathbf{z},u) = 0$. In accordance with the theory of singular perturbations [2, Chapter 11], the reduced system for (4) is

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \mathbf{A}_{\mathrm{e}}\mathbf{e}(t), && \mathbf{e}(0)=\mathbf{e}_0,\\ \dot{\mathbf{z}}(t) &= \mathbf{g}(\mathbf{e}(t)+\mathbf{x}_{\mathrm{r}}(t),\mathbf{z}(t),h(t,\mathbf{e}(t),\mathbf{z}(t))), && \mathbf{z}(0)=\mathbf{z}_0.\end{aligned}$$

With $v = u - h(t,\mathbf{e},\mathbf{z})$, and $\tau = t/\epsilon$, the boundary layer system is

$$\frac{dv}{d\tau} = -\operatorname{sign}\left(\frac{\partial f}{\partial u}\right)\tilde{f}(t,\mathbf{e},\mathbf{z},v+h(t,\mathbf{e},\mathbf{z})). \tag{7}$$

The main result of [4] for single-input systems, adapted for the generalization above, is stated below.

Theorem 1. *Assume that the following conditions hold for all* $(t,\mathbf{e},\mathbf{z},u-h(t,\mathbf{e},\mathbf{z}),\epsilon) \in [0,\infty)\times D_{\mathbf{e},\mathbf{z}}\times D_v\times[0,\epsilon_0]$ *for some domains* $D_{\mathbf{e},\mathbf{z}} \subset \mathbb{R}^n$ *and* $D_v \subset \mathbb{R}$*, which contain the origins.*

1) *On any compact subset of* $D_{\mathbf{e},\mathbf{z}}\times D_v$*, the functions* f *and* $\mathbf{g}$ *and their first partial derivatives with respect to* $(\mathbf{e},\mathbf{z},u)$*, and the first partial derivative of* f *with respect to* t *are continuous and bounded,* $h(t,\mathbf{e},\mathbf{z})$ *and* $\frac{\partial f}{\partial u}(t,\mathbf{e},\mathbf{z},u)$ *have bounded first derivatives with respect to their arguments,* $\frac{\partial f}{\partial \mathbf{e}}(t,\mathbf{e},\mathbf{z},h(t,\mathbf{e},\mathbf{z}))$ *and* $\frac{\partial f}{\partial \mathbf{z}}(t,\mathbf{e},\mathbf{z},h(t,\mathbf{e},\mathbf{z}))$ *are Lipschitz in* $\mathbf{e}$ *and* $\mathbf{z}$*, uniformly in* t*.*
2) *The origin is an exponentially stable equilibrium of the system*

$$\dot{\mathbf{z}}(t) = \mathbf{g}(\mathbf{x}_{\mathrm{r}}(t),\mathbf{z}(t),h(t,\mathbf{0},\mathbf{z}(t))).$$

The mapping $(\mathbf{e},\mathbf{z}) \mapsto \mathbf{g}(\mathbf{e}+\mathbf{x}_{\mathrm{r}}(t),\mathbf{z},h(t,\mathbf{e},\mathbf{z}))$ *is continuously differentiable and Lipschitz in* $(\mathbf{e},\mathbf{z})$ *uniformly in* t*.*
3) $(t,\mathbf{e},\mathbf{z},v) \mapsto |\frac{\partial f}{\partial u}(t,\mathbf{e},\mathbf{z},v+h(t,\mathbf{e},\mathbf{z}))|$ *is bounded from below by some positive number for all* $(t,\mathbf{e},\mathbf{z}) \in [0,\infty)\times D_{\mathbf{e},\mathbf{z}}$*.*

Then the origin of (7) *is exponentially stable. Moreover, let* Ω_v *be a compact subset of* R_v*, where* $R_v \subset D_v$ *denotes the region of attraction of the autonomous system*

$$\frac{dv}{d\tau} = -\operatorname{sign}\left(\frac{\partial f}{\partial u}\right)\tilde{f}(0,\mathbf{e}_0,\mathbf{z}_0,v+h(0,\mathbf{e}_0,\mathbf{z}_0)).$$

Then for each compact subset $\Omega_{\mathbf{e},\mathbf{z}} \subset D_{\mathbf{e},\mathbf{z}}$*, there exists a positive constant* ϵ_* *and* $T>0$ *such that* $\forall\ t\geq 0$, $(\mathbf{e}_0,\mathbf{z}_0)\in\Omega_{\mathbf{e},\mathbf{z}}$, $u_0-h(0,\mathbf{e}_0,\mathbf{z}_0)\in\Omega_v$*, and* $\forall\ \epsilon\in(0,\epsilon_*)$*, system* (1)*,* (6) *has a unique solution* $\mathbf{x}_\epsilon(t)$ *on* $[0,\infty)$ *and*

$$\mathbf{x}_\epsilon(t) = \mathbf{x}_{\mathrm{r}}(t) + O(\epsilon)$$

holds uniformly for $t\in[T,\infty)$*.*

A proof of Theorem 1 is provided in [4]. In summary, Theorem 1 states that when regularity assumptions on the system dynamics are satisfied to ensure existence and uniqueness of solutions, and system (1) is minimum phase and controllable, the ADI control signal $u(t)$ approaches that of the exact dynamic inversion solution, and the system states $\mathbf{x}(t)$ approaches and maintains within $O(\epsilon)$ of the reference model states $\mathbf{x}_{\mathrm{r}}(t)$ for a sufficiently small ϵ. See [4] for ways to verify the assumptions and further discussions.

C. Equivalent PI Controller

Here, we recall the main result of [5], which extends trivially for the above ADI generalization. For notational convenience in the sequel, let $\operatorname{sign}(\frac{\partial f}{\partial u}) = \alpha$.

Lemma 1. *For every Approximate Dynamic Inversion controller* (6) *with* $u(0)=u_0$*, there exists a* linear *Proportional-Integral* model reference *controller realization*

$$u(t) = -\frac{1}{\epsilon}\alpha\big(\mathbf{c}^{\mathrm{T}}\mathbf{e}(t)-g(t)\big), \tag{8a}$$

where

$$g(t) = \int_0^t \mathbf{c}^{\mathrm{T}}\mathbf{A}_{\mathrm{e}}\mathbf{e}(\lambda)\,d\lambda, \tag{8b}$$

and $g(0) = \mathbf{c}^{\mathrm{T}}\mathbf{e}(0)+\epsilon\alpha u_0$.

Proof: Observe that from (1), we have

$$f(\mathbf{x}(t),\mathbf{z}(t),u(t)) = \mathbf{c}^{\mathrm{T}}\dot{\mathbf{x}}(t), \tag{9}$$

and from (2), we have

$$\mathbf{c}^{\mathrm{T}}\big(\mathbf{A}_{\mathrm{r}}\mathbf{x}_{\mathrm{r}}(t)+\mathbf{B}_{\mathrm{r}}r(t)\big) = \mathbf{c}^{\mathrm{T}}\dot{\mathbf{x}}_{\mathrm{r}}(t). \tag{10}$$

Substituting (9) and (10) in the ADI control law (6) gives

$$\begin{aligned}\epsilon\dot{u}(t) &= -\alpha\mathbf{c}^{\mathrm{T}}\big(\dot{\mathbf{x}}(t)-\dot{\mathbf{x}}_{\mathrm{r}}(t)-\mathbf{A}_{\mathrm{e}}\mathbf{e}(t)\big),\\ &= -\alpha\mathbf{c}^{\mathrm{T}}\big(\dot{\mathbf{e}}(t)-\mathbf{A}_{\mathrm{e}}\mathbf{e}(t)\big),\end{aligned} \tag{11}$$

which can be integrated in time to yield (8). Setting $g(0) = \mathbf{c}^{\mathrm{T}}\mathbf{e}(0) + \epsilon\alpha u_0$ recovers the original controller initial value $u(0) = u_0$. ■

Note that $\alpha = \mathrm{sign}\big(\frac{\partial f}{\partial u}\big) \in \{-1, +1\}$ satisfy $\alpha^2 = 1$. It can be seen that the result (8) is a PI controller acting on the error between the system states and the states of the reference model. Furthermore, observe that when expressed in the error coordinates, $\mathbf{e}(t)$, the PI controller is not explicitly dependent on $\mathbf{A}_{\mathrm{r}}$ that specifies the reference model dynamics, in contrast to the form in [5]. This characteristic is the result of introducing the independent matrix $\mathbf{A}_{\mathrm{e}}$ for the error dynamics specification. Finally, it is important to note that (9) and (10) are true by definition, while (3) is a design specification that the controller attempts to achieve, so that (3) cannot be substituted in (11) for further simplifications. From (8), it is apparent that the PI controller attempts to achieve (3), which is equivalent to achieving (5).

The significance of Lemma 1 is threefold:

1) The PI controller allows a very simple *exact* realization of the ADI control law.
2) The PI controller is a *linear* realization of a (in general) nonlinear control law.
3) The PI controller realization is independent of the nonlinear function $f(\mathbf{x}(t), \mathbf{z}(t), u(t))$ in (1b), except for the sign of the control effectiveness, $\alpha = \mathrm{sign}\big(\frac{\partial f}{\partial u}\big)$.

The existence of a linear realization of a nonlinear control law hinges critically on the structure of the underlying system, reference model, error dynamics and control law. Note that the PI realization does not apply to the ADI variant in [8] as that variant uses the Jacobian map $\frac{\partial f}{\partial u}(\mathbf{x}(t), \mathbf{z}(t), u(t))$ in place of $\mathrm{sign}\big(\frac{\partial f}{\partial u}\big)$, which is constant by assumption. The main practical challenge when applying the method of [8] lies in obtaining a sufficiently accurate description of the Jacobian map, in addition to the nonlinear function $f(\mathbf{x}(t), \mathbf{z}(t), u(t))$ in (1b).

As an aside, adaptive variants to [4] have been proposed in [9]–[14] in which the nonlinear function $f(\mathbf{x}(t), \mathbf{z}(t), u(t))$ is assumed unknown. These variants attempt to estimate the *unknown* function $f(\mathbf{x}(t), \mathbf{z}(t), u(t))$, and construct an analogous ADI control law based on the estimate. Since the PI controller realizes the ADI control law exactly without explicit dependence on $f(\mathbf{x}(t), \mathbf{z}(t), u(t))$, it appears that these *approximation based* adaptive variants are unnecessary. Numerical results in [5], [15] show that for the single-input case, the PI controller achieves/exceeds the tracking performance of the adaptive variants proposed in [13], [14] respectively.

III. Equivalence for Multi-Input Systems

Here we extend the above results to multi-input nonaffine-in-control systems.

A. Approximate Dynamic Inversion for Multi-Input Systems

Here, the ADI method [4] for multi-input systems is stated with a minor generalization analogous to the single input case. Consider a multi-input nonlinear system expressed in normal form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{z}(t), \mathbf{u}(t)), \quad \mathbf{x}(0) = \mathbf{x}_0, \\ \dot{\mathbf{z}}(t) &= \mathbf{g}(\mathbf{x}(t), \mathbf{z}(t), \mathbf{u}(t)), \quad \mathbf{z}(0) = \mathbf{z}_0, \end{aligned} \tag{12a}$$

where

$$\begin{aligned} \mathbf{u}(t) &= [u_1(t), u_2(t), \ldots, u_m(t)]^{\mathrm{T}} \in \mathbb{R}^m, \\ \mathbf{x}(t) &= \left[\mathbf{x}_1^{\mathrm{T}}(t), \mathbf{x}_2^{\mathrm{T}}(t), \ldots, \mathbf{x}_m^{\mathrm{T}}(t)\right]^{\mathrm{T}} \in \mathbb{R}^{\rho}, \\ \mathbf{f}(\mathbf{x}(t), \mathbf{z}(t), \mathbf{u}(t)) &= \begin{bmatrix} \mathbf{f}_1(\mathbf{x}(t), \mathbf{z}(t), \mathbf{u}(t)) \\ \mathbf{f}_2(\mathbf{x}(t), \mathbf{z}(t), \mathbf{u}(t)) \\ \vdots \\ \mathbf{f}_m(\mathbf{x}(t), \mathbf{z}(t), \mathbf{u}(t)) \end{bmatrix} \in \mathbb{R}^{\rho}, \end{aligned} \tag{12b}$$

and for $k \in \{1, 2, \ldots, m\}$, with $\sum_{k=1}^{m} \rho_k = \rho$,

$$\begin{aligned} \mathbf{x}_k(t) &= [x_{k1}(t), x_{k2}(t), \ldots, x_{k\rho_k}(t)]^{\mathrm{T}} \in \mathbb{R}^{\rho_k}, \\ \mathbf{f}_k(\mathbf{x}(t), \mathbf{z}(t), \mathbf{u}(t)) &= \begin{bmatrix} x_{k2}(t) \\ \vdots \\ x_{k\rho_k}(t) \\ f_k(\mathbf{x}(t), \mathbf{z}(t), \mathbf{u}(t)) \end{bmatrix} \in \mathbb{R}^{\rho_k}. \end{aligned} \tag{12c}$$

Here, $[\mathbf{x}^{\mathrm{T}}(t), \mathbf{z}^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^n$ is the state of the n-th order system with m inputs, and $\mathbf{u}(t)$ is the vector of control inputs.

The objective is to design $\mathbf{u}(t)$ so that the state $\mathbf{x}(t)$ tracks the state $\mathbf{x}_{\mathrm{r}}(t)$ of the *stable* ρ-th order reference model in Brunovsky canonical form

$$\dot{\mathbf{x}}_{\mathrm{r}}(t) = \mathbf{A}_{\mathrm{r}}\mathbf{x}_{\mathrm{r}}(t) + \mathbf{B}_{\mathrm{r}}\mathbf{r}(t), \quad \mathbf{x}_{\mathrm{r}}(0) = \mathbf{x}_{\mathrm{r}0}, \tag{13a}$$

where

$$\begin{aligned} \mathbf{r}(t) &= [r_1(t), r_2(t), \ldots, r_m(t)]^{\mathrm{T}} \in \mathbb{R}^m, \\ \mathbf{x}_{\mathrm{r}}(t) &= \left[\mathbf{x}_{\mathrm{r}1}^{\mathrm{T}}(t), \mathbf{x}_{\mathrm{r}2}^{\mathrm{T}}(t), \ldots, \mathbf{x}_{\mathrm{r}m}^{\mathrm{T}}(t)\right]^{\mathrm{T}} \in \mathbb{R}^{\rho}, \end{aligned}$$

$$\mathbf{A}_{\mathrm{r}} = \begin{bmatrix} \mathbf{A}_{\mathrm{r}1} & & \\ & \ddots & \\ & & \mathbf{A}_{\mathrm{r}m} \end{bmatrix}, \quad \mathbf{B}_{\mathrm{r}} = \begin{bmatrix} \mathbf{B}_{\mathrm{r}1} & & \\ & \ddots & \\ & & \mathbf{B}_{\mathrm{r}m} \end{bmatrix}, \tag{13b}$$

and for $k \in \{1,2,\ldots,m\}$, with ρ_k as in (12),

$$\mathbf{x}_{\mathrm{r}k}(t) = [x_{\mathrm{r}k1}(t), x_{\mathrm{r}k2}(t), \ldots, x_{\mathrm{r}k\rho_k}(t)]^{\mathrm{T}} \in \mathbb{R}^{\rho_k},$$

$$\mathbf{A}_{\mathrm{r}k} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_{\mathrm{r}k1} & -a_{\mathrm{r}k2} & \cdots & -a_{\mathrm{r}k\rho_k} \end{bmatrix}, \ \mathbf{B}_{\mathrm{r}k} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_{\mathrm{r}k} \end{bmatrix}. \tag{13c}$$

Here, each $\mathbf{A}_{\mathrm{r}k}$ for $k \in \{1,2,\ldots,m\}$ is a specified Hurwitz matrix, $\mathbf{r}(t)$ is the vector of continuously differentiable reference inputs, and $\mathbf{x}_{\mathrm{r}}(t)$ is the state of the reference model.

The tracking error vector can be defined and decomposed in correspondence with the above system and reference model state variables as

$$\begin{aligned}\mathbf{e}(t) &= \mathbf{x}(t) - \mathbf{x}_{\mathrm{r}}(t) \\ &= \left[\mathbf{e}_1^{\mathrm{T}}(t), \mathbf{e}_2^{\mathrm{T}}(t), \ldots, \mathbf{e}_m^{\mathrm{T}}(t)\right]^{\mathrm{T}} \in \mathbb{R}^{\rho},\end{aligned}$$

where for $k \in \{1,2,\ldots,m\}$,

$$\begin{aligned}\mathbf{e}_k(t) &= \mathbf{x}_k(t) - \mathbf{x}_{\mathrm{r}k}(t) \\ &= [e_{k1}(t), e_{k2}(t), \ldots, e_{k\rho_k}(t)]^{\mathrm{T}} \in \mathbb{R}^{\rho_k},\end{aligned}$$

and for each $k \in \{1,2,\ldots,m\}$, $i \in \{1,2,\ldots,\rho_k\}$,

$$e_{ki}(t) = x_{ki}(t) - x_{\mathrm{r}ki}(t).$$

Let the desired *stable* error dynamics be specified by

$$\dot{\mathbf{e}}(t) = \mathbf{A}_{\mathrm{e}}\mathbf{e}(t), \tag{14}$$

where $\mathbf{A}_{\mathrm{e}}$ has identical structure as $\mathbf{A}_{\mathrm{r}}$, but with block entries $\mathbf{A}_{\mathrm{e}k}$ in place of $\mathbf{A}_{\mathrm{r}k}$ for $k \in \{1,2,\ldots,m\}$, each $\mathbf{A}_{\mathrm{e}k}$ is chosen to be Hurwitz, and has identical structure as $\mathbf{A}_{\mathrm{r}k}$, but with coefficients $a_{\mathrm{e}ki}$ in place of $a_{\mathrm{r}ki}$ for $i \in \{1,2,\ldots,\rho_k\}$. Similar to the single-input case, this extension allows the desired error dynamics to be specified independently of the reference model dynamics.

The open loop (time-varying) error dynamics are then given by the system

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \mathbf{f}(\mathbf{e}(t) + \mathbf{x}_{\mathrm{r}}(t), \mathbf{z}(t), \mathbf{u}(t)) - \mathbf{A}_{\mathrm{r}}\mathbf{x}_{\mathrm{r}}(t) - \mathbf{B}_{\mathrm{r}}\mathbf{r}(t), \\ \dot{\mathbf{z}}(t) &= \mathbf{g}(\mathbf{e}(t) + \mathbf{x}_{\mathrm{r}}(t), \mathbf{z}(t), \mathbf{u}(t)),\end{aligned} \tag{15}$$

with initial conditions $\mathbf{e}(0) = \mathbf{e}_0$, $\mathbf{z}(0) = \mathbf{z}_0$. For $k \in \{1,2,\ldots,m\}$, let

$$\mathbf{c}_k = [0,\ldots,0,1]^{\mathrm{T}} \in \mathbb{R}^{\rho_k}.$$

The ideal dynamic inversion control is found by solving for $\mathbf{u}(t)$ from the system of ρ equations

$$\mathbf{f}(\mathbf{x}(t), \mathbf{z}(t), \mathbf{u}(t)) - \mathbf{A}_{\mathrm{r}}\mathbf{x}_{\mathrm{r}}(t) - \mathbf{B}_{\mathrm{r}}\mathbf{r}(t) = \mathbf{A}_{\mathrm{e}}\mathbf{e}(t),$$

or equivalently, solving the system of m scalar equations

$$\begin{aligned}f_k(\mathbf{x}(t), \mathbf{z}(t), \mathbf{u}(t)) - \mathbf{c}_k^{\mathrm{T}}\left(\mathbf{A}_{\mathrm{r}k}\mathbf{x}_{\mathrm{r}k}(t) + \mathbf{B}_{\mathrm{r}k}r_k(t)\right)& \\ = \mathbf{c}_k^{\mathrm{T}}\mathbf{A}_{\mathrm{e}k}\mathbf{e}_k(t),&\end{aligned} \tag{16}$$

for $k \in \{1,2,\ldots,m\}$, resulting in the exponentially stable closed-loop tracking error dynamics (14). Since (16) cannot (in general) be solved explicitly for $\mathbf{u}(t)$, the approximate dynamic inversion controller for the above formulation can be given in similar form to [4] as

$$\epsilon\dot{\mathbf{u}}(t) = \mathbf{P}\tilde{\mathbf{f}}(t, \mathbf{e}(t), \mathbf{z}(t), \mathbf{u}(t)) \tag{17a}$$

where $\mathbf{P} \in \mathbb{R}^{m\times m}$ is a controller parameter, and for $k \in \{1,2,\ldots,m\}$, the k-th element of $\tilde{\mathbf{f}}(t, \mathbf{e}(t), \mathbf{z}(t), \mathbf{u}(t))$ is

$$\begin{aligned}\tilde{\mathbf{f}}_k(t, \mathbf{e}(t), \mathbf{z}(t), \mathbf{u}(t)) &= f_k(\mathbf{e}(t) + \mathbf{x}_{\mathrm{r}}(t), \mathbf{z}(t), \mathbf{u}(t)) \\ &\quad - \mathbf{c}_k^{\mathrm{T}}\left(\mathbf{A}_{\mathrm{r}k}\mathbf{x}_{\mathrm{r}k}(t) + \mathbf{B}_{\mathrm{r}k}r_k(t) + \mathbf{A}_{\mathrm{e}k}\mathbf{e}_k(t)\right),\end{aligned} \tag{17b}$$

for some initial control $\mathbf{u}(0) = \mathbf{u}_0$. Observe that for the multi-input case, there is no analogue to the sign of the control effectiveness, $\mathrm{sign}\left(\frac{\partial f}{\partial u}\right)$, and the controller design parameters are ϵ, $\mathbf{P}$, $\mathbf{A}_{\mathrm{r}k}$, $\mathbf{A}_{\mathrm{e}k}$ and $\mathbf{B}_{\mathrm{r}k}$ for $k \in \{1,2,\ldots,m\}$.

Let $\mathbf{u} = \mathbf{h}(t, \mathbf{e}, \mathbf{z})$ be an isolated root of $\tilde{\mathbf{f}}(t, \mathbf{e}, \mathbf{z}, \mathbf{u}) = \mathbf{0}$. The reduced system for (15) is

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \mathbf{A}_{\mathrm{e}}\mathbf{e}(t), & \mathbf{e}(0) &= \mathbf{e}_0, \\ \dot{\mathbf{z}}(t) &= \mathbf{g}(\mathbf{e}(t) + \mathbf{x}_{\mathrm{r}}(t), \mathbf{z}(t), \mathbf{h}(t, \mathbf{e}(t), \mathbf{z}(t))), & \mathbf{z}(0) &= \mathbf{z}_0.\end{aligned}$$

With $\mathbf{v} = \mathbf{u} - \mathbf{h}(t, \mathbf{e}, \mathbf{z})$, and $\tau = t/\epsilon$, the boundary layer system is

$$\frac{d\mathbf{v}}{d\tau} = \mathbf{P}\tilde{\mathbf{f}}(t, \mathbf{e}, \mathbf{z}, \mathbf{v} + \mathbf{h}(t, \mathbf{e}, \mathbf{z})). \tag{18}$$

The main result of [4] for multi-input systems, adapted for the above generalization, is stated below.

Theorem 2. *Let the following conditions hold for all* $(t, \mathbf{e}, \mathbf{u} - \mathbf{h}(t, \mathbf{e}, \mathbf{z}), \epsilon) \in [0, \infty) \times D_{\mathbf{e},\mathbf{z}} \times D_{\mathbf{v}} \times [0, \epsilon_0]$ *for some domains* $D_{\mathbf{e},\mathbf{z}} \subset \mathbb{R}^n$ *and* $D_{\mathbf{v}} \subset \mathbb{R}^m$ *that contain the origins.*

1) *On any compact subset of* $D_{\mathbf{e},\mathbf{z}} \times D_{\mathbf{v}}$*, the functions* $\mathbf{f}$ *and* $\mathbf{g}$ *and their first partial derivatives with respect to* $(\mathbf{e}, \mathbf{z}, \mathbf{u})$*, and the first partial derivative of* $\mathbf{f}$ *with respect to* t *are continuous and bounded,* $\mathbf{h}(t, \mathbf{e}, \mathbf{z})$ *and* $\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(t, \mathbf{e}, \mathbf{z}, \mathbf{u})$ *have bounded first derivatives with respect to their arguments,* $\frac{\partial \mathbf{f}}{\partial \mathbf{e}}(t, \mathbf{e}, \mathbf{z}, \mathbf{h}(t, \mathbf{e}, \mathbf{z}))$ *and* $\frac{\partial \mathbf{f}}{\partial \mathbf{z}}(t, \mathbf{e}, \mathbf{z}, \mathbf{h}(t, \mathbf{e}, \mathbf{z}))$ *are Lipschitz in* $\mathbf{e}$ *and* $\mathbf{z}$*, uniformly in* t*.*
2) *The origin is an exponentially stable equilibrium of the system*

$$\dot{\mathbf{z}}(t) = \mathbf{g}(\mathbf{x}_{\mathrm{r}}(t), \mathbf{z}(t), \mathbf{h}(t, \mathbf{0}, \mathbf{z}(t))).$$

The mapping $(\mathbf{e},\mathbf{z}) \mapsto \mathbf{g}(\mathbf{e}+\mathbf{x}_{\mathrm{r}}(t),\mathbf{z},\mathbf{h}(t,\mathbf{e},\mathbf{z}))$ *is continuously differentiable and Lipschitz in* $(\mathbf{e},\mathbf{z})$ *uniformly in* t.

3) *For every* $(t,\mathbf{e},\mathbf{z}) \in [0,\infty) \times D_{\mathbf{e},\mathbf{z}}$, *all the eigenvalues of*

$$\mathbf{P}\left[\frac{\partial \tilde{\mathbf{f}}}{\partial \mathbf{u}}(t,\mathbf{e},\mathbf{z},\mathbf{v}+\mathbf{h}(t,\mathbf{e},\mathbf{z}))\right]$$

have negative real parts bounded away from 0.

Then the origin of (18) *is exponentially stable. Moreover, let* $\Omega_{\mathbf{v}}$ *be a compact subset of* $R_{\mathbf{v}}$, *where* $R_{\mathbf{v}} \subset D_{\mathbf{v}}$ *denotes the region of attraction of the autonomous system*

$$\frac{d\mathbf{v}}{d\tau} = \mathbf{P}\tilde{\mathbf{f}}(0,\mathbf{e}_0,\mathbf{z}_0,\mathbf{v}+\mathbf{h}(0,\mathbf{e}_0,\mathbf{z}_0)).$$

Then for each compact subset $\Omega_{\mathbf{e},\mathbf{z}} \subset D_{\mathbf{e},\mathbf{z}}$, *there exist a positive constant* ϵ_* *and* $T>0$ *such that* $\forall t \geq 0$, $(\mathbf{e}_0,\mathbf{z}_0) \in \Omega_{\mathbf{e},\mathbf{z}}$, $\mathbf{u}_0 - \mathbf{h}(0,\mathbf{e}_0,\mathbf{z}_0) \in \Omega_{\mathbf{v}}$, *and* $\forall \epsilon \in (0,\epsilon_*)$, *system* (12), (17) *has a unique solution* $\mathbf{x}_\epsilon(t)$ *on* $[0,\infty)$ *and*

$$\mathbf{x}_\epsilon(t) = \mathbf{x}_{\mathrm{r}}(t) + O(\epsilon)$$

holds uniformly for $t \in [T,\infty)$.

In essence, Theorem 2 is the multi-input extension of Theorem 1. See [4] for ways to verify the assumptions and further discussions.

B. Equivalent PI Controller

Here, we extend the previous results to show that for the multi-input case, there also exists an equivalent PI controller.

Lemma 2. *For every Approximate Dynamic Inversion controller* (17) *with* $\mathbf{u}(0)=\mathbf{u}_0$, *there exist a* linear *Proportional-Integral* model reference *controller realization*

$$\mathbf{u}(t) = \frac{1}{\epsilon}\mathbf{P}\left(\mathbf{v}(t)-\mathbf{w}(t)\right) \tag{19a}$$

where

$$\mathbf{v}(t) = \left[\mathbf{c}_1^{\mathrm{T}}\mathbf{e}_1(t), \mathbf{c}_2^{\mathrm{T}}\mathbf{e}_2(t), \ldots, \mathbf{c}_m^{\mathrm{T}}\mathbf{e}_m(t)\right]^{\mathrm{T}} \in \mathbb{R}^m,$$

$$\mathbf{w}(t) = \begin{bmatrix} \int_0^t \mathbf{c}_1^{\mathrm{T}}\mathbf{A}_{\mathrm{e}1}\mathbf{e}_1(\lambda)\,d\lambda \\ \int_0^t \mathbf{c}_2^{\mathrm{T}}\mathbf{A}_{\mathrm{e}2}\mathbf{e}_2(\lambda)\,d\lambda \\ \vdots \\ \int_0^t \mathbf{c}_m^{\mathrm{T}}\mathbf{A}_{\mathrm{e}m}\mathbf{e}_m(\lambda)\,d\lambda \end{bmatrix} \in \mathbb{R}^m, \tag{19b}$$

and $\mathbf{w}(0) = \mathbf{v}(0) - \epsilon\mathbf{P}^{-1}\mathbf{u}_0$.

Proof: For $k \in \{1,2,\ldots,m\}$, we have from (12),

$$f_k(\mathbf{x}(t),\mathbf{z}(t),\mathbf{u}(t)) = \mathbf{c}_k^{\mathrm{T}}\dot{\mathbf{x}}_k(t), \tag{20}$$

and from (13), we have

$$\mathbf{c}_k^{\mathrm{T}}\left(\mathbf{A}_{\mathrm{r}k}\mathbf{x}_{\mathrm{r}k}(t) + \mathbf{B}_{\mathrm{r}k}r_k(t)\right) = \mathbf{c}_k^{\mathrm{T}}\dot{\mathbf{x}}_{\mathrm{r}k}(t). \tag{21}$$

Substituting (20) and (21) in the multi-input ADI control law (17) gives

$$\epsilon\dot{\mathbf{u}}(t) = \mathbf{P}\begin{bmatrix} \mathbf{c}_1^{\mathrm{T}}\left(\dot{\mathbf{x}}_1(t) - \dot{\mathbf{x}}_{\mathrm{r}1}(t) - \mathbf{A}_{\mathrm{e}1}\mathbf{e}_1(t)\right) \\ \mathbf{c}_2^{\mathrm{T}}\left(\dot{\mathbf{x}}_2(t) - \dot{\mathbf{x}}_{\mathrm{r}2}(t) - \mathbf{A}_{\mathrm{e}2}\mathbf{e}_2(t)\right) \\ \vdots \\ \mathbf{c}_m^{\mathrm{T}}\left(\dot{\mathbf{x}}_m(t) - \dot{\mathbf{x}}_{\mathrm{r}m}(t) - \mathbf{A}_{\mathrm{e}m}\mathbf{e}_m(t)\right) \end{bmatrix} = \mathbf{P}\begin{bmatrix} \mathbf{c}_1^{\mathrm{T}}\left(\dot{\mathbf{e}}_1(t) - \mathbf{A}_{\mathrm{e}1}\mathbf{e}_1(t)\right) \\ \mathbf{c}_2^{\mathrm{T}}\left(\dot{\mathbf{e}}_2(t) - \mathbf{A}_{\mathrm{e}2}\mathbf{e}_2(t)\right) \\ \vdots \\ \mathbf{c}_m^{\mathrm{T}}\left(\dot{\mathbf{e}}_m(t) - \mathbf{A}_{\mathrm{e}m}\mathbf{e}_m(t)\right) \end{bmatrix},$$

which can be integrated in time to yield (19). Setting $\mathbf{w}(0) = \mathbf{v}(0) - \epsilon\mathbf{P}^{-1}\mathbf{u}_0$ recovers the original controller initial value $\mathbf{u}(0)=\mathbf{u}_0$. Since $\mathbf{P}$ must be chosen to satisfy assumption 3 of Theorem 2, all eigenvalues of $\mathbf{P}\left[\frac{\partial\tilde{\mathbf{f}}}{\partial\mathbf{u}}\right]$ have strictly negative real parts, which implies that $\mathbf{P}\left[\frac{\partial\tilde{\mathbf{f}}}{\partial\mathbf{u}}\right]$ have full rank. This in turn implies that $\mathbf{P}$ has full rank which ensures the existence of $\mathbf{P}^{-1}$. ■

In (19), the vector $\mathbf{v}(t)$ is formed from elements of the error vector $\mathbf{e}(t)$, and $\mathbf{w}(t)$ is formed from linear combinations of the integral of the error vector. It can be seen that (19) is a multi-input PI controller that realizes the ADI controller (17). Similar to the single-input case, we note that the PI realization does not apply to the ADI variant in [8].

IV. Conclusions

An extension of the Approximate Dynamic Inversion (ADI) method for minimum-phase nonaffine-in-control systems was presented that renders the error dynamics independent of the reference model dynamics. In essence, this decouples the "steady state" response specification from the transient response specification, where the "steady state" response is specified by the reference model dynamics while the transient response is *independently* specified by the error dynamics. It was shown that every ADI control law admits an equivalent *linear* Proportional-Integral (PI) controller realization that is largely independent of the nonlinearities of the system, for both single-input and multi-input systems.

We note that this equivalence must be interpreted with caution. As shown in [7], this equivalence holds only for the time response, and only when applied to the *exact* system. In particular, even when specializing to minimum-phase linear time-invariant systems, they differ in robustness properties.

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References

[1] S. Devasia, D. Chen, and B. Paden, "Nonlinear inversion-based output tracking," *IEEE Transactions on Automatic Control*, vol. 41, no. 7, pp. 930 – 942, Jul. 1996.

[2] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, 2002.

[3] G. O. Guardabassi and S. M. Savaresi, "Approximate linearization via feedback – an overview," *Automatica*, vol. 37, no. 1, pp. 1 – 15, Jan. 2001.

[4] N. Hovakimyan, E. Lavretsky, and A. Sasane, "Dynamic inversion for nonaffine-in-control systems via time-scale separation. part I," *Journal of Dynamical and Control Systems*, vol. 13, no. 4, pp. 451 – 465, Oct. 2007.

[5] J. Teo and J. P. How, "Equivalence between approximate dynamic inversion and proportional-integral control," in *47th IEEE Conference on Decision and Control*, Cancun, Mexico, Dec. 2008, to appear.

[6] P. V. Kokotovic and R. Marino, "On vanishing stability regions in nonlinear systems with high-gain feedback," *IEEE Transactions on Automatic Control*, vol. AC-31, no. 10, pp. 967 – 970, Oct. 1986.

[7] J. Teo and J. P. How, "On approximate dynamic inversion and proportional-integral control," in *American Control Conference*, St. Louis, MO, Jun. 2009, submitted.

[8] N. Hovakimyan, E. Lavretsky, and C. Cao, "Dynamic inversion of multi-input nonaffine systems via time-scale separation," in *Proceedings of the American Control Conference*, Minneapolis, MN, Jun. 2006, pp. 3594 – 3599.

[9] E. Lavretsky and N. Hovakimyan, "Adaptive dynamic inversion for nonaffine-in-control uncertain systems via time-scale separation. part II," *Journal of Dynamical and Control Systems*, vol. 14, no. 1, pp. 33 – 41, Jan. 2008.

[10] ——, "Adaptive dynamic inversion for nonaffine-in-control systems via time-scale separation: Part II," in *Proceedings of the American Control Conference*, Portland, OR, Jun. 2005, pp. 3548 – 3553.

[11] A. Young, C. Cao, V. Patel, N. Hovakimyan, and E. Lavretsky, "Adaptive control design methodology for nonlinear-in-control systems in aircraft applications," *Journal of Guidance, Control, and Dynamics*, vol. 30, no. 6, pp. 1770 – 1782, Nov./Dec. 2007.

[12] A. Young, C. Cao, N. Hovakimyan, and E. Lavretsky, "An adaptive approach to nonaffine control design for aircraft applications," in *AIAA Guidance, Navigation and Control Conference and Exhibit*, Keystone, CO, Aug. 2006, AIAA–2006–6343.

[13] N. Hovakimyan, E. Lavretsky, and C. Cao, "Adaptive dynamic inversion via time-scale separation," in *Proceedings of the 45th IEEE Conference on Decision & Control*, San Diego, CA, Dec. 2006, pp. 1075 – 1080.

[14] E. Lavretsky and N. Hovakimyan, "Adaptive compensation of control dependent modeling uncertainties using time-scale separation," in *Proceedings of the 44th IEEE Conference on Decision and Control and the European Control Conference*, Seville, Spain, Dec. 2005, pp. 2230 – 2235.

[15] J. P. How, J. Teo, and B. Michini, "Adaptive flight control experiments using RAVEN," in *Proceedings of the 14th Yale Workshop on Adaptive and Learning Systems*, New Haven, CT, Jun. 2008, pp. 205 – 210.