TE-17

OPTIMAL CONTROL IN THE PRESENCE OF MEASUREMENT UNCERTAINTIES

by

John Jacob Deyst, Jr.

January 1967

Degree of Doctor of Science

MASS. INST. OF TECHNOLOGY
S... 14 ...7
LIBRARIES

OPTIMAL CONTROL IN THE PRESENCE OF MEASUREMENT UNCERTAINTIES

by

JOHN JACOB DEYST, JR.

B. S., Massachusetts Institute of Technology, 1958

M. S., Massachusetts Institute of Technology, 1964

SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF SCIENCE
at the
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
January 1967

Signature of Author ______________________
Department of Aeronautics and Astronautics
January 1967

Certified by ______________________
Thesis Supervisor

Certified by ______________________
Thesis Supervisor

Certified by ______________________
Thesis Supervisor

Accepted by ______________________
Chairman, Departmental Committee on Graduate Students

MASS. INST. OF TECHNOLOGY
APR 17 1968
ENGINEERING LIBRARY

Thesis
Aero
1967
No. 6

OPTIMAL CONTROL IN THE PRESENCE

OF MEASUREMENT UNCERTAINTIES

by

John Jacob Deyst, Jr.

Submitted to the Department of Aeronautics and Astronautics on January 31, 1967 in partial fulfillment of the requirements for the degree of Doctor of Science.

ABSTRACT

This research is concerned with the optimal feedback control of linear stochastic systems. Particular emphasis is placed on the solution of problems for which the cost is nonquadratic and the plant state cannot be determined by the controller without error. The plant and feedback controller are treated as discrete systems in time. It is shown that the optimal feedback control can be determined as a function of the mean plant state, conditioned on the measurements available to the controller. Recursion formulas are derived which permit the determination of this optimal control function. The theory is applied to the problem of minimum fuel midcourse guidance of spacecraft. Both fixed time of arrival and variable time of arrival guidance schemes are investigated, and an Earth-Mars mission is used as the basis for computing numerical examples. It is found that the optimal control is determined by a threshold. If the estimated state lies outside the threshold, the optimal control corrects to the threshold. Inside the threshold the optimal control is zero. Minimum fuel lateral control of a re-entry vehicle is examined and a numerical solution, based on the Apollo re-entry system is presented. It is found that the optimal re-entry control is also determined by a threshold. Quadratic cost problems are treated, producing the discrete control/estimation separation theorem. The theory is then generalized to handle problems for which the system is continuous. A partial differential equation is derived which must be satisfied by the optimal control function. Solution of a quadratic cost problem is accomplished, producing the continuous control/ estimation separation theorem. A minimum energy problem with arbitrary terminal cost is treated and a particular closed form solution is obtained. Finally, the continuous analogue of the discrete variable time of arrival problem is examined.

Thesis Supervisors:

Professor James E. Potter

Title: Assistant Professor of Aeronautics and Astronautics, MIT

Professor Wallace E. Vander Velde

Title: Professor of Aeronautics and Astronautics, MIT

Professor Arthur E. Bryson

Title: Gordan McKay Professor of Engineering, Harvard

ACKNOWLEDGMENTS

The author wishes to thank his thesis committee; Prof. James E. Potter, committee chairman, whose work provided the basis and motivation for the thesis; Prof. Wallace E. Vander Velde whose understanding, guidance and encouragement were invaluable; and Prof. Arthur E. Bryson who generously contributed many hours of fruitful discussion and criticism. Their unfailing help and inspiration are deeply appreciated.

Special thanks are due to Prof. Walter Wrigley who provided guidance and encouragement throughout the author's doctoral program.

The author is indebted to Dr. Robert Stern and the staff of the Experimental Astronomy Laboratory who gave freely of their time and energy. Association with all these people has been a most rewarding experience.

Thanks are also due to Mrs. Francis Duff and Miss Lorraine Perrotta for typing the original manuscript, and to the staff of the Publications Department of the MIT Instrumentation Laboratory for preparation of the final draft for publication.

Finally the author wishes to thank his dear wife Mary for her patient understanding, help and encouragement throughout the oourse of this endeavor.

This research was supported by a grant from the National Aeronautics and Space Administration, NsG 254-62.

The publication of this thesis does not constitute approval by National Aeronautics and Space Adminstration or by the MIT Experimental Astronomy Laboratory of the findings or the conclusions contained therein. It is published only for the exchange and stimulation of ideas.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER I

INTRODUCTION

1.1 Background

The general field of system optimization has been highly developed in two distinct, but related areas, namely stochastic estimation theory and optimal control theory. Further, optimal control theory has evolved in two separate directions, deterministic optimal control and stochastic optimal control. As a basis for further discussion, some significant results obtained by investigators in these areas will be briefly described in the following three paragraphs.

Stochastic estimation theory attempts to solve the problem of estimating the state of a system so as to minimize a specified penalty. The penalty usually takes the form of the mean value of some function of the estimation error. This theory has been notably successful in solving problems for which the system is linear and the penalty is mean square estimation error. Wiener[75] developed the mathematical basis for the theory. He showed that the weighting function of the minimum mean square linear estimator satisfies an integral equation (the Wiener-Hopf equation). By specializing to stationary problems he was able to develop the method of spectrum factorization which provides a systematic approach to the solution of this equation. The Wiener-Hopf equation and the resulting optimum linear estimator involve only the second order statistics of the system. If the system state and the measurements available to the estimator are gaussian processes, then the minimum mean square linear estimator is optimum for a wide class of penalty functions[64]. In addition, in cases where the probability density of the system is not known, but only second order statistics are available, there is good justification for using the minimum mean square estimator because it is optimum for wide classes of penalty functions and density functions. Many authors have extended Wiener's original work to handle various nonstationary and/or nonlinear problems. Significant advances were made by Kalman[40], Kalman and Bucy[41] and independently Battin[5], who developed practical methods of solving nonstationary problems. By applying the state space approach to the estimation problem, they were able to define the optimum estimator in terms of difference or differential equations. Since the solution takes this form, realization of the actual estimator is a relatively straightforward problem in computer programming or analogue circuit design. More recently, some progress has been made in the area of nonlinear filtering[18, 38, 78] for cases in which the state or the measurements are nongaussian processes. At present however, no compact systematic approach to these problems has been found. The existing methods generally involve truncated series expansions of the posterior state probability density.

Deterministic optimal control theory is an outgrowth of the calculus of variations. The theory can be used, in principle, to design a controller that will drive a plant so that some specified cost function is minimized. In recent years significant contributions have been made by many authors. Methods have been developed that permit the solution of practical control problems which could not be solved using the classical methods. Two different approaches have been used in the development of these methods. Lawden[50], Breakwell[14], and others have invented ingeneous techniques which permit the relaxation of one or more of the conditions that must be met in the application of the classical calculus of variations. In general these techniques apply to specific problems or classes of problems. The other approach, taken by Pontryagin et. al.[59] has produced a general extension of the classical calculus of variations, in the form of the well-known maximum principle. The application of this result allows solution of a wide class of optimal control problems. In addition to these analytical methods, a number of numerical techniques have been developed. Kelley[51] and Bryson[16] developed the method of steepest descents which is extremely useful for determining the optimal control as a function of time. By contrast, Bellman's method of dynamic programming[6, 22] can provide the optimal feedback control; although the computational requirements involved in this method are far more stringent than for steepest descents.

Stochastic optimal control theory attacks the problem of controlling a plant, in the presence of random input disturbances, so as to minimize a mean cost function. The parameters of the plant, its state, and the statistics of the input disturbances are assumed to be known deterministically by the controller. This theory has its roots in the theory of Markov processes. There is extensive literature on Markov process, for example, Doob[21], Wax[74] and Stratonovich[67, 69]. Formulation of the stochastic optimal control problem as a decision process was accomplished by Bellman[6, 7, 8]. He applied the well known imbedding technique of dynamic programming to derive the recursion formulas which must be satisfied by the optimal control. In his development, Bellman assumes that the plant state is a Markov process and that the state can be measured by the controller without error. As a result of these assumptions the optimal control becomes a function of the known state. Only a few computer solutions of stochastic optimal control problems have been reported. Some interesting examples are given by Aoki[1, 2], Tung and Striebel[73] and Dreyfus[22].

A very common engineering problem is the design of a feedback controller to minimize a cost function when the plant state cannot be measured perfectly. The controller must then operate with partial information and elements of both stochastic estimation theory and stochastic control theory are necessary in determining the optimum system. In such cases it is appropriate to consider a mean cost function because the plant state, the control and the cost are all random processes. By specializing to problems for which the plant is a linear system and the cost function is a quadratic form, Gunkel and Franklin(30), Joseph and Tou(39), Florentine(26), Potter(60) and Tung(71) were able to prove a separation theorem. This theorem states that the problems of estimation and control may be solved independently. The estimator is designed by the methods of Kalman and Bucy(41). The feedback controller is designed using the calculus of variations to minimize the cost function, assuming that there is no uncertainty. The cascade combination of these two systems provides the optimum over-all feedback controller. However this theorem cannot be applied to many important problems for which the cost function is nonquadratic. For such cases Fel'dbaum(24) and Stratonovich(68) have shown that if there exists a finite set of sufficient statistics, which determine the expected cost to complete the process, then the optimal control becomes a function of these sufficient statistics. Stratonovich shows that the optimal control is obtained by solving recursion formulas for the cost function in the space of the sufficient statistics.

1.2 Description of the Problem

The primary goal of this research is to develop a systematic method of determining optimal feedback controllers when the plant is a linear stochastic system, the state cannot be measured without error, and for arbitrary cost functions . In particular, solutions to problems for which the cost function is not quadratic are desired. The approach will be first, to derive the conditions that must be satisfied by the optimal control and second, to apply these conditions to some example problems in the control and guidance of spacecraft. The object in solving example problems is to demonstrate the usefulness of the theory for providing practical engineering solutions and to try to provide some physical insight.

1.3 Synopsis

As an aide to the interested reader, the remaining chapters of the work are briefly summarized below. Each summary attempts to outline the content of the chapter and point out significant results.

CHAPTER 2 - Optimal Control of Discrete Stochastic Systems

The class of systems to be handled and the cost function to be minimized, are carefully defined. Equations are developed permitting calculation of an estimate of the plant state based on the history of measurements available to the controller. This estimate turns out to be the conditional mean of the plant state, based on the measurement history available to the controller. An expression for the posterior probability density of the state is also developed and it is shown that the estimated state is a sufficient statistic for determining this posterior density. Additional properties of the estimated state are determined in preparation for development of the optimality conditions. A minimum expected value function is defined as the minimum cost to finish a partially completed process. Recursion formulas are derived which must be satisfied by the minimum expected value function and it is shown that the minimum expected value function depends only upon the estimated state and time. Solution of the recursion formulas produces the optimal control as a function of the estimated state. A useful form for the recursion formulas is determined for purposes of digital computation. The recursion formulas are generalized to handle cases for which multiple measurements are taken by the controller between control application times. Proper terminal conditions are determined for cases in which the terminal control function is specified instead of a terminal cost function.

CHAPTER 3 - Applications of the Discrete Theory

The problem of minimum fuel, variable time of arrival, midcourse spacecraft guidance is described. Errors out of the plane of the reference trajectory are ignored. The optimality conditions of Chapter 2 are applied, producing a one dimensional recursion formula for the minimum expected value function. Necessary and sufficient conditions for the optimal control are developed. It is shown that the optimal control is determined by a threshold. If the estimated state lies outside the threshold, the optimal control drives the estimated state to the threshold. If the estimated state lies within the threshold, the optimal control is zero. Equations convenient for digital computation are obtained and a step by step procedure for calculating the optimal control function is described. A computed reference trajectory is described and the numerically calculated optimal control function for this trajectory is presented. It is found that applying two partial corrections very early in the flight can save an appreciable amount of midcourse guidance fuel. Monte Carlo simulations of the optimal control system are described and the computed cost probability distribution, using the optimal control, is presented. A comparison is made between the optimum and a near optimum linear controller and it is found that the near optimum linear controller does an extremely good job as compared to the optimum. As a second example, the minimum fuel, fixed time of arrival, midcourse guidance problem is formulated. Errors out of the plane of the reference trajectory are ignored. It is shown that this problem may be reduced to two state variables and necessary and sufficient conditions for the optimal control are derived. It is also shown that the optimal control is determined by regions in the estimated state space. If the estimated state lies in a certain region $\mathfrak{Z}(n)$ the optimal control is zero. Outside $\mathfrak{Z}(n)$ the optimal control drives the estimated state, in a specified direction, to a point on the boundary of $\mathfrak{Z}(n)$. Equations convenient for digital computation are developed and the step by step procedure for determining the optimal control functions is described. The reference trajectory of the previous example is used as the basis for solving an actual numerical problem. The calculated optimal control functions are presented and methods of implementing the optimal control on a guidance computer are briefly discussed. The third problem handled in this chapter is minimum fuel lateral guidance of an atmosphere re-entry vehicle. It is shown that this problem can be reduced to two state variables. Recursion formulas for the minimum expected value function are derived and a computation procedure for determining the optimal control functions is described. Characteristics of the Apollo spacecraft re-entry system are used as the basis for calculating an actual numerical solution. It is found that the optimal control is again determined by a threshold. If, for example, the vehicle is banked to the right, so that its lift vector is pointing to the right of the target, then if the estimated miss distance at the target lies to the right of the threshold, the optimal control will roll the vehicle to the left. If, however, the estimated miss distance lies to the left of the threshold, the optimal control is zero and the lift vector remains to the right. Similar conditions hold if the vehicle is banked initially to the left. The threshold values are calculated as functions of time along the reference trajectory. The final example treated in this chapter is the general quadratic cost problem. The solution is determined in terms of a matrix difference equation and constitutes an independent proof of the well known quadratic cost control/estimation separation theorem for discrete systems.

CHAPTER 4 - Optimal Control of Continuous Linear Stochastic Systems

A class of continuous systems and the accompanying cost function to be minimized, are carefully defined. A discrete system is described which, in the limit as the time step goes to zero, has the same statistics as the continuous system. A recursion formula for the discrete minimum expected value function is derived by the methods of Chapter 2. Taking appropriate limits, as the time step approaches zero, a partial differential equation is obtained for the continuous minimum expected value function. Solution of this equation, to satisfy the appropriate terminal condition, produces the optimal control function for the continuous problem. The various terms of this differential equation are briefly discussed.

CHAPTER 5 - Applications of the Continuous Theory

Three example problems are solved. The first is the general quadratic cost problem, and its solution produces the familiar control/estimation separation theorem for continuous systems. The second problem has a single state variable plant with quadratic penalty on the control and an arbitrary terminal cost function. By specializing so that the terminal control forces the terminal estimated state to lie in a specified interval, the solution of this problem can be written in terms of error functions (erfs). Finally, the continuous analogue of the variable time of arrival midcourse guidance problem is solved; for a special case in which the control is determined by a threshold that stays constant in time.

CHAPTER 6 - Conclusions, Contributions, and Recommendations

Aspects of the theory which are felt to be of value, as design tools, are discussed and the limitations of the theory are enumerated. Those parts of the research which the author feels are original are explained. Finally, some problems which cannot be handled by the present theory are described and areas of possible future research are outlined.

CHAPTER 2

OPTIMAL CONTROL OF DISCRETE LINEAR STOCHASTIC SYSTEMS

2.1 General Discusssion

The purpose of this chapter is to derive the optimality conditions for control of a discrete linear stochastic system. It will be shown that the optimal control is a function of the mean plant state, conditioned on the measurement history. Recursion equations will be derived for the minimum expected value function and solution of these recursion equations produces the optimal control as a function of the conditional mean state.

2.2 Problem Statement

It is assumed that the dynamics of the plant may be described by discrete linear equations. Consider the transition of the plant state vector from time t_n to time t_{n+1}. The transition is described by the linear vector equation.

$$x(n+1) = \Phi(n+1, n)\, x(n) + \theta(n+1, n)\, u(n) + v(n) \tag{2-1}$$

where

$$\begin{aligned} x(n) &\equiv \text{ state vector of dimension } k \\ u(n) &\equiv \text{ control vector of dimension } p \\ \Phi(n+1, n) &\equiv \text{ state transition matrix } (k \times k) \\ \theta(n+1, n) &\equiv \text{ control influence matrix } (k \times p) \end{aligned}$$

The initial state x(0) is a k vector of normally distributed random variables with known statistics and v(n) is a k vector of normally distributed random variables, independent of x(n) and u(n), with known statistics given by

$$
\begin{aligned}
E\left[v(n)\right] &= 0 \\
E\left[v(n)\, v^T(n)\right] &= V(n) \\
E\left[v(n)\, v^T(i)\right] &= 0 \qquad i \neq n
\end{aligned}
\tag{2-2}
$$

Control sets $\mathcal{U}(n)$ are defined so that problems involving constraints on the control may be handled. $\mathcal{U}(n)$ is an arbitrary set in the space of the control vectors and control vectors which are elements of $\mathcal{U}(n)$ are said to be admissible controls. The set $\mathcal{U}(n)$ can depend upon parameters other than time; for example, it may depend in some way upon the history of the control up to time t_n. It is only required $\mathcal{U}(n)$ be known by the controller, in a deterministic sense, at the time t_n. The set $\mathcal{U}(n)$ represents all possible control vectors that can be applied at t_n. Commonly $\mathcal{U}(n)$ is a compact set.

The feedback controller has available to it a measurement process m(n). The measurement m(n) is taken and processed simultaneoulsy with the control u(n). Thus, the data received at time t_n can be incorporated into the decision process that results in the control u(n). The measurement process is described by

$$
m(n) = H(n)\ x(n) + w(n) \tag{2-3}
$$

where m(n) is an ℓ dimensional vector with

$$
H(n) \equiv \text{measurement matrix } (\ell \times k)
$$

and w(n), the measurement error, an ℓ vector of normally distributed random variables, independent of x(n) and v(n) with known statistics given by

$$
\begin{aligned}
E\left[w(n)\right] &= 0 \\
E\left[w(n)\, w^T(n)\right] &= W(n) \\
E\left[w(n)\, w^T(i)\right] &= 0 \qquad i \neq n
\end{aligned}
\tag{2-4}
$$

Measurement begins at time t_1 and ends at time t_q, so the first measurement is m(1) and the last measurement is m(q).

The plant state vector and the measurement vector are subject to random disturbances so it is appropriate to consider cost functions which are mean values of functions of the state and the control. Thus the cost J, to be minimized, is written in the form of a scalar equation

$$J = E\left[\sum_{n=1}^{q} L(x(n),\ u(n),\ n) + \phi\ (x(q+1))\right] \tag{2-5}$$

Control begins a time t_1 and the last control decision is made at time t_q. Time t_{q+1} is a specified terminal time, $L(x(n),\ u(n), n)$ is a specified scalar incremental cost at each time step and $\phi\ (x(q+1))$ is a specified scalar terminal cost function. It is to be understood that the expectation in (2-5) is conditioned on all available <u>a priori</u> information.

Using these definitions the optimization problem can be defined in specific terms as follows:

"Find the admissible control u(n), as a function of the past history of measurements up to time t_n, that drives the state x(n), so that the expected cost J is minimized".

It is important to note that the incremental cost function $L(x(n),\ u(n),\ n)$ and the terminal cost function $\phi(x(q+1))$ are not required to be quadratic in x(n), u(n) or x(q+1). Also, inherent in the definitions is the assumption that the effect of the control on the state is known without error by the controller. If the actual system has noise in the control channel, then this noise may be modelled by including it in the disturbance v(n). However, cases for which the noise in the control is statistically dependent upon either the state or the control, are specifically excluded.

2.3 Estimation and Sufficient Statistics

If there are no errors in the measurements and the measurement matrix H(n) is square and nonsingular, then the state is known perfectly and the optimal control can be specified as a function of the state. In general, however, the state cannot be determined exactly, so the control action must be based on the information available to the controller, namely, the measurement history. It will be shown that this information can be summarized in the form of a sufficient statistic. The problem of determining the sufficient statistic is approached by deriving an expression for the posterior probability density of the state, conditioned on all a priori information and the measurement history up to time t_n.

To develop an expression for the posterior probability density of the state, first define two k vectors y(n) and z(n). The vector y(n) contains all uncertainty about the state and z(n) describes the known effect of the control history upon the state. Thus y(n) and z(n) satisfy

$$y(n+1) = \Phi(n+1, n)\, y(n) + v(n) \qquad y(0) = x(0) \tag{2-6}$$

$$z(n+1) = \Phi(n+1, n)\, z(n) + \theta(n+1, n)\, u(n) \qquad z(0) = 0, \; u(0) = 0 \tag{2-7}$$

and from (2-1)

$$x(n) = y(n) + z(n) \tag{2-8}$$

Also, a pseudo measurement process r(n) is defined by the equation

$$r(n) = m(n) - H(n)\, z(n) \tag{2-9}$$

Since m(n) is known by the controller and z(n) can be calculated using (2-7), r(n) is known by the controller. Using (2-3) and (2-8) produces

$$r(n) = H(n)\, y(n) + w(n) \tag{2-10}$$

Equations (2-10) and (2-6) describe a linear system perturbed by uncorrelated normally distributed random disturbances or errors. The pseudo measurement process r(n) is composed of linear combinations of the elements of the vector y(n), plus the random measurement errors w(n). Kalman[40] and Battin[5] have shown that the minimum variance estimate of y(n), given the measurements r(n), can be calculated from the following recursion formulas.

$$
\begin{aligned}
\hat{y}(n) &= \hat{y}'(n) + P'(n)H^T(n)\left[H(n)P'(n)H^T(n) + W(n)\right]^{-1}\left[r(n) - H(n)\hat{y}'(n)\right] \\
\hat{y}'(n+1) &= \Phi(n+1, n)\,\hat{y}(n) \\
\hat{y}(0) &= E\left[x(0)\right] \qquad\qquad (2\text{-}11) \\
P(n) &= P'(n) - P'(n)H^T(n)\left[H(n)P'(n)H^T(n) + W(n)\right]^{-1}H(n)P'(n) \\
P'(n+1) &= \Phi(n+1, n)\,P(n)\,\Phi^T(n+1) + V(n) \\
P(0) &= E\left[\left(x(0) - E[x(0)]\right)\left(x(0) - E[x(0)]\right)^T\right]
\end{aligned}
$$

where

$\hat{y}(n)$ = minimum variance estimate of y(n)

P(n) = covariance matrix of errors in estimating y(n)

If the estimation error e(n) is defined as

$$e(n) = \hat{y}(n) - y(n) \qquad (2\text{-}12)$$

then,

$$E\left[e(n)\right] = 0 \qquad (2\text{-}13)$$

and

$$E\left[e(n)\,e^T(n)\right] = P(n) \qquad (2\text{-}14)$$

In order to investigate some of the statistical properties of the error e(n), define the history of pseudo measurements from the initial time up to time t_n as the $n\cdot\ell$ dimensional vector R(n). Hence

$$R(n) = \begin{bmatrix} r(1) \\ r(2) \\ \cdot \\ \cdot \\ \cdot \\ r(n) \end{bmatrix}$$

It has been shown by Kalman[40] that the minimum variance estimation error $e(n)$ must be uncorrelated with the pseudo measurement history $R(n)$.

$$E\left[e(n)\, R^T(n)\right] = 0 \tag{2-15}$$

Since $x(0)$ is normally distributed and $v(n)$ and $w(n)$ are normally distributed, it is clear from (2-6) and (2-10) that $y(n)$ and $r(n)$ must be normally distributed. It follows from (2-11) and (2-12) that $\hat{y}(n)$ and $e(n)$ must be normally distributed. Therefore, since $e(n)$ and $R(n)$ are uncorrelated and normally distributed, they must also be statistically independent.

Now define the history of actual measurements from the initial time up to time t_n as the $n \cdot \ell$ dimensional vector $M(n)$.

$$M(n) = \begin{bmatrix} m(1) \\ m(2) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ m(n) \end{bmatrix}$$

Also assume that some arbitrary admissible control function $u[\cdot]$, of the measurement history $M(n)$, is specified, so the control at time t_n becomes

$$u(n) = u\left[M(n), n\right] \tag{2-16}$$

and (2-7) may be written as

$$z(n+1) = \Phi(n+1, n)\, z(n) + \theta(n+1, n)\, u\left[M(n), n\right] \tag{2-17}$$

From equations (2-17) and (2-9) it is clear that the pseudo measurement process $r(n)$ can be considered to be a deterministic function of the actual measurement history $M(n)$. Fig. 2.1 illustrates, in block diagram form, a method by which $r(n)$ might be calculated from the measurements m(n).

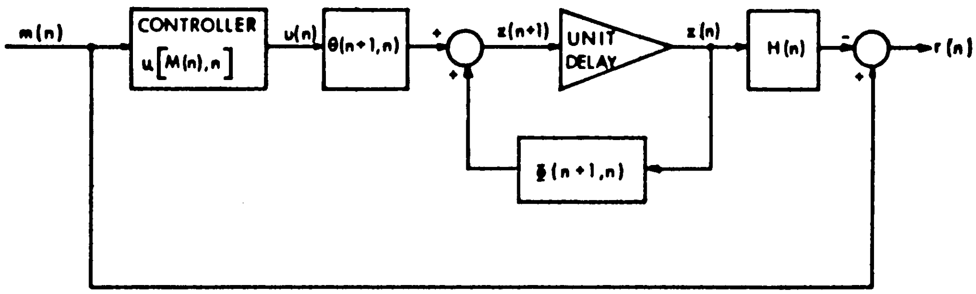

Fig. 2.1 Calculation of r(n) from m(n)

By similar reasoning, the actual measurement process m(n) can be considered to be a deterministic function of the pseudo measurement history R(n). Fig. 2.2 illustrates the calculation of m(n) from pseudo measurements r(n).

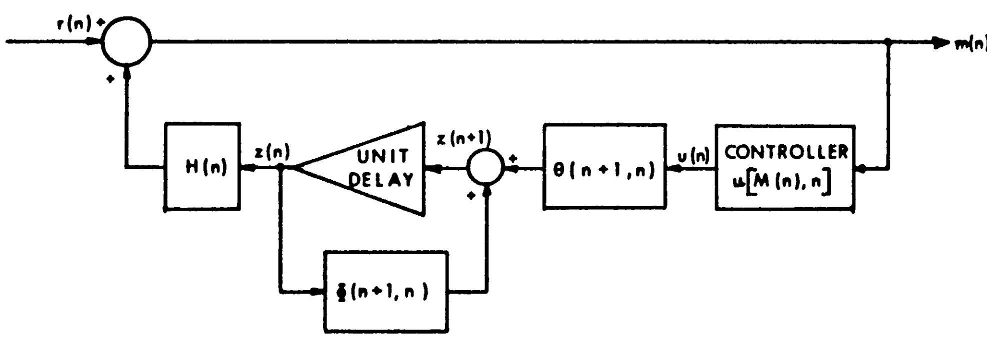

Fig. 2.2 Calculation of m(n) from r(n)

Clearly the measurement histories M(n) and R(n) contain the same statistical information. Since the error e(n) is independent of R(n), and M(n) is a deterministic function of R(n), it follows that e(n) and M(n) must be independent.

Consider an estimate of the state $\hat{x}(n)$ defined as

$$\hat{x}(n) = \hat{y}(n) + z(n) \tag{2-18}$$

From (2-7), (2-8), (2-9), (2-11) and (2-18), recursion formulas for $\hat{x}(n)$ can be derived as

$$\hat{x}(n) = \hat{x}'(n) + P'(n)H^T(n)\left[H(n)P'(n)H^T(n)+W(n)\right]^{-1}\left[m(n) - H(n)\hat{x}'(n)\right] \tag{2-19}$$

$$\hat{x}'(n+1) = \Phi(n+1, n)\hat{x}(n) + \theta(n+1, n)u(n) \tag{2-20}$$

and $P'(n)$ is calculated using the last three of equations (2-11). The error in the estimate $\hat{x}(n)$ is, from (2-8), (2-12) and (2-18)

$$\hat{x}(n) - x(n) = \left[\hat{y}(n)+z(n)\right] - \left[y(n) + z(n)\right] = e(n) \tag{2-21}$$

so the error in $\hat{x}(n)$ is identical to the error in $\hat{y}(n)$. It was shown above that e(n) and M(n) are independent so the error in the estimate $\hat{x}(n)$ is independent of the measurement history M(n). Of importance here is the fact that even through x(n) and $\hat{x}(n)$ may not be normally distributed processes, because the control function $u\left[M(n), n\right]$ may be nonlinear, the error in $\hat{x}(n)$ and the measurement history M(n) are still independent. In addition, the conditional mean of the error in $\hat{x}(n)$, given the measurement history M(n), is zero.

$$E\left[\hat{x}(n) - x(n) \middle| M(n)\right] = E\left[e(n) \middle| M(n)\right] = E\left[e(n)\right] = 0 \tag{2-22}$$

Also, since $\hat{x}(n)$ is a deterministic function of M(n), the conditional mean of the state given the measurements M(n), is the estimate $\hat{x}(n)$.

$$E\left[x(n) \middle| M(n)\right] = E\left[\hat{x}(n) - e(n) \middle| M(n)\right] = E\left[\hat{x}(n) \middle| M(n)\right] = \hat{x}(n) \tag{2-23}$$

At this point the statistical properties of the estimate $\hat{x}(n)$ and the error e(n) can be utilized to derive an expression for the posterior probability density of the state x(n). To that end, consider the state as the difference between the estimate $\hat{x}(n)$ and the error e(n)

$$x(n) = \hat{x}(n) - e(n) \tag{2-24}$$

Since $\hat{x}(n)$ is a deterministic function of the measurements and e(n) is a normally distributed random variable, independent of the measurments, with statistics given by (2-13) and (2-14), the posterior probability density for x(n) is

$$f_{x(n)}\left[\xi \middle| M(n)\right] = (2\pi)^{-\frac{k}{2}} \left|P(n)\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left[\xi - \hat{x}(n)\right]^T P(n)^{-1}\left[\xi - \hat{x}(n)\right]\right\} \tag{2-25}$$

It is assumed that the error covariance matrix P(n) can be calculated <u>a priori</u>. Since P(n) and $\hat{x}(n)$ uniquely determine the posterior state probability density and P(n) is known <u>a priori</u>, the estimate $\hat{x}(n)$ is a sufficient statistic for determining the posterior state probability density. Thus, $\hat{x}(n)$ summarizes all posterior information about the state that is obtained by the controller from the measurement history M(n), and the posterior probability density of x(n) may be written as

$$f_{x(n)}\left[\xi \middle| M(n)\right] = f_{x(n)}\left(\xi \middle| \hat{x}(n)\right) \tag{2-26}$$

2.4 Statistics of the Measurement Information

In the preceding section, recursion formulas were derived for calculating the expected value of the state conditioned on the measurement history. An expression for the posterior probability density of the state was also developed. To obtain the optimal control function, some additional properties of the estimate $\hat{x}(n)$ are necessary. Thus, define the $(k \times \ell)$ matrix K(n) and the k vector s(n) as follows

$$K(n) = P'(n)\,H^T(n)\left[H(n)\,P'(n)\,H^T(n) + W(n)\right]^{-1} \tag{2-27}$$

$$s(n) = K(n)\left[m(n) - H(n)\,\hat{x}'(n)\right] \tag{2-28}$$

and (2-19) becomes

$$\hat{x}(n) = \hat{x}'(n) + s(n) \tag{2-29}$$

The vector $\hat{x}'(n)$ is the estimated state extrapolated forward from time t_{n-1} to time t_n and s(n) represents the incremental change in the state as a result of processing the measurement m(n). Using (2-3) the vector s(n) is written as

$$s(n) = K(n)\left[H(n)\,x(n) + w(n) - H(n)\,\hat{x}'(n)\right] \tag{2-30}$$

and from (2-1) and (2-20)

$$\begin{aligned} s(n)=K(n)\Big\{ & H(n)\left[\Phi(n, n-1)\,x(n-1) + \theta(n, n-1)\,u(n-1) + v(n-1)\right] + w(n) \\ & -H(n)\left[\Phi(n, n-1)\hat{x}(n-1) + \theta(n, n-1)u(n-1)\right] \Big\} \end{aligned} \tag{2-31}$$

Employing (2-21) produces

$$s(n) = K(n)\Big\{ H(n)\left[v(n-1) - \Phi(n, n-1)e(n-1)\right] + w(n)\Big\} \tag{2-32}$$

so that mean and covariance of s(n) are, using (2-11),

$$\begin{aligned} E\left[s(n)\right] &= 0 \\ E\left[s(n)\, s^T(n)\right] &= K(n)\left[H(n)\,P'(n)\,H^T(n) + W(n)\right]K^T(n) \end{aligned} \tag{2-33}$$

and by (2-27) the covariance of s(n) becomes

$$E\left[s(n)\,s^T(n)\right] = P'(n)\,H^T(n)\left[H(n)\,P'(n)\,H^T(n) + W(n)\right]^{-1} H(n)\,P'(n) \tag{2-34}$$

Since e(n-1), v(n-1) and w(n) are normally distributed and independent of M(n-1), then from (2-32), s(n) must be normally distributed and **independent** of M(n-1). From (2-16), (2-19) and (2-20) $\hat{x}(i-1)$ **and** $\hat{x}'(i)$ are deterministic functions of M(n-1) for all $i \leqq n$; so it follows that $s(n)$ must be independent of $\hat{x}(i-1)$ and $\hat{x}'(i)$ for all $i \leqq n$. Further, from (2-28), $s(i)$ is a deterministic function of $M(n-1)$ for all $i < n$; so $s(n)$ and $s(i)$ are independent for all $i < n$. Thus, the $s(n)$ are independent increments of a gaussian process. Finally, since $e(n)$ and $M(n)$ are independent and $s(n)$ is a deterministic function of $M(n)$; $e(n)$ and $s(n)$ must be independent.

2.5 Determination of the Optimal Control Function

In this section the results obtained above are used to derive the conditions that must be satisfied by the optimal control function. Consider a partially completed process at the time t_n in the interval $t_1 \leqq t_n \leqq t_{q+1}$. Assume that some arbitrary admissible control function $u[M(i), i]$ has been used in the past and an admissible control function $u^*[M(i), i)]$ is to be used in the future. Define a minimum expected value function $C^*[M(n), n]$ as follows:

> $C^*[M(n), n] \equiv$ minimum expected cost to complete the process from time t_n, given the measurement history $M(n)$, using the admissible control function $u[M(i), i]$ in the interval $t_1 \leqq t_i < t_n$ and the admissible control function $u^*[M(i), i]$ in the interval $t_n \leqq t_i < t_{q+1}$.

By definition $u^*[M(i), i]$ is the admissible control function which, if used in the interval $t_n \leqq t_i < t_{q+1}$, will produce the minimum expected cost to complete the process. Also, define an expected value function $C[M(n), u(n), n]$ as

> $C[M(n), u(n), n] \equiv$ expected cost to complete the process from time t_n, given the measurement history $M(n)$, using the admissible control function $u[M(i), i]$ in the interval $t_1 \leqq t_i < t_n$, applying an admissible control $u(n)$ at time t_n, and using the admissible control function $u^*[M(i), i]$ in the interval $t_n < t_i < t_{q+1}$.

The difference between C and C^* at time t_n is the result of using the control $u(n)$ at time t_n instead of the control $u^*(n)$.

Now consider the values of C and C^* at the terminal time t_{q+1}. The last control decision is made at time t_q so at the terminal time the functions C and C^* are identical. Thus, from equation (2-5) and the definitions of C and C^*

$$C^*\left[M(q+1), q+1)\right] = C\left[M(q+1), q+1\right] = E\left[\phi\,(x(q+1)) \mid M(q+1)\right] \quad (2\text{-}35)$$

The expectation operation in (2-35) requires the posterior probability density for x(q + 1), given the measurement history M(q+1) and using the control function $u[M(i), i]$ for $t_1 \leqq t_i < t_{q+1}$. This probability density was derived in equations (2-25) and (2-26), so if a function $\overline{\phi}\ (\hat{x}(q+1))$ is defined as

$$\overline{\phi}\,(\hat{x}(q+1)) = \int_{-\infty}^{\infty} d\,\xi_1 \ldots \int_{-\infty}^{\infty} d\xi_k\ \phi(\xi) f_{x(q+1)}(\xi \mid \hat{x}(q+1)) \quad (2\text{-}36)$$

then the expected value functions at the terminal time become

$$C^*\left[M(q+1), q+1\right] = C\left[M(q+1), q+1\right] = \overline{\phi}\ (\hat{x}(q+1)) \quad (2\text{-}37)$$

Since the right hand side of (2-37) is a function of $\hat{x}(q+1)$ only, which is itself a function of M(q + 1), then without loss of generality the expected value functions C and C^* at time t_{q+1}, may be considered to be functions of $\hat{x}(q+1)$ instead of M(q + 1). This important change of independent variable is achieved because $\hat{x}(q+1)$ is a sufficient statistic.

Using the definition of C once again, the expected value function at time t_q can be expressed as

$$C\left[M(q),\ u(q),\ q\right] = E\left[L(x(q), u(q), q) + \phi\ (x(q+1)) \mid M(q), u(q)\right] \quad (2\text{-}38)$$

and if a function $\overline{L}(\hat{x}(n), u(u), n)$ is defined thus

$$\overline{L}(\hat{x}(n), u(n), n) = \int_{-\infty}^{\infty} d\,\xi_1 \ldots \int_{-\infty}^{\infty} d\,\xi_k \; L(\xi, u(n), n)\, f_{x(n)}(\xi \,|\, \hat{x}(n)) \tag{2-39}$$

then by the same arguments used above

$$C[M(q), u(q), q] = \overline{L}(\hat{x}(q), u(q), q) + E[\phi\,(x(q+1))\,|\,M(q), u(q)] \tag{2-40}$$

To evaluate the second term on the right of (2-40), it is necessary to determine the posterior probability density for $x(q+1)$, given the measurement history $M(q)$ and the control $u(q)$. The derivation of this probability density and an expression for the expectation in (2-40) is a somewhat tedious task. For that reason, the derivation is performed in Appendix A and the result, when substituted into (2-40), produces

$$C[M(q), u(q), q] = \overline{L}(\hat{x}(q), u(q), q) + \int_{-\infty}^{\infty} d\zeta_1 \ldots \int_{-\infty}^{\infty} d\zeta_k f_{s(q+1)}(\zeta)\,\overline{\phi}\,(\hat{x}'(q+1) + \zeta) \tag{2-41}$$

where $\hat{x}'(q+1)$ is given by (2-20) and the probability density $f_{s(q+1)}(\zeta)$ is normal with mean and covariance given by (2-33) and (2-34).

Obtaining $C^*[M(q), q]$ from (2-41) is an ordinary minimization problem. Clearly the minimum expected cost to complete the process from time t_q is given by

$$C^*[M(q), q] = \min_{u(q) \in \mathcal{U}(q)} \left\{ \overline{L}(\hat{x}(q), u(q), q) + \int_{-\infty}^{\infty} d\zeta_1 \ldots \int_{-\infty}^{\infty} d\zeta_k \, f_{s(q+1)}(\zeta)\, \overline{\phi}\,(\hat{x}'(q+1) + \zeta) \right\} \tag{2-42}$$

by

$$C^{*}(\hat{x}(q+1), q+1) = \bar{\phi}(\hat{x}(q+1) \tag{2-45}$$

Since the error covariance matrix P(n) is known _a priori_; the posterior probability densities $f_{x(n)}(\xi | \hat{x}(n))$ and $f_{s(n)}(\xi)$ are known _a priori_ as functions of ξ, $\hat{x}(n)$ and n. Therefore $\bar{L}(\hat{x}(n), u(n), n)$ and $\bar{\phi}(\hat{x}(q+1)$ can be calculated _a priori_ as functions of $\hat{x}(n)$, u(n), n and $\hat{x}(q+1)$, respectively. With these functions the system (2-44), (2-45) is closed and the solution for $C^{*}(\hat{x}(n), n)$ is realized in the $\hat{x}(n)$ space. Since the solution is obtained without leaving the $\hat{x}(n)$ space, the control that produces the minimum expected value function must itself be a function of the sufficient statistic $\hat{x}(n)$. Therefore the minimizing control function is written as

$$u^{*}[M(n), n] = u^{*}(\hat{x}(n), n) \tag{2-46}$$

Now consider the minimum expected value function at time t_0, before the first measurement is taken and before any control is applied. From (2-44) and (2-20) the minimum expected value function at the initial time is

$$C^{*}(\hat{x}(0), 0) = \int_{-\infty}^{\infty} d\zeta_1 \ldots \int_{-\infty}^{\infty} d\zeta_k f_{s(1)}(\zeta) C^{*}(\Phi(1,0)\hat{x}(0) + \zeta; 1) \tag{2-47}$$

where $\hat{x}(0)$ is the _a priori_ mean of the plant state. By definition $C^{*}(\hat{x}(0), 0)$ is the minimum expected cost to complete the process from the initial time, using control which is a function of the measurement history. Therefore $C^{*}(\hat{x}(0), 0)$ is the minimum of the cost J in equation (2-5).

$$\min[J] = C^{*}(\hat{x}(0), 0) \tag{2-48}$$

It follows that the control $u^{*}(\hat{x}(n), n)$, obtained by the solution of (2-44) and (2-45), is the optimal control function.

At this point it will be useful to write out the optimality conditions and the auxiliary equations necessary for determining the optimal control function. They are as follows:

$$C^{*}(\hat{x}(n), n) = \min_{u(n)\epsilon\, \mathcal{U}(n)} \Big\{ \overline{L}(\hat{x}(n), u(n), n) + \int_{-\infty}^{\infty} d\zeta_1 \dots \int_{-\infty}^{\infty} d\zeta_k f_{s(n+1)}(\zeta) C^{*}(\hat{x}'(n+1)+\zeta, n+1) \Big\} \tag{2-49}$$

$$\hat{x}'(n+1) = \Phi(n+1, n)\hat{x}(n) + \theta(n+1, n)\, u(n) \tag{2-50}$$

$$C^{*}(\hat{x}(q+1), q+1) = \overline{\phi}(\hat{x}(q+1)) = \int_{-\infty}^{\infty} d\xi_1 \dots \int_{-\infty}^{\infty} d\xi_k \phi(\xi)\, f_{x(q+1)}(\xi \,|\, \hat{x}(q+1)) \tag{2-51}$$

$$\overline{L}(\hat{x}(n), u(n), n) = \int_{-\infty}^{\infty} d\xi_1 \dots \int_{-\infty}^{\infty} d\xi_k\, L(\xi, u(n), n)\, f_{x(n)}(\xi \,|\, \hat{x}(n)) \tag{2-52}$$

$$f_{x(n)}(\zeta \,|\, \hat{x}(n)) = (2\pi)^{-\frac{k}{2}} \left| P(n) \right|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} \left[\zeta - \hat{x}(n)\right]^T P(n)^{-1} \left[\zeta - \hat{x}(n)\right] \right\} \tag{2-53}$$

$$f_{s(n)}(\zeta) = (2\pi)^{-\frac{k}{2}} \left| S(n) \right|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2}\, \zeta^T S(n)^{-1} \zeta \right\} \tag{2-54}$$

$$S(n) = P'(n)\, H^T(n) \left[H(n) P'(n) H^T(n) + W(n) \right]^{-1} H(n)\, P'(n) \tag{2-55}$$

Equation (2-49) is the recursion formula that must be satisfied by the minimum expected value function with the equation for $\hat{x}'(n+1)$ provided by (2-50). Equations (2-51) and (2-52) give the expected terminal cost function and the expected incremental cost functions necessary in (2-49). Equations (2-53) and (2-54) determine the probability densities for the state $x(n)$ and the processed measurement information s(n). Equation (2-55) produces the covariance matrix for s(n). Since the error covariance matrix P(n) is known a priori, the system of equations(2-49) through (2-55) is complete and the optimal control function $u^*(\hat{x}(n), n)$ can, in principle, be calculated a priori.

It is essential to realize that for most practical problems equation (2-49) must be solved by some form of approximation on a digital computer. Methods of solution utilize the techniques of dynamic programming and the concomitant requirement for large amounts of high speed storage is a significant difficulty. For a problem with k state variables, the minimum expected value function must be stored as a function of k variables and the integral on the right in (2-49) must be carried out over the entire k dimensional state space. In addition, if the control is of dimension p, the minimization on the right in (2-49) must be carried out over p variables. In general, solutions for most problems involving more than three state variables are virtually impossible with contemporary computers. There are, however, many practical problems of small dimensionality that can be handled.

2.6 A Useful Form for Digital Approximation

Basic to the usefulness of the theory developed above is the ability to obtain computer solutions. In this section a convenient method for computing C^* is described.

Let a function $C^{*\prime}$ be defined as

$$C^{*\prime}(\hat{x}, n) = \int_{-\infty}^{\infty} d\zeta_1 \cdots \int_{-\infty}^{\infty} d\zeta_k \, f_{s(n+1)}(\zeta) C^*(\hat{x}+\zeta, n+1) \tag{2-56}$$

then (2-49) may be written as

$$C^*(\hat{x}(n), n) = \min_{u(n) \epsilon \mathcal{U}(n)} \left\{ \bar{L}(\hat{x}(n), u(n), n) + C^{*\prime}(\hat{x}'(n+1), n) \right\} \tag{2-57}$$

Equation (2-56) requires a k dimensional integration over the entire state space. It can be shown (see Appendix B) that $f_{s(n+1)}(\zeta)$ is the Green's function, evaluated at $\tau = 1$, for the k dimensional diffusion equation

$$\frac{\partial D(\xi, \tau)}{\partial \tau} = \frac{1}{2} \operatorname{Tr} \left[S(n+1) \quad \frac{\partial^2 D(\xi, \tau)}{\partial \xi^2} \right] \qquad 0 \leqq \tau \leqq 1 \tag{2-58}$$

where the (k x k) matrix of second partial derivatives is defined as

$$\frac{\partial^2 D(\xi, \tau)}{\partial \xi^2} = \left[\frac{\partial^2 D(\xi, \tau)}{\partial \xi_i \, \partial \xi_j} \right] \tag{2-59}$$

If $D(\xi, \tau)$ is the solution of (2-58) with the initial condition

$$D(\xi, 0) = C^{*} \; (\xi, \; n+1) \tag{2-60}$$

then from Appendix B and equation (2-56) , $C^{*\prime}$ in (2-57) is given by

$$C^{*\prime}(\hat{x}'(n+1), n) = D(\hat{x}'(n+1), 1) \tag{2-61}$$

In many cases (see Appendix C) it is easier, from the standpoint of digital computation requirements, to approximate the solution of the diffusion equation by central differences than to approximate the k dimensional integral with quadrature formulas. If the difference equation method is chosen for the calculation of the minimum expected value function, then equation (2-49) above is replaced by equations (2-57), (2-58), (2-60) and (2-61). In a similar manner, the functions $\overline{L}(\hat{x}(n), \; u(n), n)$ and $\overline{\phi}\,(\hat{x}(q+1)$ may be determined as solutions of diffusion equations.

2.7 Multiple Measurements Between Control Applications

In some problems there will be many measurements taken between the times when control is applied. If such is the case and control is applied at time t_n and again at time t_{n+j}; where $j \geqq 1$, and j measurements are taken in the interval $t_n < t \leqq t_{n+j}$, then $\hat{x}(n+j)$ can be written as

$$\hat{x}(n+j) = \Phi(n+j, n)\hat{x}(n) + \theta(n+j, n)u(n) + \sum_{i=n+1}^{n+j} \Phi(n+j, i)\, s(i) \qquad (2\text{-}62)$$

Also, if the following are defined

$$\hat{x}'(n+j) = \Phi(n+j, n)\hat{x}(n) + \theta(n+j, n)\, u(n) \qquad (2\text{-}63)$$

$$s'(n+j) = \sum_{i=n+1}^{n+j} \Phi(n+j, i)\, s(i) \qquad (2\text{-}64)$$

then (2-62) becomes

$$\hat{x}(n+j) = \hat{x}'(n+j) + s'(n+j) \qquad (2\text{-}65)$$

Now equation (2-65) has the same form as equation (2-29) and $s'(n+j)$ is normally distributed with statistics

$$E\left[s'(n+j)\right] = 0$$

$$E\left[s'(n+j)\, s'^{T}(n+j)\right] = \sum_{i=n+1}^{n+j} \Phi(n+j, i)P'(i)H^{T}(i)\left[H(i)P'(i)H^{T}(i) + W(i)\right]^{-1} H(i)P'(i)\Phi^{T}(n+j, i) \qquad (2\text{-}66)$$

Since $s'(n+j)$ is a linear combination of the processed measurement increments $s(i)$, equations (2-49), (2-54) and (2-55) become

$$C^{*}(\hat{x}(n), n) = \min_{u(n)\epsilon\mathcal{U}(n)} \left\{ \overline{L}(\hat{x}(n), u(n), n) + \int_{-\infty}^{\infty} d\zeta_1 \cdots \int_{-\infty}^{\infty} d\zeta_k\, f_{s'(n+j)}(\zeta)\, C^{*}(\hat{x}'(n+j) + \zeta, n+j) \right\} \qquad (2\text{-}67)$$

$$f_{s'(n+j)}(\zeta) = (2\pi)^{-\frac{k}{2}} \left|S'(n+j)\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\zeta^T S'(n+j)^{-1}\zeta\right\} \tag{2-68}$$

$$S'(n+j) = \sum_{i=n+1}^{n+j} \Phi(n+j,i)P'(i)H^T(i)\left[H(i)P'(i)H^T(i) + W(i)\right]^{-1} H(i)P'(i)\Phi^T(n+j \tag{2-69}$$

with equation (2-50) replaced by equation (2-63). Thus, the theory is generalized to handle multiple measurements between control applications. When $j = 1$, these equations reduce to (2-49), (2-54) and (2-55).

2.8 Specification of the Terminal Control Function

For some problems it is more meaningful to specify the terminal control function than to specify a terminal cost function. In such cases the expected cost becomes

$$J = E\left[\sum_{n=1}^{q} L\,(x(n),\ u(n), n)\right] \tag{2-70}$$

with the terminal control $u(\hat{x}(q), q)$ specified. Then the expected cost to complete the process from time t_q is

$$C\left[M(q), u(q), q\right] = E\left[L(x(q), u(q), q) \mid M(q), u(q)\right] = \overline{L}(\hat{x}(q), u(q), q) \tag{2-71}$$

and since $u(\hat{x}(q), q)$ is specified, the expected value functions at time t_q become

$$C^*(\hat{x}(q), q) = C(\hat{x}(q), q) = \overline{L}(\hat{x}(q), u(\hat{x}(q), q), q) \tag{2-72}$$

Equation (2-72) is the terminal condition for the recursion formula (2-49). Solution of (2-49), (2-72) will provide the control functions $u^*(\hat{x}(n), n)$ that minimize (2-70), subject to the specified terminal control function $u(\hat{x}(q), q)$.

CHAPTER 3

APPLICATIONS OF THE DISCRETE THEORY

3.1 General Discussion

In Chapter 2 a theory was developed for determining optimal controllers for linear stochastic systems, when the cost function may be nonquadratic. As with any theory, its usefulness must be demonstrated by the actual solution of practical problems. Four such probelms are treated in this chapter. The first two problems require the determination of optimal midcourse velocity corrections for spacecraft on interplanetary missions. The third problem is the determination of the optimal control for an atmospheric entry vehicle. Finally, the theory is applied to the quadratic cost problem, producing the discrete control/estimation separation theorem.

3.2 Minimum Fuel Variable Time of Arrival Guidance

Consider the midcourse phase of an interplanetary spacecraft mission. Due to random errors made in injecting the spacecraft into its interplanetary trajectory, impulsive midcourse velocity corrections are necessary, if the vehicle is to hit the target with sufficient accuracy. The spacecraft uses chemical fuel rockets to perform these velocity corrections. During the midcourse phase the spacecraft is tracked by radar systems based on earth. The radars provide velocity measurements in the directions of the radius vector from Earth to the spacecraft and the measurements contain normally distributed random errors. Estimates of spacecraft position and velocity are computed from this information, using recursion formulas (2-19) and (2-20).

It is assumed that there is a reference trajectory defined which passes through the nominal point of injection and the nominal target point. At the target point there is a non-zero relative velocity between a spacecraft on the reference trajectory and the target planet. This relative velocity vector is given the symbol v_R. The guidance is variable time of arrival [4, 65] so deviations from the reference trajectory, at the nominal time of arrival, parallel to the relative velocity v_R, carry no penalty. Also, the reference trajectory is assumed to lie in a plane and spacecraft deviations out of the trajectory plane are ignored. Except for the injection errors, there are no random disturbances to the spacecraft trajectory so the covariance matrix V(n), defined in equation (2-2), is identically zero. Since out of plane errors are ignored, the deviation of the actual trajectory from the reference trajectory can be described by a four dimensional deviation state vector (2 coordinates of position and 2 coordinates of velocity). Therefore, if a velocity correction is made at time t_n, the deviation at time t_{n+1} becomes

$$\delta y(n+1) = \Phi(n+1, n)\delta y(n) + \Phi(n+1, n) \begin{bmatrix} 0 \\ I \end{bmatrix} \Delta v(n) \qquad (3\text{-}1)$$

where

$\delta y(n)$ ≡ deviation state vector before correction (4 dimensional

$\Phi(n+1, n)$ ≡ state transition matrix(4×4), evaluated along the reference trajectory

$\Delta v(n)$ ≡ velocity correction vector (2 dimensional)

$\begin{bmatrix} 0 \\ I \end{bmatrix}$ ≡ compatability matrix (4×2), first two rows are zero, last two rows are (2×2) identity matrix

and it is assumed that the state deviations and velocity corrections are sufficiently small so that linearizations about the reference trajectory are valid.

Let time t_{q+1} be the nominal time of arrival at the target.

If a transformation of state variables is defined as follows

$$\delta y'(n) = \Phi(q+1, n)\, \delta y(n) \tag{3-2}$$

then the transformed state deviation vector $\delta y'(n)$ is the deviation at time t_n, extrapolated forward to the nominal time of arrival. The vector $\delta y'(n)$ can also be thought of as the deviation from the reference trajectory, at the nominal time of arrival; given that $\delta y(n)$ is the deviation from the reference trajectory at time t_n and no velocity corrections take place for all time in the interval $t_n \leqq t \leqq t_{q+1}$. Using (3-1) and (3-2) an expression for $\delta y'(n+1)$ can be written as

$$\delta y'(n+1) = \Phi(q+1, n+1)\Phi(n+1, n)\delta y(n) + \Phi(q+1, n+1)\Phi(n+1, n) \begin{bmatrix} 0 \\ I \end{bmatrix} \Delta v(n) \tag{3-3}$$

and applying (3-2) produces

$$\delta y'(n+1) = \delta y'(n) + \Phi(q+1, n) \begin{bmatrix} 0 \\ I \end{bmatrix} \Delta v(n) \tag{3-4}$$

Equation (3-4) is the difference equation satisfied by $\delta y'(n)$. If the position components of this vector are resolved into a coordinate system such that an axis (1) lies in the trajectory plane orthogonal to the relative velocity v_R and points generally away from the sun and an axis (2) lies parallel to v_R and generally along the flight path, then deviations in position at the target are, from (3-4)

$$\delta y_1'(n+1) = \delta y_1'(n) + \Gamma_{11}(n)\, \Delta v_1(n) + \Gamma_{12}(n) \Delta v_2(n) \tag{3-5}$$

$$\delta y_2'(n+1) = \delta y_2'(n) + \Gamma_{21}(n)\, \Delta v_1(n) + \Gamma_{22}(n) \Delta v_2(n) \tag{3-6}$$

The component $\delta y_1'(n)$ is the deviation in position orthogonal to v_R and $\delta y_2'(n)$ is the deviation in position parallel to v_R with the (4 x 2) matrix $\Gamma(n)$ defined by

$$\Gamma(n) = \Phi(q+1, n) \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{3-7}$$

so $\Gamma(n)$ appears as the last two columns of $\Phi(q+1, n)$.

Since the guidance is variable time of arrival; the deviation $\delta y_2'(n)$, along the relative velocity vector v_R is ignored and each velocity correction is applied in the direction that maximizes the sensitivity of $\delta y_1'(n+1)$ to the velocity correction magnitude. This sensitivity is maximized if the velocity correction at time t_n is given by

$$\Delta v_1(n) = \left(\frac{\Gamma_{11}(n)}{\sqrt{\Gamma_{11}(n)^2 + \Gamma_{12}(n)^2}} \right) u(n) \tag{3-8}$$

$$\Delta v_2(n) = \left(\frac{\Gamma_{12}(n)}{\sqrt{\Gamma_{11}(n)^2 + \Gamma_{12}(n)^2}} \right) u(n) \tag{3-9}$$

where u(n) is a scalar that determines the magnitude and sign of the velocity correction. If the scalar state x(n) and scalar, non-zero control sensitivity $\theta(n+1, n)$ are defined as

$$x(n) = \delta y_1'(n) \tag{3-10}$$

$$\theta(n+1, n) = \sqrt{\Gamma_{11}(n)^2 + \Gamma_{12}(n)^2} = \sqrt{\Phi_{13}(q+1, n)^2 + \Phi_{14}(q+1, n)^2} \tag{3-11}$$

then equations (3-5), (3-8) and (3-9) produce the scalar relation

$$x(n+1) = x(n) + \theta(n+1, n)\, u(n) \tag{3-12}$$

Equation (3-12) is the equation of state for the minimization problem. Its simple form is a result of the transformation (3-2) and the fact that for variable time of arrival guidance, no penalty is attached to $\delta y_2'(n)$.

Using these definitions it is possible to state the optimal control problem in specific terms. It is desired to minimize the expected total fuel required to perform the midcourse maneuvers, assuming that the velocity corrections are performed at specified times. Since the spacecraft rocket engines are chemically fueled, the sum of the magnitudes of the scalar controls u(n) is simply related to the amount of fuel used. Two guidance schemes will be investigated. The first scheme has no terminal cost function but at the last correction time t_m, a total correction is made so that $\hat{x}'(q+1)$ is driven to zero. The incremental cost at each correction is

$$L(x(n), u(n), n) = |u(n)| \tag{3-13}$$

with the terminal control specified as

$$u(\hat{x}(m), m) = -\frac{\hat{x}(m)}{\theta(q+1, m)} \tag{3-14}$$

Hence $\overline{L}(\hat{x}(n), u(n), n)$ is given by

$$\overline{L}(\hat{x}(n), u(n), n) = \int_{-\infty}^{\infty} d\xi \, |u(n)| \, f_{x(n)}(\xi | \hat{x}(n)) = |u(n)| \tag{3-15}$$

and the terminal condition on the minimum expected value function is determined by (2-72) as

$$C^*(\hat{x}(m), m) = \overline{L}(\hat{x}(m), u(\hat{x}(m), m), m) = \frac{|\hat{x}(m)|}{\theta(q+1, m)} \tag{3-16}$$

Since the last correction drives the estimated state to zero, the statistics of the error in hitting the target correspond to the statistics of the estimation error at time t_m. By performing the last correction at a time when enough measurements have been taken so that the estimation error statistics satisfy the target miss distance requirements, the spacecraft will hit the target with the required accuracy. This scheme will be called total final correction guidance. The second scheme has L(x(n), u(n), n) given by

$$L(x(n), u(n), n) = |u(n)| \tag{3-17}$$

with a quadratic terminal cost function

$$\phi\,(x(q+1)) = \frac{\lambda}{2}\; x(q+1)^2 \tag{3-18}$$

In this case $\overline{L}\,(\hat{x}(n), u(n), n)$ is again given by (3-15) and the $\overline{\phi}\,(\hat{x}(q+1))$ function is

$$\overline{\phi}\,(\hat{x}(q+1)) = \frac{\lambda}{2} \int_{-\infty}^{\infty} d\xi\; \xi^2\, f_{x(q+1)}(\xi \,|\, \hat{x}(q+1)) = \frac{\lambda}{2}\,(\hat{x}(q+1)^2 + P(q+1)) \tag{3-19}$$

so in this case the terminal condition on the minimum expected value function is

$$C^*(\hat{x}(q+1), q+1) = \frac{\lambda}{2}\,(\hat{x}(q+1)^2 + P(q+1)) \tag{3-20}$$

This scheme will be called quadratic terminal cost guidance.

It is important to realize that the scalar state x(n) is the deviation from the reference trajectory extrapolated forward in time to the nominal time of arrival, and taken perpendicular to the relative velocity vector v_R. Thus $\hat{x}(n)$ is the estimated target miss distance, as calculated from the measurements and velocity corrections up to time t_n. Hence P(n) is the variance at time t_n, of the error in the estimated target miss distance and it is assumed that the measurement schedule is known <u>a priori</u> so that P(n) can be calculated <u>a priori</u>. From equation (3-12) it is clear that the state transition matrix for this problem is the scalar unity. Also, from (2-69) and the recursion formulas for P(n) in (2-11), the variance S′(n+j) can be shown to satisfy

$$S'(n+j) = P(n) - P(n+j) \tag{3-21}$$

with the probability density for s′(n+j) given by (2-68), with k = 1.

If two correction times are specified as times t_n and t_{n+j}, and there are j measurements taken in the interval $t_n < t \leqq t_{n+j}$, then the minimum expected value function must satisfy (2-67), so

$$C^*(\hat{x}(n), n) = \min_{u(n)} \left\{ |u(n)| + \int_{-\infty}^{\infty} d\zeta\; f_{s'(n+j)}(\zeta)\, C^*(\hat{x}'(n+j)+\zeta, n+j) \right\} \tag{3-22}$$

No restrictions are placed on the control so $\mathcal{U}(n)$ occupies the entire real axis. Equation (2-63) provides a scalar equation for the extrapolated estimate.

$$\hat{x}'(n+j) = \hat{x}(n) + \theta(n+j,n)u(n) \tag{3-23}$$

Let a function $C^{*\prime}$ be defined as

$$C^{*\prime}(\hat{x},n) = \int_{-\infty}^{\infty} d\zeta \; f_{s'(n+j)}(\zeta)\; C^{*}(\hat{x}+\zeta, n+j) \tag{3-24}$$

and $C^{*\prime}(\hat{x},n)$ may be interpreted as the minimum expected cost to complete the process from the point $\hat{x}$ at time t_n if no control is applied at time t_n (i.e. $u(n) = 0$). Using this definition (3-22) becomes

$$C^{*}(\hat{x}(n),n) = \min_{u(n)} \left\{ \left|u(n)\right| + C^{*\prime}(\hat{x}'(n+j),n) \right\} \tag{3-25}$$

To obtain the minimum on the right of (3-25), the derivative of the function in braces is taken with respect to u(n).

$$\frac{\partial\{\cdot\}}{\partial u(n)} = \operatorname{sgn}\left[u(n)\right] + \left[\frac{\partial C^{*\prime}(\hat{x},n)}{\partial \hat{x}}\right]_{\hat{x} = \hat{x}'(n+j)} \theta(n+j,n) \tag{3-26}$$

It can be shown (see Appendix D) that the resulting optimal control function is given by

$$u^{*}(\hat{x}(n),n) = \begin{cases} \dfrac{\operatorname{sgn}\left[\hat{x}(n)\right]\alpha(n) - \hat{x}(n)}{\theta(n+j,n)} & \text{if } \left|\hat{x}(n)\right| > \alpha(n) \\ 0 & \text{if } \left|\hat{x}(n)\right| \leqq \alpha(n) \end{cases} \tag{3-27}$$

where the positive quantity α (n) satisfies

$$\left[\frac{\partial C^{*\prime}(\hat{x},n)}{\partial \hat{x}}\right]_{\hat{x} = \alpha(n)} \theta(n+j,n) = 1 \tag{3-28}$$

Another way of describing the optimal control function at time t_n is as follows:

The value α (n) determined by (3-28) defines an interval $\mathfrak{Z}(n)$ on the $\hat{x}$ axis as $-\alpha(n) \leqq \hat{x} \leqq \alpha(n)$. If $\hat{x}(n)$ lies inside $\mathfrak{Z}(n)$ or on the boundary, no control is applied. If $\hat{x}(n)$ lies outside $\mathfrak{Z}(n)$, the optimal control drives $\hat{x}'(n+j)$ to the boundary of $\mathfrak{Z}(n)$. Application of non-zero control will move $\hat{x}'(n+j)$, but only at the expense of fuel. In $\mathfrak{Z}(n)$ the expense of applying any non-zero control is greater than the resulting saving that can be incurred by the movement of $\hat{x}'(n+j)$. Conversely, outside $\mathfrak{Z}(n)$ it is possible to apply controls at less expense than the saving attained by the resulting movement of $\hat{x}'(n+j)$. In fact, the maximum net saving is produced by applying control such that $\hat{x}'(n+j)$ is driven to the boundary of $\mathfrak{Z}(n)$. The boundaries of $\mathfrak{Z}(n)$ are determined by $\alpha(n)$ according to (3-28). Now $\theta(n+j, n)$ is the change in state per unit of applied control (fuel) and $\frac{\partial C^{*\prime}}{\partial \hat{x}}$ is the change in cost (fuel) per unit change in state. Hence, a point at which the product of these two quantities is unity, as in (3-28), is a point at which saving and expense just balance each other. Inside $\mathfrak{Z}(n)$ there is a net loss for any nonzero control and outside $\mathfrak{Z}(n)$ there is a net saving for proper application of control.

Finally, if the optimal control given by (3-27) is applied, then (3-25) becomes

$$C^*(\hat{x}(n), n) = \begin{cases} \dfrac{|\hat{x}(n)| - \alpha(n)}{\theta(n+j, n)} + C^{*\prime}(\alpha(n), n) & \text{if } |\hat{x}(n)| > \alpha(n) \\ \\ C^{*\prime}(\hat{x}(n), n) & \text{if } |\hat{x}(n)| \leqq \alpha(n) \end{cases} \tag{3-29}$$

Clearly, the optimal control problem becomes the problem of determining the set of positive numbers $\alpha(n)$, one for each velocity correction time. The solution requires digital computation and can be accomplished in the $\hat{x}$ space, which for this problem is the real axis. Calculations begin at the terminal time with the terminal condition given by (3-30), if the total final correction guidance scheme is used.

$$C^*(\hat{x}, m) = \frac{|\hat{x}|}{\theta(q+1, m)} \qquad m = n+j \tag{3-30}$$

If quadratic terminal cost guidance is used then the terminal condition is

$$C^*(\hat{x}, q+1) = \frac{\lambda}{2}(\hat{x}^2 + P(q+1)) \qquad q+1 = n+j \tag{3-31}$$

The function $C^{*\prime}(\hat{x}, n)$ is computed by approximating the solution of the diffusion equation as described in Section 2.6. The initial condition and differential equation are given by

$$D(\hat{x}, 0) = C^*(\hat{x}, n+j) \tag{3-32}$$

$$\frac{\partial D(\hat{x}, \tau)}{\partial \tau} = \frac{1}{2} S'(n+j) \frac{\partial^2 D(\hat{x}, \tau)}{\partial \hat{x}^2} \qquad 0 < \tau \leqq 1 \tag{3-33}$$

and $C^{*\prime}(\hat{x}, n)$ is given by

$$C^{*\prime}(\hat{x}, n) = D(\hat{x}, 1) \tag{3-34}$$

The value $\alpha(n)$ is realized by numerically differentiating $C^{*\prime}(\hat{x}, n)$ with respect to $\hat{x}$ and finding the point on the $\hat{x}$ axis that satisfies

$$\left[\frac{\partial C^{*\prime}(\hat{x}, n)}{\partial \hat{x}}\right]_{\hat{x} = \alpha(n)} \theta(n+j, n) = 1 \tag{3-35}$$

Finally $C^*(\hat{x}, n)$ is determined from

$$C^*(\hat{x}, n) = \begin{cases} \dfrac{|\hat{x}| - \alpha(n)}{\theta(n+j, n)} + C^{*\prime}(\alpha(n), n) & \text{if } |\hat{x}| > \alpha(n) \\ \\ C^{*\prime}(\hat{x}, n) & \text{if } |\hat{x}| \leqq \alpha(n) \end{cases} \tag{3-36}$$

Computations beginning with (3-32) are then repeated the required number of times until the initial time is reached (i.e. n = 0). In Fig. 3.1 is pictured a typical sequence of computations from time t_{n+j} to time t_n.

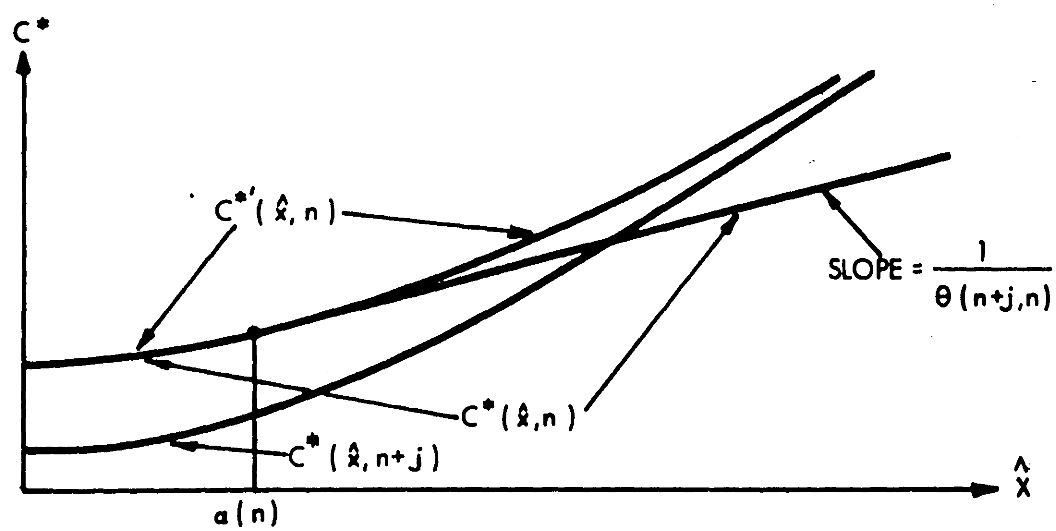

Fig. 3.1 Computation Sequence for Determining $\alpha(n)$

3.3 Numerical Example of Minumum Fuel V.T.A. Guidance

To demonstrate the actual numerical solution of the problem described in Section 3.2, a spacecraft mission to Mars was simulated. Since many body gravitation effects do not appreciably influence the minimization problem, a two body matched conic reference trajectory was computed. Two conics were used, a geocentric hyperbola in the vicinity of Earth and a heliocentric ellipse for the long interplanetary phase of the flight from the Earth sphere of influence to the Mars sphere of influence. The gravity of Mars was ignored in order to simplify the calculations. In effect the trajectory aims at a point on the Mars sphere of influence, at the nominal time of arrival. The geocentric hyperbolic trajectory begins at a point 100 miles above the Earth's surface and matches the heliocentric ellipse in position and velocity, at a point 425,400 miles from the center of Earth. The hyperbolic transfer angle is approximately 135°. The heliocentric elliptical portion of the trajectory is a 180° transfer from approximately the Earth sphere of influence to the target point near Mars. Orbital elements of the geocentric hyperbola and the heliocentric ellipse are listed in Table 3.1.

Conic	Semimajor Axis	Eccentricity
Hyperbola	31,300 mi.	1.130
Ellipse	117.3×10^6 mi.	0.208

Table 3.1 Orbital Elements of the Matched Conics

An approximate plot of the trajectory is shown in Fig. 3.2.

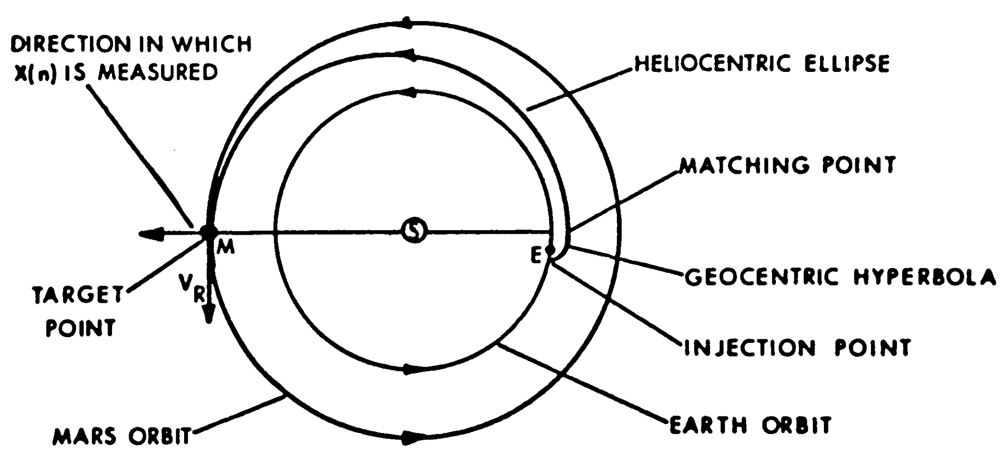

Fig. 3.2 Earth-Mars Reference Trajectory (not to scale)

Total time on the reference trajectory from injection to nominal time of arrival is 6160 hours, of which approximately 56 hours are spent on the geocentric hyperbola.

The (4 x 4) state transition matrix $\Phi(q+1, n)$ was computed as a function of time t_n along the trajectory. The matrix $\Gamma(n)$ defined by (3-7) appears as the last two columns of $\Phi(q+1, n)$. Using (3-11) the control sensitivity was calculated as a function of time t_n, and is plotted in Fig. 3.3.

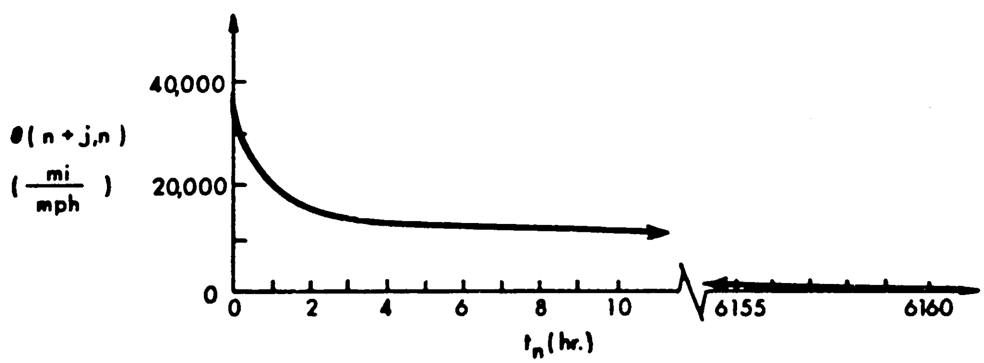

Fig. 3.3 Control Sensitivity

The spacecraft was assumed to be an unmanned probe and the variances of injection errors were chosen as typical for such a mission. It was also assumed that the cross correlations between injection errors are identically zero. Injection error variances for the flight are listed in Table 3.2.

	Altitude	Range
Position Error Variance	$1\ (\text{mi})^2$	$16\ (\text{mi})^2$
Velocity Error Variance	$400\ (\text{m.p.h.})^2$	$20(\text{m.p.h.})^2$

Table 3.2 Injection Error Variances

In the vicinity of the earth, velocity measurements are taken every hour beginning one hour after injection. It is assumed that these measurements are always taken in the direction from Earth's center to the spacecraft. The variance of errors in the measurements is 0.01 (m.p.h.)^2. Using these statistics the (4 x 4) covariance matrix, of estimation errors at the target, was calculated as a function of time(see Battin[4], Chapter 9). The upper left hand corner element of this matrix is the variance of error in estimated miss distance at the target. It is plotted in Fig. 3.4.

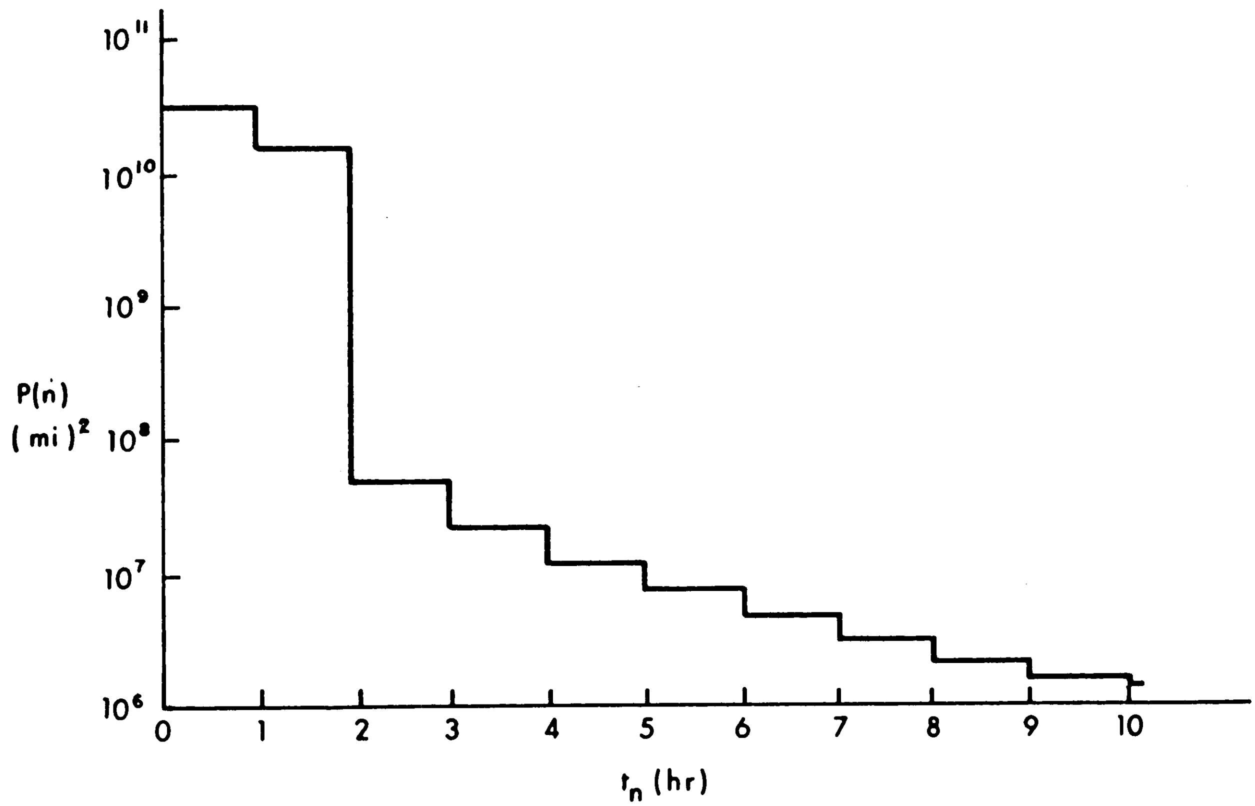

Fig. 3.4 Variance of Error in Estimated Miss Distance at the Target

From this figure it is clear that the measurement at two hours drastically reduces the error variance at the target. This occurs because between them, the two measurements at 1 and 2 hours, very accurately determine the magnitude of the spacecraft velocity, which is the important factor in determining the target miss distance.

The last velocity correction is made at 56 hr. (i.e. at the conic matching point), at which time the error variance is below 4×10^4 mi^2. Thus, the root mean square error in estimated target miss distance, at the last correction time, is less than 200 miles. Total final correction guidance is used (i.e. the estimated target miss distance is eliminated at the last correction), so the root mean square miss distance is less than 200 miles.

Using the computation method described in Section 3.2, it was soon found that optimum corrections after 2 hours and before 56 hours produce negligible savings in fuel. Physically this can be explained with reference to Figs. 3.3 and 3.4 which display relatively small decreases in control sensitivity and error variance after 2 hours and before 56 hours. In effect the terminal miss distance is known quite well after the measurement at 2 hours, so a fairly accurate correction can be made at that time. In addition, since the sensitivity does not change appreciably after 2 hours and before 56 hours, little can be gained by applying part of the correction at say 3 or 4 hours and the remainder at 56 hours. By contrast, appreciable savings can be accrued by correcting at 1 and 2 hours because the sensitivities are appreciably higher. In fact a measurement and correction before 1 hour can produce even greater savings. This possibility was not investigated, however, because it did not seem operationally feasible to require a velocity correction earlier than one hour after injection. Thus, velocity correction times were chosen as 1, 2 and 56 hours. The control sensitivities and error variances at these times are listed in Table 3.3.

Time t_n (hr)	$\theta(n+j, n)$ $(\frac{\text{mi.}}{\text{m.p.h.}})$	$P(n)$ (mi.)^2
0	35,790	$3,661 \times 10^{10}$
1	19,380	1.424×10^{10}
2	15,780	5.592×10^{7}
56	9,754	3.300×10^{4}

Table 3.3 Trajectory Data

Using these values, the variances S $'(n+j)$ were calcualted by (3-21). They are

$$S'(1) = 2.237 \times 10^{10} \text{ (mi.)}^2$$

$$S'(2) = 1.419 \times 10^{10} \text{ (mi.)}^2$$

$$S'(56) = 5.589 \times 10^{7} \text{ (mi.)}^2$$

From these data, the $\alpha(n)$ values for total final correction guidance were computed as described in Section 3.2, producing

$$\alpha(1) = 158,100 \text{ mi.}$$

$$\alpha(2) = 6,800 \text{ mi.}$$

and the optimal controls are given by (3-27) so

$$u^*(\hat{x}(1), 1) = \begin{cases} \dfrac{158,100 \operatorname{sgn}\left[\hat{x}(1)\right] - \hat{x}(1)}{19,380} & \text{if } |\hat{x}(1)| > 158,100 \text{ mi.} \\ 0 & \text{if } |\hat{x}(1)| \leqq 158,100 \text{ mi.} \end{cases} \quad (3\text{-}37)$$

$$u^*(\hat{x}(2), 2) = \begin{cases} \dfrac{6,800 \operatorname{sgn}\left[\hat{x}(2)\right] - \hat{x}(2)}{15,780} & \text{if } |\hat{x}(2)| > 6,800 \text{ mi.} \\ 0 & \text{if } |\hat{x}(2)| \leqq 6,800 \text{ mi.} \end{cases} \quad (3\text{-}38)$$

where the units of $\hat{x}$ and u^* are miles and miles per hour, respectively. Numerical results also showed that the optimum variable time of arrival guidance corrections must be applied essentially parallel to the reference trajectory velocity vectors.

The initial minimum expected value function $C^*(\hat{x}(0), 0)$ was calculated using Eq (2-47). It was assumed that the spacecraft contains an inertial measurement unit for determining the injection velocity and that this data is used to obtain $\hat{x}(0)$. In addition, the injection engine cutoff system produces errors, so

$\hat{\hat{x}}(0)$ is assumed to be a normally distributed random variable with zero mean and variance equal to the variance of the error in $\hat{x}(0)$, (i.e. 3.661×10^{10} mi^2). The minimum expected cost was calculated by integrating $C^*(\hat{\hat{x}}(0), 0)$ over the probability density for $\hat{x}(0)$, producing a value of 13.4 m.p.h. as the minimum three correction cost.

A comparison can be made between the mean cost of the optimum three correction control, and the expected fuel required to apply a single total correction at 56 hrs, which is the guidance technique usually employed in this context. The expected fuel required for a single total correction at 56 hrs is 22.1 m.p.h., or an increase of about 65% over the minimum three correction cost. The saving is, of course, due to the very early application of optimum corrections at 1 and 2 hours. Similarly, a comparison can be made between the optimum three correction controller and the mean cost of applying three total corrections. The mean cost of applying three total corrections at 1, 2 and 56 hrs is 16.7 m.p.h., an increase of about 25% over the optimum.

As a check on the accuracy of the computations, Monte Carlo runs were made using the optimal control. Three thousand runs were completed using a normal random number generater to simulate injection and measurement errors. The averaged cost for these simulated optimum trajectories was 13.2 m.p.h. or 0.2 m.p.h. less than the computed optimum. Since the standard deviation of the averaged cost is about 0.1 m.p.h., there is reasonably good agreement between the two computations. The Monte Carlo simulations also provided the probability distribution of the total cost using the optimal control. It appears as Fig. 3.5.

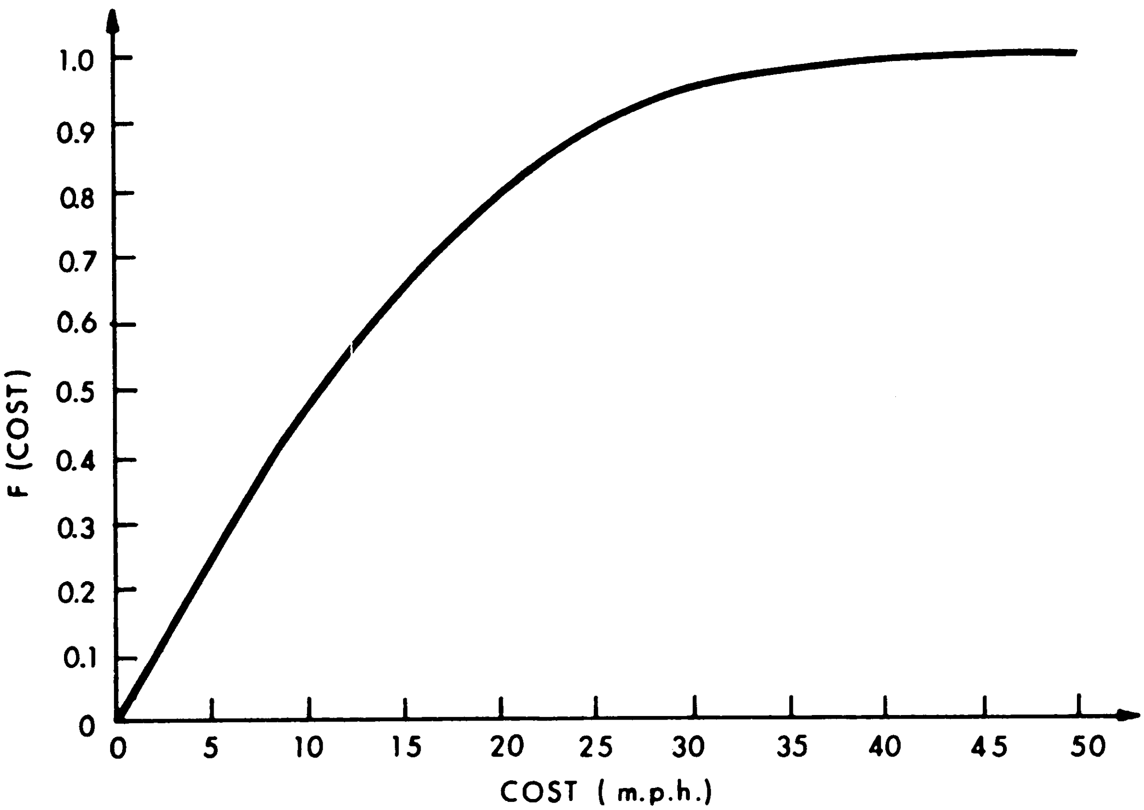

Fig. 3.5 Total Cost Probability Distribution Using Optimal Control

Some interesting statistics can be obtained from the calculated probability distribution. For example, the probability of a particular optimum trajectory costing less than the minimum mean cost (i.e. 13.2 m.p.h.) is 0.608, so about 61% of the trajectories will cost less than the mean. Further, the probability of a particular trajectory costing more than three times the mean (i.e. 39.6 m.p.h.) is 0.007, so less than 1% of the trajectories will cost more than three times the mean.

From (3-37) and (3-38) it is obvious that the optimal control is a nonlinear function of the estimated state. Breakwell and Striebel[12] have developed a method for determining a near minimum fuel control. Their theory obtains the optimum controller from the class of all linear controllers. For purposes of comparison the near optimum linear control functions were determined for corrections at 1 and 2 hours, assuming that the correction at 56 hours is a total correction. They are

$$u(\hat{x}(1), 1) = - \frac{0.31 \hat{x}(1)}{19,380} \tag{3-39}$$

$$u(\hat{x}(2), 2) = - \frac{0.97 \hat{x}(2)}{15,780} \tag{3-40}$$

The velocity correction at 1 hour decreases the estimated miss distance by 31% and the velocity correction at 2 hours decreases the estimated miss distance by 97%. It was found that Breakwell and Striebel's near optimum linear controller expended only 4% more fuel, on the average, than the true optimum.

The optimal control law for quadratic terminal cost guidance was also computed. The terminal cost function for this case is given by equation (3-20). The value λ was chosen so that the terminal cost function $\phi(x)$ has the same value at $x = 1,000$ miles, as the amount of fuel required to correct a 1,000 mile error at 56 hours. Thus, errors greater than 1,000 miles are weighed heavily and errors near zero are weighed very little. The calculated value of λ is

$$\lambda = 0.205 \times 10^{-6} \left(\frac{\text{m.p.h.}}{\text{mi}^2}\right)$$

Performing the computations as described above, the optimal control at 1 and 2 hours was determined to be identical, to within computational accuracy, to the optimal control for the total final correction case. The optimal control at 56 hours was determined as

$$u^*(\hat{x}(56), 56) = \begin{cases} \dfrac{500 \operatorname{sgn}\left[(\hat{x}(56))\right] - \hat{x}(56)}{9750} & \text{if } |\hat{x}(56)| > 500 \text{ mi} \\ 0 & \text{if } |\hat{x}(56)| \leqq 500 \text{ mi} \end{cases} \tag{3-41}$$

The estimated cost was calculated as 13.5 m.p.h.

Thus, the optimal control for quadratic terminal cost guidance is essentially the same as the optimal control for total final correction guidance, the difference appearing in the control applied at 56 hours. Since the chosen value of λ prescribes heavy weighting to terminal errors greater than 1,000 miles, the terminal correction must eliminate almost all the error (i.e., only 500 miles remain). Therefore, the functions $C^*(\hat{x}, 56)$ for both cases, are almost identical. Since the solutions are obtained by taking steps backward in time, the remaining computations for both cases do not differ appreciably. It may be concluded that for quadratic terminal cost guidance, when heavy weighting is applied to terminal errors, the optimal control approaches the optimal control for total final correction guidance.

3.4 Minimum Fuel Fixed Time of Arrival Guidance

In this section the fixed time of arrival guidance problem is examined. It closely parallels the variable time of arrival problem of Section 3.2, except that terminal deviations parallel to the relative velocity vector v_R must also be controlled. Equations (3-5) and (3-6) are the difference equations satisfied by the deviations at the target. They are repeated here

$$\delta y_1'(n+1) = \delta y_1'(n) + \Gamma_{11}(n)\,\Delta v_1(n) + \Gamma_{12}(n)\,\Delta v_2(n) \qquad (3\text{-}42)$$

$$\delta y_2'(n+1) = \delta y_2'(n) + \Gamma_{21}(n)\,\Delta v_1(n) + \Gamma_{22}(n)\,\Delta v_2(n) \qquad (3\text{-}43)$$

along with the equation for the (4 x 2) matrix $\Gamma(n)$

$$\Gamma(n) = \Phi(q+1, n)\begin{bmatrix} O \\ I \end{bmatrix} \qquad (3\text{-}44)$$

If the two dimensional state vector x(n) is defined as

$$x(n) = \begin{bmatrix} \delta y_1'(n) \\ \\ \delta y_2'(n) \end{bmatrix} \qquad (3\text{-}45)$$

and the (2 x 2) nonsingular control sensitivity matrix $\theta(n+1,n)$ is given by

$$\theta(n+1,n) = \begin{bmatrix} \Gamma_{11}(n) & \Gamma_{12}(n) \\ \Gamma_{21}(n) & \Gamma_{22}(n) \end{bmatrix} = \begin{bmatrix} \Phi_{13}(q+1,n) & \Phi_{14}(q+1,n) \\ \Phi_{23}(q+1,n) & \Phi_{24}(q+1,n) \end{bmatrix} \tag{3-46}$$

with the two dimensional control specified as

$$u(n) = \begin{bmatrix} \Delta v_1(n) \\ \Delta v_2(n) \end{bmatrix} \tag{3-47}$$

then (3-42) and (3-43) can be written as a two dimensional vector equation

$$x(n+1) = x(n) + \theta(n+1,n)\, u(n) \tag{3-48}$$

This is the equation of state for the fixed time of arrival optimization problem. It is two dimensional because two coordinates of the target miss vector are to be controlled and the out of plane errors are ignored.

The optimization problem is to find the control function $u^*(\hat{x}(n),n)$ that minimizes the expected total fuel necessary to perform the midcourse velocity corrections. Correction times are specified and the velocity corrections are made using chemically fueled rocket engines. The incremental cost at each correction time is

$$L(x(n),\, u(n),\, n) = \| u(n) \| \tag{3-49}$$

and therefore $\overline{L}(\hat{x}(n), u(n), n)$ is given by

$$\overline{L}(\hat{x}(n),\, u(n),\, n) = \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_2 \, \| u(n) \| \; f_{x(n)}(\xi \,|\, \hat{x}(n)) = \| u(n) \| \tag{3-50}$$

Total final correction guidance is used so a total final correction is made at time t_m. Thus the control at t_m is specified as

$$u(\hat{x}(m), m) = -\theta(q+1, m)^{-1}\hat{x}(m) \tag{3-51}$$

and the terminal condition for the minimum expected value function is

$$C^*(\hat{x}(m), m) = \left\| \theta(q+1, m)^{-1}\hat{x}(m) \right\| \tag{3-52}$$

From equation (3-48), the state transition matrix for this problem is the (2 x 2) identity matrix. The error covariance matrices P(n) and the measurement information covariance matrices S′(n+j) are (2 x 2). From (2-69) and the recursion formulas for P(n) in (2-11), it can be shown that

$$S'(n+j) = P(n) - P(n+j) \tag{3-53}$$

Let t_n and t_{n+j} be two correction times, so there are j measurements taken in the interval $t_n < t \leqq t_{n+j}$. The minimum expected value function must satisfy (2-67) so

$$C^*(\hat{x}(n), n) = \min_{u(n)} \left\{ \| u(n) \| + \int_{-\infty}^{\infty} d\zeta_1 \int_{-\infty}^{\infty} d\zeta_2 \, f_{S'(n+j)}(\zeta) C^*(\hat{x}'(n+j)+\zeta, n+j) \right\} \tag{3-54}$$

and no restrictions are placed on the control so $\mathcal{U}(n)$ occupies the entire two dimensional control space. The extrapolated estimate is

$$\hat{x}'(n+j) = \hat{x}(n) + \theta(n+j, n)\, u(n) \tag{3-55}$$

Now define a function $C^{*\prime}$ as

$$C^{*\prime}(\hat{x}, n) = \int_{-\infty}^{\infty} d\zeta_1 \int_{-\infty}^{\infty} d\zeta_2 \, f_{S'(n+j)}(\zeta)\, C^*(\hat{x}+\zeta, n+j) \tag{3-56}$$

so the condition for optimization becomes

$$C^*(\hat{x}(n), n) = \min_{u(n)} \left\{ \| u(n) \| + C^{*\prime}(\hat{x}'(n+j), n) \right\} \tag{3-57}$$

To find the control function that minimizes the function in braces in (3-57), it is necessary to consider separately, optimal controls which are zero and optimal controls which are non-zero. Consider first those possible cases for which the optimal control $u^*(n)$ is equal to zero. Then for any non-zero control $u(n)$ the inequality

$$\left\| u(n) \right\| + C^{*\prime}(\hat{x}(n) + \theta(n+j, n)\, u(n); n) \geqq C^{*\prime}(\hat{x}(n), n) \tag{3-58}$$

must hold. $C^{*\prime}$ is analytic in the entire $\hat{x}$ space so (3-58) can be expanded as

$$\left\| u(n) \right\| + C^{*\prime}(\hat{x}(n), n) + \left[\frac{\partial C^{*\prime}(\hat{x}, n)}{\partial \hat{x}} \right]_{\hat{x} = \hat{x}(n)} \theta(n+j, n) u(n) + \epsilon(n) \geqq C^{*\prime}(\hat{x}(n), n) \tag{3-59}$$

where the gradient of $C^{*\prime}$ is defined as the row vector

$$\frac{\partial C^{*\prime}(\hat{x}, n)}{\partial \hat{x}} = \left[\frac{\partial C^{*\prime}(\hat{x}, n)}{\partial \hat{x}_j} \right] \qquad j = 1, 2 \tag{3-60}$$

and $\epsilon(n)$ is given by

$$\epsilon(n) = \frac{1}{2} u^T(n)\, \theta^T(n+j, n) \left[\frac{\partial^2 C^{*\prime}(\hat{x}, n)}{\partial \hat{x}^2} \right]_{\hat{x} = \xi} \theta(n+j, n)\, u(n) \tag{3-61}$$

with the matrix of second partial derivatives defined as

$$\frac{\partial^2 C^{*\prime}(\hat{x}, n)}{\partial \hat{x}^2} = \left[\frac{\partial^2 C^{*\prime}(\hat{x}, n)}{\partial \hat{x}_i\, \partial \hat{x}_j} \right] \qquad i = 1, 2;\ j = 1, 2 \tag{3-62}$$

and the vector ξ given by

$$\xi = \hat{x}(n) + \gamma\, \theta(n+j, n)\, u(n) \qquad 0 \leqq \gamma \leqq 1 \tag{3-63}$$

Dividing (3-59) through by $\| u(n) \|$ yields

$$1 + \left[\frac{\partial C^{*\prime}(\hat{x}, n)}{\partial \hat{x}} \right]_{\hat{x} = \hat{x}(n)} \theta(n+j, n) \quad \frac{u(n)}{\|u(n)\|} + \frac{\epsilon(n)}{\|u(n)\|} \geqq 0 \tag{3-64}$$

Inequality (3-64) must hold for all non-zero vectors u(n). By choosing the magnitude of u(n), the third term on the left of (3-64) can be made arbitrarily small. Similarly, by choosing u(n) in the proper direction, the second term can be made negative if the gradient is non-zero. Therefore, if (3-64) is to hold, a necessary condition is

$$\left\| \left[\frac{\partial C^{*\prime}(\hat{x}, n)}{\partial \hat{x}} \right]_{\hat{x} = \hat{x}(n)} \theta(n+j, n) \right\| \leqq 1 \tag{3-65}$$

In addition, it is shown in Appendix E that the matrix of second partial derivatives (3-62) is positive definite in the entire $\hat{x}$ space. Thus, since $\theta(n+j, n)$ is nonsingular, $\epsilon(n)$ given by (3-61) must be positive for all non-zero u(n). Hence, if the optimal control $u^*(n)$ is zero, it is unique and the corresponding $\hat{x}(n)$ must lie in the region $\mathfrak{J}(n)$ determined by

$$\mathfrak{J}(n) = \left\{ \hat{x} : \left\| \frac{\partial C^{*\prime}(\hat{x}, n)}{\partial \hat{x}} \quad \theta(n+j, n) \right\| \leqq 1 \right\} \tag{3-66}$$

because (3-59) holds with strict inequality for all non-zero controls u(n).

Attention is now turned to cases for which the optimal control $u^*(n)$ is non-zero. For this situation, the inequality

$$\left\| u^*(n) \right\| + C^{*\prime} (\hat{x}(n) + \theta(n+j, n) \, u^*(n); n) \leqq C^{*\prime}(\hat{x}(n), n) \tag{3-67}$$

must hold. Expanding as before yields

$$\left\| u^*(n) \right\| + C^{*\prime}(\hat{x}(n), n) + \left[\frac{\partial C^{*\prime}(\hat{x}, n)}{\partial \hat{x}} \right]_{\hat{x} = \hat{x}(n)} \theta(n+j, n) u^*(n) + \epsilon(n) \leqq C^{*\prime}(\hat{x}(n), n) \tag{3-68}$$

where now

$$\epsilon(n) = \frac{1}{2} u^{*T}(n)\, \theta^T(n+j, n) \left[\frac{\partial^2 C^{*\prime}(\hat{x}, n)}{\partial \hat{x}^2} \right]_{\hat{x}=\xi} \theta(n+j, n) u^*(n) \tag{3-69}$$

and the vector ξ is similarly redefined. Dividing (3-68) through by $\|u^*(n)\|$ produces

$$1 + \left[\frac{\partial C^{*\prime}(\hat{x}, n)}{\partial \hat{x}} \right]_{\hat{x}=\hat{x}(n)} \theta(n+j, n) \frac{u^*(n)}{\|u^*(n)\|} + \frac{\epsilon(n)}{\|u^*(n)\|} \leqq 0 \tag{3-70}$$

Since $\epsilon(n)$ is positive, a necessary condition for (3-70) is

$$\left\| \left[\frac{\partial C^{*\prime}(\hat{x}, n)}{\partial \hat{x}} \right]_{\hat{x}=\hat{x}(n)} \theta(n+j, n) \right\| > 1 \tag{3-71}$$

Thus, if a region $\mathfrak{R}(n)$ is defined as

$$\mathfrak{R}(n) = \left\{ \hat{x} : \left\| \frac{\partial C^{*\prime}(\hat{x}, n)}{\partial \hat{x}} \theta(n+j, n) \right\| > 1 \right\} \tag{3-72}$$

then if $u^*(n)$ is non-zero, the corresponding $\hat{x}(n)$ must lie in region $\mathfrak{R}(n)$. $\mathfrak{R}(n)$ and $\mathfrak{Z}(n)$ are disjoint and together occupy the entire $\hat{x}$ space, so if $\hat{x}(n)$ lies in the region $\mathfrak{Z}(n)$, the optimal control is zero and coversely if $\hat{x}(n)$ lies in region $\mathfrak{R}(n)$, then the optimal control is non-zero.

At this point there remains the problem of actually determining the non-zero optimal control when $\hat{x}(n)$ lies in $\mathfrak{R}(n)$. Since $u^*(n)$ is non-zero, the derivative of the function in braces in (3-57) can be used. It is written as

$$\frac{\partial \{\cdot\}}{\partial u(n)} = \frac{u^T(n)}{\|u(n)\|} + \left[\frac{\partial C^{*\prime}(\hat{x}, n)}{\partial \hat{x}} \right]_{\hat{x}=\hat{x}(n)+\theta(n+j, n)u(n)} \theta(n+j, n) \qquad \|u(n)\| > 0 \tag{3-73}$$

and setting the right hand side equal to zero produces necessary conditions for the optimal control. The direction of $u^*(n)$ is given by

$$\frac{u^*(n)}{\|u^*(n)\|} = - \theta^T(n+j,n) \left[\frac{\partial C^{*\prime}(\hat{x},n)}{\partial \hat{x}} \right]^T_{\hat{x}=\hat{x}(n)+\theta(n+j,n)u^*(n)} \tag{3-74}$$

and its magnitude must be such that

$$1 = \left\| \frac{\partial C^{*\prime}(\hat{x},n)}{\partial \hat{x}} \theta(n+j,n) \right\|_{\hat{x}=\hat{x}(n)+\theta(n+j,n)u^*(n)} \tag{3-75}$$

Also, the second derivative of the function in braces in (3-57) is

$$\frac{\partial^2\{\cdot\}}{\partial u(n)^2} = \left[\frac{I - \dfrac{u(n)u^T(n)}{u^T(n)\,u(n)}}{\sqrt{u^T(n)\,u(n)}} \right] + \theta^T(n+j,n) \left[\frac{\partial C^{*\prime}(\hat{x},n)}{\partial \hat{x}^2} \right]_{\hat{x}=\hat{x}(n)+\theta(n+j,n)u(n)} \theta(n+j,n) \tag{3-76}$$

It can be easily shown that the first term on the right of (3-76) is a positive semidefinite matrix for all non-zero vectors u(n). In Appendix E it is shown that $\dfrac{\partial^2 C^{*\prime}(\hat{x},n)}{\partial \hat{x}^2}$ is positive definite for all $\hat{x}$ and since $\theta(n+j,n)$ is nonsingular, the second term on the right in (3-76) is a positive definite matrix. It follows that a local minimum exists for $u^*(n)$ satisfying (3-74) and (3-75). Furthermore, the absolute value function is semi-concave and it is shown in Appendix E that $C^{*\prime}(\hat{x},n)$ is concave, so the function in braces in (3-57) must be concave. Therefore the function in braces is unimodal (i. e., there is only one extremal). Hence, $u^*(n)$ satisfying (3-74) and (3-75) provides the absolute minimum and is the unique optimal control.

By comparing (3-75) and (3-66) it can be seen that the optimal control drives the estimated state to some point on the boundary of $\mathfrak{S}(n)$. Because of its eventual importance, define the boundary of $\mathfrak{S}(n)$ as

$$\mathfrak{B}(n) = \left\{ \hat{x} : \left\| \frac{\partial C^{*\prime}(\hat{x}, n)}{\partial \hat{x}} \, \theta(n+j, n) \right\| = 1 \right\} \tag{3-77}$$

and let vectors b designate elements of $\mathfrak{B}(n)$. Thus, if b^* is the point in $\mathfrak{B}(n)$ to which the optimal control drives the estimated state, then the optimal control direction is given by (3-74) so

$$\frac{u^*(n)}{\left\| u^*(n) \right\|} = - \theta^T(n+j, n) \left[\frac{\partial C^{*\prime T}(\hat{x}, n)}{\partial \hat{x}} \right]_{\hat{x} = b^* \epsilon \mathfrak{B}(n)} \tag{3-78}$$

To each element b of $\mathfrak{B}(n)$ there is associated an optimal control direction and an optimal trajectory direction given by the vector d(b, n) where

$$d(b, n) = - \theta(n+j, n) \, \theta^T(n+j, n) \left[\frac{\partial C^{*\prime T}(\hat{x}, n)}{\partial \hat{x}} \right]_{\hat{x} = b \epsilon \mathfrak{B}(n)} \tag{3-79}$$

Since b^* is the point in $\mathfrak{B}(n)$ to which $u^*(n)$ drives the estimated state, b^* must satisfy

$$b^* = \hat{x}(n) + \rho \, d(b^*, n) \qquad b^* \epsilon \mathfrak{B}(n) \qquad \rho = \left\| u^*(n) \right\| > 0 \tag{3-80}$$

and once b^* is known, the optimal control is determined by

$$u^*(n) = \theta(n+j, n)^{-1} \left[b^* - \hat{x}(n) \right] \tag{3-81}$$

Hence the problem of determining the optimal control becomes the problem of obtaining b^* to satisfy (3-80). The point b^* in

$\mathcal{B}(n)$ must be found such that the difference between b^* and $\hat{x}(n)$ lies in the direction $d(b^*, n)$. A typical solution is illustrated in Fig. 3.6.

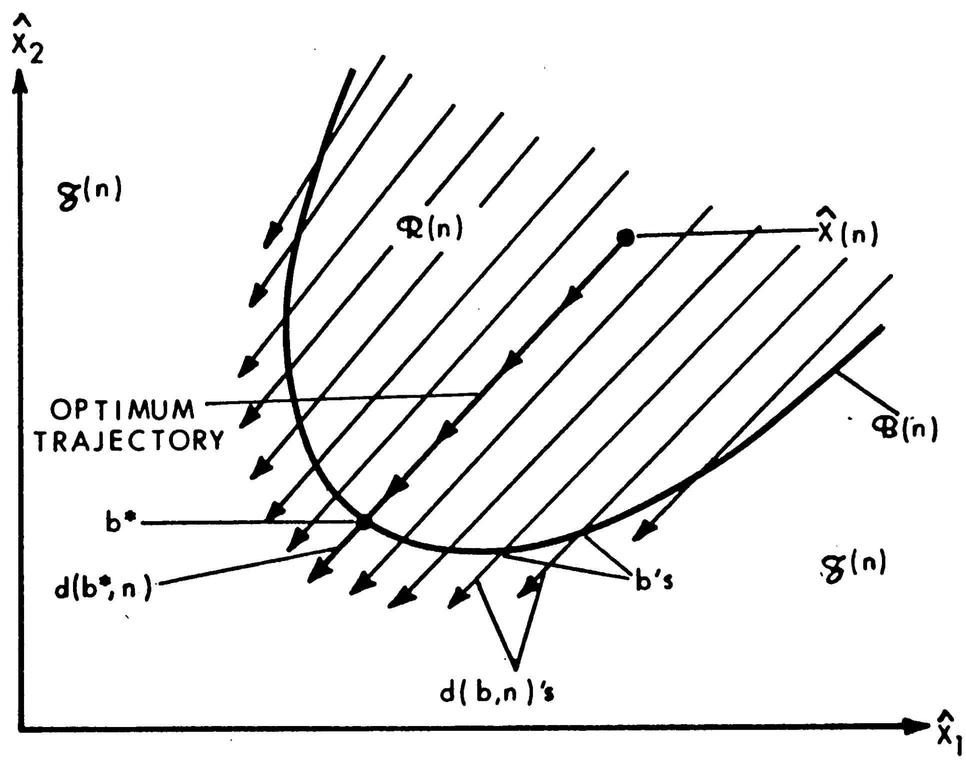

Fig. 3.6 Typical Solution for $b^*(n)$

To obtain the solution it is necessary to know the boundary $\mathcal{B}(n)$ and the vectors $d(b, n)$ as functions of points b in $\mathcal{B}(n)$. Knowing these, $\mathcal{B}(n)$ can be searched for the point b^* satisfying (3-80). The point is unique because the optimal control is unique.

To formulate an actual control function, it is necessary to determine the boundary curve $\mathcal{B}(n)$ in the $\hat{x}$ space and the vector functions d(b, n). Determining these requires digital computation. A two dimensional grid is necessary to represent the $\hat{x}$ space. Computation begins at the last correction time t_m, with the condition.

$$C^*(\hat{x}, m) = \left\| \theta(q+1, m)^{-1}\hat{x} \right\| \qquad m = n + j \tag{3-82}$$

Then an approximate solution of the two dimensional diffusion equation

$$\frac{\partial D(\hat{x}, \tau)}{\partial \tau} = \frac{1}{2}\left[S'_{11}(n+j)\frac{\partial^2 D(\hat{x},\tau)}{\partial \hat{x}_1^2} + 2\,S'_{12}(n+j)\frac{\partial^2 D(\hat{x},\tau)}{\partial \hat{x}_1 \partial \hat{x}_2} + S'_{22}(n+j)\frac{\partial^2 D(\hat{x},\tau)}{\partial \hat{x}_2^2}\right] \tag{3-83}$$

with the initial condition

$$D(\hat{x}, 0) = C^*(\hat{x}, n+j) \tag{3-84}$$

is calculated using central difference techniques, in the interval $0 \leqq \tau \leqq 1$. It follows that $C^{*\prime}(\hat{x}, n)$ is given by

$$C^{*\prime}(\hat{x}, n) = D(\hat{x}, 1) \tag{3-85}$$

The boundary curve $\mathcal{B}(n)$ is obtained by searching the two dimensional $\hat{x}$ grid to find points b that satisfy the equation

$$\left\| \frac{\partial C^{*\prime}(\hat{x}, n)}{\partial \hat{x}} \; \theta(n+j, n) \right\|_{\hat{x} = b} = 1 \tag{3-86}$$

and at each point b thus obtained, the vector d(b, n) is calculated as

$$d(b, n) = -\,\theta(n+j, n)\theta^T(n+j,n)\left[\frac{\partial C^{*\prime T}(\hat{x}, n)}{\partial \hat{x}}\right]_{\hat{x} = b} \tag{3-87}$$

By making the search fine enough, the points b will lie sufficiently close together to give an accurate representation of the boundary $\mathcal{B}(n)$ and the vector function d(b, n). Then, at each point of the $\hat{x}$ grid, the function $C^*(\hat{x}, n)$ is determined by

$$C^*(\hat{x}, n) = \begin{cases} \left\| \theta(n+j, n)^{-1} \left[b^* - \hat{x} \right] \right\| + C^{*\prime}(b^*, n) & \text{if} \quad \hat{x} \in \mathfrak{R}(n) \\ C^{*\prime}(\hat{x}, n) & \text{if} \quad \hat{x} \in \mathfrak{F}(n) \end{cases} \tag{3-88}$$

where b^* is attained by solving the transcendental equation

$$b^* - \hat{x} = \rho \, d(b^*, n) \qquad \begin{matrix} \rho > 0 \\ b^* \in \mathfrak{B}(n) \end{matrix} \tag{3-89}$$

Computations beginning with (3-83) are then repeated the required number of times until the initial time is reached. Having obtained the curves $\mathfrak{B}(n)$ and vector functions $d(b, n)$, the optimal control is given by

$$u^*(\hat{x}(n), n) = \begin{cases} \theta(n+j, n)^{-1} \left[b^* - \hat{x}(n) \right] & \text{if } \hat{x}(n) \in \mathfrak{R}(n) \\ 0 & \text{if } \hat{x}(n) \in \mathfrak{F}(n) \end{cases} \tag{3-90}$$

where b^* is determined by the solution of

$$b^* - \hat{x}(n) = \rho \, d(b^*, n) \qquad \begin{matrix} \rho > 0 \\ b^* \in \mathfrak{B}(n) \end{matrix} \tag{3-91}$$

3.5 Numerical Example of Minimum Fuel F.T.A. Guidance

In order to demonstrate the numerical solution of a fixed time of arrival problem, the Earth-Mars trajectory described in Section 3.3 is utilized once again. The reference trajectory parameters and the injection error covariance matrix are as listed in Tables 3.1 and 3.2. Optimal corrections are applied at one and two hours and total final correction guidance is used with the final correction applied at 56 hours.

To obtain the optimal control functions, the (2 x 2) control sensitivity matrices were calculated at the correction times. As a means of conveniently describing the control vectors and terminal miss vectors, separate coordinate systems were chosen at the vehicle and at the target point. The coordinate system at the vehicle has an axis (1) in the trajectory plane, orthogonal to the vehicle reference velocity, and pointing generally away from the sun, and an axis (2) pointing in the direction of the vehicle reference velocity. The target coordinate system has an axis (1) pointing in the radial direction away from the sun and an axis (2) tangential to the Mars orbit, which is assumed circular. These coordinate systems are illustrated in Fig. 3.7.

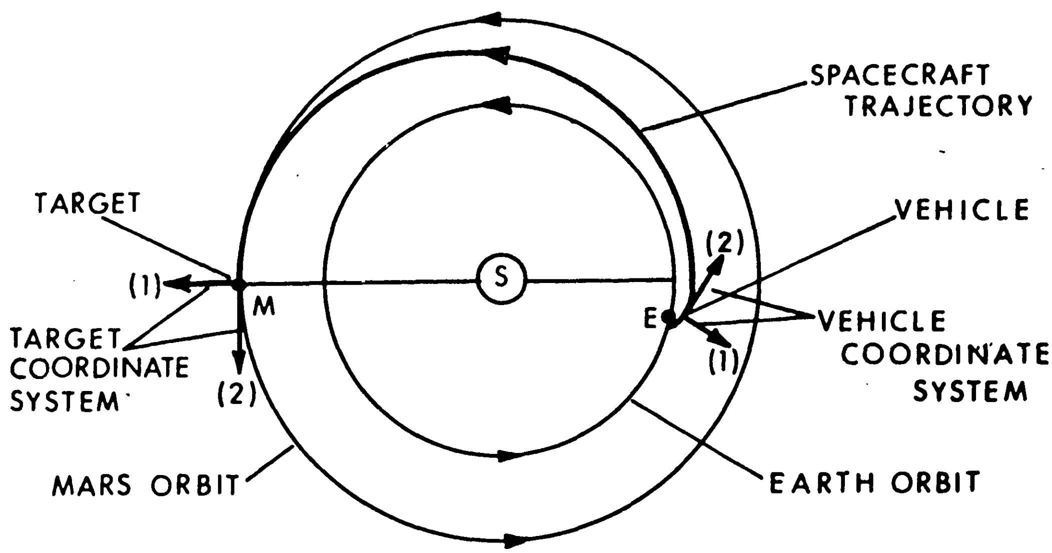

Fig. 3.7 Vehicle, Target Coordinate Systems

The control sensitivity matrices $\theta(n+j, n)$ relate control vectors resolved in the vehicle coordinate system to position changes resolved in target coordinate system. The control sensitivity matrices at the various correction times were computed as follows:

$$\theta(2,1) = \begin{bmatrix} 210 & 19,380 \\ -4,779 & -39,830 \end{bmatrix} \left(\frac{\text{mi.}}{\text{m.p.h.}}\right)$$

$$\theta(56,2) = \begin{bmatrix} 228 & 15,780 \\ -5,405 & -31,690 \end{bmatrix} \left(\frac{\text{mi.}}{\text{m.p.h.}}\right)$$

$$\theta(6160,56) = \begin{bmatrix} \cong 10^{-3} & 9,754 \\ -6,395 & -18,610 \end{bmatrix} \left(\frac{\text{mi.}}{\text{m.p.h.}}\right)$$

With the injection error variances given in Table 3.2 and assuming velocity measurement error variances of $0.01(\text{m.p.h.})^2$, the (4 x 4) estimation error covariance matrices, at the target, were computed. For purposes of determining the optimal control, however, only the position error covariance is necessary. It appears as the upper left hand corner (2 x 2) submatrix of the complete (4 x 4) error covariance matrix. The (2 x 2) position error covariance matrices, at the various times of interest, are given as follows:

$$P(0) = \begin{bmatrix} 3.661 \times 10^{10} & -8.351 \times 10^{10} \\ -8.351 \times 10^{10} & 1.960 \times 10^{11} \end{bmatrix} (\text{mi.})^2$$

$$P(1) = \begin{bmatrix} 1.424 \times 10^{10} & -2.552 \times 10^{10} \\ -2.552 \times 10^{10} & 4.574 \times 10^{10} \end{bmatrix} (\text{mi.})^2$$

$$P(2) = \begin{bmatrix} 5.592 \times 10^{7} & -6.129 \times 10^{7} \\ -6.129 \times 10^{7} & 7.515 \times 10^{7} \end{bmatrix} (\text{mi.})^2$$

$$P(56) = \begin{bmatrix} 3.300 \times 10^{4} & 9.032 \times 10^{4} \\ 9.032 \times 10^{4} & 1.819 \times 10^{6} \end{bmatrix} (\text{mi.})^2$$

From these data, the minimum expected value functions were computed as described in Section 3.4. At each correction time the $\hat{x}$ space was searched to obtain the boundary curves $\mathcal{B}(n)$ and the optimal trajectory direction vectors d(b, n). As a means of describing the $\mathcal{B}(n)$ curves and the vector directions d(b, n), the target coordinate system shown in Fig. 3.8 is utilized.

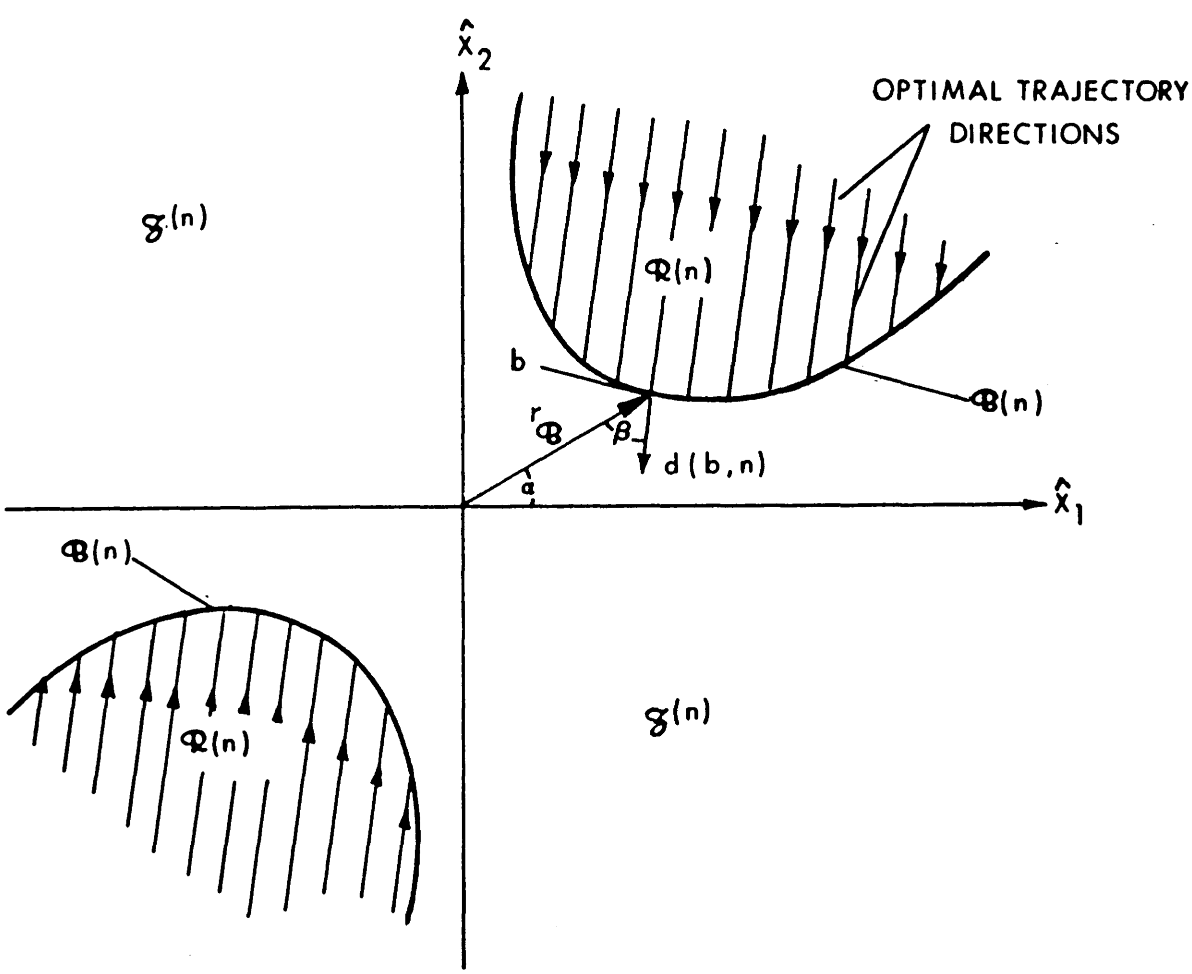

Fig. 3.8 Definition of α, β and $r_{\mathcal{B}}$

Thus, α is the polar angle measured from the $\hat{x}_1$ axis, $r_{\mathcal{B}}$ is the radial distance to the $\mathcal{B}(n)$ curve and β is the angle between the radial direction and the vector d(b, n). Therefore $r_{\mathcal{B}}$ and β as functions of α, describe the boundary curve $\mathcal{B}(n)$ and the directions of the vectors d(b, n). Table 3.4 contains the calculated values of $r_{\mathcal{B}}(\alpha)$ and $\beta(\alpha)$ for the correction at one hour. The tabulated range of α is $-90^{\circ} \leqq \alpha \leqq +90^{\circ}$, at increments of 2°. Obviously, the problem is symmetric about the origin so

$$r_{\mathcal{B}}(\alpha) = r_{\mathcal{B}}(\alpha - 180^{\circ}) \qquad 90^{\circ} < \alpha < 270^{\circ}$$

$$\beta(\alpha) = \beta(\alpha - 180^{\circ}) \qquad 90^{\circ} < \alpha < 270^{\circ}$$

The computations indicate that the $\mathcal{B}(1)$ curve has asymptotes at α values of approximately -46°, -58°, -64° and -81°. Fig. 3.9 illustrates the $\mathcal{B}(1)$ curve and the optimal trajectory directions. Similarly, Table 3.5 lists $r_{\mathcal{B}}(\alpha)$ and $\beta(\alpha)$ for the correction at two hours. Asymptotes in the $\mathcal{B}(2)$ curve occur at approximately -47° and -79°. Fig. 3.10 illustrates these tabulated values for the correction at two hours.

Calculation of the actual optimal control for cases in which $\hat{x}(n) \in \mathcal{R}(n)$ becomes the problem of determining α so that (3-80) is satisfied. To that end, define a vector $a(\alpha)$ as

$$a(\alpha) = \begin{bmatrix} \sin(\alpha + \beta(\alpha)) \\ -\cos(\alpha + \beta(\alpha)) \end{bmatrix} \tag{3-92}$$

From Fig. 3.8 it is clear that for a given value of α, the corresponding d(b, n) vector is

$$d(b, n) = \left\| d(b, n) \right\| \begin{bmatrix} -\cos(\alpha + \beta(\alpha)) \\ -\sin(\alpha + \beta(\alpha)) \end{bmatrix} \tag{3-93}$$

and it follows that

$$a^{T}(\alpha)\, d(b, n) = 0 \tag{3-94}$$

α deg	$r_{\oplus}(\alpha)$ mi	$\beta(\alpha)$ deg
+90	∞	
⋮	⋮	⋮
-46	∞	
-46.76	1.668×10^{6}	-15.63
-48	6.365×10^{5}	-13.91
-50	6.232×10^{5}	-12.13
-52	6.669×10^{5}	-10.09
-54	7.118×10^{5}	- 8.10
-56	8.865×10^{5}	- 6.01
-57.35	1.447×10^{6}	- 4.96
-58	∞	
⋮	⋮	⋮
-64	∞	
-64.72	1.335×10^{6}	0.09
-66	3.991×10^{5}	1.48
-68	3.748×10^{5}	3.22
-70	3.505×10^{5}	5.10
-72	3.287×10^{5}	6.87
-74	3.159×10^{5}	8.64
-76	2.883×10^{5}	10.52
-78	2.857×10^{5}	12.43
-80	2.894×10^{5}	14.24
-80.47	1.219×10^{6}	14.46
-82	∞	
⋮	⋮	⋮
-90	∞	

Table 3.4 $r_{\oplus}(\alpha)$ and $\beta(\alpha)$ for Correction at One Hour

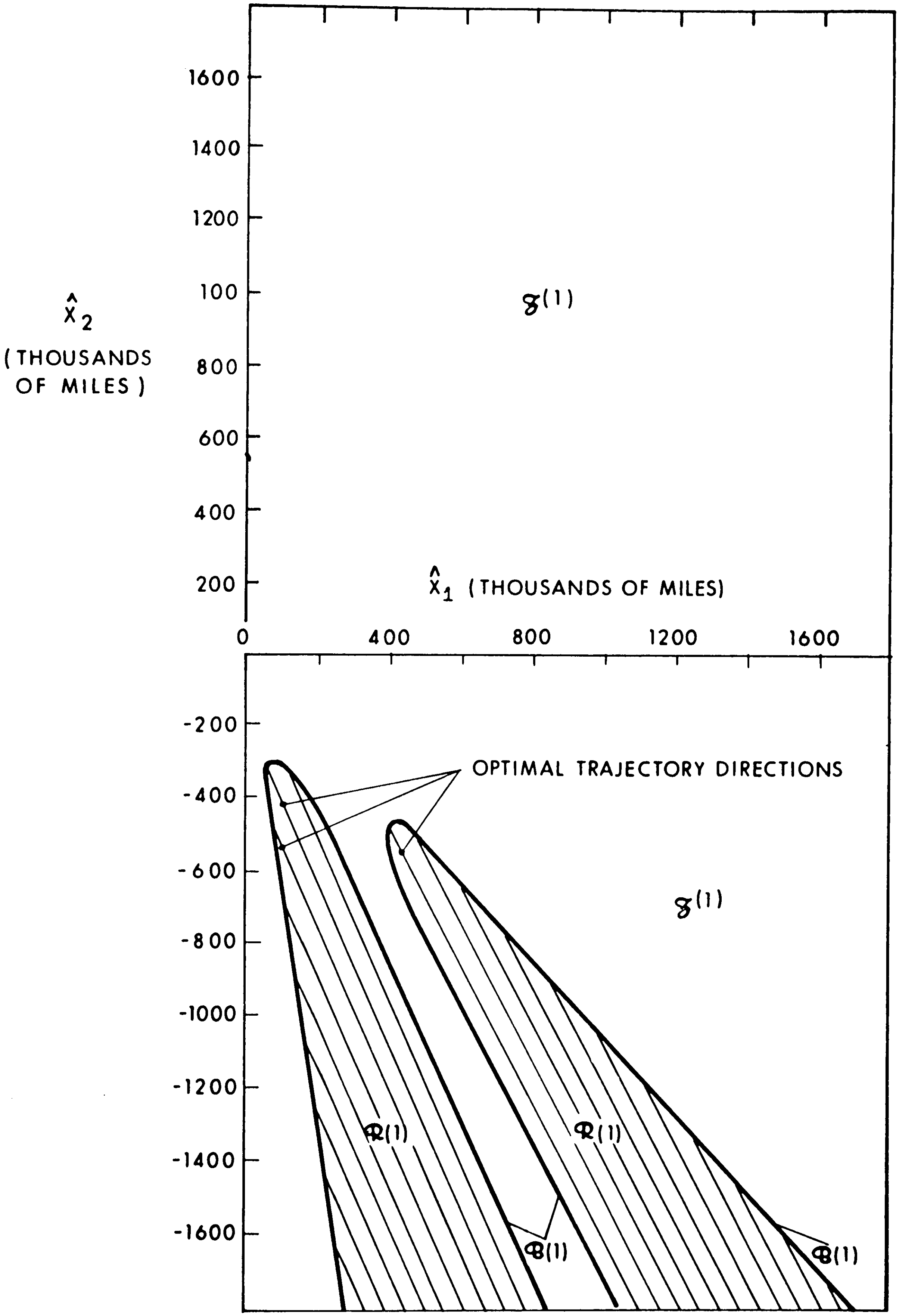

Fig. 3.9 $\mathcal{B}(1)$ and Optimal Trajectory Directions at One Hour

α deg	$r_{\oplus}(\alpha)$ mi	$\beta(\alpha)$ deg
+90	∞	
. . .	. . .	. . .
-46	∞	
-47.12	6.673×10^4	-13.48
-48	2.765×10^4	-13.00
-50	2.410×10^4	-11.52
-52	2.226×10^4	-10.06
-54	2.107×10^4	- 8.46
-56	2.019×10^4	- 6.81
-58	1.869×10^4	- 5.20
-60	1.827×10^4	- 3.54
-62	1.805×10^4	- 1.83
-64	1.739×10^4	- 0.10
-66	1.700×10^4	1.61
-68	1.715×10^4	3.35
-70	1.737×10^4	5.07
-72	1.778×10^4	6.77
-74	1.763×10^4	8.51
-76	1.925×10^4	10.16
-78	4.469×10^4	11.07
-78.16	4.907×10^4	11.24
-80	∞	
. . .	. . .	. . .
-90	∞	

Table 3.5 $r_{\oplus}(\alpha)$ and $\beta(\alpha)$ for Correction at Two Hours

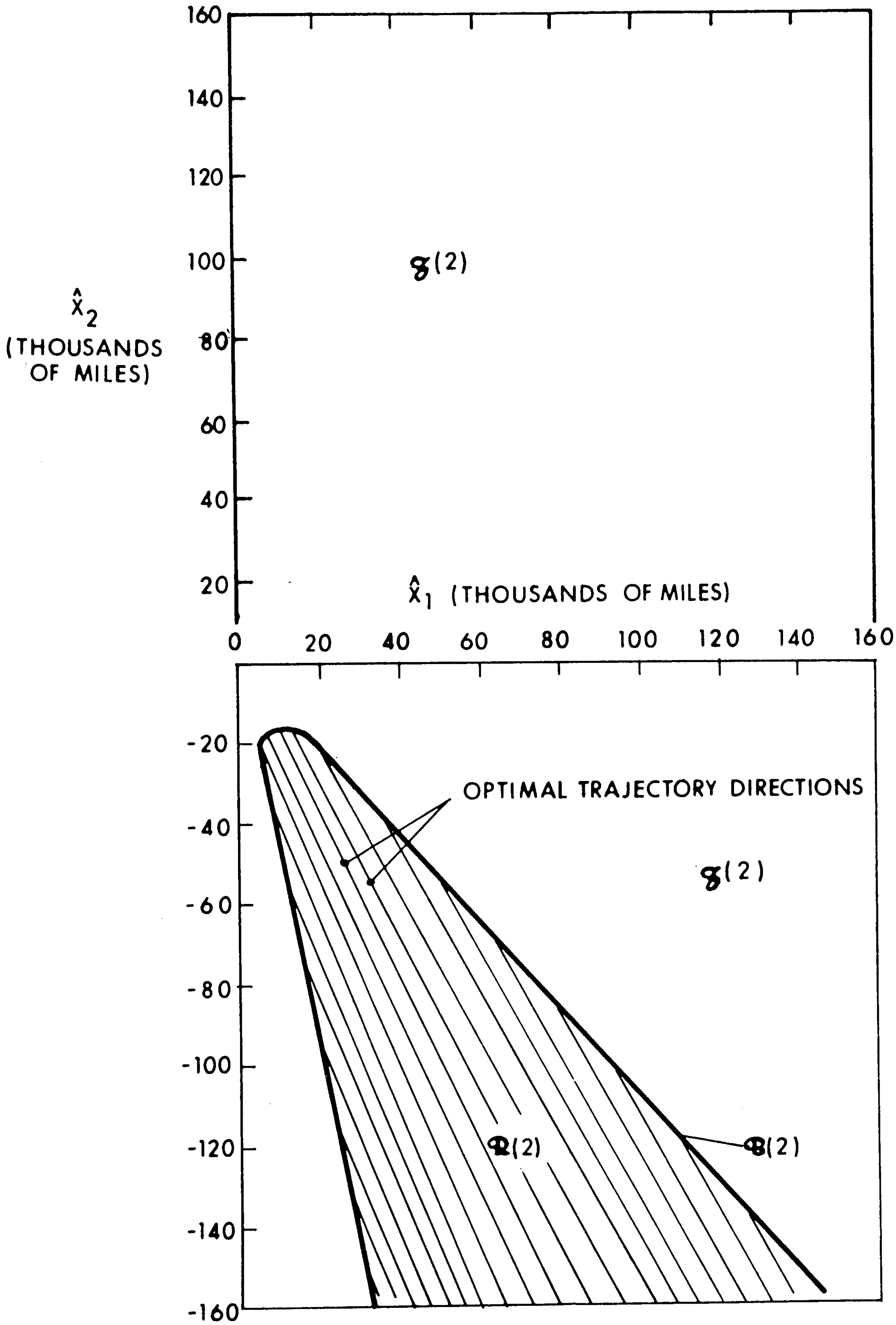

Fig. 3.10 $\mathcal{B}(2)$ and Optimal Trajectory Directions at Two Hours

so $a(\alpha)$ and the corresponding $d(b, n)$ are orthogonal. By taking the inner product of $a(\alpha)$ with both sides of equation (3-80) there results an equation that must be satisfied by the optimum value of α .

$$a^T(\alpha^*)\left[b^* - \hat{x}(n)\right] = 0 \tag{3-95}$$

Then, from Fig. 3.8 the vector b^* is given by

$$b^* = r_{\mathcal{C}}(\alpha^*) \begin{bmatrix} \cos(\alpha^*) \\ \sin(\alpha^*) \end{bmatrix} \tag{3-96}$$

so (3-95) becomes

$$\left[r_{\mathcal{C}}(\alpha^*) \cos(\alpha^*) - \hat{x}_1(n)\right] \sin(\alpha^* + \beta(\alpha^*)) - \left[r_{\mathcal{C}}(\alpha^*) \sin(\alpha^*) - \hat{x}_2(n)\right] \cos(\alpha^* + \beta(\alpha^*)) = 0 \tag{3-97}$$

Equation (3-97) is the condition to be satisfied by the optimal angle α^*. The corresponding optimal control is obtained by substituting α^* into (3-96) and applying the resulting b^* to equation (3-90). The actual calculation of α^* may be accomplished in many ways. The method used in this example was to search through the tabulated values of $r_{\mathcal{C}}(\alpha)$ and $\beta(\alpha)$ until two adjacent values of α bracketed the solution. Then linear interpolation was used to approximate the actual α^*. More sophisticated approaches could use polynominal approximations to $r_{\mathcal{C}}(\alpha)$ and $\beta(\alpha)$ etc. and iteration techniques like Newton-Raphson might be applied to obtain the approximate solution of (3-97). The most practical technique for an actual control computer would of course depend upon the required accuracies, the characteristics of the computer, etc.

From the computations of the minimum expected value functions it was found that the minimum expected fuel required to perform fixed time of arrival guidance is 31.7 m.p.h. A comparison can be made between this value and the expected cost to perform a single total correction at 56 hours. The single total correction cost is 35.6 m.p.h., an increase of about 12% .

An investigation was also made to determine the cost of eliminating one of the two early corrections. It was found that eliminating the correction at one hour produces the lesser increase in cost. If only two corrections are made, one at two hours and a total correction at 56 hours, the optimal correction at two hours is determined by the values in Table 3.5. It was found that the minimum cost for the optimal two correction case is 32.2 m.p.h., or an increase of 1.6% over the optimal three correction case. Thus the correction at one hour provides very little decrease in the expected total fuel for this example.

3.6 Minimum Fuel Atmospheric Re-entry Guidance

Consider the terminal phase of a spacecraft mission. It is desired to guide a re-entry vehicle through the atmosphere so as to land with acceptable accuracy at a target point on earth. The vehicle is a wingless, axially symmetric body with its center of gravity displaced from the axis of symmetry. The c.g. offset causes the vehicle to trim at a non-zero angle of attack, thereby providing lift. Control of the flight path is attained by rolling the vehicle about the velocity vector using attitude control jets. Since the lift vector can thus be directed anywhere in the plane perpendicular to the velocity vector, the flight path can be altered in any direction.

It is assumed that a reference trajectory exists, which passes through a nominal initial entry point and the target. Deviations in the spacecraft trajectory are measured from this reference trajectory. The problem of controlling the range deviation can be effectively decoupled from the problem of controlling the lateral deviation. Range is controlled by determining the roll angle magnitude which, if held constant will produce the necessary vertical component of lift. By rolling the spacecraft left or right to this angle, the desired range will be attained and the resulting horizontal component of lift can be used for lateral control of the spacecraft trajectory. Each roll maneuver requires fuel, however, so the ideal method for conserving fuel is to roll the spacecraft first one way, and hold the required roll angle magnitude, and then roll the other way, and hold the required roll angle magnitude. The second roll maneuver must obviously be timed precisely to hit the target.

Nongravitational accelerations of the vehicle are measured by an on board inertial measurement unit and this data is available to the controller. It processes the information to provide estimates of position and velocity, utilizing recursion formulas (2-19) and (2-20). These estimated values are then used to predict the miss vector at the target. Since the predicted miss vector contains random errors due to instrument inaccuracies and because the future path of the vehicle is perturbed by random disturbances, the ideal control scheme described in the previous paragraph is impractical. As an alternative the stochastic optimal control problem will be solved to provide an optimal feedback controller.

In what follows, only the lateral control problem will be considered. It is assumed that the range problem is handled separately and that for a particular trajectory, the required roll angle magnitude is determined at the initial time and stays essentially constant for the duration of the re-entry. Thus the roll angle magnitude will be different for each re-entry, depending upon random initial conditions, but it is known by the controller at the initial time. Define γ as the roll angle magnitude required to attain the proper range. Let the control have only discrete values of + 1, 0 or -1. Application of control +1 or -1 commands the vehicle attitude control system to roll the vehicle to the angle $+\gamma$ or $-\gamma$, respectively. Application of zero control retains the vehicle at its present roll angle. Define a switch function $x_1(n)$ to satisfy the difference equation

$$x_1(n+1) = x_1(n) + 2\,u(n) \qquad x_1(0) = \pm 1 \tag{3-98}$$

and require the control at time t_n to satisfy the following rule.

$$u(n) = \begin{cases} 0 \text{ or } -1 & \text{if } x_1(n) = +1 \\ 0 \text{ or } +1 & \text{if } x_1(n) = -1 \end{cases} \tag{3-99}$$

Therefore, $x_1(n)$ is a deterministic quantity that can have only discrete values of ± 1. If the time between control applications is longer than the time required to roll the vehicle through an angle 2γ, then at times t_n the roll angle will be either $+\gamma$ or $-\gamma$. Also, $x_1(n)$ indicates the direction in which the lift vector is pointed for if $x_1(n)$ equals +1 or -1 the roll angle at t_n will equal $+\gamma$ or $-\gamma$, respectively.

The switch function $x_1(n)$ is one state variable for the lateral control problem. A second state variable is the lateral deviation from the reference trajectory, extrapolated forward to the target. It satisfies the difference equation .

$$x_2(n+1) = x_2(n) + F(n)x_1(n) + G(n)u(n) + v_2(n) \tag{3-100}$$

with

$v_2(n)$ = normally distributed independent random disturbances with zero mean and variance $V_2(n)$

The second term on the right of (3-100) accounts for the effect of the roll angle at time t_n, on the lateral miss distance at the target. The third term accounts for the effect of a roll maneuver which may be initiated at time t_n.

With these definitions it is possible to state the minimization problem in specific terms. It is desired to minimize the expected total number of roll maneuvers plus a quadratic penalty imposed on the lateral miss distance at the target. The cost to be minimized is

$$J = E\left[\sum_{n=1}^{q} \left|u(n)\right| + \frac{\lambda}{2} x_2^2(q+1)\right] \tag{3-101}$$

where t_{q+1} is the terminal time and

$$\lambda \equiv \text{terminal cost weighting}$$

The functions $\overline{L}(\hat{x}(n), u(n)n)$ and $\overline{\phi}(\hat{x}(q+1))$ are

$$\overline{L}(\hat{x}(n), u(n), n) = |u(n)| \tag{3-102}$$

$$\overline{\phi}(\hat{x}(q+1)) = \frac{\lambda}{2}\left[\hat{x}_2^2(q+1) + P_2(q+1)\right] \tag{3-103}$$

with the definitions

$\hat{x}_2(n)$ = estimated lateral target miss distance at time t_n

$P_2(n)$ = variance of error in estimated lateral target miss distance at time t_n

If ϕ is not quadratic, the evaluation of $\overline{\phi}$ usually involves numerical integration. In any case, no real difficulty is encountered if ϕ is not quadratic. From (3-103), the terminal condition in the minimum expected value function is

$$C^*(x_1(q+1), \hat{x}_2(q+1), q+1) = \frac{\lambda}{2}\left[\hat{x}_2^2(q+1) + P_2(q+1)\right] \tag{3-104}$$

Since $x_1(n)$ is known deterministically by the controller, equations (3-100), (2-55) and the recursion formulas for $P(n)$ in (2-11), provide an expression for the variance of $s_2(n+1)$

$$S_2(n+1) = P_2(n) - P_2(n+1) + V_2(n) \tag{3-105}$$

Now if t_n and t_{n+1} are times at which control may be applied, the minimum expected value function must satisfy

$$C^*(x_1(n), \hat{x}_2(n), n) =$$

$$\min_{u(n)\in\mathcal{U}(n)}\left\{|u(n)| + \int_{-\infty}^{\infty} d\zeta\, f_{s_2(n+1)}(\zeta) C^*(x_1(n+1), \hat{x}_2'(n+1) + \zeta, n+1)\right\} \tag{3-106}$$

where

$$x_1(n+1) = x_1(n) + 2\,u(n) \tag{3-107}$$

$$\hat{x}_2{}'(n+1) = \hat{x}_2(n) + F(n)x_1(n) + G(n)u(n) \tag{3-108}$$

and the control set $\mathcal{U}(n)$ is determined by the control rule (3-99). Let the function $C^{*\prime}(x_1(n+1), \hat{x}_2{}'(n+1), n)$ be defined as

$$C^{*\prime}(x_1(n+1), \hat{x}_2{}'(n+1), n) = \int_{-\infty}^{\infty} d\zeta f_{s_2(n+1)}(\zeta) C^*(x_1(n+1), \hat{x}_2{}'(n+1) + \zeta, n+1) \tag{3-109}$$

Then (3-106) becomes

$$C^*(x_1(n), \hat{x}_2(n), n) = \min_{u(n)\in\mathcal{U}(n)} \left\{ |u(n)| + C^{*\prime}(x_1(n+1), \hat{x}_2{}'(n+1), n) \right\} \tag{3-110}$$

and since $x_1(n)$ can have only the discrete values ± 1, (3-110) may be written as two equations thus

$$C^*(1, \hat{x}_2(n), n) = \min_{u(n) = 0, -1} \left\{ |u(n)| + C^{*\prime}(1+2u(n), \hat{x}_2(n)+F(n)+G(n)u(n), n) \right\} \tag{3-111}$$

$$C^*(-1, \hat{x}_2(n), n) = \min_{u(n) = 0, +1} \left\{ |u(n)| + C^{*\prime}(-1+2u(n), \hat{x}_2(n) - F(n)+G(n)\,u(n), n) \right\} \tag{3-112}$$

By satisfying (3-111) and (3-112), the optimal control is obtained as a function of $x_1(n)$ and $\hat{x}_2(n)$.

Solutions require digital computation in the x_1, $\hat{x}_2$ space. Since x_1 can have values ± 1 and $\hat{x}_2$ can take any value on the real axis, the two quantities $C^*(1, \hat{x}_2, n)$ and $C^*(-1, \hat{x}_2, n)$ must be stored as functions of $\hat{x}_2$. Calculations begin at the terminal time with the conditions

$$C^*(1, \hat{x}_2, q+1) = C^*(-1, \hat{x}_2, q+1) = \frac{\lambda}{2}(\hat{x}_2^2 + P_2(q+1)) \tag{3-113}$$

Approximate solutions of the diffusion equations

$$\frac{\partial D(1,\hat{x}_2,\tau)}{\partial \tau} = \frac{1}{2}\, S_2(n+1)\, \frac{\partial^2 D(1,\hat{x}_2,\tau)}{\partial \hat{x}_2^{\,2}} \tag{3-114}$$

$$\frac{\partial D(-1,\hat{x}_2,\tau)}{\partial \tau} = \frac{1}{2}\, S_2(n+1)\, \frac{\partial^2 D(-1,x_2,\tau)}{\partial \hat{x}_2^{\,2}} \tag{3-115}$$

with the initial conditions

$$D(1,\hat{x}_2,0) = C^*(1,\hat{x}_2, n+1) \tag{3-116}$$

$$D(-1,\hat{x}_2,0) = C^*(-1,\hat{x}_2,n+1) \tag{3-117}$$

are calculated using central difference techniques in the interval $0 \leqq \tau \leqq 1$. Then $C^{*\prime}(1,\hat{x}_2,n)$ and $C^{*\prime}(-1,\hat{x}_2,n)$ are given by

$$C^{*\prime}(1,\hat{x}_2,n) = D(1,\hat{x}_2,1) \tag{3-118}$$

$$C^{*\prime}(-1,\hat{x}_2,n) = D(-1,\hat{x}_2,1) \tag{3-119}$$

Finally, the optimal control as a function of x_1, $\hat{x}_2$ and n is obtained by satisfying (3-120) and (3-121).

$$C^*(1,\hat{x}_2,n) = \min_{u=0,-1}\left\{ |u| + C^{*\prime}(1+2u,\hat{x}_2+F(n)+G(n)u,n)\right\} \tag{3-120}$$

$$C^*(-1,\hat{x}_2,n) = \min_{u=0,+1}\left\{ |u| + C^{*\prime}(-1+2u,\hat{x}_2-F(n)+G(n)u,n)\right\} \tag{3-121}$$

Calculations beginning with (3-114) are then repeated the required number of times until the initial time is reached.

Clearly, from purely physical reasoning, the problem must be symmetric about the origin of the $x_1, \hat{x}_2$ space so

$$C^*(-1,\hat{x}_2,n) = C^*(1,-\hat{x}_2,n)$$

$$C^{*\prime}(-1,\hat{x}_2,n) = C^{*\prime}(1,-\hat{x}_2,n)$$

and only $C^*(1,\hat{x}_2,n)$ and $C^{*\prime}(1,\hat{x}_2,n)$ must be computed and stored as described above. Typically, the optimal control is determined by a threshold region on the $\hat{x}_2$ axis. If $\hat{x}_2(n)$ lies outside the threshold then the vehicle is rolled over and if $\hat{x}_2(n)$ lies within the threshold, no action is taken. The minimizations required in (3-120) and (3-121) will produce the boundaries of these threshold regions at times t_n.

3.7 Numerical Example of Re-entry Guidance

As a means of demonstrating a numerical solution of the re-entry problem, a much simplified model of the Apollo re-entry system is used. (See (55), pps. 5-1, 5-2). The nominal initial conditions are given as follows:

altitude	= 75 mi.
velocity	= 24,700 m.p.h.
flight path angle	= - 6 deg.
lift/drag	= 0.34
weight/drag coeff. X area	= 66 psf.
latitude	= -12.7 deg.
longitude	= 122.9 deg. east
azimuth	= 61 deg.

and the desired range is 2,000 miles. Total re-entry time is approximately 700 seconds, but lateral guidance is utilized only in the final 300 seconds of the flight. The nominal roll angle magnitude to attain the proper range is about 50 degrees. Vehicle roll rate is 20 degrees per second so the time to complete a roll maneuver is about 5 seconds. Time increments between control applications are 10 seconds so

$$t_n = 400 + 10n \text{ sec.}$$

$$q + 1 = 30$$

Sensitivities F(n) and G(n) are linear functions of time determined by

$$F(n) = 3.33 \left[\frac{300 - 10n}{300}\right] \text{mi.}$$

$$G(n) = 5.00 \left[\frac{300 - 10n}{300}\right] \text{mi.}$$

Thus, for example, in the first 10 seconds of lateral control flight, an initial roll angle of + 50 deg. produces a change of + 3.33 miles at the target. Also, since it takes just half the control interval to roll the vehicle over, the value of G(n) at any time is just three halves the value of F(n). Further, it is assumed that $V_2(n)$, the variance of random disturbances to the spacecraft trajectory at the target, decreases as a quadratic function of the time to complete the trajectory. Such will be the case if the variance of random velocity disturbances at the vehicle is constant in time. The initial value of $V_2(n)$ is assumed to be 2.5 mi.^{2} so

$$V_2(n) = 2.50 \left[\frac{300 - 10n}{300}\right]^2 \text{mi.}^2$$

In order to obtain some physical feeling for the magnitude of $V_2(n)$, consider its sum over all possible values of n. It can be shown that

$$\sum_{n=0}^{29} V_2(n) \cong 25 \text{ mi.}^2$$

This sum represents the variance of the total random disturbance to the spacecraft trajectory at the target so the total r.m.s. disturbance is 5 miles. If in addition it is assumed that the inertial measurement unit does perfect measurement of the random velocity disturbances, the variance of estimation errors $P_2(n)$ is constant in time and depends only on the initial position and velocity estimation error variances. Therefore $S_2(n+1)$ in (3-105) becomes

$$S_2(n+1) = V_2(n)$$

and the assumed value of $P_2(n)$ is

$$P_2(n) = 25 \text{ mi}^2$$

Finally, the terminal cost penalty for a 5 mile error at the target is made equal to one roll maneuver. The terminal cost function is

$$\phi(x_2(30)) = 0.04\, x_2^2(30)$$

so

$$\overline{\phi}(\hat{x}_2(30)) = 0.04\left[\hat{x}_2^2(30) + 25\right]$$

The computation method described in Section 3.6 was used to calculate the optimal control functions. They are given by

$$u^*(\hat{x}_2(n), x_1(n), n) = \left\{ \begin{array}{ll} +1 & \text{if} \quad x_1(n) = -1,\ \hat{x}_2(n) < -T(n) \\ -1 & \text{if} \quad x_1(n) = +1,\ \hat{x}_2(n) > T(n) \\ 0 & \text{otherwise} \end{array} \right\}$$

So if $x_1(n)$ is $+1$ (i.e., the spacecraft has positive roll), then if $\hat{x}_2(n)$ is less than a threshold value $T(n)$, no action is taken and if $\hat{x}_2(n)$ is larger than the threshold $T(n)$, the spacecraft is rolled over. Similar conditions hold for $x_1(n) = -1$.

Digital computation produced the values of $T(n)$ listed in Table 3.6. Fig. 3.11 illustrates the thresholds, and a typical reentry trajectory is also shown. It is important to realize that the vertical axis corresponds to estimated target miss distance. The thresholds display some interesting characteristics. For values of time less than about 600 seconds, the threshold looks much like an $\hat{x}_2(n)$ trajectory with $u(n)$ and $v_2(n)$ in (3-100) set equal to zero. The threshold attempts to control $\hat{x}_2(n)$ so that it passes within a band of ± 2.5 miles at 630 seconds. For most

n	TIME(sec.)	T(n) (mi.)	n	TIME(sec.)	T(n) (mi.)
0	400	41.56	15	550	10.04
1	410	38.89	16	560	8.62
2	420	36.27	17	570	7.31
3	430	33.76	18	580	6.09
4	440	31.31	19	590	4.98
5	450	28.99	20	600	4.00
6	460	26.69	21	610	3.14
7	470	24.50	22	620	2.49
8	480	22.41	23	630	2.51
9	490	20.38	24	640	3.25
10	500	18.44	25	650	4.81
11	510	16.58	26	660	7.77
12	520	14.82	27	670	14.40
13	530	13.13	28	680	34.49
14	540	11.53	29	690	

Table 3.6 Computed Threshold Values for Re-Entry Control

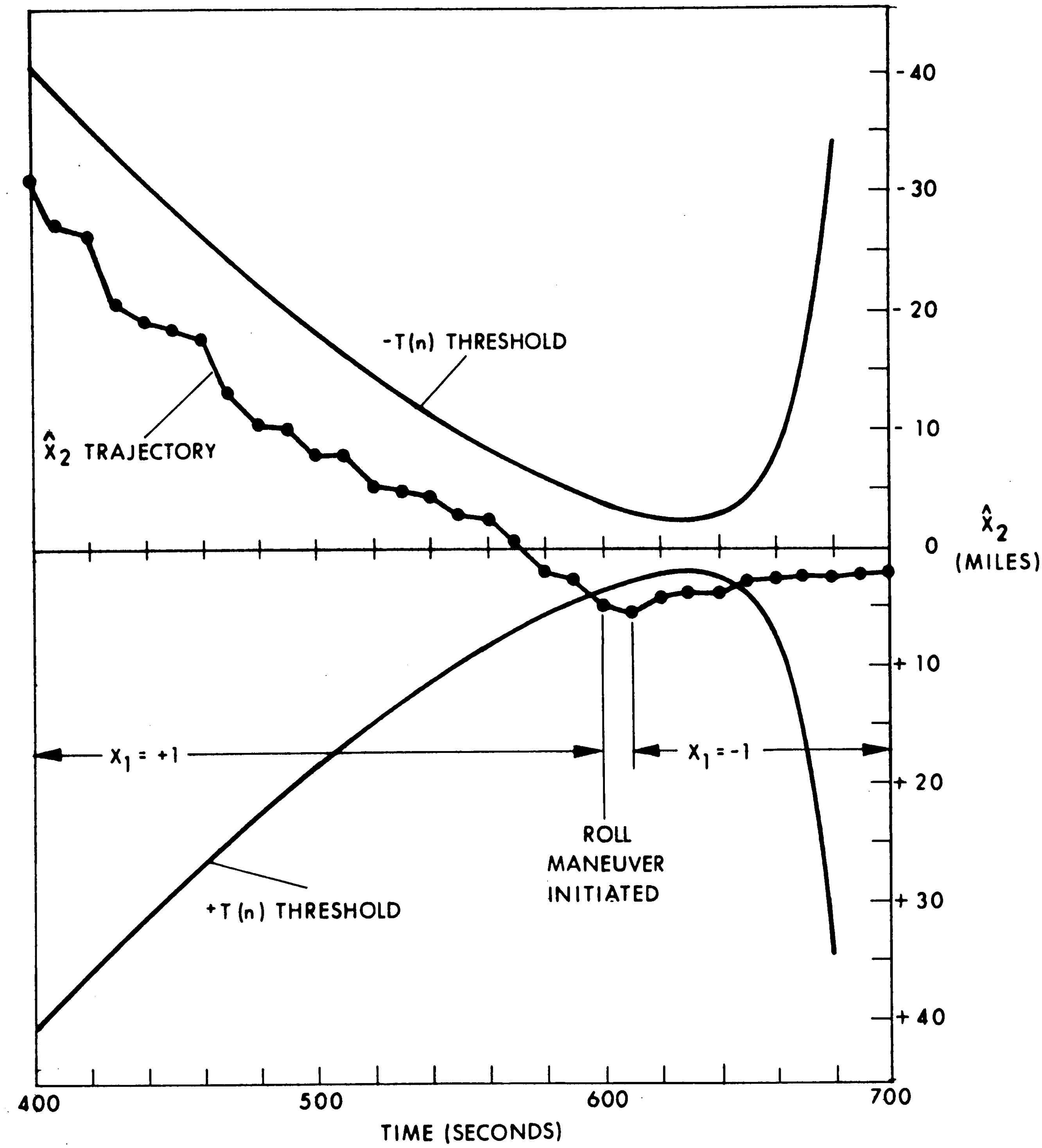

Fig. 3.11 Thresholds and Typical Re-entry Trajectory

cases, only one roll maneuver is required to do this. After 630 seconds the threshold grows rapidly with time, signifying that little reduction in the terminal miss distance can be gained for the expense of a roll maneuver. It was found that the mean cost is 2.51 roll maneuvers, including the mean terminal penalty which is equivalent to one roll maneuver.

3.8 Discrete Systems with Quadratic Cost

As a last example, the general quadratic cost problem will be solved. Assume that the plant and measurement processes may be described by equations (2-1) through (2-4). The cost is specified as a quadratic function of the state and the control so

$$J = E\left[\sum_{n=1}^{q} \frac{1}{2}\left[x^T(n)A(n)x(n)+u^T(n)D(n)u(n)\right] + \frac{1}{2}\, x^T(q+1)Rx(q+1)\right] \tag{3-122}$$

where A(n) and R are symmetric, non-negative definite, (k×k) matrices and D(n) is a symmetric, positive definite (p×p) matrix. Functions $\overline{L}(\hat{x}(n), u(n), n)$ and $\overline{\phi}(\hat{x}(q+1)$ become

$$\overline{L}(\hat{x}(n), u(n), n) = \frac{1}{2}\left[\hat{x}^T(n)A(n)\hat{x}(n)+\mathrm{Tr}\left[A(n)\,P(n)\right] + u^T(n)D(n)u(n)\right] \tag{3-123}$$

$$\overline{\phi}(\hat{x}(q+1)) = \frac{1}{2}\left[\hat{x}^T(q+1)\,R\,\hat{x}(q+1) + \mathrm{Tr}\left[R\,P(q+1)\right]\right] \tag{3-124}$$

The minimum expected value function must satisfy (2-49) which is written, for this problem, as follows:

$$C^*(\hat{x}(n), n) = \min_{u(n)}\left\{\frac{1}{2}\left[\hat{x}^T(n)\,A(n)\,\hat{x}(n) + \mathrm{Tr}\left[A(n)\,P(n)\right] + u^T(n)\,D(n)\,u(n)\right] + \int_{-\infty}^{\infty} d\zeta_1 \ldots \int_{-\infty}^{\infty} d\zeta_k\; f_{s(n+1)}(\zeta)\, C^*(\hat{x}'(n+1)+\zeta, n+1)\right\} \tag{3-125}$$

with unbounded control and $\hat{x}'(n+1)$ given by

$$\hat{x}'(n+1) = \Phi(n+1, n)\hat{x}(n) + \theta\,(n+1, n)\,u(n) \tag{3-126}$$

and the terminal condition on (3-125) as

$$C^*(\hat{x}(q+1), q+1) = \frac{1}{2}\left[\hat{x}^T(q+1) R\, \hat{x}(q+1) + \mathrm{Tr}\left[R\, P(q+1)\right]\right] \tag{3-127}$$

The solution of (3-125), (3-126), and (3-127) is assumed to be of the form

$$C^*(\hat{x}(n), n) = \frac{1}{2}\left[\hat{x}^T(n)\, K(n)\, \hat{x}(n) + g\,(n)\right] \tag{3-128}$$

where K(n) is a (k ×k) symmetric matrix, to be determined, and g(n) is a scalar, to be determined. Equation (3-127) gives terminal conditions on K(n) and g(n) so

$$K(q+1) = R \tag{3-129}$$

$$g(q+1) = \mathrm{Tr}\left[R\, P(q+1)\right] \tag{3-130}$$

Using (3-128), the integral on the right of (3-125) can be written as

$$\int_{-\infty}^{\infty} d\zeta_1 \cdots \int_{-\infty}^{\infty} d\zeta_k\; f_{s(n+1)}(\zeta)\, C^*(\hat{x}'(n+1) + \zeta, n+1) =$$

$$\frac{1}{2}\left[\hat{x}'^T(n+1) K(n+1)\, \hat{x}'(n+1) + \mathrm{Tr}\left[S(n+1) K(n+1)\right] + g(n+1)\right] \tag{3-131}$$

so (3-125) becomes

$$C^*(\hat{x}(n), n) = \min_{u(n)} \frac{1}{2}\Big\{\hat{x}^T(n) A(n) \hat{x}(n) + \mathrm{Tr}\left[A(n) P(n)\right] + u^T(n)\, D(n) u(n)$$

$$+ \hat{x}'^T(n+1) K(n+1) \hat{x}'(n+1) + \mathrm{Tr}\left[S(n+1)\, K(n+1)\right] + g(n+1)\Big\} \tag{3-132}$$

and using (3-126) to combine terms

$$C^*(\hat{x}(n), n) = \min_{u(n)} \frac{1}{2}\Big\{ \hat{x}^T(n)\Big[A(n) + \Phi^T(n+1,n)K(n+1)\Phi(n+1,n)\Big]\hat{x}(n)$$

$$+ u^T(n)\Big[D(n) + \theta^T(n+1,n)K(n+1)\theta(n+1,n)\Big]u(n)$$

$$+ 2\hat{x}^T(n)\,\Phi^T(n+1,n)K(n+1)\,\theta(n+1,n)u(n)$$

$$+ \mathrm{Tr}\Big[A(n)P(n) + S(n+1)K(n+1)\Big] + g(n+1)\Big\} \tag{3-133}$$

Taking the derivative, with respect to u(n), of the function in braces on the right of (3-133) yields

$$\frac{\partial\{\cdot\}}{\partial u(n)} = 2u^T(n)\Big[D(n) + \theta^T(n+1,n)\,K(n+1)\,\theta(n+1,n)\Big]$$

$$+ 2\hat{x}^T(n)\,\Phi^T(n+1,n)K(n+1)\theta(n+1,n) \tag{3-134}$$

and the second derivative is

$$\frac{\partial^2\{\cdot\}}{\partial u(n)^2} = 2\Big[D(n) + \theta^T(n+1,n)K(n+1)\,\theta(n+1,n)\Big] \tag{3-135}$$

Setting the right hand side of (3-134) equal to zero produces a necessary condition for the optimal control

$$u^*(n) = -\Big[D(n) + \theta^T(n+1,n)K(n+1)\,\theta(n+1,n)\Big]^{-1}\theta^T(n+1,n)K(n+1)\Phi(n+1,n)\hat{x}(n) \tag{3-136}$$

and it is assumed that the indicated inverse matrix exists. Substituting (3-136) and (3-128) into (3-133) gives

$$\hat{x}^T(n)K(n)\hat{x}(n) + g(n) = \hat{x}^T(n)\Big[A(n) + \Phi^T(n+1,n)K(n)\,\Phi(n+1,n)\Big]\hat{x}(n)$$

$$-\Big[\hat{x}^T(n)\,\Phi^T(n+1,n)K(n+1)\theta(n+1,n)\Big]\Big[D(n)+\theta^T(n+1,n)K(n+1)\theta(n+1,n)\Big]^{-1}\cdot$$

$$\Big[\theta^T(n+1,n)K(n+1)\Phi(n+1,n)\,\hat{x}(n)\Big]$$

$$+ \mathrm{Tr}\Big[A(n)\;P(n) + S(n+1)\;K(n+1)\Big] + g(n+1) \tag{3-137}$$

Since (3-137) must hold for all vectors $\hat{x}(n)$, K(n) and g(n) must satisfy

$$K(n) = A(n) + \Phi^T(n+1,n)K(n+1)\Phi(n+1,n) - \left[\Phi^T(n+1,n)K(n+1)\theta(n+1,n)\right] \cdot$$
$$\left[D(n) + \theta^T(n+1,n)K(n+1)\theta(n+1,n)\right]^{-1}\left[\theta^T(n+1,n)K(n+1)\Phi(n+1,n)\right] \quad (3\text{-}138)$$

and

$$g(n) = g(n+1) + \mathrm{Tr}\left[A(n)P(n) + S(n+1)K(n+1)\right] \quad (3\text{-}139)$$

By virtue of (3-122), with A(n) and R non-negative definite and D(n) positive definite, $C^*(\hat{x}(n),n)$ cannot be negative for any vector $\hat{x}(n)$. From (3-128), this can only occur if K(n) is non-negative definite. Thus since D(n) is positive definite the right side of (3-135) must be positive definite. The existance of the inverses in (3-136) etc. is thus assured and $u^*(n)$ given by (3-136) is the optimal control. Further, since A(n), D(n) and R are symmetric, K(n) satisfying (3-129) and (3-138) will be symmetric, as assumed at the onset.

Solution of (3-129) and (3-138) for K(n) will provide the optimal control function according to (3-136). Note that the solution for K(n) depends only upon A(n), D(n) and R so the design of the controller is independent of the statistics of the problem, as specified by the previously derived quadratic cost separation theorem[(30), (39)]. Note however that the expected cost given by (3-128) includes g(n) which is dependent upon the statistics of the problem, as shown by (3-130) and (3-139). Thus, although the control function is independent of the statistics, the cost is dependent upon the estimation error covariance P(n) and the measurement information covariance S(n).

CHAPTER 4

OPTIMAL CONTROL OF CONTINUOUS LINEAR STOCHASTIC SYSTEMS

4.1 General Discussion

Up to this point, only discrete systems have been discussed. In this chapter the discrete theory developed in Chapter 2 will be generalized to handle continuous problems. The approach is first to define the class of continuous systems under consideration and the expected cost to be minimized. Then the continuous process is approximated by a discrete process which converges to the continuous process in the limit as the time step goes to zero. The optimization theory of Chapter 2 is then applied to the discrete process, yielding the usual recursion formula for the minimum expected value function. By expanding the terms of the recursion formula and taking proper limits as the time step goes to zero, a partial differential equation is obtained which must be satisfied by the continuous minimum expected value function. Solution of this differential equation produces the optimal control as a function of the estimated state.

4.2 Problem Statement

The dynamics of the plant are described by a vector Langevin equation

$$\dot{x}(t) = F(t)\, x(t) + G(t)\, u(t) + q_1(t) \tag{4-1}$$

where

$x(t) \equiv$ state vector of dimension k

$u(t) \equiv$ control vector of dimension p

$F(t) \equiv$ system dynamics matrix (k $\times$ k), continuous in time

$G(t) \equiv$ control distribution matrix (k $\times$ p), continuous in time

The initial state x(0) is a k vector of normally distributed random variables with known statistics and $q_1(t)$ is a k vector of gaussian white noise processes with statistics given by

$$E\left[q_1(t)\right] = 0$$
$$E\left[q_1(t)\, q_1(s)\right] = Q_1(t)\, \delta(t-s) \qquad (4\text{-}2)$$

where δ is the Dirac delta function and $Q_1(t)$ is a $(k \times k)$ matrix of continuous functions. The feedback controller has available to it a measurement process m(t) described by the equation

$$m(t) = H(t)\, x(t) + q_2(t) \qquad (4\text{-}3)$$

where m(t) is an ℓ vector with

H(t) ≡ measurement matrix $(\ell \times k)$, continuous in time and $q_2(t)$, the measurement noise, is an ℓ vector of gaussian white noise processes with statistics

$$E\left[q_2(t)\right] = 0$$
$$E\left[q_2(t)\, q_2^T(s)\right] = Q_2(t)\, \delta(t-s)$$
$$E\left[q_1(t)\, q_2^T(s)\right] = 0 \qquad \text{all } s,\ t \qquad (4\text{-}4)$$

with $Q_2(t)$ an $(\ell \times \ell)$ matrix of continuous functions. Finally the cost to be minimized is written as

$$J = E\left[\int_{t_i}^{t_f} L(x(t), u(t), t)\, dt + \phi(x(t_f))\right] \qquad (4\text{-}5)$$

where L(x(t), u(t), t) is continuous in x(t), u(t) and t; and t_i, t_f are specified initial and terminal times.

4.3 Conditions for the Optimal Control

Equation (4-1) describes a linear system driven by white gaussian noise and the control u(t). If F(t), G(t) and $Q_1(t)$ are all continuous, and $u(t)$ is a piecewise continuous function, then it can be shown [21, 36] that sample functions of the random process $x(t)$ are almost all continuous (i.e., with probability one). Under these assumptions the process $x(t)$ may be approximated by a process $x_a(t)$ defined as follows

$$x_a(t_{n+1}) = x_a(t_n) + \left[F(t_n)x_a(t_n) + G(t_n)u(t_n)\right] \Delta t_n + v(t_n) \tag{4-6}$$

$$x_a(t) = x_a(t_n) \quad \text{for} \quad (t_n \leqq t < t_{n+1}) \tag{4-7}$$

$$x_a(0) = x(0) \tag{4-8}$$

and the t_n's are discrete times such that

$$t_{n+1} = t_n + \Delta t_n \tag{4-9}$$

where Δt_n is a short time step. Also $v(t_n)$ in (4-6) is a k vector of normally distributed random variables with statistics given by

$$E\left[v(t_n)\right] = 0$$

$$E\left[v(t_n)\, v^T(t_n)\right] = Q_1(t_n)\ \Delta t_n$$

$$E\left[v(t_n)\, v^T(t_i)\right] = 0 \qquad i \neq n \tag{4-10}$$

Similarly, the measurement process $m(t)$ is approximated by a process $m_a(t)$ where

$$m_a(t_n) = H(t_n)\, x(t_n) + w(t_n) \tag{4-11}$$

$$m_a(t) = m_a(t_n) \quad \text{for} \quad (t_n \leqq t < t_{n+1}) \tag{4-12}$$

and $w(t_n)$ is a vector of normally distributed random variables with statistics given by

$$E\left[w(t_n)\right] = 0$$

$$E\left[w(t_n)\, w^T(t_n)\right] = \frac{Q_2(t_n)}{\Delta t_n}$$

$$E\left[w(t_n)\, w^T(t_i)\right] = 0 \qquad i \neq n$$

$$E\left[w(t_n)\, v^T(t_i)\right] = 0 \text{ all } \; i,\, n \tag{4-13}$$

Finally, the cost function is approximated by

$$J_a = E\left[\sum_{n=1}^{q} L(x_a(t_n),\, u(t_n),\, t_n)\, \Delta t_n + \phi(x_a(t_{q+1}))\right] \tag{4-14}$$

where

$$t_1 = t_i \qquad t_{q+1} = t_f \tag{4-15}$$

By taking appropriate limits it can be shown that the statistics of the discrete system (4-6) through (4-13) converge to the statistics of the continuous system (4-1) through (4-4) in the limit as Δt_n approaches zero. Similarly, the approximate cost J_a converges to the cost J in the limit as the time steps go to zero and q approaches infinity.

Now consider an estimate $\hat{x}_a(t_n)$. With obvious changes in the notation, equations (2-19), (2-20) and the last three of equations (2-11) produce recursion formulas for the estimate $\hat{x}_a(t_n)$.

$$\hat{x}_a(t_n) = \hat{x}_a'(t_n) + P_a'(t_n)\, H^T(t_n) \left[H(t_n)\, P_a'(t_n)\, H^T(t_n) + \frac{Q_2(t_n)}{\Delta t_n} \right]^{-1} \cdot \left[m_a(t_n) - H(t_n)\, \hat{x}_a'(t_n) \right]$$

$$\hat{x}_a'(t_{n+1}) = \hat{x}_a(t_n) + \left[F(t_n)\, \hat{x}_a(t_n) + G(t_n)\, u(t_n) \right] \Delta t_n$$

$$\hat{x}_a(0) = E\left[x(0) \right]$$

$$P_a(t_n) = P_a'(t_n) - P_a'(t_n)\, H^T(t_n) \left[H(t_n)\, P_a'(t_n)\, H^T(t_n) + \frac{Q_2(t_n)}{\Delta t_n} \right]^{-1} \cdot \left[H(t_n)\, P_a'(t_n) \right]$$

$$P_a'(t_{n+1}) = P_a(t_n) + F(t_n)\, P_a(t_n)\, \Delta t_n + P_a(t_n)\, F^T(t_n)\, \Delta t_n + F(t_n)\, P_a(t_n)\, F^T(t_n)\, \Delta t_n^{\,2} + Q_1(t_n)\, \Delta t_n$$

$$P_a(0) = E\left\{ \left(x(0) - E\left[x(0) \right] \right) \left(x(0) - E\left[x(0) \right] \right)^T \right\} \tag{4-16}$$

By taking appropriate limits as Δt_n approaches zero, the differential equations for continuous estimation of $x(t)$ are derived as follows:*

$$\dot{\hat{x}}(t) = F(t)\, \hat{x}(t) + G(t)\, u(t) + P(t)\, H^T(t)\, Q_2(t)^{-1} \left[m(t) - H(t)\, \hat{x}(t) \right]$$

$$\hat{x}(0) = E\left[x(0) \right]$$

$$\dot{P}(t) = F(t)\, P(t) + P(t)\, F^T(t) - P(t)\, H^T(t)\, Q_2(t)^{-1} H(t)\, P(t) + Q_1(t)$$

$$P(0) = E\left\{ \left(x(0) - E\left[x(0) \right] \right) \left(x(0) - E\left[x(0) \right] \right)^T \right\} \tag{4-17}$$

*In cases for which $Q_2(t)$ is singular, the methods of Deyst[20] or Bryson and Johanson[15] must be used. All succeeding results derived in this chapter are applicable, with small modification, to such cases.

Equations (4-17) are the familiar continuous estimation formulas of Kalman and Bucy[41].

If, as in Chapter 2, a processed measurement vector $s_a(t_n)$ is defined as

$$s_a(t_n) = P'_a(t_n)\, H^T(t_n) \left[H(t_n)\, P'_a(t_n)\, H^T(t_n) + \frac{Q_2(t_n)}{\Delta t_n} \right]^{-1} \cdot$$

$$\left[m_a(t_n) - H(t_n)\, \hat{x}'_a(t_n) \right] \tag{4-18}$$

then the first two of equations (4-16) may be written as

$$\hat{x}_a(t_{n+1}) = \hat{x}_a(t_n) + \left[F(t_n)\, \hat{x}_a(t_n) + G(t_n)\, u(t_n) \right] \Delta t_n + s_a(t_{n+1})$$

$$= \hat{x}'_a(t_{n+1}) + s_a(t_{n+1}) \tag{4-19}$$

and it was shown in Chapter 2 that the elements of $s_a(t_n)$ are normally distributed with statistics given by

$$E\left[s_a(t_n) \right] = 0$$

$$E\left[s_a(t_n)\, s_a^T(t_n) \right] = S_a(t_n)$$

$$E\left[s_a(t_n)\, s_a^T(t_i) \right] = 0 \qquad i \neq n \tag{4-20}$$

where $S_a(t_n)$ is

$$S_a(t_n) = P'_a(t_n)\, H^T(t_n) \left[H(t_n)\, P'_a(t_n)\, H^T(t_n) + \frac{Q_2(t_n)}{\Delta t_n} \right]^{-1} H(t_n)\, P'_a(t_n)$$

$$(4\text{-}21)$$

With these definitions, a recursion formula for the discrete minimum expected value function $C_a^*(\hat{x}_a(t_n), t_n)$ can be derived by the methods of Chapter 2. The result is

$$C_a^*(\hat{x}_a(t_n), t_n) = \min_{u(t_n) \in \mathcal{U}(t_n)} \Big\{ \overline{L}(\hat{x}_a(t_n), u(t_n), t_n) \Delta t_n$$

$$+ \int_{-\infty}^{\infty} d\rho_1 \cdots \int_{-\infty}^{\infty} d\rho_k \, f_{s_a(t_{n+1})}(\rho) \, C_a^*(\hat{x}'_a(t_{n+1}) + \rho, t_{n+1}) \Big\}$$

(4-22)

and substituting an expectation operator for the integral in (4-22) produces

$$C_a^*(\hat{x}_a(t_n), t_n) = \min_{u(t_n) \in \mathcal{U}(t_n)} \Big\{ \overline{L}(\hat{x}_a(t_n), u(t_n), t_n) \Delta t_n$$

$$+ E\left[C_a^*(\hat{x}_a(t_{n+1}), t_{n+1}) \,\middle|\, \hat{x}_a(t_n), u(t_n) \right] \Big\} \quad (4\text{-}23)$$

The terminal condition on C_a^* is

$$C_a^*(\hat{x}_a(t_{q+1}), t_{q+1}) = \overline{\phi}(\hat{x}_a(t_{q+1})) \quad (4\text{-}24)$$

and the functions $\overline{L}(\hat{x}_a(t_n), u(t_n), t_n)$ and $\overline{\phi}(\hat{x}_a(t_{q+1}))$ are defined by equations (2-51) and (2-52) with obvious changes in notation.

It is assumed that a continuous minimum expected value function $C^*(\hat{x}, t)$ can be constructed. This function coincides with the discrete function $C_a^*(\hat{x}, t_n)$ at the discrete time points t_n and is continuous in time. That is

$$C^*(\hat{x}, t_n) = C_a^*(\hat{x}, t_n) \qquad \|\hat{x}\| < \infty \quad (4\text{-}25)$$

$$C^*(\hat{x}, t) = \text{continuous function of } t \quad (4\text{-}26)$$

Further, if the partial derivatives $\frac{\partial C^*}{\partial t}$, $\frac{\partial C^*}{\partial \hat{x}_i}$ and $\frac{\partial^2 C^*}{\partial \hat{x}_i \partial \hat{x}_j}$ exist and are continuous, then $C_a^*(\hat{x}_a(t_{n+1}), t_{n+1})$ in (4-23) can be written as a Taylor series expansion about $\hat{x}_a(t_n)$ and t_n, thus

$$
\begin{aligned}
C_a^*(\hat{x}_a(t_n), t_n) = \min_{u(t_n)\in \mathcal{U}(t_n)} \Bigg\{ & \overline{L}(\hat{x}_a(t_n), u(t_n), t_n)\,\Delta t_n \\
& + E\Bigg(C_a^*(\hat{x}_a(t_n), t_n) + \left[\frac{\partial C^*}{\partial t}\right]_{\hat{x}_a(t_n), t_n} \Delta t_n \\
& + \left[\frac{\partial C^*}{\partial \hat{x}}\right]_{\hat{x}_a(t_n), t_n} \left[(F(t_n)\hat{x}_a(t_n) + G(t_n)u(t_n))\Delta t_n + s_a(t_{n+1})\right] \\
& + \frac{1}{2}\left[(F(t_n)\hat{x}_a(t_n) + G(t_n)u(t_n))\Delta t_n + s_a(t_{n+1})\right]^T \left[\frac{\partial^2 C^*}{\partial \hat{x}^2}\right]_{\hat{x}_a(t_n), t_n} \cdot \\
& \left[(F(t_n)\hat{x}_a(t_n) + G(t_n)u(t_n))\Delta t_n + s_a(t_{n+1})\right] \\
& + \ldots \,\Bigg|\, \hat{x}_a(t_n), u(t_n)\Bigg)\Bigg\}
\end{aligned}
\tag{4-27}
$$

with the row vector of first partial derivatives and the square matrix of second partial derivatives defined as

$$\frac{\partial C^*(\hat{x}, t)}{\partial \hat{x}} = \left[\frac{\partial C^*(\hat{x}, t)}{\partial \hat{x}_j}\right]$$

$$\frac{\partial^2 C^*(\hat{x}, t)}{\partial \hat{x}^2} = \frac{\partial^2 C^*(\hat{x}, t)}{\partial \hat{x}_i\, \partial \hat{x}_j}$$

Note that the expectation in (4-27) is conditioned on $\hat{x}_a(t_n)$ and $u(t_n)$. With the help of (4-20) the conditional expectations in (4-27) can be evaluated so

$$0 = \min_{u(t_n)\in \mathcal{U}(t_n)} \Bigg\{ \overline{L}(\hat{x}_a(t_n), u(t_n), t_n)\,\Delta t_n + \left[\frac{\partial C^*}{\partial t}\right]_{\hat{x}_a(t_n),\, t_n} \Delta t_n$$

$$+ \left[\frac{\partial C^*}{\partial \hat{x}}\right]_{\hat{x}_a(t_n),\, t_n} \left[F(t_n)\hat{x}_a(t_n)+G(t_n)u(t_n)\right] \Delta t_n$$

$$+ \frac{1}{2}\,\mathrm{Tr}\left(S_a(t_{n+1}) \left[\frac{\partial^2 C^*}{\partial \hat{x}^2}\right]_{\hat{x}_a(t_n),\, t_n} \right) + o(\Delta t_n) + HM(s_a) \Bigg\}$$

(4-28)

where o (Δt_n) represents higher order terms in Δt_n such that

$$\lim_{\Delta t_n \to 0} \frac{o(\Delta t_n)}{\Delta t_n} = 0 \qquad (4\text{-}29)$$

and $HM(s_a)$ represents terms containing moments of $s_a(t_{n+1})$ higher than the second and terms with the second moments of $s_a(t_{n+1})$ multiplied by Δt_n. Dividing (4-28) through by Δt_n and taking the limit as Δt_n approaches zero yields

$$0 = \min_{u(t_n)\in\mathcal{U}(t_n)} \Bigg\{ \overline{L}(\hat{x}_a(t_n), u(t_n), t_n) + \left[\frac{\partial C^*}{\partial t}\right]_{\hat{x}_a(t_n),\, t_n}$$

$$+ \left[\frac{\partial C^*}{\partial \hat{x}}\right]_{\hat{x}_a(t_n),\, t_n} \left[F(t_n)\, \hat{x}_a(t_n) + G(t_n)\, u(t_n)\right]$$

$$+ \frac{1}{2}\, \mathrm{Tr} \left(\lim_{\Delta t_n \to 0} \left[\frac{S_a(t_{n+1})}{\Delta t_n}\right] \left[\frac{\partial^2 C^*}{\partial \hat{x}^2}\right]_{\hat{x}_a(t_n),\, t_n} \right)$$

$$+ \lim_{\Delta t_n \to 0} \left[\frac{HM(s_a)}{\Delta t_n}\right] \Bigg\} \tag{4-30}$$

First consider the limit containing $S_a(t_{n+1})$. From (4-21)

$$\lim_{\Delta t_n \to 0} \left[\frac{S_a(t_{n+1})}{\Delta t_n}\right] = \lim_{\Delta t_n \to 0} \Bigg(P_a'(t_{n+1})\, H^T(t_{n+1}) \Big[H(t_{n+1}) \cdot$$

$$P_a'(t_{n+1})\, H^T(t_{n+1})\, \Delta t_n + Q_2(t_{n+1}) \Big]^{-1} H(t_{n+1})\, P_a'(t_{n+1}) \Bigg) \tag{4-31}$$

and since $P_a'(t_{n+1})$ converges to $P(t)$ as Δt_n approaches zero

$$\lim_{\Delta t_n \to 0} \left[\frac{S_a(t_{n+1})}{\Delta t_n}\right] = P(t)\, H^T(t)\, Q_2(t)^{-1}\, H(t)\, P(t) \tag{4-32}$$

and therefore $S_a(t_{n+1})$ is of order Δt_n. Since $s_a(t_n)$ is normally distributed all higher moments must be either zero or of order $o(\Delta t_n)$ so

$$\lim_{\Delta t_n \to 0} \left[\frac{HM(s_a)}{\Delta t_n} \right] = 0 \tag{4-33}$$

Thus, if the (k × k) matrix B(t) is defined as

$$B(t) = P(t)\ H^T(t)\ Q_2(t)^{-1}\ H(t)\ P(t) \tag{4-34}$$

then (4-30) becomes

$$0 = \min_{u \in \mathcal{U}(t)} \left\{ \overline{L}(\hat{x},\ u,\ t) + \frac{\partial C^*(\hat{x},\ t)}{\partial t} + \frac{\partial C^*(\hat{x},\ t)}{\partial \hat{x}} \left[F(t)\ \hat{x} + G(t)\ u \right] + \frac{1}{2} \operatorname{Tr} \left[B(t) \frac{\partial^2 C^*(\hat{x},\ t)}{\partial \hat{x}^2} \right] \right\} \tag{4-35}$$

Equation (4-35), with the terminal condition

$$C^*(\hat{x},\ t_f) = \overline{\phi}(\hat{x}) \tag{4-36}$$

is the partial differential equation that must be satisfied by the minimum expected value function. Functions $\overline{\phi}(\hat{x})$ and $\overline{L}(\hat{x},\ u,\ t)$ are determined by

$$\overline{\phi}(\hat{x}) = \int_{-\infty}^{\infty} d\xi_1 \ldots \int_{-\infty}^{\infty} d\xi_k\ \phi(\xi)\ f_{x(t_f)}(\xi | \hat{x}) \tag{4-37}$$

and

$$\overline{L}(\hat{x},\ u,\ t) = \int_{\infty}^{\infty} d\xi_1 \ldots \int_{\infty}^{\infty} d\xi_k\ L(\xi,\ u,\ t)\ f_{x(t)}(\xi | \hat{x}) \tag{4-38}$$

with

$$f_{x(t)}(\xi|\hat{x}) = (2\pi)^{-\frac{k}{2}} |P(t)|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left[\xi - \hat{x}\right]^T P(t)^{-1}\left[\xi - \hat{x}\right]\right\} \tag{4-39}$$

It is quite interesting to consider separate combinations of the terms in (4-35). For example, if the last term on the right is missing then (4-35) becomes the familiar Hamilton Jacobi equation for deterministic problems. The last term accounts for the randomness in the processed measurement information which is a gaussian white noise. If the $\bar{L}$ term is zero and u is a function of $\hat{x}$, then (4-35) becomes a Kolmogorov equation. Further, if F, G and $\bar{L}$ are zero, the diffusian equation, familiar from Chapter 2 and Appendix B is obtained. Finally the entire equation is well known from the theory of optimal control of Markov processes with perfect measurements and has been called the stochastic Hamilton-Jacobi equation[77].

Solution of (4-35) through (4-39) produces the optimal control as a function of $\hat{x}$ and t. Solutions are generally quite difficult to obtain and in most practical cases, numerical approximations must be made and the solution calculated using the discrete formulas of Chapter 2. The following chapter illustrates some few solutions which are obtainable.

CHAPTER 5

APPLICATIONS OF THE CONTINUOUS THEORY

5.1 General Discussion

The purpose of this chapter is to illustrate some methods that are useful in solving the stochastic Hamilton Jacobi equation. The first problem is solution of the quadratic cost case and the results yield the previously derived separation theorem[60]. The second problem involves quadratic weighting of the control and an arbitrary terminal cost. Finally, the third problem is the continuous analog of the discrete variable time of arrival problem of Chapter 3.

5.2 Continuous Systems with Quadratic Cost

The plant and measurement processes are described by equations (4-1) through (4-4). The cost to be minimized is quadratic in the state and the control, thus

$$J = E\left[\int_{t_i}^{t_f} \frac{1}{2}\left[x^T(t)A(t)x(t)+u^T(t)D(t)u(t)\right]dt + \frac{1}{2}x^T(t_f)R\,x(t_f)\right] \tag{5-1}$$

where $A(t)$ and R are symmetric, non-negative definite $(k \times k)$ matrices and $D(t)$ is a symmetric, positive definite $(p \times p)$ matrix. Functions $\overline{L}\,(\hat{x}(t), u(t), t)$ and $\overline{\phi}\,(\hat{x}(t_f))$ are, from (4-37) and (4-38)

$$\overline{L}(\hat{x}(t), u(t), t) = \frac{1}{2}\left[\hat{x}^T(t)\; A(t)\hat{x}(t) + \mathrm{Tr}\left[A(t)P(t)\right] + u^T(t)\,D(t)\,u(t)\right] \tag{5-2}$$

$$\overline{\phi}(\hat{x}(t_f)) = \frac{1}{2}\left[\hat{x}^T(t_f)R\,\hat{x}(t_f) + \mathrm{Tr}\left[R\,P(t_f)\right]\right] \tag{5-3}$$

The minimum expected value function satisfies equation (4-35) which is written, for this problem, as follows:

$$0 = \min_{u} \left\{ \frac{1}{2}\left[\hat{x}^T A(t)\hat{x} + \mathrm{Tr}\left[A(t)P(t)\right] + u^T D(t) u\right] + \frac{\partial C^*(\hat{x},t)}{\partial t} \right.$$
$$\left. + \frac{\partial C^*(\hat{x},t)}{\partial \hat{x}}\left[F(t)\hat{x} + G(t)u\right] + \frac{1}{2}\,\mathrm{Tr}\left[B(t)\,\frac{\partial^2 C^*(\hat{x},t)}{\partial \hat{x}^2}\right]\right\} \tag{5-4}$$

With B(t) given by (4-34), unbounded control, and the terminal condition

$$C^*(\hat{x},t_f) = \frac{1}{2}\left[\hat{x}^T R\,\hat{x} + \mathrm{Tr}\left[R\,P(t_f)\right]\right] \tag{5-5}$$

The solution of (5-4), (5-5) is assumed to be of the form

$$C^*(\hat{x},t) = \frac{1}{2}\left[\hat{x}^T K(t)\hat{x} + g(t)\right] \tag{5-6}$$

where K(t) is a $(k \times k)$ symmetric matrix, to be determined, and g(t) is a scalar, to be determined. If $C^*(\hat{x},t)$ is given by (5-6), then its derivatives with respect to time and space are

$$\frac{\partial C^*(\hat{x},t)}{\partial t} = \frac{1}{2}\left[\hat{x}^T \dot{K}(t)\,\hat{x} + \dot{g}(t)\right] \tag{5-7}$$

$$\frac{\partial C^*(\hat{x},t)}{\partial \hat{x}} = \hat{x}^T K(t) \tag{5-8}$$

$$\frac{\partial^2 C^*(\hat{x},t)}{\partial \hat{x}^2} = K(t) \tag{5-9}$$

and substituting (5-7) through (5-9) into (5-4) produces

$$0 = \min_{u} \left\{ \frac{1}{2}\left[\hat{x}^T A(t)\hat{x} + \mathrm{Tr}\left[A(t)P(t)\right] + u^T D(t) u\right] \right.$$
$$+ \frac{1}{2}\left[\hat{x}^T \dot{K}(t)\hat{x} + \dot{g}(t)\right] + \hat{x}^T K(t)\left[F(t)\hat{x} + G(t)\,u\right]$$
$$\left. + \frac{1}{2}\,\mathrm{Tr}\left[B(t)\;K(t)\right]\right\} \tag{5-10}$$

If the derivative of the function in braces on the right of (5-10) is taken with respect to u then

$$\frac{\partial\{\cdot\}}{\partial u} = u^T D(t) + \hat{x}^T K(t) G(t) \tag{5-11}$$

and the second derivative is

$$\frac{\partial^2\{\cdot\}}{\partial u^2} = D(t) \tag{5-12}$$

which is positive definite for all u, so an absolute minimum occurs if (5-11) is set equal to zero, producing

$$u^* = - D(t)^{-1} G^T(t) K(t) \hat{x} \tag{5-13}$$

Substituiòn of (5-13) into (5-10) produces

$$\begin{aligned} 0 = \frac{1}{2} &\left[\hat{x}^T A(t) \hat{x} + \mathbf{Tr}\left[A(t) P(t)\right] + \hat{x}^T K(t) G(t) D(t)^{-1} G^T(t) K(t) \hat{x}\right] \\ &+ \frac{1}{2}\left[\hat{x}^T \dot{K}(t) \hat{x} + \dot{g}(t)\right] + \hat{x}^T K(t)\left[F(t) \hat{x} - G(t) D(t)^{-1} G^T(t) K(t) \hat{x}\right] \\ &+ \frac{1}{2}\ \mathbf{Tr}\left[B(t) K(t)\right] \end{aligned} \tag{5-14}$$

and since all terms on the right of (5-14) are scalars, it may be written as

$$\begin{aligned} 0 = \frac{1}{2} &\left[\hat{x}^T A(t) \hat{x} + \mathbf{Tr}\left[A(t) P(t)\right] + \hat{x}^T K(t) G(t) D(t)^{-1} G^T(t) K(t) \hat{x}\right] \\ &+ \frac{1}{2}\left[\hat{x}^T \dot{K}(t) \hat{x} + \dot{g}(t)\right] + \frac{1}{2}\left[\hat{x}^T K(t) F(t) \hat{x} + \hat{x}^T F^T(t) K(t) \hat{x}\right] \\ &- \hat{x}^T K(t) G(t) D(t)^{-1} G^T(t) K(t) \hat{x} + \frac{1}{2}\ \mathrm{Tr}\left[B(t) K(t)\right] \end{aligned} \tag{5-15}$$

Now (5-15) **can** be satisfied for all values of $\hat{x}$ if the Riccati equation

$$\dot{K}(t) + K(t) F(t) + F^T(t) K(t) + A(t) - K(t) G(t) D(t)^{-1} G^T(t) K(t) = 0 \tag{5-16}$$

is satisfied with the terminal condition determined by (5-5) and (5-6) as

$$K(t_f) = R \qquad (5\text{-}17)$$

Clearly since A(t), D(t) and R are symmetric the matrix K(t) satisfying (5-16) and (5-17) will by symmetric, as assumed at the onset. Further the **scalar** function g(t) satisfies the differential equation.

$$\dot{g}(t) + \mathbf{Tr}\left[A(t)\,P(t) + B(t)\,K(t)\right] = 0 \qquad (5\text{-}18)$$

with the terminal condition

$$g(t_f) = \mathrm{Tr}\left[R\; P(t_f)\right] \qquad (5\text{-}19)$$

Solution of (5-16), (5-17) for K(t) will provide the optimal control function according to (5-13). Note that (5-16) and (5-17) do not contain the matrix B(t), which determines the statistical characteristics of the system. Thus, the design of the controller is independent of the design of the estimator, as specified by the previously derived quadratic cost separation theorem[60]. In addition, (5-13), (5-16) and (5-17) are identical to the equations that are obtained by assuming that there is no measurement or process noise and applying the classical methods of the calculus of variations or the maximum principle[3]. Finally, from (5-6), (5-18) and (5-19) it is clear that the minimum cost is dependent upon the statistics of the problem.

5.3 A Single State Variable Problem

Consider a single state, stationary plant described by the scalar equation

$$\dot{x}(t) = F\,x(t) + G\,u(t) + q_1(t) \qquad (5\text{-}20)$$

where F and G are scalars, the scalar control u(t) is unbounded and $q_1(t)$ is a stationary gaussian white noise with statistics

$$E\left[q_1(t)\right] = 0$$

$$E\left[q_1(t)\,q_1(s)\right] = Q_1\,\delta\,(t-s) \qquad (5\text{-}21)$$

The scalar measurement process m(t) is available to the controller, where

$$m(t) = H\,x(t) + q_2(t) \tag{5-22}$$

and $q_2(t)$ is a stationary gaussian white noise with statistics

$$E\left[q_2(t)\right] = 0$$

$$E\left[q_2(t)\,q_2(s)\right] = Q_2\,\delta\,(t - s) \tag{5-23}$$

Assume that estimation of x(t) has proceeded for a sufficiently long time, before control is applied, so that P(t), the estimation error variance, can be considered to be constant. Then, from (4-17) it can be shown that

$$P = \lim_{t \to \infty} P(t) = \frac{Q_2}{H^2}\left[F + \sqrt{F^2 + \frac{H^2 Q_1}{Q_2}}\right] \tag{5-24}$$

and with (4-34), B is determined as

$$B = \lim_{t \to \infty} B(t) = \frac{Q_2}{H^2}\left[F + \sqrt{F^2 + \frac{H^2 Q_1}{Q_2}}\right]^2 \tag{5-25}$$

The expected cost to be minimized involves the control energy and a terminal cost function. It is given by the equation

$$J = E\left[\int_{t_i}^{t_f} \frac{a}{2}\,u^2(t)\,dt + \phi\,(x(t_f))\right] \tag{5-26}$$

where the positive constant a and the terminal cost function $\phi(x(t_f))$ are arbitrary. Functions $\bar{\phi}$ and $\bar{L}$, defined by (4-37) and (4-38), become

$$\bar{\phi}(\hat{x}) = \int_{-\infty}^{\infty} d\xi\;\phi(\xi)\;f_{x(t_f)}\,(\xi\,|\,\hat{x}) \tag{5-27}$$

$$\overline{L}(\hat{x}, u, t) = \frac{a}{2} u^2 \tag{5-28}$$

where

$$f_{x(t)} (\xi | \hat{x}) = (2\pi P)^{-\frac{1}{2}} \exp\left\{- \frac{(\xi - \hat{x})^2}{2P}\right\} \tag{5-29}$$

By substituting the proper values into (4-35), the differential equation for C^* is obtained as

$$0 = \min_u \left\{ \frac{a}{2} u^2 + \frac{\partial C^*}{\partial t} + \frac{\partial C^*}{\partial \hat{x}} \left[F\hat{x} + Gu\right] + \frac{B}{2} \frac{\partial^2 C^*}{\partial \hat{x}^2} \right\} \tag{5-30}$$

and taking the first and second derivatives, with respect to u, of the function in braces in (5-30), produces

$$\frac{\partial\{\cdot\}}{\partial u} = au + G \frac{\partial C^*}{\partial \hat{x}} \qquad \frac{\partial^2\{\cdot\}}{\partial u^2} = a \tag{5-31}$$

Since a is positive, the optimal control is given by

$$u^* = - \frac{G}{a} \frac{\partial C^*}{\partial \hat{x}} \tag{5-32}$$

and upon substitutuion of (5-32) into (5-30), there results

$$0 = \frac{\partial C^*}{\partial t} - \frac{G^2}{2a} \left(\frac{\partial C^*}{\partial \hat{x}}\right)^2 + \frac{\partial C^*}{\partial \hat{x}} F\hat{x} + \frac{B}{2} \frac{\partial^2 C^*}{\partial \hat{x}^2} \tag{5-33}$$

Define a function $A(\hat{x}, t)$, as

$$A(\hat{x}, t) = \exp\left\{- \frac{G^2}{aB} C^*(\hat{x}, t)\right\} \tag{5-34}$$

so the derivatives of C^* may be expressed as

$$\frac{\partial C^*}{\partial t} = - \frac{aB}{G^2 A} \frac{\partial A}{\partial t} \tag{5-35}$$

$$\frac{\partial C^*}{\partial \hat{x}} = - \frac{aB}{G^2 A} \frac{\partial A}{\partial \hat{x}} \tag{5-36}$$

$$\frac{\partial^2 C^*}{\partial \hat{x}^2} = \frac{a B}{G^2 A}\left[\frac{1}{A}\left(\frac{\partial A}{\partial \hat{x}}\right)^2 - \frac{\partial^2 A}{\partial \hat{x}^2}\right] \tag{5-37}$$

and the differential equation (5-33) becomes a linear equation in $A(\hat{x}, t)$

$$\frac{\partial A(\hat{x}, t)}{\partial t} + \frac{\partial A(\hat{x}, t)}{\partial \hat{x}} F \hat{x} + \frac{B}{2} \frac{\partial^2 A(\hat{x}, t)}{\partial \hat{x}^2} = 0 \tag{5-38}$$

Now consider a transformation of state variables such that

$$\hat{y} = \boldsymbol{\Phi}(t_f, t)\, \hat{x} \tag{5-39}$$

where $\Phi(t_f, t)$ is the weighting function of the system (5-20), so

$$\frac{d\, \Phi(t_f, t)}{dt} = - F\, \Phi(t_f, t) \qquad \Phi(t_f, t_f) = 1 \tag{5-40}$$

and therefore

$$\boldsymbol{\Phi}(t_f, t) = e^{F\left[t_f - t\right]} \tag{5-41}$$

Using this transformation, equation (5-38) becomes

$$\frac{\partial A'(\hat{y}, t)}{\partial t} + \frac{\Phi^2(t_f, t) B}{2} \frac{\partial^2 A'(\hat{y}, t)}{\partial \hat{y}^2} = 0 \tag{5-42}$$

which has the solution

$$A'(\hat{y}, t) = \int_{-\infty}^{\infty} d\zeta\, f_{s(t)}(\zeta)\, A'(\hat{y} + \zeta, t_f) \qquad t < t_f \tag{5-43}$$

with

$$f_{s(t)}(\zeta) = \left|2\pi R(t)\right|^{-\frac{1}{2}} \exp\left\{-\frac{\zeta^2}{2R(t)}\right\} \tag{5-44}$$

and

$$R(t) = B \int_t^{t_f} \Phi^2(t_f, \tau) d\tau = \frac{B}{2F} \left[e^{2F[t_f - t]} - 1 \right] \qquad (5\text{-}45)$$

Transforming coordinates back to the $\hat{x}$ system produces the solution for $A(\hat{x}, t)$

$$A(\hat{x}, t) = \int_{-\infty}^{\infty} d\zeta\, f_{s(t)}(\zeta) A(\Phi(t_f, t)\hat{x} + \zeta ; t_f) \qquad (5\text{-}46)$$

Finally, the optimal control function may be written in terms of $A(\hat{x}, t)$, using (5-32) and (5-36)

$$u^*(\hat{x}, t) = \frac{B}{G\, A(\hat{x}, t)} \frac{\partial A(\hat{x}, t)}{\partial \hat{x}} \qquad (5\text{-}47)$$

and the derivative of $A(\hat{x}, t)$ is

$$\frac{\partial A(\hat{x}, t)}{\partial \hat{x}} = \Phi(t_f, t) \int_{-\infty}^{\infty} d\zeta\, f_{s(t)}(\zeta) D(\Phi(t_f, t)\hat{x} + \zeta ; t_f) \qquad (5\text{-}48)$$

where

$$D(\hat{x}, t_f) = \frac{\partial A(\hat{x}, t_f)}{\partial \hat{x}} \qquad (5\text{-}49)$$

The terminal condition $A(\hat{x}, t_f)$ required in (5-46) and (5-49) is determined by (5-34) as

$$A(\hat{x}, t_f) = \exp\left\{ - \frac{G^2}{aB} C^*(\hat{x}, t_f) \right\} = \exp\left\{ - \frac{G^2}{aB} \bar{\phi}(\hat{x}) \right\} \qquad (5\text{-}50)$$

An interesting closed form solution can be obtained by altering the problem slightly. Instead of specifying the terminal cost function $\phi(\hat{x}(t_f))$, consider a specification placed on the terminal control. It is desired to force the terminal estimated state $\hat{x}(t_f)$ to lie in a target interval such that

$$b_1 \leqq \hat{x}(t_f) \leqq + b_2 \qquad b_1 < b_2 \qquad (5\text{-}51)$$

where the values b_1 and b_2 are specified. Consider the control over a short interval from t' to t_f where

$$t_f = t' + \Delta t \tag{5-52}$$

and Δt is small. Assume that a control decision is made at time t' and that the control is held constant over the interval Δt. The control decision at t' is specified according to the following rule.

$$u(\hat{x}(t'), t') = \begin{cases} \dfrac{b_2 - \hat{x}(t')}{G\,\Delta t} & \text{if} \quad \hat{x}(t') > b_2 \\[2ex] 0 & \text{if} \quad b_1 \leqq \hat{x}(t') \leqq b_2 \\[2ex] \dfrac{b_1 - \hat{x}(t')}{G\,\Delta t} & \text{if} \quad \hat{x}(t') < b_1 \end{cases} \tag{5-53}$$

From (5-26) the cost to complete the process from time t' is

$$C^*(\hat{x}(t'), t') = \int_{t'}^{t_f} \frac{a}{2}\, u^2(t)\, dt \tag{5-54}$$

and since $u(t)$ is constant in the interval and given by(5-53), equation (5-54) becomes

$$C^*(\hat{x}(t'), t') = \begin{cases} \dfrac{a}{2}\dfrac{\left[b_2 - \hat{x}(t')\right]^2}{G^2\,\Delta t} & \text{if} \quad \hat{x}(t') > b_2 \\[2ex] 0 & \text{if} \quad b_1 \leqq \hat{x}(t') \leqq b_2 \\[2ex] \dfrac{a}{2}\dfrac{\left[b_1 - \hat{x}(t')\right]^2}{G^2\,\Delta t} & \text{if} \quad \hat{x}(t') < b_1 \end{cases} \tag{5-55}$$

If the time interval Δt approaches zero and $\hat{x}(t')$ lies outside the target interval, the control rule (5-53) will supply a delta function of proper strength to drive $\hat{x}(t_f)$ to the boundary of the target interval. However, if $\hat{x}(t')$ lies within the target interval, no control, is applied. Also, the terminal condition on C^* can be obtained as the limit of (5-55) as Δt approaches zero. Thus

$$C^*(\hat{x}, t_f-) = \begin{Bmatrix} +\infty & \text{if} & \hat{x} > b_2 \\ 0 & \text{if} & b_1 \leqq \hat{x} \leqq b_2 \\ +\infty & \text{if} & \hat{x} < b_1 \end{Bmatrix} \tag{5-56}$$

The terminal condition on $A(\hat{x}, t)$ can then be obtained from (5-34) as

$$A(\hat{x}, t_f) = \begin{Bmatrix} 0 & \text{if} & \hat{x} > b_2 \\ 1 & \text{if} & b_1 \leqq \hat{x} \leqq b_2 \\ 0 & \text{if} & \hat{x} < b_1 \end{Bmatrix} \tag{5-57}$$

and with (5-49)

$$D(\hat{x}, t_f) = \delta\,(\hat{x} - b_1) - \delta(\hat{x} - b_2) \tag{5-58}$$

Applying (5-57) and (5-58) to (5-46) and (5-48) yields

$$A(\hat{x}, t) = \frac{1}{2}\ \text{erf}\left[\frac{b_2 - \Phi(t_f, t)\hat{x}}{\sqrt{2R(t)}}\right] - \frac{1}{2}\ \text{erf}\left[\frac{b_1 - \Phi(t_f, t)\hat{x}}{\sqrt{2R(t)}}\right] \tag{5-59}$$

and

$$\frac{\partial A(\hat{x}, t)}{\partial \hat{x}} = \Phi(t_f, t)\left[f_{s(t)}(b_1 - \Phi(t_f, t)\hat{x}) - f_{s(t)}(b_2 - \Phi(t_f, t)\hat{x})\right] \tag{5-60}$$

where erf(x) is the error function of probability theory and is tabulated[42].

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\zeta^2}\, d\zeta \qquad (5-61)$$

The optimal control is obtained by substituting (5-59) and (5-60) into (5-47).

An actual physical example to which this solution might be applied is rate control of a vehicle in the presence of damping. Measurements of vehicle rate are inaccurate, so the state must be estimated. It is desired that the estimated state lie within a specified interval at the terminal time. The solution given above will accomplish this task with minimum mean expenditure of control energy.

5.4 Continuous Minimum Fuel V.T.A. Guidance

Consider the variable time of arrival guidance problem of Section 3.2. If measurements and corrections are made at very short intervals, the spacecraft guidance may be analysed as a continuous system. Assuming that state variables are transformed to the terminal time, the state satisfies the scalar equation

$$\dot{x}(t) = g(t)u(t) \qquad (5-62)$$

where u(t) is the control (acceleration) and g(t) is the control sensitivity (rate of change of terminal state per unit of vehicle acceleration). The cost to be minimized is the mean total fuel plus the weighted mean square terminal miss

$$J = E\left[\int_{t_i}^{t_f} |u(t)|\, dt + \frac{\lambda}{2}\, x^2(t_f)\right] \qquad (5-63)$$

so functions $\bar{L}$ and $\bar{\phi}$ become

$$\bar{L}(\hat{x}, u, t) = |u| \qquad (5-64)$$

$$\bar{\phi}(\hat{x}) = \frac{\lambda}{2}\left[\hat{x}^2 + P(t_f)\right] \qquad (5\text{-}65)$$

and a bound is placed on the control so

$$|u(t)| \leqq b \qquad (5\text{-}66)$$

In section 4.2 the differential equation satisfied by the continuous minimum expected value function $C^*(\hat{x}, t)$, was obtained as (4-35). The derivation of (4-35) involved the limit of the discrete recursion formula (4-23) as the time step Δt_n approached zero. In taking this limit it was assumed that the first and second derivatives of the discrete minimum expected value function $C_a^*(\hat{x}, n)$, exist and are continuous. For the discrete v.t.a. problem, this assumption holds everywhere except at the points $\pm\alpha$., where the second derivative of $C_a^*(\hat{x}, n)$ is discontinuous. Therefore, Eq(4-35) will apply everywhere except possibly at the points $\pm\alpha$. Hence from (4-35), (5-62), (5-64) and (5-65)

$$0 = \min_{|u| \leqq b}\left\{|u| + \frac{\partial C^*(\hat{x}, t)}{\partial t} + \frac{\partial C^*(\hat{x}, t)}{\partial \hat{x}} g(t)u + \frac{B(t)}{2}\frac{\partial^2 C^*(\hat{x}, t)}{\partial \hat{x}^2}\right\} \qquad \hat{x} \neq \pm\alpha \qquad (5\text{-}67)$$

with B(t) determined by (4-34) and (4-17) as

$$B(t) = -\dot{P}(t) \qquad (5\text{-}68)$$

and P(t) is the variance of error in estimated target miss distance at time t. The optimal control is obtained by minimizing the right side of (5-67), hence

$$u^* = \left\{\begin{array}{lll} b & \text{if} & \frac{\partial C^*}{\partial \hat{x}} g < -1 \\ 0 & \text{if} & \left|\frac{\partial C^*}{\partial \hat{x}} g\right| < 1 \\ -b & \text{if} & \frac{\partial C^*}{\partial \hat{x}} g > 1 \end{array}\right\} \qquad (5\text{-}69)$$

and C^* satisifes three different equations in different intervals of the $\hat{x}$ space

$$0 = b + \frac{\partial C^*(\hat{x},t)}{\partial t} + \frac{\partial C^*(\hat{x},t)}{\partial \hat{x}} g(t)b + \frac{B(t)}{2} \frac{\partial^2 C^*(\hat{x},t)}{\partial \hat{x}^2} \quad \text{if } \frac{\partial C^*}{\partial \hat{x}} g < -1 \tag{5-70}$$

$$0 = \frac{\partial C^*(\hat{x},t)}{\partial t} + \frac{B(t)}{2} \frac{\partial^2 C^*(\hat{x},t)}{\partial \hat{x}^2} \quad \text{if } \left|\frac{\partial C^*}{\partial \hat{x}}\right| g < 1 \tag{5-71}$$

$$0 = b + \frac{\partial C^*(\hat{x},t)}{\partial t} - \frac{\partial C^*(\hat{x},t)}{\partial \hat{x}} g(t)b + \frac{B(t)}{2} \frac{\partial^2 C^*(\hat{x},t)}{\partial \hat{x}^2} \quad \text{if } \frac{\partial C^*}{\partial \hat{x}} g > 1 \tag{5-72}$$

No bound is placed on the control for the discrete v.t.a. problem of Section 3.2, so it is necessary to investigate the limit as b goes to infinity. If (5-70) and (5-72) are divided through by b and limits taken as b approaches infinity; (5-70), (5-71) and (5-72) become

$$0 = \frac{\partial C^*(\hat{x},t)}{\partial t} + \frac{B(t)}{2} \frac{\partial^2 C^*(\hat{x},t)}{\partial \hat{x}^2} \quad \text{if } \left|\frac{\partial C^*}{\partial \hat{x}}\right| g < 1 \tag{5-73}$$

$$0 = 1 \pm \frac{\partial C^*(\hat{x},t)}{\partial \hat{x}} g(t) \quad \text{otherwise} \tag{5-74}$$

Also, since this problem is to be the continuous equivalent of the discrete v.t.a. problem, it may be inferrred that there is a function α (t), determined by

$$\left[\frac{\partial C^*(\hat{x},t)}{\partial \hat{x}}\right]_{\hat{x}=\alpha(t)} g(t) = 1 \tag{5-75}$$

so that (5-73) and (5-74) are written as

$$\frac{\partial C^*(\hat{x},t)}{\partial \hat{x}} = -\frac{1}{g(t)} \qquad \hat{x} < -\alpha(t) \tag{5-76}$$

$$\frac{\partial C^*(\hat{x},t)}{\partial t} = -\frac{B(t)}{2}\frac{\partial^2 C^*(\hat{x},t)}{\partial \hat{x}^2} \qquad |\hat{x}| < \alpha(t) \tag{5-77}$$

$$\frac{\partial C^*(\hat{x},t)}{\partial \hat{x}} = \frac{1}{g(t)} \qquad \hat{x} > \alpha(t) \tag{5-78}$$

Now if the derivative of $C^*(\hat{x},t)$ is defined as

$$G^*(\hat{x},t) = \frac{\partial C^*(\hat{x},t)}{\partial \hat{x}} \tag{5-79}$$

and both sides of (5-77) are differentiated with respect to $\hat{x}$, then (5-76), (5-77) and (5-78) become

$$G^*(\hat{x},t) = -\frac{1}{g(t)} \qquad \hat{x} < -\alpha(t) \tag{5-80}$$

$$\frac{\partial G^*(\hat{x},t)}{\partial t} = -\frac{B(t)}{2}\frac{\partial^2 G^*(\hat{x},t)}{\partial \hat{x}^2} \qquad |\hat{x}| < \alpha(t) \tag{5-81}$$

$$G^*(\hat{x},t) = \frac{1}{g(t)} \qquad \hat{x} > \alpha(t) \tag{5-82}$$

In addition, since this problem is the continuous equivalent of the discrete v.t.a. problem, $G^*(\hat{x},t)$ must be continuous across the boundaries $\pm\alpha(t)$. Also, if the second derivative of $C^*(\hat{x},t)$ is defined as

$$H^*(\hat{x},t) = \frac{\partial^2 C^*(\hat{x},t)}{\partial \hat{x}^2} = \frac{\partial G^*(\hat{x},t)}{\partial \hat{x}} \tag{5-83}$$

then differentiating (5-80), (5-81) and (5-82) produces

$$H^*(\hat{x},t) = 0 \qquad |\hat{x}| > \alpha(t) \tag{5-84}$$

$$\frac{\partial H^*(\hat{x}, t)}{\partial t} = - \frac{B(t)}{2} \frac{\partial^2 H^*(\hat{x}, t)}{\partial \hat{x}^2} \qquad |\hat{x}| < \alpha(t) \tag{5-85}$$

At this point the problem is further restricted by assuming that α (t) is constant. With this assumption it is shown in Appendix F that if the control sensitivity g(t) has a derivative for all values of time, then $H^*(\hat{x}, t)$ is continuous across the boundaries $\pm\alpha$. Thus boundary conditions on (5-85) are obtained as

$$H^*(\alpha^-, t) = H^*(-\alpha^+, t) = 0 \tag{5-86}$$

The terminal condition on H^* is yet to be determined.

The terminal condition on C^* is

$$C^*(\hat{x}, t_f) = \bar{\phi}(\hat{x}) = \frac{\lambda}{2} \left[\hat{x}^2 + P(t_f)\right] \tag{5-87}$$

By analogy once again with the discrete v.t.a. problem, the minimum expected cost to complete the process from time t_f^- is obtained with the help of (3-29) as

$$C^*(\hat{x}, t_f^-) = \begin{cases} \dfrac{|\hat{x}| - \alpha}{g(t_f)} + C^*(\alpha, t_f) & \text{if } |\hat{x}| > \alpha \\ \dfrac{\lambda}{2} (\hat{x}^2 + P(t_f)) & \text{if } |\hat{x}| \leq \alpha \end{cases} \tag{5-88}$$

where α is determined by

$$\left[\frac{\partial C^*(\hat{x}, t_f)}{\partial \hat{x}}\right]_{\hat{x} = \alpha} g(t_f) = \lambda\, \alpha\, g(t_f) = 1 \tag{5-89}$$

so

$$\alpha = \frac{1}{\lambda\, g(t_f)} \tag{5-90}$$

Taking the second derivative of (5-88) produces the terminal condition on $H^*(\hat{x}, t)$

$$H^*(\hat{x}, t_f^-) = \begin{Bmatrix} 0 & \text{if } |\hat{x}| > \alpha \\ \lambda & \text{if } |\hat{x}| < \alpha \end{Bmatrix} \tag{5-91}$$

Equations (5-84), (5-85), (5-86) and (5-91) completely determine the solution for $H^*(\hat{x}, t)$. By applying the usual method of separation of variables, the solution of (5-85) with the boundary conditions (5-86) and terminal condition (5-91), is obtained as

$$H^*(\hat{x}, t) = \frac{2\lambda}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+\frac{1}{2})} \cos\left(\frac{(k+\frac{1}{2})\pi\hat{x}}{\alpha}\right) \exp\left[-\frac{(k+\frac{1}{2})^2 \pi^2 \int_t^{t_f} B(\tau)d\tau}{2\alpha^2}\right]_{|\hat{x}| < \alpha} \tag{5-92}$$

and $G^*(\hat{x}, t)$ is the integral

$$G^*(\hat{x}, t) = \int_0^{\hat{x}} H^*(\xi, t)\, d\xi$$

$$= \frac{2\lambda\alpha}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+\frac{1}{2})^2} \sin\left(\frac{(k+\frac{1}{2})\pi x}{\alpha}\right) \exp\left[-\frac{(k+\frac{1}{2})^2 \pi^2 \int_t^{t_f} B(\tau)d\tau}{2\alpha^2}\right]_{|\hat{x}| < \alpha} . \tag{5-93}$$

Since $G^*(\hat{x}, t)$ is continuous across the boundary α, evaluating (5-93) at $\hat{x}$ equal to α and applying (5-82) produces an equation that must be satisfied by B(t) and g(t). Thus if α is constant B(t) and g(t) satisfy

$$\frac{2\lambda\alpha\, g(t)}{\pi^2} \sum_{k=0}^{\infty} (k+\tfrac{1}{2})^{-2} \exp\left[-\frac{(k+\frac{1}{2})^2 \pi^2 \int_t^{t_f} B(\tau)d\tau}{2\alpha^2}\right] = 1 \tag{5-94}$$

with α determined by (5-90). Since this solution requires that α be constant, the result is not very practical because of the restriction placed on B(t) and g(t) by Eq (5-94). However, the result of Appendix F which shows that the second derivative of C^* is continuous across the boundary α, may be very useful for purposes of numerical computation.

CHAPTER 6

CONCLUSIONS, CONTRIBUTIONS AND RECOMMENDATIONS

6.1 Conclusions

A method is developed by which optimal feedback controllers may be determined for discrete linear stochastic systems, when the system state cannot be measured without error. The theory admits cost functions which are nonquadratic in the state and/or the control and recognizes constraints on the control variables. The optimal control is a function of an estimate of the plant state, which is shown to be the mean state conditioned on the measurement history. Calculation of an optimal feedback control function involves solution of recursion formulas via the method of dynamic programming.

A variable time of arrival midcourse spacecraft guidance problem is postulated, ignoring errors out of the trajectory plane. By transforming state variables to the terminal time, this problem is simplified to one state variable. The optimal control is determined by an interval $\mathfrak{Z}(n)$ of values of the estimated state. If the estimate lies outside $\mathfrak{Z}(n)$, the optimal control drives the estimated state to the boundary of $\mathfrak{Z}(n)$; otherwise the optimal control is zero. Equations suitable for numerical determination of the intervals $\mathfrak{Z}(n)$ are derived and an actual numerical solution is obtained. A comparison made between the optimal controller and a near optimum linear controller, shows that there is negligible increase in cost if the near optimum controller is used.

Minimum fuel fixed time of arrival spacecraft guidance, ignoring out of plane errors, is also treated. Transformation of state variables to the terminal time simplifies this problem to two state variables. Necessary and sufficient conditions for the optimal control function are derived and the form of the optimal control is explained. It is found that the optimal control is determined by threshold curves in the two dimensional space of estimated state variables. A numerical example is used to illustrate an actual solution. Determination of the optimal control by a guidance computer involves a one parameter search of precalculated values to compute the optimum velocity correction vector.

A minimum fuel re-entry guidance problem is formulated. The optimal control is determined by threshold values of the estimated miss distance at the target. Equations suitable for numerical determination of the threshold are derived and the numerical solution of an example is illustrated.

The discrete quadratic cost problem is also investigated. Its solution determines the optimal control as a linear function of the estimated state. This result represents an independent derivation of the control/estimation separation theorem for discrete systems.

Under suitable conditions of differentiability of the minimum expected value function, the discrete recursion formulas are generalized to handle continuous problems. The result is a stochastic Hamilton-Jacobi equation, to be satisfied by the continuous minimum expected value function. Solution of this equation produces the optimal control as a function of the estimated state.

The theory is applied to the continuous quadratic cost problem. Solution of this problem produces the control/estimation separation theorem for continuous systems. The optimum controller is linear with time varying gains determined by the solution of a matrix Riccati equation.

Solution of a minimum control energy problem with arbitrary terminal cost function is also obtained. The result is written in terms of integrals. By specifying the terminal control to insure that the estimated terminal state lies within a target interval, the solution is obtained in terms of error functions (erfs).

Finally, the continuous variable time of arrival spacecraft guidance problem is formulated. By specializing to a particular case for which the control thresholds remain constant in time, a solution is obtained which specifies the relationship between the control sensitivity and the measurement information rate. An important result of this solution is the fact that the second derivative of the minimum expected value function is continuous across the control threshold.

A significant limitation to application of the discrete theory developed above is the requirement for large amounts of high speed computer memory in the solution of even the simplest problems. Calculation of the minimum expected value function must take place on a grid of dimension equal to the dimension of the state. For example, the one dimensional midcourse guidance problem of Section 3.2 required 60 points on the real line to adequately determine the minimum expected value function. Because of the symmetry of the problem however, calculations were made for only 30 positive values of the estimated state. The analogous two dimensional problem required a grid of 30 $\times$ 60 points. In general, the number of points required is $\frac{N^k}{2}$ where N is the number of points in each axis direction and k is the dimension of the problem. Thus, for a three dimensional problem with 60 points along each axis, the memory requirement for storage of the minimum expected value function is 108,000 locations. Clearly, the requirements grow out of hand for problems of even three or four dimensions. There is, of course, the possibility of increasing the interval between points, thereby decreasing N. However, this leads to decreased resolution and larger errors in approximating the optimal control function.

The computational difficulties outlined above are inherent in the dynamic programming method itself. For deterministic problems the calculas of variations or maximum principle can provide optimal solutions without resort to dynamic programming. With this approach two point boundary value problems must be solved. Although quite formidable in themselves, these solutions are usually much easier than their dynamic programming counterpart and problems of higher dimension can generally be handled. However, the maximum principle approach is possible only because the deterministic problems possess unique characteristic curves. No such unique characteristics exist for the class of stochastic problems considered above, so if a solution is desired, it must be obtained via dynamic programming.

6.2 Contributions

This section contains descriptions of those results of the research which are felt, by the author, to be either original developments or independent derivations of previously published results. Clearly, such statements are easily disputed, and often with some justification since it is quite difficult to determine just what constitutes original work. Thus, with some hesitation, the contributions made in this research are described in the following paragraph.

Recursion formulas are derived for the discrete minimum expected value function, for cases in which the plant state cannot be determined without error and the cost may be nonquadratic. Solution of these equations produces the optimal control as a function of the estimated state. This derivation is original, although the results have also been obtained by Striebel[70]. Minimum fuel, variable time of arrival, midcourse spacecraft guidance is developed. This solution is quite similar to some work of Tung and

Striebel[73]. Solutions of the fixed time of arrival midcourse guidance problem and the optimum re-entry guidance problem are original and to the best of the author's knowledge, do not appear elsewhere. Derivations of the control/estimation separation theorem are also included in the work, although these results are well known. The stochastic Hamilton-Jacobi equation for continuous systems with measurement errors is also derived; however, this result was brought to the attention of the author by Potter[61] and appears in a paper by Kushner[48]. Finally, the method of solving the recursion formulas by approximating the solution of diffusion equations is original.

6.3 Recommendations

In this Section some areas of possible future research will be briefly outlined. In the course of the discussion some partial results will be explained and some shortcomings of the existing theory will be enumerated.

6.3.1 Steady State Problems - If the estimation problem becomes stationary after a sufficiently long time, it is possible to pose a stationary control problem. For this case there is no terminal time and therefore no terminal cost function ϕ. Rather the mean expenditure per unit time is to be minimized by an appropriate choice of the control function. Since the problem is stationary the minimum expected value function will not change with time so for discrete problems, it should be possible to show that C^* satisfies the following equation

$$C^*(\hat{x}(n)) = \min_{u(n)\,\epsilon\,\mathcal{U}(n)} \Big\{ \overline{L}(\hat{x}(n),\ u(n)) \quad \ldots$$

$$+ \int_{-\infty}^{\infty} d\zeta_1 \ldots \int_{-\infty}^{\infty} d\zeta_k f_s(\zeta) C^*(\hat{x}'(n+1)+\zeta) \Big\} \qquad (6\text{-}1)$$

If numerical techniques can be devised to determine $C^*(\hat{x}(n))$

satisfying (6-1), then the optimal steady state control function will also be obtained.

Similarly, it should be possible to show that for continuous stationary problems, the minimum expected value function satisfies (4-35), with the second term on the right set equal to zero.

6.3.2 Variable Terminal Time Problems - If the estimation problem becomes stationary as time approaches infinity, then a variable terminal time problem can be posed such that sample trajectories terminate when the estimated state enters a target set or region $\hat{\mathcal{S}}$ in the $\hat{x}$ space. For this problem the minimum expected value function satisfies the usual recursion formula

$$C^*(\hat{x}(n), n) = \min_{u(n)\,\epsilon\,\mathcal{U}(n)} \Big\{ \overline{L}(\hat{x}(n), u(n), n) \quad \ldots$$

$$+ \int_{-\infty}^{\infty} d\zeta_1 \ldots \int_{-\infty}^{\infty} d\zeta_k \, f_{s(n+1)}(\zeta)\, C^*(\hat{x}'(n+1)+\zeta, n+1) \Big\} \tag{6-2}$$

Since each sample trajectory terminates when $\hat{x}(n)$ enters the target set $\hat{\mathcal{S}}$, it should be possible to show that the boundary condition

$$\lim_{\hat{x}(n) \to \hat{x} \epsilon \hat{\mathcal{S}}} C^*(\hat{x}(n), n) = 0 \tag{6-3}$$

must be satisfied at the boundary of $\hat{\mathcal{S}}$. In addition, if $\overline{L}(\hat{x}(n), u(n), n)$ and $\mathcal{U}(n)$ become stationary for large values of n, then since the estimation problem also becomes stationary at large n, it should be possible to show that the terminal condition on C^* is given by (6-1). Thus, solution of the system (6-1), (6-2) and (6-3) should produce the optimal control function for this class of problems.

Another type of variable terminal time problem can be posed by requiring that sample trajectories terminate when the actual state enters a target set or region $\mathbf{S}$. At first glance it may seem that this problem might be solved by some straightforward extension of the theory developed above. This is not the case, however, because $\hat{x}(n)$, based on the measurement history, is no longer a sufficient statistic for determining the posterior state probability density. In fact, at any time before a sample trajectory terminates, in addition to the measurement history there is another piece of information available; namely that the state cannot lie in the region $\mathbf{S}$. Thus the posterior state probability density cannot be normal because it must be identically zero in $\mathbf{S}$. It follows that $\hat{x}(n)$, calculated using the recursion formulas of Section 2.3, is no longer a sufficient statistic and the developments of Section 2.5 cannot be readily applied.

6.3.3 Specification of a Terminal Statistic - In some optimization problems, specification of a terminal cost function may not be desirable. This situation is best illustrated by the re-entry problem of Sections 3.6 and 3.7. For that problem the terminal cost function was made quadratic with arbitrary choice of the weighting factor λ. It would probably be more useful, in this case, to be able to determine the control function that minimizes the mean fuel required to attain a specified value of some statistic of the terminal state. For example, from the class of all admissible controls, as functions of the measurement history, find the control that will produce a given value of the variance of the terminal state, with the minimum total mean expenditure of fuel. This problem cannot be handled with the existing theory.

APPENDIX A

AN EXPRESSION FOR $E\left[\phi\,(x(q+1))\,\middle|\,M(q), u(q)\right]$

In order to evaluate the last term on the right of (2-40) an expression for the posterior probability density of $x(q+1)$ given $M(q)$ and $u(q)$, is necessary. First an expression for the terminal state is written as

$$x(q+1) = \hat{x}(q+1) - e(q+1) = \hat{x}'(q+1) - e(q+1) + s(q+1) \qquad \text{(A-1)}$$

Define a k dimensional vector $a(q+1)$ as

$$a(q+1) = \hat{x}'(q+1) - e(q+1) \qquad \text{(A-2)}$$

so (A-1) becomes

$$x(q+1) = a(q+1) + s(q+1) \qquad \text{(A-3)}$$

The error $e(q+1)$ is normally distributed and independent of $\hat{x}'(q+1)$. Also $\hat{x}'(q+1)$ is a deterministic function of $M(q)$ and $u(q)$ according to (2-19) and (2-20). Therefore, the posterior probability density of $a(q+1)$ can be written as

$$f_{a(q+1)}\left[\xi\,\middle|\,M(q), u(q)\right] = (2\pi)^{-\frac{k}{2}}\left|P(q+1)\right|^{-\frac{1}{2}} \cdot \exp\left\{-\frac{1}{2}\left[\xi - \hat{x}'(q+1)\right]^T P(q+1)^{-1}\left[\xi - \hat{x}'(q+1)\right]\right\} \qquad \text{(A-4)}$$

where

$$\hat{x}'(q+1) = \Phi(q+1, q)\hat{x}(q) + \theta(q+1, q)u(q) \qquad \text{(A-5)}$$

Since $P(q+1)$ is known a priori, $\hat{x}'(q+1)$ is a sufficient statistic and the posterior probability density of $a(q+1)$ given $M(q)$ and $u(q)$ can be written as

$$f_{a(q+1)}\left[\xi \,\middle|\, M(q), u(q)\right] = f_{a(q+1)}(\xi \mid \hat{x}'(q+1)) \tag{A-6}$$

The processed measurement information $s(q+1)$ is normally distributed with density function given by

$$f_{s(q+1)}(\zeta) = (2\pi)^{-\frac{k}{2}} \left| S(q+1) \right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\zeta^T S(q+1)^{-1}\zeta\right\} \tag{A-7}$$

and the covariance matrix determined from

$$S(q+1) = P'(q+1)H^T(q+1)\left[H(q+1)P'(q+1)H^T(q+1)+W(q+1)\right]^{-1} H(q+1)P'(q+1) \tag{A-8}$$

It was shown in Section 2.5 that $s(q+1)$ is independent of $\hat{x}'(q+1)$ and $e(q+1)$ so $s(q+1)$ and $a(q+1)$ are independent. Since $x(q+1)$ is the sum of $a(q+1)$ and $s(q+1)$, the posterior probability density of $x(q+1)$, given $M(q)$ and $u(q)$, can be determined by a convolution integral

$$f_{x(q+1)}(\xi \mid \hat{x}'(q+1)) = \int_{-\infty}^{\infty} d\zeta_1 \dots \int_{-\infty}^{\infty} d\zeta_k \, f_{a(q+1)}(\xi - \zeta \mid \hat{x}'(q+1)) f_{s(q+1)}(\zeta) \tag{A-9}$$

Using this density function, the second term on the right of (2-40) is written as

$$E\left[\phi(x(q+1)) \,\middle|\, M(q), u(q)\right] = \int_{-\infty}^{\infty} d\xi_1 \dots \int_{-\infty}^{\infty} d\xi_k \, \phi(\xi) f_{x(q+1)}(\xi \mid \hat{x}'(q+1)) \tag{A-10}$$

Substituting (A-9) into (A-10) and reversing the order of integration obtains

$$E\left[\phi\,(x(q+1))\,\middle|\,M(q),\,u(q)\right] =$$

$$\int_{-\infty}^{\infty} d\zeta_1 \ldots \int_{-\infty}^{\infty} d\zeta_k\, f_{s(q+1)}(\zeta) \int_{-\infty}^{\infty} d\xi_1 \ldots \int_{-\infty}^{\infty} d\xi_k\, \phi\,(\xi)\, f_{a(q+1)}(\xi - \zeta \,|\, \hat{x}'(q+1)) \tag{A-11}$$

Now, by comparing (A-4) to (2-25) and (2-26) it is clear that (A-11) may be written thus

$$E\left[\phi\,(x(q+1))\,\middle|\,M(q),\,u(q)\right] =$$

$$\int_{-\infty}^{\infty} d\zeta_1 \ldots \int_{-\infty}^{\infty} d\zeta_k\, f_{s(q+1)}(\zeta) \int_{-\infty}^{\infty} d\xi_1 \ldots \int_{-\infty}^{\infty} d\xi_k\, \phi\,(\xi)\, f_{x(q+1)}(\xi - \zeta \,|\, \hat{x}'(q+1)) \tag{A-12}$$

Finally, by the definition of $\overline{\phi}$ given in (2-36)

$$E\left[\phi\,(x(q+1))\,\middle|\,M(q),\,u(q)\right] = \int_{-\infty}^{\infty} d\zeta_1 \ldots \int_{-\infty}^{\infty} d\zeta_k\, f_{s(q+1)}(\zeta)\, \overline{\phi}\,(\hat{x}'(q+1)+\zeta) \tag{A-13}$$

which is the expression required for (2-41).

APPENDIX B

GREEN'S FUNCTION FOR THE DIFFUSION EQUATION

The k dimensional diffusion equation of interest is given by

$$\frac{\partial D(\xi, \tau)}{\partial \tau} = \frac{1}{2} \operatorname{Tr} \left[S(n+1) \; \frac{\partial^2 D(\xi, \tau)}{\partial \xi^2} \right] \tag{B-1}$$

where the (k$\times$k) matrix of second derivatives is

$$\frac{\partial^2 D(\xi, \tau)}{\partial \xi^2} = \left[\frac{\partial^2 D(\xi, \tau)}{\partial \xi_i \, \partial \xi_j} \right] \tag{B-2}$$

If $g(\xi, \tau)$ is the k dimensional Green's function corresponding to (B-1), then by definition $g(\xi, \tau)$ must satisfy the inhomogeneous equation

$$\frac{\partial g(\xi, \tau)}{\partial \tau} - \frac{1}{2} \operatorname{Tr} \left[S(n+1) \; \frac{\partial^2 g(\xi, \tau)}{\partial \xi^2} \right] = \delta(\xi)\, \delta(\tau) \qquad 0 \leqq \tau \leqq 1 \tag{B-3}$$

where $\delta(\xi)$ is the k dimensional Dirac delta function. Now $\delta(\xi)$ can be written as a k dimensional Fourier integral

$$\delta(\xi) = \frac{1}{(2\pi)^k} \int_{-\infty}^{\infty} dp_1 \cdots \int_{-\infty}^{\infty} dp_k \, \exp\left\{ i\, p^T \xi \right\} \tag{B-4}$$

where p is a k dimensional vector and $i = \sqrt{-1}$.

Also, if $\gamma(p, \tau)$ is the k dimensional Fourier transform of $g(\xi,\tau)$ then

$$g(\xi,\tau) = \frac{1}{(2\pi)^k} \int_{-\infty}^{\infty} dp_1 \ldots \int_{-\infty}^{\infty} dp_k \exp\left\{i p^T \xi\right\} \gamma(p,\tau) \tag{B-5}$$

and the terms on the left of (B-3) become

$$\frac{\partial g(\xi,\tau)}{\partial \tau} = \frac{1}{(2\pi)^k} \int_{-\infty}^{\infty} dp_1 \ldots \int_{-\infty}^{\infty} dp_k \exp\left\{ i p^T \xi\right\} \frac{\partial \gamma(p,\tau)}{\partial \tau} \tag{B-6}$$

$$\mathrm{Tr}\left[S(n+1) \frac{\partial^2 g(\xi,\tau)}{\partial \xi^2}\right] = -\frac{1}{(2\pi)^k} \int_{-\infty}^{\infty} dp_1 \ldots \int_{-\infty}^{\infty} dp_k \exp\left\{i p^T \xi\right\} \gamma(p,\tau) \mathrm{Tr}\left[S(n+1) p p^T\right]$$

$$= -\frac{1}{(2\pi)^k} \int_{-\infty}^{\infty} dp_1 \ldots \int_{-\infty}^{\infty} dp_k \exp\left\{i p^T \xi\right\} \gamma(p,\tau) p^T S(n+1) p \tag{B-7}$$

Substituting (B-4), (B-6) and (B-7) into (B-3) produces a differential equation for $\gamma(p,\tau)$

$$\frac{\partial \gamma(p,\tau)}{\partial \tau} + \frac{p^T S(n+1) p}{2} \gamma(p,\tau) = \delta(\tau) \tag{B-8}$$

with the solution

$$\gamma(p,\tau) = \exp -\left\{\frac{1}{2} p^T S(n+1) p \tau\right\} \tag{B-9}$$

Upon substituion of (B-9) into (B-5) there results

$$g(\xi,\tau) = \frac{1}{(2\pi)^k} \int_{-\infty}^{\infty} dp_1 \ldots \int_{-\infty}^{\infty} dp_k \exp\left\{i p^T \xi - \frac{1}{2} p^T S(n+1) p \tau\right\} \tag{B-10}$$

The matrix $S(n+1)$ is symmetric so an orthogonal transformation A can be found that diagonalizes $S(n+1)$

$$A\ S(n+1)A^T = \begin{bmatrix} \lambda_1 & & & O \\ & \ddots & & \\ & & \ddots & \\ O & & & \lambda_k \end{bmatrix} \tag{B-11}$$

If the orthogonal transformation of coordinates

$$\rho = Ap \tag{B-12}$$

is made, then (B-10) becomes

$$g(\xi,\tau) = \frac{1}{(2\pi)^k} \int_{-\infty}^{\infty} d\rho_1 \ldots \int_{-\infty}^{\infty} d\rho_k \exp\left\{ i\,\rho^T A\,\xi - \frac{1}{2}\rho^T A\,S(n+1)A^T \rho\,\tau \right\} \tag{B-13}$$

Define the k vector y thus

$$y = A\xi \tag{B-14}$$

so (B-13) yields

$$g(\xi,\tau) = \frac{1}{(2\pi)^k} \int_{-\infty}^{\infty} d\rho_1 \ldots \int_{-\infty}^{\infty} d\rho_k \exp\left\{ \sum_{j=1}^{k} (i\,\rho_j y_i - \frac{1}{2}\rho_j^2\,\lambda_j\,\tau) \right\} \tag{B-15}$$

or

$$g(\xi,\tau) = \frac{1}{(2\pi)^k} \prod_{j=1}^{k} \int_{-\infty}^{\infty} d\rho_j \exp\left\{ i\rho_j\,y_j - \frac{1}{2}\rho_j^2\,\lambda_j\,\tau \right\} \tag{B-16}$$

The exponent in (B-16) can be written as

$$\begin{aligned} i\,\rho_j y_j - \frac{1}{2}\rho_j^2\lambda_j\,\tau &= -\left(\rho_j\sqrt{\frac{\lambda_j\,\tau}{2}} - \frac{i\,y_j}{\sqrt{2\,\lambda_j\,\tau}}\right)^2 - \frac{y_j^2}{2\,\lambda_j\,\tau} \\ &= -\left(\frac{\lambda_j\,\tau\,\alpha_j^2}{2}\right) - \left(\frac{y_j^2}{2\,\lambda_j\,\tau}\right) \end{aligned} \tag{B-17}$$

where

$$\alpha_j = \rho_j - \frac{i y_j}{\lambda_j \tau} \tag{B-18}$$

Thus the integral on the right in (B-16) becomes

$$\int_{-\infty}^{\infty} d\rho_j \exp\left\{i \rho_j y_j - \frac{1}{2} \rho_j^2 \lambda_j \tau\right\} = \int_{-\infty}^{\infty} d\rho_j \exp\left\{- \frac{\lambda_j \tau \alpha_j^2}{2} - \frac{y_j^2}{2\lambda_j \tau}\right\}$$

$$= \exp\left\{- \frac{y_j^2}{2\lambda_j \tau}\right\} \int_{-\infty}^{\infty} d\alpha_j \exp\left\{- \frac{\lambda_j \tau \alpha_j^2}{2}\right\}$$

$$= \sqrt{\frac{2\pi}{\lambda_j \tau}} \exp\left\{- \frac{y_j^2}{2\lambda_j \tau}\right\} \tag{B-19}$$

and substituting into (B-16) yields

$$g(\xi, \tau) = \frac{1}{(2\pi)^k} \left(\frac{2\pi}{\tau}\right)^{\frac{k}{2}} \left(\prod_{j=1}^{k} \lambda_j\right)^{-\frac{1}{2}} \exp\left\{- \frac{1}{2\tau} \sum_{j=1}^{k} \frac{y_j^2}{\lambda_j}\right\}$$

$$= (2\pi\tau)^{-\frac{k}{2}} \left|S(n+1)\right|^{-\frac{1}{2}} \exp\left\{- \frac{1}{2\tau} y^T \begin{bmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_k \end{bmatrix}^{-1} y\right\}$$

$$= (2\pi\tau)^{-\frac{k}{2}} \left|S(n+1)\right|^{-\frac{1}{2}} \exp\left\{- \frac{1}{2\tau} \xi^T A^T \begin{bmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_k \end{bmatrix}^{-1} A\xi\right\}$$

(B-20)

Using (B-11) produces the desired expression for the Green's function.

$$g(\xi, \tau) = (2\pi\tau)^{-\frac{k}{2}} \left| S(n+1) \right|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2\tau} \xi^T S(n+1)^{-1} \xi \right\} \tag{B-21}$$

Finally at $\tau = 1$, the Green's function for the diffusion equation becomes the probability density of s(n+1)

$$g(\xi, 1) = (2\pi)^{-\frac{k}{2}} \left| S(n+1) \right|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} \xi^T S(n+1)^{-1} \xi \right\} = f_{s(n+1)}(\xi) \tag{B-22}$$

APPENDIX C

NUMERICAL EVALUATION OF EXPECTATIONS

In order to apply the theory of Chapter 2, it is necessary to numerically evaluate integrals of the form

$$C^{*\prime}(\xi, n) = \int_{-\infty}^{\infty} d\zeta_1 \ldots \int_{-\infty}^{\infty} d\zeta_k \, f_{s(n+1)}(\zeta) C^*(\xi + \zeta, n+1) \tag{C-1}$$

It was shown in Appendix B that the probability density $f_{s(n+1)}(\zeta)$ is the Green's function for a k dimensional diffusion equation. Thus (C-1) can be numerically evaluated in two ways, either by approximating the integral via quadrature formulas or by approximating the solution of the diffusion equation

$$\frac{\partial D(\xi, \tau)}{\partial \tau} = \frac{1}{2} \mathrm{Tr} \left[S(n+1) \frac{\partial^2 D(\xi, \tau)}{\partial \xi^2} \right] \tag{C-2}$$

$$D(\xi, 0) = C^*(\xi, n+1) \tag{C-3}$$

$$C^{*\prime}(\xi, n) = D(\xi, 1) \tag{C-4}$$

using central difference methods.

As a means of comparison of these two methods, consider a **problem** with two state variables. Define subscripts i and j, indicating the row (i) and the column(j) of a point in a two dimensional mesh representing the $\hat{x}$ space. Thus, $C^*(\xi_{ij}, n)$ is the minimum expected value function evaluated at ξ_{ij}, where i represents the row and j the column of the point ξ_{ij} in the mesh. If h is the interval between mesh points, then the lowest order quadrature formula for $C^{*\prime}$ is

$$C^{*\prime}(\xi_{ij}, n) = h^2 \sum_{\ell=-a}^{a} \sum_{m=-b}^{b} f_{s(n+1)}(\xi_{\ell m})\, C^*(\xi_{(i+\ell)(j+m)}, n+1) \tag{C-5}$$

where ℓ and m range over a sufficiently large set of values so that all points with non-negligible values of the product on the right in (C-5) are included. If, for example, σ_1^2 and σ_2^2 are the eigenvalues of the covariance matrix S(n+1); then the range of ℓ and m can usually be chosen to encompass a square region with each side greater than six times the square root of the larger eigenvalue

$$ah = bh > 3 \max(\sigma_1, \sigma_2) \tag{C-6}$$

Further, if the point ξ_{ij} is close to the boundary of the mesh, then points $\xi_{(i+\ell)(j+m)}$ will lie outside the mesh for certain values of ℓ and m. Thus, some sort of approximation of C^* must be used, usually involving a functional description of C^* outside the boundary. In addition, the probability density

$$f_{s(n+1)}(\xi_{\ell m}) = (2\pi)^{-\frac{k}{2}} \left|S(n+1)\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\, \xi_{\ell m}^T\, S_{(n+1)}^{-1}\, \xi_{\ell m}\right\} \tag{C-7}$$

must be evaluated at the points $\xi_{\ell m}$. Calculation of these values can be done once at the beginning of the computations and stored for later use. The stored values are multiplied by the appropriate values of C^*, according to (C-5) and summed over ℓ and m to produce each new value of $C^{*\prime}$. This procedure is repeated for each point ξ_{ij} of the mesh. For example, in the two dimensional fixed time of arrival problem of Section 3.5, a mesh of points (30×60) is used. Because of the symmetry of C^*, only half of a (60×60) mesh is necessary. It was found that the limits a and b in (C-5) must be at least equal to 10, so the computations range over a (20×20) grid and the sums in (C-5) involve 400 points. Thus for each point ξ_{ij}, the computation of $C^{*\prime}(\xi_{ij}, n)$ requires about 400 multiplications and 400 additions. Since there are 1800 points in the mesh, 720,000 multiplications and 720,000 additions are necessary.

For the equivalent computation by approximating the solution of the diffusion equation (C-2), the lowest order central difference equation is given by

$$D(\xi_{ij}, \tau + \Delta\tau) = D(\xi_{ij}, \tau) + \frac{1}{2}\left[S_{11}(n+1)\,\alpha + 2\,S_{12}(n+1)\beta + S_{22}(n+1)\gamma\right]\Delta\tau \tag{C-8}$$

where α, β and γ are

$$\alpha = \left[D(\xi_{(i+1)(j)}, \tau) + D(\xi_{(i-1)(j)}, \tau) - 2\,D(\xi_{ij}, \tau)\right]/h^2 \tag{C-9}$$

$$\beta = \left[D(\xi_{(i+1)(j+1)}, \tau) + D(\xi_{(i-1)(j-1)}, \tau) - D(\xi_{(i+1)(j-1)}, \tau) - D(\xi_{(i-1)(j+1)}, \tau)\right]/4h^2 \tag{C-10}$$

$$\gamma = \left[D(\xi_{(i)(j+1)}, \tau) + D(\xi_{(i)(j-1)}, \tau) - 2D(\xi_{ij}, \tau)\right]/h^2 \tag{C-11}$$

On the boundaries of the mesh, some approximation must be made to the values of D just outside the boundaries, so that α, β and γ can be computed. For the problem of Section 3.5, with a (30×60) point mesh, it was found that about 30 steps in $\Delta\tau$ should be taken (i.e., $\Delta\tau = \frac{1}{30}$). By properly rearranging terms, the computations involved in (C-8)-(C-11) can be accomplished with 3 multiplications and 12 additions. With 30 steps in $\Delta\tau$, the total number of operations involved in computing one point $C^{*\prime}(\xi_{ij}, n)$ is 90 multiplications and 360 additions. Thus, for the (30×60) mesh, 162,000 multiplications and 648,000 additions are required.

For most computers, the time required to do multiplications is much greater than the time to perform additions. Since the central difference formulas require only about one fourth as many multiplications as the quadrature formulas and approximately the same number of additions, the central difference method should be about four times faster than the quadrature method.

An even more significant difficulty with the quadrature formulas is the problem of approximating C^* outside the boundaries of the mesh. Some form of approximation is necessary to produce C^* as a function of ξ_{ij} in those regions. By comparison, the central difference equation requires values of C^* just one interval outside the boundary, at each step $\Delta\tau$. The accumulated error over n steps will be propagated into the mesh a distance $(n-1)h$ from the boundary. The approximation used for the two degree of freedom problem of Section 3.5 is to assume that the incremental change in D, at a point on the boundary, is the same as the change in D at the nearest point inside the boundary. This technique will provide exact answers on the boundary if D is a quadratic form. Typically D is not quadratic however, but looks more like a linear function of the radial distance to the origin, so this approximation is conservative in the sènse that the calculated incremental change in D will be larger than the true value. In general it is much easier, from the standpoint of computer programming, to handle the boundary approximations on a step by step basis, than to calculate C^* outside the boundary with approximation formulas. In addition, the central difference method will always provide a conservative (larger) estimate of the true value; whereas the quadrature approximation may not necessarily be conservative.

Finally, there is the possibility of using higher order quadrature formulas with the hope of increasing the permissable computation interval h. Similarly, higher order central differences may permit larger intervals h. In this Appendix only the lowest order quadratüre and central difference formulas were compared. It is felt that the conclusions drawn from this comparison will also be valid for the higher order formulas because the quadrature formulas always involve many more multiplications than the central difference formulas.

APPENDIX D

OPTIMAL CONTROL FOR THE V.T.A. GUIDANCE PROBLEM*

To formulate the v.t.a. optimal control function, it is necessary to perform the minimization required in (D-1).

$$C^*(\hat{x}(n), n) = \min_{u(n)} \left\{ |u(n)| + \int_{-\infty}^{\infty} d\zeta \, f_{s'(n+j)}(\zeta) \, C^*(\hat{x}'(n+j)+\zeta, n+j) \right\} \tag{D-1}$$

where

$$f_{s'(n+j)}(\zeta) = (2\pi)^{-\frac{1}{2}} \left| S'(n+j) \right|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} \zeta^T S'(n+j)^{-1} \zeta \right\} \tag{D-2}$$

and

$$\hat{x}'(n+j) = \hat{x}(n) + \theta(n+j, n) \, u(n) \tag{D-3}$$

If the derivative of the function in braces on the right of (D-1) is taken with respect to u(n), there results

$$\frac{\partial\{\cdot\}}{\partial u(n)} = \operatorname{sgn}[u(n)] + \theta(n+j, n) \int_{-\infty}^{\infty} d\zeta \, f_{s'(n+j)}(\zeta) \, G^*(\hat{x}(n)+\theta(n+j,n)u(n)+\zeta; n+j) \tag{D-4}$$

with the derivative of the minimum expected value function defined as

$$G^*(\hat{x}, n+j) = \frac{\partial C^*(\hat{x}, n+j)}{\partial \hat{x}} \tag{D-5}$$

Assume that $G^*(\hat{x}, n+j)$ is a piecewise continuous, anti-symmetric, monatonically nondecreasing function of $\hat{x}$ such that

---

*This analysis closely follows the work of Tung and Striebel(73)

$$- \frac{1}{\theta(n+j+k, n+j)} \leqq G^{*}(\hat{x}, n+j) \leqq \frac{1}{\theta(n+j+k, n+j)}$$

$$\lim_{\hat{x} \to +\infty} G^{*}(\hat{x}, n+j) = \frac{1}{\theta(n+j+k, n+j)}$$

$$\lim_{\hat{x} \to -\infty} G^{*}(\hat{x}, n+j) = - \frac{1}{\theta(n+j+k, n+j)} \tag{D-6}$$

where θ (n+j +k, n+j) is the effect of the control at time t_{n+j} upon the state at the following control time t_{n+j+k}. Index k implies that k measurements are taken in this interval. Fig. (D-1) illustrates a typical function $G^{*}(\hat{x}, n+j)$

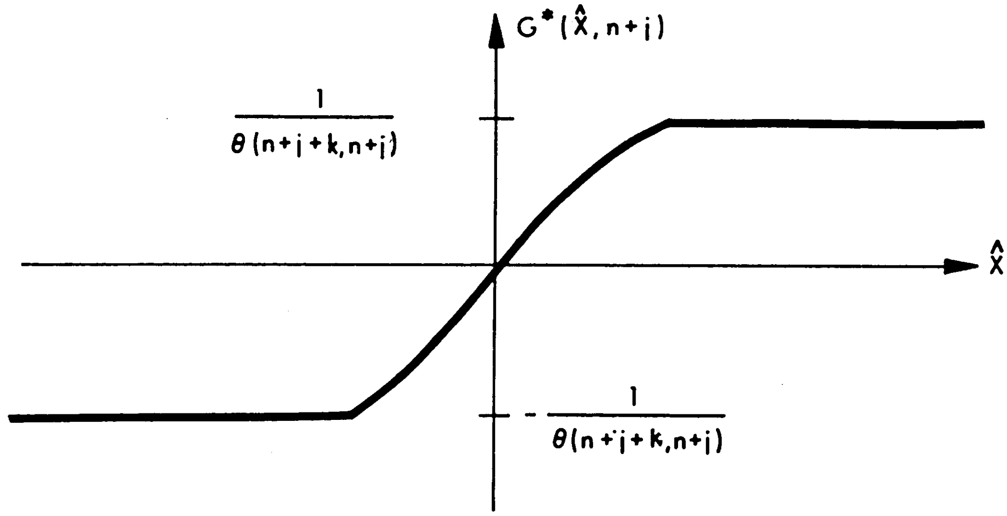

Fig. (D-1) Typical Graph of $G^{*}(\hat{x}, n+j)$

Further assume that the control sensitivities decrease with time so that

$$\frac{\theta(n+j, n)}{\theta(n+j+k, n+j)} > 1 \tag{D-7}$$

This is a reasonable assumption because θ(n+j, n) is the sensitivity of the terminal state to a velocity correction applied at time t_n.

Case #1

Consider values of $\hat{x}(n)$ in the set $\mathfrak{R}^{+}(n)$ where

$$\mathfrak{R}^{+}(n) = \left\{ \hat{x}(n) : \theta(n+j,n) \int_{-\infty}^{\infty} d\zeta\, f_{s'(n+j)}(\zeta)\, G^{*}(\hat{x}(n)+\zeta, n+j) > +1 \right\} \tag{D-8}$$

Setting the right hand side of (D-4) equal to zero produces a necessary condition for the minimum in (D-1). If $\hat{x}(n)$ is in $\mathfrak{R}^{+}(n)$, then this condition will be satisfied by a negative value of $u^{*}(n)$ such that

$$-1 + \theta(n+j,n) \int_{-\infty}^{\infty} d\zeta\, f_{s'(n+j)}(\zeta) G^{*}(\hat{x}(n)+\theta(n+j,n)u^{*}(n)+\zeta\ ; n+j) = 0 \tag{D-9}$$

A unique solution of (D-9) always exists if the assumptions about θ and G^{*} are valid and because $f_{s'(n+j)}(\zeta)$ is a positive, symmetric, analytic function. A locally sufficient condition is obtained by taking the second derivative of the function in braces in (D-1) and evaluating it at $u^{*}(n)$

$$\left[\frac{\partial^2 \{\cdot\}}{\partial u(n)^2}\right]_{u(n)=u^{*}(n)} = \delta(u^{*}(n)) + \theta^2(n+j,n) \int_{-\infty}^{\infty} d\zeta\, f_{s'(n+j)}(\zeta)\, H^{*}(\hat{x}(n)+\theta(n+j,n)u^{*}(n)+\zeta, n+j) \tag{D-10}$$

where the second derivative of the minimum expected value function is defined as

$$H^{*}(\hat{x}, n+j) = \frac{\partial G^{*}(\hat{x}, n+j)}{\partial \hat{x}} = \frac{\partial^2 C^{*}(\hat{x}, n+j)}{\partial \hat{x}^2} \tag{D-11}$$

Because of the form assumed for G^{*}, the integral on the right of (D-10) must be positive. Further $u^{*}(n)$ is nonzero so $\delta(u^{*}(n))$ is zero. Therefore, a local minimum exists for $u^{*}(n)$ satisfying (D-9). Clearly positive values of u(n) cannot satisfy the necessary condition if $\hat{x}(n)$ is in $\mathfrak{R}^{+}(n)$ and the only other possibility occurs for u(n) equal zero, at which point there is a simple discontinuity in (D-4). However, if $\hat{x}(n) \in \mathfrak{P}^{+}(n)$ there is no change in the sign of (D-4) across the discontinuity so no minimum can occur at zero. Finally, since the solution of (D-9) is unique, $u^{*}(n)$ gives the absolute minimum.

Case #2

Now consider values of $\hat{x}(n)$ in the set $\mathfrak{Z}(n)$ where

$$\mathfrak{Z}(n) = \left\{ \hat{x}(n) : -1 \leqq \theta(n+j, n) \int_{-\infty}^{\infty} d\zeta \, f_{s'(n+j)}(\zeta) \, G^{*}(\hat{x}(n) + \zeta, n+j) \leqq +1 \right\} \tag{D-12}$$

For u(n) less than zero, the derivative (D-4) is

$$\left[\frac{\partial\{\cdot\}}{\partial u(n)}\right]_{u(n)<0} = -1 + \theta(n+j, n) \int_{-\infty}^{\infty} d\zeta \, f_{s'(n+j)}(\zeta) \quad G^{*}(\hat{x}(n) + \theta(n+j, n) u(n) + \zeta \, ; n+j) < 0 \tag{D-13}$$

because of the assumed form of G^{*}. Also, for u(n) greater than zero

$$\left[\frac{\partial\{\cdot\}}{\partial u(n)}\right]_{u(n)>0} = 1 + \theta(n+j, n) \int_{-\infty}^{\infty} d\zeta \, f_{s'(n+j)}(\zeta) \quad G^{*}(\hat{x}(n) + \theta(n+j, n) u(n) + \zeta \, ; n+j) > 0 \tag{D-14}$$

Thus since the sign of the derivative is negative for u(n) less than zero and positive for u(n) greater than zero, an absolute minimum must occur for $u^{*}(n)$ equal to zero.

Case #3

Finally consider values of x(n) in the set $\mathfrak{P}^{-}(n)$ so that

$$\mathfrak{P}^{-}(n) = \left\{ \hat{x}(n) : \theta(n+j, n) \int_{-\infty}^{\infty} d\zeta \, f_{s'(n+j)}(\zeta) G^{*}(\hat{x}(n) + \zeta, n+j) < -1 \right\} \tag{D-15}$$

With reasoning completely analogous to that used in Case #1 above, it can be shown that $u^*(n)$ must be positive and satisfy

$$1 + \theta(n+j, n) \int_{-\infty}^{\infty} d\zeta\, f_{s'(n+j)}(\zeta)\, G^*(\hat{x}(n)+\theta(n+j, n)u^*(n) + \zeta; n+j) = 0 \qquad \text{(D-16)}$$

From these results it is clear that if the positive quantity $\alpha(n)$ is defined to satisfy

$$\theta(n+j, n) \int_{-\infty}^{\infty} d\zeta\, f_{s'(n+j)}(\zeta)\, G^*(\alpha(n) + \zeta, n+j) = 1 \qquad \text{(D-17)}$$

then if the conditions on $G^*(\hat{x}, n+j)$ are satisfied, the optimal control must drive the estimated state to $+\alpha(n)$ if $\hat{x}(n) > \alpha(n)$, to $-\alpha(n)$ if $\hat{x}(n) < -\alpha(n)$ and must be zero if $-\alpha(n) \leqq \hat{x}(n) \leqq +\alpha(n)$. These conditions are equivalent to (3-27). Further, the recursion formula for $C^*(\hat{x}(n), n)$ is given by

$$C^*(\hat{x}(n), n) = \begin{cases} \dfrac{|\alpha(n) - \hat{x}(n)|}{\theta(n+j, n)} + \displaystyle\int_{-\infty}^{\infty} d\zeta\, f_{s'(n+j)}(\zeta) C^*(\alpha(n) + \zeta, n+j) & \text{if } \hat{x}(n) > +\alpha(n) \\[2ex] \displaystyle\int_{-\infty}^{\infty} d\zeta\, f_{s'(n+j)}(\zeta)\, C^*(\hat{x}(n) + \zeta, n+j) & \text{if } -\alpha(n) \leqq x(n) \leqq +\alpha(n) \\[2ex] \dfrac{|-\alpha(n) - \hat{x}(n)|}{\theta(n+j, n)} + \displaystyle\int_{-\infty}^{\infty} d\zeta\, f_{s'(n+j)}(\zeta)\, C^*(-\alpha(n) + \zeta, n+j) & \text{if } \hat{x}(n) < -\alpha(n) \end{cases} \qquad \text{(D-18)}$$

so $G^*(\hat{x}(n), n)$ satisfies

$$G^*(\hat{x}(n), n) = \begin{cases} \dfrac{1}{\theta(n+j, n)} & \text{if } \hat{x}(n) > +\alpha(n) \\[2ex] \displaystyle\int_{-\infty}^{\infty} d\zeta\, f_{s'(n+j)}(\zeta) G^*(\hat{x}(n) + \zeta, n+j) & \text{if } -\alpha(n) \leqq \hat{x}(n) \leqq +\alpha(n) \\[2ex] -\dfrac{1}{\theta(n+j, n)} & \text{if } \hat{x}(n) < -\alpha(n) \end{cases} \qquad \text{(D-19)}$$

Therefore, if $G^*(\hat{x}, n+j)$ is piecewise continuous, monatonically nondecreasing, antisymmetric, and satisfies (D-6), these conditions also hold for $G^*(\hat{x}, n)$. Clearly the conditions will always be satisfied if they are satisfied by the terminal derivative. Consider first total final correction guidance. The minimum expected value function at the last correction time is given by (3-30) so the terminal derivative is

$$G^*(\hat{x}, m) = \frac{\mathrm{sgn}[\hat{x}]}{\theta(q+1, m)} \tag{D-20}$$

which obviously satisfies the assumed conditions. Second, consider quadratic terminal cost guidance with the terminal condition (3-31). The terminal derivative is

$$G^*(\hat{x}, q+1) = \lambda \hat{x} \tag{D-21}$$

Since the terminal sensitivity $\theta(q+1, q+1)$ is zero and λ is positive, this function also satisfies the conditions. Therefore, the optimal control law is given by (3-27) if (D-7) is satisifed. In cases for which (D-7) is not satisfied, the optimal control is always identically zero at time t_n.

APPENDIX E

CONCAVITY OF THE MINIMUM EXPECTED VALUE FUNCTION FOR THE F.T.A. PROBLEM

Consider the function $C^{*\prime}(\hat{x}, n)$ given by (3-56) and repeated here

$$C^{*\prime}(\hat{x}, n) = \int_{-\infty}^{\infty} d\zeta_1 \int_{-\infty}^{\infty} d\zeta_2 f_{s'(n+j)}(\zeta) C^{*}(\hat{x}+\zeta, n+j) \qquad \text{(E-1)}$$

Assume that the function $C^{*}(\hat{x}, n+j)$ is semi-concave. That is to say, for any two points $\hat{x}_A$ and $\hat{x}_B$, the inequality

$$C^{*}(\mu \hat{x}_A + (1-\mu)\hat{x}_B, n+j) \leqq \mu C^{*}(\hat{x}_A, n+j) + (1-\mu) C^{*}(\hat{x}_B, n+j)$$
$$0 < \mu < 1 \qquad \text{(E-2)}$$

must hold. Equation (E-2) implies that the line segment joining $C^{*}(\hat{x}_A, n+j)$ and $C^{*}(\hat{x}_B, n+j)$ cannot lie below its projection onto the $C^{*}(\hat{x}, n+j)$ surface. For appropriate choices of $\hat{x}_A$, $\hat{x}_B$ and μ, a point $\hat{x}$ may be determined as

$$\hat{x} = \mu \hat{x}_A + (1-\mu)\hat{x}_B \qquad 0 < \mu < 1 \qquad \text{(E-3)}$$

so (E-1) becomes

$$C^{*\prime}(\mu \hat{x}_A + (1-\mu)\hat{x}_B, n) =$$

$$\int_{-\infty}^{\infty} d\zeta_1 \int_{-\infty}^{\infty} d\zeta_2 f_{s'(n+j)}(\zeta)\, C^{*}(\mu(\hat{x}_A + \zeta) + (1-\mu)(\hat{x}_B + \zeta), n+j) \qquad \text{(E-4)}$$

Since $f_{s'(n+j)}(\zeta)$ is positive, equations (E-2) and (E-4) yield the inequality

$$C^{*\prime}(\mu\hat{x}_A + (1-\mu)\hat{x}_B, n) \leqq \mu \int_{-\infty}^{\infty} d\zeta_1 \int_{-\infty}^{\infty} d\zeta_2 \, f_{s'(n+j)}(\zeta) C^*(\hat{x}_A + \zeta, n+j)$$

$$+ (1-\mu) \int_{-\infty}^{\infty} d\zeta_1 \int_{-\infty}^{\infty} d\zeta_2 \, f_{s'(n+j)}(\zeta) C^*(\hat{x}_B + \zeta, n+j) \tag{E-5}$$

or by (E-1)

$$C^{*\prime}(\mu\hat{x}_A + (1-\mu)\hat{x}_B, n) \leqq \mu C^{*\prime}(\hat{x}_A, n) + (1-\mu) C^{*\prime}(\hat{x}_B, n) \tag{E-6}$$

Since (E-6) holds for all points $\hat{x}_A, \hat{x}_B$; $C^{*\prime}(\hat{x}, n)$ is semiconcave. In addition, the integral in (E-4) is taken over the entire space, so if there is a region (set) of ζ, of nonzero area (measure) such that

$$C^*(\mu(\hat{x}_A + \zeta) + (1-\mu)(\hat{x}_B + \zeta), n+j) < \mu C^*(\hat{x}_A + \zeta, n+j) + (1-\mu) C^*(\hat{x}_B + \zeta, n+j) \tag{E-7}$$

Then (E-5) holds with strict inequality and (E-6) becomes

$$C^{*\prime}(\mu\hat{x}_A + (1-\mu)\hat{x}_B, n) < \mu C^{*\prime}(\hat{x}_A, n) + (1-\mu) C^{*\prime}(\hat{x}_B, n) \quad 0 < \mu < 1 \tag{E-8}$$

For the class of problems considered in Section 3.4, there are always regions of ζ for which (E-7) holds. In fact it can be shown that if either $\hat{x}_A + \zeta$ or $\hat{x}_B + \zeta$ is an interior point of $\mathfrak{F}(n+j)$, given by equation (3-66), then (E-7) holds. Since (E-8) holds for all $\hat{x}_A$, $\hat{x}_B$, the function $C^{*\prime}(\hat{x}, n)$ is said to be concave.

Now consider the function $C^*(\hat{x}, n)$ given by (3-57) and repeated here in slightly different form as

$$C^*(\hat{x}, n) = \min_{u} \left\{ \| u \| + C^{*\prime}(\hat{x}', n) \right\} \tag{E-9}$$

where

$$\hat{x}' = \Phi(n+j, n)\hat{x} + \theta(n+j, n)u \tag{E-10}$$

For appropriate choices of $\hat{x}_A$, $\hat{x}_B$ and μ, a point $\hat{x}$ may be determined as

$$\hat{x} = \mu \hat{x}_A + (1-\mu)\hat{x}_B \qquad 0 < \mu < 1 \tag{E-11}$$

Also, appropriate vectors u_A and u_B may be chosen so that

$$u = \mu u_A + (1-\mu) u_B \tag{E-12}$$

Hence (E-10) becomes

$$\hat{x}' = \mu \hat{x}'_A + (1-\mu)\hat{x}'_B \tag{E-13}$$

where

$$\hat{x}'_A = \Phi(n+j, n)\hat{x}_A + \theta(n+j, n) u_A \tag{E-14}$$

$$\hat{x}'_B = \Phi(n+j, n)\hat{x}_B + \theta(n+j, n) u_B \tag{E-15}$$

The choice of u_A and u_B determines u according to (E-12), so (E-9) may be written as

$$C^*(\mu\hat{x}_A + (1-\mu)\hat{x}_B, n) = \min_{u_A, u_B} \left\{ \| \mu u_A + (1-\mu)u_B \| + C^{*\prime}(\mu\hat{x}'_A + (1-\mu)\hat{x}'_B, n) \right\} \tag{E-16}$$

The magnitude function on the right in (E-16) is semiconcave and $C^{*\prime}(\hat{x}, n)$ is concave so (E-16) yields the inequality

$$C^*(\mu\hat{x}_A + (1-\mu)\hat{x}_B, n) \leqq \min_{u_A, u_B} \left\{ \mu\left[\| u_A \| + C^{*\prime}(\hat{x}'_A, n)\right] + (1-\mu)\left[\| u_B \| + C^{*\prime}(\hat{x}'_B, n)\right] \right\} \tag{E-17}$$

Equality in (E-17) can occur when u_A and u_B are applied in the same direction and drive the estimated states $\hat{x}'_A$ and $\hat{x}'_B$ to the same point. In fact this is precisely what happens in region $\mathfrak{R}(n)$, defined by (3-72), when $\hat{x}_A$ and $\hat{x}_B$ lie along a common optimal trajectory direction d(b, n), as given by (3-79). The first term inside the braces in (E-17) is not a function of u_B and the second term is not a function of u_A. Clearly the minimizations over u_A and u_B may be done separately so

$$C^*(\mu\hat{x}_A + (1-\mu)\hat{x}_B, n) \leqq \mu\left[\min_{u_A}\left\{\left\| u_A \right\| + C^{*\prime}(\hat{x}'_A, n)\right\}\right] + (1-\mu)\left[\min_{u_B}\left\{\left\| u_B \right\| + C^{*\prime}(\hat{x}'_B, n)\right\}\right] \tag{E-18}$$

and applying (E-9)

$$C^*(\mu\hat{x}_A + (1-\mu)\hat{x}_B, n) \leqq \mu C^*(\hat{x}_A, n) + (1-\mu) C^*(\hat{x}_B, n) \qquad 0 < \mu < 1 \tag{E-19}$$

Therefore, since (E-19) applies for all $\hat{x}_A$, $\hat{x}_B$; $C^*(\hat{x}, n)$ is semiconcave.

Thus far, by assuming that $C^*(\hat{x}, n+j)$ is semiconcave, it has been shown that $C^{*\prime}(\hat{x}, n)$ is concave and $C^*(\hat{x}, n)$ is semiconcave. If it can be shown that these conditions hold at the terminal time, then they must be true for all n. The terminal condition is given by (3-82) and repeated here as

$$C^*(\hat{x}, m) = \left\| \theta(q+1, m)^{-1}\hat{x} \right\| \qquad m = n+j \tag{E-20}$$

The magnitude function in (E-20) is semiconcave along radial lines from the origin and concave otherwise. At the next previous correction time $C^{*\prime}(\hat{x}, n)$ is given by

$$C^{*\prime}(\hat{x}, n) = \int_{-\infty}^{\infty} d\zeta_1 \int_{-\infty}^{\infty} d\zeta_2\, f_{s'(m)}(\zeta) \left\| \theta(q+1, m)^{-1}\hat{x} + \zeta \right\| \tag{E-21}$$

and in the same manner as above, it can be shown that (E-8) applies at the next to last correction time. Hence $C^{*\prime}(\hat{x}, n)$ is concave for all n.

Because of the integral in (E-1), $C^{*\prime}(\hat{x}, n)$ is an analytic function of $\hat{x}$ and its matrix of second partial derivatives exists and is continuous in the entire $\hat{x}$ space. Thus $C^{*\prime}(\hat{x}, n)$ may be expanded about a point $\hat{x}$ as

$$C^{*\prime}(\hat{x}+\Delta, n) = C^{*\prime}(\hat{x}, n) + \frac{\partial C^{*\prime}(\hat{x}, n)}{\partial \hat{x}} \Delta + \frac{1}{2} \Delta^T \frac{\partial^2 C^{*\prime}(\hat{x}, n)}{\partial \hat{x}^2} \Delta + 0(|\Delta|^3) \tag{E-22}$$

and similarly

$$C^{*\prime}(\hat{x}-\Delta, n) = C^{*\prime}(\hat{x}, n) - \frac{\partial C^{*\prime}(\hat{x}, n)}{\partial \hat{x}} \Delta + \frac{1}{2} \Delta^T \frac{\partial^2 C^{*\prime}(\hat{x}, n)}{\partial \hat{x}^2} \Delta + 0(|\Delta|^3) \tag{E-23}$$

Adding (E-22) and (E-23), dividing by two and rearranging terms produces

$$C^{*\prime}(\hat{x}, n) = \frac{1}{2} C^{*\prime}(\hat{x}+\Delta, n) + \frac{1}{2} C^{*}(\hat{x}-\Delta, n) - \frac{1}{2} \Delta^T \frac{\partial^2 C^{*\prime}(\hat{x}, n)}{\partial \hat{x}^2} \Delta + 0(|\Delta|^3) \tag{E-24}$$

By choosing $|\Delta|$ the fourth term on the right on (E-24) can be made arbitrarily small. Since Δ can be chosen in any direction and (E-8) must be satisfied, it is clear that the matrix of second partial derivatives cannot have negative eignevalues. Also, this matrix cannot have zero eigenvalues in any region (set) of non-zero area (measure). To show this, assume that there is a region $\mathcal{S}$, of non-zero area, in which an eigenvalue of the matrix is zero. Choose $\hat{x}$ in $\mathcal{S}$ and choose Δ so it is an eigenvector of the matrix corresponding to the zero eigenvalue. Further choose the magnitude of Δ so that $\hat{x} \pm\Delta$ lie in $\mathcal{S}$. Under these conditions, the last two terms on the right in (E-24) are identically zero, and (E-8) is violated. Therefore, the matrix of second partial derivatives must be positive definite, almost everywhere in the

$\hat{x}$ space. Now consider, once again, the integral in (E-1). It may be broken down into two separate integrals, thus

$$A(\hat{x}) = \int_{-\infty}^{\infty} d\zeta_1 \int_{-\infty}^{\infty} d\zeta_2 \, f_a(\zeta) \, C^*(\hat{x} + \zeta, n+j) \tag{E-25}$$

$$C^{*\prime}(\hat{x}, n) = \int_{-\infty}^{\infty} d\zeta_1 \int_{-\infty}^{\infty} d\zeta_2 \, f_a(\zeta) \, A(\hat{x} + \zeta) \tag{E-26}$$

where

$$f_a(\zeta) = (2\pi)^{-1} \left| \frac{S'(n+j)}{2} \right|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} \zeta^T \left[\frac{S'(n+j)}{2} \right]^{-1} \zeta \right\} \tag{E-27}$$

From the results above, $A(\hat{x})$ is analytic and its matrix of second partial derivatives is positive definite, almost everywhere. Taking the second derivative of (E-26) produces

$$\frac{\partial^2 C^{*\prime}(\hat{x}, n)}{\partial \hat{x}^2} = \int_{-\infty}^{\infty} d\zeta_1 \int_{-\infty}^{\infty} d\zeta_2 \, f_a(\zeta) \, \frac{\partial^2 A(\hat{x} + \zeta)}{\partial \hat{x}^2} \tag{E-28}$$

The right side of (E-28) may be interpreted as the limit of a sum of matrices. Since the matrices are positive definite almost everywhere, the left side of (E-28) must be positive definite for all $\hat{x}$ and n. This is the required result .

APPENDIX F

CONTINUITY OF DERIVATIVES FOR THE CONTINUOUS V.T.A. PROBLEM

In this appendix the continuous analogue of the v. t. a. problem of Section 3. 3 is examined. In particular, the continuity of the second derivative of C^* is investigated.

For the discrete v. t. a. problem the optimal control function is determined by the derivative of C^* as in (3-27), (3-28). If that derivative is defined as

$$G^*(\hat{x}, n) = \frac{\partial C^*(\hat{x}, n)}{\partial \hat{x}} \tag{F-1}$$

then it is shown in Appendix D that $G^*(\hat{x}, n)$ is continuous and satisfies the recursion formulas

$$G^*(\hat{x}, n) = \left\{ \begin{array}{ll} \dfrac{1}{\theta(n+1, n)} & \text{if } \hat{x} > \alpha(n) \\ G^{*\prime}(\hat{x}, n) & \text{if } -\alpha(n) \leqq \hat{x} \leqq \alpha(n) \\ -\dfrac{1}{\theta(n+1, n)} & \text{if } \hat{x} < -\alpha(n) \end{array} \right\} \tag{F-2}$$

$$G^{*\prime}(\hat{x}, n) = \int_{-\infty}^{\infty} d\zeta\, f_{s'(n+1)}(\zeta) G^*(\hat{x}+\zeta, n+1) \tag{F-3}$$

and similarly

$$G^*(\hat{x}, n+1) = \begin{cases} \dfrac{1}{\theta(n+2, n+1)} & \text{if } \hat{x} > \alpha(n+1) \\ G^{*\prime}(\hat{x}, n+1) & \text{if } -\alpha(n+1) \leqq \hat{x} \leqq \alpha(n+1) \\ -\dfrac{1}{\theta(n+2, n+1)} & \text{if } \hat{x} < -\alpha(n+1) \end{cases}$$

(F-4)

Using (F-3) and (F-4), the equation for $G^{*\prime}(\hat{x}, n)$ becomes

$$G^{*\prime}(\hat{x}, n) = -\frac{1}{\theta(n+2, n+1)}\int_{-\infty}^{-\hat{x}-\alpha(n+1)} d\zeta\, f_{s(n+1)}(\zeta) + \int_{-\hat{x}-\alpha(n+1)}^{-\hat{x}+\alpha(n+1)} d\zeta\, f_{s(n+1)}(\zeta) G^{*\prime}(\hat{x}+\zeta, n+1)$$

$$+ \frac{1}{\theta(n+2, n+1)}\int_{-\hat{x}+\alpha(n+1)}^{\infty} d\zeta\, f_{s(n+1)}(\zeta)$$

(F-5)

Now $G^{*\prime}(\hat{x}, n+1)$ is an analytic function of $\hat{x}$ so the second integral on the right of (F-5) may be written as an expansion about $\hat{x}$ producing

$$G^{*\prime}(\hat{x}, n) = \frac{1}{\theta(n+2, n+1)}\left[\int_{-\hat{x}+\alpha(n+1)}^{\infty} d\zeta\, f_{s(n+1)}(\zeta) - \int_{-\infty}^{-\hat{x}-\alpha(n+1)} d\zeta\, f_{s(n+1)}(\zeta)\right]$$

$$+ \int_{-\hat{x}-\alpha(n+1)}^{-\hat{x}+\alpha(n+1)} d\zeta\, f_{s(n+1)}(\zeta)\left[G^{*\prime}(\hat{x}, n+1) + \frac{\partial G^{*\prime}(\hat{x}, n+1)\zeta}{\partial \hat{x}} + \frac{\partial^2 G^{*\prime}(\hat{x}, n+1)}{\partial \hat{x}^2}\frac{\zeta^2}{2} + o(\zeta^2)\right]$$

(F-6)

Consider the very special case for which the values $\alpha(n)$ are constant for all n. In other words, the control sensitivities and variances of the problem vary in just the proper way so that the boundary $\alpha(n)$ stays constant. Then (F-6) evaluated at α becomes

$$G^{*\prime}(\alpha, n) = \frac{1}{\theta(n+2, n+1)} \left[\int_0^{\infty} d\zeta\, f_{s(n+1)}(\zeta) - \int_{-\infty}^{-2\alpha} d\zeta\, f_{s(n+1)}(\zeta) \right]$$

$$+ \int_{-2\alpha}^{0} d\zeta\, f_{s(n+1)}(\zeta) \left[G^{*\prime}(\hat{x}, n+1) + \frac{\partial G^{*\prime}(\hat{x}, n+1)}{\partial \hat{x}} \zeta + \frac{\partial^2 G^{*\prime}(\hat{x}, n+1)}{\partial \hat{x}^2} \frac{\zeta^2}{2} + o(\zeta^2) \right]_{\hat{x}=\alpha} \tag{F-7}$$

to evaluate these integrals, the probability density

$$f_{s(n+1)}(\zeta) = \frac{1}{\sqrt{2\pi\, S(n+1)}} \exp\left\{ -\frac{1}{2} \frac{\zeta^2}{S(n+1)} \right\} \tag{F-8}$$

is required. If the time between control applications is small so that

$$t_{n+1} = t_n + \Delta t_n \tag{F-9}$$

Then it is shown in Section 4.3 that to first order the variance S(n+1) is given by

$$S(n+1) = B(t_n)\, \Delta t_n + o(\Delta t_n) \tag{F-10}$$

where $B(t_n)$ is defined by (4-34). Thus to first order in Δt_n, the probability density for s(n+1) is

$$f_{s(n+1)}(\zeta) = \frac{1}{\sqrt{2\pi\, B(t_n)\, \Delta t_n}} \exp\left\{ -\frac{1}{2} \frac{\zeta^2}{B(t_n)\, \Delta t_n} \right\} \tag{F-11}$$

Let the time step Δt_n be sufficiently small so that

$$B(t_n)\,\Delta t_n \ll \alpha$$

If such is the case, then (F-7) may be approximated as

$$G^{*\prime}(\alpha, n) \cong \frac{1}{2\,\theta(n+2, n+1)} + \int_{-\infty}^{0} d\zeta\; f_{s(n+1)}(\zeta)\left[G^{*\prime}(\hat{x}, n+1) + \frac{\partial G^{*\prime}}{\partial \hat{x}}(\hat{x}, n+1)\zeta + \frac{\partial^2 G^{*\prime}(\hat{x}, n+1)}{\partial \hat{x}^2}\frac{\zeta^2}{2} + o(\zeta^2)\right]_{\hat{x}=\alpha} \qquad \text{(F-12)}$$

and by the definition of α, (3-28)

$$G^{*\prime}(\alpha, n) = \frac{1}{\theta(n+1, n)} \qquad \text{(F-13)}$$

$$G^{*\prime}(\alpha, n+1) = \frac{1}{\theta(n+2, n+1)} \qquad \text{(F-14)}$$

so (F-12) becomes

$$\frac{1}{\theta(n+1, n)} \cong \frac{1}{\theta(n+2, n+1)} + \int_{-\infty}^{0} d\zeta\; f_{s(n+1)}(\zeta)\left[\frac{\partial G^{*\prime}}{\partial \hat{x}}(\hat{x}, n+1)\,\zeta + \frac{\partial^2 G^{*\prime}}{\partial \hat{x}^2}(\hat{x}, n+1)\frac{\zeta^2}{2} + o(\zeta^2)\right]_{\hat{x}=\alpha} \qquad \text{(F-15)}$$

or

$$\frac{1}{\theta(n+1, n)} - \frac{1}{\theta(n+2, n+1)} \cong -\left[\frac{\partial G^{*\prime}(\hat{x}, n+1)}{\partial \hat{x}}\right]_{\hat{x}=\alpha}\left(\frac{B(t_n)\Delta t_n}{2\pi}\right)^{\frac{1}{2}} + \left[\frac{\partial^2 G^{*\prime}}{\partial \hat{x}^2}(\hat{x}, n+1)\right]_{\hat{x}=\alpha}\frac{B(t_n)\Delta t_n}{4} + o(\Delta t_n) \qquad \text{(F-16)}$$

It is assumed that the control sensitivity is a positive continuous function of time. Therefore, the limit

$$\lim_{\Delta t_n \to 0} \left[\frac{\frac{1}{\theta(n+2, n+1)} - \frac{1}{\theta(n+1, n)}}{\Delta t_n} \right] \qquad \text{(F-17)}$$

must exist for all t_n. However, the first term on the right in (F-16) contains the square root of Δt in the numerator. Hence if the limit (F-17) is to exist, $\left[\frac{\partial G^{*\prime}(\hat{x}, n+1)}{\partial \hat{x}}\right]_{\hat{x}=\alpha}$ must go to zero at least as fast as the sqaure root of Δt_n. Hence

$$\lim_{\Delta t_n \to 0} \left[\frac{\partial G^{*\prime}(\hat{x}, n+1)}{\partial \hat{x}} \right]_{\hat{x}=\alpha} = 0$$

and by virture of (F-4), it must be true that

$$\lim_{\Delta t_n \to 0} \left[\frac{\partial G^{*}(\hat{x}, n+1)}{\partial \hat{x}} \right]_{\hat{x}=\alpha^-} = 0$$

Finally, since

$$\left[\frac{\partial G^{*}(\hat{x}, n+1)}{\partial \hat{x}} \right]_{\hat{x}=\alpha^+} = 0$$

the second derivative of C^* must be continuous across the boundaries $\pm\,\alpha$ and equal zero at those points.

BIBLIOGRAPHY

This bibliography consists of references collected as a result of an extensive literature search and other publications which were useful during the course of the research.

In the area of deterministic optimal control theory, useful references concerned with the maximum principle are Athans and Falb (3) and Pontryagin (59). Other approaches are taken by Breakwell, Speyer and Bryson (13), Breakwell (14), Lawden (50), and Leitman (51).

For stochastic optimal control theory, when the plant state can be observed perfectly, there is Aoki (1, 2), Bellman (7, 8), Fleming (25), Florentine (27), Fuller (28), Kushner (46, 47), Tung and Strebel (73), and Wonham (76). In cases for which the cost function is quadratic and there are errors in measurements, see Florentine (26), Gunkel and Franklin (30), Joseph and Tou (39), Potter (60), and Tung (71, 72). If the cost function is nonquadratic, with measurement errors, there is Feldbaum (24), Kushner (48), Orford (57), Stratonovich (68) and Striebel (70). For a discussion of the stochastic maximum principle consult Kushner and Schweppe (43) and Kushner (45).

The classical filter theory of Wiener may be found in Laning and Battin (49), and Wiener (75). Modern filter theory for application to nonstationary linear systems is developed in Battin (5), Bryson and Johansen (15), Deyst (20), Kalman (40), Kalman and Bucy (41). Some interesting applications of the theory are McLean, Schmidt and McGee (53); and Rauch, Tung and Striebel (63). Nonlinear filter theory is found in Bucy (18), Jazwinski (38) and Wonham (78). A discussion of the criteria satisfied by minimum variance estimators is given by Sherman (64).

Some general information about astronautical guidance and celestial mechanics may be obtained from Battin (4), Breakwell (11), (12), Hollister (34), Morth (55) and Stern (65).

The theory of dynamic programming and its applications is explained in Bellman (6), Boltyanskii (10), Dreyfus (22) and Howard (35).

Numerical techniques are found in Bryson and Denham (16), Hildebrand (32), and Milne (54).

Probability theory and statistics are the subjects treated in Blackwell and Girschick (9), Cramer (19), Doob (21), Dynkin (23), Ito (36), Kushner (44), Loeve (52), Papoulis (58), Raiffa and Schlaifer (62), Stratonovich (66, 67, 69) and Wax (74).

Finally, some additional texts in applied mathematics are Hancock (31), Hildebrand (33), Korn and Korn (42) and Morse and Feshbach (56).

1. Aoki, M., "Dynamic Programming Approach to a Final-Value Control System", IRE Trans., PGAC, AC-5, 4, 1960, p 270.

2. Aoki, M., "Stochastic Time-Optimal Control Systems", Trans. AIEE, 80 (II), 1961, p 41.

3. Athans, M. and Falb, P., Optimal Control, McGraw-Hill, New York, N.Y., 1966.

4. Battin, R.H., Astronautical Guidance, McGraw-Hill Book Co., Inc., New York, N.Y., 1964.

5. Battin, R.H., "A Statistical Optimization Navigation Procedure for Space Flight", ARS Journal, Vol 32, #11, November 1962, p 1681.

6. Bellman, R., Adaptive Control Theory-A Guided Tour, Princeton, 1961.

7. Bellman, R., "Dynamic Programming and Stochastic Control Processess", Information and Control,1 (3), 1958, pp 228-239.

8. Bellman, R., "A Markovian Decision Process", Journal Math. Mech., 6, 1957, p 679.

9. Blackwell, D. and Girshick, M.A., Theory of Games and Statistical Decisions, Wiley, N.Y., 1954.

10. Boltyanskii, V.G., "Sufficient Conditions for Optimality and the Justification of the Dynamic Programming Method", SIAM Journal on Control, 1966, Vol 4, #2, p 326.

11. Breakwell, J.V., "The Spacing of Corrective Thrust in Interplanetary Navigation", Advances in Astronautical Sciences, 1961, Vol 7, p 219.

12. Breakwell, J.V. and Striebel, C.T., "Minimum Effort Control in Interplanetary Guidance", presented IAS Metting, New York, 1963, Preprint 63-80.

13. Breakwell, J.V., Speyer, J.C., and Bryson, A.E., "Optimization and Control of Nonlinear Systems Using the Second Variation", SIAM Journal on Control, Vol 1 #2, p 193.

14. Breakwell, "The Optimization of Trajectories", SIAM Journal, Vol 7, #2, June 1959.

15. Bryson, A.E. and Johansen, D.E., "Linear Filtering for Time-Varying Systems Using Measurements Containing Colored Noise", IEEE Transactions on Automatic Control, Vol AC-10, Number 1, January 1965.

16. Bryson, A.E. and Denham, W.F., "A Steepest Ascent Method for Solving Optimum Programming Problems", Report BR-1303, The Raytheon Company, August 1961.

17. Bryson, A.E., Denham, W.F., and Dreyfus, S.E., "Optimal Programming with Inequality Constraints", AIAA Paper #11, p 2544, November 1963.

18. Bucy, R.S., "Nonlinear Filtering Theory", IEEE Trans. on Automatic Control, AC-10 (2), p 198, April 1965.

19. Cramer, H., Mathematical Methods of Statistics, Princeton University Press, 1954.

20. Deyst, J.J., "Optimum Continuous Estimation of Nonstationary Random Variables", (M.S. Thesis) MIT, January 1964.

21. Doob, J.L., Stochastic Processes, Wiley, New York, 1953.

22. Dreyfus, S.E., Dynamic Programming and the Calculus of Variations, Academic Press, New York and London, 1965.

23. Dynkin, E.B., "Necessary and Sufficient Statistics for a Family of Probability Distributions", Selected Transl. Math. Statis. Probab., 1, 1961, 17-40.

24. Fel'dbaum, A.A., "Dual Control Theory I-IV", Automation and Remote Control, 21 (9), 1960, p 1240; 21 (11), 1960, p 1453; 22 (1), 1961, p 3; 22 (2), 1961, p 129.

25. Fleming, W.H., "Some Markovian Optimization Problems", J. Math. Mech., 12 (1963), p 131.

26. Florentine, J.J., "Partial Observability and Optimal Control", J. Electronics and Control, 13 (3), 1962, pp 263-279.

27. Florentine, J.J., "Optimal Control of Continuous-Time, Markov, Stochastic Systems", J. Electronics and Control, 10, 1961, pp 473-488.

28. Fuller, A.T., "Optimization of Nonlinear Control Systems with Random Inputs", Journal Electronic Control, April 1960.

29. Gelfand, I. M. and Fomin, S. V., Calculus of Variations, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963.

30. Gunkel, T. L., III and Franklin, G. F., "A General Solution for Linear Sampled Data Control", Trans. ASME, J. Basic Eng., June 1963, Vol 85-D, p 197.

31. Hancock, H., Theory of Maxima and Minima, Dover, New York, N.Y., 1960.

32. Hildebrand, F. B., Introduction to Numerical Analysis, McGraw-Hill, New York, 1965.

33. Hildebrand, F. B., Methods of Applied Mathematics, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1952.

34. Hollister, W. M., "The Mission for a Manned Expedition to Mars", Sc. D. Thesis, MIT Experimental Astronomy Laboratory Report, TE-4, 1963.

35. Howard, R. A., Dynamic Programming and Markov Processes, MIT Press, 1960.

36. Ito, K., "On Stochastic Differential Equations", Memos of the American Mathematical Society, 4, p 51, 1951.

37. Jazwinski, "Optimal Trajectories and Linear Control of Nonlinear Systems", AIAA Journal, August 1964, Vol 2, No. 8, p 1371.

38. Jazwinski, "Nonlinear Filtering with Discrete Observations", AIAA Paper #66-38.

39. Joseph, P. D. and Tou, J. T., "On Linear Control Theory", Trans. AIEE, pt 2 (Applications and Industry), September 1961, Vol 80, p 193.

40. Kalman, R. E., "A New Approach to Linear Filtering and Prediction Problems", Transactions of the ASME, Journal of Basic Engineering, March 1960.

41. Kalman, R. E. and Bucy, R. S., "New Results in Linear Filtering and Prediction Theory", Transactions of the ASME, Journal of Basic Engineering, March 1961.

42. Korn, G. and Korn, T., Mathematical Handbook for Scientists and Engineers, McGraw-Hill, New York, N.Y., 1961.

43. Kushner, H. J. and Schweppe, F. C., "A Maximum Principle for Stochastic Control Systems", Journal Math. Anal. and Appl., Vol 8, #2, 1964.

44. Kushner, H.J., "On the Differential Equations Satisfied by Conditional Probability Densities of Markov Processes, with Applications", SIAM Journal, Series A: Control, 1964, Vol 2, #1, p 106.

45. Kushner, H.J., "On the Stochastic Maximum Principle with 'Average' Constraints," J. Math. Anal. Appl., 12 (1965), p 13.

46. Kushner, H.J., "Near-Optimal Control in the Presence of Small Stochastic Perturbations", Proc. JACC, June 1964.

47. Kushner, H.J., "On the Existence of Optimal Stochastic Controls", SIAM, Series A: Control, 3 (3), 1965.

48. Kushner, H.J., "Optimal Stochastic Control", Correspondence, IRE Trans. on Automatic Control, October 1962, p 120.

49. Laning, J.H. and Battin, R.H., Random Processes in Automatic Control, McGraw-Hill, New York.

50. Lawden, D.F., Optimal Trajectories for Space Navigation, Butterworth, London, 1963.

51. Leitman, G., Editor, Optimization Techniques, Academic Press, New York, N.Y., 1963.

52. Loeve, M., Probability Theory, 3rd ed., Van Nostrand, Princeton, 1963.

53. McLean, J.D., Schmidt, S.F., and McGee, L.A., "Optimal Filtering and Linear Prediction Applied to a Midcourse Navigation System for the Circumlunar Mission", NASA TN D-1208, March 1962.

54. Milne, W.E., Numerical Solution of Differential Equations, John Wiley and Sons, Inc., London, 1963.

55. Morth, R., "Reentry Guidance for Apollo", MIT Instrumentation Laboratory, Report R-532, Cambridge, Mass., 1966.

56. Morse, P.M. and Feshback, Methods of Theoretical Physics, 2 Parts, McGraw-Hill, New York, N.Y., 1953.

57. Orford, R.J., "Optimal Stochastic Control Systems", Journal of Mathematical Analysis and Applications, 6, pp 419-429, June 1963.

58. Papoulis, A., Probability, Random Variables, and Stochastic Processes, McGraw-Hill, 1965.

59. Pontryagin et. al., The Mathematical Theory of Optimal Processes, John Wiley and Sons, Inc., New York, 1962.

60. Potter, J.E., "A Guidance-Navigation Separation Theorem", Report RE-11, Experimental Astronomy Laboratory, MIT, August 1964.

61. Potter, J.E., Private Communication, November 1965.

62. Raiffa, H. and Schlaifer, R., "Applied Statistical Decision Theory", Division of Research, Harvard Business School, 1961.

63. Rauch, H.E., Tung, F., and Striebel, C.T., "Maximum Likelihood Estimates of Linear Dynamic Systems", AIAA Journal, August 1965, Vol 3, #8, p 1445.

64. Sherman, S., "Non-Mean-Square Error Criteria", Trans. IRE Prof. Group on Inf. Theory, IT-4, 1958, p 125.

65. Stern, R.G., "Interplanetary Midcourse Guidance Analysis", Sc. D. Thesis, MIT Experimental Astronomy Laboratory Report TE-5, 1963.

66. Stratonovich, R.L., "A New Representation for Stochastic Integrals and Equations", SIAM Journal on Control, 1966, Vol 4, #2, p 362.

67. Stratonovich, R.L., Topics on the Theory of Random Noise, Vol I, Gordon and Breach, New York-London, 1963.

68. Stratonovich, R.L., "On the Theory of Optimal Control, Sufficient Coordinates", Automation and Remote Control, 23 (7), 1962.

69. Stratonovich, "Conditional Markov Processes", Theor. Probability Appl., 5 (1962).

70. Striebel, C.T. "Sufficient Statistics on the Optimal Control of Stochastic Systems", Journal of Mathematical Analysis and Applications, 12, 576-592, December 1965.

71. Tung, F., "Linear Control Theory Applied to Interplanetary Guidance", IEEE Transactions on Automatic Control, Vol AC-9, Number 1, January 1964.

72. Tung, F., "An Optimal Discrete Control Strategy for Interplanetary Guidance", IEEE Transactions on Automatic Control, July 1965, Vol AC-10, #3, p 328.

73. Tung, F. and Striebel, C.T., "A Stochastic Optimal Control Problem and its Applications", Journ. Math. Anal. and Appl., Vol 2, No. 2, October 1965.

74. Wax, N., Selected Papers on Noise and Stochastic Processes, Dover Publications, New York, 1954.

75. Wiener, N., The Extrapolation, Interpolation and Smoothing of Stationary Time Series, John Wiley and Sons, Inc., New York, N.Y., 1949.

76. Wonham, W.M., "Stochastic Analysis of a Class of Nonlinear Control Systems with Random Inputs", JACC Preprints, 1962.

77. Wonham, W.M., Class Notes from Course on Stochastic Optimal Control Theory at MIT, 1966.

78. Wonham, W.M., "Some Applications of Stochastic Differential Equations to Optimal Nonlinear Filtering", SIAM Journal, Series A: Control, 1964, Vol 2, #3, p 347.

BIOGRAPHY

John J. Deyst Jr. was born July 26, 1936 in Teaneck, New Jersey. He attended public school in Teaneck and graduated from Teaneck High School in June 1954. He received the degree of Bachelor of Science in Aeronautical Engineering from the Massachusetts Institute of Technology in 1958.

Upon graduation from MIT, Mr. Deyst was employed as a junior engineer by the Eclipse Pioneer Division of the Bendix Aviation Corporation. His work at Bendix included analysis and simulation of elements of the autopilot system for the B-58 aircraft. In December 1958 he became an engineer at the Marine Division of the Sperry Gyroscope Co. While at Sperry he performed analytical work on the Polaris Submarine navigation system. In September 1962 he returned to MIT to work as an engineer at the MIT Instrumentation Laboratory. His work there included analytical studies and computer simulations of the Apollo guidance and navigation system.

While at the Instrumentation Laboratory, Mr. Deyst continued his studies on a part time basis, at MIT. In September 1963 he became a full time research assistant and he received the degree of Master of Science in Aeronautics and Astronautics from MIT in February 1964. He worked at the Instrumentation Laboratory from February to September 1964, at which time he again became a full time student at MIT. While a graduate student he held the Aviation Week and Space Technology Fellowship. During the summer of 1964 Mr. Deyst returned to the Instrumentation Laboratory to work on the Apollo system. In September 1964 he became a research assistant at the MIT Experimental Astronomy Laboratory, doing research in the areas of spacecraft navigation and control.

Mr. Deyst is married to the former Maryanne Dowd of Canton, Massachusetts and has two small children.