COMMUNICATION THROUGH NOISY, RANDOM-MULTIPATH CHANNELS

by

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S. B., Massachusetts Institute of Technology
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Submitted to the Department of Electrical Engineering on May 14, 1956 in partial fulfillment of the requirements for the degree of Doctor of Science

ABSTRACT

Statistical methods are applied in this paper to the problem of communication through a multipath channel which has random (or unknown) path characteristics, and which has additive random noise present at the receiver end. In an introductory chapter, the transmitter of a system for use with such a channel is defined as one which encodes the output of an information source into a sequence of selections from a finite set of message waveforms, and transmits this sequence into the channel. The receiver is specified as one which, on reception of the channel-perturbed transmitted signal, computes a posteriori probabilities of the possible transmitted message-waveform sequences, and, on the basis of these, supplies guesses at the information source output to an information user.

The first problem with which the paper concerns itself is that of establishing an a priori statistical model of the channel. It is shown that this a priori model is often inadequate, and measurement techniques for obtaining further (a posteriori) information about the channel are discussed.

Next, the problem of determining the operational form of the receiver's probability computer is investigated. Since this form depends on the amount of information about the channel which is available to the receiver, several results are obtained, one for each of several assumptions concerning the state of the receiver's knowledge of the channel. Probabilities of error corresponding to two of these probability-computer forms are evaluated for the special case in which there are only two equi-energy, equiprobable message waveforms and only one path; and for which the receiver makes its guess, for each message waveform in a sequence, by choosing the waveform which is a posteriori most probable.

The problem of generation of an optimal set of message waveforms is then considered, in particular, for the special case described above. In this case, the optimization condition consists in the adjustment of the cross-correlation coefficient of the two message waveforms.

A commentary on possible future extensions of the present work concludes the paper.

Thesis Supervisor: Dr. W. B. Davenport, Jr.
Title: Assistant Division Leader, Lincoln Laboratory, M.I.T.

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May, 1956 G. L. T.

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CONTENTS (Cont.)

LIST OF COMPLEX FUNCTIONS

LIST OF IMPORTANT SYMBOLS

Symbol	Definition	Page Introduced
a_i	strength of i^{th} path	13
D	decision variable	46
E	energy of sounding signal	17
E_m	energy of m^{th} message waveform	35
f_o	carrier frequency of bandpass waveform	7
g_{mi}	$g_m(\tau_i)$	35
L	number of paths	14
M	number of message waveforms	3
N_o	white noise power-density	11
P_e, P'_e	probability of error	45, 55
P_m	_a priori_ probability of m^{th} message waveform	32
Pr[ ]	"probability of"	9
pr [ ]	"probability density of"	9
s_i	strength of random component of i^{th} path	13
T	duration of message waveform	33
T'	duration of output of multipath medium	71
W	transmission bandwidth	11
W_N	bandwidth of noise	11
α_i	strength of fixed component of i^{th} path	13
β_{mi}	$2\sigma_i^2 E_m/N_o$	38
γ_i	α_i/σ_i	49
δ_i	phase-shift of fixed component of the i^{th} path	13
ϵ_i	phase-shift of random component of i^{th} path	13
θ_i	phase-shift of i^{th} path	13
λ	complex cross-correlation coefficient of two message waveforms	48
Λ_m	likelihood corresponding to m^{th} message waveform	34
σ_i	$(1/\sqrt{2})$ times r.m.s. strength of random component of i^{th} path	14
τ_i	modulation delay of i^{th} path	12
$\wedge$	"real part of"	7
$\sim$	"imaginary part of"	7

CHAPTER I: INTRODUCTION

In recent years the problem of the design of systems for communication through randomly-disturbed channels has been approached from a more fundamental point of view than was previously employed. It was recognized that the essential feature of the problem is the statistical nature of the disturbance, and that, hence, statistical methods must be used in system design. The vanguard of this approach included Wiener[1], Shannon[2, 3], and Woodward and Davies[4, 5], to name but a very few.

Attention has heretofore been focused almost exclusively on additively-disturbed channels, that is, channels in which the only random disturbance is one added to the transmitted signal. The more difficult problem of communication through channels which are not purely additively disturbed has, on the other hand, received relatively little attention. It is the purpose of this paper to consider, from a statistical point of view, the problem of communication through one type of non-additively-disturbed channel: a random-multipath channel, in which a signal may travel from transmitter to receiver by way of many paths, which have randomly-distributed characteristics. The channel will also be considered to be noisy, with the noise added at the receiver end. Other statistical investigations relating to this type of channel have been made by Price[6, 7, 8] and by Root and Pitcher[9]. The present work, in fact, is closely related to that of Price in some aspects.[‡]

A. Model of System to be Considered.

The generic model of the communication system to be considered is shown in Figure 1-1. Like Shannon's model[10], it consists of an information source, a transmitter, a channel, a receiver, and an information user. We shall take

‡ See especially Chapter III, Sections A and B.

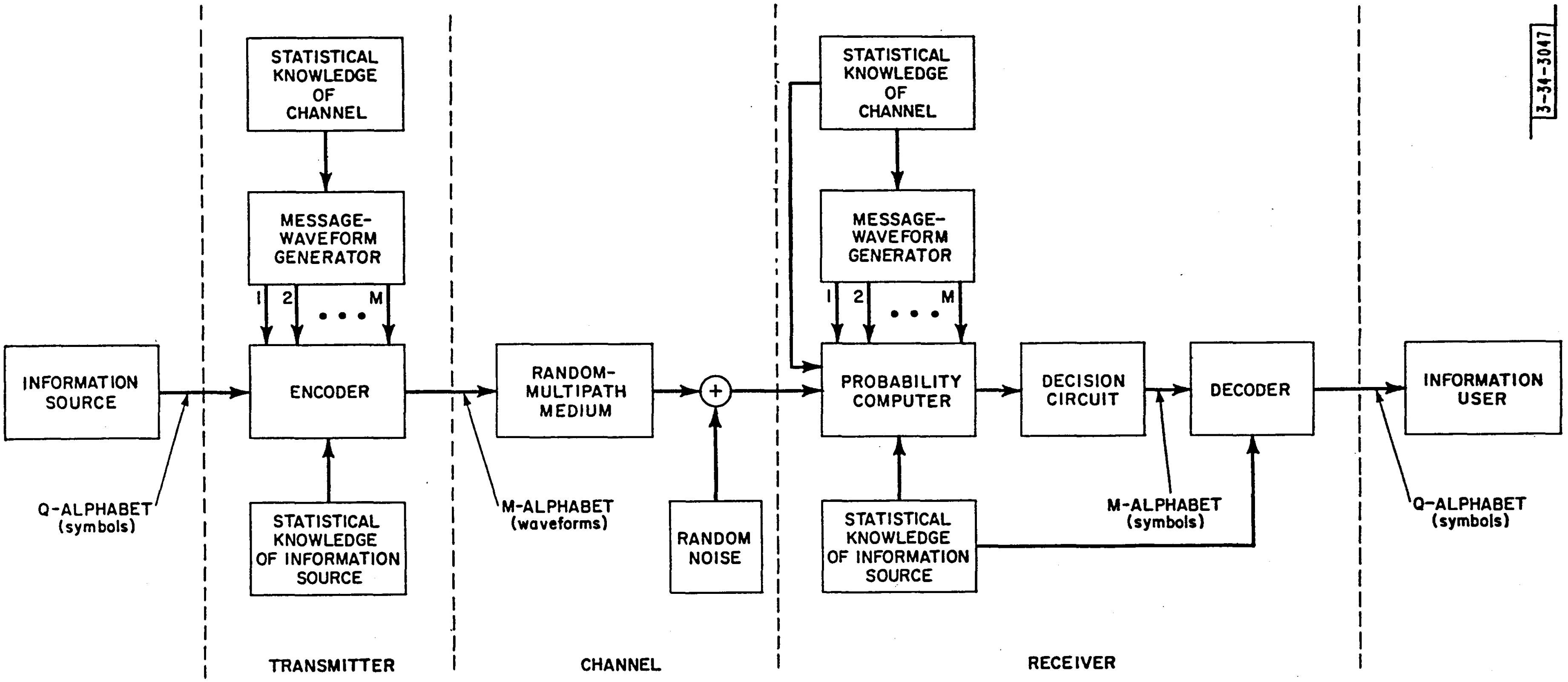
STATISTICAL KNOWLEDGE OF CHANNEL
MESSAGE-WAVEFORM GENERATOR
1
2
M
INFORMATION SOURCE
ENCODER
Q-ALPHABET (symbols)
STATISTICAL KNOWLEDGE OF INFORMATION SOURCE
RANDOM-MULTIPATH MEDIUM
M-ALPHABET (waveforms)
RANDOM NOISE
STATISTICAL KNOWLEDGE OF CHANNEL
MESSAGE-WAVEFORM GENERATOR
1
2
M
PROBABILITY COMPUTER
DECISION CIRCUIT
DECODER
STATISTICAL KNOWLEDGE OF INFORMATION SOURCE
M-ALPHABET (symbols)
INFORMATION USER
Q-ALPHABET (symbols)
TRANSMITTER
CHANNEL
RECEIVER
3-34-3047

Fig. 1–1

the information source to be one which produces as its output a sequence of symbols drawn from a finite alphabet; it may be, for example, a printed English text. We define the function of the transmitter to be, first, the encoding of the information source output into a sequence of message waveforms, which are drawn from a finite set of finite-duration message waveforms; and, second, the transmission of this message-waveform sequence into the channel. The set of message waveforms may be, for example, a set of pulsed sine-waves of different frequencies, as in frequency-shift telegraphy. In the channel the transmitted message-waveform sequence, or signal, is distorted, first by transmission through a random-multipath medium, and then by the addition of random noise at the receiver input. The received signal, then, generally does not provide the receiver with an unequivocal indication of the transmitted message-waveform sequence. The receiver must make do with an imperfect situation by guessing at the transmitted sequence on the basis of some operation on the received signal. This guess, or perhaps a set of guesses, is presented to the information user.

A closer analysis of the encoding function of the transmitter leads to its division into two parts. The first of these, which is performed by the encoder of Figure 1-1, consists in the one-to-one translation of the information source output, which is a sequence of symbols drawn from an alphabet of say Q letters, into a sequence of new symbols drawn from an alphabet of say M letters. Possible purposes of this translation may be, for instance: reduction of alphabet size, reduction of redundancy[11], insertion of error-detection or error-correction symbols. An example of the first purpose is the teletype code, in which written text is translated into a two-symbol ("mark" and "space") alphabet. The second purpose is illustrated by the optimum code of Huffman(12). An example of the third is given in a paper by Hamming(13). Generally, the encoder may require a statistical knowledge of the information source (e.g., letter frequencies, digram frequencies, etc.), and this is indicated in Figure 1-1.

Up until now we have considered only abstract alphabets. That is, the symbols of both the M- and Q-alphabets are so far merely abstract ink-marks, and as such are hardly eligible for transmission through the channel. The second part of the encoding function, then, consists in the generation of a set of M distinct physical waveforms to take the place of the abstract symbols of the M-alphabet. This is by no means a trivial operation, for not any arbitrary set of waveforms will do; they must be chosen to combat the perturbing effect of the channel effectively. For example, a set of sine waves of the same frequency, but of different phases, would be an absurd selection if the channel contains a random phase-shifting device. Essentially, the second part of the encoding function of the transmitter is concerned with the generation of a suitable "codebook" for use by the encoder. This is done by the <u>message-waveform generator</u>, which requires a statistical knowledge of the channel, as shown in Figure 1-1.

From the redundancy point of view, the entire encoding operation consists in the replacement of the natural redundancy of the information-source sequence by redundancy of a kind more suitable for use with the given channel. In this sense, the function of the transmitter is to "match" the information source to the channel.

The function of the receiver may also be divided into several parts. The first of these is postulated to be the computation of the <u>a posteriori</u> probabilities of the various possible message-waveform sequences. As is noted by Woodward and Davies[4], all of the information in the received signal is implicit in these probabilities, so that this computation merely reduces the available data (i.e., the received signal) to an alternate form; there is no loss of information involved. In order to carry through the computation, the probability computer must have available a statistical knowledge of the source (i.e., the *a priori* probabilities of the various possible message-waveform sequences) and of the channel, and a copy of the transmitter's "codebook" of message waveforms.

The complete *a posteriori* distribution as available at the output of the probability computer is not in itself useful to the information user; the user requires a receiver output which is in the same form as the transmitter input--for example, printed English text. Thus, the receiver is called upon to guess at the transmitted message-waveform sequence on the basis of its *a posteriori* knowledge. This is done by the decision circuit, whose output is hence a sequence of symbols from the abstract M-alphabet, or perhaps a set of a few highly-probable sequences. This guessing operation of course involves a loss of information.

Finally, the output of the decision circuit is translated back into a sequence (or sequences) of Q-alphabet symbols by the decoder, which is the exact inverse of the encoder. The decoder output is supplied to the information user.

B. System Design Problems.

The above description of the functions of the transmitter and receiver automatically brings to mind certain questions, which may be phrased as follows:

1) How does one design the encoder (and hence the decoder)?

2) What are "suitable" message-waveforms, or equivalently, how does one design the message-waveform generator?

3) What is the operational form of the probability computer?

4) On what basis should the decision circuit make its guesses?

There is a fifth question which, although not as immediately apparent, is just as important:

5) In what form is statistical knowledge of the channel available to the receiver and transmitter?

These questions are not really completely independent of one another, as might be implied from their being asked separately. However, as in most mathematical investigations, one finds it expedient first to break the problem up into nearly-independent parts, and to investigate each one separately, in each case assuming solutions to the others. In this way, one hopes to gain an insight into the overall problem. The parts defined by the above questions seem to be natural ones for our present problem.

We shall not investigate questions (1) and (4) in this work. However, whenever it is necessary, we shall assume that the encoder and the decision circuit satisfy, respectively, the following conditions:

1) the symbols of the encoder output-sequence are independent of each other, and

2) the decision circuit chooses the <u>a posteriori</u> most probable sequence.

As for the remaining questions, they will be considered in the order (5), (3), (2). Chapter II will be devoted to the establishment of a model for the channel, and of ways of measuring the characteristics of this model. Chapter III will be concerned with the determination of the operational form of the probability computer under various assumptions concerning the amount of information about the channel which is available at the receiver. The probability of error corresponding to two of these probability-computer forms will be derived in Chapter IV for a simple special case. The question of message-waveform generation will then be discussed in Chapter V for this special case. Finally, Chapter VI will be devoted to a commentary on the results obtained, and to suggestions for future investigations.

C. Notation.

1. Complex representation of physical waveforms.

In this paper we shall, for the most part, use the complex representation of physical waveforms which is described by Woodward.[14] The part of this notation which is of immediate concern to us is that relating to the representation of narrow-band-pass waveforms. This is essentially a generalization of the familiar practice of representing A cos $(2\pi ft+\phi)$ by A $e^{j2\pi ft}$, where A is complex. More generally, a narrow-band-pass waveform is represented as the product of a complex low-pass modulating waveform, x(t), and a cisoidal carrier:*

$$\xi(t) = x(t)\, e^{j2\pi f_o t} \tag{1.01}$$

f_o is a suitably-defined carrier frequency--for example, the centroid of the energy- or power-density spectrum of the waveform. As in the cosinusoidal case, the actual physical waveform is represented by the real part of $\xi(t)$:

$$\hat{\xi}(t) = \hat{x}(t) \cos 2\pi f_o t - \tilde{x}(t) \sin 2\pi f_o t \tag{1.02}$$

---

* Following Woodward, we shall use Greek letters for complex band-pass waveforms and English letters for low-pass modulating waveforms.

where we have used a circumflex (^) to denote "real part of" and a tilde (~) to denote "imaginary part of". Equation (1.02) is, in fact, a representation which has been in use for some time[15]; (1.01) is merely a more convenient and compact way of writing it.

x(t) represents both an amplitude and phase modulation. Its magnitude,

$$|x(t)| = \sqrt{\hat{x}^2(t) + \tilde{x}^2(t)} = |\xi(t)| \tag{1.03a}$$

is the amplitude, or envelope, and its angle,

$$\measuredangle\, x(t) = \tan^{-1} \frac{\tilde{x}(t)}{\hat{x}(t)} \tag{1.03b}$$

the phase deviation, of the carrier.

The cross-correlation function of two complex transient waveforms, $\xi(t) = x(t)e^{j2\pi f_o t}$ and $\eta(t) = y(t)e^{j2\pi f_o t}$, is defined as the complex function

$$\psi(\tau) = \int \xi^*(t)\, \eta(t-\tau)dt = e^{-j2\pi f_o \tau} \int x^*(t)\, y(t-\tau)\, dt \tag{1.04}$$

where the asterisk denotes "complex conjugate". It is easily shown* that the real part of $\psi(t)$ is twice the cross-correlation function of the actual physical waveforms $\hat{\xi}(t)$ and $\hat{\eta}(t)$. Similarly, the magnitude of $\psi(t)$,

$$|\psi(\tau)| = \left| \int x^*(t)\, y(t-\tau)\, dt \right| \tag{1.05}$$

may be shown to be twice the envelope of the physical cross-correlation function.*

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* See Appendix I.

We shall find it necessary to use the sampling theorem for band-limited waveforms. In complex notation this takes the following form.[14] Suppose that $\xi(t)$ contains no frequency components outside of the band $f_c - \frac{W}{2} \leq f \leq f_c + \frac{W}{2}$ $(f_c \geq \frac{W}{2})$. Then $\xi(t)$ is completely specified by complex samples $\xi_k \equiv \xi(t_k)$ taken at intervals of $\frac{1}{W}$ seconds. Furthermore,

$$W \int |\xi(t)|^2 \, dt = \sum_k |\xi_k|^2 \tag{1.06}$$

2. Fourier transforms.

In this paper we shall use Fourier transforms in which the frequency-domain variable is a cyclic, rather than a radian, frequency. That is, the transform pair

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} \, df \tag{1.07a}$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} \, dt \tag{1.07b}$$

will be used, where x(t) is the time-domain function, and X(f) its frequency-domain mate.

3. Probability and probability density.

We shall use the notation Pr[x] for "probability of x" and pr[x] for "probability density of x", and shall reserve the letters P and p (usually with subscripts) to denote particular probability (density) distributions.

CHAPTER II: THE CHANNEL

We now turn our attention to answering the fifth question of the last chapter, "In what form is statistical knowledge of the channel available to the receiver and transmitter?" We may partially answer this by stating, "In the form of probability distributions of channel characteristics", but this is not very satisfactory, for one is immediately led to ask, "What are the pertinent characteristics, and what are their probability distributions?" This chapter is devoted to answering the latter question.

Knowledge of the channel may be divided into two types: *a priori* and *a posteriori*. The former type may be based on a physical model of the channel; on the other hand, it may merely reflect ignorance of the channel, and thus be better labelled *a priori* "misknowledge".[*] *A posteriori* knowledge is based on channel soundings, i.e., on measurements of channel characteristics. We shall discuss the two types of knowledge in the order mentioned.

A. *A Priori* Knowledge.

1. A model for the additive noise.

Let us first dispose of the question of a model for the additive noise at the receiver end of the channel by assuming that the noise is statistically independent of the multipath medium, and is statistically stationary, Gaussian, and has a flat power-density spectrum, at least over a range of frequencies which covers the transmission band. If we are considering a radio communication system, for example, such a model might correspond to the thermal noise in the receiving antenna and receiver, and to the shot noise in the receiving tubes.

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[*] This will be discussed later in more detail; suffice it to say now that the design of the receiver and transmitter depends not on what the channel *actually is*, but on what the receiver and transmitter *think it is*.

We may, without loss of generality, assume that the noise power-density spectrum is constant and equal to N_o (watts/c.p.s.) over a bandwidth W_N, which at least covers the transmission bandwidth W, and is zero elsewhere.[‡] Then from the sampling theorem of Section I.C.1, a noise waveform, $\nu(t)$, is completely specified by complex samples, ν_k, taken at intervals of $1/W_N$ seconds. The real and imaginary parts of each sample are Gaussianly distributed, and may be easily shown, using a result of Rice[(15)], to be independent and to have a common variance of $W_N N_o$. Furthermore, since the auto-correlation function of the noise has zeros every $1/W_N$ seconds, and uncorrelated Gaussian numbers are independent, the samples are independent of one another. Then, a sample of noise T seconds long is specified by approximately[‡‡] TW_N independent complex samples, whose real and imaginary parts are Gaussian and independent. The joint probability-density distribution of these samples is thus

$$\mathrm{pr}\left[\nu_1, \ldots, \nu_{TW_N}\right] = \frac{1}{(2\pi W_N N_o)^{TW_N}} \exp\left[-\frac{1}{2W_N N_o} \sum_{k=1}^{TW_N} |\nu_k|^2\right] \qquad (2.01)$$

Using equation (1.06), we may write for the "probability-density" of a noise waveform, $\nu(t)$, of duration T:

$$\mathrm{pr}\left[\nu(t)\right] = \frac{1}{(2\pi W_N N_o)^{TW_N}} \exp\left[-\frac{1}{2N_o} \int_0^T |\nu(t)|^2 \, dt\right] \qquad (2.02)$$

---

‡ Strictly speaking, the noise cannot be truly random (i.e., unpredictable from its past) under these conditions (cf. reference 1, Sec. 2.4). Practically speaking, however, we may ignore the inherent predictability, for it would be difficult, if not impossible, to utilize.

‡‡ We say "approximately", for a waveform cannot be simultaneously of finite bandwidth and finite time duration. For $T \gg 1/W_N$, the approximation is very good.

Equation (2.02) is sufficient, for our purposes, to characterize the additive noise. We assume that N_o is known.

2. A model for the multipath medium.

We shall describe the multipath medium in terms of elementary "sub-paths", which we shall group together in a certain way to form "paths". A sub-path is defined by a strength, b_{ik}, and a delay, t_{ik}, such that, if a signal, $\xi(t)$, is transmitted, the output of the sub-path is $b_{ik}\,\xi(t-t_{ik})$.[+] (The subscript ik, indicates that we are considering the k^{th} sub-path of the i^{th} path.) A path is in turn defined as a group of sub-paths whose delays differ from one another by amounts much less than the reciprocal of the transmission bandwidth, W. The contribution of the i^{th} path to the multipath-medium output is the sum of its sub-path contributions:

$$\eta_i(t) \equiv \sum_k b_{ik}\, x(t-t_{ik})\, e^{j2\pi f_o(t-t_{ik})} \tag{2.03}$$

where we have written, as in equation (1.01), $\xi(t) = x(t)\, e^{j2\pi f_o t}$. Now, since $x(t)$ is a low-pass function which does not vary appreciably over intervals much less than $\frac{1}{W}$, we may, to a very good approximation, set $x(t-t_{ik}) = x(t-\tau_i)$ for all k. Then $\eta_i(t)$ becomes

$$\eta_i(t) = x(t-\tau_i) \sum_k b_{ik}\, e^{j2\pi f_o(t-t_{ik})} \tag{2.04a}$$

We shall call τ_i the modulation delay of the i^{th} path. It is a vaguely-defined quantity which, practically speaking, may be set equal to any one of the t_{ik}'s.

We shall assume that, generally, the terms of the summation of equation

---

[+] This implies the assumption that the multipath medium is linear and its physical properties do not vary appreciably across the transmission band.

(2.04a) are of two types: those terms for which b_{ik} and t_{ik} are fixed quantities, and those for which b_{ik} and t_{ik} are randomly time-varying quantities. Summing over these two types separately, we may rewrite equation (2.04a) in the form:

$$\eta_i(t) = x(t-\tau_i)\Big[\underbrace{\alpha_i e^{-j\delta_i}}_{\text{fixed}} + \underbrace{s_i e^{-j\epsilon_i}}_{\text{random}}\Big] e^{j2\pi f_0 t} \tag{2.04b}$$

The first bracketed term in equation (2.04b) we shall call the fixed component of the i^{th} path, for α_i and δ_i are fixed quantities. The second term we shall call the random component; s_i and ϵ_i are randomly time-varying quantities. These terms may be represented vectorially (see Figure 2-1): the resultant of the fixed and random components is a vector of length a_i and phase θ_i. Correspondingly, equation (2.04b) may be rewritten as

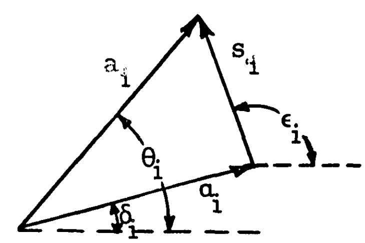

FIGURE 2-1

$$\eta_i(t) = a_i\, x(t-\tau_i)\, e^{j(2\pi f_o t - \theta_i)} \tag{2.04c}$$

We shall call a_i the strength, and θ_i the carrier phase-shift, of the i^{th} path.

We have thus reduced our description of the i^{th} path to that of three characteristics: a_i, θ_i, and τ_i.‡ These are generally random functions of time, and we may therefore describe the i^{th} path in terms of probability-density distributions of the three characteristics. It will be seen later that, for the purposes of this paper, it will be sufficient to know only the first-order joint distribution, pr $[a_i, \theta_i, \tau_i]$; we shall not require higher-order distributions.

---

‡We are justified in speaking of τ_i and θ_i separately, for, because of the manner of definition of τ_i, we may consider variations in θ_i (i.e., in the t_{ik}'s) while considering τ_i to be fixed. This will prove to be a convenient technique.

In establishing the expression for this first-order distribution, we first write $pr[a_i, \theta_i, \tau_i] = pr[\tau_i]\ pr[a_i, \theta_i/\tau_i]$. We then assume that for a "fixed" τ_i, the random sub-paths combine in such a way that the strength, s_i, of the i^{th} random path-component is Rayleigh distributed with mean square $2\sigma_i^2$; and its phase-shift, ϵ_i, is completely random (i.e., distributed evenly over the interval $(-\pi, \pi)$).* Now, Rice[16] gives an expression for the joint distribution of amplitude and phase of the sum of a fixed vector and a vector with Rayleigh-distributed amplitude and completely-random phase; using this we may write:**

$$pr[a_i, \theta_i/\tau_i] = \begin{cases} \dfrac{a_i}{2\pi\sigma_i^2} \exp\left[- \dfrac{a_i^2 + \alpha_i^2 - 2\alpha_i a_i \cos(\theta_i - \delta_i)}{2\sigma_i^2} \right] & \left\{ \begin{array}{l} 0 \leq a_i \leq \infty \\ -\pi \leq \theta_i - \delta_i \leq \pi \end{array} \right\} \\ 0 & \text{elsewhere} \end{cases} \tag{2.05}$$

The dependence of $pr[a_i, \theta_i/\tau_i]$ on τ_i, if any, will be through the parameters $\alpha_i, \sigma_i, \delta_i$. We shall leave the question of an <u>a priori</u> distribution for τ_i, $pr[\tau_i]$, until a later chapter.

The multipath medium will generally contain many paths such as the one described above, say L of them. The total output of the medium will then be

$$\eta(t) = \sum_{i=1}^{L} \eta_i(t) = \sum_{i=1}^{L} a_i\, x(t-\tau_i)\, e^{j(2\pi f_o t - \theta_i)} \tag{2.06}$$

---

*This assumption implies that the quadrature components of the random vector, s_i, are independent, Gaussian variables, with zero means and equal variances. These conditions obtain in many physical situations (see page 16 for examples) in which the number of sub-paths is great enough for the central limit theorem to apply.

**The marginal distribution of the path strength, a_i (see equation (A8-8) of Appendix VIII) is Rayleigh for $(\alpha_i/\sigma_i)=0$, and is essentially Gaussian of mean α_i and variance σ_i^2 for $(\alpha_i/\sigma_i) \to \infty$ (Cf. reference 16). For curves of the marginal distribution of θ_i as a function of α_i/σ_i, see reference 17.

and the medium will be completely described, for our purposes, by the joint first-order distribution of the three sets of characteristics: (a_i), (θ_i), and (τ_i). We shall assume that all paths are conditionally independent, so we may write

$$\text{pr}\left[(a_i),\ (\theta_i)/(\tau_i)\right] = \prod_{i=1}^{L} \text{pr}\left[a_i,\ \theta_i \mid \tau_i\right] \tag{2.07}$$

In order to complete the a priori description of the medium, we need the joint distribution of the τ_i's, pr $\left[(\tau_i)\right]$; again, we postpone consideration of this until a later chapter.

One may perhaps obtain a greater insight into the above description of the multipath medium by referring to Figure 2-2, in which the output of a three-path medium is depicted on the assumption that a pulse is transmitted which has unit height and a width approximately equal to the reciprocal of the bandwidth. The output of the multipath medium then consists of three pulses, also of width

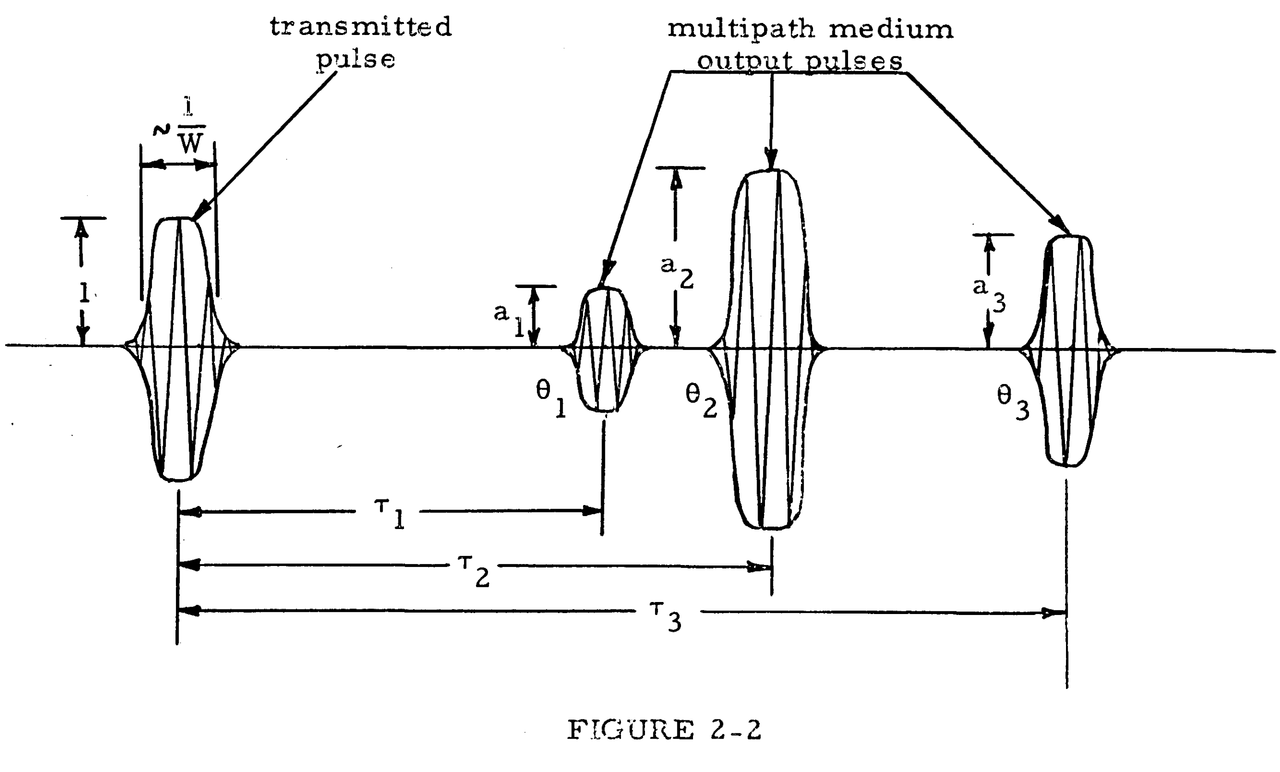

FIGURE 2-2

approximately equal to $\frac{1}{W}$, whose heights, modulation delays, and carrier phases (relative to the carrier phase of the transmitted pulse) are a_i, τ_i, and θ_i, respectively (i=1, 2, 3). The carrier period has of course been greatly exaggerated. We shall postpone discussion of the discreteness of the output pulses until the next section (see the discussion following equation (2.20)).

We may establish the following correspondences between the above description of the multipath medium and actual physical phenomena. The fixed path-components may be attributed to reflections from fixed objects or from stationary refracting layers. The random components may be attributed to scattering from turbulent regions of a medium[18], reflections from groups of randomly-moving reflectors[19], or diffraction from a randomly-moving diffraction screen[20]. Using these correspondences, our multipath model may, in many cases, be applied to such situations as radio transmission via the ionosphere (either below or above the MUF) or via tropospheric scattering; or to sonic or super-sonic transmission through fluid media. Experimental verifications of the validity of the probability-density distribution of equation (2.05) in the ionospheric and tropospheric cases are reported in the literature[21, 22, 23].

So much for the model of the multipath medium. Its failing is quite evident, for although we have established a conditional distribution for the path strengths and phase-shifts, we see that this is dependent on the sets of parameters (α_i), (σ_i), and (δ_i), and these are generally unknown _a priori_. The parameters may in fact, in an actual physical situation, vary slowly with time, thus making (2.05) and(2.07) only quasi-stationary. It is also likely that the actual probability-density distribution of the τ_i's will be unknown _a priori_. Thus, the _a priori_ physical description of the multipath medium is generally incomplete, and we are reduced to one of the two alternatives: either we may accept our ignorance, and make "educated guesses" at the unknowns, i.e., assign rather arbitrary probability distributions to them; or we may try to measure the characteristics, (a_i), (θ_i), and (τ_i), directly.[+] We shall consider the first alternative in a later chapter, the latter in the next section.

B. *A Posteriori* Knowledge.

1. *A posteriori* distribution of characteristics of medium.

Let us consider a measurement system in which a sounding signal, $\xi(t) = x(t)\, e^{j2\pi f_0 t}$, of bandwidth W, duration T, and energy E, is transmitted. The received signal, $\zeta(t) = z(t)\, e^{j2\pi f_0 t}$, is the sum of the multipath medium output, $\eta(t)$, and a noise waveform, $\nu(t)$:

$$\zeta(t) = \eta(t) + \nu(t) \tag{2.08}$$

We shall assume that T is small enough so that the multipath medium may be considered fixed (i.e., (a_i), (θ_i), (τ_i) constant) for the duration of the transmission.[++]

Because the measurement is made in the presence of noise, our *a posteriori* knowledge will not be exact; all we can expect is an *a posteriori* distribution of the characteristics of the medium, to be obtained by some operation on $\zeta(t)$, assuming that the receiver has an exact replica of $\xi(t)$. We denote this distribution by $\mathrm{pr}\left[(a_i), (\theta_i), (\tau_i)/\zeta, \xi\right]$.

Before deriving the expression for this *a posteriori* distribution, let us write down expressions for the auto-correlation function of $\xi(t)$ and the cross-correlation

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[+] The possibility of measuring just the parameters, (α_i), (σ_i), and (δ_i), of (2.07) may be eliminated, for these may, in general, vary with time and thus cannot be measured once and for all. If we are going to the trouble of repetitive measurements, however, we might just as well attempt to measure the characteristics, (a_i), (θ_i), and (τ_i), themselves.

[++] In the ionospheric case, for example, this limits us practically to T's of the order of fractions of seconds or less.

function of $\xi(t)$ and $\zeta(t)$. In complex notation (cf. equation (1.04)), the auto-correlation is*

$$\phi(\tau) = \int \xi^*(t)\, \xi(t-\tau)dt = e^{-j2\pi f_o \tau} \int x^*(t)\, x(t-\tau)\, dt \tag{2.09}$$

The cross-correlation is

$$\psi(\tau) = \int \zeta^*(t)\, \xi(t-\tau)\, dt = e^{-j2\pi f_o \tau} \int z^*(t)\, x(t-\tau)\, dt \tag{2.10}$$

The second part of (2.10) may also be written as

$$\psi(\tau) = g(\tau)\, e^{-j2\pi f_o \tau} \tag{2.11}$$

where $g(\tau)$ is equal to the integral containing z and x, which is a complex low-pass function.

As we have noted in Chapter I, the real part of a complex correlation function is twice the correlation function of the physical waveforms, and the magnitude, twice the envelope of the physical correlation function. Thus, in (2.11), the real part

$$\hat{\psi}(\tau) = \hat{g}(\tau) \cos 2\pi f_o \tau + \tilde{g}(\tau) \sin 2\pi f_o \tau \tag{2.12}$$

represents twice the correlation function of $\hat{\zeta}(t)$ and $\hat{\xi}(t)$; and $|\psi(\tau)| = |g(\tau)|$, twice the envelope of this correlation function.

We may write the <u>a posteriori</u> distribution of the characteristics of the medium in the form

$$\mathrm{pr}\left[(a_i), (\theta_i), (\tau_i)/\zeta, \xi\right] = \mathrm{pr}\left[(\tau_i)/\zeta, \xi\right]\, \mathrm{pr}\left[(a_i), (\theta_i)/(\tau_i), \zeta, \xi\right] \tag{2.13}$$

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*All integrals extend over regions of non-zero integrand.

We shall first derive an expression for the second factor on the right. By Bayes' equality,

$$\mathrm{pr}\ \left[(a_i),\ (\theta_i)/(\tau_i),\ \zeta,\ \xi\right] = \frac{\mathrm{pr}\left[(a_i),\ (\theta_i)/(\tau_i)\right]\ \mathrm{pr}\left[\zeta/(a_i),\ (\theta_i),\ (\tau_i),\ \xi\right]}{\mathrm{pr}\left[\zeta/(\tau_i),\ \xi\right]} \tag{2.14}$$

In this expression, $\mathrm{pr}\left[(a_i),\ (\theta_i)/(\tau_i)\right]$ is an <u>a priori</u> distribution. We shall assume that it is of the form of (2.07), and leave the question of the values of the parameters, (α_i), (σ_i), and (δ_i), open for the moment. $\mathrm{pr}\left[\zeta/(\tau_i),\ \xi\right]$ is just a normalizing factor; it ensures that the integral of (2.14), taken over all values of the a_i's and θ_i's, is unity.

The remaining factor, $\mathrm{pr}\left[\zeta/(a_i),\ (\theta_i),\ (\tau_i),\ \xi\right]$, we shall call the likelihood function of ζ. It is just the probability that, from (2.08),

$$\nu(t) = \zeta(t) - \eta(t) \tag{2.15}$$

given $\eta(t)$ in the form of (2.06), with (a_i), (θ_i), and (τ_i) known. Using (2.02) for the probability of $\nu(t)$, and assuming that

$$\frac{1}{2E}\ \left|\phi(\tau_i - \tau_k)\right| \ll 1 + \frac{N_o}{2\sigma_i^2 E}\ ,\ \text{all } i \neq k \tag{2.16}$$

we easily obtain an expression for the likelihood function, which, when used with (2.07) and (2.14), yields for the <u>a posteriori</u> probability[‡]

$$\mathrm{pr}\ \left[(a_i),\ (\theta_i)/(\tau_i),\ \zeta,\ \xi\right] = \begin{cases} \displaystyle\prod_{i=1}^{L} \frac{a_i}{2\pi\sigma'^2}\ \exp\left[-\frac{a_i^2 + \alpha_i'^2 - 2\alpha_i' a_i \cos(\theta_i - \delta_i')}{2\sigma_i'^2}\right] & \begin{Bmatrix} 0 \leq a_i \leq \infty \\ -\pi \leq \theta_i - \delta_i' \leq \pi \end{Bmatrix} \\ 0 & \text{elsewhere} \end{cases} \tag{2.17}$$

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‡See Appendix II.

In this equation

$$a_i' = \sigma_i'^2 \left\{ \frac{|g(\tau_i)|^2}{N_o^2} + \left(\frac{a_i}{\sigma_i^2}\right)^2 + \frac{2\,a_i}{\sigma_i^2 N_o} \, Re\left[g(\tau_i)\, e^{-j\delta_i}\right] \right\}^{\frac{1}{2}} \tag{2.18a}$$

$$\sigma_i'^2 = \left[\frac{2E}{N_o} + \frac{1}{\sigma_i^2}\right]^{-1} \tag{2.18b}$$

$$\delta' = \tan^{-1} \frac{\frac{a_i}{\sigma_i^2} \sin \delta_i + \frac{\tilde{g}(\tau_i)}{N_o}}{\frac{a_i}{\sigma_i^2} \cos \delta_i + \frac{\hat{g}(\tau_i)}{N_o}} \tag{2.18c}$$

Thus, the a posteriori conditional distribution of (a_i) and (θ_i) is of the same form as the a priori conditional distribution (cf. equations (2.05) and (2.07)). The new parameter sets, (a_i'), (σ_i'), and (δ_i'), depend only on the noise power-density, the old parameters, and sampled values of the cross-correlation function of the incoming signal and the replica of the sounding signal stored in the receiver. Thus, the factor $Re\left[g(\tau_i)\, e^{-j\delta_i}\right]$ of (2.18a) will be recognized from (2.12) to be twice the cross-correlation, sampled at modulation delay τ_i, in carrier phase δ_i;[‡] similarly, $\hat{g}(\tau_i)$ and $\tilde{g}(\tau_i)$ of (2.18c) are twice the correlation at τ_i, in carrier phases 0 and $\frac{\pi}{2}$, respectively. $|g(\tau_i)|$ is twice the envelope of the cross-correlation, sampled at delay τ_i.

Let us now return to the question of the a priori parameters, (a_i), (σ_i), and (δ_i). Because of the equivalence of the forms of (2.07) and (2.17), we see that these parameters may indeed be the a posteriori parameters of a previous measurement. If, however, there has been no previous measurement, we must do the best we can by choosing the parameters in such a way as to register our a priori ignorance. Now,

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[‡]We here again invoke the property of narrow-band (correlation) functions, that we may speak of modulation delay and carrier phase separately; that is, we may speak of the correlation at various phases for a given delay.

roughly, we may think of the σ_i's as measures of this ignorance --- the larger a particular σ_i is, the more uncertain is our a priori knowledge of the strength and phase-shift of that path; $\sigma_i = \infty$ indicates complete a priori ignorance. In this latter case, we note from (2.18) that the a posteriori parameters are independent of the a priori ones; (α_i'), (σ_i'), and (δ_i') depend then only on the measured cross-correlation function.

The result of the measurement, we hope, will be to make a substantial increase in our knowledge of the multipath medium. For this to be true, we must have $\sigma_i' \ll \sigma_i$. From (2.18b), this means that

$$\frac{2\sigma_i^2 E}{N_o} \gg 1 \tag{2.19}$$

That is, for a useful measurement, either our a priori knowledge must be small (σ_i large), so that any a posteriori knowledge is helpful, or the signal-to-noise ratio, $\frac{E}{N_o}$, must be large.

Suppose that (2.19) is satisfied, i.e., the measurement is a useful one. Then assumption (2.16) becomes

$$\frac{1}{2E} \left|\phi(\tau_i - \tau_k)\right| \ll 1 , \quad \text{all } i \neq k \tag{2.20}$$

The left-hand side of this inequality is the envelope of the normalized auto-correlation function of the sounding signal. Now, this signal has bandwidth W, and we know that the auto-correlation function of a signal of bandwidth W is small for values of argument greater than the order of $\frac{1}{W}$. Thus, (2.20) will be satisfied if

$$\left|\tau_i - \tau_k\right| > o\left(\frac{1}{W}\right) \tag{2.21}$$

We conclude, then, that if the measurement is useful at all (i.e., (2.19) holds), the distribution (2.17) is valid only if the multipath medium is made up of paths whose modulation delays differ by amounts greater than the order of $\frac{1}{W}$. This is not too much of a restriction on the generality of (2.17), however, for we have already seen that, for all practical purposes, a group of paths (sub-paths) whose delays differ by amounts less than the order of $\frac{1}{W}$ can be considered a priori as one path by vectorially adding their strengths. We can thus reduce the number of paths under consideration in such a way that (2.21) is automatically satisfied, at least to a good approximation.

We may also interpret (2.20) from the point of view of resolvability of paths. Let us first write down the expression for the a posteriori probability distribution of the τ_i's, which can be derived in a manner similar to the derivation of (2.17).[‡]

$$\mathrm{pr}\left[(\tau_i)/\zeta,\xi\right] = C\,\mathrm{pr}\left[(\tau_i)\right]\prod_{i=1}^{L}\exp\left[\frac{1}{2}\left(\frac{a_i'}{\sigma_i'}\right)^2\right] \tag{2.22}$$

C is a normalizing constant which ensures that the integral of (2.22), over all configurations of the τ_i's, is unity. pr $\left[(\tau_i)\right]$ is an a priori distribution of the τ_i's.

We see that the only operation upon which the a posteriori distribution of the τ_i's is based is again the cross-correlation function of the received signal with the stored replica of the sounding signal. In particular, for the case $\sigma_i \to \infty$, all i, we require only the envelope of this cross-correlation (cf. equation (2.18a)). Now, for a reasonably high signal-to-noise ratio, E/N_o (which is required for a good measurement), the cross-correlation function will be essentially the auto-correlation function of the sounding signal, added to itself several times with different delays, once for each path (cf. equations (2.06), (2.08), (2.09), and (2.10)). If (2.20) is satisfied, then the contribution to the

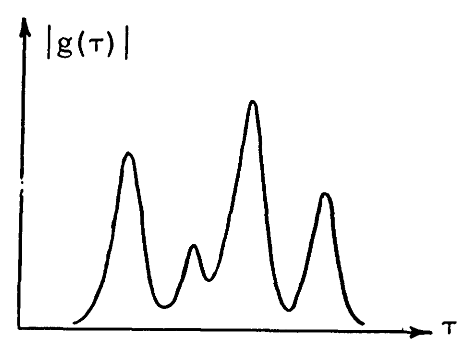

FIGURE 2-3

[‡]See Appendix II.

cross-correlation due to any one path will be essentially nil at the peak of the contribution due to another path, so that the envelope of the cross-correlation will consist of L distinct pulses, as in Figure 2-3. On the other hand, if (2.20), and hence (2.21), does not hold for a pair of paths, the corresponding pair of pulses in Figure 2-3 would merge, i.e., be unresolvable by the measuring equipment. But the two paths may also be considered as sub-paths of a composite path (at least to a good approximation), so that our method of grouping sub-paths is equivalent to saying that we need only consider as distinct those paths which can be resolved by the receiver.

Now consider the case for which we have no a priori knowledge of the multipath medium, not even of the number of paths which exist. By the above reasoning, we can find an effective path number by counting the number of resolved pulses in $|g(\tau)|$. We may then use (2.17) and (2.22) to obtain an a posteriori distribution of the characteristics of these effective paths, for condition (2.21) will automatically be satisfied.

This reasoning of course breaks down if extended too far. We have assumed that we can group sub-paths in a reasonable manner, so that Figure 2-3 is an accurate representation, i.e., is composed of discrete pulses of width very close to $\frac{1}{W}$. If there is a continuum of sub-paths, however, we should be able neither to group sub-paths in a reasonable way a priori, nor to resolve any discrete pulses in $|g(\tau)|$. We shall henceforth assume that the possibility of a continuum of paths does not exist; we shall restrict ourselves to the discrete - path case.

As a final point related to (2.20) and (2.21), we consider the power-transmission spectrum of the multipath medium. If there are more than one path, and (2.21) holds, the paths will interfere constructively at some frequencies in the band and destructively at others, as in Figure 2-4; that is, the medium will be frequency selective. On the other hand, the interference of the component sub-paths of a path is not frequency selective; it is for this reason that we are able to group sub-paths and represent the resultant path by the frequency-independent parameters, α_i, σ_i, δ_i.

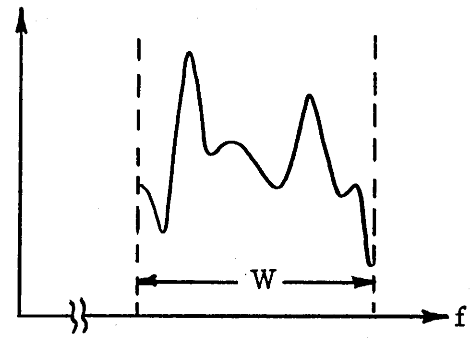

FIGURE 2-4

We have noticed that the only operation which is required to obtain an a posteriori distribution of the multipath characteristics is the cross-correlation of the received signal with the stored replica of the sounding signal. Now, as is well known,[5] this operation can be performed with a linear filter which is "matched"[25] to the sounding signal; that is, one whose impulse response is the same as the sounding signal, but reversed in time. This is easily seen by writing the convolution integral, giving the output of a linear filter in terms of its input and its unit-impulse response function. In complex notation, this is‡

$$\beta(t) = \frac{1}{2} \int \zeta^*(\tau)\,\mu(t-\tau)\,d\tau \tag{2.23}$$

where β is the output; ζ, the input; and μ, the impulse-response function. If we let $\mu(t) = \xi(-t)$ and $\beta(t) = \frac{1}{2}\psi(t)$, (2.23) becomes formally identical with (2.10). $\mu(t)$ may be made physically realizable ($\xi(-t)$ is not) by allowing a time delay of T seconds, that is, setting $\mu(t) = \xi(T-t)$. The times at which the filter output is sampled must be correspondingly delayed. For large $\frac{E}{N_o}$, the envelope of the matched-filter output will look like Figure 2-3 (four-path case), where τ is now the time variable.

2. Minimum-mean-square-error estimation of impulse response of medium.

Instead of obtaining the complete a posteriori distribution of the characteristics of the medium, we might only be interested in obtaining some definite estimate of

‡For an explanation of the factor of 1/2, cf. Appendix I.

the characteristics. One type of estimation would be to find those values of the a_i's, θ_i's, and τ_i's which maximize (2.17) and (2.22); that is, we may choose the a posteriori most probable characteristics. We shall describe in this section another type of estimation of which we shall call minimum-mean-square-error estimation.

Let us think of the medium as a randomly time-varying linear filter; and let us try to estimate its impulse response by some linear operation on the received signal, when we have transmitted a sounding signal, $x(t)$ ǂ , of duration T. We again assume that the medium, and hence its impulse response, stays fixed for the duration of the transmission.

The measurement system may be depicted as in Figure 2-5, where we have indicated the sounding signal as the output of a linear filter with impulse response $x(\tau)$, when a unit impulse, $\delta(t)$, is applied to its input. We require the impulse response, $h_e(\tau)$, of the linear estimating filter which makes a minimum-mean-

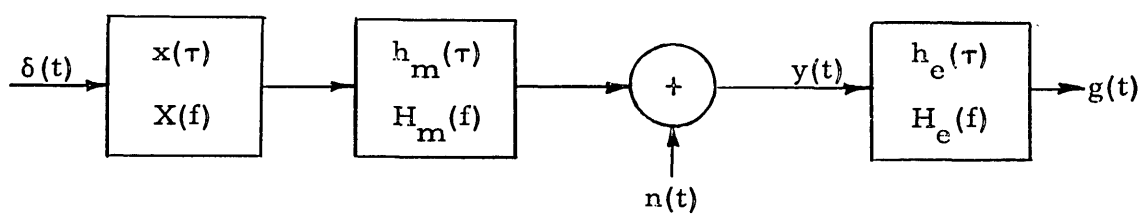

FIGURE 2-5

square-error estimate of the impulse response of the medium, $h_m(\tau)$; this estimate is made in the presence of additive noise, $n(t)$.

Suppose that we know that the impulse response of the medium lies essentially inside of some interval, say $0 \leq \tau \leq \Delta$. Then the expression for the mean-square

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ǂWe abandon here the complex notation, for the results in this section may be applied more generally than to just narrow-band-pass situation.

error of the output of the estimating filter may be defined as

$$\epsilon \equiv E_{N,M}\left[\frac{1}{\Delta}\int_0^{\Delta}\left\{g(t) - h_m(t)\right\}^2 dt\right] \tag{2.24}$$

where g(t) is the estimating filter output, and $E_{N,M}$ denotes a statistical average over the ensembles of possible noises and possible impulse responses of the medium. We minimize this error by varying the estimating-filter impulse-response; that is, we set

$$\delta\epsilon = 0 \tag{2.25}$$

where the variation is with respect to the estimating-filter impulse-response.

It is shown in Appendix III that the transfer function of the estimating filter (i.e., the Fourier transform of $h_e(f)$) which satisfies (2.25) is

$$H_{e_{opt}}(f) = \frac{1}{\Delta}\cdot\frac{X^*(f)}{\dfrac{N(f)}{\overline{|H_m(f)|^2}} + \dfrac{|X(f)|^2}{\Delta}} \tag{2.26}$$

where the asterisk denotes "complex conjugate". In this expression, X(f) is the sounding-signal voltage-density spectrum; $\overline{|H_m(f)|^2}$, the average power-transmission function of the medium; and N(f), the noise power-density spectrum.[‡] Thus, the only statistic of the medium which one must have _a priori_ is $\overline{|H_m(f)|^2}$; if we have no _a priori_ knowledge of the medium, we set this equal to a constant.

The filter of (2.26) may not be physically realizable, because the condition that the impulse response of a realizable filter must be zero for negative arguments was not used in the derivation. However, this is not a great problem; we can usually accept some delay in obtaining our estimate of $h_m(\tau)$, and $H_{e_{opt}}(f)$ can usually be

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[‡]In the interest of generality, we shall temporarily neglect our white-noise assumption; we shall assume, however, that N(f) is known to the receiver.

made realizable, at least to a very good approximation, by introducing sufficient delay. This delay will usually not be more than the order of T seconds.

For the no-noise case, i.e., $N(f) \equiv 0$, the solution reduces to the inverse filter

$$H_{e_{opt}}(f) = \frac{1}{X(f)} \tag{2.27}$$

This is, of course, to be expected, for the voltage-density spectrum at the channel output is $H_m(f)\,X(f)$, and the filter of (2.27) restores this to just $H_m(f)$, which is Fourier transform of $h_m(\tau)$. Thus, in the no-noise case, $h_m(\tau)$ is estimated without error.

For $N(f) \not\equiv 0$, comparison of (2.26) and (2.27) shows that the optimum filter may be expressed (see Figure 2-6) as the cascade of the inverse filter of (2.27) and a filter with transfer function

$$H_c(f) = \frac{\frac{|X(f)|^2}{\Delta}}{N_r(f) + \frac{|X(f)|^2}{\Delta}} \tag{2.28}$$

where we have set

$$N_r(f) = \frac{N(f)}{|H_m(f)|^2} .$$

($N_r(f)$ is thus the noise power-density spectrum, referred back to the transmitter through the <u>average</u> medium.) The output of the inverse filter in Figure 2-6 contains $h_m(\tau)$, but it also contains a large amount of noise, especially at those frequencies where $X(f)$ is small. The second filter attenuates the noise, but in doing so, smears $h_m(\tau)$. The optimization procedure may be thought of as one which makes the best compromise between eliminating noise and keeping $h_m(\tau)$ undistorted. $H_c(f)$ is a zero-phase filter, as one would expect, for any other phase (except a linear one, which is a trivial exception) would distort the desired output, $h_m(\tau)$, without helping to attenuate the effect of the noise, which has random phase anyway.

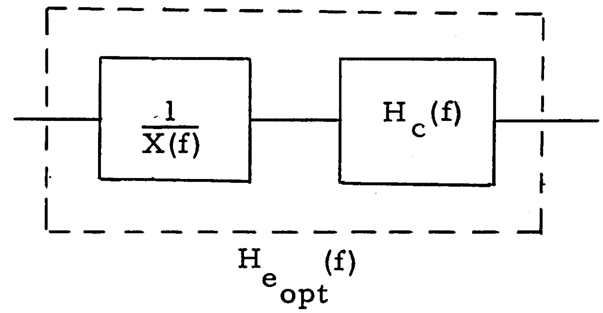

FIGURE 2-6

If Δ is of the order of magnitude of T, then the second term in the denominator of (2.26) is roughly the transmitted power-density spectrum. If this is small, for all f, compared to $N_r(f)$ (or, equivalently, if the average received-signal power-density spectrum is small compared to N(f)), then (2.26) becomes

$$H_{e_{opt}}(f) = \frac{1}{\Delta} \frac{X^*(f)}{N_r(f)} \tag{2.29}$$

If $N_r(f)$ is constant with frequency, which will occur, for example, if the noise is white and we have no a priori knowledge of the medium, then the filter of (2.29) is matched[25] to the sounding signal; for taking the conjugate of a frequency spectrum implies reversing the time function.

So far we have considered X(f) to be arbitrary. Now let us, while keeping $H_e(f) = H_{e_{opt}}(f)$, solve for the X(f) which minimizes ϵ, subject to the constraint that the energy in x(t) be fixed and that X(f) lie within a given band. That is, let us set

$$\int_{-\infty}^{\infty} |X(f)|^2 \, df = K \tag{2.30}$$

and

$$X(f) \equiv 0 \quad \text{for f not in } F_1 \tag{2.31}$$

where F_1 is the permitted band of frequencies, and solve the equation

$$\delta(\epsilon_m + \lambda K) = 0 \tag{2.32}$$

where λ is some constant.(26) ϵ_m is the minimum mean-square error for an arbitrary X(f).

The solution to (2.32) is shown in Appendix III to be

$$X(f) = e^{-j\beta(f)} \begin{cases} \left[N(f)\Delta\right]^{\frac{1}{4}} \left[\frac{1}{\sqrt{\lambda}} - \frac{\sqrt{N(f)\Delta}}{|H_m(f)|^2}\right]^{\frac{1}{2}} & \text{f in } F_2 \\ 0 & \text{f in } F_2' \end{cases} \tag{2.33}$$

In this equation, $\beta(f)$ is an arbitrary phase function, F_2 is the set of all frequencies in F_1 for which

$$\frac{1}{\sqrt{\lambda}} \geqslant \frac{\sqrt{N(f)\Delta}}{|H_m(f)|^2} ,$$

and F_2' is the set of all frequencies not in F_2. The constant, λ, is adjusted to satisfy the energy constaint, (2.30). The interpretation of (2.33) confirms one's intuitive notions. The first factor, $\left[N(f)\Delta\right]^{\frac{1}{4}}$, indicates that if the noise power at some frequency, f_1, is very small, then little signal energy is needed at that frequency to determine $H_m(f_1)$. On the other hand, the second factor indicates that if the noise power at f_1 is very large, or the average power transmission of the medium is very small, then it is a waste of the limited available energy to put much, if any, energy at f_1; the energy may be used to more advantage elsewhere.

If we place (2.33) in (2.26), we obtain the optimum estimating-filter transfer-function corresponding to the optimum X(f):

$$H_{e_{opt}}(f) = \sqrt{\lambda}\, e^{+j\beta(f)} \begin{cases} \left[N(f)\,\Delta\right]^{-\frac{1}{4}} \left[\dfrac{1}{\sqrt{\lambda}} - \dfrac{N(f)\,\Delta}{|H_m(f)|^2}\right]^{\frac{1}{2}} & f \text{ in } F_2 \\ 0 & f \text{ in } F_2' \end{cases} \tag{2.34}$$

We have assumed here that N(f) is non-zero at all frequencies. The optimum filter of (2.34), as we should expect, has large gain at frequencies with small noise, and small, or zero, gain at frequencies with large noise.

The mean-square error corresponding to (2.33) and (2.34) is, from Appendix III:

$$\epsilon_{mm} = \frac{1}{\Delta}\left[\int_{F_2} \sqrt{N(f)\,\Delta\,\lambda}\; df + \int_{F_2'} \overline{|H_m(f)|^2}\; df\right] \tag{2.35}$$

The first term in this expression is the contribution to the error of the noise and smear components of the estimate which arise within the passband of $H_{e_{opt}}(f)$. The second is the smear contribution arising from the complete lack of an estimate of $H_m(f)$ in the stopband of $H_{e_{opt}}(f)$.

We are actually usually interested in estimating $H_m(f)$ only within the transmission band, F_1, instead of for all f as we have done, for F_1 is the band we shall use for communication. That is, we are interested in estimating the instantaneous impulse response of an equivalent medium which has zero transmission outside of F_1. This imposes no additional problem, however; it is easy to see that the result of (2.34) is optimum in this case also, since it is independent of values of $H_m(f)$ outside of the transmission band. The error of (2.35) also obtains in this case, except that the second integral is now taken over only those frequencies in F_2' which are within the transmission band (i.e., the intersection of F_1 and F_2').

If the noise is white, i.e., N(f) is constant, at least over the transmission band, we see that the optimum estimator of (2.34) is proportional to the complex conjugate of the sounding-signal spectrum of (2.33). That is, the optimum estimator is matched to the sounding signal. Thus, we use the same device in making a minimum-mean-square-error estimation as in measuring the a posteriori probability distribution of channel characteristics (cf. preceding section). It seems reasonable to assume that, this being the case, the spectrum of (2.33) will also give the least equivocal a posteriori distribution, that is, the largest ratio $\frac{a_i'}{\sigma_i'}$ in (2.18).

CHAPTER III: THE PROBABILITY COMPUTER.

We shall in this chapter derive expressions for the operational form of the probability computer. Let us first restate the problem. The transmitter transmits a sequence of message waveforms chosen independently, with probabilities, P_m, from a set of M message waveforms, $\xi_m(t) = x_m(t)e^{j2\pi f_o t}$ $(m = 1, 2, \ldots, M)$; these waveforms and probabilities are known to the receiver. The receiver receives a signal, $\zeta(t) = z(t)e^{j2\pi f_o t}$, which is the sum of a noise waveform, $\nu(t)$, and the output, $\eta(t)$, of the multipath medium. The probability computer is asked, on the basis of its knowledge of the channel and of the *a priori* waveform probabilities, P_m, to operate on $\zeta(t)$ in such a way as to obtain *a posteriori* probabilities of the various possible transmitted sequences.

The analysis becomes particularly straightforward if we make two simplifying restrictions; the effect of these is to allow us to describe the multipath medium completely in terms of the first-order joint distributions of (a_i), (θ_i), and (τ_i) given in the last chapter, with no higher-order distributions[#] required.

First, we shall require, as we did of the sounding signal of the last chapter that the message-waveform durations be small enough so that we may consider that the multipath medium remains essentially fixed during the transmission of a message waveform. In the case of an ionospheric medium, for example, this requirement limits the waveforms to durations of the order of fractions of seconds or less.

Second, we shall restrict the receiver to per-waveform operation. That is, we shall require that the receiver consider each waveform in the received

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I.e., joint distributions of the values of the variables (a_i), (θ_i), and (τ_i) at many different times.

sequence as an event which is independent of each other waveform. This independence does not in fact exist, for although we have assumed that the waveforms of the transmitted sequence are independent of one another, it is clear that the perturbed waveforms of the received sequence are not. This follows from the fact that the characteristics of the multipath medium have been assumed to change only very slowly from waveform to waveform of a sequence, and hence, the multipath-caused perturbations of successive message waveforms are not independent.*

The assumption of per-waveform operation implies two other assumptions: that all message waveforms have the same duration (say, T), so that the starting time of each member of the sequence does not depend on the past history of the sequence; and that either enough time is allowed between the transmission of successive message waveforms of a sequence so that the waveforms do not overlap at the output of the multipath medium because of the spread of path delays, or that any overlapping is small enough to be neglected (i.e., the spread of delays of the medium is small compared to the duration of a message waveform).

Using Bayes' equality, we may write for the per-waveform a posteriori probability of the m^{th} waveform:

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* The overlooking of the inter-waveform dependences which is entailed in per-waveform operation results in a loss of information. One may, by the following argument, gain an insight into the way in which consideration of these dependences could lead to additional information about the transmitted message. It is evident that we may use the message waveforms as channel-sounding signals as well as information-bearing signals; that is, on the hypothesis, say, that $\xi_m(t)$ was sent, we may compute an a posteriori distribution of channel characteristics just as discussed in Chapter II. Now, since these characteristics are assumed to vary very slowly, we should, for example, consider a sequence of message waveforms which all give similar distributions of characteristics more probable than a sequence for which successively-obtained distributions are vastly dissimilar. Per-waveform operation does not utilize such information.

$$\Pr\left[\xi_m/\zeta\right] = \frac{P_m \mathrm{pr}\left[\zeta/\xi_m\right]}{\mathrm{pr}\left[\zeta\right]} \tag{3.01}$$

Now, the a priori probabilities, P_m, are known, and pr $[\zeta]$ is just a normalizing factor independent of m, so the problem of computing $\Pr[\xi_m/\zeta]$ reduces to an evaluation of the "likelihoods", $\Lambda_m = \mathrm{pr}[\zeta/\xi_m]$. Λ_m is just the probability that the noise waveform, $\nu(t)$, is

$$\nu(t) = \zeta(t) - \eta^{(m)}(t) \tag{3.02}$$

where $\eta^{(m)}(t)$ is given by (2.06), on the hypothesis $x(t) = x_m(t)$. Using the assumption that the multipath medium stays essentially fixed for the duration of a message waveform, this probability may be written as:

$$\Lambda_m = \underbrace{\int \cdots \int}_{3L \text{ times}} \mathrm{pr}[(\tau_i)]\mathrm{pr}[(a_i),(\theta_i)/(\tau_i)]\mathrm{pr}[\nu = \zeta - \eta^{(m)}/(a_i),(\theta_i),(\tau_i)]\, da_1 \ldots da_L d\theta_1 \ldots d\theta_L d\tau_1 \ldots d\tau_L \tag{3.03}$$

We shall devote the rest of this chapter to the evaluation of (3.03).

For reference, we write down the complex cross-correlation function of the received signal and the m^{th} message waveform:

$$\psi_m(\tau) = \int \zeta^*(t)\, \xi_m(t-\tau)\, dt = g_m(\tau)\, e^{-j2\pi f_o \tau} \tag{3.04}$$

where

$$g_m(\tau) \equiv \int z^*(t)\, x_m(t-\tau)\, dt \tag{3.05}$$

A. (τ_i) Known: (δ_i) Known.

Let us first assume either that the modulation delays, (τ_i), are known a priori, or that their a posteriori distribution, (2.22), indicates that they are contained, with high probability, in intervals which are small compared to $\frac{1}{W}$. Then the (τ_i)-integrations of (3.03) are unnecessary. We also assume that the parameters of $pr\left[(a_i), (\theta_i)/(\tau_i)\right]$ are known a priori (equations (2.05) and (2.07)), or have been evaluated by measurement (equations (2.17) and (2.18)). We shall use the unprimed a priori parameters for convenience.

Then, assuming the noise-waveform distribution of (2.02), and further that the resolvability condition*

$$\frac{1}{2E_m} \left|\phi_m(\tau_i - \tau_k)\right| << 1 \qquad \text{all } i \neq k \tag{3.06}$$

holds for all m, where $\phi_m(\tau)$ is the complex auto-correlation function of the m^{th} message waveform, it can be shown** that the 2L-fold integration on the a_i's and θ_i's in (3.03) reduces to

$$\Lambda_m = C \prod_{i=1}^{L} \frac{1}{1 + \frac{2\sigma_i^2 E_m}{N_o}} \exp\left[\frac{\frac{\sigma_i^2}{N_o}\left|g_{mi}\right|^2 + 2a_i \operatorname{Re}\left[g_{mi}\, e^{-j\delta_i}\right] - 2a_i^2 E_m}{2N_o\left[1 + \frac{2\sigma_i^2 E_m}{N_o}\right]}\right] \tag{3.07}$$

In this equation, C is a constant,** E_m is the energy of the m^{th} message waveform, N_o is the (white) noise power-density, and we have written $g_{mi} = g_m(\tau_i)$.

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* See the discussion following equation (2.20) for an explanation of the meaning of this condition.

** See Appendix IV.

We thus see that the only operations performed by the probability computer on the received signal consist in 1) the cross-correlation of this signal with the M message waveforms; 2) the sampling of these correlations at delays τ_i, in carrier phases δ_i (cf. equation (3.04)); and 3) the sampling of the envelopes of the correlations at delays τ_i. As we have noted before,* the correlation operation may be performed by matched filters. Thus, the probability computer contains a set of M matched filters, one matched to each message waveform, and also a sampling device to sample the outputs and the output envelopes of these filters.**

Let us now consider (3.07) for the case where either we know *a priori* that the medium contains no random path components, or the receiver has *exact a posteriori* knowledge of the medium. Then $\sigma_i = 0$, all i, and

$$\Lambda_m = C \exp \frac{1}{N_o}\left[\sum_{i=1}^{L} a_i \mathrm{Re}\left[g_{mi}\, e^{-j\delta_i}\right] - \sum_{i=1}^{L} a_i^2 E_m\right] \tag{3.08}$$

We notice that the samples of the *envelope* of the cross-correlation function have disappeared, and that only the samples of the cross-correlation itself, $\mathrm{Re}\left[g_{mi}\, e^{-j\delta_i}\right]$, remain. This, of course, makes sense, for envelope sampling is of use only when there is phase uncertainty, and here we know the path phase-shifts *exactly*.

Now, the multipath-medium output in the case $\sigma_i = 0$, all i, is

$$\eta^{(m)}(t) = \sum_{i=1}^{L} a_i x_m(t-\tau_i)\, e^{j(2\pi f_o t - \delta_i)} \tag{3.09}$$

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* See page 24.

** An early interpretation of a correlation receiver in terms of a set of matched filters was made by Fano(27).

We therefore see, using (3.04) and (3.05), that the first term of the exponent of (3.08) is just the real part of the complex correlation

$$\psi'_m = \int \zeta^*(t)\,\eta^{(m)}(t)\,dt \tag{3.10}$$

That is, we may think of the likelihood computer of (3.08) as one which operates on the basis of the cross-correlation between the received signal and the set of known signals, $\eta^{(m)}(t)$ $(m = 1, 2, \ldots, M)$, which may appear at the output of the medium. This is just the correlation receiver of Woodward and Davies[5], as may be expected; for from the receiver's point of view the (non-random) medium may be considered part of the transmitter, and the channel then is perturbed just by additive noise.

In the other extreme, when we know a priori that the medium has no fixed path-components, and we make no channel measurements, we have $\alpha_i = 0$, all i. Then

$$\Lambda_m = C \prod_{i=1}^{L} \frac{1}{1 + \frac{2\sigma_i^2 E_m}{N_o}} \exp\left[\frac{\sigma_i^2 |g_{mi}|^2}{2N_o^2\left[1 + \frac{2\sigma_i^2 E_m}{N_o}\right]}\right] \tag{3.11}$$

We notice here that only samples of the correlation function envelope appear. This is intuitively justified, for if we were required to sample the correlation function itself, we should have to do this at a given set of carrier phases. But we are completely ignorant of the path phase-shifts in this case; i.e., we have no reason to prefer one set of carrier phases over another. (3.11), with a much different notation, has also been obtained by Price.[8a]

The behavior of (3.07) for large noise is of interest: as $N_o \to \infty$, it converges to the fixed-multipath computer of (3.08). Essentially, this implies that, in the limit, the information which is transferred through the channel is conveyed exclusively by the fixed path-components. This makes sense, for it stands to reason that the capacity of that part of the channel which is disturbed by path fluctuations as well as by noise should vanish more rapidly with increasing noise than the capacity of the part which is distrubed by noise only. In fact, Price has explicitly shown this for a special case.[6]

B. (τ_i) Known: (δ_i) Unknown.

We have thus far assumed that the parameters (δ_i) are known. This may very well be the case if they are a posteriori parameters. But on an a priori basis, although we may know the strength parameters, (α_i) and (σ_i), of the various paths, it is unlikely that we know the phase-shift parameters, (δ_i). Let us, in fact, assume that the δ_i's are a priori completely random from the receiver's viewpoint (i.e., distributed evenly over the interval $(-\pi, \pi)$), and independent. Then, averaging Λ_m of (3.07) over the δ_i's, we obtain[#] for the likelihoods

$$\Lambda'_m = C \prod_{i=1}^{L} \frac{1}{1+\beta_{mi}} \exp\left[\frac{\frac{\sigma_i^2}{N_o}|g_{mi}|^2 - 2\alpha_i^2 E_m}{2N_o(1+\beta_{mi})}\right] I_o\left[\frac{\alpha_i |g_{mi}|}{N_o(1+\beta_{mi})}\right] \tag{3.12}$$

where we have written

$$\beta_{mi} = \frac{2\sigma_i^2 E_m}{N_o} \tag{3.13}$$

I_o is the zeroth-order modified Bessel function of the first kind.

Equation (3.12) depends just on the envelope of the cross-correlation function of the received signal and the m^{th} stored message waveform; as in

[#] See Appendix IV.

the case of equation (3.11), this is due to the receiver's complete lack of knowledge of the multipath-medium phase-shifts. The message waveforms may now be stored with arbitrary phase, since phase shifts of the correlation function will of course not affect its envelope.

One might be tempted to argue that, conversely, equation (3.12) applies to any receiver in which the message waveforms are stored with arbitrary phase. This is not true, for although a random phase shift of the m^{th} stored message-waveform transforms g_{mi} of equation (3.07) into $g_{mi}e^{-j\mu_m}$, where μ_m is random, μ_m is independent of i; that is, instead of having to average equation (3.07) over L independent random variables, as in the derivation of equation (3.12), we must in this case average over just one random variable. Writing equation (3.07) as the exponential of a sum, and assuming that μ is evenly distributed over the interval $(-\pi, \pi)$, we may easily evaluate this average:‡

$$\Lambda''_m = C\left[\prod_{i=1}^{L} \frac{1}{1+\beta_{mi}}\right] \exp\left[\sum_{i=1}^{L} \frac{\frac{\sigma_i^2}{N_o}|g_{mi}|^2 - 2a_i^2 E_m}{2N_o(1+\beta_{mi})}\right] I_o\left[\left|\sum_{i=1}^{L} \frac{a_i g_{mi} e^{-j\delta_i}}{N_o(1+\beta_{mi})}\right|\right] \tag{3.14}$$

In equation (3.14), the argument of the I_o factor contains the L correlation samples, which have first been summed coherently (i.e., in the proper phase relationships) before envelope detection. Equation (3.14), in fact, also applies to the case where the δ_i's are not known exactly, but their differences are. In this case, the $e^{-j\delta_i}$ of equation (3.07) is transformed into $e^{-j\delta_i}\, e^{-j\mu}$, where μ is again random, and independent of i; this is clearly equivalent to the case we have just considered.

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‡ See Appendix IV.

C. (τ_i) Unknown.

Let us finally consider the possibility that the receiver does not have exact knowledge of the modulation delays, (τ_i), as we have assumed up until now, or that, in the hope of equipment simplification, we choose not to use this knowledge. Then, from the receiver's point of view, the τ_i's are random variables with some distribution, $pr\left[(\tau_i)\right]$. In this case, we generally will not have, or will choose to ignore, any knowledge of the δ_i's, so that we may obtain a new expression for the likelihoods by averaging equation (3. 12) over the τ_i's. Let us first set

$$F_{mi}(x) = \frac{1}{1+\beta_{mi}} \exp\left[\frac{\frac{\sigma_i^2}{N_o}x^2 - 2a_i^2E_m}{2N_o(1+\beta_{mi})}\right] I_o\left[\frac{a_i x}{N_o(1+\beta_{mi})}\right] \qquad (3.15)$$

Then, using (3. 15) in (3. 12), and averaging over the τ_i's, we obtain immediately:

$$\Lambda_m''' = C\underbrace{\int\cdots\int}_{L \text{ times}} pr\left[(\tau_i)\right]\prod_{i=1}^{L} F_{mi}\left[\left|g_m(\tau_i)\right|\right]d\tau_1\ldots d\tau_L \qquad (3.16)$$

where the integrations extend over all possible values of the τ_i's.

Strictly speaking, $pr\left[(\tau_i)\right]$ cannot be an arbitrary distribution because of the restriction imposed by the resolvability condition, (3.06), upon which equation (3. 16) is based. Let us at this point, however, neglect the resolvability condition, and assume that the paths are independent, i.e., that $pr\left[(\tau_i)\right] = \prod_{i=1}^{L} pr\left[\tau_i\right]$; and further that

$$pr\left[\tau_i\right] = \begin{cases} \frac{1}{B-A} & A \le \tau_i \le B \\ & & \text{all } i \\ 0 & \text{elsewhere} \end{cases} \qquad (3.17)$$

If the number of paths is small, and/or the interval (A, B) large, the total probability of the cases in which (3.17) contradicts the resolvability condition will be small. In such a case, we may expect that the contradiction between (3.06) and (3.17) will have little effect on the validity of subsequent derivations.

Let us further assume that $\alpha_i = \alpha$ and $\sigma_i = \sigma$ for all i, so that

$$F_{mi}(x) = F_m(x) \qquad \text{all } i \tag{3.18}$$

Equations (3.17) and (3.18), taken together, imply that, from the receiver's point of view, all paths are statistically identical. That is, <u>as far as the receiver knows *a priori*</u>, all paths have the same strength distribution, and all configurations of path delays in the interval (A, B) are equally probable. Using equations (3.17) and (3.18) in (3.16), it follows that

$$\Lambda'''_m = C\left[\frac{1}{B-A}\int_A^B F_m\left[|g_m(\tau)|\right]d\tau\right]^L \tag{3.19}$$

In many cases we require only the order, rather than the values, of the *a posteriori* probabilities of equation (3.01). If all of the *a priori* probabilities, P_m, are equal, this reduces to requiring the order of the likelihoods; that is, we ask whether, say, Λ'''_p is greater than or less than Λ'''_q. Since the term in brackets in equation (3.19) is positive, we may then equally well ask whether or not

$$\int_A^B F_p\left[|g_p(\tau)|\right]d\tau > \int_A^B F_q\left[|g_q(\tau)|\right]d\tau \tag{3.20}$$

It is important to note that in order to answer this question, the receiver does not need to know how many paths there are.

A receiver which works on the basis of the operations of (3.20) is much more ignorant of the state of the multipath medium than ones which operate on the basis of (3.07), (3.12), or (3.14), and we should expect a correspondingly inferior performance. On the other hand, the operations of (3.20) may be implemented much more easily than those of (3.07), (3.12), and (3.14), since no sampling equipment is required in the former. In fact, the "probability computer" of (3.20) consists simply of M units like that in Figure 3-1.[#] In this figure, we have indicated a matched filter as the correlation operator. The output of this filter, the cross-correlation function of the received signal and the m^{th} message waveform as a function of time, is envelope detected, giving $\frac{1}{2}\,|g_m(t)|$. This in turn is fed into a nonlinear device with transfer characteristic $F_m(x)$. In general, $F_m(x)$ may be time-varying, for α and σ may be functions of τ; that is, the receiver may know a priori that, say, paths at one end of the interval (A, B) will be stronger, on the average, than paths at the other end. Practically speaking, however, we may wish to ignore this information, so that we may make $F_m(x)$ non-time-varying. The output of the nonlinear device is passed through an integrator, which completes the operations required by (3.20).

Figure 3-2 shows a typical nonlinear transfer characteristic, $F_m(x)$. It increases slowly for small values of x, and very rapidly for large values of x. The effect of such a nonlinear operation on the cross-correlation function envelope is greatly to accentuate the peaks of this envelope. That is, it weights heavily those parts of the cross-correlation envelope which are most probably due to the presence of a signal, while it suppresses, relatively, those

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[#] The possibility of a solution in the general form of Figure 3-1 was originally suggested to the author by R.M. Fano.

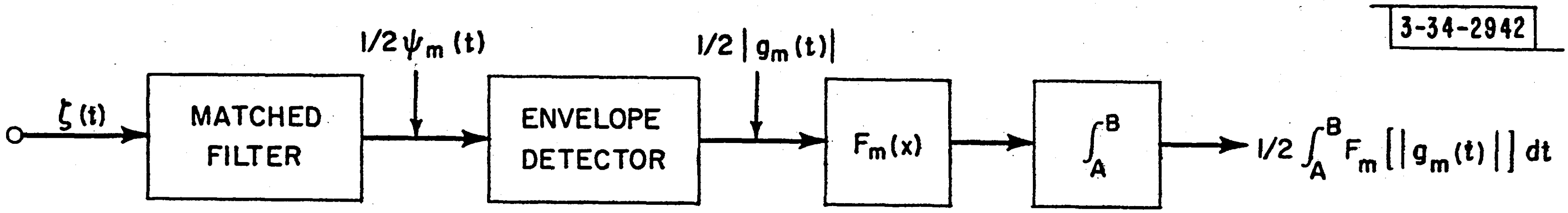
3-34-2942
ζ(t)
MATCHED FILTER
1/2 ψm (t)
ENVELOPE DETECTOR
1/2 | gm (t)|
Fm(x)
∫A B
1/2 ∫A B Fm [| gm(t) |] dt

Fig. 3-1

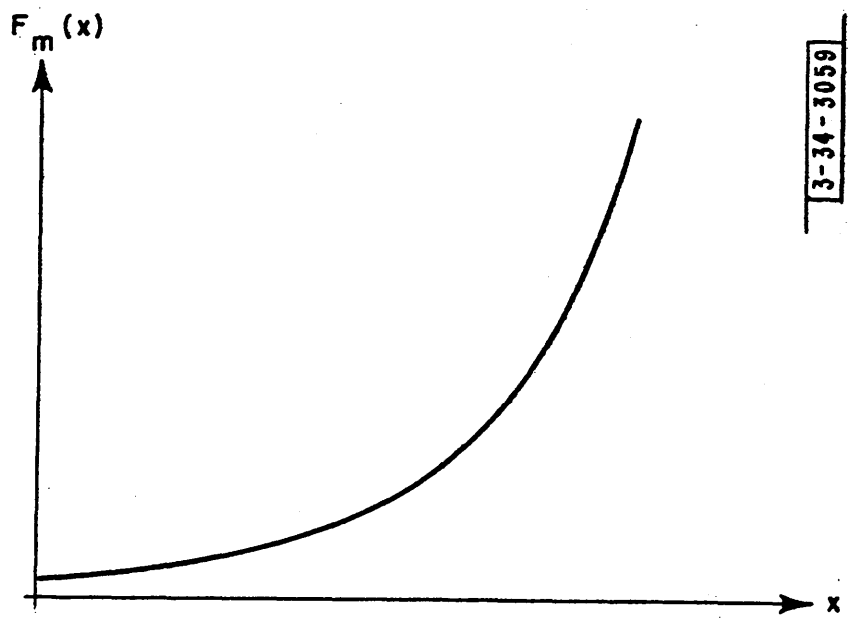
Fm (x)
3-34-3059
x

Fig. 3-2

parts which most probably originate from noise. Essentially, the nonlinear operator expresses the fact that the receiver grows rapidly more sure that a cross-correlation envelope peak is significant, the greater is the magnitude of that peak. The same sort of reasoning may be applied to the results in Sections A and B of this chapter; in these results, however, nonlinear operations are applied to sampled values of the cross-correlation functions and their envelopes, rather than to the complete functions.

CHAPTER IV: PROBABILITY OF ERROR: SPECIAL CASE

Let us assume that the receiver makes a guess at which message waveform was transmitted by choosing the one with the greatest a posteriori probability. Under this assumption, we shall evaluate the probabilities of error of the receivers containing the likelihood computers of equations (3.07) and (3.12). We shall do this for what is essentially the simplest non-trivial case, in which there are only two message waveforms (M=2) of equal energy ($E_1 = E_2 = E$) and equal a priori probability ($P_1 = P_2 = \frac{1}{2}$), and only a single path (L=1). The methods we shall use can be extended to more general cases, at least for the system containing the computer of equation (3.07), but only at the expense of rapidly increasing complexity and difficulty of computation.

For the case under consideration, because the a priori probabilities are equal, we see from equation (3.01) that the receiver may make its guess by choosing the message waveform corresponding to the greatest likelihood. Thus, we may write the total probability of error as

$$P_e = P_1 \Pr^{(1)}\left[\Lambda_2 > \Lambda_1\right] + P_2 \Pr^{(2)}\left[\Lambda_1 > \Lambda_2\right] \tag{4.01}$$

where the first term is evaluated on the hypothesis that $\xi_1(t)$ was transmitted, and the second, on the hypothesis that $\xi_2(t)$ was transmitted. It is easily seen from considering the symmetry of our special case that these two terms are equal, so that we may write:

$$P_e = \Pr^{(1)}\left[\Lambda_2 > \Lambda_1\right] \tag{4.02}$$

That is, we shall evaluate the probability that $\Lambda_2 > \Lambda_1$ on the hypothesis that $\xi_1(t)$ was transmitted.

In evaluating equation (4.02), we shall assume that the receiver has only *a priori* knowledge of the channel, and that this knowledge is correct. That is, we shall assume that the parameters, a, σ, and N_o, of equations (3.07) and (3.12), and δ of (3.07), are in fact equal to the corresponding parameters of the channel.*

A. τ Known: δ Known.

We shall first evaluate P_e for the receiver which has phase information, equation (3.07). We may assume, without loss of generality, that $\delta=0$. Then, from (3.07), $\Lambda_2 > \Lambda_1$ if

$$D \equiv \left[\frac{\sigma^2}{N_o^2} |g_1|^2 + \frac{2a}{N_o} \hat{g}_1\right] - \left[\frac{\sigma^2}{N_o^2} |g_2|^2 + \frac{2a}{N_o} \hat{g}_2\right] < 0 \qquad (4.03)$$

where we have, of course, dropped the path index, i. Noting that $|g_1|^2 = \hat{g}_1^2 + \tilde{g}_1^2$, and similarly for $|g_2|^2$, and writing

$$\begin{aligned} w_1 &= \frac{\sigma}{N_o} \hat{g}_1 + \frac{a}{\sigma} \\ w_2 &= \frac{\sigma}{N_o} \tilde{g}_1 \\ w_3 &= \frac{\sigma}{N_o} \hat{g}_2 + \frac{a}{\sigma} \\ w_4 &= \frac{\sigma}{N_o} \tilde{g}_2 \end{aligned} \qquad (4.04)$$

we may rewrite (4.03) as

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* Cf. footnote, page 10.

$$D = (w_1^2 + w_2^2) - (w_3^2 + w_4^2) < 0 \tag{4.05}$$

We shall call D the decision variable.

Now, from equation (3.05) we may write

$$\begin{aligned} g_1 &= \int z^*(t)\, x_1(t)\, dt \\ g_2 &= \int z^*(t)\, x_2(t)\, dt \end{aligned} \tag{4.06}$$

where we have set $\tau=0$ for convenience. By hypothesis, the modulation waveform of the transmitted signal is $x_1(t)$, so, from equations (2.06) and (2.08), we have for the modulation waveform of the received signal:

$$z(t) = a x_1(t) e^{-j\theta} + n(t) \tag{4.07}$$

where n(t) is the modulation waveform of the additive noise. Now, a and θ share a joint distribution as in equation (2.05); hence, the real and imaginary parts of the first term in (4.07) are Gaussianly distributed.[16] The real and imaginary parts of n(t) are also Gaussianly distributed.[15] Hence, the two parts of z(t) are Gaussianly distributed. It follows that the real and imaginary parts of g_1 and g_2, and hence w_1, w_2, w_3, and w_4, share a joint Gaussian distribution.[28] Thus, the decision variable is a quadratic form of dependent Gaussian variables.

It is easily shown[*] that the characteristic function of a quadratic form of Gaussian variables is given by

[*] See Appendix V. For the special case for which $\overline{W}$ is a zero matrix, this result has been given by Whittle.[31]

$$F_D(ju) \equiv \overline{e^{juD}} = \frac{1}{\left|I-2juMQ\right|^{1/2}} \exp\left[-\frac{1}{2}\overline{W}_t M^{-1}\left\{I-(I-2juMQ)^{-1}\right\}\overline{W}\right] \tag{4.08}$$

where I is the unit matrix, Q is the matrix of the quadratic form, M is the moment matrix of the variables, $\overline{W}$ is the (column) matrix of the means of the variables, "t".denotes "transpose of", and $|\ldots|$ denotes "determinant of". The probability-density distribution of D is given by the Fourier transform of F_D:

$$\mathrm{pr}\left[D\right] = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} F_D(s)\, e^{-sD}\, ds \tag{4.09}$$

Now, the probability of error is

$$P_e = \mathrm{Pr}\left[D<0\right] = \int_{-\infty}^{0} \mathrm{pr}\left[D\right] dD \tag{4.10}$$

Substituting (4.09) in (4.10), changing the order of integration, and integrating on D, we obtain

$$P_e = -\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{F_D(s)}{s}\, ds \tag{4.11}$$

The path of integration in (4.11) is taken to be indented to the left at all j-axis singulatities.

Let us now evaluate equations (4.08) and (4.11) for our special case. It is first useful to define the complex correlation coefficient of the two message waveforms:

$$\lambda = \frac{1}{2E}\int \xi_1^*(t)\, \xi_2(t)\, dt = \frac{1}{2E}\int x_1^*(t)\, x_2(t)\, dt \tag{4.12}$$

The real and imaginary parts of this, $\hat{\lambda}$ and $\tilde{\lambda}$, are, respectively, the values of the normalized physical cross-correlation function of the message waveforms at the origin, and at a displacement of $1/4f_o$. The magnitude, $|\lambda|$, is the value of the envelope of the normalized physical cross-correlation function at the origin. It is easily shown, using the Schwarz inequality, that $|\lambda| \leq 1$.

It may be shown[‡] that the moment matrix M, the elements of which are

$$m_{ij} = \overline{w_i w_j} - \overline{w}_i \overline{w}_j \tag{4.13}$$

is

$$M = \beta \begin{bmatrix} (\beta+1) & 0 & \hat{\lambda}(\beta+1) & \tilde{\lambda}(\beta+1) \\ 0 & (\beta+1) & -\tilde{\lambda}(\beta+1) & \hat{\lambda}(\beta+1) \\ \hat{\lambda}(\beta+1) & -\tilde{\lambda}(\beta+1) & \beta|\lambda|^2+1 & 0 \\ \tilde{\lambda}(\beta+1) & \hat{\lambda}(\beta+1) & 0 & \beta|\lambda|^2+1 \end{bmatrix} \tag{4.14}$$

It may also be shown[‡] that

$$\overline{W} = \begin{bmatrix} \gamma(\beta+1) \\ 0 \\ \gamma(\beta\hat{\lambda}+1) \\ \gamma\beta\tilde{\lambda} \end{bmatrix} \tag{4.15}$$

where $\gamma = \frac{a}{\sigma}$. From (4.05) it is seen that the matrix of the quadratic form is:

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‡ See Appendix VI for outline of method.

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{4.16}$$

Substituting equations (4.14) through (4.16) in (4.08), we obtain, after considerable matrix algebra[#]:

$$F_D(s) = \frac{\exp\left[\dfrac{k_1 s(1+k_2 s)}{1 - k_3 s(1+k_2 s)}\right]}{1 - k_3 s(1+k_2 s)} \tag{4.17}$$

where we have written

$$\begin{aligned} k_1 &= \beta\gamma^2\left[\beta(1-|\lambda|^2) + 2(1-\hat{\lambda})\right] \\ k_2 &= 2(\beta+1) \\ k_3 &= 2\beta^2(1-|\lambda|^2) \end{aligned} \tag{4.18}$$

The integral obtained by substituting equation (4.17) into (4.11) apparently cannot be evaluated in closed form except in the special cases $\sigma = 0$ and $\alpha = 0$. In the former case,[##]

$$P_e = \frac{1}{2}\left[1 - \mathrm{erf}\left\{\sqrt{\frac{(1-\hat{\lambda})\alpha^2 E}{2N_o}}\right\}\right] \tag{4.19}$$

where

See Appendix VI for outline of method.

See Appendix VII.

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz \tag{4.20}$$

This, as should be expected, is the probability of error for a simple correlation detector operating in the face of white, Gaussian noise,[34] since in this case the path has no random component.

For $\alpha = 0$,*

$$P_e = \frac{r_1}{r_1 - r_2} \tag{4.21}$$

where r_1 and r_2 are the roots of

$$k_2 y^2 + y - \frac{1}{k_3} = 0 \tag{4.22}$$

r_1 is the positive root.

In the general case ($\alpha \neq 0$, $\sigma \neq 0$), we may put the probability of error in a form which is more convenient for numerical evaluation. This is*:

$$P_e = k_6 \int_0^{\pi/2} \frac{e^{-k_7 \sin^2\theta}}{1 + k_4^2 \tan^2\theta}\, d\theta \tag{4.23}$$

where

$$\begin{aligned} k_4 &= \sqrt{1 + (4k_2/k_3)} \\ k_5 &= (k_1/k_3) \\ k_6 &= \left[(k_4^2 - 1)/\pi k_4\right] \exp\left[-k_5/k_4^2\right] \\ k_7 &= \left[(k_4^2 - 1)/k_4\right] k_5 \end{aligned} \tag{4.24}$$

---

* See Appendix VII.

It is to be noted that the only characteristics of the message waveforms on which the probability of error depends are their energy and their complex correlation coefficient. (Note especially the dependence on the quadrature component of correlation, $\tilde{\lambda}$, as well as the in-phase component, $\hat{\lambda}$; this dependence on $\tilde{\lambda}$ derives from the phase instability of the path.) For any given set of channel parameters, α, σ, N_o, there is a value of λ which minimizes P_e; this value, λ_{opt}, indicates the relationship between the two message waveforms for optimum system performance. We shall evaluate λ_{opt} as a function of the channel parameters in the next chapter. Substituting the values of λ_{opt} obtained there into equation (4.23), we may compute values of P_e for a system with optimally-related message-waveforms. These are presented as the unbroken-line curves of Figure 4-1. The family parameter of these curves may be written as

$$\beta(2+\gamma^2) = \frac{2(2\sigma^2+\alpha^2)E}{N_o} \tag{4.25}$$

Now, it is easily shown[16] that the mean-square value of a, the path strength, is given by

$$\overline{a^2} = 2\sigma^2 + \alpha^2 \tag{4.26}$$

Hence, the family parameter of the curves of Figure 4-1 is the ratio of the average received signal energy to the noise power-density. In progressing along any one curve, this ratio is held constant, while the ratio, $2/\gamma^2 = 2\sigma^2/\alpha^2$, of the average energy received via the random path-component to that received via the fixed path-component is varied. The effect of an increase in the average strength of the random component, at the expense of a corresponding decrease in the strength of the fixed component, shows up as an increase in probability of error. It is interesting to note how rapidly the performance of the system deteriorates as the path changes from one which is completely fixed ($2/\gamma^2 = 0$) to one with the same average total strength but with equal fixed and random components ($2/\gamma^2 = 1$).

The effect of path disturbances on the performance of the system is strikingly illustrated by considering P_e for the large signal-to-noise ratio in the two limiting cases, $\sigma = 0$ and $\alpha = 0$. In the former case, letting $\lambda = \lambda_{opt} = -1$ (see Figure 5-1), we have from equation (4.19)[35]:

$$P_e \xrightarrow[\frac{E}{N_o} \to \infty]{} \frac{\exp\left[-\frac{\alpha^2 E}{N_o}\right]}{2\sqrt{\frac{\pi\alpha^2 E}{N_o}}} \tag{4.27}$$

In the latter case, letting $\lambda = \lambda_{opt} = 0$, we have from equations (4.21) and (4.22)

$$P_e \xrightarrow[\frac{E}{N_o} \to \infty]{} \frac{1}{\frac{2\sigma^2 E}{N_o}} \tag{4.28}$$

As would be expected, the probability of error approaches zero with increasing signal-to-noise ratio very much more slowly when the channel is perturbed by severe path disturbances as well as by additive noise, than when it is perturbed solely by additive noise.[‡] It is interesting to note, however, that the system is capable of operating without error in the absence of noise <u>even when the channel has random path disturbances</u>.

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‡ To estimate the rate of approach of P_e to zero for large E/N_o for other values of $2/\gamma^2$ than the ones considered above (0 and ∞), see Figure 4-1.

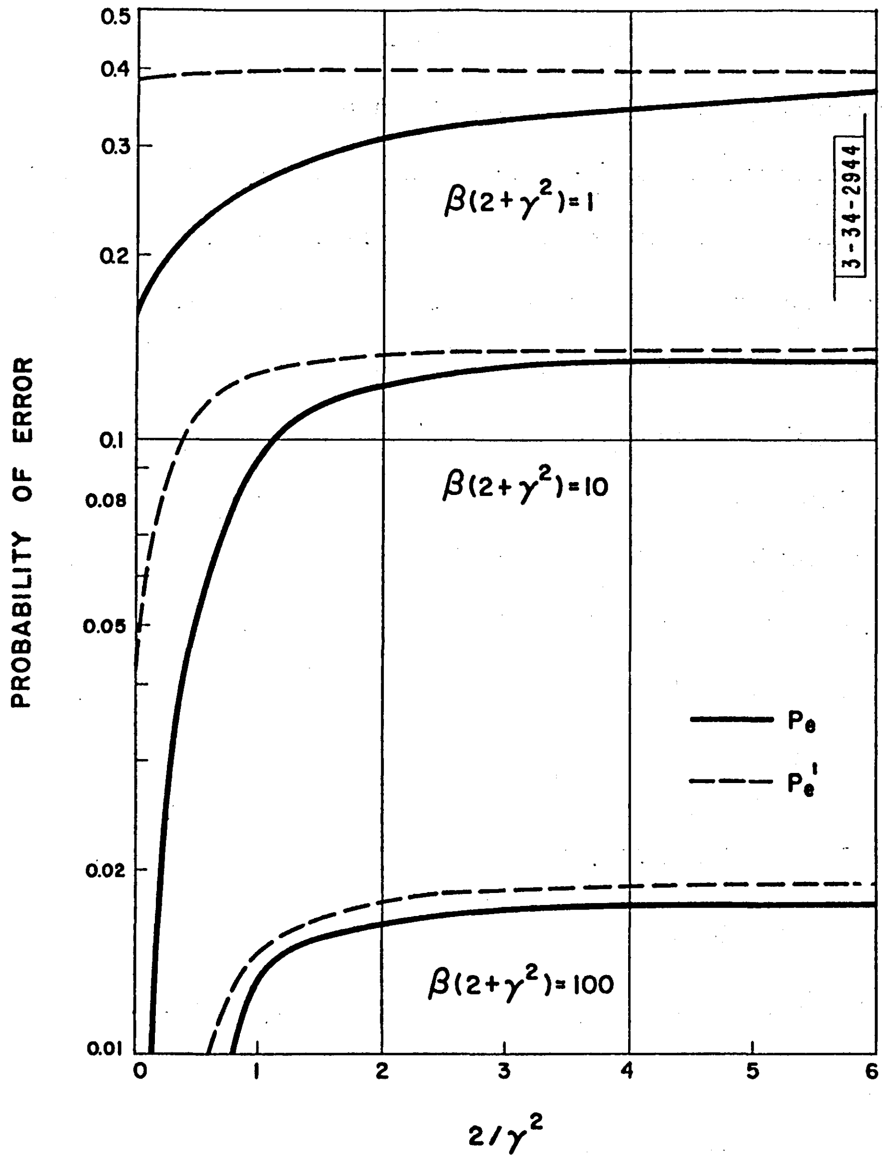
3-34-2944
PROBABILITY OF ERROR
0.5
0.4
0.3
0.2
0.1
0.08
0.05
0.02
0.01
$\beta(2+\gamma^2)=1$
$\beta(2+\gamma^2)=10$
$\beta(2+\gamma^2)=100$
P_e
P_e'
0
1
2
3
4
5
6
$2/\gamma^2$

Fig.4-1

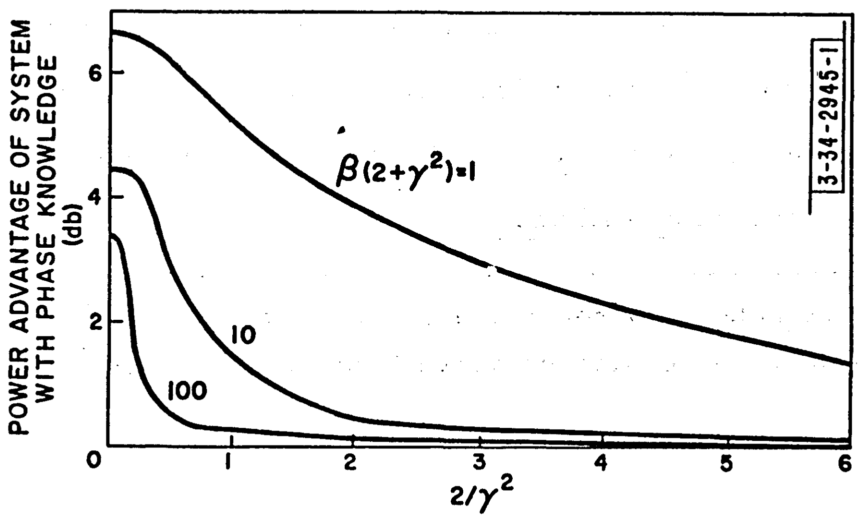
3-34-2945-1
POWER ADVANTAGE OF SYSTEM WITH PHASE KNOWLEDGE (db)
6
4
2
0
$\beta(2+\gamma^2)=1$
10
100
1
2
3
4
5
6
$2/\gamma^2$

Fig.4-2

B. τ Known: δ Unknown.

The probability of error for a receiver with the likelihood computer of equation (3.12) is much simpler to determine. Since Λ'_m is a monotonic function of $|g_m|$, it is evident that equation (4.02) reduces to

$$P'_e = Pr^{(1)}\left[\,|g_2| > |g_1|\,\right] \tag{4.29}$$

where we have primed P_e so as to distinguish it from the probability of error calculated in the last section. In evaluating equation (4.29) we shall assume that $\lambda = 0$; this is the optimum correlation coefficient for a system which has no phase information.*

Let us first assume that the path strength, a, is known to the receiver. It is easily shown** that the probability of error, conditional upon knowing a, is

$$P'_e(a) = \frac{1}{2}\exp\left[-\frac{a^2E}{2N_o}\right] \tag{4.30}$$

The total probability of error is then $P'_e(a)$, averaged over a:

$$P'_e = \int_0^\infty pr[a]\, P'_e(a)\, da \tag{4.31}$$

pr[a] may be obtained from equation (2.05) by integrating it over θ. Using this result in (4.31), we immediately obtain***

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* See Section V.B, page 62.

** See Appendix VIII. For the general case of this result, for $\lambda \neq 0$, see equation (37) of reference 34.

*** See Appendix VIII.

$$P'_e = \frac{1}{\beta+2} \exp\left[- \frac{\beta\gamma^2}{2(\beta+2)} \right] \tag{4.32}$$

Equation (4.32) is plotted as the broken-line curves of Figure 4-1, along with the curves of equation (4.23). As we might expect, the advantage of the receiver which has knowledge of the mean path phase-shift, δ, is not great, except in the region of high path phase stability (small $2/\gamma^2$). This advantage is expressed in terms of power in Figure 4-2, in which the ordinate gives the increase in power which would be required in the system without phase knowledge in order to reduce its probability of error to that of the system which has phase knowledge. The family-parameter values of Figure 4-2 (average received-signal-energy to noise-power-density ratio) are those for the latter system.

Equation (4.32) again illustrates the change, between the limiting cases $\sigma = 0$ and $\alpha = 0$, of the rate at which the probability of error approaches zero with increasing signal-to-noise ratio. For $\sigma = 0$ (and, hence, $\beta = 0$), the approach is exponential:

$$P'_e = \frac{1}{2} \exp\left[- \frac{\alpha^2 E}{2N_o} \right] \tag{4.33}$$

while for $\alpha = 0$ (and, hence, $\gamma = 0$), the approach is inverse, as in (4.28). For other values of $\gamma = \alpha/\sigma$, the approach to zero is roughly exponential for small values of E/N_o (small β), and roughly inverse for large values of E/N_o (large β); the point at which the change in behavior occurs depends on the values of the path-strength parameters, α and σ.

CHAPTER V: THE MESSAGE WAVEFORMS: SPECIAL CASE

In this chapter we shall turn our attention to the second question we asked in Section B of Chapter II: "What are 'suitable' message waveforms for use with the channel under consideration?" That is, we wish to determine the set of message waveforms which optimizes, in some sense, the performance of the system. As an optimization criterion, we shall choose the minimization of probability of error, for this is the yardstick of system performance which is of primary importance to the communications engineer.

As in the last chapter, we shall only consider receivers which make a decision by choosing the a posteriori most probable message waveform; and these receivers only for the simple special case in which there are but two equi-energy, equiprobable message waveforms, and only one path. We shall postpone comments on the more general, multi-message-waveform, multipath case until the concluding chapter.

A. τ Known: δ Known.

As we have noted in the last chapter (see equations (4.18), (4.23), and (4.24)), the only characteristics of the two message waveforms on which the probability of error depends in the special case under consideration are their (common) energy, E, and their normalized complex cross-correlation coefficient, λ (see equation (4.12)). Thus, for a given energy, specification of an optimal set of message waveforms consists merely in specifying that value of cross-correlation coefficient which minimizes the probability of error. That is, we require the value of $\lambda = \hat{\lambda} + j\tilde{\lambda}$ for which the conditions

$$\frac{\partial P_e}{\partial \hat{\lambda}} = 0 \tag{5.01a}$$

$$\frac{\partial P_e}{\partial \tilde{\lambda}} = 0 \tag{5.01b}$$

are simultaneously satisfied, and for which

$$\frac{\partial^2 P_e}{\partial \hat{\lambda}^2} > 0 \tag{5.02a}$$

$$\frac{\partial^2 P_e}{\partial \tilde{\lambda}^2} > 0 \tag{5.02b}$$

In addition, if we obtain more than one solution for (5.01) and (5.02), we require the one which yields the absolute minimum of P_e.

Now, it is easily seen from equations (4.18) that P_e depends on the quadrature component, $\tilde{\lambda}$, of the cross-correlation coefficient only in the square. Therefore, we may write (5.01b) as

$$\frac{\partial P_e}{\partial \tilde{\lambda}} = \frac{\partial P_e}{\partial (\tilde{\lambda}^2)} \cdot \frac{\partial (\tilde{\lambda}^2)}{\partial \tilde{\lambda}} = 2\tilde{\lambda} \frac{\partial P_e}{\partial (\tilde{\lambda}^2)} = 0 \tag{5.03}$$

We thus have immediately a possible solution for $\tilde{\lambda}$:

$$\tilde{\lambda} = 0 \tag{5.04}$$

It is difficult to show precisely that this solution is indeed the one which we require, i.e., that it satisfies (5.02b), and yields, in conjunction with some solution of (5.01a) and (5.02a), the absolute minimum of P_e. We may, however, construct a plausible, but not rigorous, argument that this is so.

The probability of error is the probability, on the hypothesis that $\xi_1(t)$ was sent, that the decision variable,*

$$D = \frac{\sigma^2}{N_o^2} \left(|g_1|^2 - |g_2|^2 \right) + \frac{2a}{N_o} \left(\hat{g}_1 - \hat{g}_2 \right) \tag{5.05}$$

is less than the decision threshold, zero. Now, roughly, we should expect that

*Cf. equation (4.03).

the larger D is on the average, the smaller will be this probability that D is less than the threshold. Thus, the problem of minimization of P_e with respect to λ would seem to be related to that of maximizing $\overline{D}$ with respect to λ. Of course, we should not expect in general that there will be a rigid relationship between the two problems --- one may envisage special situations in which a change in λ, while increasing the average value of D, may so increase, say, the variance of D, that P_e would also increase, instead of decreasing. But we are interested at this point not in exact solutions, but in trends, and for this purpose an investigation of the problem of maximization of $\overline{D}$ with respect to λ would seem to be justified.

The average value of D may be obtained easily from the characteristic function, $F_D(s)$, by use of the relationship(36)

$$\overline{D} = \left. \frac{d F_D(s)}{ds} \right|_{s=o} \tag{5.06}$$

Applying (5.06) to (4.17), we obtain

$$\overline{D} = k_1 + k_3 = \beta^2 (2+\gamma^2)(1 - |\lambda|^2) + 2\beta\gamma^2(1 - \hat{\lambda}) \tag{5.07}$$

Now, remembering that $|\lambda|^2 = \hat{\lambda}^2 + \tilde{\lambda}^2$, we see that setting $\tilde{\lambda}$ equal to anything other than zero will cause a decrease in the first term of (5.07) while not affecting the second term; that is, $\overline{D}$ is maximum with respect to $\tilde{\lambda}$ for $\tilde{\lambda} = 0$. We may perhaps obtain a better understanding of this result by using the fact that the first and second terms of (5.07) are, respectively, the averages of the corresponding terms in (5.05):‡ on this basis we see that setting $\tilde{\lambda}$ equal to anything other than zero decreases, on the average, the difference of the squares of the correlation envelopes in (5.05) while leaving the difference of the correlations themselves un-

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‡This may be shown with the help of equations (A6-6), (A6-7), (A6-9), (A6-12), and (A6-13) of Appendix VI.

affected, again on the average. It is therefore plausible that the solution $\tilde{\lambda} = 0$ corresponds to the minimum probability of error for any given value of $\hat{\lambda}$.

On the other hand, we may guess from (5.07) that the optimum value of $\hat{\lambda}$ lies somewhere between 0 and -1. For, although the first term decreases as $\hat{\lambda}$ progresses from 0 to -1, the second term increases, and $\overline{D}$ will be maximum somewhere between these extremes, depending on the values of β and γ. We should, in fact, expect from (5.07), as well as from physical reasoning, that the following results will obtain:

(1) for a path with no random component, i.e., with stable phase ($\sigma = 0$, $\gamma = \infty$, $\beta\gamma^2$ finite): $\hat{\lambda}_{opt} = -1$

(2) for a path with no fixed component, i.e., with completely-random phase ($\alpha = 0$, $\gamma = 0$): $\hat{\lambda}_{opt} = 0$

(3) for a very noisy channel ($N_o \to \infty$, $\beta \to 0$): $\hat{\lambda}_{opt} \to -1$

(4) for a noiseless channel ($N_o = 0$, $\beta = \infty$): $\hat{\lambda}_{opt} = 0$

For the first and third of these conditions, the decision variable, equation (5.05), contains only the correlations, $\hat{g}_1$ and $\hat{g}_2$; their difference is maximized on the average by making the two message waveforms antipodal, that is, by setting $\hat{\lambda} = -1$. For the second and fourth conditions, on the other hand, the decision variable contains only the correlation envelopes, $|g_1|$ and $|g_2|$; their difference is maximized on the average by making the two message waveforms orthogonal, that is, by setting $\hat{\lambda} = 0$.

The results predicted above may indeed be shown to be true by minimizing the probability of error with respect to $\hat{\lambda}$, with $\tilde{\lambda} = \tilde{\lambda}_{opt} = 0$. The optimum value of $\hat{\lambda}$ thus found is shown in Figure 5-1 as a function of the channel parameters, $\frac{2}{\gamma^2}$ and $\beta\gamma^2$. The first of these parameters, it will be recalled, is the ratio of the average strength of the random path-component to the strength of the fixed path-component.

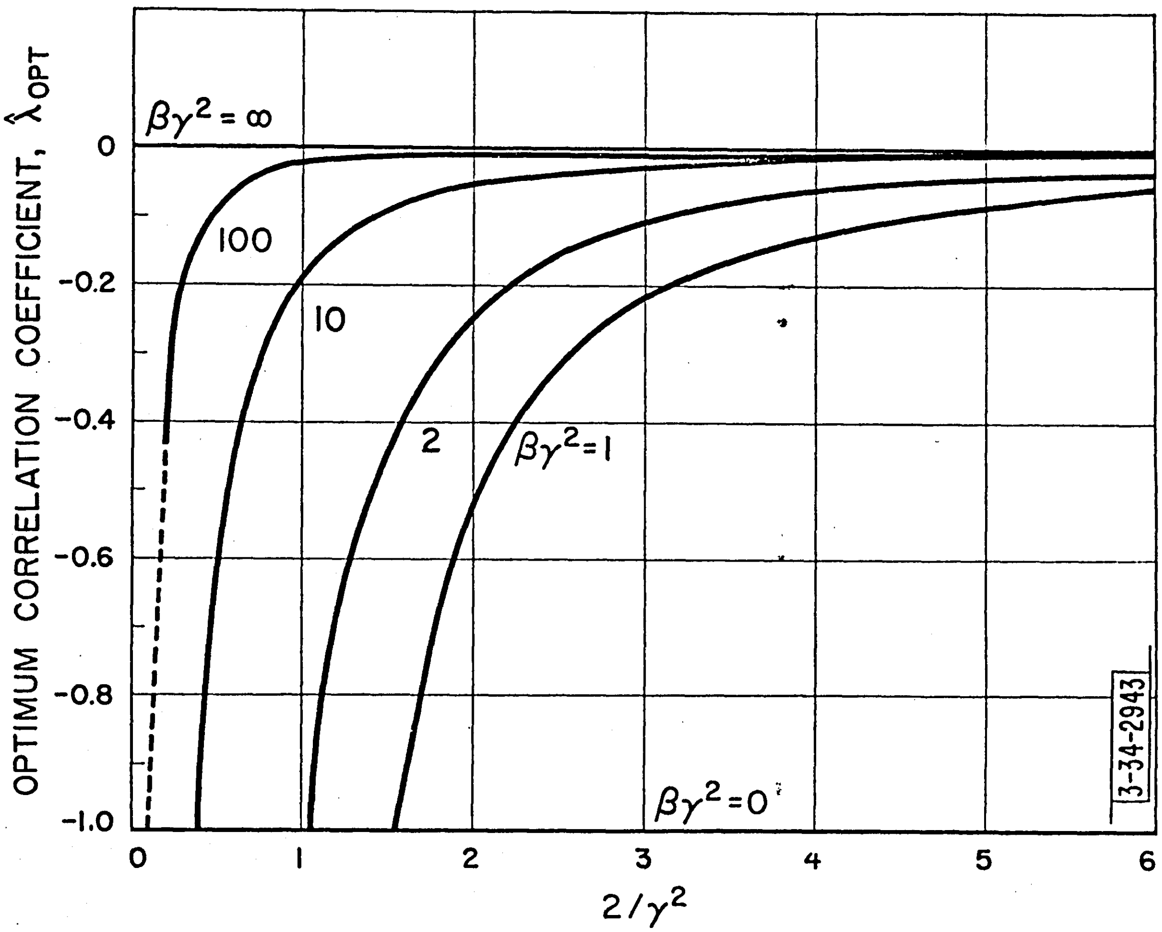
OPTIMUM CORRELATION COEFFICIENT, $\hat{\lambda}_{OPT}$
$\beta\gamma^2 = \infty$
100
10
2
$\beta\gamma^2 = 1$
$\beta\gamma^2 = 0$
0
-0.2
-0.4
-0.6
-0.8
-1.0
0
1
2
3
4
5
6
$2/\gamma^2$
3-34-2943

Fig.5-1

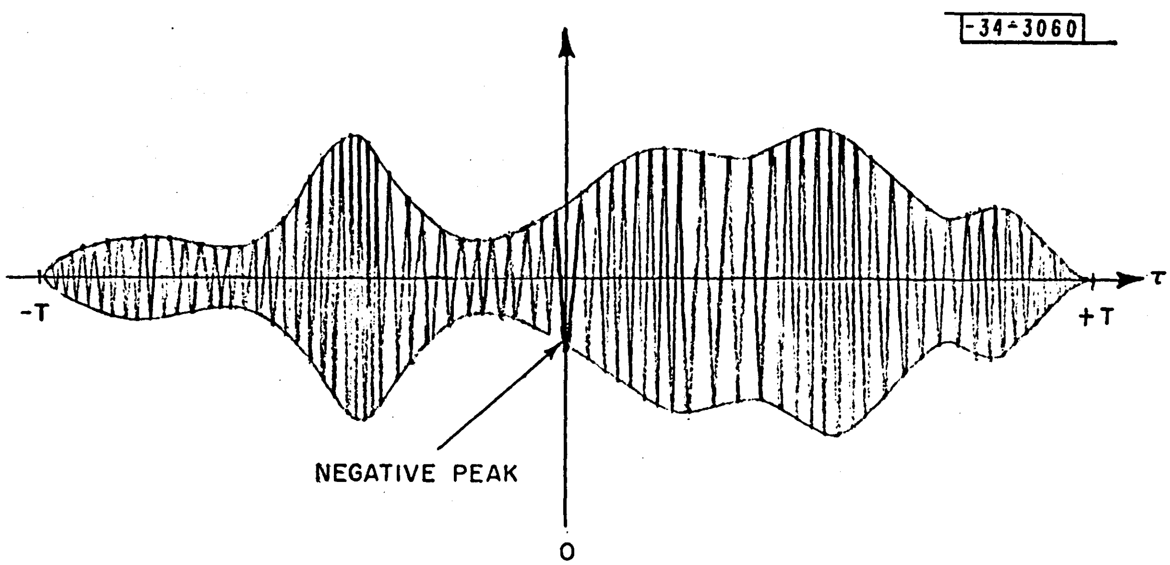
-34-3060
τ
-T
+T
NEGATIVE PEAK
0

Fig.5-2

The second, $\beta\gamma^2 = \frac{2\sigma^2 E}{N_o}$, is the ratio of the average energy received via the random path-component to the noise power-density. Figure 5-1 was obtained by numerical minimization of equation (4.23).

Physically, the above results may be illustrated as in Figure 5-2, which shows a possible physical cross-correlation function of the two message waveforms.[+] The fact that $\tilde{\lambda}_{opt} = 0$ indicates that this function passes through zero one-quarter of a carrier period from the origin, and hence through an r.f. peak at the origin. The fact that $\hat{\lambda}_{opt}$ is negative indicates that this r.f. peak is negative. For channel conditions (2) and (4) above, the correlation function is zero at the origin as well as at displacements of one-quarter period from the origin; hence the correlation function envelope is also zero at the origin.

We have until now assumed for convenience that the modulation delay, τ, and the mean fixed-component phase-shift, δ, of the path are zero. If this is not so, the foregoing results still apply; the receiver restores the problem to that we have just considered by introducing an identical delay, τ, and phase-shift, δ, into the stored message waveforms (cf. equations (3.05) and (3.07)).

B. τ Known: δ Unknown.

In this case, when δ is unknown, the receiver makes its decision according as the difference, $|g_1| - |g_2|$, is greater than or less than zero (see Section IV.B, page 55). In the light of the discussion in the last section, it is clear that this difference is maximized on the average by setting $\lambda = 0$, so it is plausible (as well as satisfying to the intuition) that this value of λ will minimize the probability of error. Thus, optimally, the envelope of the correlation function in Figure 5-2 goes to zero at the origin.

[+]Of course, both the carrier period and the amount of phase modulation in this Figure are exaggerated for the sake of clarity.

CHAPTER VI: CONCLUDING REMARKS.

In answering the questions we asked in Section I. B, we have made the following restrictions and assumptions:

(1) The additive noise is stationary, white, Gaussian, and statistically independent of the multipath medium (see page 10);

(2) The paths of the multipath medium are statistically independent of one another (see equation (2.07), page 15, and the discussion preceding equation (3.17), page 40);

(3) The paths are resolvable, i.e., their modulation delays satisfy condition (3.06) (see page 35);

(4) The medium is non-time-varying, at least for the duration of a message waveform (see page 32);

(5) The system performs on a per-waveform basis (see page 32).

In addition, in evaluating the performance of the various systems derived in Chapter III, and in determining optimal relations among the message waveforms, we have assumed that

(6) The receiver's knowledge of the multipath medium is _a priori_ knowledge, and correctly represents the medium (see page 46);

and we have restricted ourselves to consideration of the special case in which there are only

(7) Two equi-energy, equiprobable message waveforms, and one path, (see page 45).

Avenues for future work are immediately suggested by these assumptions and restrictions; that is, we may ask for the solutions to the problems we have investigated, but with any or all of the above assumptions and restrictions removed. We shall comment briefly here on these various possible extensions of the present work.

The assumption that the additive noise is Gaussian would seem to be realistic enough in most practical cases, and extension of the analysis to non-Gaussian noises would not at present seem to be worth the concomitant severe complication of the mathematics. Similarly, the assumption of the independence of the noise and the multipath medium is in most cases justified. For, even if some or all of the noise arrives at the receiver from distant noise sources by way of the multipath medium, it would almost certainly traverse a different region of the medium than that traversed by the signal, and these two regions would in general be statistically independent. It is of course this independence of the noise and the signal-utilized region of the multipath medium which we have in mind in the last part of assumption (1).

A clue to the extension of our probability-computer results to the non-white noise case may be taken from the equivalent analysis for a channel which is disturbed by noise only [37, 38, 9]. In this case it has been shown that the received signal is correlated not with the stored message-waveforms directly, but with the stored message-waveforms after their modification by linear filtering. In particular, for $T \gg \frac{1}{W}$, the modifying filters have a (common) transfer function which is equal to the reciprocal of the noise power-density spectrum, N(f). Intuitively, one would expect the same results to apply in our case.

We may easily extend our previous work for the case of stationary noise to the quasi-stationary case, in which it is assumed that the noise is stationary at least for the duration of a message waveform. For this latter case, our previous results still apply, but now the parameter N_o, or more generally, the noise spectrum N(f), will vary from message waveform to message waveform, along with the other channel parameters, a_i, σ_i, δ_i, τ_i. The more general non-stationary case is of course considerably more complicated, and of dubious practical interest.

The assumption of independence of paths is most probably a realistic one, for different paths generally pass through independent regions of the multipath medium. One exception to this, which is of possible interest, is the case in which some or all paths have a common random attenuation (e.g., in the case of an ionospheric medium, where some paths may pass through a common region of an absorbing layer; say, the "D" layer). This case would probably lend itself easily to analysis. Besides this special case, however, it is doubtful whether a generalization to the dependent-path case would be worth the labor.

As we have noted in Chapter II (see page 23), assumption (3) may be almost automatically satisfied, for if two paths are completely unresolvable, they may be considered a priori as a single path. To be sure, there is a no-man's-land in which two paths neither can be considered to be completely unresolvable nor can strictly satisfy (3.06); this is the small region in which the difference of modulation delays is approximately $\frac{1}{W}$. In most cases, however, one would expect few delay differences to fall into this category, and one would feel that our results would apply with no great error if we established a sharp line of demarcation between "unresolvable" and "resolvable" delay differences, say, $\frac{1}{W}$; we would then arbitrarily consider as a single path all paths whose modulation delays differ by less than this amount, and consider as completely resolvable all paths whose delays differ by more than this amount. The only important case in which this technique would be suspect, and in which a more general analysis which does not make use of assumption (3) would be in order, is the case where there is a continuum of paths; for, in this case, large numbers of delay differences would fall into the no-man's-land category, that is, near the line of demarcation. The more general analysis of the probability computer has in fact been performed by Price for the special case in which the paths have no fixed components[8a].

Assumptions (4) and (5) were made in order to avail ourselves of the simplicity of analysis which evolves from use of only the first-order joint distribution of the multipath characteristics. The obvious generalization is to eliminate the necessity for these assumptions by taking into account higher-order distributions.‡ Again, Price has done this for the special case in which there are no fixed path-components[8a]. Extension of his excellent work to the case in which fixed path-components are present would be of great interest.

Elimination of assumption (6) would allow the determination of the effect on system performance of errors in the receiver's a priori knowledge of the multipath medium. It would also allow the evaluation of the performance of the system when the receiver's knowledge of the medium is based on measurements, and would enable a comparison to be made between system performances with and without the benefit of measurements.

Extension of the analyses of Chapters IV and V to more general cases than that of assumption (7) would be of great interest, but also, unfortunately, of great difficulty. A simple extension of the probability-of-error analysis of Chapter IV which would give an insight into the relative effects on system performance of the different paths of the medium would be the evaluation of P_e for the case in which there are two paths of equal total energy: it would be interesting to determine how rapidly P_e increases as one path changes from a purely-fixed to a purely-random one, while the other path remains, say, purely fixed.

In regard to the extension of the work in Chapter V to the general M-waveform, L-path case, it seems clear that the minimization of probability of error would be with respect to $\frac{M(M-1)}{2} \cdot \left[L^2 - (L-1)\right]$ complex variables instead of just one as in Chapter V. These are the values, at the $L^2 - (L-1)$ distinct delays, $\tau_i - \tau_k$ (i, k = 1, ..., L), of the $\frac{M(M-1)}{2}$ complex cross-correlation function modulation-waveforms,‡‡ $\int x_p^*(t-\tau_i)\, x_m(t-\tau_k)\, dt$ $(p \neq m)$. When the

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‡ See footnote, page 32.

‡‡ We say M(M-1)/2 modulation waveforms, instead of M(M-1), since these waveforms come in complex-conjugate pairs.

fixed-component phase-shifts, (δ_i), are unknown, one would expect that minimum probability of error would occur when all of these variables are zero, that is, when the envelopes of all the message-waveform cross-correlations go to zero at all delay differences, $\tau_i - \tau_k$.

When the paths are not completely resolvable, and a more general probability computer which is not based on assumption (3) is employed, an additional $M \frac{L(L-1)}{2}$ variables are added to the minimization problem: the values, at the $\frac{L(L-1)}{2}$ modulation delays,⁺ $\tau_i - \tau_k$ $(\tau_i > \tau_k)$, of the M complex auto-correlation function modulation-waveforms, $\int x_m^*(t-\tau_i)\, x_m(t-\tau_k)\, dt$. These are assumed to be zero in the resolvable-path case. Finally, if the waveform energies are not considered to be given originally, an additional M energy variables $(\int |x_m(t)|^2\, dt)$ enter into the problem.

Another problem of interest is the evaluation of the probability of error for the case where the τ_i's are unknown (Section III. C). Minimization of probability of error in this case would involve specification of the complete $\frac{M^2+M}{2}$ auto- and cross-correlation functions of the M message waveforms, or, more precisely, their envelopes. Ideally, one would like to have all of the cross-correlation envelopes vanish for all values of their arguments; but there are probably physical-realizability constraints which prevent this, and these would have to be taken into account in the minimization problem.

In Chapter I we split our analysis problem up into nearly-independent parts, but noted that this was for convenience of analysis, and that, more strictly, the problem should be considered as an integrated whole. Perhaps the first step that should be taken in the direction of integration is that of considering the channel-measurement and probability-computer problems together. That there should be some intimate connection between the two problems would seem to

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⁺ We say L(L-1)/2 values, instead of L(L-1), since the complex auto-correlation functions are (Hermitian) symmetric about the origin.

be indicated by comparison of the results of Section II. B. 1 and of Chapter III; these show that precisely the same operations of correlation and sampling are used for channel measurement and for probability computation. One may ask such questions as: Should the message waveforms themselves first be considered by the receiver as channel-sounding signals, and then the hypothetical channel-characteristic distributions so obtained used for probability computation?[+] Or, perhaps, should known channel-sounding signals and message waveforms be sent alternately, and if so, what proportion of the transmission time should be allotted to each? Is there some form of measurement already implicit in the probability computers of Chapter III, as some of Price's work[(8a)] would suggest?

Besides the questions relating to integration of the problems we have investigated separately, there is a whole group of questions relating to problems we have not even considered. How, for example, can we supply the transmitter and receiver with identical information about the channel, as we have assumed to be the case? By transmission of the transmitter-to-receiver sounding data, which is available at the receiver, back to the transmitter? (How will channel disturbances affect this transmission?) Or, perhaps, by establishing a separate receiver-to-transmitter sounding link ? (Is the channel reciprocal?)

We may put the gist of this chapter in just a few words: there are still many, many questions which must be answered before we may say that we have a thorough understanding of the problem of multipath communication.

+ Cf. footnote, page 33.

APPENDIX I: THE COMPLEX CORRELATION FUNCTION

The real part of equation (1.04) is

$$\hat{\psi}(\tau) = \int \left[\hat{\xi}(t)\ \hat{\eta}(t-\tau) + \tilde{\xi}(t)\ \tilde{\eta}(t-\tau) \right] dt \qquad \text{(A1-1)}$$

in which

$$\hat{\xi}(t) = \hat{x}(t)\ \cos 2\pi f_o t - \tilde{x}(t)\ \sin 2\pi f_o t \qquad \text{(A1-2a)}$$

$$\tilde{\xi}(t) = \hat{x}(t)\ \sin 2\pi f_o t + \tilde{x}(t)\ \cos 2\pi f_o t \qquad \text{(A1-2b)}$$

and similar expressions hold for $\hat{\eta}(t)$ and $\tilde{\eta}(t)$. From (A1-2b)

$$\tilde{\xi}\left(t + \frac{1}{4f_o}\right) = \hat{x}\left(t + \frac{1}{4f_o}\right) \cos 2\pi f_o t - \tilde{x}\left(t + \frac{1}{4f_o}\right) \sin 2\pi f_o t \qquad \text{(A1-3)}$$

Now, it has been assumed that $\xi(t)$ represents a narrow-band waveform. This implies that x(t) remains essentially constant over many cycles of carrier, so that we may write $x\left(t + \frac{1}{4f_o}\right) = x(t)$, and hence

$$\tilde{\xi}\left(t + \frac{1}{4f_o}\right) = \hat{\xi}(t) \qquad \text{(A1-4)}$$

Similarly,

$$\tilde{\eta}\left(t + \frac{1}{4f_o}\right) = \hat{\eta}(t) \qquad \text{(A1-5)}$$

The cross-correlation function of the physical waveforms, $\hat{\xi}(t)$ and $\hat{\eta}(t)$, is

$$\varphi(\tau) \equiv \int \hat{\xi}(t)\ \hat{\eta}(t-\tau)dt \tag{A1-6}$$

But from (A1-4) and (A1-5) we have also

$$\varphi(\tau) = \int \tilde{\xi}(t)\ \tilde{\eta}(t-\tau)dt \tag{A1-7}$$

from which it follows that

$$\hat{\psi}(\tau) = 2\,\varphi(\tau) \tag{A1-8}$$

which was to be proved.

By inserting (A1-2a) and a similar expression for $\hat{\eta}(t)$ in (A1-6), one obtains, after some trigonometric manipulations and the use of the narrow-band assumption to eliminate integrals of double-frequency terms:

$$\varphi(\tau) = A(\tau)\cos 2\pi f_o\tau + B(\tau)\sin 2\pi f_o\tau \tag{A1-9}$$

where

$$A(\tau) = \frac{1}{2}\int\left[\hat{x}(t)\hat{y}(t-\tau) + \tilde{x}(t)\tilde{y}(t-\tau)\right]dt \tag{A1-10a}$$

$$B(\tau) = -\frac{1}{2}\int\left[\hat{x}(t)\tilde{y}(t-\tau) - \tilde{x}(t)\hat{y}(t-\tau)\right]dt \tag{A1-10b}$$

The envelope of $\varphi(t)$ is just $\sqrt{A^2+B^2}$. But the magnitude of $\psi(t)$, equation (1.05), is just twice this, or $2\sqrt{A^2+B^2}$, which was to be shown.

APPENDIX II: A POSTERIORI DISTRIBUTION OF MULTIPATH CHARACTERISTICS

Using (2.02), (2.15), and (2.07), (2.14) may be written as

$$\mathrm{pr}\left[(a_i),(\theta_i)/(\tau_i),\zeta,\xi\right] = K\left[\prod_i a_i\right] \exp\left[-\frac{1}{2N_o}\int_o^{T'}\left|\zeta(t)-\eta(t)\right|^2 dt - \sum_i \frac{a_i^2-2\alpha_i a_i \cos(\theta_i-\delta_i)}{2\sigma_i^2}\right] \quad \text{(A2-1)}$$

where the factor

$$K = \frac{1}{(2\pi W_N N_o)^{T'W_N}} \cdot \frac{1}{\mathrm{pr}[\zeta/(\tau_i),\xi]} \cdot \prod_i \frac{\exp\left[-\frac{\alpha_i^2}{2\sigma_i^2}\right]}{2\pi\sigma_i^2} \quad \text{(A2-2)}$$

is independent of (a_i) and (θ_i). T' is the duration of $\eta(t)$, the output of the multipath medium; it is greater than T, the duration of the sounding signal, because of the spread of the delays in the medium. Using (2.06) for $\eta(t)$ and writing $\zeta(t) = z(t)e^{j2\pi f_o t}$, the first term of the exponent of (A2-1) may be written (dropping, for convenience, the limits on the integral) as

$$\frac{1}{2N_o}\int\left|z(t) - \sum_i a_i x(t-\tau_i)\epsilon^{-j\theta_i}\right|^2 dt \quad \text{(A2-3)}$$

We write, as in going from (2.10) to (2.11),

$$\int z^*(t)x(t-\tau_i)dt = g(\tau_i) \quad \text{(A2-4)}$$

and also

$$\int x^{*}(t-\tau_i)x(t-\tau_k)dt = f(\tau_i-\tau_k) \tag{A2-5}$$

(cf. equation (2.09)). Using (A2-4) and (A2-5), we may rewrite (A2-3):

$$\frac{1}{2N_o}\left[\int|z(t)|^2\,dt - 2\sum_i a_i \mathrm{Re}\left[g(\tau_i)e^{-j\theta_i}\right] + \sum_i\sum_k a_i a_k f(\tau_i-\tau_k)e^{j(\theta_i-\theta_k)}\right] \tag{A2-6}$$

Substituting (A2-6) into (A2-1), we get

$$\mathrm{pr}\left[(a_i),(\theta_i)/(\tau_i),\zeta,\xi\right] = K'\left[\prod_i a_i\right]\exp\left[-\frac{1}{2}\sum_i\sum_k c_{ik}a_i a_k + \sum_i d_i a_i\right] \tag{A2-7}$$

where

$$K' = K\exp\left[-\frac{1}{2N_o}\int|z(t)|^2 dt\right] \tag{A2-8}$$

$$c_{ik} = \frac{1}{N_o}\, f(\tau_i-\tau_k)\, e^{j(\theta_i-\theta_k)} + \frac{1}{\sigma_i^2}\,\varepsilon_{ik} \tag{A2-9}$$

$$d_i = \mathrm{Re}\left[\left\{\frac{g(\tau_i)}{N_o} + \frac{\alpha_i}{\sigma_i^2}\, e^{j\delta_i}\right\} e^{-j\theta_i}\right] \tag{A2-10}$$

ε_{ik} in (A2-9) is unity for i=k, zero for $i \neq k$. In order for (A2-7) to be useful to us, we should like it in the form of the product of first-order distributions. This will occur if the double summation can be written as a single summation for all values of (a_i), (θ_i), and (τ_i), for then (A2-7) is a product of exponentials. Thus, we require that $c_{ik} = 0$

$(i \neq k)$ for all (a_i), (θ_i), and (τ_i), or at least $c_{ik} \ll c_{ii}$ $(i \neq k)$. Noting that $f(0) = \varphi(o) = 2E$, where E is the energy of the sounding signal, this last condition leads to

$$\frac{1}{N_o} \left| f_i(\tau_i - \tau_k) \right| = \frac{1}{N_o} \left| \varphi(\tau_i - \tau_k) \right| \ll \frac{2E}{N_o} + \frac{1}{\sigma_i^2} \qquad \text{all } i \neq k \tag{A2-11}$$

from which (2.16) follows immediately. Using (A2-9), (A2-10), and (A2-11) in (A2-7), we have, after some algebraic manipulation,

$$\mathrm{pr}\left[(a_i), (\theta_i)/(\tau_i), \zeta, \xi\right] = K' \prod_i a_i \exp\left[- \frac{a_i^2 - 2\alpha_i' a_i \cos(\theta_i - \delta_i')}{2\sigma_i'^2} \right] \tag{A2-12}$$

where α_i', σ_i', and δ_i' are given by equations (2.18). To find K', we integrate (A2-12) over all values of (a_i) and (θ_i) and equate the result to unity.*

$$\frac{1}{K'} = \prod_i \int_o^{\infty} \int_{-\pi+\delta}^{\pi+\delta} a_i \exp\left[- \frac{a_i - 2\alpha_i' a_i \cos(\theta_i - \delta_i')}{2\sigma_i'^2} \right] d\theta_i \, da_i = \prod_i (2\pi\sigma_i'^2) \exp\left[\frac{1}{2} \left(\frac{\alpha_i'}{\sigma_i'} \right)^2 \right]$$

(A2-13)

(2.17) follows directly from (A2-12) and (A2-13).

In order to derive (2.22) we note that

$$\mathrm{pr}\left[(\tau_i)/\zeta, \xi\right] = \frac{\mathrm{pr}\left[(\tau_i)\right] \mathrm{pr}\left[\zeta/(\tau_i), \xi\right]}{\mathrm{pr}\left[\zeta/\xi\right]} \tag{A2-14}$$

But from (A2-2), (A2-8), and (A2-13)

$$\mathrm{pr}\left[\zeta/(\tau_i), \xi\right] = K'' \prod_i \exp\left[\frac{1}{2} \left(\frac{\alpha_i'}{\sigma_i'} \right)^2 \right] \tag{A2-15}$$

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* For the evaluation of (A2-13), cf. reference 24.

where K'' is independent of (τ_i). Equation (2.22) results from inserting (A2-15) in (A2-14), and letting $C = \frac{K''}{pr[\zeta/\xi]}$.

APPENDIX III: MINIMUM-MEAN-SQUARE-ERROR MEASUREMENT OF AN UNKNOWN IMPULSE RESPONSE

We desire the solution of equation (2.25):

$$\delta\epsilon = \delta E_{N,M}\left[\frac{1}{\Delta}\int_0^\Delta \left\{g(t) - h_m(t)\right\}^2 dt\right] = 0 \tag{A3-1}$$

Expanding this, we obtain

$$E_{N,M}\left[\int_0^\Delta g(t)\,\delta g(t)dt\right] = E_{N,M}\left[\int_0^\Delta h_m(t)\,\delta g(t)dt\right] \tag{A3-2}$$

since $\delta h_m(t) = 0$. Now g(t), as indicated in Figure 2-5, is the output of the estimating filter, whose unit-impulse response is $h_e(\tau)$. If y(t) is the filter input, then

$$g(t) = \int_{-\infty}^{\infty} h_e(\tau)y(t-\tau)d\tau \tag{A3-3}$$

y(t) is, in turn, the sum of the noise, n(t) and the output of the unknown filter, whose unit-impulse response, $h_m(\tau)$, is to be estimated. That is

$$y(t) = \int_{-\infty}^{\infty} h_m(\tau)x(t-\tau)d\tau + n(t) \tag{A3-4}$$

where x(t) is the channel input (sounding signal). Combining (A3-3) and (A3-4):

$$g(t) = \iint_{-\infty}^{\infty} h_e(\tau)h_m(\sigma)x(t-\tau-\sigma)d\sigma d\tau + \int_{-\infty}^{\infty} h_e(\tau)n(t-\tau)d\tau \tag{A3-5}$$

The variation of g(t) is thus

$$\delta g(t) = \iint_{-\infty}^{\infty} h_m(\sigma)\, x(t-\tau-\sigma)\delta h_e(\tau)d\sigma d\tau + \int_{-\infty}^{\infty} n(t-\tau)\delta h_e(\tau)d\tau \qquad \text{(A3-6)}$$

Remembering that n(t) and $h_m(\tau)$ are independent, and assuming that $E_N[n(t)] = 0$, we obtain for the right-hand side of (A3-2):

$$E_M\left[\int_0^\Delta \iint_{-\infty}^{+\infty} h_m(t)h_m(\sigma)x(t-\tau-\sigma)\delta h_e(\tau)d\sigma d\tau dt\right] \qquad \text{(A3-7)}$$

Similarly, the left-hand side of (A3-2) is

$$E_M\left[\int_0^\Delta \iiiint_{-\infty}^{\infty} h_e(\tau')h_m(\sigma')h_m(\sigma)x(t-\tau'-\sigma')x(t-\tau-\sigma)\delta h_e(\tau)d\sigma' d\tau' d\sigma d\tau dt\right]$$
$$+E_M\left[\Delta \iint_{-\infty}^{\infty} h_e(\tau')\varphi_N(\tau-\tau')\delta h_e(\tau)d\tau' d\tau\right] \qquad \text{(A3-8)}$$

In deriving (A3-8) we have assumed statistically-stationary noise with auto-correlation function

$$\varphi_N(\tau) = E_N\left[n(t)\; n(t+\tau)\right] \qquad \text{(A3-9)}$$

Since we have assumed (cf. page 25) that Δ is essentially greater than the duration of $h_m(\tau)$, the limits on the first integral sign in (A3-7) may be extended to $(-\infty, +\infty)$ without changing the value of the integral. The same statement may be made approximately about the first term in (A3-8); for this term is the product of the integral of the " signal" component* of the estimate of $h_m(\tau)$ and the variation of this component,

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*I.e., the first term in (A3-5).

and one would not expect the signal component to last appreciably longer than $h_m(\tau)$ itself.

Thus extending the limits, and equating (A3-7) and (A3-8), we get

$$\int_{-\infty}^{\infty} \delta h_e(\tau)d\tau \cdot E_M\left[\iiiint_{-\infty}^{\infty}\int h_e(\tau')h_m(\sigma')h_m(\sigma)x(t-\tau'-\sigma')x(t-\tau-\sigma)d\sigma' d\tau' d\sigma dt + \Delta \int_{-\infty}^{\infty} h_e(\tau')\varphi_N(\tau-\tau')d\tau' - \iint_{-\infty}^{\infty} h_m(t)h_m(\sigma)x(t-\tau-\sigma)d\sigma dt\right. = 0 \tag{A3-10}$$

We neglect the physical realizability condition, $h_e(\tau) = 0$ for $\tau < 0$.* Then $\delta h_e(\tau)$ is arbitrary for all τ, and in order for (A3-10) to be satisfied, the factor which multiplies $\delta h_e(\tau)$ must be zero for all τ. Setting it equal to zero, Fourier transforming the resulting equation**, and averaging over the ensemble of all possible unknown filters, we obtain

$$H_{e_{opt}}(f)\ \overline{|H_m(f)|^2}\ |X(f)|^2 + \Delta\, H_{e_{opt}}(f)N(f) - \overline{|H_m(f)|^2}\ X^*(f) = 0 \tag{A3-11}$$

Equation (2.26) follows from this immediately. In order to show that this solution yields a minimum, rather than a maximum or inflectional, error, one merely finds the second variation of ϵ, and shows that this is positive for $H_e(f) = H_{e_{opt}}(f)$.

To derive equation (2.33) for the optimum spectrum of x(t), we start with equation (2.24) for the mean-square error. Using (A3-5) in

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*See discussion, page 26.

** Cf. equation (1.07b), Chapter I.

this, and remembering that the average of the bracketed expression in (A3-10) is identically zero, we obtain for the minimum mean-square error

$$\epsilon_m = \frac{1}{\Delta} E_M \left[\int_{-\infty}^{\infty} h_m^2(t)dt - \iiint_{-\infty}^{\infty} h_m(t)h_m(\sigma)h_{e_{opt}}(\tau)x(t-\tau-\sigma)d\sigma d\tau dt \right] \qquad \text{(A3-12)}$$

ϵ_m is also expressible in terms of frequency domain functions; using Parseval's theorem in (A3-12) and averaging:

$$\epsilon_m = \frac{1}{\Delta} \left[\int_{-\infty}^{\infty} \overline{|H_m(f)|^2} df - \int_{-\infty}^{\infty} H_{e_{opt}}(f) \overline{|H_m(f)|^2} X(f)df \right] \qquad \text{(A3-13)}$$

Using equation (2.26) for $H_{e_{opt}}(f)$, (A3-13) becomes

$$\epsilon_m = \frac{1}{\Delta} \int_{-\infty}^{\infty} \overline{|H_m(f)|^2} \left[1 - \frac{|X(f)|^2}{N_r(f)\cdot\Delta + |X(f)|^2} \right] df \qquad \text{(A3-14)}$$

where $N_r(f) = \dfrac{N(f)}{|H_m(f)|^2}$. We now constrain the energy in the transmitted waveform to be constant:

$$\int_{-\infty}^{\infty} |X(f)|^2 df = K \qquad \text{(A3-15)}$$

In order to find the optimum $X(f)$, we must solve the variational problem[26]

$$\delta(\epsilon_m + \lambda K) = 0 \qquad \text{(A3-16)}$$

where λ is some constant. Using (A3-14) and (A3-15), (A3-16) becomes

$$\int_{-\infty}^{\infty} \left\{ \lambda - \frac{N_r(f)\ \overline{|H_m(f)|^2} \cdot \Delta}{\left[N_r(f)\cdot\Delta + |X(f)|^2\right]^2} \right\} \delta |X(f)|^2 df = 0 \qquad \text{(A3-17)}$$

If we constrain $|X(f)|^2$ to be zero outside a certain band, F_1 (cf. equation (2.31)), then $\delta|X(f)|^2$ is also zero there, and (A3-17) is satisfied for those frequencies. For frequencies within the band, on the other hand, we must try to set the bracketed term in the integrand equal to zero.* This leads to the equation

$$\lambda\ |X(f)|^4 + 2\lambda\Delta N_r(f)|X(f)|^2 + N_r(f)\Delta\left[\lambda\Delta N_r(f) - \overline{|H_m(f)|^2}\right] = 0 \qquad \text{(A3-18)}$$

If (A3-18) and (A3-15) can be simultaneously satisfied within the band F_1 by a non-negative function $|X(f)|^2$, then the solution is complete. If, however, there are frequencies at which $|X(f)|^2$ would be negative, then the correct solution is simultaneously to satisfy (A3-18) and (A3-15) at all frequencies for which $|X(f)|^2$ turns out non-negative, and to set $|X(f)|^2$ equal to zero at all other frequencies, as indicated in equation (2.33). That this is indeed the correct solution may be shown by taking the second variation of $(\epsilon_m + \lambda K)$:

$$\delta^2(\epsilon_m + \lambda K) = 2 \int_{-\infty}^{\infty} \frac{N_r(f)\ \overline{|H_m(f)|^2}}{\left[N_r(f)\cdot\Delta + |X(f)|^2\right]^3} \left[\delta|X(f)|^2\right]^2 df \qquad \text{(A3-19)}$$

* One might be tempted to obtain another solution, $X(f) = 0$, by writing $\delta|X(f)|^2 = 2|X(f)|\ \delta|X(f)|$. This is a spurious solution, however, for the problem is actually phrased completely in terms of $|X(f)|^2$ (cf. (A3-14) and (A3-15)). The second solution would disappear if we replaced $|X(f)|^2$ by, say, $S(f)$.

(A3-19) is positive for all variations of $|X(f)|^2$. This implies first that the solution of (A3-18) is in fact a minimum, and second that the further $|X(f)|^2$ is varied from this solution, the larger is the error. Therefore, one must satisfy (A3-18) as closely as possible, and for those frequencies for which the solution is negative this is achieved by setting $|X(f)|^2$ equal to zero.

Finally, equation (2.35) follows directly on substitution of (2.33) into (A3-14).

APPENDIX IV: DERIVATION OF LIKELIHOODS.

Using (2.02), (2.05), and (2.07), the integrand of the (a_i)-and (θ_i)-integrations of (3.03) may be written as

$$\frac{1}{(2\pi W_N N_o)^{T'W_N}} \left[\prod_{i=1}^{L} \frac{a_i}{2\pi\sigma_i^2}\right] \exp\left[-\frac{1}{2N_o}\int_0^{T'} |\zeta(t)-\eta^{(m)}(t)|^2\,dt - \sum_{i=1}^{L}\frac{a_i^2+\alpha_i^2-2\alpha_i a_i\cos(\theta_i-\delta_i)}{2\sigma_i^2}\right] \tag{A4-1}$$

where T' is the duration of $\eta^{(m)}(t)$, which is given by (2.06), with $x(t) = x_m(t)$. The expansion of the first term of the exponent of (A4-1) follows exactly the derivation from (A2-3) to (A2-6) of Appendix II, with the waveform index, m, inserted at the appropriate places. Assuming the validity of (3.06), and noting that $f_m(0) = \varphi_m(0) = 2E_m$, (A4-1) becomes, using (A2-6):

$$C\prod_{i=1}^{L}\frac{a_i\exp\left[-\frac{\alpha_i^2}{2\sigma_i^2}\right]}{2\pi\sigma_i^2}\exp\left\{-\left(\frac{E_m}{N_o}+\frac{1}{2\sigma_i^2}\right)a_i^2+\mathrm{Re}\left[\left(\frac{g_{mi}}{N_o}+\frac{\alpha_i}{\sigma_i^2}e^{j\delta_i}\right)e^{-j\theta_i}\right]a_i\right\} \tag{A4-2}$$

where

$$C = \frac{1}{(2\pi W_N N_o)^{T'W_N}}\exp\left[-\frac{1}{2N_o}\int_0^{T'}|z(t)|^2dt\right] \tag{A4-3}$$

is independent of (a_i), (θ_i), and m, and we have written $g_{mi} = g_m(\tau_i)$ (cf. equations (3.04) and (3.05)). We obtain equation (3.07) by integrating (A4-2) over the variables (a_i) and (θ_i). For these integrations we need the results[24]:

$$\frac{1}{2\pi} \int_{-\pi+c_2}^{\pi+c_2} \exp\left[\mathrm{Re}\,(c_1 e^{-j\theta})\right] d\theta = I_o\,(|c_1|) \tag{A4-4}$$

and

$$\int_o^\infty a\, e^{-c_3 a^2}\, I_o(c_4 a)\, da = \frac{1}{2c_3} \exp\left[\frac{c_4^2}{4c_3}\right] \tag{A4-5}$$

In (A4-4), c_1 is generally complex. In using (A4-4) in (A4-2) we note that

$$\left|\left(\frac{g_{mi}}{N_o} + \frac{\alpha_i}{\sigma_i^2}\, e^{j\delta_i}\right) a_i\right| = \left[\frac{|g_{mi}|^2}{N_o^2} + \left(\frac{\alpha_i}{\sigma_i^2}\right)^2 + \frac{2\alpha_i}{N_o \sigma_i^2}\, \mathrm{Re}\left[g_{mi}\, e^{-j\delta_i}\right]\right]^{\frac{1}{2}} a_i \tag{A4-6}$$

The derivation of equation (3.12) from equation (3.07) involves merely a straightforward application of equation (A4-4).

In deriving equation (3.14), we first write equation (3.07) in the form

$$\Lambda_m = C\left[\prod_{i=1}^{L} \frac{1}{1+\beta_{mi}}\right] \exp\left[\sum_{i=1}^{L} \frac{\frac{\sigma_i^2}{N_o}\,|g_{mi}|^2 - 2\alpha_i^2 E_m}{2N_o(1+\beta_{mi})}\right] \exp\left[\mathrm{Re}\left\{e^{-j\mu_m} \sum_{i=1}^{L} \frac{\alpha_i g_{mi} e^{-j\delta_i}}{N_o(1+\beta_{mi})}\right\}\right] \tag{A4-7}$$

where μ_m is the random phase-shift of the m^{th} message waveform. Then, (3.14) follows directly upon use of (A4-4).

APPENDIX V: CHARACTERISTIC FUNCTION OF A QUADRATIC FORM OF GAUSSIAN VARIABLES.

We are given a quadratic form,

$$D = W_t \, Q \, W \tag{A5-1}$$

of n variables,

$$W = \begin{bmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ \cdot \\ w_n \end{bmatrix} \tag{A5-2}$$

which share a joint Gaussian distribution,[29]

$$\mathrm{pr}\left[W\right] = \frac{1}{(2\pi)^{\frac{n}{2}} \left|M\right|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(W-\overline{W})_t M^{-1}(W-\overline{W})\right] \tag{A5-3}$$

Q is the matrix of the quadratic form, $\overline{W}$ is the matrix of the means of the variables:

$$\overline{W} = \begin{bmatrix} \overline{w}_1 \\ \overline{w}_2 \\ \cdot \\ \cdot \\ \cdot \\ \overline{w}_n \end{bmatrix} \tag{A5-4}$$

and M is the moment matrix of the variables, the typical element of which is

$$m_{ij} = \overline{(w_i-\overline{w}_i)(w_j-\overline{w}_j)} = \overline{w_i w_j} - \overline{w}_i\overline{w}_j \tag{A5-5}$$

"t" denotes " transpose of", and $|\ldots|$, " determinant of".

We require the characteristic function of the quadratic form:

$$F_D(ju) = \overline{e^{juD}} = \underbrace{\int_{-\infty}^{\infty}\cdots\int}_{n\ \text{times}} e^{juD}\ pr\left[W\right] dw_1\ldots dw_n \tag{A5-6}$$

Noting that M, and hence M^{-1}, is symmetric, we may write

$$W_t M^{-1}\overline{W} = (\overline{W}_t M^{-1} W)_t = \overline{W}_t M^{-1} W \tag{A5-7}$$

The last equality follows from the fact that $\overline{W}_t M^{-1} W$ is a one-element matrix. Using (A5-7), we find

$$(W-\overline{W})_t\ M^{-1}\ (W-\overline{W}) = W_t M^{-1} W - 2\overline{W}_t M^{-1} W + \overline{W}_t M^{-1}\overline{W} \tag{A5-8}$$

Then placing (A5-1) and (A5-3) in (A5-6), and making use of (A5-8), we obtain

$$F_D(ju) = \frac{\exp\left[-\frac{1}{2}\,\overline{W}_t M^{-1}\,\overline{W}\right]}{(2\pi)^{\frac{n}{2}}\ \left|M\right|^{\frac{1}{2}}} \underbrace{\int_{-\infty}^{\infty}\cdots\int}_{n\ \text{times}} \exp\left[\overline{W}_t M^{-1} W\right] \exp\left[-\tfrac{1}{2}W_t(M^{-1}-2juQ)W\right] dw_1\ldots dw_n \tag{A5-9}$$

Using a result given by Cramér[30], (A5-9) becomes

$$F_D(ju) = \frac{\exp\left[-\frac{1}{2}\,\overline{W}_t M^{-1} W\right]}{\left|M\right|^{\frac{1}{2}}} \cdot \frac{\exp\left[\frac{1}{2}\,\overline{W}_t M^{-1}(M^{-1}-2juQ)^{-1}M^{-1}\overline{W}\right]}{\left|M^{-1}-2juQ\right|^{\frac{1}{2}}} \qquad \text{(A5-10)}$$

Finally, noting that

$$(M^{-1}-2juQ)^{-1}\,M^{-1} = (I-2juMQ)^{-1} \qquad \text{(A5-11)}$$

we may immediately obtain equation (4.08) from (A5-10). I is the unit matrix.

APPENDIX VI: EVALUATION OF MATRICES OF EQUATION (4.08).

Substituting equation (4.07) in (4.06), we obtain

$$g_1 = ae^{j\theta} \int |x_1(t)|^2 dt + \int n^*(t) x_1(t)\, dt$$

$$g_2 = ae^{j\theta} \int x_1^*(t) x_2(t) dt + \int n^*(t) x_2(t) dt \qquad \text{(A6-1)}$$

Noting that (cf. equation (4.12)

$$\int |x_1(t)|^2 dt = \int |x_2(t)|^2 dt = 2E$$

$$\int x_1^*(t)\, x_2(t) dt = 2E\lambda \qquad \text{(A6-2)}$$

and letting

$$p = 2aEe^{j\theta} \qquad \text{(A6-3)}$$

and

$$q_i = \int n^*(t) x_i(t)\, dt \qquad i = 1,2 \qquad \text{(A6-4)}$$

we may rewrite (A6-1) as

$$g_1 = p + q_1$$

$$g_2 = p\lambda + q_2 \qquad \text{(A6-5)}$$

In order to obtain the means and second self- and cross-moments of the w-variables of equation (4.04), which are required for $\overline{W}$- and M-matrices, it is obvious that we must have the means and second self- and cross-moments of the real and imaginary parts of g_1 and g_2. These real and imaginary parts are, from (A6-5):

$$\begin{aligned} \hat{g}_1 &= \hat{p} + \hat{q}_1 \\ \tilde{g}_1 &= \tilde{p} + \tilde{q}_1 \\ \hat{g}_2 &= \hat{p}\hat{\lambda} - \tilde{p}\tilde{\lambda} + \hat{q}_2 \\ \tilde{g}_2 &= \hat{p}\tilde{\lambda} + \tilde{p}\hat{\lambda} + \tilde{q}_2 \end{aligned} \qquad \text{(A6-6)}$$

Since we have assumed (without loss of generality) that $\delta = 0$ in the joint distribution of a and θ of equation (2.05), we may write immediately for the various moments of $\hat{p}$ and $\tilde{p}$, using results of Rice[16]:*

$$\begin{aligned} \overline{\hat{p}} &= 2\alpha E \\ \overline{\tilde{p}} &= 0 \\ \overline{\hat{p}^2} &= 4(\alpha^2+\sigma^2)E^2 \\ \overline{\tilde{p}^2} &= 4\sigma^2 E \\ \overline{\hat{p}\,\tilde{p}} &= 0 \end{aligned} \qquad \text{(A6-7)}$$

From (A6-4), we may write

$$\begin{aligned} \hat{q}_i &= \int \left[\hat{n}(t)\hat{x}_i(t) + \tilde{n}(t)\tilde{x}_i(t)\right] dt \\ \tilde{q}_i &= \int \left[\hat{n}(t)\tilde{x}_i(t) - \tilde{n}(t)\hat{x}_i(t)\right] dt \end{aligned} \qquad i = 1,2 \qquad \text{(A6-8)}$$

---

* Cf. Fig. 2-2, in which let $\delta = 0$, and multiply all vectors by 2E.

Noting that $\overline{\hat{n}(t)} = \overline{\tilde{n}(t)} = 0$, we have immediately

$$\overline{\hat{q}_i} = \overline{\tilde{q}_i} = 0 \qquad i = 1,2 \qquad \text{(A6-9)}$$

Now , for all practical purposes, we may consider the bandwidth of the noise, W_N, very much larger than the transmission bandwidth, W. Then, to a very good approximation, we may consider that

$$\overline{\hat{n}(t)\ \hat{n}(s)} = \overline{\tilde{n}(t)\ \tilde{n}(s)} = N_o\ \delta(t-s) \qquad \text{(A6-10)}$$

where $\delta(x)$ is the Dirac delta-function. We also note that[15]

$$\overline{\hat{n}(t)\ \tilde{n}(s)} = 0 \qquad \text{(A6-11)}$$

Using (A6-2), (A6-10), and (A6-11), we obtain for the various second self- and cross-moments of the real and imaginary parts of q_1 and q_2 of equation (A6-8):

$$\begin{aligned}
&\overline{\hat{q}_1^2} = \overline{\tilde{q}_1^2} = \overline{\hat{q}_2^2} = \overline{\tilde{q}_2^2} = 2EN_o \\
&\overline{\hat{q}_1\tilde{q}_1} = \overline{\hat{q}_2\tilde{q}_2} = 0 \\
&\overline{\hat{q}_1\hat{q}_2} = \overline{\tilde{q}_1\tilde{q}_2} = 2\hat{\lambda}EN_o \\
&\overline{\hat{q}_1\tilde{q}_2} = -\overline{\tilde{q}_1\hat{q}_2} = 2\tilde{\lambda}EN_o
\end{aligned} \qquad \text{(A6-12)}$$

Finally, because of the assumed independence of the noise and the multipath medium, we may write for the cross-moments of the real and imaginary parts of p and q_1 and q_2:

$$\begin{aligned} \overline{\hat{p}\,\hat{q}_i} &= \overline{\hat{p}}\;\overline{\hat{q}_i} = 0 \\ \overline{\hat{p}\,\tilde{q}_i} &= \overline{\hat{p}}\;\overline{\tilde{q}_i} = 0 \\ \overline{\tilde{p}\,\hat{q}_i} &= \overline{\tilde{p}}\;\overline{\hat{q}_i} = 0 \\ \overline{\tilde{p}\,\tilde{q}_i} &= \overline{\tilde{p}}\;\overline{\tilde{q}_i} = 0 \end{aligned} \qquad i = 1,2 \tag{A6-13}$$

We have thus evaluated (equations (A6-7), (A6-9), (A6-12), and (A6-13)) the means and the various second self- and cross-moments of the real and imaginary parts of p, q_1, and q_2. Using these, we may obtain the means and second self- and cross-moments of the real and imaginary parts of g_1 and g_2, which in turn may be used to evaluate the $\overline{w}_i$'s and the m_{ij}'s of equation (4.13).

In order to obtain the inverse of the moment matrix, which is required in equation (4.08), we apply the following identity to equation (4.14):

$$\begin{bmatrix} a & o & b & c \\ o & a & -c & b \\ b & -c & d & o \\ c & b & o & d \end{bmatrix} = \frac{1}{(ad-b^2-c^2)} \begin{bmatrix} d & o & -b & -c \\ o & d & c & -b \\ -b & c & a & o \\ -c & -b & o & a \end{bmatrix} \tag{A6-14}$$

Post-multiplication of the M-matrix by the Q-matrix changes the signs of the elements of the last two columns of the M-matrix. Multiplication of the MQ-matrix by the scalar, 2ju, and subtraction of the result from the unit matrix leads to:

$$I-2juMQ = -2ju\beta \begin{bmatrix} \beta+1-\frac{1}{2ju\beta} & 0 & -\hat{\lambda}(\beta+1) & -\tilde{\lambda}(\beta+1) \\ 0 & \beta+1-\frac{1}{2ju\beta} & \tilde{\lambda}(\beta+1) & -\hat{\lambda}(\beta+1) \\ \hat{\lambda}(\beta+1) & -\tilde{\lambda}(\beta+1) & -\beta|\lambda|^2-1-\frac{1}{2ju\beta} & 0 \\ \tilde{\lambda}(\beta+1) & \hat{\lambda}(\beta+1) & 0 & -\beta|\lambda|^2-1-\frac{1}{2ju\beta} \end{bmatrix} \qquad \text{(A6-15)}$$

In inverting the (I - 2juMQ) -matrix, as required in equation (4.08), we again make use of (A6-14). The determinant of the (I - 2juMQ) - matrix may be evaluated through the use of the relation

$$\left| K \begin{bmatrix} a & o & b & c \\ o & a & -c & b \\ b & -c & d & o \\ c & b & o & d \end{bmatrix} \right| = K^4(ad-b^2-c^2)^2 \qquad \text{(A6-16)}$$

APPENDIX VII: DERIVATION OF EXPRESSIONS FOR P_e.

We start with equation (4.11), in which we substitute equation (4.17) for $F_D(s)$:

$$P_e = -\frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{\exp\left[\frac{k_1 s(1+k_2 s)}{1-k_3 s(1+k_2 s)}\right]}{s\left[1-k_3 s(1+k_2 s)\right]}\, ds \tag{A7-1}$$

where k_1, k_2, and k_3 are given by equations (4.18). The path of integration is taken to be indented to the left at the origin. We may immediately obtain P_e for the special case, $\sigma = 0$, by noting that, in this case, $k_2 = 2$ and $k_3 = 0$, so that

$$P_e = -\frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{e^{2k_1 s^2}}{s}\, e^{k_1 s}\, ds \tag{A7-2}$$

Equation (A7-2) is in the form of the Fourier transform of $-\frac{e^{2k_1 s^2}}{s}$, which is available in tables[32]. Thus, we obtain for $\sigma = 0$*:

$$P_e = \frac{1}{2}\left[1 - \text{erf}\left(\tfrac{1}{2}\sqrt{k_1/2}\right)\right] \tag{A7-3}$$

---

* The tables referred to actually give the right-hand side of (A7-3) as the sum of the transforms of $-e^{2k_1 s^2}/s$ and $1/s$. However, the path of integration in the tables is taken to be indented to the right at the origin. Now, the left-indented integral in (A7-2) may be written as the sum of two other integrals having the same integrand: one along the j-axis with an indentation to the right at the origin, and one along a closed contour of infinitesimal radius which encircles the origin. The first of these is the right-indented transform of $-e^{2k_1 s^2}/s$, and the second may easily be shown to be equal to the right-indented transform of $1/s$. Thus, the right-hand side of (A7-3) is also equal to the left-indented transform of $-e^{2k_1 s^2}/s$, equation (A7-2).

This, with equations (4.18), leads directly to equation (4.19).

We may evaluate P_e for the case $\alpha = 0$ by the method of residues. For this case, $k_1 = 0$. Writing (A7-1) in terms of the roots, r_1 and r_2, of the quadratic, $1 - k_3 s(1 + k_2 s)$, we have

$$P_e = -\frac{r_1 r_2}{2\pi j} \int_{-j\infty}^{j\infty} \frac{1}{s(s-r_1)(s-r_2)}\, ds \qquad \text{(A7-4)}$$

where

$$\left.\begin{matrix} r_1 \\ \\ r_2 \end{matrix}\right\} = \frac{-1 \pm \sqrt{1 + \frac{4k_2}{k_3}}}{2k_2} \qquad \text{(A7-5)}$$

Since the integrand of the integral in (A7-4) vanishes as $\frac{1}{s^3}$ as $s \to \infty$, we may close the path of integration by enclosing either the left or the right half-plane, without changing the value of the integral. Since the j-axis path is taken to be indented to the left at the origin, the former contour encloses only one pole of the integrand (see Figure A7-1), that at $s = r_2 (r_2 < 0)$, and the integral may be evaluated by calculating the residue, R, in that pole, and multiplying it by $2\pi j$[33]:

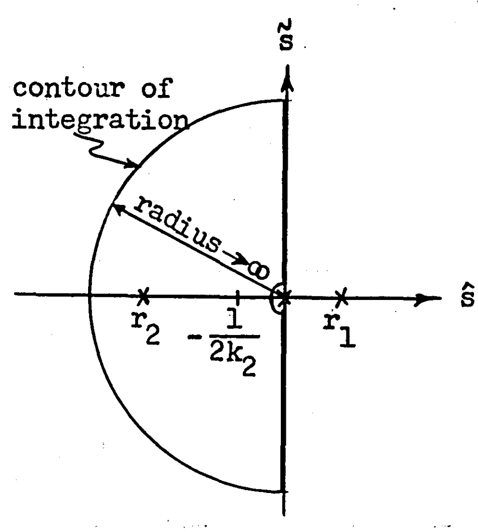

FIGURE A7-1

$$P_e = -r_1 r_2 R \qquad \text{(A7-6)}$$

R is easily calculated:

$$R = \frac{1}{r_2(r_2-r_1)} \tag{A7-7}$$

Equation (4.21) results from the substitution of (A7-7) into (A7-6).

In the general case ($\sigma \neq 0$, $\alpha \neq 0$), equation (A7-1) may be put in a better form for numerical computation by the following method. We first shift the path of integration from the j-axis to the left by an amount $\frac{1}{2k_2}$; this does not change the value of the integral because no singularities of the integrand are crossed in the process (see Figure A7-1; the singularities at r_1 and r_2 are now essential singularities). Then, making the change of variables, $s = \frac{1}{2k_2}(jz-1)$, and noting that the imaginary part of the resulting integrand is odd about $z = 0$, and the real part even, we obtain

$$P_e = \frac{k_4^2-1}{\pi} \int_0^{\infty} \frac{\exp\left[-\frac{k_5(z^2+1)}{z^2+k_4^2}\right]}{(z^2+1)(z^2+k_4^2)} \tag{A7-8}$$

where k_4 and k_5 are given by equations (4.24). Now, making the further change of variable, $z^2 = k_4^2 \tan^2\theta$, and noting that $1 + \tan^2\theta = \sec^2\theta$, we obtain the desired result, equation (4.23).

APPENDIX VIII: DERIVATION OF EXPRESSION FOR P'_e.

We write, as in equation (A6-5) of Appendix VI:

$$g_1 = 2aEe^{j\theta} + q_1$$
$$g_2 = q_2 \qquad \text{(A8-1)}$$

where we have used the assumption that $\lambda = 0$. g_1 and g_2 may be represented vectorially as in Figure A8-1. Now, the real and imaginary parts of q_1 and q_2 can be expressed in terms of linear operations on the Gaussian functions $\hat{n}(t)$ and $\tilde{n}(t)$. (Cf. equation (A6-8).) Hence[28], $\hat{q}_i$ and $\tilde{q}_i$ are also Gaussian. Furthermore, we have from (A6-9) and (A6-12):

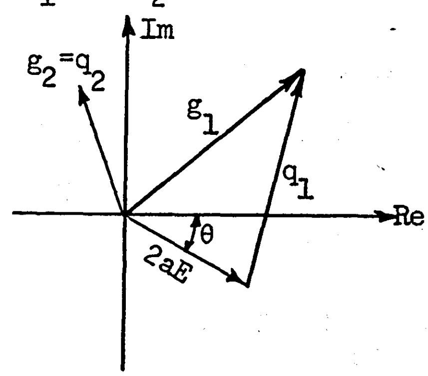

FIGURE A8-1

$$\overline{\hat{q}_i} = \overline{\tilde{q}_i} = 0$$
$$\overline{\hat{q}_i^2} = \overline{\tilde{q}_i^2} = 2EN_o \qquad i = 1,2 \qquad \text{(A8-2)}$$
$$\overline{\hat{q}_i \tilde{q}_i} = 0$$

That is, the real and imaginary parts of q_i are independent, Gaussian, of zero mean, and common variance, $2EN_o$; and an identical statement applies to q_2. Thus, using a result of Rice[16], we may write

$$\mathrm{pr}\left[|q_i|\right] = \frac{|q_i|}{2EN_o} \exp\left[-\frac{|q_i|^2}{4EN_o}\right] \qquad i = 1,2 \qquad \text{(A8-3)}$$

That is, the lengths of the vectors q_1 and q_2 in Figure A8-1 are Rayleigh distributed. Now, we first assume that the path strength, a, is known to the receiver. Then the vector $2aEe^{j\theta}$ in Figure A8-1 is of fixed length. Again invoking a result of Rice for the length of the sum of a vector of fixed length and a vector whose length is Rayleigh distributed(16), we obtain

$$\mathrm{pr}\left[|g_1|/a\right] = \frac{|g_1|}{2EN_o} \exp\left[-\frac{|g_1|^2+4a^2E^2}{4EN_o}\right] I_o\left[\frac{a|g_1|}{N_o}\right] \qquad \text{(A8-4)}$$

We may finally show, using (A6-12) and the assumption that $\lambda = 0$, that

$$\overline{\hat{q}_1\hat{q}_2} = \overline{\tilde{q}_1\tilde{q}_2} = \overline{\hat{q}_1\tilde{q}_2} = \overline{\tilde{q}_1\hat{q}_2} = 0 \qquad \text{(A8-5)}$$

Since uncorrelated Gaussian variables are independent, we infer from (A8-5) that q_1 and q_2, and hence g_1 and g_2, are independent. Then we may write

$$P_e'(a) = \Pr\left[|g_2| > |g_1|/a\right]$$

$$= \int_0^\infty \mathrm{pr}\left[|g_1|/a\right] d|g_1| \int_{|g_1|}^\infty \mathrm{pr}\left[|g_2|/a\right] d|g_2| \qquad \text{(A8-6)}$$

Since, from (A8-1), $g_2 = q_2$, we may write $\mathrm{pr}\left[|g_2|/a\right] = \mathrm{pr}\left[|g_2| = |q_2|\right]$. Using (A8-3), the $|g_2|$ - integration of (A8-6) becomes exactly $\exp\left[-\frac{|g_1|^2}{4EN_o}\right]$. Using this result with equation (A8-4) in the remaining $|g_1|$-integration, we have

$$P_e'(a) = \frac{\exp\left[-\frac{a^2 E}{N_o}\right]}{2EN_o} \int_o^\infty |g_1| \exp\left[-\frac{|g_1|^2}{2EN_o}\right] I_o\left[\frac{a|g_1|}{N_o}\right] d|g_1| \qquad \text{(A8-7)}$$

With the help of equation (A4-5) of Appendix IV, equation (A8-7) reduces immediately to equation (4.30).

The marginal distribution, pr[a], of the path strength, obtained by integrating equation (2.05) over θ, is[16]:

$$pr[a] = \frac{a}{\sigma^2} \exp\left[-\frac{a^2+\alpha^2}{2\sigma^2}\right] I_o\left[\frac{\alpha a}{\sigma^2}\right] \qquad \text{(A8-8)}$$

Substituting (4.30) and (A8-8) into (4.31), we have:

$$P_e' = \frac{\exp\left[-\frac{\alpha^2}{2\sigma^2}\right]}{2\sigma^2} \int_o^\infty a \exp\left[-\frac{a^2}{2}\left(\frac{1}{\sigma^2} + \frac{E}{N_o}\right)\right] I_o\left[\frac{\alpha a}{\sigma^2}\right] da \qquad \text{(A8-9)}$$

Again using (A4-5), and remembering that $\gamma = \frac{\alpha}{\sigma}$ and $\beta = \frac{2\sigma^2 E}{N_o}$, we may obtain equation (4.32) directly from equation (A8-9).

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BIOGRAPHICAL NOTE

George Lewis Turin was born in New York City on January 27, 1930. After having obtained his primary and secondary educations in the public schools of that city, he entered the Massachusetts Institute of Technology in 1947. At M.I.T. he participated in the Cooperative Course in Electrical Engineering with Philco Corporation, and was awarded the S. B. and S. M. degrees in Electrical Engineering in 1952. During the summer of 1952 he was an M.I.T. Overseas Summer Fellow at Marconi's Wireless Telegraph Company in England.

In the fall of 1952 he became a Staff Member of Lincoln Laboratory, M.I.T., in which position he remained for the next two years, engaged in the design and development of new types of communication systems.

From the fall of 1954 to the present he has been a Research Assistant at the Department of Electrical Engineering, M.I.T., on assignment to Lincoln Laboratory. During this period he has completed his doctoral studies. He has also during the past two years done consulting work for the Boston firm of Edgerton, Germeshausen and Grier.

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