Optimal Execution for Portfolio Transactions

by

Alexander Fadeev

B.A.'s in Economics, Computer Science from Oberlin College (1995)

MBA in Finance from SMU (2005)

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Signature of Author ______________________________

Alexander Fadeev
System Design and Management Program
January 2007

Certified by ______________________________

John C. Cox
Thesis Supervisor
Nomura Professor of Finance, MIT Sloan School of Management

Certified by ______________________________

Patrick Hale
Director
System Design & Management Program

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ABSTRACT

In my thesis I explore the problem of optimizing trading strategies for complex portfolio transitions. Institutional investors run into this issue during periodic portfolio rebalancing or transition between asset managers. The costs of rebalancing can be broadly broken into trading costs (both the transaction cost and the market impact) and the opportunity costs of delaying the execution and bearing the risk of current-to-target portfolio divergence. This thesis proposes a methodology for measuring the opportunity cost as well as a strategy that minimizes the proposed measure through optimal portfolio transition execution. The benefits from the proposed trading strategy are benchmarked against the industry standard portfolio trading practices.

ACKNOWLEDGEMENTS

I would like to take a moment to thank all of my MIT professors who have contributed, directly or indirectly, to this exploration. Chief among them, I would like to thank Mark Kritzman for taking the time to share his knowledge with the students of the Financial Technology Option program and pointing me in the right direction and class and during this research effort. His viral intellectual curiosity has inspired me and many others to search for the practical answers to the not so obvious. I would next like to thank John Cox and Andrew Lo for both creating the FTO program in the first place and inviting practitioners and thought leader like Mark Kritzman to share their insights with the MIT students.

I would also like to thank one of the professors from my previous academic endeavors and professional endeavors; Rex Thompson, for inspiring me to direct my curiosity towards the field of quantitative finance, and Art Selender, my first boss in the financial services industry, who admitted me to the club of finance practitioners.

Finally, I would like to acknowledge the assistance I had received from two Managing Directors who have helped me to collect the relevant data for this study and guided me to pursue the most relevant questions: Scott McLellan of State Street and Heiko Ebens of Merrill Lynch.

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1. Introduction

Institutional investors periodically reallocate their portfolios to shift the asset mix or reshuffle investment managers. They face a variety of costs when they undertake these transitions, including commissions, bid-ask spreads, opportunity cost, and market impact.[i] Commissions and crossing bid-ask spread are relatively well researched and are straight-forward to quantify. Opportunity cost refers to adverse changes in price arising from exogenous market forces, while market impact refers to adverse price movements that occur in response to the purchase and sale of securities. Opportunity cost and market impact are especially interesting both because they represent the largest share of total trading cost and because investors can influence their magnitude by how they trade.

I explore an algorithm for determining the optimal sequence and size of trades that minimize opportunity cost for portfolio transitions. The proposed algorithm differs from the examined industry-standard practices in that it aims to minimize the opportunity cost. Our primary concern is with minimizing the value deviation between the legacy ("from") and the target ('to") portfolios rather then the absolute performance of a single security or a group of securities targeted for a purchase or a sale. As standard in the asset custody business, I assume the investor does not use leverage to execute the transition and hence the trades are constrained to be self-financing.

1.1. Portfolio Manager Transitions

Institutional investment committees and plan sponsors manage or oversee taxable corporate and tax-exempt endowment, pension or foundation funds. Their investment decisions are bound by the fiduciary responsibility and code of conduct standards stemming from the 1974 ERISA federal statue. Among those, is the requirement to review the investment performances at reasonable intervals and evaluate if the fiduciary should continue using the current mix of service providers or look for replacements[ii].

The emphasis on the decisions of the institutional investors stems from the dominant position of the institutional equity and bond funds in the US asset management marketplace. As of December 2000, institutional funds represented $6,646 B compared to $4,770 B of the retail equity and bond holdings. The institutional portfolios are far more concentrated in size and more actively supervised, evaluated and managed. This presents the challenge and the opportunity of minimizing transition costs whenever an investment committee decides to re-allocate their assets among its managers or the universe of investment strategies and styles.

The details of the investment evaluation and subsequent allocation decisions have been explored by Heisler et al (2006)[iii] as well as Goyal and Wahal (2006)[iv], the latter of whom had built a unique dataset of asset managers' hiring and firing decisions by 3,591 plan sponsors from 1994 to 2003. The plan sponsors took into account managers' performance, investment style as well as own risk tolerance and forward-looking market views in allocating funds among 7,153 equity bond and hybrid mutual funds (as of 2003).

Allocations to Hedge Funds and Private Equity partnerships fall outside our domain of analysis due to the fact that such transitions are almost always based on cash transactions (either withdrawals or investments) and would not trigger portfolio rebalancing trades. Rather, such investments would likely lead to portfolio liquidation challenges studied at length by Almgren and Chriss (1999, 2000, 2003), Liu and Loewenstein (2002) and Rosu and Lo (2004) among others.

In particular, Goyal and Wahal's study describes 9,214 manager transition events over 10 years involving $636 B, or roughly 35 portfolio transitions per week. If we could minimize the transition cost by even a few bips, the value to the asset management community would be economically meaningful $63.6 M per bip of savings over the past ten years alone.

1.2. Research goal

I explore a methodology for measuring and optimizing one of the three costs that constrain optimal portfolio rebalancing – the opportunity cost. The goals of this study are:

- to analyze the present industry-standard portfolio rebalancing and trading strategies
- propose a measure of the opportunity cost
- derive a portfolio trading optimization (PTO) methodology that would improve on the present industry practices
- measure and quantify the improvement

2. Portfolio Trading Research Review

Most of the existing Portfolio Trading research and analysis has been concentrated on the task of estimating the liquidation value and the cost of trading individual equity positions or portfolio consisting of these individual equities. The classical papers on the subject are those by Robert Almgren and Neil Criss[v] (1997, 2000), Bertsimas and Lo[vi] (1998), Barra Market Impact Handbook (1997)[vii] and André Perold[viii] (1988).

My research has been inspired by the work of Mark Kritzman, Simon Myrgren and Sébastien Page on "Optimal Execution for Portfolio Transitions" (2005 draft, upcoming in JPM in the summer of 2006).

3. Proposed Optimal Portfolio Trading Methodology

Kritzman, Myrgren and Page paper[ix] proposes a portfolio trading optimization (PTO) method based on the principal of minimizing portfolio trading tracking error. The tracking error (squared) is defined as a function of the securities in the current ("from"), the target ("to") portfolios and the proposed trades, as shown in Equation (1): [1]

$$TE^2 = (w^k_{CURRENT} + w_{MN}) * \Sigma * (w^k_{TARGET} + w_{MN}) \qquad (1)$$

-or –

(w-legacy)*Sigma* (w-legacy + w-delta) = (w-legacy) * Sigma *

[1] My algorithm for portfolio transition is motivated by Sharpe (1987). Sharpe showed that by calculating the derivatives of quadratic utility with respect to portfolio weights and then continually shifting the portfolio weights from the security with the lowest derivative to the one with the highest derivative, one arrives at the mean-variance efficient portfolio when all the derivatives are equal. We apply this insight to derive our algorithm for minimizing tracking error as a function of the sequence and size of trades.

Where: w_R^k = relative portfolio weights of the securities in the legacy and target portfolios after the k[th] trade. Sum of the weights should add up to one.

w_{MN} = the vector of relative weights corresponding to each proposed transaction. For example, buying security #1 and selling identical dollar amount of security #3 would be represented as $w_{MN} = [w \quad 0 \quad -w \quad ...]$ Net sum equality of relative weights with zero (0) ensures that all trades are self financing.

Σ = covariance matrix of the securities in the two portfolios

$(w_{CURRENT}^k + w_{MN})$ is effectively a vector of delta portfolio weights, defined as ($W_{Target} - W_{Legacy}$)

Taking the partial derivative of TE with respect to the proposed trades will give us the ranking of the sensitivities of the tracking errors with respect to the potential trades. The derivative with the highest values will reduce the TE the most and the corresponding pair-trade should be executed first.

$$\frac{dTE^2}{dw_{MN}} = 2\Sigma w_R^k + 2\Sigma w_{MN} \qquad (2)$$

To find the optimal trade size for the security with the highest TE derivatives, we find the point where the two marginal TE derivative values are equal ($dTE_A^2 = dTE_B^2$) on both the long and short legs of the proposed trades. Taking partial derivatives of two sample positions (i.e.: assets ‘A’ and ‘B’) w.r.t. individual security weights, we can rewrite equation [2] as:

$$dTE_A^2 = \alpha_A + \beta_A w \tag{3}$$

$$dTE_B^2 = \alpha_B + \beta_B w \tag{4}$$

Solving the two equalities for the trade size yields a linear relationship and the optimal trade amount where the marginal TE decays are equal:

$$\alpha_B + \beta_B w = \alpha_A + \beta_A w \tag{5}$$

$$w_{OPTIMAL} = \frac{\alpha_B - \alpha_A}{\beta_A - \beta_B} \tag{6}$$

After we identify the optimal long and short legs of the pair trade and their corresponding optimal trade quantities, we need to impose an additional constraint of trading the smallest amount of the two quantities. As a result, the larger quantity will remain the optimal trade leg for the next round of trade selection, but its opposite optimal leg of the pair trade will be different in the next round.

As we execute the proposed trades, we adjust the “from” and the “to” portfolio weights by the optimal trade size of both legs of the pair trade. This optimization process will continue iteratively until all outstanding trades are prioritized and quantified, bringing the “from” and the “to” portfolio vector weights into equality.

It is conceivable that the trade sizes will be either too large (and would require significant market impact penalty to execute) or too small (trade thrashing). A simple solution to this problem would be to further constraint both the maximum (as a % of average daily volume) and the minimum trade size to avoid these issues.

This trading strategy need not to be optimized and executed in real-time. Instead, we can pre-optimize the trading path and queue the trades through the execution algorithm of choice while the trader could add value by actively watching the market for temporary pricing inefficiencies that could be exploited outside of the recommended trading strategy. While the human trader is capable of exploiting alpha-generating trading opportunities, our strategy can be used to re-optimize the remaining portfolio weights as they experience unplanned shifts due to human interventions. The methodology is flexible enough to support both real-time trading optimization, pre-optimized trading strategy derivation as well as the combination of the two.

4. Data

The data for much of the list of publicly traded securities and their daily price attributes came from Bloomberg.

The initial data download produced the list of 4916 publicly listed equity tickers, their descriptive data (market capitalization, GICS[x] industry designation, 360-day annualized volatility) and daily closing prices for the period of 90 trading days preceding August 7th 2006. After the price quote pull was completed, additional 186 equities with missing pricing data points were dropped reducing the total population to 4730 publicly listed companies. Where price data was available but descriptive information missing, the missing data was populated from either Bloomberg's own industry descriptive fields or from finance.yahoo.com and studio.financialcontent.com

Sector-specific return matrices were calculated on the equal-weighted (arithmetic average) basis of available returns per each GICS sector.

The samples of complex portfolio transaction tasks came from real-world portfolio re-balancing orders courtesy of Ross McLellan of the State Street Global Markets.

Complete data sets of historical prices and sample trade portfolio allocations are available upon request.

4.1. Data overview

The sample "buy" and "sell" portfolios containing 4,020 securities served as a model for analyzing the real-world implications of the portfolio trading method explored in this paper. These portfolios were derived from a real world rebalancing scenario on a roughly $1B client portfolio.

From my universe of 4730 publicly traded US equities, I collected empirical observations on two types of portfolio transitions.

The first comparison was based of a real-world portfolio obtained from a well known Boston-area asset custody firm. The portfolio's original trade list contained 122 "buys" representing 19.26% of the overall portfolio value of $965,944,646.00. The 1,983 "sells" represented 18.88% of the overall portfolio value. The difference in quantity between the numbers of equities comprising "buy" vs. "sell" portfolios is purely coincidental and is the result of client's asset allocation change.

	Buys	Sells
Portfolio Size	$965,462,368	
# of equities	122	1983
fraction portfolio value	0.192596472	0.188799943
$ starting value	$185,944,646.18	$182,279,240.21
average order (#, shares)	60,170	4,883
min order (#, shares)	800	100
max order (#, shares)	242,197	117,800
average order ($$)	$ 1,524,136	$ 91,921
min order ($$)	$ 328,563	$ 1,364
max order ($$)	$ 3,680,585	$ 653,250

Of the original 122 "buys", 2 equity trades were missing identifying tickers and were dropped from the sample portfolio. This had the cumulative effect of reducing the "buy" portfolio fractional value by 0.12%, or $1,164,756. Likewise, 33 "sell" tickers were missing and were dropped. The dropped "sell" trades represented 0.15%, or $1,461,957 of the portfolio. The net of these changes were inconsequential to the test performed.

The second comparison algorithms' performance comparison was based off random portfolios of varying size, drawn from the equity population and assigned random weights. The random portfolio construction methodology has allowed me to verify the observations from the one-off real-world portfolio and explore the effects of portfolio size and composition on the OPT strategy performance. The results were similar to those derived from examining a sample real-world portfolio and are reported below.

4.2. Data limitations

I have concentrated my analysis on the publicly traded US equities. To the extent that portfolio transactions called for trades in foreign-listed equities, they were dropped from the sample portfolio and their relative economic value redistributed among the remaining assets.

My analysis is based on the data collected and applied to the US public markets. However, there is nothing in the proposed method, that we are aware of, that would limit its application to foreign equity markets. In fact, this method could be logically extended to work on any publicly traded securities (i.e.: fixed income and derivate products). However, the data availability and the sampling error in analyzing relatively illiquid asset classes would require careful consideration.

For the purpose of exploring the effect of substituting individual equity covariance observations with those of its sector, the sector covariance was derived from an equal-weighted (arithmetic average) of returns for each GICS sector. An alternative metric of market-weighted returns could be computed but would be expected to produce similar covariance values. In practical terms, computing market-weighted index required relying on the availability of the market weights for each of the 4730 equities in my universe. 282 further market cap values were missing from my universe. While I have no reason to suspect a biased omission of that market-cap data, such a suspicion could not have been cleared without further data analysis.

The task of testing and applying this method to international portfolios denominated in multiple currencies would present another order of complexity in adding the currency exposure and discontinuous trading horizons (time-zones) risks to the optimization. The former could be simplified by assuming we would treat our international portfolio as US$ denominated by hedging-away the currency risk at the portfolio level. The latter presents a structural problem of trading in the markets were one can not execute the long and the short legs of a pair trade at one continuous point in time. One could imagine short-term hedging TE execution risk through international markets' futures. However, the method of executing optimized pair trades would be severely compromised in practice if such a trade can not be put on at a singular point in time.

5. Numerical Analysis

In quantifying the practical benefits of the proposed portfolio trading strategy, I need to benchmark it against the existing industry norms. This will allow me to measure the benefit out the proposed method in minimizing the Tracking Error (TE) relative to the status quo. The second benefit is in freeing me from the practical complications of selecting and fitting a market model. For the purpose of this research effort, I will concentrate my analysis on estimating the relative improvement derived from employing the proposed PTO method from the industry standard approach of minimizing sector imbalances during transition.

The industry-standard practice calls for prioritizing trades around the goal of minimizing intra-sector imbalances by dollar-value exposure. In this way, the largest dollar imbalances will get traded away ahead of the positions requiring the smallest net dollar adjustments.

As the benchmark, we will rely on the industry standard approach of striving for balanced sector weights between the "from" and the "to" portfolios.

5.1. Implementation Roadmap

The PTO method takes three key inputs: the covariance matrix of individual securities and the relative weights of the "from" and the "to" portfolios. The sum of initial weights of the two portfolios is standardized to add up to one.

I construct my covariance matrix based off the daily closing prices of publicly listed US equities taken during the 90 day period ending on August 7^{th} 2006.

The "to be traded" or "delta weights" portfolio is defined as the vector weight difference between the "From" and the "To" portfolios). The "sell" or "over-weight" positions are represented by negative delta weights. The "buy" or "under-weight" positions have positive delta weight values.

The portfolio of delta weights will gradually decay to zero with each trade as the portfolio transition is progressing.

I than proceed to construct three (3) trading paths:

1. The baseline relying on the balanced-sector approach
2. The proposed PTO method
3. The modified-PTO method that proxies individual equities' covariance with the covariance attributes of their corresponding sectors. This third approach could be employed for the assets that are missing historical data or which are highly illiquid, resulting in biased record of historical prices.

After each trade, I update the "delta weights" vector and measure the estimated tracking error (TE) of the remaining positions until the portfolio transition is complete. The cumulative TE measures are tracked for all three paths so that the TE-improvement effect from all three strategies can be measured.

In the end, I plot all three TE decay paths as a function of portfolio transition progress. The transition progress is measured by the percentage reduction in the delta weights portfolio from the original amount that is defined as the sum of absolute values of all delta weights.

5.2. Practical Implementation Details and Assumptions

Computing the historical covariance matrix for the individual equities is an area open to further research. For the purpose of this experiment, I relied on 90 days of historical daily closing prices. For the equities whose historical return streams were unavailable, I set the default historical return stream to zeros. As a practical matter, the optimal covariance matrix estimation can be implemented as a stand-alone module that would refresh the data on the period basis and cached in memory to minimize the run-time complexity.

Given two portfolios' vector weights: "from" and the "to", I compute the delta transition positions' vector that becomes my "to be traded" requirements. The weights in the vector are computed as fractions of one (1) representing the relative weight of those positions to the overall portfolio value. The weights can be either positive (the positions need to be accumulated) or negative (reduced).

All three methods (PTO, modified- PTO and the baseline) will generate trading lists that side-step the issue of optimal trading size for the reasons stated earlier. For practical purposes, these lists can be fed to the trading algorithm of choice. It is my expectations that the trading strategy and the resulting market impact between the two trading strategies will be roughly equivalent, ceteris peribus. The alternative of simultaneously optimizing for both the minimum TE and Market Impact is both computationally unbounded and requires extensive Market Impact modeling and calibration.

The high-level architectural outline of the software implementation is shown in the appendices.

5.3. Computational Efficiency Analysis

Both the OPT (modified-OPT is computationally equivalent to core OPT) and the baseline methods generate recommended trade sequences. The computational complexity of generating this sequence for the baseline (minimal sector imbalance) method is linear due to the fact that the trade size for each asset is bounded only by the size of that asset in the delta weights portfolio. The baseline method never fragments that asset trades and therefore produces an N-order computational complexity (where N is the length of the delta weights vector).

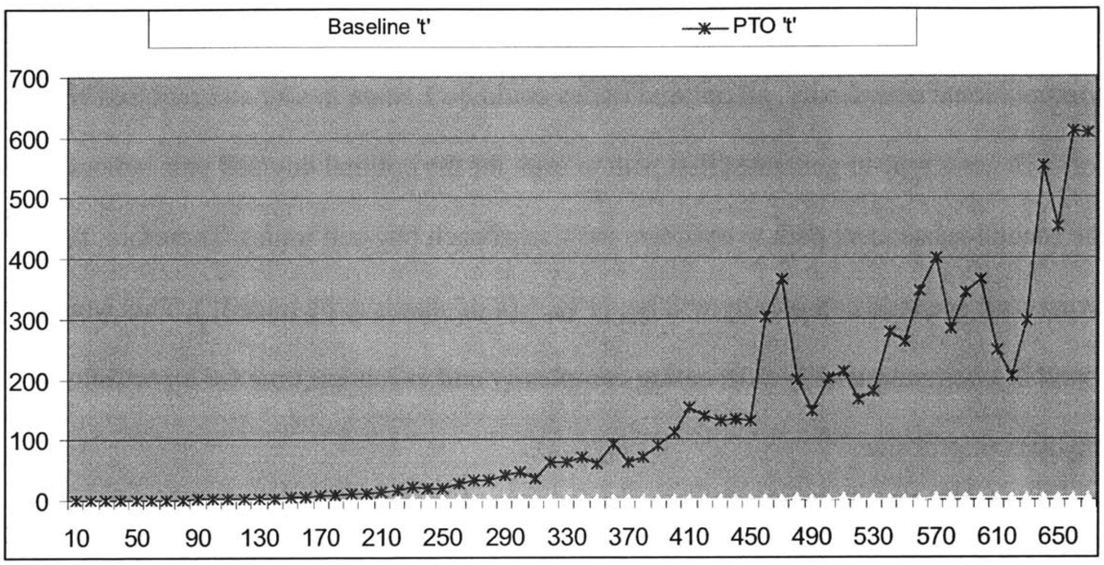
The OPT method does fragment the asset trades at the exact weight where the marginal TE of a given asset is superseded by that of another. Therefore, the computational complexity of the OPT method is significantly greater for at least two core computational loads:

1. Calculating the threshold of marginal TE improvement for alternative trades (worst case scenario order-N for each path through the delta weights portfolio)
2. Re-optimizing the trade for the same asset due to the fact that preceding trades only reduced it delta weight by the "optimal" amount as computed in item #1.

From the computational efficiency perspective, the OPT method theoretically unbounded outside of the minimum trade size constraint of 1 share. At the extreme end of the computational complexity, all optimal trades could be 1 share in size and required N^2 for each asset type to generate (first path to look for the optimal buy/sell pair indices and the second subsequent path to optimize the size of each buy/sell trade). Therefore, the worst-case scenario complexity will be: N^(2 * [# of shares to be traded]). This would result is a computationally suffocating complexity and execution time for a portfolio of any meaningful size.

However, the worst-case situation is neither the likely outcome, nor is it inevitable. One simple solution is introducing a minimum trade size constraint that would require trading of a certain amount of shares once the asset is picked for trading. For example, we could require trading at least 0.1% of average daily volume (ADV) or 1 % of assets size in delta weight portfolio. This way the worst-case scenario computational complexity will be bounded by: N^(2 * [1 / (average minimum asset size fraction)]). For the worst case of a portfolio of equally distributed asset sizes (all delta weights equal to each other) and the hypothetical minimum limit of 1% of individual asset's weight, the computational complexity will be: N ^(N/50). Still not pretty (N^N never is), but at least a well-defined outcome and a significant improvement on the previous worst-case scenario.

In reality, the empirical computational complexity, as measured by time required to generating a trade list as a function of portfolio size, growth as follows:



As demonstrated by the graph above, the time it takes to generate the optimal PTO strategy (as the number of trades that comprise it – see Appendix 8.1 for details) grows exponentially with the number of securities comprising the portfolio. The growth path graphed above is a bit jagged due to the sample size of one for each portfolio size increment.

Another measure of variability of the computational complexity is how the trade list length and the time it takes to compute it vary from one equal-sized portfolio to the next.

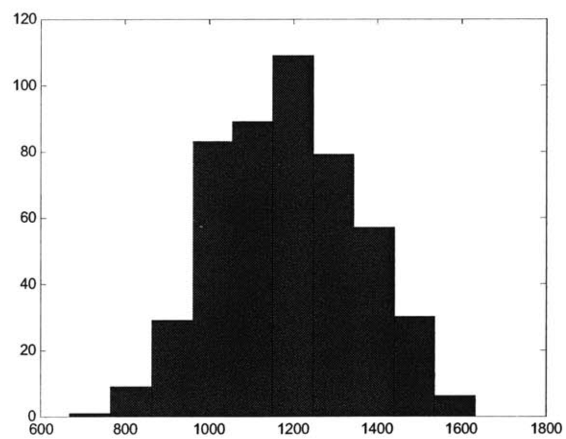
Below is an output from just such a test on 50 optimization runs of 100-equity large portfolios with the "minimum trade size" constraint set to 10^-6 (roughly $1,000 trade on $1B portfolio):

Portfolio Size: 100	Baseline		OPT		mod-OPT	
Sample Size: 100	time	# trades	time	# trades	time	# trades
Mean	0.0195	100	1.761522	1190.184	0.157988	126.6741
Stand Dev	0.0034	0.0	0.287327	168.379	0.018514	8.093

The variability of the length of the trading list is by far the greatest with our OPT strategy.

Size:	Baseline		OPT		mod-OPT	
100	time	# trades	time	# trades	time	# trades
Mean	0.02031	100	1.781082	1181.906	0.161645	126.263
Stand Dev	0.01257	0.0	0.372172	166.894	0.032108	7.748

Since the histogram of OPT-generated # of trades is roughly normal, I ran a t-test to compare the means of the two tests. The test failed to reject Ho that the two populations are different with significance of 0.4713 and confidence interval of [-28.6503, 13.2601]



Leaving all else equal but raising the "minimum trade size" constraint set to 10^-5 (roughly $10,000 trade on $1B portfolio) yields no statistically significant improvement in the trade quantity fragmentation.

6. Results

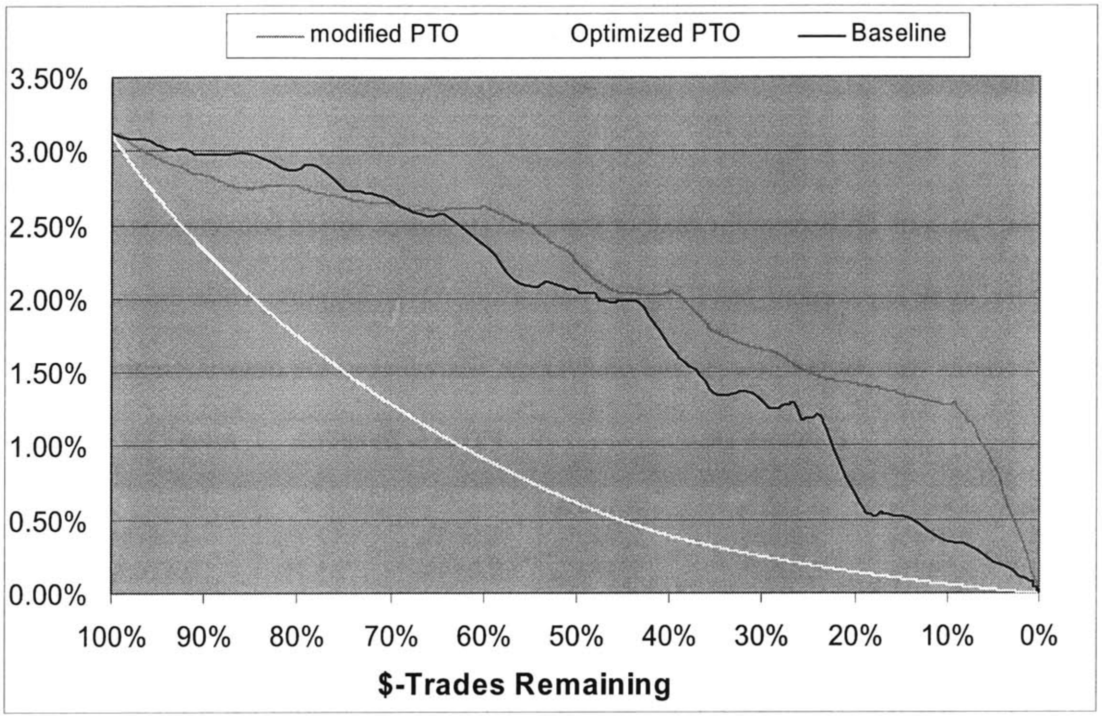
The results reported below are based on two categories of observations: random portfolio samples drawn from the population of equities and a sample real-world portfolio.

6.1. Simulation results

First, I ran a simulation on 50 portfolios (sample size = 50) of 100 randomly selected real-world equities with randomly assigned trading requirements (delta weights). I measured the tracking error (TE) decay for each of those 50 portfolios between the PTO, the modified-PTO (using sector returns and covariances in place of actual equity returns and covariances) and the sector-neutrality baseline method. The sample statistics for the PTO vs. Baseline and modified-PTO vs. Baseline cumulative TE decay out-performance are summarized in the table below:

	Baseline – PTO	Baseline –Mod-PTO
Min	46.32%	-72.03%
Mean	59.26%	-16.66%
Max	71.37%	16.93%
Std.Dev	6.91%	26.81%
t-test	60.68	-4.39
% > 0	100%	36%
Median	58.70%	-15.78%

The proposed PTO method has consistently and statistically significantly outperformed the Baseline method in every single experiment, by an average TE improvement of 59.26%. That is to say that over the sample space of 50 experiments, the average cumulative TE exposure of the PTO strategy was 59.26% lower than that of the baseline. Observing a graph of a one sample run below (one can not graph an "average" run since all 50 random portfolios had different TE decay paths and number of trades), the area below the PTO [yellow] curve is 59.26% less than the area below the [blue] baseline curve, on average.



Modified-PTO strategy is a statistically significant under-performer at the 5% confidence level, though it had randomly out-performed the Baseline on 36% sample runs. Modified-PTO relied on the strategy covariance in place of individual equity covariance. However, it used the actual equity covariances to calculate the tracking error. Not surprisingly, the modified-PTO method is "flying blind" and is consistently failing to deliver an improvement on the baseline. Its performance, as measured by TE decay, averaged out to be -16.66% that of the baseline. As measured by the TE decay, modified-PTO had underperformed all three trading strategies. However this outcome is a predictable result of the experiment set-up. The impetus to explore proxying individual covariances comes from the difficulty of estimating forward-looking covariance values even for the continuously trading securities. For illiquid securities, or the ones with unavailable historical data requires, such estimates may be hard to derive. Modified-PTO approach will need to be re-examined during the future in-/out-of-sample covariance estimation experiments.

The exact path of TE decays for each of the three strategies varied with each experiment. However, in all experiment, the PTO had outperformed the baseline while the modified-PTO's results were unpredictable and on average, the worst of the three strategies.

	Average TE			PTO vs. Baseline		mod-PTO vs. Baseline	
	Baseline	PTO	mod-PTO	Absolute	Relative	Absolute	Relative
0%	3.126%	3.126%	3.126%				
10%	3.044%	2.658%	2.990%	0.386%	12.7%	0.054%	1.8%
20%	2.942%	2.027%	2.768%	0.914%	31.1%	0.174%	5.9%
30%	2.760%	1.507%	2.688%	1.253%	45.4%	0.072%	2.6%
40%	2.573%	1.094%	2.613%	1.479%	57.5%	-0.039%	-1.5%
50%	2.116%	0.759%	2.464%	1.356%	64.1%	-0.348%	-16.5%
60%	1.992%	0.493%	2.071%	1.499%	75.3%	-0.079%	-4.0%
70%	1.420%	0.316%	1.810%	1.104%	77.8%	-0.391%	-27.5%
80%	1.194%	0.193%	1.518%	1.001%	83.8%	-0.324%	-27.2%
90%	0.510%	0.099%	1.380%	0.411%	80.6%	-0.870%	-170.6%
100%	0.077%	0.014%	0.889%	0.063%	81.5%	-0.811%	-1049.9%
		Net	Improvement:		50.82%		-13.76%
# trades	100	30,127	23,408				

Average number of trades was predictably equal to the size of the portfolio for the baseline method (order N), and roughly order 3*N^(N/50) for the PTO method. Modified PTO's complexity falls between the two.

6.2. Real-world results

The second type of empirical analysis was based on quantifying TE decay on a real-world portfolio transition consisting of 4020 individual equities from a $1B portfolio in custody of a Boston-area asset management firm. As this portfolio represented both a real-world trading task as well as a large-scale trading optimization problem, it served as an interesting test of the how well my sampling methodology would predict a real world situation.

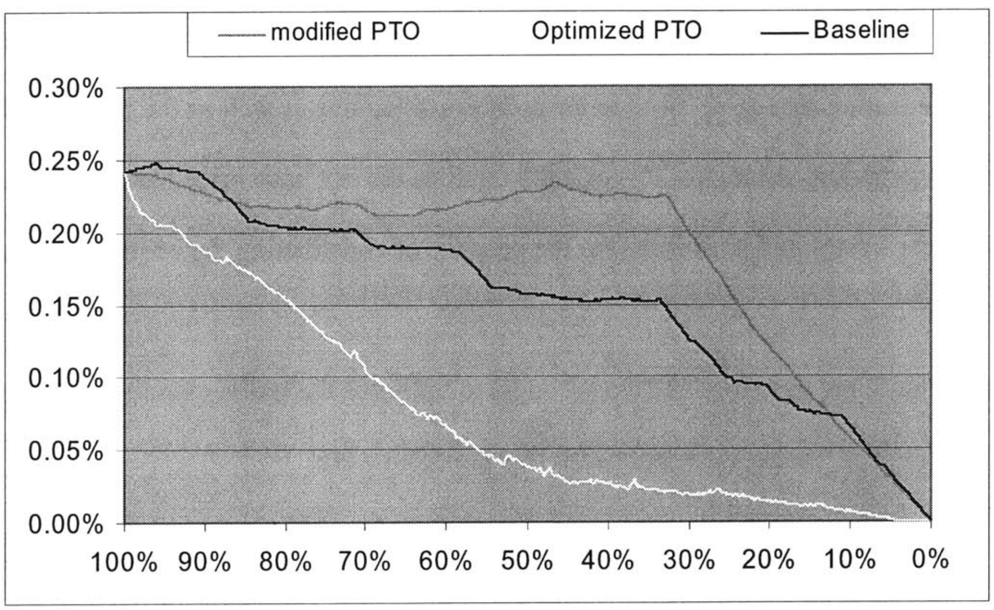
This was one of the larger portfolio optimization tasks that I have run through my three trade optimization strategies. Both in terms of portfolio size as well as the time it took to optimize the trading strategies. 395 of 4020 equities did not have price history and their returns were substituted with zeros for the purpose of constructing the covariance matrix.

The computational time required (in seconds) to optimize the trading strategies and the resulting number trading instructions for this portfolio were:

	Time (sec)	Time (min)	# of Trades
Baseline	475.46	7.9	4020
PTO	5323.9	88.7	4021
Mod-PTO	4695	78.3	4021

It took 5323.9 seconds (88.7 minutes) for PTO to generate the optimal trading strategy on a 1,700 Mhz Intel Pentium M processor with 600 Mhz bus and 1.5 Gigs of RAM. The resulting PTO trade list contained fewer than expected 4,021 individual pair trade instructions. As a reference point, the baseline portfolio took 7.9 seconds to generate and predictably consisted of 4020 pair trades.

The performance of the three trading strategies is graphed below. The graph of TE decay looks similar to those produced by the random control sample experiments. Namely, the PTO strategy outperforms the baseline throughout by the cumulative 28.36%. The performance of the modified-PTO is inferior by -19.69%.



Progress:	Average TE Baseline	PTO	mod-PTO	PTO vs. Baseline Absolute	Relative	mod-PTO vs. Baseline Absolute	Relative
0%	0.242%	0.242%	0.242%				
10%	0.244%	0.205%	0.238%	0.038%	15.7%	0.005%	2.2%
20%	0.220%	0.165%	0.220%	0.055%	25.2%	0.000%	-0.1%
30%	0.201%	0.136%	0.218%	0.065%	32.4%	-0.017%	-8.7%
40%	0.189%	0.120%	0.213%	0.070%	36.9%	-0.023%	-12.2%
50%	0.174%	0.114%	0.222%	0.060%	34.3%	-0.048%	-27.8%
60%	0.154%	0.116%	0.228%	0.038%	24.4%	-0.074%	-47.9%
70%	0.145%	0.128%	0.223%	0.016%	11.4%	-0.078%	-53.9%
80%	0.108%	0.087%	0.162%	0.021%	19.6%	-0.054%	-49.6%
90%	0.077%	0.028%	0.084%	0.049%	63.3%	-0.006%	-8.2%
100%	0.024%	0.001%	0.031%	0.023%	95.3%	-0.007%	-28.5%
			Net Improvement	0.436%	28.36%	-0.302%	-19.69%

The principal difference between these real-world example and the randomly constructed sample portfolios is the relative imbalances between a few "accumulate" orders (122 of them) and the glut of small "reduce" orders (1983 in all). The resulting relative out-performance over the baseline falls well within one standard deviation improvement range around the test-sample mean.

7. Conclusion

The proposed Portfolio Trading Optimization method is an improvement on the industry standard portfolio trading strategy of minimizing intra-sector dollar imbalances. Across the sample of randomly drawn portfolio rebalances, the average reduction in the tracking error was on the order of 32.95% improvement over baseline.

The vast majority of the improvement is obtained in the first 50% of the trading by dollar volume.

The modified-PTO that proxies equity returns statistics with those of their sectors did not live up to the expectations and failed to outperform the baseline.

The PTO method is easy to apply interactively with acceptable run-time performance for portfolios up-to 1,000 assets. The exponential growth in complexity may limit the potential for live trading strategy re-calculations on portfolios in access of 5,000 assets without appreciable investment in computational hardware. Given the raid decline in marginal benefits in the second half of trading, it may be computationally attractive to explore a combination of early PTO-optimized trading with a transition to sector-neutral trading list prioritization for the remainder of the portfolio once the majority of the benefits have been realized.

8. Potential Future Research Directions

Areas of future research can be sub-divided into two broad categories. The first involves calibrating and optimizing the current model. The second extends it to incorporate market impact estimation. The latter will allow me to estimate and "firm quote" the absolute value of all costs for a potential portfolio transition, not just the relative improvement over a baseline.

8.1. Optimizations to the Current model

Developing the present model into a production-quality trading optimization engine will require calibrating baseline pre-trade estimates with the post-trade observations. The key input in our model is the individual equities' covariance matrix. At this time, I am using a

90 daily return data points to estimate the future covariance of the returns. In practice, it may prove worthwhile to explore more elaborate covariance sampling and estimation methods based on historical return patterns of varying length weighted in non-linear fashion. Further work in this direction will involve minimizing estimation error by comparing in- and out-of-sample TE decay performance. The currently discredited approach of proxying missing or highly volatile equity covariance estimates with sector or industry characteristics may prove to be worthwhile in the presence of estimation error.

Another area of current modeling optimization may involve reducing the computation complexity that hinders optimizations of 1,000+ element portfolios. Some of the computational complexity can be contained by marginally reducing the thoroughness of our optimization algorithm. For example, the second-order of magnitude traversing of the available trades during optimal trade size evaluation could be curtailed through the use of heuristics that would come up with a less precise but more computationally efficient estimator of optimal trade size. One extreme end of such simplification would be identifying the optimal trading pair and executing the maximum trading size allowable for the pair (min of the two dollar-value legs of the trade). This extreme simplification would obviously lead to decay in algorithm's TE improvement over baseline, but the exact magnitude of the decay may or may not prove to be a tolerable price to pay for return of real-time performance.

One could also explore the optimal calibration of the "minimal trade size" constraint. The benefit would come from the ability to solve larger portfolios and implement real-time updates to the trade-list as opportunistic trades are pursued by the human traders. The cost would come from a marginally less optimal outcome. The exact magnitude as well as the marginal costs and benefits of such a trade-off could be computed empirically.

A more elegant approach may involve estimating the maximum market-impact-free volume of trading that can be executed within a given time interval (i.e.: % of average daily volume per day) and allowing trade allocation of that magnitude without any further optimization. Since trading in the same security may appear at different points of the optimal trading path, combining and executing them within a discrete period of time would result in no real-world reduction in performance as long as such combinations would not produce additional market impact costs. However, for this approach to be utilized, we will need to extend our model to incorporate rudimentary market impact estimation.

8.2. Logical Extensions to the Current model

The far larger area of applying this research on the trading floor would require estimating total cost of portfolio transition, not just relative improvements over the baseline. This logical extension would require modeling, developing and calibrating temporary and permanent market impact models and optimizing the absolute cost of executing a proposed vector of trades. This effort would need to combine the research into optimal portfolio trading with that of market impact estimation and pre-/post-trade analysis. My humble efforts in pursuit of this research direction were constrained by both the complexity of modeling market impact modeling and the proprietary nature of the data used to calibrate and validate these models. Coupled with the time limitations of writing a master's thesis and graduating over a finite time horizon, this noble effort is left for the pursuit by those less constrained in their access to time, market infrastructure data and positive cash flows.

The most interesting task of trading off the market impact for tracking errors remains elusive due to the "curse of dimensionality" in attempting to simultaneously optimize for both the tracking error and the market impact across time. As evidenced by the exponential growth in complexity involved in optimizing the proposed trading strategy alone, and second order of computational complexity may prove to be unsustainable.

9. Appendices

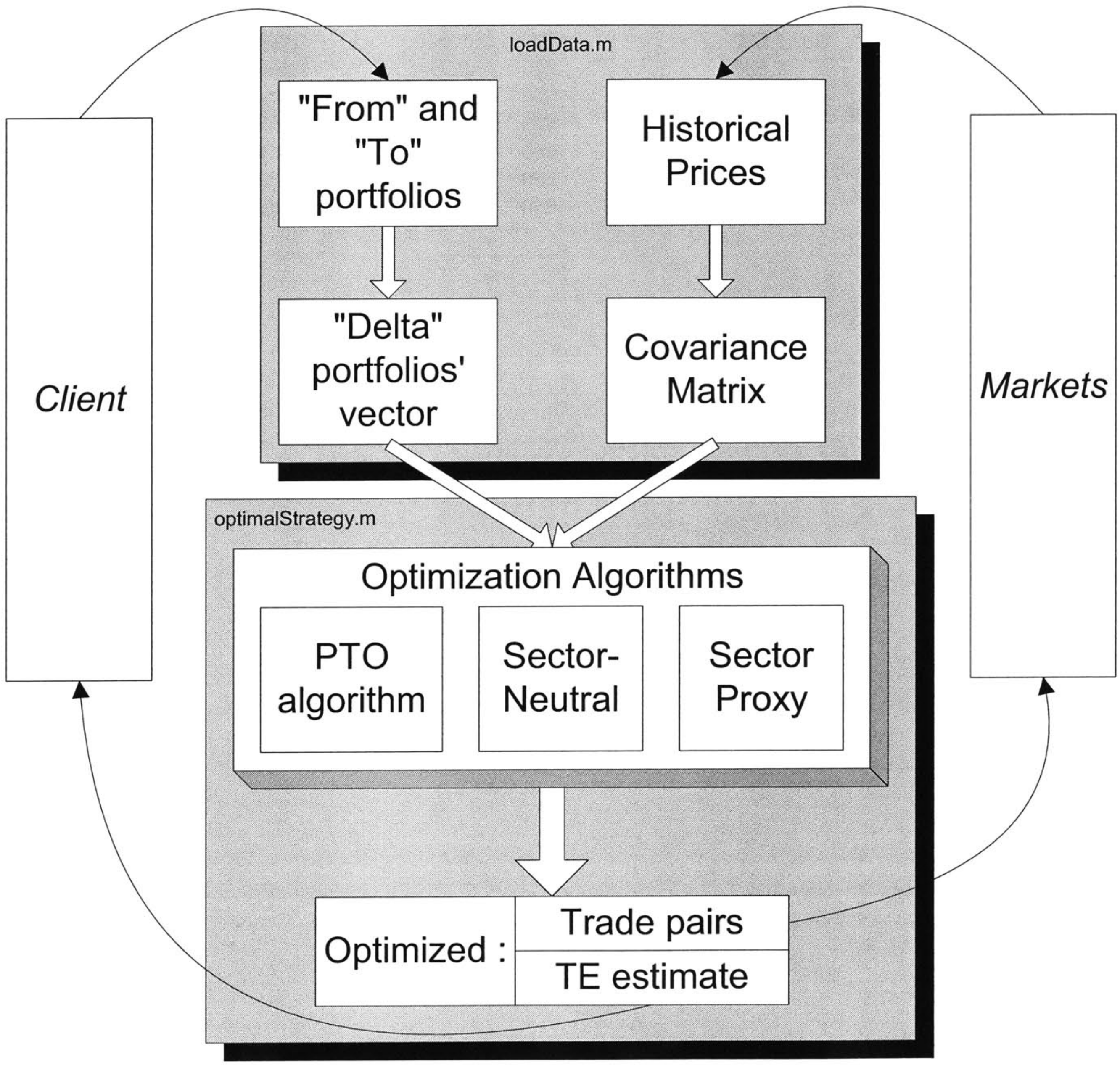
9.1. Computational complexity growth with Portfolio Size

Portfolio Size	Baseline 't'	Baseline # trades	PTO 't'	PTO # trades
100	0.02	100	1.762	1122
110	0.02	110	2.404	1470
120	0.02	120	2.934	1557
130	0.03	130	3.065	1415
140	0.03	140	3.705	1623
150	0.03	150	5.929	2344
160	0.03	160	6.319	2227
170	0.05	170	6.82	2209
180	0.08	180	8.853	2533
190	0.05	190	11.296	3234
200	0.06	200	11.426	3005
210	0.081	210	12.598	3008
220	0.07	220	16.534	3591
230	0.08	230	21.551	4749
240	0.09	240	19.177	3807
250	0.11	250	18.486	3224
260	0.1	260	26.729	4133
270	0.14	270	33.298	4817
280	0.18	280	33.609	5201
290	0.16	290	40.499	6123
300	0.19	300	46.737	6503
310	0.21	310	36.302	4572
320	0.24	320	62.45	7243
330	0.281	330	62.82	6438
340	0.311	340	70.391	7013
350	0.35	350	59.886	5330
360	0.362	360	92.894	9597
370	0.391	370	63.271	6008
380	0.45	380	70.902	6175
390	0.48	390	88.628	7182
400	0.551	400	110.368	8354
410	0.57	410	152.92	10582
420	0.621	420	138.75	9782
430	0.641	430	130.097	8728
440	0.699	440	132.997	8563
450	0.751	450	130.939	8097
460	0.832	460	301.419	13982
470	0.961	470	366.287	12173
480	2.173	480	200.118	10577
490	0.942	490	147.822	7522
500	1.142	500	201.008	9871
510	1.853	510	211.364	10441
520	1.102	520	166.389	8753
530	1.162	530	179.878	9364

540	1.172	540	276.838	13720
550	1.282	550	262.417	12990
560	1.332	560	345.234	15321
570	1.492	570	400.516	17581
580	1.472	580	282.136	12304
590	1.472	590	342.491	15981
600	1.582	600	365.095	16339
610	1.582	610	248.717	10884
620	1.672	620	204.234	9112
630	1.712	630	296.296	12419
640	1.862	640	551.794	14738
650	2.203	650	453.332	17208
660	1.883	660	610.382	20482

9.2. Code Architecture Outline

Rough outline of the program's architecture



9.3. PTO Trading Algorithm (MatLab implementation)

```
function [trades, strategy] = tradingStrategy(weightFrom, weightTo, teCV, cv, tickers, ...
   max_num_trades, min_trade, verbose)
%Do input values' error checking here...
% ...........
% ...........
%end-of error checking
strategy = zeros(max_num_trades,2);          %Init the strategy tracker array
lastTrade = 0;                               %Local Boolean
teFlag = 0;                                  %Print-out TE increase debug statements

% calculate the relative weigths
weights = weightFrom - weightTo;

% Calculate the derivatives
dW = 2 * cv * weights';                      %TE derivatives' = largest => trade first
te = sqrt(weights * teCV * weights');

trades(1).tickers = tickers;
trades(1).te = te;
trades(1).currentAUM = sum(abs(weights));
%trades(1).weights = weights;
trades(1).buy = {''};
trades(1).sell = {''};

strategy(1,1) = te;
strategy(1,2) = sum(abs(weights));

if(verbose)
   msg = ['TE: ' num2str(te,'%1.6f') ' Weights:' num2str(sum(abs(weights)),'%1.4f') ' Left:
' num2str(weights)];
   disp(msg);
else
   msg = ['Processed trade # :    0 TE: ' num2str(te,'%1.6f') ' Weights:'
num2str(sum(abs(weights)),'%1.4f')];
   disp(msg);
end

count = 2;              %Sadly, MatLab does not like index 0 references
                  %...and I subsequently refer to (count-1) for comparison
while sum(abs(weights)) > 0 && count <= max_num_trades
   if( rem(count,1000) == 0)
      msg = ['Processed trade # : ' num2str(count) ' TE: ' num2str(te,'%1.6f') ' Weights:'
num2str(sum(abs(weights)),'%1.4f')];
      disp(msg);
   end
```

```
try
   % calculate the current market impact (zeros now, could model later)
   mi = zeros(1, size(weights,2));

   ow = find(weights > 0);        %All Overweight/TO SELL indices
   uw = find(weights < 0);        %All Underweight/TO BUY indices

   [tmp, ow_deriv_sort] = sort(dW(ow));     %Overweight derivative indices, sorted
   [tmp, uw_deriv_sort] = sort(dW(uw));     %Underweight derivative indices, sorted


   % Step 1. Determine which assets to SELL...
   %        => pick highest Deriv's amongst Over/Under-weight available

   % if there is more than 1 asset left to SELL... && ow_deriv
   if length(ow) > 1 && abs(dW(ow(ow_deriv_sort(end))) -
dW(ow(ow_deriv_sort(end-1)))) < 0
      % sorting did not work?
      sell_index = ow(ow_deriv_sort(max(intersect(find(weights(ow(ow_deriv_sort)) >
0), find(strcmpi(tickers(ow(ow_deriv_sort)),trades(count-1).sell) == 0)))));
   else
      % sell the most overweight (last) one on the sort 'ow' array
      if (length(ow) > 0)
         %...if there is one to be sold...
         sell_index = ow(ow_deriv_sort(end));
      else
         %Exception case: we are left with a 1-leg trade...
         %...so pick any known 0-weight asset
         zeroWeights = find(weights == 0);
         sell_index = zeroWeights(end);  %as good as any
         lastTrade = 1;
      end
   end

   % if there is more than 1 asset left to BUY...
   if length(uw) > 1 && abs(dW(uw(uw_deriv_sort(1))) - dW(uw(uw_deriv_sort(2))))
< eps
      % sorting did not work?
      buy_index = uw(uw_deriv_sort(min(intersect(find(weights(uw(uw_deriv_sort)) <
0), find(strcmpi(tickers(uw(uw_deriv_sort)),trades(count-1).buy) == 0)))));
   else
      % buy the most underweight (first) one on the sort 'uw' array
      if (length(uw) > 0)
         %...if there is one to be sold...
         buy_index = uw(uw_deriv_sort(1));
```

```
    else
       %Exception case: we are left with a 1-leg trade...
       %...so pick any known 0-weight asset
       zeroWeights = find(weights == 0);
       buy_index = zeroWeights(end);   %as good as any
       lastTrade = 1;
    end
 end


 % Step 2. Determine optimal trade SIZE for BUY/SELL orders
 %       => pick weight of the highest Deriv's amongst Over/Under-weight available
 %       => adjust until Marginal Deriv w.r.t others == 0

 %INDEXES of ALL other ow/SELL candidates'
 adjacents = ow(find(abs(dW(ow) - dW(sell_index)) > 0));
 if length(adjacents) > 0
    % True, unless the LAST TRADE !!!
    sellShift = weights(sell_index);   %Initial Optimal SELL size
    if all(dW(adjacents) < dW(sell_index) + mi(sell_index))
       % <-- ALWAYS TRUE w/MKT Impact = 0
       m_fac = .5;
    else
       m_fac = 0;
    end
    for i = 1:length(adjacents)
       shift = min( ...
                 min( -(weights*(cv(sell_index,:)-cv(adjacents(i),:))' +
m_fac*mi(sell_index)) / (cv(sell_index,buy_index) - cv(sell_index,sell_index) -
cv(adjacents(i)...
                    ,buy_index) + cv(adjacents(i),sell_index)), weightFrom(sell_index)),
...
                 min(  -(weights*(cv(sell_index,:)-
cv(adjacents(i),:))'+m_fac*mi(sell_index)) / (cv(sell_index,buy_index)-
cv(sell_index,sell_index)-cv(adjacents(i),buy_index)+cv(adjacents(i)...
                    ,sell_index)), weightTo(buy_index)));
       if shift > 0 && shift < sellShift
          sellShift = shift;  % Marginal derivative adjustment threshold
       end
    end
 else
    % If last trade => sell it all...
    sellShift = weights(sell_index);
 end

 %INDEXES of ALL other ow/SELL candidates'
```

```
    % (based on how much thier derivs' are behind the top guy)
    adjacents = uw(find(abs(dW(uw) - dW(buy_index)) > 0));
    if length(adjacents) > 0
       % True, unless the LAST TRADE !!!
       buyShift = -weights(buy_index);   %Optimal BUY size
       if all(dW(adjacents) > dW(buy_index) + mi(buy_index))
         % <-- ALWAYS TRUE w/MKT Impact = 0
         m_fac = .5;
       else
         m_fac = 0;
       end
       for i = 1:length(adjacents)
           shift = min(min( -(weights*(cv(buy_index,:)-
cv(adjacents(i),:))'+m_fac*mi(buy_index))/(cv(buy_index,buy_index)-
cv(sell_index,buy_index)-cv(adjacents(i),buy_index)+cv(adjacents(i),sell_index)),
weightFrom(sell_index)), ...
              min( -(weights*(cv(buy_index,:)-cv(adjacents(i),:))' +
m_fac*mi(buy_index))/(cv(buy_index,buy_index)-cv(sell_index,buy_index)-
cv(adjacents(i),buy_index)+cv(adjacents(i),sell_index)), weightTo(buy_index)));
          if shift > 0 && shift < buyShift
             buyShift = shift;  % Marginal derivative adjustment threshold
          end
       end
    else
       % If last 'uw' position => last trade => buy it all...
       buyShift = -weights(buy_index);
    end

    % We <MAY> want to avoid tiny trades...
    % ... bump them up to min_trade size if a valid trade
    % ... (beyond optimal derivative threshold = Marginal Deriv above that of all others)
    sellShift = max(sellShift, min(min_trade, abs(weights(sell_index))));
    buyShift = max(buyShift, min(min_trade, abs(weights(buy_index))));

    % Execute the trade
    if (~lastTrade)
       maxTradeSize = min(buyShift, sellShift);
       weights(buy_index) = weights(buy_index) + maxTradeSize;  %Buy leg
       weights(sell_index) = weights(sell_index) - maxTradeSize; %Sell leg
    else
       weights(buy_index) = weights(buy_index) + buyShift;      %Buy leg
       weights(sell_index) = weights(sell_index) - sellShift;   %Sell leg
    end


    %Trading is done. Now record => don't mind errors here
```

```
        try
          % Record the strategy
          dW = 2 * cv * weights';
          te = sqrt(weights * teCV * weights');
          trades(count).buy = tickers(buy_index);
          trades(count).sell = tickers(sell_index);
          trades(count).tradeSize = maxTradeSize;
          %trades(count).weights = weights;
          trades(count).currentAUM = sum(abs(weights));
          trades(count).te = te;

          if(te > trades(count-1).te & teFlag & verbose)  %debug moment
            msg = ['TE alert: trade # ' num2str( count-1 ) ' => TE was: ' num2str(
trades(count-1).te ) ' is: ' num2str(te) ];
            disp(msg);
          end

          strategy(count,1) = te;
          strategy(count,2) = sum(abs(weights));

          if(verbose)
            msg = ['TE: ' num2str(te,'%1.6f') ' Weights:'
num2str(sum(abs(weights)),'%1.6f')...
              ' Trade size: ' num2str( maxTradeSize ) ' Left: ' num2str(weights)];
            disp(msg);
          end
        catch
                        [msgstr, msgid] = lasterr;
        end %finished recording

        count = count + 1;

      catch
      [msgstr, msgid] = lasterr;
      %s = lasterror(lasterr);
      msg = ['Error reading data on pass : ' int2str(count-1) ' : ' msgstr];
      disp(msg);
      count = count + 1;   %move on
      end

end %while loop

%Convert Strategy AUM values to % of original Total AUM
%strategy(:,2) = strategy(:,2)/strategy(1,2);
strategy = strategy(1:count-2, :);
clear tmp cv teCV;
```

9.4. Baseline Trading Algorithm (MatLab implementation)

```
function [trades, strategy] = tradingStrategy(weights, cv, tickers, sectors, ...
  max_num_trades, verbose)
% Default strategy optimizes for sector neutrality:
% 1. Rank all sectors w.r.t thier intra-sector imbalances
% 2. Initiate largest $-value pair-trades from the sector with the largest
%    imbalance. If a pair trade is not possible - execute single trade
% 3. Re-rank and repeat
%

counter = 1;

try
  while sum(abs(weights)) > eps && counter <= max_num_trades
    te = sqrt(weights' * cv * weights);
    sumTrades = sum( abs(weights) );

    strategy(counter,1) = te;
    strategy(counter,2) = sumTrades;

    %Record trade...
    trades(counter).counter = counter;
    trades(counter).te = te;
    trades(counter).sumTrades = sumTrades;
    trades(counter).positions = weights';

    if(verbose)
      msg = ['TE: ' num2str(te,'%1.8f') ' Left: ' num2str(weights')];
      disp(msg);
    end

    [s, maxImbalanceSector] = PrioritizeSectors(weights, sectors);

    currentSectorIndices = find(sectors == maxImbalanceSector);   %Over-weight/BUY
indices from equity array

    %Identify the largest Under- and Over-weight positions
    [tmp, rankedSectorWeights] = sort(weights(currentSectorIndices));

    smallestIndex = currentSectorIndices(rankedSectorWeights(1)); %largest Buy index
(as orders are filled, deltas converge to 0
    largestIndex = currentSectorIndices(rankedSectorWeights(end)); %largest Sell index
```

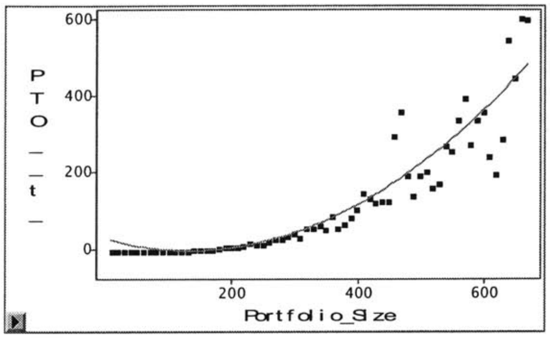
```
    %Execute trade
    if ( weights(smallestIndex) < 0 && weights(largestIndex) > 0 )
       %do pairs' trade
       maxTradeSize = min( abs(weights(smallestIndex)), abs(weights(largestIndex)) );

       weights(smallestIndex) = weights(smallestIndex) + maxTradeSize;   %Buy leg
       weights(largestIndex) = weights(largestIndex) - maxTradeSize;     %Sell leg

       %...Record trade
       trades(counter).size = maxTradeSize;
       trades(counter).sectorId = maxImbalanceSector;
       trades(counter).sectorName = convertSectorId(maxImbalanceSector);
       trades(counter).buyTicker = tickers(smallestIndex);
       trades(counter).sellTicker = tickers(largestIndex);
    else
       %do single trade (no offsetting leg available)
       if ( weights(smallestIndex) < 0)
          maxTradeSize = abs(weights(smallestIndex));
          weights(smallestIndex) = 0;   %Buy all

          %...Record trade
          trades(counter).size = maxTradeSize;
          trades(counter).sectorId = maxImbalanceSector;
          trades(counter).sectorName = convertSectorId(maxImbalanceSector);
          trades(counter).buyTicker = tickers(smallestIndex);
          trades(counter).sellTicker = '';

       elseif ( weights(largestIndex) > 0)
          maxTradeSize = abs(weights(largestIndex));
          weights(largestIndex) = 0;     %Sell all

          %...Record trade
          trades(counter).size = maxTradeSize;
          trades(counter).sectorId = maxImbalanceSector;
          trades(counter).sectorName = convertSectorId(maxImbalanceSector);
          trades(counter).buyTicker = '';
          trades(counter).sellTicker = tickers(largestIndex);

       end
    end

    counter = counter + 1;

 end %while loop

 strategy(counter,1) = sqrt(weights' * cv * weights);
```

```
    strategy(counter,2) = sum( abs(weights) );

    %Convert Strategy AUM values to % of original Total AUM
    %strategy(:,2) = strategy(:,2)/strategy(1,2);

catch
    [msgstr, msgid] = lasterr;
    msg = ['Error reading data on pass : ' int2str(counter)];
    disp(msg);
end
```

9.5. Empirical computational complexity of PTO algorithm

PTO_t_ = Portfolio_Size	
Response Distribution:	Normal
Link Function:	Identity

Model Equation
PTO_t_ = - 106.521 + 0.6979 Portfolio_Size



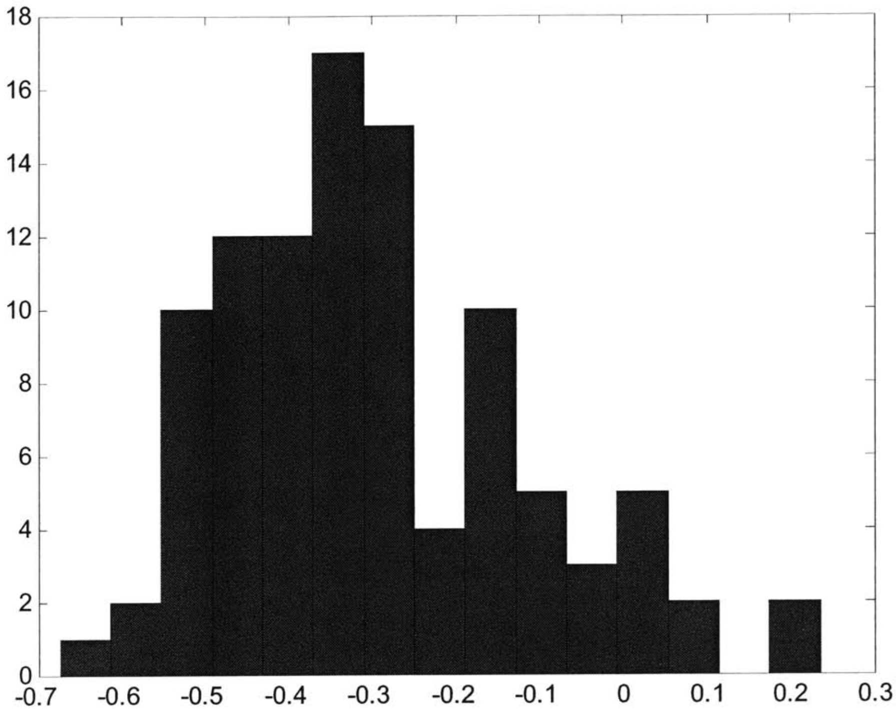
Parametric Regression Fit								
		Model		Error				
Curve	Degree(Polynomial)	DF	Mean Square	DF	Mean Square	R-Square	F Stat	Pr > F
	2	2	718576.320	64	3192.5229	0.8755	225.08	<.0001

Summary of Fit			
Mean of Response	130.7730	R-Square	0.7436
Root MSE	80.4708	Adj R-Sq	0.7396

Analysis of Variance					
Source	DF	Sum of Squares	Mean Square	F Stat	Pr > F
Model	1	1220563.79	1220563.79	188.49	<.0001
Error	65	420910.315	6475.5433		
C Total	66	1641474.10			

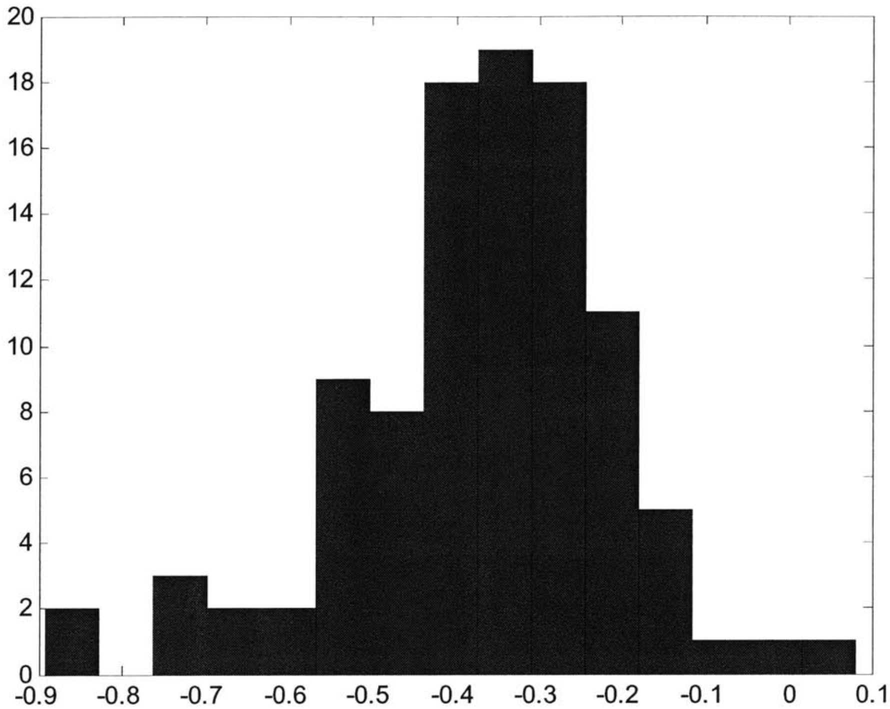
9.6. Histogram of % improvements of OPT vs. Baseline

(100-equity portfolio, 500 sample size)



9.7. Histogram of % improvements of modified-OPT vs. Baseline

(100-equity portfolio, 50 sample size)



9.8. Statistics for % improvements of OPT and modified-OPT vs. Baseline

	Baseline – PTO	Baseline – Mod-PTO
Min	46.3%	-72.0%
Mean	59.3%	-16.7%
Max	71.4%	16.9%
Std.Dev	6.9%	26.8%
t-test	60.68	(4.39)
% > 0	100%	36%
Median	58.7%	-15.8%

	OPT vs. Baseline	Modified-OPT vs. Baseline	
mean	-0.30078	-0.38291	mean
median	-0.31653	-0.36707	median
stdev	0.185125	0.179596	stdev
skew	0.474524	-1.19204	skew
kurtosis	0.587255	4.079812	kurtosis

9.9. Sample Baseline portfolio trading instructions

Sample 10-asset Portfolio (delta weights):

-0.077184	0.075589	0.028394	0.022737	-0.13134	-0.074829	0.056929	0.13146	-0.084622	0.052859

Trading flow:

TE: 0.00566935 Left:	-0.077184	0.075589	0.028394	0.022737	-0.13134	-0.074829	0.056929	0.13146	-0.084622	0.052859
TE: 0.00554969 Left:	-0.077184	0.075589	0.028394	0.022737	-0.13134	-0.074829	0.056929	0	-0.084622	0.052859
TE: 0.00463694 Left:	-0.077184	0.075589	0.028394	0.022737	0	-0.074829	0.056929	0	-0.084622	0.052859
TE: 0.00455377 Left:	-0.077184	0.075589	0.028394	0.022737	0	-0.074829	0	0	-0.084622	0.052859
TE: 0.00389197 Left:	-0.077184	0	0.028394	0.022737	0	-0.074829	0	0	-0.084622	0.052859
TE: 0.00329516 Left:	-0.077184	0	0.028394	0.022737	0	-0.074829	0	0	0	0.052859
TE: 0.00287678 Left:	-0.077184	0	0.028394	0.022737	0	0	0	0	0	0.052859
TE: 0.00215737 Left:	-0.054447	0	0.028394	0	0	0	0	0	0	0.052859
TE: 0.00098478 Left:	0	0	0.028394	0	0	0	0	0	0	0.052859
TE: 0.00087032 Left:	0	0	0.028394	0	0	0	0	0	0	0

...DEFAULT trading strategy done in: <10> trades

9.10. Sample OPT portfolio trading instructions

TE: 0.005669	Weights:0.7359 Left:	-0.077184	0.075589	0.028394	0.022737	-0.13134	-0.074829	0.056929	0.13146	-0.084622	0.052859
TE: 0.005073	Weights:0.6927 Left:	-0.05556	0.053965	0.028394	0.022737	-0.13134	-0.074829	0.056929	0.13146	-0.084622	0.052859
TE: 0.005073	Weights:0.6927 Left:	-0.055559	0.053964	0.028394	0.022737	-0.13134	-0.074829	0.056929	0.13146	-0.084622	0.052859
TE: 0.004711	Weights:0.6587 Left:	-0.038572	0.036977	0.028394	0.022737	-0.13134	-0.074829	0.056929	0.13146	-0.084622	0.052859
TE: 0.004711	Weights:0.6587 Left:	-0.038571	0.036976	0.028394	0.022737	-0.13134	-0.074829	0.056929	0.13146	-0.084622	0.052859
TE: 0.004664	Weights:0.6533 Left:	-0.035873	0.034278	0.028394	0.022737	-0.13134	-0.074829	0.056929	0.13146	-0.084622	0.052859
TE: 0.004442	Weights:0.6191 Left:	-0.018749	0.017154	0.028394	0.022737	-0.13134	-0.074829	0.056929	0.13146	-0.084622	0.052859
TE: 0.004442	Weights:0.6191 Left:	-0.018748	0.017153	0.028394	0.022737	-0.13134	-0.074829	0.056929	0.13146	-0.084622	0.052859
TE: 0.004428	Weights:0.6158 Left:	-0.017104	0.015509	0.028394	0.022737	-0.13134	-0.074829	0.056929	0.13146	-0.084622	0.052859
TE: 0.004417	Weights:0.6129 Left:	-0.015679	0.014084	0.028394	0.022737	-0.13134	-0.074829	0.056929	0.13146	-0.084622	0.052859
TE: 0.004365	Weights:0.5871 Left:	-0.002757	0.001162	0.028394	0.022737	-0.13134	-0.074829	0.056929	0.13146	-0.084622	0.052859
TE: 0.004364	Weights:0.5848 Left:	-0.0015949	0	0.028394	0.022737	-0.13134	-0.074829	0.056929	0.13146	-0.084622	0.052859
TE: 0.004349	Weights:0.5816 Left:	0	0	0.026799	0.022737	-0.13134	-0.074829	0.056929	0.13146	-0.084622	0.052859
TE: 0.004282	Weights:0.5761 Left:	0	0	0.024083	0.022737	-0.12862	-0.074829	0.056929	0.13146	-0.084622	0.052859
TE: 0.003798	Weights:0.5316 Left:	0	0	0.0018137	0.022737	-0.10635	-0.074829	0.056929	0.13146	-0.084622	0.052859
TE: 0.003791	Weights:0.5309 Left:	0	0	0.00145	0.022737	-0.10599	-0.074829	0.056929	0.13146	-0.084622	0.052859
TE: 0.003765	Weights:0.5280 Left:	0	0	0	0.022737	-0.10454	-0.074829	0.056929	0.13146	-0.084622	0.052859

TE: 0.002698 Weights:0.4246 Left:	0	0	0	0.022737	-0.052869	-0.074829	0.056929	0.079796	-0.084622	0.052859
TE: 0.002698 Weights:0.4246 Left:	0	0	0	0.022737	-0.052868	-0.074829	0.056929	0.079795	-0.084622	0.052859
TE: 0.002695 Weights:0.4244 Left:	0	0	0	0.022737	-0.052725	-0.074829	0.056929	0.079651	-0.084622	0.052859
TE: 0.002431 Weights:0.3859 Left:	0	0	0	0.022737	-0.03352	-0.074829	0.056929	0.060446	-0.084622	0.052859
TE: 0.002431 Weights:0.3859 Left:	0	0	0	0.022737	-0.033519	-0.074829	0.056929	0.060445	-0.084622	0.052859
TE: 0.002333 Weights:0.3647 Left:	0	0	0	0.022737	-0.022914	-0.074829	0.056929	0.04984	-0.084622	0.052859
TE: 0.002262 Weights:0.3189 Left:	0	0	0	0.022737	0	-0.074829	0.056929	0.026927	-0.084622	0.052859
TE: 0.001964 Weights:0.2650 Left:	0	0	0	0.022737	0	-0.074829	0.056929	0	-0.057695	0.052859
TE: 0.001852 Weights:0.2528 Left:	0	0	0	0.016609	0	-0.074829	0.056929	0	-0.051567	0.052859
TE: 0.001852 Weights:0.2528 Left:	0	0	0	0.016608	0	-0.074829	0.056929	0	-0.051566	0.052859
TE: 0.001702 Weights:0.2290 Left:	0	0	0	0.0047079	0	-0.074829	0.056929	0	-0.039666	0.052859
TE: 0.001673 Weights:0.2196 Left:	0	0	0	0	0	-0.074829	0.056929	0	-0.034958	0.052859
TE: 0.001312 Weights:0.1497 Left:	0	0	0	0	0	-0.074829	0.056929	0	0	0.017901
TE: 0.000998 Weights:0.1139 Left:	0	0	0	0	0	-0.056929	0.056929	0	0	0
TE: 0.000000 Weights:0.0000 Left:	0	0	0	0	0	0	2.7756e-016	0	0	0
TE: 0.000000 Weights:0.0000 Left:	0	0	0	0	0	0	0	0	0	0

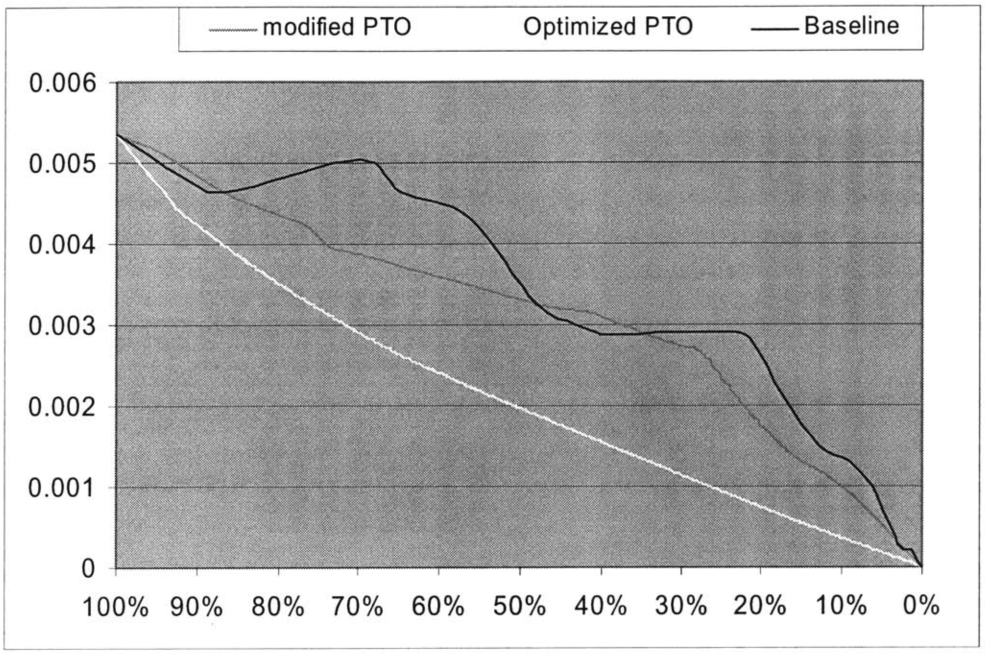
...OPTIMIZED trading strategy done in: <33> trades

9.11. Sample Modified OPT portfolio trading instructions

TE: 0.005669 Weights:0.7359 Left:	-0.077184	0.075589	0.028394	0.022737	-0.13134	-0.074829	0.056929	0.13146	-0.084622	0.052859
TE: 0.005355 Weights:0.6905 Left:	-0.077184	0.075589	0.028394	0	-0.1086	-0.074829	0.056929	0.13146	-0.084622	0.052859
TE: 0.005284 Weights:0.5847 Left:	-0.077184	0.075589	0.028394	0	-0.1086	-0.02197	0.056929	0.13146	-0.084622	0
TE: 0.005254 Weights:0.5781 Left:	-0.077184	0.075589	0.028394	0	-0.10528	-0.02197	0.053614	0.13146	-0.084622	0
TE: 0.005274 Weights:0.5342 Left:	-0.077184	0.075589	0.028394	0	-0.10528	0	0.031643	0.13146	-0.084622	0
TE: 0.005159 Weights:0.5014 Left:	-0.077184	0.075589	0.028394	0	-0.088878	0	0.015238	0.13146	-0.084622	0
TE: 0.005129 Weights:0.4903 Left:	-0.077184	0.075589	0.028394	0	-0.083328	0	0.0096875	0.13146	-0.084622	0
TE: 0.005129 Weights:0.4903 Left:	-0.077184	0.075589	0.028394	0	-0.083327	0	0.0096865	0.13146	-0.084622	0
TE: 0.005087 Weights:0.4709 Left:	-0.077184	0.075589	0.028394	0	-0.07364	0	0	0.13146	-0.084622	0
TE: 0.004525 Weights:0.3236 Left:	-0.077184	0.075589	0.028394	0	0	0	0	0.057823	-0.084622	0
TE: 0.003888 Weights:0.2080 Left:	-0.077184	0.075589	0.028394	0	0	0	0	0	-0.026799	0
TE: 0.003648 Weights:0.1544 Left:	-0.077184	0.075589	0.0015949	0	0	0	0	0	0	0
TE: 0.003611 Weights:0.1512 Left:	-0.075589	0.075589	0	0	0	0	0	0	0	0
TE: 0.000000 Weights:0.0000 Left:	0	2.7756e-016	0	0	0	0	0	0	0	0
TE: 0.000000 Weights:0.0000 Left:	0	0	0	0	0	0	0	0	0	0

...MODIFIED OPT trading strategy done in: <15> trades

9.12. Three Sample Strategies' comparative performance



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