Control of Mobile Networks Using Dynamic Vehicle Routing

by

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Abstract

This thesis considers the Dynamic Pickup and Delivery Problem (DPDP), a dynamic multi-stage vehicle routing problem in which each demand requires two spatially separated services: pickup service at its source location and then delivery service at its destination location. The Dynamic Pickup and Delivery Problem arises in many practical applications, including taxi and courier services, manufacturing and inventory routing, emergency services, mobile sensor networks, Unmanned Aerial Vehicle (UAV) routing, and delay tolerant wireless networks.

The main contribution of this thesis is the quantification of the delay performance of the Dynamic Pickup and Delivery Problem as a function of the number of vehicles, the total arrival rate of messages, the required message service times, the vehicle velocity, and the network area. Two lower bounds are derived. First, the Universal Lower Bound quantifies the impact of spatially separated service locations and system loading on average delay. The second lower bound is derived by reducing the two-stage Dynamic Pickup and Delivery Problem to the single-stage Dynamic Traveling Repairperson Problem (DTRP). Policies are then presented for which these lower bounds are tight as a function of the system scaling parameters (up to a constant). The impact of information and inter-vehicle relays is also studied.

The last part of this thesis examines the application of the Dynamic Pickup and Delivery Problem to mobile multi-agent wireless networks from a physical layer perspective, seeking insights for the control of the network to achieve trade-offs between throughput and delay.

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Chapter 1

Introduction

Mobile networks are characterized by a set of servers that travel throughout a given region to collectively perform a set of tasks. The motion and service activity of the mobile servers is to be controlled to optimize some performance measure based on the completion of tasks and consumption of server resources. Several varieties of tasks may be defined, including surveillance coverage of the region, convergence to a fixed vehicle formation, and service of a series of externally generated demands. This thesis focuses on the latter type of mobile network as an instance of a Dynamic Vehicle Routing Problem.

Vehicle Routing Problems (VRPs) constitute a class of well-studied problems in the Operations Research and Applied Mathematics literature. The classical example of a VRP is the Traveling Salesperson Problem (TSP) in which a single server is to visit each member of a fixed set of locations such that the total travel cost is minimized. Each location may be viewed as a demand which is served when the server passes through that location. Many vehicle routing problems may be viewed as extensions of this classical static and deterministic VRP. Several extensions may be envisioned, including more complex types of demand service, dynamic arrival of demands, and the use of multiple servers.

In this thesis, we consider the Dynamic Pickup and Delivery Problem (DPDP), a dynamic multi-stage VRP in which each demand requires service at each of several spatially separated locations, specifically pickup service at its source location and then delivery service at its destination location elsewhere in region. The DPDP problem arises in many practical applications. For example, consider a scenario where people are demands who telephone a cab-service exchange to request a ride. The cab-service exchange is to decide which cab picks up (and delivers) each person and at what time, such that each customer is completely served with minimum average delay. This problem is also known as Dial-a-Ride problem (DARP). Other applications include courier services, manufacturing and inventory routing, less-than-truckload (LTL) trucking, emergency services, mobile sensor networks, and Unmanned Aerial Vehicle (UAV) routing. Surveys [13] and [25] contain references to several of these applications.

Of particular interest to this thesis is the quantification of the performance of the network as a function of several scaling parameters, including the number of vehicles, the total arrival rate of messages, the required service times, and to a lesser extent, the vehicle velocity and network area. Such analysis exists for a single-stage problem known as the Dynamic Traveling Repairperson Problem. Our results for the two-stage DPDP and the four-stage DPDP (with Relays) are the first of their kind. Besides analyzing system performance as a function of the scaling parameters, we also examine the impact of several other system qualities, including information structure and service type. Our results provide general methods which are different than those in the existing literature.

1.1 Literature Review

We first review the previous research on dynamic vehicle routing problems, including the DPDP, in the context of operations research. We then address existing research on other types of mobile networks, including pickup and delivery networks in which services may be performed remotely via wireless transmission.

1.1.1 Dynamic Vehicle Routing

The relevant research in dynamic vehicle routing may be grouped into three areas that address specific aspects of the Dynamic Pickup and Delivery Problem as an extension of the canonical vehicle routing problem, the Traveling Salesperson Problem (TSP). After reviewing previous research on the TSP, we first consider the impact of demand uncertainty in dynamic and stochastic problems. Next, we look at methods for incorporating multiple vehicles via a demand assignment component. Finally, we examine the impact of multi-stage demand service.

The Traveling Salesperson Problem (TSP)

In the classical static and deterministic vehicle routing problem, a fixed set of demands is specified a priori and the solution is a route through these demands that minimizes some collective cost of service, usually expected total travel time. The most well-studied static and deterministic vehicle routing problem is the Traveling Salesperson Problem, in which a salesman must determine the shortest route through a fixed set of cities in his territory. Classically, the TSP is formulated on an undirected graph with distances between cities denoted by edge weights between nodes. The TSP is known to be NP-complete. Various heuristics have been developed to find approximately optimal polynomial-time solutions to the TSP [19, 31] and its natural counterpart, the directed TSP [29].

In the Euclidean TSP, cities are taken to be points arbitrarily distributed in $\mathbb{R}^2$ with distances corresponding to the Euclidean distance measure. In the case that there are N cities uniformly distributed in a region of area $\mathcal{A}$, the expected length of the TSP tour scales as $L_N \approx \beta\sqrt{A}\sqrt{N}$ when $N \rightarrow \infty$. This asymptotic result is originally due to Beardwood, Halton, and Hammersley [3], but has been more recently studied in [17]. This theorem will be important to our analysis and will be stated precisely in Section 3.3. Heuristics for computing approximately optimal polynomial-time solutions to the Euclidean TSP are presented in [2].

Static to Dynamic

In a static and deterministic VRP, the set of demands is fixed a priori, and the vehicle control consists of computing a single route through these demands to optimize a given cost function. In a dynamic and stochastic VRP, new demands arrive according to a stochastic process over time, and the solution is a control policy that that determines how the vehicles' routes evolve as a function of the demands in the system. A common objective is to minimize the average message time in system over an infinite time horizon rather than completion time of the fixed set.

Dynamic vehicle routing problems have received much less attention than their static counterparts. Recent surveys on dynamic vehicle routing problems include [13, 24]. Common solution methods for the dynamic problem include the reduction of the dynamic problem to a series of static problems via periodic reoptimization or batching.

From a theoretical standpoint, the most significant analysis of dynamic vehicle routing problems is the the work on the single-stage Dynamic Traveling Repairperson Problem (DTRP) by Bertsimas and van Ryzin [4, 5, 6]. These papers obtain several policies which achieve order-optimal average delay for the problem.

Intuitively, the DPDP seems similar to the DTRP as both the pickup and delivery of messages in the DPDP could be treated as separate requests in the DTRP problem setup. This thesis shows that this is indeed the case when messages may be relayed between vehicles. However, when we include the restriction that the vehicle that picks up a message must also deliver it, the pickup and delivery services of a single message are strongly linked, making our problem significantly distinct from the DTRP. As we shall see, the optimal solutions to these problems are both qualitatively and quantitatively different.

Single-vehicle to Multi-vehicle

When there are multiple vehicles to perform the system services, the optimal solution becomes more complex as not only service order but also service assignments to the vehicles must be determined.

Typical assignment methods for the static single-stage problem rely on the intuition that demands that are located close together ought to be served by the same vehicle. Two popular algorithms incorporate this intuition via a two-step process. In partitioning algorithms, an optimal TSP tour is found through all of the demands and then the tour is partitioned such that each vehicle travels the TSP route through a subset of the demands and the total vehicle travel cost is minimized. Clustering algorithms take the reverse approach, first assigning each vehicle to a subset of points and then leaving each vehicle to determine the TSP tour through just its own subset [25].

This intuition does not extend to the Pickup and Delivery Problem when the vehicle that picks up a message must also deliver it. Even if source locations are geographically close, destination locations may be spread throughout the region. Balancing the clustering of source locations and destination locations served by a single vehicle in a multi-vehicle setup will be an important intuitive insight in our work on the multi-vehicle DPDP.

Repair to Pickup and Delivery

Recent survey papers on Pickup and Delivery Problems (PDPs), both static and dynamic, include [9, 25]. We briefly summarize some of these results below.

The static single-vehicle pickup and delivery problem expands upon the TSP to include a delivery requirement. Not only does this delivery requirement double the number of locations to be visited, but it also imposes a precedence constraint on the order in which the locations may be visited. In the case that the objective is to minimize the completion time of a fixed set of demands, this may be formulated as a directed TSP. A straightforward dynamic program may also be used to solve for the optimal solution associated with other cost functions. Other solution methods include branch and bound algorithms that incorporate the precedence constraints. Approximations to the static PDP include clustering and routing algorithms similar to those for the TSP, although the analysis is much more complex for the two-stage problem (see [25]).

A problem similar to the Dynamic Pickup and Delivery Problem has been studied as the Online Dial-a-Ride Problem (OLDARP). In this problem, like the DTRP problem, demands arrive according to a Poisson process of time intensity λ. The messages need to be picked up by vehicles from a random arrival location and dropped off at random destination location. This problem has been studied by [10, 21]. Vehicles are usually assumed to have unit or finite capacity, that is, each vehicle can transport only one or finitely many messages at a time. The goal of the OLDARP problem is usually to minimize the service completion time of a collection of messages arriving during a finite time interval. Recent analysis in this problem has focused on competitive analysis to compare the performance of periodic reoptimization methods with static and deterministic solutions subject to time constraints.

The single-vehicle Dynamic Pick-up and Delivery Problem (DPDP) was analyzed in [32]. In this setup, a single service vehicle is responsible for picking up and delivering all messages that arrive. The goal is to minimize the average delay experienced by all messages. Analysis is performed for vehicles with both finite and infinite capacity, and several policies are analyzed in both the heavily and lightly loaded cases, using methods similar to those for the DTRP. In our work, we develop novel proof methods to study the multi-vehicle Dynamic Pickup and Delivery Problem.

1.1.2 Other Mobile Networks

The single-stage, multi-vehicle dynamic vehicle routing problem (i.e. the DTRP) has received attention in the controls community of late, motivated by applications to Unmanned Aerial Vehicles (UAVs) and other mobile sensing networks. A decentralized method for computing locally optimal solutions to the m-vehicle DTRP was presented in [11]. This work applied previous research on decentralized algorithms for the optimal placement of sensors to provide full coverage of a region with event locations represented by a continuous distribution [7]. The optimal sensor placement problem is closely related to the m-median problem and the optimal idle positioning of the vehicles in a lightly-loaded DTRP system. Other work has considered the impact of vehicle constraints, such as finite turning radius, on the solution of the static and deterministic Repairperson Problem [26, 27].

In general, there are several possible extensions of the classical dynamic vehicle routing problem, the Dynamic Traveling Repairperson Problem. Likewise, the Dynamic Pickup and Delivery Problem may be viewed as a two-stage extension of the single-stage DTRP.

1.1.3 Throughput and Delay in Wireless Networks

For many wireless network applications, including video and voice transmission, the goal is to provide a point-to-point path between each pair of nodes in the network such that any message that arises may be routed immediately to its destination. In contrast, in delay-tolerant networks, such a point-to-point path need not exist at each time instant, but the nodes may store the messages and forward them to the required connections over time as they move closer to other nodes in the network.

The main application we consider is a delay-tolerant communication network in which messages orginating in a geographic region must be delivered to their destinations elsewhere in the region. This service is carried out by a number of mobile nodes or vehicles. This differs from usual vehicle routing problems in that each node is capable of transmitting messages wirelessly to the vehicle and to other nodes.

A distinct characteristic of wireless transmission is *interference*: a wireless transmission by one node may adversely affect other transmissions occuring simultaneously. Interference generally implies that to allow the greatest number of nodes to transmit simultaneously, wireless transmissions should only take place over short distances to avoid creating interference over large regions. Since messages may be destined to locations far from their origin, several wireless transmissions may be required for delivery unless there is another method of transport. If the nodes are mobile, messages may also be physically carried over distances in the region. Physical transport of messages has the advantage that it does not create interference and many messages may be carried at one time by a single node. However, node velocity is typically much less than the speed of electromagnetic propagation of wireless transmissions. Therefore, physical transport is a much slower method of delivering messages.

Two important performance measures characterizing wireless networks are throughput and delay. Delay is defined, as in the DPDP, to be the time from message arrival to delivery. Throughput is defined to be the average number of bits to be delivered to their destinations each time unit.

Several previous works have considered varying the amounts of wireless transmission and physical transport of messages in communication networks to study the effect of these methods of message delivery on the throughput and delay characteristics of the network. Gupta and Kumar [16] introduced a random network model to study throughput scaling of fixed wireless networks in which nodes are not mobile and thus messages are delivered solely by wireless transmission. They showed that, under the random network model, the maximal achievable throughput per node, $T(n)$ scales as $\Theta(1/\sqrt{n \log n})$ for a network of n nodes (see Section 2.1 for a definition of order notation). That is, as the number of nodes and the traffic they bring to the network increases, the throughput achievable by each node goes to 0. Subsequently, Grossglauser and Tse [15] showed that by using node mobility, it is possible to achieve optimal per node throughput scaling, $T(n) = \Theta(1)$. That is, by using physical transport to carry messages without creating wireless interference, the throughput per node remains constant regardless of the number of nodes.

These results, however did not address the issue of delay performance. El Gamal, Mammen, Prabhakar and Shah [12] posed the question of achievable throughput and delay tradeoff. They obtained the following optimal tradeoff: (a) for fixed random networks, the throughput per node, $T(n)$ and delay per packet, $D(n)$ are related as $T(n) = \Theta(D(n)/n)$ for $T(n) = O(1/\sqrt{n \log n})$; and (b) for mobile networks with each node performing independent random walk, for most of the throughput, the delay scales $D(n) = \Theta(n \log n)$. The result of [12] for mobile networks provides a pessimistic conclusion: even at the loss of significant throughput, the delay can not be reduced under a random walk based mobility model.

In search of better delay scaling, various authors [20, 22, 33] have suggested different mobility models. While some of these models provide significant delay reduction at the loss of throughput, they are far from being realistic. Many ignore the physical constraint on the velocity of node. Most assume that node motion is completely random, regardless of the current message delivery requirements. In summary, most of the previous results assume a certain mobility model in order to study delay and throughput of network. Because they consider specific models, they are not able to make statements on optimal performance achievable under any mobility model.

The work of [28] has recently analyzed the minimum worst case delay in a delay-tolerant network with controlled mobility. Other than that paper, little attention has been given to the throughput/delay tradeoff problem with controlled mobility.

1.2 Results

The contributions of this thesis are divided into four main chapters. The first main chapter provides a general lower bound on the delay performance of any Pickup and

Delivery problem. The next two chapters are divided according to whether messages can be relayed between vehicles. The no relay problem is more closely related to classical vehicle routing as described above. Finally, the fourth main chapter contains some preliminary work on the mobile wireless network problem.

The optimality of our results is stated in terms of order optimality, that is, optimal scaling of performance as a function of the system parameters, such as arrival rate, number of vehicles, and the area of the region. We do not seek the optimal solution for a specific realization of the stochastic network. The nontriviality of the order optimal bounds we derive reveals the complexity of finding complete optimal solutions.

1.2.1 Universal Lower Bounds for Dynamic Pickup and Delivery

In the Dynamic Pickup and Delivery Problem, vehicles must pause at a service location for the duration of the message pickup and delivery service times. This implies that while vehicles are traveling, they are not performing work, in the sense of directly servicing a single message. This restriction implies a lower bound on message delay which we will call a *Universal Lower Bound.* This lower bound makes an important connection between dynamic vehicle routing problems and non-work-conserving queueing systems. This bound may also be generalized for analysis of other multi-stage systems.

1.2.2 Dynamic Pickup and Delivery with No Relays

The Dynamic Pickup and Delivery Problem with No Relays refers to the case in which the vehicle that picks up a message must be the one to deliver it. The goal is to find bounds on the minimum average message delay achievable by any valid control policy for the DPDP.

Control policies are divided into two categories based on the information structure in place for making the control decisions. In the Source Only structure, only message source locations are known before the message is picked up. In the Source and Destination structure, both the source and destination locations of messages are known as soon as the message arrives. We will prove lower bounds on the average message delays achieveable by control policies from these two groups. We will further propose policies that adhere to these information structures and will show that the order of the asymptotic delay scaling demonstrated by these policies matches that of the lower bounds for all scaling ranges of the arrival rate λ as a function of n. Therefore these policies are order optimal and the lower bounds may be achieved.

The lower bound results are achieved by formulating the multi-vehicle control policies as a collection of joint source and destination densities that capture the assignment policy for each of the vehicles. Existing results in the single-stage Dynamic Traveling Repairperson Problem (DTRP) and joint constraints on the probability densities of valid control policies form an optimization problem which may then be used to lower bound the delay for any control policy. The effects of the information are reflected in the joint constraints. Upper bounds are computed by computing the average delay for specific batching policies which are found to be order optimal. From a system design standpoint, these scalings quantify the perfomance improvements achievable by adding additional information gathering capabilities to the vehicles.

1.2.3 Dynamic Pickup and Delivery with Relays

As long as vehicles are required to perform physical pickups and deliveries at the source and destination locations, the DTRP lower bound serves as a lower bound on the DPDP problem. We show that this lower bound can be achieved by removing the restriction that the same vehicle that picks up a message is the one that delivers it. In fact, we show that this order optimal delay may be achieved with each message being relayed only once, and therefore additional relays cannot improve performance.

1.2.4 Throughput-Delay Tradeoff in Wireless Networks with Controlled Mobility

Previous analysis in throughput scaling as a function of n, the number of nodes in a wireless network, has focused on networks with fixed nodes or nodes with random mobility. In practice, one expects nodes to have control of their movement, and in fact, we might assume that the primary task of each node is to provide network infrastructure. Thus, we may use some combination of wireless transmission and dynamic vehicle routing to find an improved tradeoff. We consider using controlled mobility models in which vehicles (nodes) decide how to service the arriving messages.

As preliminary work towards this problem, we consider controlling a single vehicle to pick up streams of messages arriving at two locations. We answer several questions regarding the impact of vehicle motion on delay and stability. This preliminary analysis suggests a strong connection between the DPDP and the minimization of delay for high throughput networks. Analysis of low throughput networks requires an extension of the Dynamic Pickup and Delivery network model.

1.3 Organization of Thesis

The remainder of this thesis is organized as follows. Chapter 2 details the problem formulation. Chapter 3 provides some results from the existing literature that will be useful in the subsequent analysis. The main theorems of the Dynamic Pickup and Delivery Problem are contained in Chapters 4-6. The wireless DPDP is addressed in Chapter 7. Finally, discussion and conclusions are contained in Chapter 8.

Chapter 2

Problem Formulation

2.1 Model

Let there be n vehicles, indexed by i, in a geographic area $\mathcal{A} \subset \mathbb{R}^2$, which is a convex, compact set with area A. For simplicity, we consider $\mathcal{A} = [0, \sqrt{A}]^2$, with the understanding that these results may be extended to other convex environments with the same area. Each vehicle may move in any direction at any time with a velocity of magnitude $\leq v$.

Messages are generated according to a Poisson process with rate $\lambda(n)$. Each message, indexed by j, requires a fixed deterministic onsite service time $\bar{s}(n)$ at each of two locations: pick up at its source $s(j)$ and delivery at its destination $d(j)$. Associated with each message are source and destination locations denoted by $s(j) \in \mathcal{A}$ and $d(j) \in \mathcal{A}$ respectively. Source locations are independently and identically distributed (IID) in $\mathcal{A}$ according to the distribution density $\phi_s : \mathcal{A} \to \mathbb{R}^+$. Similarly, destination locations are IID with density $\phi_d : \mathcal{A} \to \mathbb{R}^+$. In this paper, we assume that both source and destination locations have uniform distribution on $[0, \sqrt{A}]^2$, that is $\phi_s(\zeta) = \phi_d(\zeta) = \frac{1}{A}, \forall \zeta \in \mathcal{A}$.

The task of the vehicles is to pick up messages from their source locations and deliver them to their destinations. A vehicle picks up a message by spending $\bar{s}(n)$ at the message's source location, after which it is said to be carrying the message. A vehicle delivers a message that it is carrying when it spends another $\bar{s}(n)$ at the message's delivery location. Each vehicle can carry an unlimited number of messages at any time.

For a given system, $\lambda(n)$ and $\bar{s}(n)$ are fixed constants, but are expressed as a function of n to emphasize the connection between the arrival rate $\lambda(n)$, the number of servers n, and the maximum onsite service time that may be supported in a stable system. Further discussion of the stability condition may be found in Section 2.3. We will use the following order notation to express the scaling of $\lambda(n)$ and $\bar{s}(n)$ as a function of n:

(*i*) $f(n) = O(g(n))$ means that $\exists$ a constant c and integer N such that $f(n) \leq cg(n), \forall n \geq N$.

(*ii*) $f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$.

(*iii*) $f(n) = \Theta(g(n))$ means that $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

(*iv*) $f(n) = o(g(n))$ means that $f(n) = O(g(n))$ but $g(n) \neq O(f(n))$, that is, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

For ease of notation, we will sometimes use λ for $\lambda(n)$ and $\bar{s}$ for $\bar{s}(n)$.

A vehicle that is carrying a message may either carry the message all the way to its destination or it may relay the message to another vehicle for delivery. To perform a relay, both vehicles involved in the relay (sender and receiver) must be co-located at an arbitrary service location for a full $\bar{s}(n)$ service time to complete the relay. Assume that for safety or other reasons, the colocated pair may perform the relay only if there are no other vehicle pairs within distance r of the relay service location. In this thesis, we will consider two special cases of the relay problem.

No Relays Messages may not be transferred between vehicles after they have been picked up. That is, the vehicle that picks up a message must be the one that delivers it.

Single Relay Each message may be transferred between vehicles exactly once between pickup and delivery. That is, exactly two vehicles are involved in the service of each message.

2.2 Control Policies

A control policy, π, is a set of decision making rules that decides the pickup and delivery schedule of arriving messages, based on a set of constraints on the information available to the vehicle. In this thesis, we consider policies $\pi = (\pi_A, \pi_S)$ that can be decomposed into two components, assignment and service. An assignment policy, π_A, describes how a centralized controller assigns arriving messages to vehicles for pickup, relay, and delivery on a real-time basis. A service policy, π_S, describes how each vehicle performs the pickup and delivery of its assigned messages. We assume that neither the vehicles nor the centralized assignment controller have any knowledge of individual messages before they arrive although the overall message arrival process and source and destination distributions are known.

2.2.1 Assignment Policies

Upon the arrival of a message, the centralized controller immediately assigns it to a single vehicle for pickup and also specifies which vehicle it will eventually be relayed to, if any. The message is not officially *assigned* to the relay vehicle until the moment the pickup and relay vehicles simultaneously begin the onsite relay service. The below assignment policy descriptions are valid for policies with at most one relay per message.

We limit our attention to time-invariant and spatially-based assignment policies where π_A is described by a collection of scaled densities $\{p_{i,l}(x,y)\}_{i=1,l=1}^{n,n}$ with the following property:

$$\sum_{l=1}^{n}\sum_{i=1}^{n} p_{i,l}(x,y) = \phi_S(x)\phi_D(y) = \frac{1}{A^2}, \forall x, y. \tag{2.1}$$

Informally, $p_{i,l}(x,y)$ is the probability that a randomly selected message arrives at x destined for y and is picked up by vehicle i and relayed to vehicle l. The precise operational meaning of $\{p_{i,l}(x,y)\}_{i=1,l=1}^{n,n}$ is defined below. First, we have the following two definitions:

$$\begin{aligned} p_{s(i,l)}(x) &= \int_{\mathcal{A}} p_{i,l}(x,y)dy, \\ p_{d(i,l)}(y) &= \int_{\mathcal{A}} p_{i,l}(x,y)dx. \end{aligned}$$

We restrict the set of assignment policies according to the information available to the controller in making message assignments for pickup and delivery. In particular, we consider two types of information structure: Source Only Information and Source and Destination Information.

Source only information

When a message arrives, its source location is known to the centralized controller, but vehicles do not know the destination of messages until they pick them up. When a message arrives at location x, the centralized controller randomly assigns the message to one of the pickup/relay pairs, with each assignment occuring with probability

$$P(\text{arrives at } x, \text{ assigned to } i \text{ and relayed to } l) = p_{s(i,l)}(x)/\phi_S(x) = p_{s(i,l)}(x)A.$$

Each assignment is made independently of all previous assignments.

Because the message assignment is made independent of the message destination, the source and destination locations served by a single vehicle are independent. That is, there exist two marginal densities $f_{i,l}(x)$ and $g_{i,l}(y)$ such that

$$p_{i,l}(x,y) = f_{i,l}(x)g_{i,l}(y).$$

Again, because destination information may not be exploited by the message assignments, the density of destination locations served by each vehicle pair must be the same as the overall density of destinations, that is, the marginal density

$$p_{d(i,l)}(y) = \phi_D(y) = \frac{1}{A}, \forall y, \forall i.$$

Using these facts, we may solve for $f_{i,l}(x)$ and $g_{i,l}(y)$ as follows:

$$\begin{aligned}
p_{d(i,l)}(y) &= \int_A f_{i,l}(x)g_{i,l}(y)dx = \int_A f_{i,l}(x)dx g_{i,l}(y) \\
&= g_{i,l}(y) = \phi_D(y) = \frac{1}{A}, \\
p_{s(i,l)}(x) &= \int_A f_{i,l}(x)g_{i,l}(y)dy = f_{i,l}(x)\int_A g_{i,l}(y)dy \\
&= f_{i,l}(x).
\end{aligned}$$

Therefore, $p_{i,l}(x,y)$ has the form

$$p_{i,l}(x,y) = p_{s(i,l)}(x)\phi_D(y) = p_{s(i,l)}(x)\frac{1}{A}, \forall x, y \in A, \forall i, l.$$

Let Π_{SO} denote the set of all policies that satisfy the assignment properties above and use Source Only information in making message assignments. Then Π_{SO} is described by

$$\Pi_{SO} = \{(\pi_A, \pi_S) | \sum_{l=1}^{n}\sum_{i=1}^{n} p_{i,l}(x,y) = \frac{1}{A^2}, p_{i,l}(x,y) = p_{s(i,l)}(x)\frac{1}{A}, \forall x, y, \forall i, l\}. \quad (2.2)$$

Source-destination information

When a message arrives, both its source and also its destination location are known to the centralized controller. The densities are used to make the message assignments in the following way. When a message arrives at location x that is destined for location y, the centralized controller randomly assigns the message to one of the vehicles, with each assignment occuring with the following probability:

$$\begin{aligned} P(\text{arrives at } x, \text{ destined for } y, \text{ assigned to } i, \text{ relayed to } l) &= p_{i,l}(x,y)/\phi_S(x)\phi_D(y) \\ &= p_{i,l}(x,y)A^2. \end{aligned}$$

Each assignment is made independently of all previous assignments. Under the Source and Destination information structure, destination information may be used to shape the destination density and therefore Equation (2.1) remains the only restriction on the assignment policy.

Let Π_{SD} denote the set of all policies satisfying the properties above and using only information available in the Source and Destination information structure. Then Π_{SD} is described by

$$\Pi_{SD} = \{(\pi_A, \pi_S) | \sum_{i=1}^{n} \sum_{l=1}^{n} p_{i,l}(x,y) = \frac{1}{A^2}, \forall x, y\}. \tag{2.3}$$

Special Case - No Relay

When no relays are allowed between vehicles, we may say that $p_{i,l}(x,y) = 0, \forall x, y, \forall i \neq l$. Dropping the l notation, the above expressions (2.2) and (2.3) simplify to

$$\Pi_{SO} = \{(\pi_A, \pi_S) | \sum_{i=1}^{n} p_i(x,y) = \frac{1}{A^2}, p_i(x,y) = p_{s(i)}(x)\frac{1}{A}, \forall x, y, \forall i\}, \tag{2.4}$$

$$\Pi_{SD} = \{(\pi_A, \pi_S) | \sum_{i=1}^{n} p_i(x,y) = \frac{1}{A^2}, \forall x, y\}. \tag{2.5}$$

2.2.2 Service Policies

While many of the lower bounds presented in this thesis are independent of the service policy specifics, some stronger results are available when a *batching* policy is used.

Definition 1 (Batch). *A batch is a set of requests for service, such that 1) all service requests within a single batch are assigned to a single vehicle, and 2) once a vehicle* *begins service of one of the requests in the batch, it completely serves all the requests in the batch, oblivious to other demands in the system.*

Definition 2 (Batching Policy). *Under a batching policy, each request for service is buffered at a batch processor upon arrival. Service requests are assigned to batches in some arbitrary way, and a request remains at the batch processor until the batch it is assigned to is released into the batch queue. Once a batch is released to the batch queue, no new service requests may be added or removed from the batch. Vehicles serve the batches from the batch queue one at a time.*

Requests for service may include the full service of a message as it arrives externally to the system, or the service policy may divide requests into a series of subrequests, each of which are treated individually under the batching policy. The defining characteristic of a batching policy is that a set of services is fixed and then carried out without interruption, deletions, or additions.

Batches are numbered in order of the release of the batch to the batch queue and are indexed by k. Let B_k be the number of requests contained in the k^{th} batch. The service of each batch has two components: 1) the onsite service of the B_k individual messages, and 2) travel and overhead time required to complete the batch service. Denote this overhead time by I_k. The total time to service the k^{th} batch is denoted as T_k, which is a function of B_k and I_k.

2.3 Performance Metrics

There are two main performance measures to be defined: stability and average delay. Informally, a system is *stable* if the messages arriving to the system have finite average delay between arrival and final delivery. The precise average delay differentiates the performance between various stable policies. Before precisely defining average delay and stability, we introduce some preliminary definitions.

2.3.1 Preliminary Definitions

Message j arrives at time t_j, completes pickup service at time v_j, and completes delivery service and departs the system at time y_j. With this notation, the arrival process is equivalent to

$$\Lambda(t) = \max\{j | t_j < t\}, \tag{2.6}$$

Further, we define counting processes associated with the cumulative pickup and delivery services, respectively. Because messages are not always served in the order in which they are received, these service counting processes are the cardinality of the given sets.

$$\mathcal{V}(t) = card\{j | v_j < t\} \tag{2.7}$$

$$D(t) = card\{j | y_j < t\} \tag{2.8}$$

Message j is said to be *assigned* to vehicle i at time t if either 1) the message is waiting for pickup by vehicle i or has already been picked up by vehicle i but not yet relayed or delivered or 2) the relay transmission of the message to vehicle i has been intiated and the message has not yet been completely serviced by that vehicle.

$$\mathbf{1}_{j,i}(t) = \begin{cases} 1 & \text{if message } j \text{ is assigned to vehicle } i \text{ at time } t \\ 0 & \text{else} \end{cases}$$

A vehicle is *traveling* if it is moving between service locations. A vehicle is in *onsite* service when it is stopped at a service location and performing pickup, relay or delivery service.

$$\mathbf{1}_{i,T}(t) = \begin{cases} 1 & \text{if vehicle } i \text{ is traveling at time } t \\ 0 & \text{else} \end{cases}$$

$$\mathbf{1}_{i,O}(t) = \begin{cases} 1 & \text{if vehicle } i \text{ is in onsite service at time } t \\ 0 & \text{else} \end{cases}$$

We assume that at any time there are messages in the system, the vehicle is either traveling or in onsite service, i.e.

$$\mathbf{1}_{j,i}(t) = \mathbf{1}_{j,i}(t)\left[\mathbf{1}_{i,T}(t) + \mathbf{1}_{i,O}(t)\right].$$

These indicator functions may be used to define various measures of delay for an individual message j, along with their limiting expectation.

We also introduce the following notation: let $\mathbb{E}_\theta[g(\cdot)]$ denote the Lebesgue integral of $g(\cdot)$ with respect to the variable θ. When $g(\cdot)$ has a single argument, the θ notation will be dropped. $\mathbb{E}[\cdot]$ is defined for a function of nonrandom arguments. This notation differs from $E[g(\cdot)]$ which will denote the expected value of a function of a random variable.

2.3.2 Average Delay

The total time that message j is in the system is defined to be

$$W(j) = y_j - t_j. \tag{2.9}$$

We may define several notions of average delay. Let Ω denote the set of all realizations of the system behavior. For a given realization $\omega \in \Omega$ of the system, we may define the following limit if it exists:

$$\overline{W}(\omega) = \lim_{j\to\infty} \frac{\sum_{j=1}^{J} W(j,\omega)}{J}.$$

Under suitable assumptions, $\overline{W}(\omega)$ exists and is equal to $\overline{W}(\omega) = \overline{W}$ with probability 1, and we have the following time average delay. The time average delay over all messages that pass through the system is defined to be

$$\overline{W} \;=\; \limsup_{J\to\infty} \frac{\sum_{j=1}^{J} W(j)}{J}. \tag{2.10}$$

Under a stronger set of assumptions, the following limiting distribution for the variable $\tilde{W}$ may be shown to exist,

$$P(\tilde{W} \leq w) \;\triangleq\; \lim_{j\to\infty} P(W(j) \leq w), \tag{2.11}$$

that is, $W(j)$ converges in distribution to $\tilde{W}$. Under suitable conditions, such as uniform integrability of the set $\{W(j)\}_{j=1}^{\infty}$, this convergence in distribution implies convergence in expectation when $W \triangleq E[\tilde{W}] < \infty$ exists (see [14], pp. 316 and 351).

$$W \triangleq E[\tilde{W}] = \lim_{j\to\infty} E[W(j)]. \tag{2.12}$$

Under one more set of assumptions, $\overline{W} = E[\tilde{W}]$. We shall assume that when $W = E[\tilde{W}]$ exists, it is equal to $\overline{W}$. In general, we will assume that the distributional expression for W is well defined and use this as our measure of average delay. Where this assumption is not required, we will use the $\bar{\cdot}$ notation to denote time average.

We will often be interested in the behavior of the above defined variables when the limits are taken over only the messages that are served by a single vehicle i. In this case, an additional subscript i will be used to denote the appropriate function for vehicle i. Define the subsequence of messages served by vehicle i as $\{(i_1, i_2, ..., i_j, ...) : W_i(i_j) > 0\}$. Then the time average delay at vehicle i for messages served by vehicle i is:

$$\overline{W}_i \;=\; \lim_{J\to\infty} \frac{\sum_{j=1}^{J} W(i_j)}{J} \tag{2.13}$$

when this limit exists. The limiting distributions and expected values W_i are defined similar to W above.

We will also be interested in differentiating the delay of a message while it is assigned to a vehicle that is traveling from the delay while it is assigned to a vehicle performing onsite service. The delay of message j at vehicle i while i is traveling is

$$W_{T,i}(j) = \int_0^\infty \mathbf{1}_{j,i}(\tau)\mathbf{1}_{i,T}(\tau)d\tau.$$

Likewise, the delay of message j at vehicle i while i is in onsite service of the message itself as well as any other services that occur while j is assigned to vehicle i is

$$W_{O,i}(j) = \int_0^\infty \mathbf{1}_{j,i}(\tau)\mathbf{1}_{i,O}(\tau)d\tau.$$

Therefore, the delay of a single message j while it is either in service or in queue for vehicle i is

$$W_i(j) = W_{T,i}(j) + W_{O,i}(j).$$

The total delay of a single message is then equivalent to

$$\begin{aligned} W(j) &= \sum_{i=1}^{n} W_i(j) \\ &= W_T(j) + W_O(j), \end{aligned}$$

where $W_T(j) = \sum_{i=1}^n W_{T,i}(j)$ and $W_O(j) = \sum_{i=1}^n W_{O,i}(j)$. For the No Relay DPDP, $W_i(j)$ is nonzero for exactly one i.

With these definitions, we have the following main definition of average delay that will be used in this thesis.

Definition 3 (Average Delay and Stability). *If the time average limits in* (2.10) *and* (2.13) *exist and are finite, the system is defined to be* stable. *Further, the total delay*

$\overline{W}$ is composed of two parts:

$$\overline{W} = \overline{W}_T + \overline{W}_O$$

where

$$\begin{aligned} \overline{W}_T &= \lim_{J\to\infty} \frac{\sum_{j=1}^{J} W_T(j)}{J} \\ \overline{W}_O &= \lim_{J\to\infty} \frac{\sum_{j=1}^{J} W_O(j)}{J} \end{aligned}$$

when these limits exist. Under suitable assumptions, $W = \overline{W}$ and we will drop the $\bar{\cdot}$ notation.

2.3.3 Arrival Rates and Alternative Representations of Delay

With an assignment policy $\pi_A = \{p_{i,l}(x,y)\}_{i=1,l=1}^{n,n}$, a randomly selected message is assigned to vehicle i and relayed to vehicle l with the following probability:

$$\begin{aligned} P(j,(i,l)) &= P(\text{message } j \text{ assigned to vehicle } i \text{ and relayed to vehicle } l) \\ &= \int_A \int_A p_{i,l}(x,y)dxdy. \end{aligned}$$

Because external arrivals are Poisson and assignments are independent, by the Poisson splitting property, the assignment process to each pair of vehicles is an independent Poisson process. Because the assignment to the pickup vehicle is immediate, messages arrive for pickup via a Poisson process of rate $\lambda \sum_{l=1}^{n} \int_A \int_A p_{i,l}(x,y)dxdy$. For the No Relay DPDP, this describes the complete arrival process and we define the arrival rate to vehicle i as

$$\text{No Relay:} \quad \lambda_i(\pi) = \lambda_i(\pi_A) = \lambda \int_A \int_A p_i(x,y)dxdy.$$

Combining equations (2.1) and (2.14) for No Relays implies that for any valid set of densities $\{p_i(x,y)\}_{i=1}^n$,

$$\text{No Relay:} \quad \sum_{i=1}^{n} \lambda_i(\pi) = \lambda. \tag{2.14}$$

The single relay DPDP includes internal arrivals due to message relaying in addition to the Poisson process of external arrivals described above. A more general definition of λ_i is required. Let $\Lambda_i(t)$ be the number of arrivals to vehicle i in the interval $[0,t]$, including both external arrivals to the system at vehicle i (new pickups) and also messages that are relayed to vehicle i for further service. Define the time average rate of arrivals to vehicle i to be

$$\lambda_i = \lim_{t\to\infty} \frac{\Lambda_i(t)}{t}. \tag{2.15}$$

where this limit is assumed to exist. The total arrival rate to the system (not including the internal arrivals) λ may be similarly defined as

$$\lambda = \lim_{t\to\infty} \frac{\Lambda(t)}{t}. \tag{2.16}$$

For the single relay system, it is possible for a message to be relayed from one vehicle to itself if the same vehicle handles both the pickup and the delivery of the message. For ease of exposition, this relay is counted as a new internal arrival to the vehicle when the message relay is initiated. Messages which will eventually be relayed to l arrive according to a Poisson process at rate $\lambda \sum_{i=1}^{n} \int_A \int_A p_{i,l}(x,y)dxdy$. Although the timing of the assignment of the relayed messages to the vehicles depends on the service policy in place, we assume that condition (2.15) holds for any policy under consideration. Combining these two types of arrivals, messages arrive for either pickup or delivery to a single vehicle with rate

$$\text{Single Relay:} \quad \lambda_i(\pi) = \lambda_i(\pi_A) = \lambda \left[\sum_{l=1}^{n} \int_A \int_A p_{i,l}(x,y)dxdy + \sum_{l=1}^{n} \int_A \int_A p_{l,i}(x,y)dxdy \right]. \tag{2.17}$$

Because each message is handled by exactly two vehicles, combining (2.1) and (2.17) yields

$$\text{Single Relay:}\quad \sum_{i=1}^{n} \lambda_i = 2\lambda. \tag{2.18}$$

Noting that the limits and the definitions of λ and λ_i in (2.16) and (2.15) respectively apply regardless of relay for Poisson processes as well, we have the following equivalent representation for W.

$$\begin{aligned}
\overline{W} &= \lim_{J\to\infty} \frac{\sum_{j=1}^{J} W(j)}{J} = \lim_{t\to\infty} \frac{\sum_{j=1}^{\Lambda(t)} W(j)}{\Lambda(t)} \\
&= \lim_{t\to\infty} \sum_{i=1}^{n} \frac{\Lambda_i(t)}{\Lambda(t)} \frac{\sum_{j=1}^{\Lambda_i(t)} W(i_j)}{\Lambda_i(t)} \\
&= \lim_{t\to\infty} \sum_{i=1}^{n} \frac{\Lambda_i(t)}{t} \frac{t}{\Lambda(t)} \frac{\sum_{j=1}^{\Lambda_i(t)} W(i_j)}{\Lambda_i(t)} \\
&= \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} \overline{W}_i.
\end{aligned} \tag{2.19}$$

Similar expressions may also be obtained for W_O and W_T.

$$\overline{W}_O = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} \overline{W}_{O,i} \tag{2.20}$$

$$\overline{W}_T = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} \overline{W}_{T,i} \tag{2.21}$$

Assume that $\overline{W}_i$ and $\overline{W}_{i,O}$ are both increasing functions of λ_i. That is, $\lambda_i \geq \lambda_{i'} \Rightarrow \overline{W}_i \geq \overline{W}_{i'}$ and similarly for $\overline{W}_{i,O}$. When a vehicle serves proportionally more messages, the average delay seen by messages served by that vehicle also increases. This is a natural assumption in the case where service locations are uniformly distributed and all onsite service times are iid.

2.3.4 Number in System

Define $N(t)$ to be the number of the messages in the system at time t. Because each message in the system is awaiting exactly one of two kinds of service at any time t, we further define $N_1(t)$ be the number that have arrived but have not been picked up, and let $N_2(t)$ be the number that have been picked up but not yet delivered. These three processes are defined in terms of the arrival and service counting processes as below.

$$N(t) = \Lambda(t) - D(t) = N_1(t) + N_2(t) \tag{2.22}$$

$$N_1(t) = \Lambda(t) - \mathcal{V}(t) \tag{2.23}$$

$$N_2(t) = \mathcal{V}(t) - D(t) \tag{2.24}$$

As above, there are multiple notions of delay for the number in system process. These definitions are shown for $N(t)$; $N_1(t)$ and $N_2(t)$ are similar.

$$\overline{N} \triangleq \lim_{t\to\infty} \frac{\int_0^t N(\zeta)d\zeta}{t} \tag{2.25}$$

$$P(\tilde{N} = k) \triangleq \lim_{t\to\infty} P(N(t) = k) \tag{2.26}$$

$$N = E[\tilde{N}] = \lim_{t\to\infty} E[N(t)] \tag{2.27}$$

It will also be useful to define limiting distributions for the number in system seen by an arrival or a departure where they exist:

$$P(\tilde{N}^- = k) \triangleq \lim_{j\to\infty} P(N(t_j) = k),$$
$$P(\tilde{N}^+ = k) \triangleq \lim_{j\to\infty} P(N(y_j) = k).$$

$\tilde{N}_1^-, \tilde{N}_1^+, \tilde{N}_2^-, \tilde{N}_2^+$ and their limiting expectations are defined similarly.

2.3.5 System Utilization and Stability

We may view the network of vehicles as a non-work-conserving, n-server system with service times defined to be onsite service only. A necessary stability condition is derived by comparing each vehicle to a work-conserving $GI/G/1$ queue with arrival rate λ_i and expected total service time $2\bar{s}(n)$ (see Section 3.2.2). The average utilization for this system is $\rho_i = \lambda_i 2\bar{s}(n)$, the product of the arrival rate and the service time per message. If either the service times or the arrival process is non-deterministic, then by classical queueing theory, a necessary condition for the stability of this system is

$$\rho_i \triangleq \lambda_i 2\bar{s}(n) < 1, \quad \forall i. \tag{2.28}$$

We define the total system utilization as

$$\rho = \frac{\sum_{i=1}^{n} \lambda_i \bar{s}}{n} = \frac{\sum_{i=1}^{n} \rho_i}{n}. \tag{2.29}$$

We demonstrate by construction in Section 5.2 that the following is a sufficient condition for the existence of a stable policy for the DPDP with No Relays:

$$\rho = \frac{2\lambda(n)\bar{s}(n)}{n} < 1. \tag{2.30}$$

For the single relay DPDP, due to the doubling of the total arrival rate, the equivalent sufficient condition as proven in Section 6.2 is

$$\rho = \frac{4\lambda(n)\bar{s}(n)}{n} < 1. \tag{2.31}$$

Note that while ρ is a function of the system parameters and the number of relays per message, the individual ρ_i are also a function of the specific control policy in effect.

As noted in the analysis of the DTRP in [4], the stability condition does not depend on the geometry of the system, i.e. the placement of the message service locations, but only on the net arrival rate of onsite workload. This stability condition extends for our case as well. Further, this sufficiency in independent of v as long as $v > 0$.

2.4 Problem Statement

We will call the above defined control problem the Dynamic Pickup and Delivery Problem (DPDP). The goal is to compute a tight lower bound on the average message delay, W, under any valid stable control policy for the DPDP for all ranges of the scaling parameters, $\lambda(n), n$, and $\bar{s}(n)$.

The separable policies considered here may be described by a collection of densities $\{p_{i,l}(x,y)\}_{i=1,l=1}^{n,n}$ with the information constraints described in the set description, plus the description of single vehicle service policies. The assignment policy is captured by the following optimization problem. For emphasis, the dependence of the various terms on the assignment and service policies is given explicitly.

$$\mathcal{OPT}: \min_{(\pi_A,\pi_S)\in\Pi_I} W(\pi) = \min_{(\pi_A,\pi_S)\in\Pi_I} \sum_{i=1}^{n} \frac{\lambda_i(\pi_A)}{\lambda} W_i(\pi_A,\pi_S)$$

The tightness of the lower bounds is demonstrated by the construction of a valid control policy for each vehicle that decides the pickup, relay, and delivery schedule of arriving messages such that the average message delay is of the same order as that of the lower bound.

Chapter 3

Preliminary Technical Details

In this chapter, we present several results from the existing literature in probability, queuing theory and vehicle routing that will be used throughout the remaining chapters.

3.1 General Probability

3.1.1 Jensen's Inequality

If $g : \mathbb{R}^d \to \mathbb{R}$ is a convex function and X is a random variable taking values in $\mathbb{R}^d$, then

$$E[g(X)] \geq g(E[X]). \tag{3.1}$$

3.1.2 Implications of Positive Correlation

The covariance of two random variables, X and Y, is defined as

$$cov(X,Y) = E\left[(X - E[X])(Y - E[Y])\right].$$

X and Y are said to be positively correlated if $cov(X, Y) > 0$ and uncorrelated if $cov(X, Y) = 0$. If X and Y are either positively correlated or uncorrelated, then

$$E[XY] \geq E[X]E[Y]. \tag{3.2}$$

Suppose we are given a finite sequence of n pairs of real numbers, $\{(X_i, Y_i)\}_{i=1}^n$, such that $X_i \geq X_j \Rightarrow Y_i \geq Y_j$. Consider a random variable $\mathcal{I}$ which selects an index i, each with probability $1/n$. Then applying (3.2) to the positively correlated random variables $X_\mathcal{I}$ and $Y_\mathcal{I}$, the following general relation holds:

$$\sum_{i=1}^{n} X_i Y_i \geq \frac{\sum_{i=1}^n X_i \sum_{i=1}^n Y_i}{n}. \tag{3.3}$$

3.1.3 Geometric Probability

Given two uniformly and independently distributed points, $X_1, X_2 \in [0, \sqrt{A}]^2$, the expected distance between these points is

$$E[\|X_1 - X_2\|] = c_1\sqrt{A} \tag{3.4}$$

where $c_1 \approx 0.52$ (see [18], p. 135).

3.2 Queueing Theory

Unless otherwise noted, the definitions and results in this section may be found in [34].

3.2.1 Queueing Notation

Define a $GI/G/1$ queuing system as follows. Messages are generated according to a stationary renewal process $\Lambda(t)$. That is, message interarrival times are i.i.d. with expected interarrival time $1/\lambda$ and interarrival variance σ_A^2. All demands require an i.i.d. service time with mean $E[s]$ and variance σ_s^2.

3.2.2 Stability of $GI/G/1$ Queues and Queuing Networks

The utilization of a single $GI/G/1$ server is defined to be

$$\rho \triangleq \lambda E[s]. \tag{3.5}$$

A necessary condition for the stability of the queue is $\rho \leq 1$. If either the arrival process or the service times is non-deterministic, no queue can be stable when $\rho = 1$, and therefore, the necessary condition becomes $\rho < 1$. If a work-conserving policy is used (that is, the server is in service anytime there are demands in the system), then $\rho < 1$ is also sufficient.

Now consider a network of n $GI/G/1$ queues such that server i has arrival rate λ_i and service time mean $E[s_i]$. A necessary condition for the stability of the queuing network is

$$\rho_i = \lambda_i E[s_i] < 1, \quad \forall i.$$

If service times are identically distributed across all servers, $E[s_i] = E[s], \forall i$, the following is a less restrictive necessary condition for the total network utilization:

$$\rho \triangleq \frac{\sum_{i=1}^{n} \rho_i}{n} = \frac{\lambda E[s]}{n} < 1. \tag{3.6}$$

3.2.3 Upper Bound on Waiting Time in $GI/G/1$ Queue

There is no simple explicit bound on the system time for a general $GI/G/1$ queue. However, Kingman's bound proves that

$$W \leq \frac{\lambda \left(\sigma_a^2 + \sigma_s^2\right)}{2(1-\rho)} + E[s]. \tag{3.7}$$

This bound is asymptotically exact as $\rho \to 1$. Further, for the $M/G/1$ queue with Poisson arrivals, the waiting time is exactly

$$W = \frac{\lambda E[s^2]}{2(1-\rho)} + E[s]. \tag{3.8}$$

This is known as the Pollacek-Khinchin formula. For the $M/D/1$ queue with Poisson arrivals and deterministic service times, this simplifies to

$$W = \frac{E[s]\rho}{2(1-\rho)} + E[s]. \tag{3.9}$$

3.2.4 Little's Law

Little's Law holds more generally than for a GI/G/1 queue with renewal arrivals. In its most general form, all that is required is the existence of the following two limits for a system realization $\omega \in \Omega$:

$$\overline{\lambda(\omega)} = \lim_{t\to\infty} \frac{\Lambda(t,\omega)}{t}, \tag{3.10}$$

$$\overline{W(\omega)} = \lambda = \lim_{t\to\infty} \frac{W(t,\omega)}{t}. \tag{3.11}$$

The sample path version Little's Law says that when (3.10) and (3.11) hold, $N(\omega) = \lim_{t\to\infty} N(t,\omega)$ exists and is equal to

$$\overline{N(\omega)} = \overline{\lambda(\omega)}\overline{W(\omega)}. \tag{3.12}$$

When time averages exist with probability 1, then $\overline{N} = \lambda\overline{W}$. If $\Lambda(t)$, $N(t)$ and W_j are regenerative, then the expected value version of Little's Law says that

$$N = \lambda W. \tag{3.13}$$

3.2.5 Arrival and Departure Distributions

If a stochastic counting process $N(t)$ increases by one at times $t_j, j \geq 1$ and decreases by one at times $y_j, j \geq 1$ with probability 1, then for fixed k, if the limits $P(L^- = k) = \lim_{j \to \infty} P(L(t_j) = k)$ and $P(L^+ = k) = \lim_{j \to \infty} P(L(y_j) = k)$ exist, then

$$P(L^- = k) = P(L^+ = k). \tag{3.14}$$

This is a general version of the Burke's Theorem and may be found in [23].

3.3 Euclidean TSP Tour Length

3.3.1 Asymptotic Performance

Let $X_1, \ldots, X_N$ be independendently and uniformly distributed in a square of area A and let L_N denote the length of an minimum length tour through these points. In the case that N, the number of locations to be visited on the tour, is large, the length of the TSP tour may be bounded with the following asymptotic result originally due to Beardwood, Halton, and Hammersley [3] (see also [17]):

Theorem 1. *Given N points uniformly distributed over a region of area A, and denoting the expected length of the optimal TSP tour through these points as L_N, there exists a constant $0 < \beta < \infty$ such that:*

$$\lim_{N \to \infty} \frac{L_N}{\sqrt{N}} = \beta\sqrt{A}$$

with probability 1. β has been estimated through simulation to be $\beta \approx 0.72$. Furthermore, the variance of the length of the optimal tours scales as

$$var(L_N) = O(1). \tag{3.15}$$

That is, for N large, $E[L_N] \approx \beta\sqrt{NA}$ and further, $\lim_{N\to\infty} \frac{var(L_N)}{N} = 0$.

The following result is a generalization of Theorem 1 found in [30].

Theorem 2. *If $X_1, \ldots, X_N$ are identically and independently distributed (i.i.d.) according to a general absolutely continuous distribution with density $f(x)$ and compact support $\mathcal{A}$, then the following limit holds:*

$$\lim_{N\to\infty} \frac{L_N}{\sqrt{N}} = \beta \int_{\mathcal{A}} f^{1/2}(x)dx.$$

3.3.2 Worst-Case Performance

In the case that N is not large, the average length of a TSP tour through N points may be bounded by considering a fixed worst case tour through these points.

Lemma 1. *Given N locations arbitrarily located in a square region of area B, there exists a tour through these points of length at most $2\sqrt{2NB}$.*

Proof. [Lemma 1] First note that if $N = 1$, the total time to visit the location and then return to the starting point, starting from anywhere in the region, is at most $2\sqrt{2B}$, so the bound in the theorem holds. The following is for $N \geq 2$.

Divide the region into N cells of area B/N. Consider a tour that begins at an arbitary location, then travels directly to the center of the upper-leftmost cell. The tour then travels between the centers of all the cells in a row-by-row manner, working across and then down through the region. Once all cell centers have been visited, the vehicle returns to the starting point to complete the tour. Such a tour through the cell centers takes at most time $N\sqrt{\frac{B}{N}} + \sqrt{2B} = \sqrt{(N+2)B} \leq \sqrt{2NB}$ for $N \geq 2$.

The tour through the N arbitrarily located points is performed by following the cell tour above, but stopping in each cell to visit all of the required locations that are located within that cell. To visit each location, the vehicle travels from the cell center to the the location and then back to the cell center. Each of these visits takes at most $\sqrt{2B/N}$. Since there are N locations to visit in this way, the location visits take a total of at most $\sqrt{2BN}$ in addition to the cell tour.

Combining this with the cell tour length above, the total tour through the N locations takes at most $2\sqrt{2NB}$. □

That is, even when N is not large, the scaling of the TSP tour length is bounded in terms of N and A in the same way as in Theorem 1, with the scaling constant β increased to $2\sqrt{2}$.

3.4 Dynamic Traveling Repairperson Problem

The DTRP considers the case in which demands arrive to a convex environment $\mathcal{A}$ of area A according to some arrival process with demands being randomly located in the region according to some distribution. A demand is serviced when a vehicle arrives to the demand location and spends a random amount of onsite service time, s, to service the demand. To perform these services, there are n vehicles that travel with bounded velocity $\leq v$ within $\mathcal{A}$. The average system utilization is defined in the standard queueing theory sense to be $\rho = \lambda E[s]/n$. The demands are to be serviced in such a way that all demands are eventually serviced and average delay between arrival and service of the demands, W, is minimized.

In the case that demands arrive according to a Poisson process with rate λ and demand locations are independently and identically uniformly distributed in $\mathcal{A}$, the average delay of message in the system is:

Theorem 3. (Theorem 2 in [5])

$$W \geq \gamma^2 \frac{\lambda A}{n^2 v^2 (1-\rho)^2} - \frac{n\,(1-2\rho)}{2\lambda}$$

for constant $\gamma = 2/3\sqrt{2\pi}$.

[6] treats the more general case of non-Poisson arrivals and nonuniform iid demand distributions. They consider two classes of policies: spatially unbiased and spatially

biased. Spatially unbiased policies require that the average expected delay of a message is the same regardless of the demand location, and spatially biased policies simply remove this restriction. Therefore, if we are not concerned about the notion of spatial biasedness, the results on spatially biased policies provide the strongest result. Below we state two versions of the result in [6] on the average delay over all messages that arrive according to demand distribution $f(\zeta)$ and are served under a spatially biased policy. The first version is the theorem as stated in [6] for the limit as $\rho \to 1$, and the second version is a slightly modified proof that is valid for the limit as $\lambda/n \to \infty$.

Theorem 4. (a) Theorem 2 from [6]

$$\lim_{\rho \to 1} (1-\rho)^2 W \geq 2\gamma^2 \frac{\lambda(\mathbb{E}[f^{2/3}])^3}{v^2 n^2}$$

where $\gamma \geq \frac{2}{3\sqrt{2\pi}}$.

(b) Theorem 2 from [6] (modified) *If both* $\frac{\lambda \mathbb{E}[\sqrt{f}]}{vn} \to \infty$ *and also* $\frac{\lambda \mathbb{E}[\sqrt{f}]^2}{v^2 n} \to \infty$, *then*

$$W = \Omega\left(\frac{\lambda(\mathbb{E}[f^{2/3}])^3}{v^2(1-\rho)^2 n^2}\right).$$

Theorem 4(b) follows with a slight modification of the proof in [6], which may be found in the appendix.

Chapter 4

Universal Lower Bounds

With the preliminary details in place, we turn to the first major result of this thesis. In the Dynamic Pickup and Delivery Problem, vehicles must pause at a service location for the duration of the message pickup and delivery service times. This implies that while vehicles are traveling, they are not performing work, in the sense of directly servicing a single message. This restriction implies a lower bound on message delay. This lower bound will be valid for any scaling of the parameters $\lambda(n), n$ and ρ and therefore we call it a *Universal Lower Bound.*

Two versions of this Universal Lower Bound are presented in this chapter. Both lower bounds take the same form, but the proofs are much different depending on the assumptions required. The first version holds when the arrivals to each vehicle are Poisson and various limiting distributions of delay and number in system hold. Because the internal arrivals in the Single Relay system are not Poisson, the second version was developed to hold for general arrivals. In fact, this results holds in the absence of these limiting distributions for general arrivals, but requires the use of a batching service policy.

The Universal Lower Bounds will be combined with lower bounds derived from the DTRP and the onsite service time in the main theorems of this thesis in Chapters 5 and 6.

4.1 Universal Lower Bound for Poisson Arrivals

Theorem 5. *For any stable policy for the No Relay DPDP for which the following properties hold:*

1. *Arrivals to each vehicle are independent Poisson processes,*
2. *onsite message service can only occur when a vehicle is stopped at the message service location,*
3. W_i, N_1^-, N_1^+, N_2^-, N_2^+ *have limiting distributions for all* i,
4. $\lambda_i \geq \lambda_{i'} \implies W_i \geq W_{i'}$, *and*
5. $\rho_i = 2\lambda(n)\bar{s}(n)/n < 1$

the expected message delay W *is lower bounded as follows:*

$$W \geq \frac{c_1\sqrt{A}}{v(1-\rho)}$$

where $c_1 \approx 0.52$.

This proof requires two main steps. First, a lemma relating the delay while the vehicle is in onsite service time to the total delay is proven. Then, this is combined with travel delay to derive the result.

4.1.1 Preliminary Lower Bound for Single Vehicle

First, we have the follow Lemma bounding $E[W_{O,i}]$ for a single vehicle with arrivals of rate λ_i.

Lemma 2. *When each of the following distributions exist for a single vehicle* i*:* $\tilde{W}_i$, $\tilde{W}_{O,i}$, $\tilde{N}_{1,i}^-$, $\tilde{N}_{1,i}^+$, $\tilde{N}_{2,i}^-$, $\tilde{N}_{2,i}^+$, *the onsite service time and total service time of messages served by that vehicle are related as*

$$W_{O,i} \geq \rho_i W_i$$

where $\rho_i = 2\lambda_i \bar{s}(n)$.

Proof. For this proof, we shall drop the reference to the vehicle index i and assume that the limits over the message index j are taken only for j that are served by vehicle i, that is, $j \in \{(i_1, i_2, ..., i_j, ...) : W_i(i_j) > 0\}$. First, we find the relation between $W_O(j)$ and $W(j)$ for an individual message, and then we take the appropriate limits. For a work-conserving system, these two measures are the same, that is, the system is always in onsite service while there are messages in the system waiting to be served.

$W_{O,i}(j)$ is equal to the sum of three terms: 1) the time, denoted by $R(t_j)$, to complete the service of the message (if any) in service when message j arrives, 2) the total number of complete pickups and deliveries completed in the interval $[t_j, y_j)$ of length $W(j)$, multiplied by the service time $\bar{s}(n)$, and 3)the message's own final delivery service. If $R(t_j) = 0$, that is, there is no message in service at time t_j, then in terms of the service completion processes, $\mathcal{V}(t)$ and $D(t)$, $W_O(j)$ is defined as

$$W_O(j) = \bar{s}(n)\left[(\mathcal{V}(y_j) - \mathcal{V}(t_j)) + (D(y_j) - D(t_j)) + 1\right].$$

If $R(t_j) > 0$, then the completion of the message in service at time t_j is already included in the difference $(\mathcal{V}(y_j) - \mathcal{V}(t_j)) + (D(y_j) - D(t_j))$. To add $R(t_j)$, we must first subtract this service. Adding in the final service of the selected message itself yields

$$W_O(j) = \bar{s}(n)\left[(\mathcal{V}(y_j) - \mathcal{V}(t_j)) + (D(y_j) - D(t_j))\right] + R(t_j).$$

In either case, because we are looking for a lower bound, we may ignore the residual terms and use the following bound:

$$W_O(j) \geq \bar{s}(n)\left[(\mathcal{V}(y_j) - \mathcal{V}(t_j)) + (D(y_j) - D(t_j))\right]. \tag{4.1}$$

We may compute the number of services by relating them to the number in system processes and the arrival process.

$$\mathcal{V}(y_j) - \mathcal{V}(t_j) = \Lambda(y_j) - \Lambda(t_j) - N_1(y_j) - N_1(t_j) \tag{4.2}$$

$$D(y_j) - D(t_j) = \mathcal{V}(y_j) - \mathcal{V}(t_j) - N_2(y_j) - N_2(t_j) \tag{4.3}$$

Because $N_1(t)$ and $N_2(t)$ are both unit increment/decrement processes, $\tilde{N}_1^- \stackrel{d}{=} \tilde{N}_1^+$ (and likewise for $\tilde{N}_2$) when these distributions exist. In particular, $E[\tilde{N}_1^-] = E[\tilde{N}_1^+]$. Further, the Poisson arrival rate implies that, for each interval $[t_j, y_j]$,

$$E[(\Lambda(y_j) - \Lambda(t_j))] = \lambda E[(y_j - t_j)]. \tag{4.4}$$

Combining these two facts, and taking limits, we have

$$\begin{aligned} \lim_{j\to\infty} E[(V(y_j) - V(t_j))] &= \lim_{j\to\infty} E[(\Lambda(y_j) - \Lambda(t_j))] \\ &= \lambda \lim_{j\to\infty} E[(y_j - t_j)] = \lambda W, \qquad (4.5) \\ \lim_{j\to\infty} E[(D(y_j) - D(t_j))] &= \lim_{j\to\infty} E[(V(y_j) - V(t_j))] \\ &= \lambda \lim_{j\to\infty} E[(y_j - t_j)] = \lambda W. \qquad (4.6) \end{aligned}$$

Therefore, combining these with equation (4.1) and adding back in the i notation

$$\begin{aligned} W_{O,i} &\geq \bar{s}(n)\,[\lambda_i W_i + \lambda_i W_i] \\ &\geq \rho_i W_i. \qquad (4.7) \end{aligned}$$

□

4.1.2 Universal Lower Bound for Multi-Vehicle

With this Lemma in place, we complete the proof of Theorem 5.

Proof of Theorem 5. Consider the total waiting time $W(j)$ of a randomly tagged message j. Because onsite service can occur only when the vehicle is not traveling, the waiting time may be divided into two parts: $W_T(j)$, the time that the vehicle is traveling between message locations, and $W_O(j)$, the time the vehicle spends in onsite service. $W_O(j)$ includes the onsite service time of the tagged message as well at the onsite service times of any other messages served between the tagged message's arrival and final delivery service.

Recall that $W = W_T + W_O$. The travel time may be bounded by the time to travel the expected distance between the source and destination locations of the randomly tagged message . The actual time in travel may include deviations from this straight line distance, and so this term is a lower bound on $W_T(j)$. Because sources and destinations are independently and uniformly distributed, this distance is $c_1\sqrt{A}$, where the constant $c_1 \approx 0.52$ (see equation (3.4)). Therefore

$$W_T \geq \frac{c_1\sqrt{A}}{v}.$$

For the onsite waiting time, we have the following claim:

$$W_O \geq \rho W. \tag{4.8}$$

This does not follow immediately from Lemma 2 which was proven for a single vehicle only. However, taking the weighted sum of these terms for each vehicle and applying the definitions in equations (2.19), (2.20), and (2.29),

$$\begin{aligned}
\sum_{i=1}^{n} \frac{\lambda_i}{\lambda} W_{O,i} &\geq \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} \rho_i W_i, & (4.9)\\
W_O &\geq \frac{\sum_{i=1}^{n} \rho_i}{n} \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} W_i & (4.10)\\
&\geq \rho W. & (4.11)
\end{aligned}$$

The implication of (4.10) from (4.9) is given by the assumption that W_i and $W_{i,O}$ are both increasing functions of λ_i (and therefore of ρ_i).

Combining these two formulas

$$\begin{aligned} W &\geq \frac{c_1\sqrt{A}}{v} + \rho W \\ &\geq \frac{c_1\sqrt{A}}{v(1-\rho)}. \end{aligned} \tag{4.12}$$

□

4.2 Universal Lower Bound for Batching Policies

The above proof requires Poisson message arrivals and the existence of certain limiting distributions. For the single relay DPDP, the arrival condition may not be met for the internal arrivals that are relayed between vehicles. Modifications of this proof for more general arrivals require the use of a batching service policy. This proof will also be valid in the absence of limiting distributions.

Theorem 6. *For any stable batching policy for either the No Relay DPDP or the Single Relay DPDP for which all of the following properties hold:*

1. *onsite message service can only occur when a vehicle is stopped at the message service location,*
2. $\lambda_i \geq \lambda_{i'} \implies \overline{W}_i \geq \overline{W}_{i'}$,
3. *the number in the batch and the batch overhead time are related by* $\frac{\overline{B^2}}{\overline{B}} \geq \frac{\overline{BI}}{\overline{I}}$, *and*
4. *the appropriate stability condition holds:*
   (a) *No Relay:* $\rho = 2\lambda\bar{s}(n) < 1$,
   (b) *Single Relay:* $\rho = 4\lambda\bar{s}(n) < 1$,

the time average system delay is bounded as

$$\overline{W} = \frac{c_1\sqrt{A}}{2v(1-\rho)}$$

where $c_1 \approx 0.52$.

4.2.1 Batching Policies Preliminaries

Assume that messages are assigned to batches in such a way that the time averages of B_k, B_k^2, B_kI_k, and I_k exist and are finite with probability 1, i.e.

$$\begin{aligned}
\overline{B} &\triangleq \lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{K} B_k < \infty, \quad w.p.1 \\
\overline{B^2} &\triangleq \lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{K} B_k^2 < \infty, \quad w.p.1 \\
\overline{BI} &\triangleq \lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{K} B_kI_k < \infty, \quad w.p.1 \\
\overline{I} &\triangleq \lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{k} I_k < \infty, \quad w.p.1.
\end{aligned}$$

For technical reasons, we will require the number in the batch and the idle time to be related by

$$\frac{\overline{B^2}}{\overline{B}} \geq \frac{\overline{BI}}{\overline{I}}. \tag{4.13}$$

The proof of Theorem 6 rests on the analysis of a reduced batch system which is coupled to the original DPDP system in that delay in the reduced system is a lower bound on the delay in the DPDP system.

Definition 4 (Reduced Batch System). *The reduced batch system is a system under a batching policy in which messages at the batch processor or in the batch queue are ignored. That is, in the reduced system, service requests do not arrive until just before the batch to which they belong begins service. Comparing this to the original system, delay between the time a message arrives and is assigned to a vehicle and the beginning of its batch service is ignored.*

The following proof comprises an analysis of the reduced system.

4.2.2 Preliminary Lower Bound for Single Vehicle

Analogous to Lemma 2, we have the following Lemma bounding the time spent in onsite service.

Lemma 3. *If inequality* (4.13) *holds for a batching policy then for a single vehicle in the reduced system*

$$\overline{W}_{O,i} \geq \frac{\rho_i}{2-\rho_i}\overline{W}_i$$

with $\rho = 2\lambda(n)\bar{s}(n)/n < 1$ *for No Relay DPDP and* $\rho = 4\lambda(n)\bar{s}(n)/n < 1$ *for the Single Relay DPDP.*

Before proving Lemma 3, we present and prove Lemma 4 below which makes a statement about the number in system at individual vehicles. The proof of Lemma 3 will then relate the number in system to the delay at a single vehicle. First, define $\overline{N}_{O,i}$ to be the time average number in the vehicle queue when the vehicle is in onsite service and $\overline{N}_{I,i}$ to be the time average number when the vehicle is not in service of an individual message (either traveling or idling). Assume that both averages exist and are finite.

Lemma 4. *The following inequality holds for the reduced system under any batching policy which satisfies* (4.13)*:*

$$\overline{N}_{O,i} \geq \frac{\overline{N}_{I,i}}{2}.$$

for all vehicles i in the multi-vehicle system. That is, the average number assigned to a vehicle when the vehicle is not in service is no more than twice the average number when the vehicle is in service.

Proof. [Lemma 4] Examine a single batch k_i that is served by vehicle i with $B_{k,i}$ messages and $I_{k,i}$ total travel time between service locations to fully serve the batch. For ease of notation, we will drop the references to the vehicle i for the remainder of the proof of this Lemma.

By the definition of the batching policy, all of the B_k messages must be assigned to the vehicle before any deliveries of these messages can occur. Because subsequent arrivals are ignored in the reduced system, this implies that the number in the reduced system is exactly B_k at the beginning of the batch and at most B_k for the duration of the batch. Therefore, letting $N_{I,k}$ be the number of messages in the system during the travel time in this batch, $N_{I,k} \le B_k$.

Further, during the delivery phase, the number in the system falls from exactly B_k to 0 in unit decrements. The decrement from value $B_k - j + 1$ to $B_k - j$ occurs immediately after $\bar{s}(n)$ time has been spent serving the jth message. Averaging over each of these increments, the average value is exactly half of the peak value, that is, $N_{O,k} = B_k/2$. Therefore, the inequality (4.14) holds over each of the batches individually.

Now compute the time averages of $\overline{N}_O$ and $\overline{N}_I$. Because $2\bar{s}(n)$ service time is required for each message, the total time in onsite service is in each batch exactly $N_{O,k}2\bar{s}(n)$. Therefore, each batch in $\overline{N}_O$ sum is weighted by the number of messages served by that batch, B_k.

$$\begin{aligned}
\overline{N}_O &\ge \lim_{K\to\infty} \sum_{k=1}^{K} \frac{B_k}{2} \frac{B_k}{\sum_{k=1}^{K} B_k} \\
&= \lim_{K\to\infty} \sum_{k=1}^{K} \frac{B_k^2}{2\sum_{k=1}^{K} B_k} \\
&= \frac{\overline{B^2}}{2\overline{B}} \qquad (4.14)
\end{aligned}$$

On the other hand, in $\overline{N}_I$, the batches are weighted by the time spent traveling. $\overline{N}_I$ also includes idle time between batches that is not accounted for in the sums below. Accounting for this idle time would only further reduce the actual value of $\overline{N}_I$.

$$\begin{aligned}
\overline{N}_I &\le \lim_{K\to\infty} \sum_{k=1}^{K} B_k \frac{I_k}{\sum_{k=1}^{K} I_k} \\
&= \frac{\overline{BI}}{\overline{I}} \qquad (4.15)
\end{aligned}$$

Combining equations (4.14) and (4.15), and recalling the inequality (4.13), we have

$$\overline{N}_O = \frac{1}{2}\frac{\overline{B^2}}{\overline{B}} \geq \frac{\overline{BI}}{\overline{I}} \geq \frac{\overline{N}_I}{2}.$$

Therefore, we have proven that when (4.13) holds, (4.14) holds as well. $\square$

Finally, we prove Lemma 3.

Proof. [Lemma 3] Define $T_{O,i}$ to be the total time in the interval $[0, T]$ in which the vehicle i is in service.

$$\begin{aligned}
\overline{N}_i = \lim_{T\to\infty} \frac{\int_0^T N_i(t)dt}{T} &\leq \lim_{T\to\infty} \frac{\int_0^T N_i(t)\left[\mathbf{1}_{O,i}(t) + \mathbf{1}_{I,i}(t)\right]dt}{T} \\
&= \lim_{T\to\infty} \frac{\int_0^T N_i(t)\mathbf{1}_{O,i}(t)dt}{T_{O,i}} \left(\frac{T_{O,i}}{T}\right) + \\
&\qquad \frac{\int_0^T N_i(t)\mathbf{1}_{I,i}(t)dt}{T - T_{O,i}} \left(\frac{T - T_{O,i}}{T}\right) \\
&\leq N_{O,i} \lim_{T\to\infty} \frac{T_{O,i}}{T} + N_{I,i} \lim_{T\to\infty} \frac{T - T_{O,i}}{T}.
\end{aligned}$$

We have the following bounds on $T_{O,i}$:

$$(\Lambda_i(T) - N_i(T))2\bar{s} \leq T_{O,i} \leq \Lambda_i(T)2\bar{s}.$$

The left hand side bound is the total time spent to service all messages that have already departed, and the right hand side is the time spent to service all messages that have arrived. Dividing by T and taking limits on both sides yields

$$\lim_{T\to\infty} \frac{(\Lambda_i(T) - N_i(T))2\bar{s}}{T} \leq \lim_{T\to\infty} \frac{T_{O,i}}{T} \leq \lim_{T\to\infty} \frac{\Lambda_i(T)2\bar{s}(n)}{T}. \tag{4.16}$$

Because we assume that the limit $\overline{N}_i$ exists and each message is in the system for at least $2\bar{s} > 0$ time, we must have $\lim_{T\to\infty} N_i(T)/T = 0$ (see Lemma 15 in Appendix B for an equivalent proof). Therefore the left hand and right hand limits of (4.16) are equal and

$$\lim_{T\to\infty} \frac{T_{O,i}}{T} = 2\lambda_i \bar{s}(n) = \rho_i. \tag{4.17}$$

$\overline{N}_i$, the unconditioned average number assigned to vehicle i, is given by

$$\overline{N}_i \leq \rho_i \overline{N}_{O,i} + (1-\rho_i)\overline{N}_{I,i}.$$

Substituting in the result of Lemma 4 above yields

$$\begin{aligned} \overline{N}_i &\leq \rho_i \overline{N}_{O,i} + 2(1-\rho_i)\overline{N}_{O,i} \\ &\leq (2-\rho_i)\overline{N}_{O,i}, \end{aligned}$$

and therefore

$$\overline{N}_{O,i} \geq \frac{\overline{N}_i}{2-\rho_i}. \tag{4.18}$$

Little's Law may be applied to the single vehicle system relating $\overline{N}_i$ and $\overline{N}_{O,i}$ to $\overline{W}_i$ and $\overline{W}_{O,i}$ respectively. Given the definition of λ_i in (2.15) and the assumption of the existence of the time average $\overline{N}_i$, then $\overline{N}_i = \lambda \overline{W}_i$. The application of Little's Law to the onsite system formed by deleting all times in which the system is not in service is less straightforward. See Appendix B for the full details of this proof. Briefly, because messages are always being served in the onsite system, messages complete service in the onsite system at a fixed rate of $1/\bar{s}$ with no idling. Because half of these service completions are departures, this implies that the departure rate from the onsite system is $1/2\bar{s}$. For stability, the arrival rate to the onsite system must also be $1/2\bar{s}$, and the corresponding Little's Law result is $\overline{N}_{O,i} = \overline{W}_{O,i}/2\bar{s}$. Therefore, the following is equivalent to equation (4.18):

$$\begin{aligned} \frac{1}{2\bar{s}}\overline{W}_{O,i} &\geq \frac{\lambda_i \overline{W}_i}{2-\rho_i} \\ \overline{W}_{O,i} &\geq \frac{\rho_i}{2-\rho_i}\overline{W}_i \end{aligned}$$

for a single vehicles i. $\square$

4.2.3 Universal Lower Bound for Multi-Vehicle, Batching Policies

Similar to the final proof of Theorem 5, these single vehicle onsite service results may be combined with the expected travel time to provide a lower bound on the overall delay.

Proof. [Proof of Theorem 6] Again recall that because a vehicle is stationary whenever it is performing onsite service, message delay may be partitioned into time spent traveling plus time spent in onsite service, i.e. $\overline{W} = \overline{W}_T + \overline{W}_O$.

The average total travel time, $\overline{W}_T$, is lower bounded by the expected straight line distance between the message's source and destination. As before,

$$W_T = \overline{W}_T \geq \frac{c_1\sqrt{A}}{v}.$$

To compute $\overline{W}_O$, take the weighted sum of the terms of Lemma 3 and apply the definitions in equations (2.19), (2.20), and (2.29),

$$\begin{aligned}
\sum_{i=1}^{n} \frac{\lambda_i}{\lambda}\overline{W}_{O,i} &\geq \frac{1}{2}\sum_{i=1}^{n}\left[\frac{\lambda_i}{\lambda}\rho_i\left(\overline{W}_i + \overline{W}_{O,i}\right)\right], \\
\overline{W}_O &\geq \frac{1}{2}\frac{\sum_{i=1}^{n}\rho_i}{n}\left[\sum_{i=1}^{n}\frac{\lambda_i}{\lambda}\overline{W}_i + \sum_{i=1}^{n}\frac{\lambda_i}{\lambda}\overline{W}_{O,i}\right] \\
&\geq \frac{1}{2}\rho(\overline{W} + \overline{W}_O) \\
&\geq \frac{\rho}{2-\rho}\overline{W}.
\end{aligned}$$

Combining this with the expected travel time we have

$$
\begin{aligned}
\overline{W} &\geq \overline{W}_T + \overline{W}_O \\
&\geq c_1\sqrt{A}/v + \frac{\rho}{2-\rho}\overline{W} \\
&= \frac{2-\rho}{2}\frac{c_1\sqrt{A}}{v(1-\rho)} \\
&\geq \frac{1}{2}\frac{c_1\sqrt{A}}{v(1-\rho)}
\end{aligned}
$$

where $\rho = \frac{2\lambda(n)s(n)}{n}$ or $\frac{4\lambda(n)s(n)}{n}$ as appropriate. The final equation is the desired result of Theorem 6. $\square$

4.2.4 Universal Lower Bound Corollary for Batching Policies

Before interpreting the previous results, we provide the following Corollary which provides some more intuitive conditions under which (4.13) and Theorem 6 hold.

Corollary 1.

$$\overline{W} = \frac{c_1\sqrt{A}}{2v(1-\rho)}$$

for any batching policy such that $\rho = \frac{4\lambda\bar{s}}{n} < 1$, *the limiting expectations* $E[B_k], E[B_k^2]$, $E[B_kI_k]$, *and* $E[I_k]$ *exist, and at least one of the following conditions is satisfied:*

(a) I_k *and* B_k *are uncorrelated or negatively correlated random variables.*

(b) $E[B_k|I_k]$ *is either a constant or a linear function of* I_k, *i.e.* $E[B_k|I_k] = \delta$ *or* $E[B_k|I_k] = \gamma I_k$ *for some* $\delta, \gamma \in [0,\infty)$.

(c) $E[I_k|B_k]$ *is an affine function of* B_k^α *for some* $\alpha \in [0,1]$, *i.e.* $E[I_k|B_k] = \gamma B_k^\alpha + \delta$ *for some* $\delta, \gamma \in [0,\infty), \alpha \in [0,1]$.

Proof. [Corollary 1] If condition **(a)** holds, then $E[B_kI_k] \leq E[B_k]E[I_k]$. Then equation (4.13) is equivalent to

$$\frac{E[B_k^2]}{E[B_k]} \geq \frac{E[B_k]E[I_k]}{E[I_k]} \Leftrightarrow E[B_k^2] \geq E[B_k]^2 \tag{4.19}$$

since $E[X^2] \geq E[X]^2$ for any random variable X.

If condition **(b)** holds, then equation (4.13) is equivalent to either

$$\frac{N_B^2}{N_B} \geq \frac{N_B E[I_k]}{E[I_k]} \Leftrightarrow N_B \geq N_B \tag{4.20}$$

or

$$\frac{\gamma^2 E[I_k^2]}{\gamma E[I_k]} \geq \frac{\gamma E[I_k^2]}{E[I_k]}. \tag{4.21}$$

To show the proof when condition **(c)** holds, we first have the following set of equalities implied by the bounds on $E[I_k|B_k]$:

$$\begin{aligned} E[B_k^2]E[I_k] &= E[B_k^2]E[E[I_k|B_k]] = \gamma E[B_k^2]E[B_k^\alpha] + E[B_k^2]\delta \\ E[B_k I_k]E[B_k] &= E[E[B_k I_k|B_k]]E[B_k] = \gamma E[B_k^{1+\alpha}]E[B_k] + E[B_k]^2\delta. \end{aligned}$$

Again, because $E[B_k^2] \geq E[B_k]^2$, the δ terms cancel and it remains to show

$$E[B_k^2]E[B_k^\alpha] \geq E[B_k^{1+\alpha}]E[B_k] \tag{4.22}$$

for $0 \leq \alpha \leq 1$.

Let $p_q = P(B_k = q)$. Then, expanding (4.22) in terms of the p_q yields

$$\begin{aligned} E[B_k^2]E[B_k^\alpha] &= \left(\sum_{q=1}^{\infty} q^2 p_q\right)\left(\sum_{q=1}^{\infty} q^\alpha p_q\right) \\ &= \sum_{q=1}^{\infty}\sum_{r=1}^{\infty} q^2 r^\alpha p_q p_r \\ E[B_k^{1+\alpha}]E[B_k] &= \left(\sum_{q=1}^{\infty} q^{1+\alpha} p_q\right)\left(\sum_{q=1}^{\infty} q p_q\right) \\ &= \sum_{q=1}^{\infty}\sum_{r=1}^{\infty} q^{1+\alpha} r p_q p_r. \end{aligned}$$

Matching the terms multiplied by $p_q p_r = p_r p_q$, it remains to show

$$q^2 r^\alpha + r^2 q^\alpha \geq q^{1+\alpha} r + r^{1+\alpha} q$$

for all pairs $(q, r) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. Without loss of generality, assume $q \geq r$. For $\alpha \in [0, 1]$, we have the following series of equivalent inequalities:

$$\begin{aligned}
q\left(q^{1-\alpha} - r^{1-\alpha}\right) &\geq r\left(q^{1-\alpha} - r^{1-\alpha}\right), \\
q^{2-\alpha} - qr^{1-\alpha} &\geq rq^{1-\alpha} - r^{2-\alpha}, \\
q^{2-\alpha} + r^{2-\alpha} &\geq rq^{1-\alpha} + qr^{1-\alpha}, \\
q^\alpha r^\alpha \left(q^{2-\alpha} + r^{2-\alpha}\right) &\geq q^\alpha r^\alpha \left(rq^{1-\alpha} + qr^{1-\alpha}\right), \\
q^2 r^\alpha + r^2 q^\alpha &\geq q^{1+\alpha} r + r^{1+\alpha} q.
\end{aligned}$$

Therefore, the theorem holds when condition **(c)** is true. □

Remark: Note that the corollary holds when I_k is affine in B_k. However this does not hold if B_k is affine in I_k. In that case

$$\begin{aligned}
E[(aI_k + b)^2]E[I_k] &= a^2 E[I_k^2]E[I_k] + 2abE[I_k]^2 + b^2 E[I_k] \\
E[(aI_k + b)I_k](aE[I_k] + b) &= a^2 E[I_k^2]E[I_k] + abE[I_k]^2 + abE[I_k^2] + b^2 E[I_k] \\
\text{but} \quad abE[I_k]^2 &\leq abE[I_k^2].
\end{aligned}$$

4.3 Conclusions

In this chapter, we have proven two versions of a Universal Lower Bound for the DPDP. The Poisson proof was fairly straightforward, but has relatively strong assumptions, requiring Poisson arrivals and the existence of certain limiting distributions. The batching proof held under weaker system assumptions but required the use of a specific type of service policy. A proof holding under these weaker assumptions is required for the Single Relay DPDP because the relay arrivals between vehicles destroys the property of Poisson arrivals to a single vehicle.

Each of the two proofs was based on partitioning total delay into two parts: delay while traveling and delay while in onsite service. In both bounds, the travel delay is bounded by the time to travel the expected distance between the source and destination locations of a given message. The onsite time was bounded in terms of the system utilization and the total delay, but the specifics of this bound differed in the two versions of the proof. In either case, this proof method linked the classical analysis of work-conserving queues to the inherently non-work-conserving systems of vehicle routing.

We call the lower bounds in this chapter *universal* because they hold for any scaling of the arrival and message service parameters and under relatively weak requirements. We will provide other lower bounds in the following chapter that are not universal in the sense that they only apply for certain scalings of the system parameters. These new lower bounds will be tighter than the universal lower bound in the regime in which they apply, and so will be useful in constructing a tight lower bound over all parameter scalings.

Chapter 5

No Relay DPDP

In this chapter, we provide complete lower bounds on the Dynamic Pickup and Delivery Problem in the case that the vehicle that picks up a message must be the one to deliver it. In particular, we prove the following two theorems:

Theorem 7. **(a)** *For any policy in Π_{SO} under the Source Only information structure, the average delay per message is finite only if $\rho = 2\lambda(n)\bar{s}(n)/n < 1$ and it is lower bounded as*

$$W_{SO} \geq \max\left\{\gamma^2\left(\frac{\lambda(n)A}{v^2(1-\rho)^2 n}\right) - \frac{n(1-2\rho)}{2\lambda(n)}, \frac{c_1\sqrt{A}}{v(1-\rho)}, 2\bar{s}(n)\right\}$$

with constants $\gamma = 2/3\sqrt{2\pi}$ and $c_1 \approx 0.52$.

(b) *Further, if $\rho < 1$ then there exists a policy using Source Only information, for which the average delay is finite and is upper bounded as*

$$W_{SO} = O\left(\frac{\lambda(n)A}{v^2(1-\rho)^2 n}\right) + O\left(\frac{\sqrt{A}}{v(1-\rho)}\right) + O(\bar{s}(n))$$

for all $\lambda(n)$. Therefore the lower bound scaling is achievable, and $\rho < 1$ is necessary and sufficent for stability.

Theorem 8. **(a)** *For any policy in Π_{SD} under the Source-Destination information structure, the average delay per message is finite only if $\rho = 2\lambda(n)\bar{s}(n)/n < 1$.*

In that case, the following lower bounds hold. If both $\frac{\lambda(n)A}{v^2 n^{3/2}} \to \infty$ *and* $\frac{\lambda(n)\sqrt{A}}{vn^{5/4}} \to \infty$*, then*

$$W_{SD} = \Omega\left(\frac{\lambda(n)A}{v^2(1-\rho)^2 n^{3/2}}\right) + \Omega\left(\frac{\sqrt{A}}{v(1-\rho)}\right) + \Omega\left(\bar{s}(n)\right).$$

(b) *Further, if* $\rho < 1$ *then there exists a policy using Source and Destination information for which the average delay is finite and is upper bounded as*

$$W_{SD} = O\left(\frac{\lambda(n)A}{v^2(1-\rho)^2 n^{3/2}}\right) + O\left(\frac{\sqrt{A}}{v(1-\rho)}\right) + O(\bar{s}(n))$$

for all $\lambda(n)$*. Therefore the lower bound scaling is achievable and* $\rho < 1$ *is necessary and sufficient for stabilty.*

5.1 Lower Bounds on Average Delay

In this section we prove the claimed lower bounds of Theorems 7(a) and 8(a) for arbitrary policies. Policies achieving these lower bounds will be described in section 5.2.

Each lower bound comprises three terms. The first term of each is proven in the following sections. The second and third terms are the Poisson Universal Lower Bound from Chapter 4 and the trivial lower bound provided by the service time of a single message respectively.

Both lower bound proofs in this chapter follow the same general method based on successive relaxations of the main optimization problem $\mathcal{OPT}$. First, by fixing certain qualities of the assignment distribution of an individual vehicles, we may lower bound the delay over all valid service policies by relating the single-vehicle two-stage DPDP to a corresponding single-vehicle single-stage DTRP. With these single vehicle lower bounds, a new optimization problem may be defined over the collection of vehicles as a function of assignment policies only. This optimization problem may be further reduced such that the optimization is in terms of λ_i only, ignoring all other details of the assignment distributions.

5.1.1 A Relaxation of $\mathcal{OPT}$

We first formulate a relaxation of the main optimization problem, $\mathcal{OPT}$, for the No Relay DPDP. Every control policy $\pi = (\pi_A, \pi_S)$ induces an average delay function for each vehicle, $W_i(\pi_A, \pi_S) : \Pi_A \times \Pi_S \to [0, \infty)$. Now suppose we have another set of single vehicle functions that are functions of the assignment policy only, $W_i^*(\pi_A) : \Pi_A \to [0, \infty)$, such that $W_i^*(\pi_A) \leq W_i(\pi_A, \pi_S),\ \forall \pi_S \in \Pi_S$. $W_i^*(\pi_A)$ provides a lower bound on the delay achievable by service policy for a given assignment policy. For example, $W_i^*(\pi_A) \equiv 0, \forall \pi_A$ satisfies these conditions, but we will use results on the DTRP to construct tighter service lower bound functions.

Combining a given service lower bound function $W_i^*(\pi_A)$ with the constraints on π_A provided by the relevant information structure, we have the following relaxation of the optimization $\mathcal{OPT}$.

$$
\begin{aligned}
\mathcal{OPT}^* : \min_{\{p_i(x,y)\}_{i=1}^n} \quad & \sum_{i=1}^n \frac{\lambda_i(\pi_A)}{\lambda} W_i^*(\pi_A) \\
s.t. \quad & W_i^*(\pi_A) \leq W_i(\pi_A, \pi_S),\ \forall \pi_S \\
& \sum_{i=1}^n \lambda_i(\pi_A) = \lambda \\
& \lambda_i(\pi_A) = \lambda \int_A \int_A p_i(x,y)\,dx\,dy, \forall i \\
& \sum_{i=1}^n p_i(x,y) = \frac{1}{A^2},\ \forall x, y \in \mathcal{A} \\
& p_i(x,y) = \frac{1}{A} p_i(x),\ \forall y, i \quad \text{(If Source Only)}
\end{aligned}
\tag{5.1}
$$

$\mathcal{OPT}_1$ minimizes the weighted sum of lower bounds on the average delays over all vehicles by the selection of a valid assignment policy. The weights are given according to a joint constraint on the policies used by the individual vehicles. The lower bounds arise by bounding the delay that may be achieved by any service policy given the fixed assignment policy. If the minimum is finite, each of the $W_i(\pi)$ must be finite as well and the system is stable.

To compute lower bounds on delay at a single vehicle as a function of π_A, we must also take into account the constraints on $p_i(x, y)$ for a single vehicle i. Equations (2.1) and (2.14) provide characterizations of valid collections of densities $\{p_i(x, y)\}_{i=1}^n$ for the No Relay DPDP. These equations also provide useful bounds on the density $p_i(x, y)$ for a single vehicle i.

Recall the following notation: let $\mathbb{E}_\theta[g(\cdot)]$ denote the Lebesgue integral of $g(\cdot)$ with respect to the variable θ. Assume that the arrival rate of messages to be served by vehicle i is fixed to be λ_i. Then, from the definition of λ_i in equation (2.14) we have,

$$\mathbb{E}_x[\mathbb{E}_y[p_i(x, y)]] = \frac{\lambda_i}{\lambda}. \tag{5.2}$$

That is, each $p_i(x, y)$ is a scaled probability density with scaling λ_i/λ.

By definition, $p_i(x, y) \geq 0, \ \forall (x, y)$. Further, from the defining equation (2.1), $p_i(x, y) \leq \frac{1}{A^2}$. This implies

$$p_i(x, y) \in \left[0, \frac{1}{A^2}\right]. \tag{5.3}$$

In the following two sections, results from the DTRP will be used to bound $W_i^*(\pi_A)$ as a function of the arrival rate λ_i only, providing further relaxation of $\mathcal{OPT}_1$. This relaxation will lead directly to the derivation of the main lower bounds.

5.1.2 Lower Bound: Source Only

Because destination locations are not known immediately upon message arrival, this information may not be exploited when assigning messages to vehicles. This implies that the performance of each single vehicle system may be lower bounded by a single vehicle DTRP with uniform service locations corresponding to delivery only.

Proof of Theorem 7(a). Consider a fixed stable assignment and service policy in Π_{SO}. Each message is assigned to its vehicle immediately upon arrival. Consider the queue of message assigned to vehicle i which arrive according to a Poisson process of rate λ_i.

To lower bound the average delay of messages at vehicle i, we consider a simplified system in which the same message assignment process holds, but messages arrive directly at the vehicle. For consistency of the ρ notation, let the onsite service time for delivering each message be $2\bar{s}(n)$. This delivery problem may be formulated as a single-vehicle Dynamic Traveling Repairperson Problem with a Poisson arrival process of rate λ_i and onsite service time $2\bar{s}(n)$ for delivery. Because vehicles do not spend any time traveling to pickup messages, this simplified system naturally has lower delay than the original DPDP system. To apply the DTRP results and compute $W_i^*(\pi_A)$, it remains to compute the service location distribution associated with a single vehicle under any policy in Π_{SO}.

Since the distribution of destination locations is independent of the source locations and may not be exploited by the message assignment policy, the distribution of the destination locations of the messages assigned to a single vehicle is the same as that of the overall destination process for any assignment policy. That is, for any policy in Π_{SO}, each vehicle will service messages with destination locations distributed uniformly in $\mathcal{A}$. Thus, to obtain a lower bound on the single vehicle DPDP, it is sufficient to compute a lower bound on the delay of a Dynamic Traveling Repairperson Problem with a Poisson arrival process of rate λ_i, uniformly distributed service locations, and onsite service time $2\bar{s}(n)$. Applying the DTRP results of Theorem 3 to this formulation, we obtain the average delay for messages served by a single vehicle with message arrival rate λ_i:

$$W_i \geq \gamma^2 \left(\frac{\lambda_i A}{v^2(1-\rho_i)^2} \right) - \frac{1-2\rho_i}{2\lambda_i}. \tag{5.4}$$

This DTRP lower bound is a function of the message arrival rate λ_i only; no other details of the assignment distribution are required. Therefore, we may then bound the solution of $\mathcal{OPT}^*$ by further optimizing over the collection of $\{\lambda_i\}_{i=1}^n$ of valid assignment policies:

$$\begin{aligned} \mathcal{OPT}_{SO}: \min_{\{\lambda_i\}_{i=1}^n} \quad & \sum_{i=1}^n \frac{\lambda_i}{\lambda} W_i \geq \min_{\{\lambda_i\}_{i=1}^n} \sum_{i=1}^n \frac{\lambda_i}{\lambda}\left(\gamma^2\left(\frac{\lambda_i A}{v^2(1-\rho_i)^2}\right) - \frac{1-2\rho_i}{2\lambda_i}\right) \quad (5.5) \\ s.t. \quad & \sum_{i=1}^n \lambda_i = \lambda. \end{aligned}$$

The optimization over the set of all $\{p_i(x,y)\}_{i=1}^n$ has been replaced by the relaxed restriction on the sum of the λ_i.

Removing constant terms and noting that $\sum_{i=1}^n \rho_i = \sum_{i=1}^n 2\lambda_i \bar{s}(n) = n\rho$, this is equivalent to:

$$\begin{aligned} \min_{\{\lambda_i\}_{i=1}^n} \quad & \sum_{i=1}^n \frac{\lambda_i^2}{(1-\rho_i)^2} = \sum_{i=1}^n \frac{\lambda_i^2}{(1-2\lambda_i\bar{s})^2} \\ s.t. \quad & \sum_{i=1}^n \lambda_i = \lambda. \end{aligned} \tag{5.6}$$

This optimization is straightforward to solve. Briefly, this is a minimization of an equally weighted sum of convex single vehicle functions, subject to an equally weighted sum constraint. By the symmetry of the weights, we find that the optimal solution is $\lambda_i = \frac{\lambda}{n}, \forall i$. In this case, $\rho_i = 2\lambda(n)\bar{s}(n)/n, \forall i$. By the symmetry of this solution, the weighted average lower bound is the same at the lower bound for an individual vehicle. Substituting $\lambda_i = \lambda(n)/n$ into (5.4), we have the following lower bound on the average delay over all vehicles:

$$W_{SO} \;\geq\; \gamma^2\left(\frac{\lambda(n)A}{v^2(1-\rho)^2 n}\right) - \frac{n(1-2\rho)}{2\lambda(n)}. \tag{5.7}$$

To complete the proof, we combine (5.7) with the universal lower bound in Theorem 5 and the trivial lower bound given by the total service time per message.

$\square$

5.1.3 Lower Bound: Source and Destination

If both the Source and Destination locations are known upon message arrival, assignment policies may exploit this information to limit the area covered by each vehicle in making its pickups and deliveries. This has the effect of reducing the minimum average delay of messages in the system.

Proof of Theorem 8(a). Consider a fixed stable assignment and service policy in Π_{SD}. As before, we will consider a simplified system in which the same message assignment process holds and each message requires a single service of length $2\bar{s}(n)$. Now however, we let the DTRP demand location associated with each message be selected uniformly at random between the source $s(j)$ and the destination $d(j)$ of the message. That is, instead of performing delivery only as above, this DTRP visits exactly one of the pickup and delivery locations for each message, with either location being chosen with probability 1/2. The distribution of demand locations arriving to this DTRP queue is the uniform mixture of pickup and delivery locations served by vehicle i, $f_i(\zeta)$.

$$f_i(\zeta) \triangleq \frac{1}{2}\frac{\lambda}{\lambda_i}\left[\mathbb{E}_x[p_i(\cdot,\zeta)] + \mathbb{E}_y[p_i(\zeta,\cdot)]\right]. \tag{5.8}$$

This DTRP queue fits the framework of the single vehicle Dynamic Traveling Repairperson Problem with generalized demand distributions. Then, according to Theorem 4, we have the following bound on minimum delay for a single vehicle policy with demand distribution $f_i(\zeta)$, arrival rate λ_i, and $\rho = 2\lambda_i\bar{s}$. If both $\frac{\lambda_i \mathbb{E}[\sqrt{f_i}]^2}{v^2} \to \infty$ and $\frac{\lambda_i \mathbb{E}[\sqrt{f_i}]}{v} \to \infty$, then

$$W_i = \Omega\left(\frac{\lambda_i (\mathbb{E}[f_i^{2/3}])^3}{v^2(1-\rho)^2}\right). \tag{5.9}$$

For now, we will assume that the scaling conditions hold and check for the required conditions when we have derived a valid f_i.

Because W_i is dependent on the distribution of the service locations, not just the net arrival rate, an extra step is required to define a relaxation of $\mathcal{OPT}$ in terms of λ_i only. The definition of $f_i(\zeta)$ in (5.8) implies the following two lower bounds:

$$\begin{aligned}
\mathbb{E}[f_i^{2/3}] &\geq \left(\frac{\lambda}{2\lambda_i}\right)^{2/3} \mathbb{E}_\zeta[\mathbb{E}_x[p_i(x,\zeta)]^{2/3}] && (5.10)\\
&\geq \left(\frac{\lambda}{2\lambda_i}\right)^{2/3} \mathbb{E}_\zeta[\mathbb{E}_y[p_i(\zeta,y)]^{2/3}]. && (5.11)
\end{aligned}$$

Each of the individual $p_i(x,y)$ have the following basic constraints (see (5.2) and (5.3)):

$$\begin{aligned}
p_i(x,y) &\in \left[0, \frac{1}{A^2}\right], && (5.12)\\
\mathbb{E}_x[\mathbb{E}_y[p_i]] &= \frac{\lambda_i}{\lambda}. && (5.13)
\end{aligned}$$

We may combine the two lower bounds above to form the following single vehicle optimization problem $\mathcal{OPT}_1$ which will then be used to lower bound the delay of a single vehicle policy with fixed arrival rate λ_i:

$$\begin{aligned}
\mathcal{OPT}_1 : \min_{p_i(x,y)} \quad & \frac{1}{2}\left(\mathbb{E}_\zeta[(\mathbb{E}_x[p_i(x,\zeta)])^{2/3}] + \mathbb{E}_\zeta[(\mathbb{E}_y[p_i(\zeta,y)])^{2/3}]\right)\\
\text{subject to} \quad & p_i(x,y) \in \left[0, \frac{1}{A^2}\right],\\
& \mathbb{E}_x[\mathbb{E}_y[p_i(x,y)]] = \mathbb{E}_y[\mathbb{E}_x[p_i(x,y)]] = \frac{\lambda_i}{\lambda}.
\end{aligned}$$

We now show that $\mathcal{OPT}_1$ is a concave optimization over a convex set. Consider a convex combination of two densities satisfying (5.12) and (5.13), i.e. $p_i^3(x,y) = \alpha p_i^1(x,y) + (1-\alpha)p_i^2(x,y), \forall x,y \in \mathcal{A}$. It is easy to see that the set of valid probability distributions satisfying (5.12) and (5.13) is convex. Then, by the concavity of $(\cdot)^{2/3}$,

$$\begin{aligned}
(\mathbb{E}_x[p_i^3(x,\zeta)])^{2/3} &= (\alpha\mathbb{E}_x[p_i^1(x,\zeta)] + (1-\alpha)\mathbb{E}_x[p_i^2(x,\zeta)])^{2/3} \\
&\geq \alpha(\mathbb{E}_x[p_i^1(x,\zeta)])^{2/3} + (1-\alpha)(\mathbb{E}_x[p_i^2(x,\zeta)])^{2/3}, \\
\mathbb{E}_\zeta[(\mathbb{E}_x[p_i^3(x,\zeta)])^{2/3}] &\geq \alpha\mathbb{E}_\zeta[(\mathbb{E}_x[p_i^1(x,\zeta)])^{2/3}] + (1-\alpha)\mathbb{E}_\zeta[(\mathbb{E}_x[p_i^2(x,\zeta)])^{2/3}].
\end{aligned}$$

Therefore both of the lower bounds (5.10) and (5.11) are concave in $p_i(x,y)$ and so is their sum. Thus, $\mathcal{OPT}_1$ is a concave minimization over a convex set. Hence, it must attain its optima on the boundary of the feasible bounded convex set.

The boundary of the constraint set defined by (5.12)-(5.13) implies that $p_i(x,y) \in \{0, 1/A^2\}$ for all (x,y) (almost surely w.r.t. Lebesgue measure). Condition (5.13), along with this implication, will provide the following complete characterization of boundary:

$$p_i(x,y) = \begin{cases} \frac{1}{A^2} & \text{for all } x \in \mathcal{A}_1^i, y \in \mathcal{A}_2^i \\ 0 & \text{otherwise} \end{cases} \tag{5.14}$$

for some regions $\mathcal{A}_1^i, \mathcal{A}_2^i \subset \mathcal{A}$ with areas such that $A_1^i A_2^i = \frac{A^2\lambda_i}{\lambda}$.

To minimize the cost function in $\mathcal{OPT}_2$, we must select the boundary points where the areas of A_1^i and A_2^i are equal, i.e. both are equal to $A\sqrt{\frac{\lambda_i}{\lambda}}$.

For any p_i satisfying the above properties we have:

$$\mathbb{E}_x[p_i(x,\zeta)] = A\sqrt{\frac{\lambda_i}{\lambda}}\frac{1}{A^2} = \frac{1}{A}\sqrt{\frac{\lambda_i}{\lambda}}$$

and

$$\mathbb{E}_\zeta[\mathbb{E}_x[p_i(x,\zeta)]^{2/3}] = A\sqrt{\frac{\lambda_i}{\lambda}}\left(\frac{1}{A}\sqrt{\frac{\lambda_i}{\lambda}}\right)^{2/3} = A^{1/3}\left(\frac{\lambda_i}{\lambda}\right)^{5/6},$$

and therefore the bound (5.10) on $\mathbb{E}[f_i^{2/3}]$ becomes

$$\mathbb{E}[f_i^{2/3}] \geq \left(\frac{\lambda_i}{2\lambda}\right)^{2/3} \mathbb{E}_\zeta[\mathbb{E}_x[p_i(x,\zeta)]^{2/3}] = \frac{1}{2^{2/3}} A^{1/3} \left(\frac{\lambda_i}{\lambda}\right)^{1/6}. \qquad (5.15)$$

Cubing this and then substituting this bound into equation (5.9), we thus have the following scaling for W_i:

$$W_i = \Omega\left(\frac{\lambda_i A}{v^2(1-\rho_i)^2}\sqrt{\frac{\lambda_i}{\lambda}}\right). \qquad (5.16)$$

The condition required for this to hold is for both $\frac{\lambda_i \mathbb{E}[\sqrt{f_i}]}{v} \to \infty$ and also $\frac{\lambda_i \mathbb{E}[\sqrt{f_i}]^2}{v^2} \to \infty$. As above, we may bound

$$\mathbb{E}[\sqrt{f_i}] \geq \frac{1}{2^{1/2}} A^{1/2} \left(\frac{\lambda_i}{\lambda}\right)^{1/4}. \qquad (5.17)$$

Squaring this, the conditions required for (5.16) to hold are $\frac{\lambda_i^{5/4}\sqrt{A}}{v\lambda^{1/4}} \to \infty$ and also $\frac{\lambda_i^{3/2} A}{v^2 \lambda^{1/2}} \to \infty$.

This result lower bounds the delay achievable by any service policy for a single vehicle in terms of λ_i only. We may again lower bound the solution of $\mathcal{OPT}_1$ by further optimizing over the collection of $\{\lambda_i\}_{i=1}^n$.

$$W_{SD} \;\geq\; \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} W_i. \qquad (5.18)$$

We then construct $\mathcal{OPT}_{SD}$.

$$\begin{aligned} \mathcal{OPT}_{SD} : \min_{\{\lambda_i\}_{i=1}^n} \quad & \sum_{i=1}^{n} \frac{\lambda_i}{\lambda}\frac{\gamma^2}{4}\frac{\lambda_i A}{v^2(1-\rho_i)^2}\sqrt{\frac{\lambda_i}{\lambda}} \\ s.t. \quad & \sum_{i=1}^{n} \lambda_i = \lambda. \end{aligned} \qquad (5.19)$$

Repeating the analysis that led to the optimization problem (5.6), the corresponding optimization here is:

$$\begin{aligned} \min_{\{\lambda_i\}_{i=1}^n} \quad & \sum_{i=1}^{n} \frac{\lambda_i^{5/2}}{(1-2\lambda_i \bar{s})^2} \\ s.t. \quad & \sum_{i=1}^{n} \lambda_i = \lambda. \end{aligned} \tag{5.20}$$

For the same reasons as before, this average delay is minimized with all λ_i equal to $\lambda(n)/n$ and again $\rho_i = 2\lambda(n)\bar{s}(n)/n, \forall i$. The scaling condition is then $\frac{\lambda(n)\sqrt{A}}{vn^{5/4}} \to \infty$ and $\frac{\lambda(n)A}{v^2 n^{3/2}} \to \infty$ and the delay scaling is

$$W_{SD} = \Omega\left(\frac{\lambda(n)A}{v^2(1-\rho)^2 n^{3/2}}\right). \tag{5.21}$$

To complete the proof, we take a convex combination of (5.21), the universal lower bound in Theorem 6, and the trivial lower bound given by the total service time per message.

□

5.2 Policies

In this section, we describe two policies that achieve the delay performance claimed in Theorems 7 and 8 for Source Only and Source Destination information respectively. These policies provide additional insight into the effect of information structure on achievable delay. Furthermore, both policies achieve the lower bounds presented in the previous section.

5.2.1 Source Only Policy

Recall that in the Source Only information structure, vehicles do not know the destination of messages before they are picked up, thus this information may not be used by vehicles in deciding which messages to pick up. In fact, in the source only policy described below, each message is assigned to any of the vehicles at random. We note that "smarter" message assignments are possible to minimize the vehicles' time spent in picking up messages. For example, a vehicle could be assigned all messages that arrive in a given limited area. However, since the vehicles must still traverse the whole region to deliver messages, regardless of assignment policy, the vehicle deliveries will dominate the delay and no message assignment process with only source information can improve the order of the performance for large arrival rates.

A complete description of the policy is given below.

(a) *Message Assignment.* Upon arrival, each message is assigned to one of the vehicles uniformly at random. The message is not immediately picked up, but the vehicle is notified of the message assignment and records the source location information of this message. Since the message assignment is a uniform splitting of the Poisson message arrival process, the assignment of messages to each vehicle is Poisson with an expected arrival rate of λ/n. All messages assigned to a single vehicle that arrive in the interval $[kT, (k+1)T)$ form a batch, where T, the batch time interval, is a parameter to be determined. Each batch is deposited into a queue for its assigned vehicle upon formation at time $(k+1)T$ for appropriate k.

(b) *Message Service.* Batches for each vehicle are served in First Come, First Serve order from the vehicle's batch queue. Pickups are performed along a TSP tour through the source locations which is computed at the beginning of the interval. Once pickups are complete and destination information is collected, a TSP tour through the delivery locations is constructed and the deliveries are performed accordingly. To perform each service, the vehicle stops at the source (destination) location for $\bar{s}(n)$ time to pickup (deliver) the associated message.

Proof of Theorem 7(b). Consider the queue of batches assigned to an arbitrary vehicle i. Note that by the symmetry of the vehicle policies, the average delay of messages at a single vehicle is the same as the average delay over all vehicles. Since the batch interarrival time is fixed at T, the batches form a D/G/1 queue. This batching protocol is stable if and only if the expected time to service each batch of messages, T_B, is less than T, the expected time between batch arrivals. The first part of the proof bounds T in terms of the system parameters so that this stability condition is met.

The batch service time requires two TSP tours, one for pickup and one for delivery, plus the associated onsite service times to perform each service. Let N_T be the number of messages arriving in $[kT,(k+1)T)$ that are assigned to vehicle i. Therefore, using Theorem 1 to bound the travel time required for each of the shortest paths (pickup and delivery), the total expected service time required to service the messages accumulated in $[kT,(k+1)T)$ is:

$$
\begin{aligned}
E[T_B] &\leq 2E[\frac{1}{v}E[L_{N_T}|N_T] + N_T\bar{s}(n)] && (5.22)\\
&= 2E[\beta\frac{1}{v}\sqrt{A}\sqrt{N_T} + N_T\bar{s}(n)] && (5.23)\\
&\leq 2\beta\frac{1}{v}\sqrt{A}\sqrt{E[N_T]} + 2E[N_T]\bar{s}(n) && (5.24)\\
&= 2\beta\frac{1}{v}\sqrt{A}\sqrt{\frac{\lambda T}{n}} + 2\frac{\lambda T}{n}\bar{s}(n) && (5.25)
\end{aligned}
$$

where (5.23) is by Theorem 1, (5.24) is by concavity of $\sqrt{\cdot}$, and (5.25) is given by the Poisson distribution of N_T.

Therefore the following bound on T is sufficient for stability:

$$
\begin{aligned}
T &> 2\beta\sqrt{\frac{\lambda A}{v^2 n}}\sqrt{T} + \rho T \geq E[T_B]\\
\Longrightarrow T &> \frac{4\beta^2\lambda A}{v^2(1-\rho)^2 n}. && (5.26)
\end{aligned}
$$

Because we are not interested in the tightness of the constants, we may upper bound the TSP tour time by a worst case tour and let $\beta = 2\sqrt{2}$.

The second part of the proof uses the batch interval time T to compute the average message delay. For the remainder of the proof, fix T to be:

$$T = \kappa \frac{4\beta^2 \lambda A}{v^2(1-\rho)^2 n} \tag{5.27}$$

for some $\kappa > 1$.

Message delay has four components: 1) time waiting for batch to form, 2) time batch spends in queue, 3) time waiting for service of other vehicles in batch, and 4) time of own service. Since batch interarrival time $= T$, each message waits at most T for its batch to form, bounding 1). Letting T_Q denote the expected amount of time the batch spends in queue, 2) may be bounded using the following lemma:

Lemma 5. *For the policy in Theorem 7(b) with batch time* $T = \kappa \frac{4\beta^2\lambda A}{v^2(1-\rho)^2 n}$ *for some* $\kappa > 1$*, the delay of the batch in the queue is bounded by*

$$T_Q = O(T).$$

The proof uses Kingman's Bound from queueing theory and is largely a matter of algebra. The proof can be found in Appendix C.

Now consider a randomly selected message. Delay components 3) and 4) may be bounded by bounding the expected total batch service time for the batch in which this message arrives. We might expect this batch service time to be upper bounded by $T > E[T_B]$ as well, however this does not take into account that the expected batch time includes the possibility of batches of size 0 when no messages arrive in $[kT, (k+1)T)$ which do not contribute to the average delay over all messages that do arrive. Therefore, we compute the expected batch service time by first conditioning on the size of the batch in which a message arrives and then taking the expectation over this batch size. The upper bound on batch service time will be divided in to two cases: $\frac{\lambda T}{n} > 1$ and $\frac{\lambda T}{n} \leq 1$. Both cases begin the same way as below.

If a message arrives in a batch of size B, according to the worst case TSP tour discussion in section 3.3.2, the total service time of the batch may be bounded by

$$S_B \le 2\left(\frac{2\sqrt{2}\sqrt{A}\sqrt{B}}{v} + \bar{s}B\right). \tag{5.28}$$

By the law of random incidence, a randomly selected message arrives in a batch of size B with probability

$$P\{\text{message arrives in batch of size } B\} = \frac{BP\{\text{batch has size } B\}}{E[B]}. \tag{5.29}$$

where the batch sizes are Poisson distributed with parameter $\frac{\lambda T}{n}$.

Therefore, the expected batch service time is

$$\begin{aligned}
E[S_B] &= \frac{4\sqrt{2}\sqrt{A}}{v}\sum_{k=1}^{\infty}\frac{k^{3/2}\left(\frac{\lambda T}{n}\right)^k e^{-\frac{\lambda T}{n}}}{\frac{\lambda T}{n}k!} + 2\bar{s}(n)\sum_{k=1}^{\infty}\frac{k^2\left(\frac{\lambda T}{n}\right)^k e^{-\frac{\lambda T}{n}}}{\frac{\lambda T}{n}k!} && (5.30)\\
&= \frac{4\sqrt{2}\sqrt{A}}{v}\sum_{k=0}^{\infty}\frac{(k+1)^{1/2}\left(\frac{\lambda T}{n}\right)^k e^{-\frac{\lambda T}{n}}}{k!} + 2\bar{s}(n)\sum_{k=0}^{\infty}\frac{(k+1)\left(\frac{\lambda T}{n}\right)^k e^{-\frac{\lambda T}{n}}}{k!} && (5.31)\\
&= \frac{4\sqrt{2}\sqrt{A}}{v}E[(B+1)^{1/2}] + 2\bar{s}(n)E[B+1] && (5.32)\\
&\le \frac{4\sqrt{2}\sqrt{A}}{v}\sqrt{\left(\frac{\lambda T}{n}+1\right)} + 2\bar{s}(n)\left(\frac{\lambda T}{n}+1\right) && (5.33)\\
&= \frac{4\sqrt{2}\sqrt{A}}{v}\sqrt{\left(\frac{\lambda T}{n}+1\right)} + \rho T + 2\bar{s}(n) && (5.34)
\end{aligned}$$

where the first term of (5.33) is by the concavity of $(\cdot)^{1/2}$.

For $\frac{\lambda T}{n} > 1$, $\frac{\lambda T}{n} + 1$ can be bounded by $2\frac{\lambda T}{n}$. In that case, (5.34) can be bounded by

$$\begin{aligned}
E[S_B] &\leq 8\sqrt{\frac{A\lambda T}{nv^2}} + \rho T + 2\bar{s}(n) &(5.35)\\
&= 8\sqrt{\kappa 4\beta^2 \frac{A^2\lambda^2}{v^4(1-\rho)^2 n^2}} + \rho T + 2\bar{s}(n) &(5.36)\\
&= 16\sqrt{\kappa}\beta \frac{\lambda A}{v^2(1-\rho)n} + \rho T + 2\bar{s}(n) &(5.37)\\
&= O\left((1-\rho)T\right) + \rho T + 2\bar{s}(n) &(5.38)\\
&= O(T) + O(\bar{s}(n)). &(5.39)
\end{aligned}$$

Combining this with delay components 1) and 2) above, for $\frac{\lambda T}{n} > 1$,

$$W_{SO} = O(T) + O(\bar{s}(n)). \tag{5.40}$$

For $\frac{\lambda T}{n} \leq 1$, $\frac{\lambda T}{n} + 1$ is bounded above by 2. In that case, (5.34) is bounded by

$$E[S_B] \leq 8\sqrt{2}\frac{A}{v} + \rho T + 2\bar{s}(n). \tag{5.41}$$

Since $\rho < 1$, $\frac{\sqrt{A}}{v} < \frac{\sqrt{A}}{v(1-\rho)}$. Note also that $\frac{\lambda T}{n} \leq 1$ implies that for some constants c_1 and c_2

$$\begin{aligned}
&\frac{\lambda^2 A}{v^2(1-\rho)^2 n^2} \leq c_1^2 &(5.42)\\
\Rightarrow\ &\frac{\lambda\sqrt{A}}{v(1-\rho)n} \leq c_1 &(5.43)\\
\Rightarrow\ &\frac{\lambda}{n} \leq c_1 \frac{v(1-\rho)}{\sqrt{A}} &(5.44)\\
\Rightarrow\ &T < c_2 \frac{v(1-\rho)}{\sqrt{A}} \frac{A}{v^2(1-\rho)^2} = O\left(\frac{\sqrt{A}}{v(1-\rho)}\right). &(5.45)
\end{aligned}$$

Therefore, combining with delay components 1) and 2) above, for $\frac{\lambda T}{n} \leq 1$

$$W_{SO} = O\left(\frac{\sqrt{A}}{v(1-\rho)}\right) + O(\bar{s}(n)). \tag{5.46}$$

Therefore,

$$W_{SO} = O\left(\frac{\lambda(n)A}{v^2(1-\rho)^2 n}\right) + O\left(\frac{\sqrt{A}}{v(1-\rho)}\right) + O(\bar{s}(n))$$

for all $\lambda(n)$, and Theorem 7(b) is proven.

□

5.2.2 Source and Destination Policy

In the Source-Destination information structure, destination information may be used by vehicles in deciding which messages to pick up. By exploiting this information, vehicles need not traverse the entire geographical region when servicing messages, but may instead only pick up messages that have both source and destination locations in a limited area. In the source destination policy described below, each vehicle is assigned a pickup region and a delivery region. Messages are not assigned to a random vehicle as above, but are instead assigned to the vehicle that has the message's source location in its pickup region and the message's destination location in its delivery region. Even though the message service policy is similar to that used in the Source Only policy above, Theorem 8(b) shows that the change in assignment policy made possible by using both source and destination information has a significant effect on message delay.

A more complete description of the policy is given below.

(a) *Message Assignment.* Divide the geographical region into an $\sqrt{\frac{A}{\sqrt{n}}} \times \sqrt{\frac{A}{\sqrt{n}}}$ grid of subregions, each of area $\frac{A}{\sqrt{n}}$. To each of the n ordered pairs of subregions, assign exactly one vehicle to service that pair. Each vehicle is assigned to pickup all messages that originate in the first subregion of its assigned ordered pair that have a destination location in second assigned subregion. As before, all messages assigned to a single vehicle that arrive in the interval $[kT, (k+1)T)$ form a batch, where T, the batch time interval, is a parameter to be determined. Each batch is deposited into a queue for its assigned vehicle upon formation at time $(k+1)T$ for appropriate k.

(b) *Message Service.* As before, batches for each vehicle are served in First Come, First Serve order from the vehicle's batch queue. Batch pickups and deliveries are performed in the same way as in the policy with Source only information with the notable addition of possible interregion travel time between source region and destination region.

Proof of Theorem 8(b). Service of assigned messages is the same as in the Source only policy described above except that the TSP tours are performed over possibly distinct subregions of the environment. Each TSP tour now ranges over a subset of the geographical region $\mathcal{A}$ with area $A/\sqrt{n}$. Travel time between subregions must also be included in the batch service time analysis. Since the total geographical region is a square of area A, this interregion travel time may be upper bounded by $2\frac{1}{v}\sqrt{A}$.

Therefore, as before, the total expected service time required to service the messages accumulated in $[kT, (k+1)T]$ is:

$$\begin{aligned} E[T_B] &\leq 2E[\frac{1}{v}E[L_{N_T}|N_T] + N_T\bar{s}(n) + 2\frac{1}{v}\sqrt{A}] \\ &= 2\beta\frac{1}{v}\sqrt{\frac{A}{\sqrt{n}}}\sqrt{\frac{\lambda T}{n}} + 2\frac{\lambda T}{n}\bar{s}(n) + 4\frac{1}{v}\sqrt{A}. \end{aligned} \tag{5.47}$$

Again, we may upper bound the TSP tour time by a worst case tour and let $\beta = 2\sqrt{2}$.

Therefore the following bound on T is sufficient for stability:

$$T > 2\beta\frac{1}{v}\sqrt{\frac{\lambda A}{n^{3/2}}}\sqrt{T} + \rho T + 4\frac{1}{v}\sqrt{A} \geq E[T_B]. \tag{5.48}$$

This equation is quadratic in $\sqrt{T}$ and may be easily solved for T. Specifically, consider the quadratic

$$a(\sqrt{T})^2 + b\sqrt{T} + X \geq 0 \tag{5.49}$$

for some a, b and X. Using the quadratic equation, this is satisfied for

$$T \geq \left(\frac{-b + \sqrt{b^2 - 4aX}}{2a}\right)^2. \tag{5.50}$$

Note that for any parameters α and μ

$$\alpha^2 + \mu \leq (\alpha + \sqrt{\mu})^2 \leq 2(\alpha^2 + \mu). \tag{5.51}$$

Using this to bound the right hand side of (5.50) and simplifying

$$T \geq \frac{b^2}{a^2} + \frac{2X}{a} \tag{5.52}$$

is sufficient for (5.50) to be satisfied.

Finally, substituting in for a, b and X in the original quadratic of equation (5.48)

$$T \geq \frac{4\beta^2 \lambda A}{v^2(1-\rho)^2 n^{3/2}} + \frac{2\sqrt{A}}{v(1-\rho)} \tag{5.53}$$

is sufficient for stability.

As before, the total message delay as a function of the batch time T may be bounded by fixing a batch scaling constant κ and then using Kingman's bound with $A = A/\sqrt{n}$ and $T = \kappa\left(\frac{4\beta^2\lambda A}{v^2(1-\rho)^2 n^{3/2}} + \frac{2\sqrt{A}}{1-\rho}\right)$. Therefore, skipping several steps which parallel the completion of the Source Only proof with the altered scaling of the area A and the addition of the constant interregion travel time,

$$W_{SD} = O\left(\frac{\lambda A}{v^2(1-\rho)^2 n^{3/2}}\right) + O\left(\frac{\sqrt{A}}{v(1-\rho)}\right) + O(\bar{s}). \tag{5.54}$$

□

5.3 Conclusions

In this chapter, we have obtained lower and upper bounds on the scaling of the average message delay for the DPDP with No Relays. Each bound is the sum of three terms. The three terms of the lower bounds were derived individually. The first term was derived in this chapter by reducing each two-stage DPDP problem to a single-stage DTRP with a given demand distribution depending on the information structure in place. We saw that the information that is available in making assignment decisions has a significant effect on the delay scaling. The second term, the Universal Lower Bound, was derived in Chapter 4. The final term is the trivial lower bound given by the total service time of a single message.

The upper bounds were derived by constructing appropriate policies and computing upper bounds on their delay performance. Because the three terms of the upper bounds had the same order as those of the lower bounds, we say that the lower bounds derived in this chapter are tight.

Further discussion on the scaling behavior of the three bound terms as a function of the system parameters may be found in Chapter 8.

We note that as long as vehicles are required to perform physical pickups and deliveries at the source and destination locations, the DTRP lower bound serves as a lower bound on the DPDP problem. Even with Source and Destination information, the DTRP bound is still stronger than the DPDP bound by a factor of $1/\sqrt{n}$. We will see in the next chapter that the DTRP delay bound can be achieved by removing the restriction that the same vehicle that picks up a message is the one that delivers it via the use of relays.

Chapter 6

Single Relay DPDP

In this chapter, we consider control policies in which each message may be relayed exactly once. We will see that such policies approach the lower bound provided by the single-stage DTRP.

Theorem 9. **(a)** *For any Single Relay batching policy, the average delay per message is finite only if* $\rho = 4\lambda(n)\bar{s}(n)/n < 1$ *and it is lower bounded as*

$$W_{1R} \geq \max\left\{\gamma^2\left(\frac{\lambda(n)A}{v^2(1-\rho)^2n^2}\right) - \frac{n(1-2\rho)}{2\lambda(n)}, \frac{c_1\sqrt{A}}{v(1-\rho)}, 2\bar{s}(n)\right\}$$

with constants $\gamma = 2/3\sqrt{2\pi}$ *and* $c_1 \approx 0.52$.

(b) *Further, if* $\rho < 1$ *then there exists a synchronous single-relay batching policy for which the average delay is finite and is bounded as*

$$W_{1R} = O\left(\frac{\lambda(n)A}{v^2(1-\rho)^2n^2}\right) + O\left(\frac{\sqrt{A} + n^{3/2}r}{v(1-\rho)}\right) + O(\bar{s}(n)).$$

Therefore, $\rho < 1$ *is sufficient for stability. Further, if the interrelay distance* $r = 0$*, the lower bound scaling is achievable.*

6.1 Lower Bound on Average Delay

For any instance of the Dynamic Pickup and Delivery Problem, with or without relays, the result for the n-vehicle DTRP is a natural lower bound. For consistency of the ρ notation, we let $s = 4\bar{s}(n)$ and apply Theorem 3. The universal lower bound and the straightforward lower bound of the total service time are lower bounds as well. Combining these three results yields Theorem 9(a).

6.2 Upper Bounds for Single-Relay Policies

In this section, we demonstrate the tightness of the lower bound of Theorem 9(a) for the four-stage single relay DPDP. We present and analyze a policy which uses a synchronous vehicle rendezvous schedule to relay messages directly between vehicles. Under certain assumptions on the vehicle rendezvous locations, this policy achieves delay with the same order as that of the lower bound.

6.2.1 Synchronous Single-Relay Policy

This policy has two general components, Assignment and Service, with the Service component being carried out in three phases. A spatially based assignment policy is used to allow arriving messages to be assigned to vehicles without any real-time communication between the vehicles. This assignment policy may be initialized by a centralized controller and then implemented in a decentralized manner by each of the vehicles.

(a) *Message Assignment.* The region is divided into a $\sqrt{n} \times \sqrt{n}$ grid of cells, each of area A/n. Exactly one vehicle is assigned to each cell and is responsible for performing all of the pickups and deliveries in that cell. Upon arrival from outside the system, a message is assigned to the vehicle responsible for the cell in which the message's source location lies. When a message is relayed from the pickup to the delivery vehicle, it is immediately assigned to the vehicle responsible for the cell containing the message's destination.

(b) *Message Service.* Each vehicle has the same basic service policy, differing only by assignment region. The following service policy is described for a single vehicle. This policy for each vehicle has three basic steps: Pickup Batching and Service, Relay and Delivery Batching, and Delivery Service. Each vehicle cycles through these steps as long as there are messages to be served. In cycle k, two kinds of batches are defined: B_k, the pickup batch, and B_k^*, the delivery batch.

1. *Pickup Batching and Service:* In order to maintain a synchronous vehicle rendezvous schedule, messages are batched in such a way that the total time to service each batch may be deterministically upper bounded. For this, each vehicle maintains n source-destination queues of messages, one for each of the n cells in which the destination locations of arriving messages may occur. The k^{th} pickup batch, B_k, is formed by collecting up to the first N_n messages from each of the n queues to form a batch, where N_n is a parameter to be determined. The total number of messages to be serviced in each pickup batch is then at most $B_k \leq nN_n$.

   To service a batch of messages, the vehicle computes and then traverses a worst case Traveling Salesperson (TSP) tour through the source locations of all of the messages contained in the batch, pausing at each service location to pickup the corresponding message. A worst case tour is used to maintain the deterministic synchronicity between the vehicles.

2. *Relay and Delivery Batching:* To relay messages to their delivery vehicles, a pre-determined synchronous schedule is used such that each vehicle meets up with every other vehicle during each batch service time to hand off the appropriate messages (see Lemma 6). The rendezvous points at which the vehicles meet are predetermined and are distributed throughout the region. Assume that for safety or other reasons, the minimum interpoint distance between valid rendezvous points is r.

As the messages are being relayed, the delivery batch is collected by receiving at most N_n messages from each of the other vehicles. Therefore, the number in the delivery batch is deterministically bounded by $B_k^* \leq nN_n$ as well.

3. *Delivery Service:* Once the inter-vehicle meetings are complete and all messages to be delivered have been received, another worst-case TSP tour is performed through the destination locations of the messages in the pickup batch.

These three stages of batch service occur within constant time-length $T_k = T$. N_n and T are the policy parameters to be determined.

The following theorem bounds the delay of the above policy where T and N_n are stated in terms of an arbitrary constant ϵ. With appropriate selection of ϵ, this upper bound is of the same order as the lower bound in Theorem 9(a) for $r = 0$.

Theorem 10. *For the No-Depot policy described above, the delay scales as*

$$W = O\left(\frac{\lambda(n)A}{v^2(1-(1+\epsilon)\rho)^2 n^2}\right) + O\left(\frac{\sqrt{A} + n^{3/2}r}{v(1-(1+\epsilon)\rho)}\right) + O\left(\frac{1}{\epsilon}\right)$$

for any $\epsilon > 0$.

Proof. [Theorem 10] To establish validity of the above described policy as well as analyze its performance, we need the following Lemmas which establishes existence of a synchronous schedule for rendevous between vehicles.

Lemma 6. *Given n vehicles, there exists a schedule of length n such that each vehicle visits all other $n-1$ vehicles at least once.*

Proof. Consider a complete bipartite graph of $2n$ nodes, where each vehicle is represented by one node on the left and one on the right. An edge between node i on left and node j on right represents the requirement that vehicle i must meet vehicle j.

Now color the edges of this graph such that no two edges connected to the same node have the same color. By assigning color k to the edge between vehicle i on the left and vehicle $(i+k) \mod n$ on the right, this may be accomplished using n colors. The schedule is then constructed by letting each color represent a time slot in which the two vehicles are assigned to meet and transfer messages. $\square$

Next, we use these lemmas to obtain appropriate values of T, N_T so that all the arriving messages are eventually delivered to their destinations and we will evaluate the induced delay. Given any $\epsilon > 0$, let

$$N_n = \frac{(1+\epsilon)\lambda(n)}{n^2}T.$$

Let T_{TSP} be the worst-case travel time it takes to tour-through pickup or delivery locations of nN_n messages in cell of area A/n. Then by Lemma 1

$$T_{TSP} \leq \sqrt{\frac{8nN_nA}{n}} = \sqrt{\frac{8(1+\epsilon)\lambda(n)A}{n^2}T}.$$

To complete the description of the rendezvous schedule guaranteed by Lemma 6, the locations of the vehicle rendezvous must be specified. If there is no restriction on the separation of rendezvous points, all vehicles may be assumed to travel to the center of the region and perform the handoffs without any need for further travel. If the pairs of vehicles must be separated by at least r, then the time for each vehicle to visit each of the rendezvous points is at least nr/v. If $r = 0$, all vehicles may perform the rendezvous at the center of the region for a total travel time upper-bounded by $\sqrt{2A}/v$. At the other extreme, if the rendezvous point for each vehicle pairing represented may be taken to the be the center of the cell assigned to the vehicle on the left hand side in the graph, then the distance between points is at most $\sqrt{2A}$ and the total travel may be upper-bounded by $n\sqrt{2A}/v$, or $r = \sqrt{2A}$. If each pairing must be separated by distance r, we may construct a $\sqrt{n} \times \sqrt{n}$ grid of n rendezvous locationsn separated by r. The maximum distance between any two locations is $O(\sqrt{n}r)$. Letting each vehicle pair be assigned arbitrarily to one of the rendezvous point in each of the $n-1$ relay steps, the maximum total distance traveled through these grid points is $O\left(n^{3/2}r\right)$. Adding in the travel time to the grid at the center of the region, $O\left(\frac{\sqrt{A}+n^{3/2}r}{v}\right)$ is an upper bound on the travel time through these locations.

The total time to exchange the messages during the rendevous of vehicles (both relaying and receiving) is $2nN_n\bar{s}(n)$. Hence, the total batch time T can be bounded above as

$$\begin{aligned} T &\leq 2T_{TSP} + 4nN_n\bar{s}(n) + \frac{\sqrt{2A}+n^{3/2}r}{v} \\ &= \sqrt{\frac{32(1+\epsilon)\lambda(n)A}{n^2}T} + (1+\epsilon)\rho T + \frac{\sqrt{2A}+n^{3/2}r}{v}. \end{aligned} \tag{6.1}$$

From (6.1) and some manipulation similar to the solution of the quadratic in the analysis of the Source and Destination policy will lead to the conclusion that it is sufficient to have T such that

$$T = O\left(\frac{\lambda(n)A}{(1-(1+\epsilon)\rho)^2 n^2}\right) + O\left(\frac{\sqrt{A}+n^{3/2}r}{(1-(1+\epsilon)\rho)}\right). \tag{6.2}$$

To complete the proof, we examine each of the arrival queues, show that they are stable with N_n as given, and then compute the time a message spends waiting to be collected into a batch. Note that in time T, in a given cell $\lambda(n)T/n^2$ messages arrive that are destined for any other cell. In the above described scheme with the selection of T as in (6.2), each vehicle serves up to $N_n = (1+\epsilon)\lambda(n)T/n^2$ messages for a given pair of cells. Thus, we have a service rate higher than the arrival rate and hence by standard queueing argument, each queue must be stable.

Finally, we compute the average delay per message in this scheme. To this end, note that each message has the following types of delays: (a) waiting time to be serviced in a cell after arrival and (b) the batch time T. Now, the T is bounded above as (6.2). To bound (a), note that messages are queued separately depending on their destination cells. Consider one particular queue for a destination cell. The arrivals to this queue happen at rate $\lambda(n)/n^2$ while every T units of time, $N_n = N_T/n$ of them get served. Delay through this queue can be upper bounded by T plus the delay through an M/D/1 queue with arrival rate $\lambda(n)/n^2$ and deterministic service requirement of $T/N_n = \frac{n^2}{(1+\epsilon)\lambda(n)}$.

The Pollacek-Khinchin formula (3.9) may be applied to compute the average delay in this M/D/1 queue to be $O(1/\epsilon)$. Therefore the average delay experienced by message between arrival and delivery is $O(T) + O(1/\epsilon)$. □

To complete the proof of Theorem 9(b), note that the batching policy above meets the criteria for Theorem 9 to apply. That is, $E[I_k|B_k] = c_1\sqrt{A}\sqrt{B_k}+2\sqrt{2}\sqrt{A}/v$ which satisfies condition **(c)** of the Corollary.

Letting $\epsilon = (1-\rho)/2$,

$$\frac{1}{\epsilon} = \frac{1}{1-(1+\epsilon)\rho} \leq \frac{2}{1-\rho}.$$

That is, this choice of ϵ increases the upper bound on the delay performance by only a constant factor with respect to the desired lower bound.

Finally, note that for $r = 0$, the delay of the policy above approaches that of the lower bound (up to a constant).

6.2.2 Other Relay Policies - The 1-Depot Policy

In addition to the synchronous policy described above, we may also consider a policy in which messages may be dropped at a depot for asynchronous relay service. For this, we assume the existence of a fixed-location depot at which each vehicle may drop an unlimited number of messages at any time. Messages remain at the depot until they are picked up at the depot for delivery by other vehicles. It is assumed that multiple vehicles may transmit simultaneously to and from the depot while at the depot location.

The assignment and service components of the 1-Depot policy are similar to those of the synchronous policy in Section 6.2 with $r = 0$ which we have already seen to be order optimal. Allowing asynchrous relays implies that variable size batches may be used and $a-$optimal TSP tours may be used to visit the service locations instead of worst-case.

The asynchronous nature of the depot relay makes the analysis of this policy more complicated, but the delay of this policy can only be smaller than that of the synchronous policy in which worst case batching and tours are used. In practice, if a depot is available, it can reduce the delay below that of the synchronous policy, although both are of the same order as the lower bound.

6.3 Conclusions

In this chapter, we analyzed a synchronous single relay policy. The computed upper bound on the delay achievable by this policy had three terms. The three terms of the upper bound on the delay performance were of the same order as the DTRP lower bound, the Universal Lower Bound (for $r = 0$), and the service time lower bound respectively.

The main difference between the relay policies and the no relay policies is in the assignment subregion served by each vehicle. When no relays are allowed, each vehicle must serve two regions: pickup and delivery. Because messages arriving to a single subregion are destined throughout the entire region, vehicle pickup regions must overlap. This implies that each vehicle must cover a region of area at least $A/\sqrt{n}$. In the relay policy, pickup regions need not overlap, and each vehicle services only a region of area A/n, decreasing the area that must be covered in the pickup and delivery tours, thus decreasing delay.

However note that if $\rho > 1/2$ in the DPDP with no relays, the addition of even a single relay will make the system unstable due to the additional onsite service times induced by the relay and the corresponding change in effective ρ. That is

$2\lambda(n)\bar{s}(n)/n > 1/2$ for No Relay implies that $4\lambda(n)\bar{s}(n)/n > 1$ for Single-Relay. Therefore, in heavily loaded systems, relays may not be possible and the DTRP bound may not be achievable.

6.3.1 Optimality of Single Relay

We may extend the analysis of the single-relay policies to multiple-relay systems to show that a single-relay is sufficient to achieve the optimal order performance. Assume that $\rho = 4\lambda(n)\bar{s}(n)/n << 1$, there are no stability concerns associated with additional relays.

The lower bound of the multiple relay problem would be the same as for the single relay problem, aside from a possible increase in ρ. To see this, note that the Universal Lower Bound holds for any Dynamic Pickup and Delivery Problem with a general batching policy, and the services associated with a single demand may be arbitrarily split between several batches. The DTRP bound holds for any multi-stage problem by simply ignoring all services except the delivery. Therefore, since all Pickup and Delivery Problems have this same lower bound which the single relay policy achieves, multiple relays cannot offer any improvement in terms of order.

The impact that multiple relays could have is in decreasing the total time each vehicle has to travel in a single batch to perform a relay, for example if each vehicle needed only to relay to nearby vehicles in each batch time. In the single relay problem, this total relay time is represented by the numerator of the ULB-like upper bound, $\sqrt{A} + n^{3/2}r$. For $r = 0$, this matches the Universal Lower Bound. However, in a multiple-relay system, any savings in an individual batch time would be lost when each message is required to go through multiple relays in multiple batch times to reach its destination.

Chapter 7

Wireless DPDP

One particular application of interest is the Wireless Dynamic Pickup and Delivery Problem in which the vehicles may pickup and deliver the messages via a wireless transmission. Wireless transmissions allow messages to be picked up and delivered remotely, reducing the net amount of travel time per message. Wireless transmission may also reduce the amount of service time required to pickup/deliver a message the closer the vehicle approaches the node. A major question is that of *interference*: a wireless transmission by one node may adversely affect other transmissions occuring simultaneously. In the DPDP presented in the preceding chapters, vehicles perform independently of each other, but interference may imply that not all vehicles can perform service simulataneously.

This application is motivated by recent work in the communication field examining the tradeoff between delay and throughput, where throughput is defined to be the number of messages that are delivered each time unit. Fundamental bounds exist in the case that vehicles are stationary or when vehicles have particular random walk trajectories. Our interest is in deriving fundamental bounds for networks in which vehicles have full control of their motion.

When adding control to vehicle mobility, we need to consider communication and message models different than those usually studied. For example, in [15], messages arrive in a stream to a single node and this stream must be transmitted to exactly one partner node. If node mobility is fully controlled, the trivial optimal solution is to line the nodes up such that each is close to its transmission partner. This policy may achieve a constant rate of throughput with delay only constrained by packet size and rate of wireless transmission.

Besides the message origination model, many other modelling questions arise as well. In particular, there are at least three very different types of wireless transmission: node to vehicle (pickup), vehicle to vehicle (relay), and vehicle to node (delivery). Because vehicles may be much different than the message-generating nodes, each type of service may be characterized by different power and interference constraints.

The vehicle routing analysis we have performed thus far takes a network approach to the pickup and delivery problem. Other vehicle routing problems that extend in a network sense include application of the DPDP to the neighborhood or generalized TSP. The main question is how to model the neighborhoods to capture a meaningful notion of interference. In this chapter, we take a closer look at the Wireless DPDP from a physical layer perspective, seeking bounds on performance for picking up within a neighborhood of relatively few competing demands. We seek general results on the nature of policies which achieve the throughput and delay tradeoff and examine the impact of constructive interference which allows each vehicle to perform service for multiple messages at once.

7.1 Model and Problem Statement

7.1.1 Nodes, Messages, and Vehicles

There is a single vehicle with position $z(t) \in [-A, A] \subset \mathbb{R}$ at time t. The vehicle velocity is bounded by v, that is, $|\dot{z}(t)| \leq v$. There are two nodes located at $-A$ and A and labeled 1 and 2, respectively. Messages comprising a single bit arrive to each node with a deterministic interarrival time of $1/\lambda$ seconds per bit.

The messages are to be transmitted to the vehicle. Assume that the vehicle may only receive messages while it is stationary. When the vehicle is fixed at position $z(t)$, transmission occurs at rate $R_1(z(t))$ from node 1 and $R_2(z(t))$ from node 2 where $R_1(z(t))$ and $R_2(z(t))$ are rate functions to be defined.

7.1.2 Wireless Model

Assume that nodes 1 and 2 transmit over a shared AWGN channel with noise variance 1. That is, at time t, node 1 (node 2) transmits a Gaussian codeword with power $p_1(t)$ (respectively $p_2(t)$). Assume a maximum power constraint, that is,

$$p_j(t) \leq P, \forall t, j = 1, 2.$$

The total signal received by the vehicle is $Y_z(t) \triangleq \sum_j X_{jz}(t) + W(t)$ where $W(t)$ is Gaussian white noise of variance 1. The received signal from nodes 1 and 2 are $X_{1z}(t)$ and $X_{2z}(t)$ with power

$$\begin{aligned} P_{1z}(t) &= p_1(t)|A + z(t)|^{-\alpha}, \\ P_{2z}(t) &= p_2(t)|A - z(t)|^{-\alpha}, \end{aligned}$$

respectively where $\alpha \geq 1$ is the power attenuation constant.

The messages are decoded by the vehicle in one of two ways: Independent Decoding or Successive Interference Cancellation.

Independent Decoding: All transmitting nodes create interference for all of the other nodes. The maximum information theoretic rates as a function of the received powers are

$$\begin{aligned} R_{1z}(t) &= \log\left(1 + \frac{P_{1z}(t)}{1 + P_{2z}(t)}\right), \\ R_{2z}(t) &= \log\left(1 + \frac{P_{2z}(t)}{1 + P_{1z}(t)}\right). \end{aligned}$$

The rate maximizing solution for Independent Decoding at a fixed point is to only allow the user with the largest received power to transmit at full power. The other weaker user has zero transmission power. See [8], Chapter 10.

Successive Decoding and Cancellation: Users are decoded sequentially. The signal used to decode a given user is the original signal minus the decoded signals of users already decoded. The capacity region is the set of rates that satisfy the following constraints:

$$\begin{aligned} R_{1z}(t) &\leq \log\left(1 + P_{1z}(t)\right), \\ R_{2z}(t) &\leq \log\left(1 + P_{2z}(t)\right), \\ R_{1z}(t) + R_{2z}(t) &\leq \log\left(1 + P_{1z}(t) + P_{2z}(t)\right). \end{aligned}$$

The rate maximizing solution in this case is for both nodes transmit at full power. The information theoretic sum rate is achieved by decoding the strongest user first while treating the weakest as interference. The decoded signal is subtracted off and then the weakest user is decoded with no interference. See [8], pp. 378-9.

The following table collects the set of rates for each decoding scheme when the capacity maximizing solution is used.

	Independent Decoding (ID)	Successive Interference Cancellation (SIC)
$z(t) < 0$	$R_{1z}(t) = \log\left(1 + P\lvert A + z(t)\rvert^{-\alpha}\right)$ $R_{2z}(t) = 0$	$R_{1z}(t) = \log\left(1 + \frac{P\lvert A+z(t)\rvert^{-\alpha}}{1+P\lvert A-z(t)\rvert^{-\alpha}}\right)$ $R_{2z}(t) = \log\left(1 + P\lvert A - z(t)\rvert^{-\alpha}\right)$
$z(t) = 0$	$R_{1,0}(t) = R_{2,0}(t)$ $= \frac{1}{2}\log\left(1 + PA^{-\alpha}\right)$	$R_{1,0}(t) = R_{2,0}(t)$ $= \frac{1}{2}\log\left(1 + 2PA^{-\alpha}\right)$
$z(t) > 0$	$R_{1z}(t) = 0$ $R_{2z}(t) = \log\left(1 + P\lvert A - z(t)\rvert^{-\alpha}\right)$	$R_{1z}(t) = \log\left(1 + P\lvert A + z(t)\rvert^{-\alpha}(t)\right)$ $R_{2z}(t) = \log\left(1 + \frac{P\lvert A-z(t)\rvert^{-\alpha}}{1+P\lvert A+z(t)\rvert^{-\alpha}}\right)$

Table 7.1: Detailed Rate Profiles for Decoding Schemes

The coding theorem for Gaussian channels states that the above capacity rates may be achieved with arbitarily long block lengths. In this chapter, we would like to examine the transmission rate over a finite fixed block length. For this, we note that the converse to the coding theorem for Gaussian channels states that transmission rates greater than the above capacity rates are not achieveable, even for arbitrarily long block lengths, that is, the capacity rate is an upper bound on any achievable transmission rate. Therefore, performing the analysis with the capacity rates provides a lower bound on the actual delay seen by a bit in a realistic system.

7.1.3 Control Policies

A control policy controls the vehicle position $z(t)$, subject to the velocity constraint. We assume that the node transmission powers $p_1(t)$ and $p_2(t)$, and the decoding order for Successive Interference Cancellation are chosen as described above to maximize instantaneous total capacity.

In this paper, we examine a particular control policy, the Symmetric Waypoint Policy, which is characterized by two parameters r and K as follows: Assume that pickups occur at two locations (for 2 nodes in 1D), r and $-r$, for some $r \in [0, A]$ to be determined. Assume that the vehicle's initial condition is 0. Messages are served in batches of K bits from each node. Each batch requires the following steps.

1. Vehicle travels from 0 to r,
2. Picks up from 1 with rate $R_1(r)$, from 2 with rate $R_2(r)$,
3. Vehicle travels from r to $-r$,
4. Picks up from 1 with rate $R_1(-r)$, from 2 with rate $R_2(-r)$), and
5. Vehicle returns to 0.

Let $R(r)$ be the total pickup rate when the vehicle is fixed at location r (or equivalently at $-r$). By the assumptions of symmetric traffic and service, when the vehicle may switch infinitely quickly between r and $-r$, the average rate of service received by each node is $R(r)/2$. When vehicles transmit with optimal powers,

$$R_{ID}(r) = \log(1 + P|A - r|^{-\alpha}), \tag{7.1}$$

$$R_{SIC}(r) = \log(1 + P|A - r|^{-\alpha} + P|A + r|^{-\alpha}). \tag{7.2}$$

Let T_K be the total time to collect a batch of K bits from each node and return to the starting position A.

$$T_K = \frac{4r}{v} + \frac{2K}{R(r)}$$

7.1.4 Performance Measures

Throughput λ is said to achievable if the messages that are arriving at rate λ to each each node are served with finite delay.

Delay is the total time to serve a single batch of K messages from each node as defined by T_K. This delay ignores queuing delay at the node as well as any additional decoding time required to achieve the capacity rate due to small batch sizes.

7.1.5 Problem Statement

We would like to answer several general questions: If it is possible to pickup without moving, does not moving provide the minimum delay? If it is not possible to pickup with out moving, what is the optimal distance from the node for pickup? When is it best to travel directly to the node?

More specifically, we will examine the following three problems.

Problem 1 - Stabilizing Region Let $\mathcal{R}(\lambda) \subset [0, A]$ be the set of all r such that there exists a K such that a symmetric waypoint policy parameterized by r and K achieves throughput λ. Find $\mathcal{R}(\lambda)$ as a function of the rate profile $R(r)$.

Problem 2 - Optimal Batch Scaling and Delay as a function of r For a given throughput λ and $r \in \mathcal{R}(\lambda)$, find the minimum batch size K such that the symmetric waypoint policy achieves throughput λ. With the minimum batch size K, find a lower bound on the minimum batch time, $T_{min}(r, \lambda)$. Characterize $T_{min}(r, \lambda)$.

Problem 3 - Optimal Throughput-Delay Tradeoff For given throughput λ, find the minimum delay $T^*(\lambda)$ such that there exists $r \in \mathcal{R}(\lambda)$ such that $T_{min}(r, \lambda) =$

$T^*(\lambda)$. This is this a lower bound on the minimum achievable delay for the given throughput.

Test Cases Plots for various test cases will be used to illustrate the results. In particular, we consider $A = 2, P = 1$ and $v = 1$ for a range of arrival rates λ.

7.1.6 Organization

The remaining sections of this chapter will address Problems 1-3 above and Problem 4 will be addressed in the discussion. The stability analysis of Section 7.2 with provide an analytical solution for $\mathcal{R}(\lambda)$ for the two rate profiles: Independent Decoding (ID) and Successive Interference Cancellation (SIC). The stability regions are also plotted for the test case for illustration. Section 7.3 provides a solution for $T_{min}(r, \lambda)$ in terms of $R(r)$. A few characterizations of $T_{min}(r, \lambda)$ are derived analytically, but due to its highly nonlinear form, further graphical characterizations are provided as well. In particular, the a closed form solution for the minimum of $T_{min}(r, \lambda)$ over r does not exist, but this minimum may be computed graphically over a range of λ to give some intuition into the throughput and delay tradeoff of this system. This analysis is performed in Section 7.4. Finally, the discussion may be found in Section 7.5.

7.2 Stability Analysis

Ignoring the travel constraints, the system may be viewed as a $D/D/1$ queue where the demands are bits and the service is the vehicle's wireless reception of these bits. In the case that the vehicle may only receive when stationary at r or $-r$, the service rate of the vehicle is at most the wireless reception rate associated with that location, i.e. $R(r)$. For the stability of the queue, the service rate must exceed the arrival rate, and therefore, the condition for stability is

$$\mathcal{R}(\lambda) = \{r | R(r) \geq 2\lambda, r \in [0, A]\}. \tag{7.3}$$

If $R(0) \geq 2\lambda$, then $\mathcal{R} = [0, A]$. If $R(0) < 2\lambda$, then $\mathcal{R}(\lambda) = [r^*(\lambda), A]$ where

$$r^*(\lambda) = 2\lambda. \tag{7.4}$$

Note this is only a necessary condition for stability, as some rate may be lost while the vehicle is traveling between waypoints and is not receiving messages. Further, for a more general G/G/1 queue with stochastic arrivals and/or service times, the necessary condition becomes $R(r) > \rho$ and the stability intervals are half-open to the left.

Applying (7.3) to the rate profile for Independent Decoding,

$$R(r) = \log(1 + P(A-r)^{-\alpha}) \geq 2\lambda \implies r \leq A - \left(\frac{P}{2^{2\lambda}-1}\right)^{\frac{1}{\alpha}},$$

$$\mathcal{R}_{ID}(\lambda) = \begin{cases} [0, A] & \text{if } A \leq \left(\frac{P}{4^\lambda - 1}\right)^{\frac{1}{\alpha}} \\ [r^*, A] & \text{else, with } r^* = A - \left(\frac{P}{4^\lambda-1}\right)^{\frac{1}{\alpha}}. \end{cases} \tag{7.5}$$

Similarly, for Successive Interference Cancellation,

$$\begin{aligned} R(r) &= \log(1 + P(A-r)^{-\alpha} + P(A+r)^{-\alpha}) \geq 2\lambda \\ &\implies (A+r)^{-\alpha} + (A-r)^{-\alpha} \geq \left(\frac{2^{2\lambda}-1}{P}\right). \end{aligned}$$

This is an implicit function of r. To check if the midpoint is in $\mathcal{R}$, let $r = 0$. Then

$$\mathcal{R}_{SIC}(\lambda) = \begin{cases} [0, A] & \text{if } A \leq \left(\frac{1}{2}\right)^{\frac{1}{\alpha}} \left(\frac{P}{4^\lambda-1}\right)^{\frac{1}{\alpha}} \\ [r^*, A] & \text{else, with } r^* \text{ s.t. } (A+r^*)^{-\alpha} + (A-r^*)^{-\alpha} = \left(\frac{4^\lambda - 1}{P}\right). \end{cases} \tag{7.6}$$

Note that the stability region is wider with SIC decoding, but that this benefit decreases as the power attenuation constant α increases. SIC decoding can receive messages from both nodes simultaneously, but this benefit has less impact as the received power drops off more quickly with a larger attenuation constant. Large power attenuations has the benefit of rapid rate increase as the vehicle approaches the node, but small power attenuations permit meaningful rates to be achieved at the center of the region.

In Figure 7-1, the boundary of the stability region is plotted for both decoding schemes and $\alpha = 1$ and 3. The stability region for each lies above the associated curve. This highlights the increase in stability region for SIC decoding over ID. This difference is more pronounced for small α.

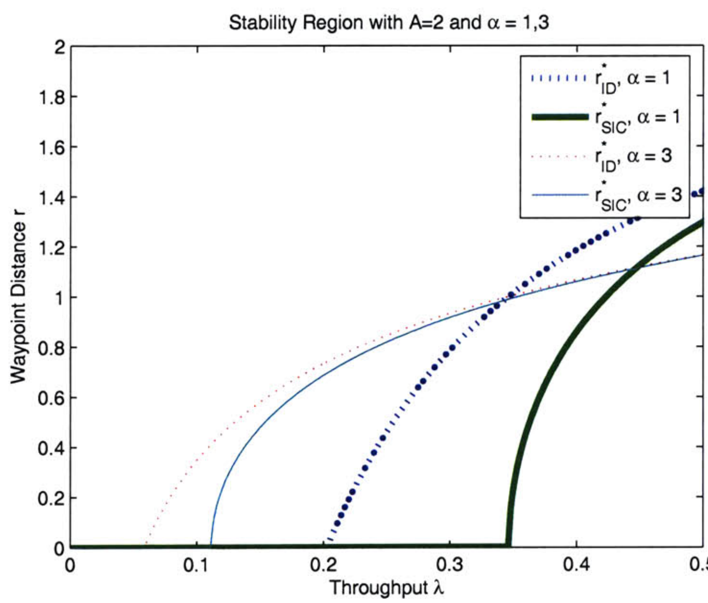

Figure 7-1: Stability Region for test case

Figure 7-2 contains the same data as Figure 7-1 for a wider range of power attenuations. For small throughputs, the stability region is narrower for larger power attenuation. This relation is reversed at higher throughputs.

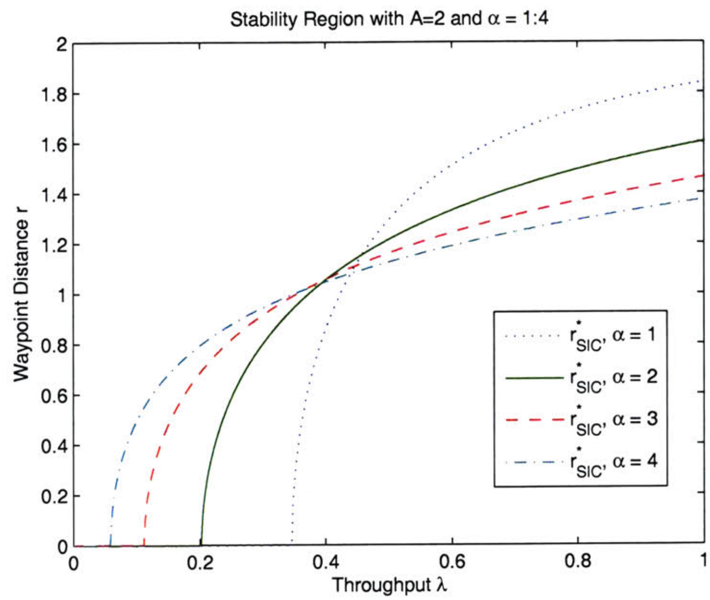

Figure 7-2: Stability Region for test case - $\alpha = 1, 2, 3, 4$

7.3 Optimal Batch Scaling and Delay for fixed r and λ

For a fixed batch size K and waypoint r, the total time to pickup K messages from each node with waypoints r and $-r$ is

$$T_K = \frac{4r}{v} + 2K\frac{1}{R(r)}. \tag{7.7}$$

Since $2K$ total messages are picked up in this time, the average service rate for a fixed K is

$$\text{average service rate} = \frac{2K}{T_K} = \frac{2KvR(r)}{4rR(r) + 2Kv}. \tag{7.8}$$

Note that as K or v becomes large, this average service rate approaches $R(r)$.

Similar to the queueing analysis above, for the system to be stable, this service rate must be greater than the total arrival rate 2λ. Some algebra reveals that the required condition on K is

$$\begin{aligned} K &\geq \max\left\{\frac{4\lambda r R(r)}{v(R(r)-2\lambda)}, 1\right\} \\ &= \max\left\{\frac{4\lambda r}{v\left(1-\frac{2\lambda}{R(r)}\right)}, 1\right\} \\ &= \max\left\{\frac{4\lambda r}{v\,(1-\rho)}, 1\right\} \end{aligned} \tag{7.9}$$

where

$$\rho = \frac{2\lambda}{R(r)}. \tag{7.10}$$

Because $1/R(r)$ is the average service time at the waypoint r to pickup a message, the quantity $\frac{2\lambda}{R(r)}$ is the analogue of ρ for the DPDP or any G/G/1 queue. K is bounded below by 1 to prevent the computation of nonmeaningful delays when ρ, and the resulting K, is small.

The minimum achievable batch delay for fixed r is achieved by substituting this bound on K into the formula for T_K.

$$T_{min}(r,\lambda) = \max\left\{\frac{4r}{v(1-\rho)}, \frac{4r}{v} + \frac{2}{R(r)}\right\} \tag{7.11}$$

Note the relation to Little's Law where the $K > 1$, i.e. $2K = 2\lambda T$.

With this formula, we then have the following expression for $\mathcal{R}_T(\lambda)$

$$\mathcal{R}_T(\lambda) = \{r \mid T_{min}(r,\lambda) \leq T\}. \tag{7.12}$$

7.3.1 Analytical Characterization of $T_{min}(r,\lambda)$

For either of the rate functions under consideration, (7.12) is a complicated function due to the nonlinearity of $R(r)$. To gain some insight into the general shape of this curve as a function of r, we may examine its derivative with respect to r:

$$\frac{dT_{min}(r,\lambda)}{dr} = \frac{4}{v}\left(1 + \frac{2\lambda}{R(r)-2\lambda} - \frac{2\lambda r \frac{dR(r)}{dr}}{(R(r)-2\lambda)^2}\right). \tag{7.13}$$

We can gain some insight about the r at which the minimum delay is achieved by examining the sign of the derivative at the endpoints of the stability region: $r^*(\lambda)$ and A.

Travel directly to node, $r = A$

First consider $r \to A$, that is, the vehicle travels all the way to the node to perform the pickup. The rate $R(r) \to \infty$, which sends the second term (7.13) to 0, but additional information about $R(r)$ is required to evaluate the sign of the third term. Note that for either of the rate profiles under consideration here $R(r) \approx \log((A-r)^{-\alpha})$ for $r \approx A$. In this limiting case, $dR(r)/dr = O(1/(A-r))$ and the third term of (7.13) goes to $1/[(A-r)\log(A-r)] \to -\infty$. This negative term dominates and A is a local minimum. Once the vehicle has traveled far enough, it is best to continue traveling directly to the node.

Remain stationary, $r = 0 \in \mathcal{R}(\lambda)$

Examine the derivative at the other endpoint, $r^*(\lambda)$. If $r^*(\lambda) = 0$, i.e. $0 \in \mathcal{R}(\lambda)$, $R(r)$ is finite, but since $r = 0$, the third term of (7.13) disappears. Since this was the only negative term, the derivative is positive and 0 is at least a local minimum.

Travel to closest point in $\mathcal{R}(\lambda)$, $r = r^*(\lambda) \neq 0$

If $r^*(\lambda) > 0$, then $R(r^*(\lambda)) - 2\lambda \to 0$. Assuming that $dR(r)/dr$ is finite at $r^*(\lambda)$, the third term of (7.13) dominates. In fact, the derivative approaches $-\infty$, so $r^*(\lambda)$ is definitely a local maximum. This makes sense because the batch sizes must become very large to compensate for the small amount of extra throughput alloted to travel time.

Optima in $\mathcal{R}(\lambda)$: Summary

We have established that when $0 \in \mathcal{R}(\lambda)$, both 0 and A are local minima. It is easy to test which of these extreme cases has lower delay by comparing $T_{min}(0, \lambda) = \frac{2}{R(r)}$ and $T_{min}(A, \lambda) = \frac{4A}{v}$. When $0 \notin \mathcal{R}(\lambda)$, A is a local minima, but $r * (\lambda)$ is a maximum.

Because we are interested in finding a global maximum, the question remains as to whether there are any other local minima. We see below that the answer to this question is yes, sometimes.

7.3.2 Graphical Characterization of $T_{min}(r, \lambda)$

Figures 7-3 and 7-4 plot $T_{min}(r, \lambda)$ as a function of r for some test cases of λ. Test cases were chosen to highlight the behavior when the various radii are within the stability region.

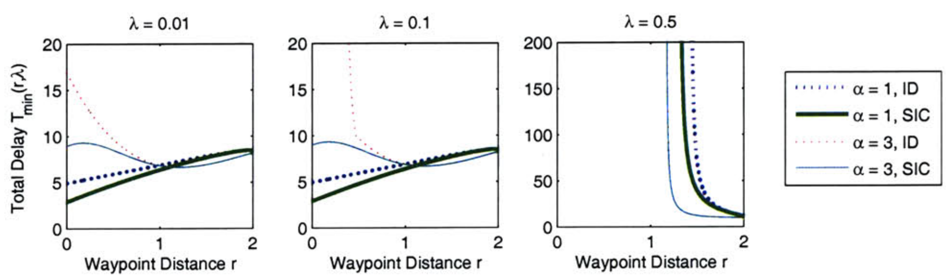

Figure 7-3: Delay as a function of r with optimal batch size

First examine 7-3. For $\lambda = .01$, all four test cases have $0 \in \mathcal{R}(\lambda)$, and therefore it is feasible to pick up the messages without moving. For both $\alpha = 1$ cases, staying at 0 is indeed optimal. For larger α however, although 0 is stable, the associated rate is so slow (due to the larger power attenuation) that it is better to move towards the node before picking up.

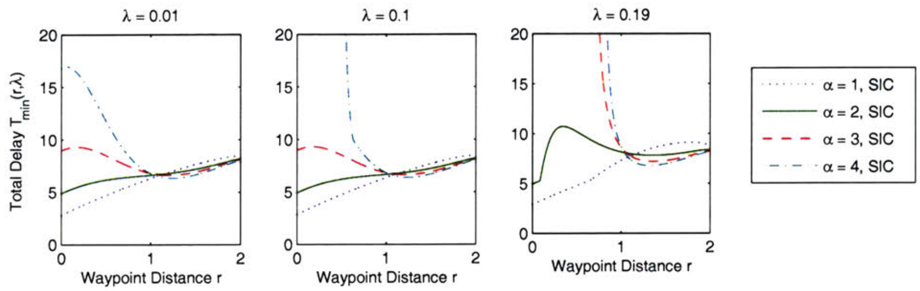

Figure 7-4: Delay as a function of r with optimal batch size, $\alpha = 1, 2, 3, 4$

With $\alpha = 3$, the optimum is just over halfway to the node and is approximately the same for both decoding schemes. In general, using Independent Decoding results in a longer delay than Successive Interference Cancellation because constructive interference is used in SIC to pick up messages from both nodes simultaneously. As the power received from the more distant node decreases as the vehicle approaches the dominant node, the performance of the ID and SIC schemes converge.

Moving to $\lambda = 0.1$, 0 is no longer within the stability region for ID and larger α. As expected, the delay is very large for r near $r^*(\lambda)$ due to the large required batch sizes. For $\lambda = 0.5$, $0 \notin \mathcal{R}(\lambda)$ for any of the test cases. Lower delays are achieved for larger α as the vehicle approaches the node. This is also highlighted in 7-4.

Though it is hard to see from these plots, both 0 and A are always local minima, but neither is always a global minimum. Particularly for larger α, the optimal r may fall somewhere in between the two extremes.

7.4 Optimal Throughput/Delay Tradeoff

We can repeat the above analysis over a range of λ and compute the minimum delays achievable for each throughput λ and the r (and associated K) at which this minimum is achieved. Define

$$T_{min}(\lambda) = \min_{r \in \mathcal{R}(\lambda)} T_{min}(r, \lambda).$$

The delay curves are plotted in Figure 7-5. Each delay curve is characterized by two limiting values: the delay when picking with $K = 1$ and the delay when traveling all the way and picking up instantaneously. The lower limit is an artifact of the $K \geq 1$ bounding, but the upper limit represents the situation in which the vehicle travels to the node and picks up messages infinitely quickly. Increased batch sizes for stability have no impact on delay.

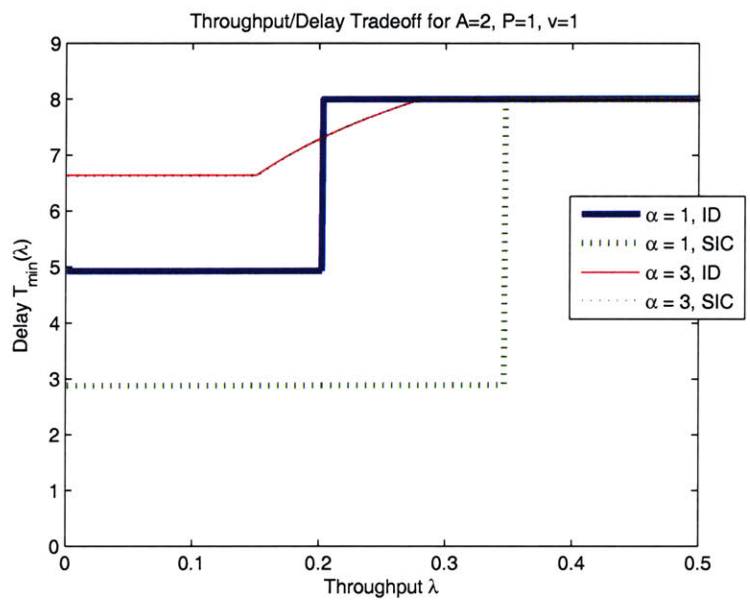

Figure 7-5: Optimal Delay as a function of λ, Comparison

As we saw in the figures above, for large α, the optimum with $K = 1$ is not necessarily to remain at 0. The optimum delay includes both travel time and service time components, but this point remains the minimum for a large range of λ, including some λ for which $0 \notin \mathcal{R}(\lambda)$.

For large α, the optimum r changes slowly with λ with a concave delay between two limiting values. For small α, the change between the two step points is nearly instantaneous when λ increases to the point that $0 \notin \mathcal{R}(\lambda)$. The reason for this step change may be illustrated by Figure 7-6. This plot illustrates the behavior of the delay as a function of r near the point $\lambda \approx 0.2$ where 0 is no longer in the stability region for the ID policy with $\alpha = 1$. There is no significant minimum between 0 and

A for this policy. For larger α, we have already seen that there is a minimum between 0 and A and that this minimum is approximately the same for both SIC and ID. This also accounts for the similarity between SIC and ID for $\alpha = 3$ in Figure 7-5.

Also note the linear scaling for small r. This is an artifact of the $K \geq 1$ bounding. When $K > 1$ is not restricted to integer values, the increase in delay is strictly concave, accounting for increase both in travel time and also batch size.

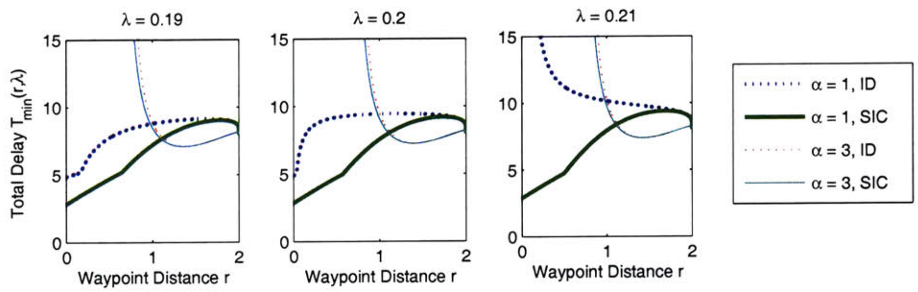

Figure 7-6: Delay as a function of r, λ near Stability Region Critical Point

7.5 Conclusions

Finally, a few comments are in order about the optimality of the above analysis.

First, in the above, messages were assumed to be continously arriving, and the batch size K was varied to create batches of optimum size. If fixed-size batches are used (for a static, one-time pickup, for example), this may weight the transmission time more heavily than the travel time and the optima as a function of r may not hold. Further, if arrivals are taken to be Poisson, there is an additional queuing time to be taken into account, both in computing delay and also in constructing optimum batch size. Batch sizes larger than the minimum batch size computed above can minimize this impact.

Returning to our questions of Section 7.1.5, even when not moving provides a stable solution, sometimes it is better to travel, especially if either the vehicle velocity or the power attenuation constant is large. There is always a balance between linear travel time and exponential rate increase, but finding the right r to achieve this balance is not trivial in general.

These results for the simplified two-node, one-vehicle network provide some intuition for control of general m-node wireless networks. In this chapter, we have shown that large throughputs may only be achieved when the vehicle travels directly to each node. This suggests that the control of a high throughput m-node network is equivalent to the Dynamic Pickup and Delivery Problem as studied in this thesis. Very low throughput networks may be served by placing immobile receivers at fixed locations throughout the network.

To tradeoff throughput with delay between these two extremes, different models of the Dynamic Pickup and Delivery Problem are required. For example, we have seen in this chapter that delay is minimized for moderate throughputs by traveling part-way to the node and picking up remotely. Remote pickup may be incorporated into a DPDP model using results on the Generalized TSP or TSP with Neighborhoods [1]. This is a subject for future work.

Chapter 8

Discussion

In this final chapter, we provide an interpretation of the main results, explore the significance of the control policy limitations, describe how our methods may be extended, and discuss other directions for future work.

8.1 Scaling Interpretation of Results

First, we interpret the results of the three main theorems of Chapters 5 and 6. Each bound was made up of three terms:

$$W = \Theta\left(\frac{\lambda(n)A}{v^2(1-\rho)^2 n^\sigma}\right) + \Theta\left(\frac{\sqrt{A}}{v(1-\rho)}\right) + \Theta(\bar{s}(n))$$

for some exponent σ that was different for each DPDP setup. For the two No Relay policies, $\sigma = 1$ and $\sigma = 3/2$ for Source Only and Source and Destination respectively. For the Single Relay policy, $\sigma = 2$. The individual terms of the theorems dominate over different regions of the parameter scaling.

The service time term can dominate when $\sqrt{A}/v$ and λ/n^σ are small and ρ is of moderate size. When the onsite service times are modest and most of the delay is due to travel time, one of the first two terms will dominate. The first term, the DTRP

	$\frac{\lambda\sqrt{A}}{nv} = o(1)$	$\frac{\lambda\sqrt{A}}{nv} = \Omega(1)$ and $\frac{\lambda\sqrt{A}}{n^{3/2}v} = o(1)$	$\frac{\lambda\sqrt{A}}{n^{3/2}v} = \Omega(1)$ and $\frac{\lambda\sqrt{A}}{n^2v} = o(1)$	$\frac{\lambda\sqrt{A}}{n^2v} = \Omega(1)$
Source Only	$\Theta\left(\frac{\sqrt{A}}{v(1-\rho)}\right)$	$\Theta\left(\frac{\lambda A}{v^2(1-\rho)^2n}\right)$	$\Theta\left(\frac{\lambda A}{v^2(1-\rho)^2n}\right)$	$\Theta\left(\frac{\lambda A}{v^2(1-\rho)^2n}\right)$
Source and Destination	$\Theta\left(\frac{\sqrt{A}}{v(1-\rho)}\right)$	$\Theta\left(\frac{\sqrt{A}}{v(1-\rho)}\right)$	$\Theta\left(\frac{\lambda A}{v^2(1-\rho)^2n^{3/2}}\right)^\dagger$	$\Theta\left(\frac{\lambda A}{v^2(1-\rho)^2n^{3/2}}\right)^\dagger$
Single Relay	$\Theta\left(\frac{\sqrt{A}}{v(1-\rho)}\right)$	$\Theta\left(\frac{\sqrt{A}}{v(1-\rho)}\right)$	$\Theta\left(\frac{\sqrt{A}}{v(1-\rho)}\right)$	$\Theta\left(\frac{\lambda A}{v^2(1-\rho)^2n^2}\right)$

† Requires also $\lambda(n)A/v^2n^{3/2} = \Omega(1)$

Table 8.1: Average Delay scaling over various ranges of Scaling Parameters

term, dominates when

$$\frac{\lambda A}{v^2(1-\rho)^2n^\sigma} >> \frac{\sqrt{A}}{v(1-\rho)} \Rightarrow \frac{\lambda\sqrt{A}}{v(1-\rho)n^\sigma} >> 1.$$

The results of the main theorems are provided in terms of the dominance of the first two terms in Table 8.1.

To understand the scaling behavior of these terms, note that the delay of a message is made up of three components: 1)time the vehicle to which the message is assigned spends traveling to serve other messages, 2)time spent traveling directly to the source and destination locations of the message , and 3) onsite service times. For ease of exposition, assume $\sqrt{A}/v = \Omega(1)$, that is, the time to cross the entire region is constant or increasing.

When $\lambda(n) = o(n)$ and $1-\rho$ is of moderate size, the onsite service time can dominate. In all other cases, the impact of the onsite service times is generally captured by the $(1 - \rho)$ terms in the denominators of the other two terms. The scaling of $\lambda(n)$ determines which of the other two travel time terms dominates.

For $\lambda(n) = \Omega(n)$, the arrival rate of messages per vehicle, $\lambda(n)/n$, shrinks as the number of vehicle increases. Therefore, most of the delay is accumulated during the travel associated with the message's own service. Roughly, this is captured by the universal lower bound term, which is a straightforward function of the average distance between the source and destination locations of an individual message. Note that this scaling is not a function of the number of vehicles n but is intrinsic to the pickup and delivery requirements of the messages.

For $\lambda(n) = o(n^{3/2})$ and $\lambda(n) = \Omega(n)$, the DTRP-based bound is increasing as a function of n for the Source Only policy, but shrinks for the Source and Destination policy. Briefly, this is because the number of vehicles n can affect both the average rate of messages arriving to the vehicle, $\lambda(n)/n$, and also the area that a vehicle must cover to service those messages. The Source and Destination case performs significantly better, because as n increases, destination information may be exploited to shrink the region over which each vehicle must travel (roughly $A/\sqrt{n}$). The region shrinks faster than the rate at which the number of messages to be served per vehicle is growing ($\lambda(n)/n$). Due to this, again the travel time between the source and destination of an individual message dominates. The Source Only policies cannot exploit this information to reduce their travel time, and therefore the delay in traveling between other messages dominates. For $\lambda(n) = o(n^2)$ and $\lambda(n) = \Omega(n^{3/2})$, a similar argument holds for the difference between the Single Relay policy (with area A/n) and the best No Relay policy (with area $A/\sqrt{n}$).

For $\lambda(n) = \Omega(n^2)$, the number of messages per vehicle is always growing faster than any valid policy can shrink the service regions for the vehicles. Therefore, in this regime, the travel time for other messages dominates.

8.2 Significance of Policy Restrictions

Note that the set of policies under consideration is somewhat restrictive. We consider only policies with separable assignment and service policies. Assignments are made by a centralized controller independent of the current service requirements associated with each of the vehicles. Further, the use of the $\{p_i(x,y)\}_{i=1}^n$ for the assignment policy fails to include any policies in which batches of requests are collected into a G/G/n queue at a centralized depot and served in FCFS order. Such policies were proposed for the DTRP in [4]. The assignments of consecutive messages are likely to be correlated due to their collection into a single batch, and therefore the independent assignment property fails to hold. Comparing the delay of the G/G/n to the average delay of a collection of n G/G/1 queues as in Section 6.2, the G/G/n assignment and service policy actually has a lower average delay than the n G/G/1 queues. This difference disappears as the traffic increases and the probability of vehicle idleness approaches 0. Therefore, in the limit, our policy restriction does not seem to hurt us in terms of finding the minimum delay scaling, at least for G/G/n policies.

We also note that the centralized assignment policies presented in Section 6.2 may be decentralized given appropriate assumptions on inter-vehicle communication. In the Source Only case, each message may be assigned the vehicle closest to it upon its arrival. Each vehicle needs only to communicate with its neighboring vehicles to resolve which vehicle will pick up each message. In the Source and Destination case, message assignments are based only on the locations associated with each message. After a centralized initialization period in which vehicles are assigned to pickup and delivery regions, no centralized decision making or inter-vehicle communication is required.

The restriction to a batching policy appears limiting, but the conditions required are quite natural. First, most vehicle routing policies that are used in practice entail some batching. Further, we have computed lower bounds without the batching assumption and have shown that batching policies achieve these lower bounds. Finally, the time average conditions for Theorem 6 are quite natural. Intuitively, condition **(c)** in Corollary 1 means that the overhead time per batch does not grow faster than linearly with the number in batch. This is a natural condition for batch service as it implies that there is some economy of scale associated with grouping demands into batches for service.

8.3 Extensions of Methods

Under certain natural assumptions, the bound in Theorem 6 would hold for the general multi-stage system with two differences. First, if μ services are required per demand, ρ would increase to $\rho = \mu\lambda(n)\bar{s}(n)/n$. We have already seen this increase in ρ seen in the progression from DTRP to DPDP to single relay DPDP. Second, the numerator in equation (6) would be replaced by the appropriate minimum travel time per message. This may be bounded by computing a minimum expected TSP time through the service locations of a single demand. Therefore, with minor changes, the lower bound methods presented in this paper may be adapted for other multi-stage problems. Similar batching policies may also be implemented, although the nature and performance of these policies naturally depend on the application involved.

The density optimization methods used to solve for a lower bound on the Source and Destination problem may also be extended to non-uniform and non-independent source and destination distributions as long as the locations associated with different messages are still independent. More complicated distribution constraints would lead to more complicated and possibly asymmetric assignment distributions, but would be solvable in theory.

8.4 Future Work in Wireless

The preliminary wireless results may be extended to an arbitrary m-node network in several different ways. First, it may be possible to develop heuristics for optimal transmission distance based upon our analysis of the impact of transmission distance on delay for a given throughput. Generally, we have shown that for high throughput, the optimal solution is a DPDP problem. For low throughputs, fixed vehicle locations are optimal. For throughputs in between, remote pickup is optimal. It would be useful to estimate the critical values differentiating high, low, and in-between throughput. Further, remote pickups may be incorporated into the existing DPDP model using the generalized TSP.

More rigorously, it may be possible to directly extend the analysis of the 2-node problem to a generic m-node problem with a specific symmetric placement of nodes. Similar results may extend to certain asymmetric networks as well.

8.5 Conclusion

In this thesis, we have presented a dynamic vehicle routing problem, the Dynamic Pickup and Delivery Problem (DPDP), and obtained tight lower and upper bounds on the scaling of the average message delay for three variants of the DPDP: No Relays with Source Only information, No Relays with Source and Destination information, and Single-Relay. These results are a significant extension of the existing results on the DPDP.

Appendix A

Extended proof of DTRP

In this appendix, we prove Theorem 4(b) which is a modified version of Theorem 2 in [6].

The Dynamic Traveling Repairperson (DTRP) problem refers to the following setup: demands arrive to a closed and bounded region $\mathcal{A}$ of area A according to a stationary renewal process. Demands are independently and identically distributed according to the demand distribution $f(x)$. There are n vehicles traveling in the region with bounded velocity v to service these demands. A demand is serviced when a vehicle arrives at the demand location and spends a random service time s at that location. The goal is to service the demands with the minimum average delay W between message arrival and service.

The following assumptions are required for the DTRP proof as written in this appendix.

1. $\mathcal{A} \subset \mathbb{R}^2$ is closed and bounded.
2. Interarrival times are Poisson.
3. Onsite service times deterministic of length $\bar{s}$.
4. Demand locations are i.i.d. and distributed according to density $f(x)$.

5. $f(x)$ is bounded from above and is K-Lipschitz, i.e.

$$\begin{aligned} 0 &\leq f(x) \leq \overline{f} < \infty, \quad &(A.1) \\ |f(x) - f(y)| &\leq K|x - y|, \ \forall x, y \in \mathcal{A}. \quad &(A.2) \end{aligned}$$

6. The waiting time conditioned on message service location, $W(x) = [W(j)|x(j) = x]$, exists and is bounded from above and K-Lipschitz.

$$\begin{aligned} \Psi(x) &= \frac{W(x)}{W}, \quad &(A.3) \\ 0 &\leq \Psi(x) \leq \overline{\Psi} < \infty, \quad &(A.4) \\ |\Psi(x) - \Psi(y)| &\leq K|x - y|, \ \forall x, y \in \mathcal{A}. \quad &(A.5) \end{aligned}$$

In [6], the assumptions are slightly different in that general interarrival and service times with finite first and second moments are allowed, and $f(x)$ and $\Psi(x)$ are required to be bounded away from 0.

Before proving lower bounds on this average delay, [6] provides a few additonal definitions and assumptions. First, with every subset $\mathcal{S} \in \mathbb{R}^2$, associate a queue $\mathcal{S}$, viewed as a black box that has arrivals and departures according to the arrival and service of demands in $\mathcal{S}$. Let $N(\mathcal{S})$ denote the time average number of customers in $\mathcal{S}$ and assume that this time average exists for all $\mathcal{S}$. Then $N = N(\mathcal{A})$ denotes the time average number of customers in the whole system. The queue occupancy density is defined to be

$$\phi(x) = f(x)\Psi(x). \quad (A.6)$$

Because service locations are i.i.d. and arrivals are Poisson, the arrival rate of messages to $\mathcal{S}$ is $\lambda(\mathcal{S}) = \lambda \int_{\mathcal{S}} \phi(x)dx$. Little's Law may be used to show that an equivalent definition of the time average number in system is $N(\mathcal{S}) = N \int_{\mathcal{S}} \phi(x)dx$.

The proof of Theorem 4 belows follows the same sequence of lemmas as in the proof of Theorem 2 in [6]. While our proofs are modified appropriately, the full proof is given only where it differs significantly from that in [6].

Lemma 7. *For any stable property satisfying the above properties, the queue occupancy density function satisfies*

$$\begin{aligned} \int_{\mathcal{A}} \phi(x)dx &= 1, \\ 0 \leq \phi(x) &\leq \overline{\phi} < \infty, \\ |\phi(x) - \phi(y)| &\leq K|x-y|, \quad \forall x, y \in \mathcal{A}. \end{aligned}$$

Proof. The proof follows directly from the properties of $f(x)$ and $\Psi(x)$. □

Let $\mathcal{B}_z(x) = \{y|\ |x-y| \leq z\}$ denote a closed ball of radius z around location x.

Lemma 8. *For any stable policy satisfying the above properties,*

$$N^+(\mathcal{B}_z(x)) = N\phi(x)\pi z^2 + No(z^2)$$

for any x such that $f(x) > 0$.

Proof. Because arrivals are Poisson, by PASTA, $N^+(\mathcal{B}_z(x)) = N(\mathcal{B}_z(x))$. From the queue occupancy interpretation of $\phi(x)$, $N(\mathcal{B}_z(x)) = N\int_{\mathcal{B}_z(x)} \phi(x)dx$. Applying the mean value theorem for integrals as well as the Lipschitz property of $\phi(x)$ yields

$$\begin{aligned} N\int_{\mathcal{B}_z(x)} \phi(x)dx &\leq N\left(\phi(x) + Kz\right)\pi z^2 && \text{(A.7)} \\ &= N\phi(x)\pi z^2 + NK\pi z^3 && \text{(A.8)} \\ &= N\phi(x)\pi z^2 + No(z^2). && \text{(A.9)} \end{aligned}$$

The last line follows because $z^3 = o(z^2)$ for $z \rightarrow 0$. □

The total service time associated with a demand is defined to be the onsite service time s plus the incremental travel time between the demand and the next demand to be serviced. Denoting the distance to be traveled after the jth demand as d_j, the total service time associated with demand j is then $d_j/v + s_j$.

Lemma 9. *The average interdemand travel time is related to $E[Z^*]$, the expected minimum distance between any two active demands, according to*

$$E[Z^*] \triangleq \lim_{j\to\infty} E[Z^*(j)] \leq \lim_{j\to\infty} E[d_j] \triangleq \bar{d}.$$

Proof. Straightforward. □

Lemma 10. *$E[Z^*]$, the expected minimum distance between active demands, is related to the system parameters as follows:*

$$\lim_{N\to\infty} \sqrt{N}E[Z^*] \geq \gamma \int_A \phi^{-1/2}(x)f(x)dx$$

where $\gamma \geq \frac{2}{3\sqrt{\pi}}$.

Proof. Because the number left behind by a departure is integer,

$$P(Z^* \leq z|x(j) = x) \leq N^+(\mathcal{B}_z(x)). \tag{A.10}$$

Integrating to find the expected value conditioned on x

$$\begin{aligned} E[Z^*|x(j) = x] &\geq \int_A \max\{0, 1 - N^+(\mathcal{B}_z(x)\}dz & \text{(A.11)} \\ &= \gamma\phi^{-1/2}(x)N^{-1/2} - o\left(N^{-1/2}\right). & \text{(A.12)} \end{aligned}$$

Unconditioning by integrating over $df(x)$ and taking the limit as $N \to \infty$ yields the result. □

Note that as there are more demands are in the queue on average, the interpoint distance decreases, thus decreasing the service time associated with each demand. It is this shrinking of the service time with increased queue occupancy that makes the stability of the DTRP a function of the onsite utilizations only.

In [6], Lemma 5 is then stated as follows:

Lemma 5 from [6]

$$\lim_{\rho \to 1} W(1-\rho)^2 \geq \gamma^2 \frac{\lambda[\int_A \phi^{-1/2}(x) f(x) dx]^2}{v^2 n^2}.$$

As we are interested in the case with scaling of parameters other than ρ, we instead prove the following result:

Lemma 11. (Lemma 5 from [6] (modified)) *If both* $\frac{\lambda \mathbb{E}[\sqrt{f}]}{vn} \to \infty$ *and also* $\frac{\lambda \mathbb{E}[\sqrt{f}]^2}{v^2 n} \to \infty$*, then*

$$W \geq \Omega\left(\frac{\lambda[\int_A \phi^{-1/2}(x) f(x) dx]^2}{v^2(1-\rho)^2 n^2}\right).$$

Proof. Consider the following necessary condition for stability

$$\bar{s} + \frac{\bar{d}}{v} \leq \frac{n}{\lambda}. \tag{A.13}$$

Using the fact that $E[Z^*] \leq \bar{d}$, multiplying the second term on the left hand side above by $\frac{\sqrt{N}}{\sqrt{N}}$ and rearranging implies

$$\sqrt{N} \geq \frac{\lambda \sqrt{N} E[Z^*]}{v(1-\rho)n}. \tag{A.14}$$

We show in Lemma 13 below that $N \to \infty$ as both $\frac{\lambda \mathbb{E}[\sqrt{f}]}{vn} \to \infty$ and also $\frac{\lambda \mathbb{E}[\sqrt{f}]^2}{v^2 n} \to \infty$. Therefore, with this scaling,

$$\sqrt{N} = \Omega\left(\frac{\lambda \int_A \phi^{-1/2}(x) f(x) dx}{v(1-\rho)n}\right). \tag{A.15}$$

Squaring both sides of (A.15) and applying Little's Theorem, $N = \lambda W$, we then have

$$W = \Omega\left(\frac{\lambda[\int_A \phi^{-1/2}(x) f(x) dx]^2}{v^2(1-\rho)^2 n^2}\right) \tag{A.16}$$

and the modified lemma is proven.

□

To complete the proof of Theorem 4, we use the proof of Theorem 2 in [6] as originally written. This proof solves for $\min[\int_{\mathcal{A}} \phi^{-1/2}(x)f(x)dx]^2$ as a function of $f(x)$. Theorem 4 here differs from Theorem 2 in [6] only in the restatement of the limiting terms as in the modified lemma.

To complete our modifed proof, we must show that $N \to \infty$ when both $\frac{\lambda \mathbb{E}[\sqrt{f}]}{vn} \to \infty$ and also $\frac{\lambda \mathbb{E}[\sqrt{f}]^2}{v^2 n} \to \infty$ in the DTRP system. We first prove a preliminary lemma on the scaling of the system workload in a DTRP queue where workload is defined as in the standard definition of workload in the context of networks:

Definition 5 (Workload). *The workload in the system at time t, $V(t)$, is the amount of time it takes the n vehicles to serve all of the messages currently in the system at time t.*

To show that the average work in system goes to ∞ as both $\frac{\lambda \mathbb{E}[\sqrt{f}]}{vn} \to \infty$ and also $\frac{\lambda \mathbb{E}[\sqrt{f}]^2}{v^2 n} \to \infty$, we have the following lemma.

Lemma 12. *For $\frac{\lambda \mathbb{E}[\sqrt{f}]}{vn} \to \infty$, the average workload in the system V scales as:*

$$V = \Omega\left(\frac{\lambda \mathbb{E}[\sqrt{f}]^2}{v^2 n}\right).$$

Proof. Assume the vehicle started serving at time $-\infty$. Now consider any time, say 0. Let $V(0)$ denote the amount of workload in the system at time 0. Since time 0 is arbitrary, $V(0)$ is distributed like the stationary distribution of workload. Let $A(s)$ denote the minimal amount of time it takes to serve messages arriving in interval $[-t, 0]$. Then, it is easy to see that

$$V(0) \;\; \geq \;\; (A(t) - nt)^{+}. \qquad \text{(A.17)}$$

That is, the work in system is greater than difference between the amount of arrived work in an interval of length t and the maximum possible work completed by the n vehicles in the interval. The equation (A.17) is true for all t. Further, the time 0 is a randomly chosen time and hence represents the stationary time. Hence, we obtain the time average of workload in the system, $E[V]$, is lower bounded as

$$E[V] \;\geq\; E[A(t)] - nt,\ \forall t \geq 0. \tag{A.18}$$

Thus, to compute lower bound on average workload V, we need to compute $E[A(t)]$. That is, we need to compute the average minimal time required to serve messages arriving to the system in an interval of length t. Let $\Lambda(t)$ be random number of arrivals happening in time interval of length t. Then, $A(t)$ can be lower bounded by the length of shortest path connecting all source and destination locations of these $\Lambda(t)$ messages. The length of a shortest path through a set of locations is no longer than twice the length of the shortest cycle through these points, the TSP tour. Similarly, note that the TSP tour is no more than twice the length of the shortest path through these points. Hence, to obtain lower bound $A(t)$, it is sufficient to consider the length of TSP tour through the source and destination location of $\Lambda(t)$ points.

Recall the BHH Theorem of Theorem 2 which bounds the length of a TSP tour. Let L_N denote the length of the TSP tour through N points independently and identically distributed according to probability density $f(\cdot)$. Then, for any $\epsilon > 0$, there exists a $\bar{N}$ such that

$$E[L_N] \geq \beta_{TSP}\sqrt{N}\mathbb{E}[\sqrt{f}] - \epsilon\sqrt{N}, \tag{A.19}$$

where β_{TSP} is a finite positive constant. In particular, choose $\epsilon = \frac{1}{2}\beta_{TSP}\mathbb{E}[\sqrt{f}]$. Then, Theorem 1 implies that there exists a $\bar{N}$ such that for all $N \geq \bar{N}$, the following holds:

$$E[L_N] \geq \frac{\beta_{TSP}}{2}\sqrt{N}\mathbb{E}[\sqrt{f}]. \tag{A.20}$$

We would like to apply (A.20) to $N = \Lambda(t)$ for t sufficiently large. Note that $E[\Lambda(t)] = \lambda t$. Due to the Poisson property of the arrival process, $\Lambda(t) \geq \lambda t/2$ with probability at least $1/2$ for large enough λ. Therefore, $P(\sqrt{\Lambda(t)} \geq \sqrt{\lambda t/2}) \geq 1/2$ and

$$E[\sqrt{\Lambda(t)}] \geq \frac{1}{2}\sqrt{\frac{\lambda t}{2}}. \tag{A.21}$$

Assume that t is sufficiently large so that $\lambda t \geq \bar{N}$ and (A.20) holds. Substituting in (A.21), we may lower bound A as

$$E[A(t)] \;\geq\; \hat{\beta}\frac{\sqrt{\lambda t}E[\sqrt{f}]}{v}, \tag{A.22}$$

where $\hat{\beta} = \frac{\beta_{TSP}}{8\sqrt{2}}$. From (A.18) and (A.22), we obtain

$$E[V] \;\geq\; \hat{\beta}\frac{\sqrt{\lambda t}\mathbb{E}[\sqrt{f}]}{v} - nt. \tag{A.23}$$

Consider $t^* = \frac{\lambda\hat{\beta}^2\mathbb{E}[\sqrt{f_i}]^2}{2v^2n^2}$. Note that the condition of the lemma that $\lambda\mathbb{E}[\sqrt{f}]/vn \to \infty$ implies that $\lambda t^* \to \infty$ as required for (A.20) to hold. Then, from (A.23) we obtain

$$E[V] \;\geq\; \frac{\sqrt{2}-1}{2}\frac{\hat{\beta}^2\lambda\mathbb{E}[\sqrt{f}]^2}{v^2n} \tag{A.24}$$

for $\lambda\mathbb{E}[\sqrt{f_i}]/vn$ sufficiently large, and Lemma 12 is proven. □

Lemma 13. *If both* $\frac{\lambda\mathbb{E}[\sqrt{f}]}{vn} \to \infty$ *and also* $\frac{\lambda\mathbb{E}[\sqrt{f}]^2}{v^2n} \to \infty$*, the average number in queue* $N \to \infty$ *as well.*

Proof. The first condition of the Lemma 13, $\frac{\lambda\mathbb{E}[\sqrt{f}]}{vn} \to \infty$, implies that Lemma 12 holds. With this lemma, the second condition, $\frac{\lambda\mathbb{E}[\sqrt{f}]^2}{v^2n} \to \infty$, implies $E[V] \to \infty$ as well.

The work associated with each message is upper bounded by the diameter of the region plus the onsite service time, $\sqrt{2}\sqrt{A} + 2\bar{s}(n)$. Therefore, because the average work in the system is going to ∞ and the work associated with each message is finite, the average number of messages in the system, N, must be going to ∞ as well.

□

This completes the proof of Theorem 4.

Appendix B

Proof of Little's Law for the onsite system

The onsite system for each vehicle is defined to be the system formed by deleting all time in which the vehicle is not in onsite service. Messages that arrive while the vehicle is in service arrive immediately to the onsite system. Messages that arrive while the vehicle is not in service arrive to the onsite system as soon as a new message begins service. Recall that each message requires two services by the vehicle, and so messages depart the system when they receive their second service by the vehicle. The following relation between the average number in the onsite system at vehicle i, $\overline{N}_{O,i}$ and the average waiting time in this system, $\overline{W}_{O,i}$, is used in the proof of Lemma 3. Note that with $N_{O,i}(\tau)$ and $W_{O,i}(\tau)$ regenerative, the theorem holds in expected value as well.

Lemma 14 (Little's Law for the Onsite System.). *For the reduced onsite system, the following relation holds with probability 1:*

$$\overline{N}_{O,i} = \frac{\overline{W}_{O,i}}{2\bar{s}(n)},$$

The proof of the lemma is adapted directly from the proof of the sample path version of Little's Law found in [34] (pp.286-8). The main change made here is to state

Little's Law in terms of departure rate instead of arrival rate. In this appendix, the reference to the vehicle index i is dropped. Further, we will use the time index τ instead of t to denote time in the onsite system.

Recall the definition of the onsite system time for a stable system:

$$\overline{W}_O = \lim_{J\to\infty} \frac{\sum_{j=1}^{J} W_O(j)}{J} < \infty. \tag{B.1}$$

$N_O(\tau)$ is defined to be the number in the onsite system at time τ. We assume that the time average number in the system while the system is in service, N_O, is also well defined as

$$\overline{N}_O = \lim_{\tau\to\infty} \frac{\int_0^{\tau} N_O(\zeta)d\zeta}{\tau}. \tag{B.2}$$

Further, we show that the following pointwise limit exists with probability 1:

Lemma 15.

$$\lim_{\tau\to\infty} \frac{N_O(\tau)}{\tau} = 0.$$

Proof. Suppose not. Then for any $\epsilon > 0$, there exists a τ_ϵ such that $N_O(\tau) > \epsilon\tau$ for all $\tau \geq \tau_\epsilon$. Fix some $\epsilon > 0$. Let $(\tau_m)_{m=1}^{\infty}$ be a strictly increasing sequence of epochs such that $N_O(\tau_m) > \epsilon\tau_m, \forall m$. At most one message may leave the onsite system during the interval $[\tau_m, \tau_m + \bar{s}]$, therefore, $N_O(\tau) > \epsilon\tau_m - 1$ over the entire interval. That is, associated with each τ_m we have the following integral:

$$\frac{\int_{\tau_m}^{\tau_m+\bar{s}} N_O(\zeta)d\zeta}{\tau_m} > \frac{\bar{s}(\epsilon\tau_m - 1)}{\tau_m} = \bar{s}\epsilon - \frac{1}{\tau_m}.$$

Because $\tau_m \to \infty$ as $m \to \infty$, the limit (B.2) is equivalent to

$$\begin{aligned}
\lim_{m\to\infty} \frac{\int_0^{\tau_m+\bar{s}} N_O(\zeta)d\zeta}{\tau_m + \bar{s}} &= \lim_{m\to\infty} \frac{\int_0^{\tau_m} N_O(\zeta)d\zeta}{\tau_m}\frac{\tau_m}{\tau_m+\bar{s}} + \frac{\int_{\tau_m}^{\tau_m+\bar{s}} N_O(\zeta)d\zeta}{\tau_m}\frac{\tau_m}{\tau_m+\bar{s}}, \\
\overline{N}_O &= \overline{N}_O + \bar{s}\epsilon.
\end{aligned}$$

Since this is true for any $\epsilon > 0$, this is a contradiction, and (B.2) is proven. $\square$

Let τ_j be the arrival time of the j^{th} message to the onsite system. The departure time of this message is then $\tau_j + W_O(j)$. Define the departure process, $D(\tau)$, to be the number of messages that have departed the onsite system in the interval $[0, \tau]$, that is, $D(\tau) = card(\{j \mid \tau_j + W_O(j) \leq \tau\})$. Let $\mathcal{V}(\tau)$ be the number of messages that have completed one full vehicle service, but have not yet left the system (recall that each message must receive two services by the vehicle).

Because the vehicle is continuously serving messages, each requiring deterministic service time $\bar{s}(n)$, the total service rate is given by

$$\lim_{\tau \to \infty} \frac{D(\tau) + \mathcal{V}(\tau)}{\tau} = \frac{1}{\bar{s}(n)}.$$

Because every message must receive at least one service before departing, $D(\tau) \leq \mathcal{V}(\tau), \forall \tau$. Further, because the messages in the onsite system may include messages that have not yet received any service, $\mathcal{V}(\tau) \leq D(\tau) + N_O(\tau), \forall \tau$. Combining these and taking limits yields

$$\limsup_{\tau \to \infty} \frac{2D(\tau)}{\tau} \leq \lim_{\tau \to \infty} \frac{D(\tau) + \mathcal{V}(\tau)}{\tau} \leq \liminf_{\tau \to \infty} \frac{2D(\tau) + N_O(\tau)}{\tau}.$$

From Lemma 15, $N_O(\tau)/\tau \to 0$, and the right-hand and left-hand limits imply that

$$\lim_{\tau \to \infty} \frac{D(\tau)}{\tau} = \frac{1}{2\bar{s}(n)}. \tag{B.3}$$

With the departure rate precisely established, we move to computing bounds on $\overline{N}_O$.

Bound $\overline{N}_O = \int_0^t N_O(\tau) d\tau$ in the following way:

$$\sum_{j=1}^{D(\tau)} W_O(j) \leq \int_0^{\tau} N_O(\zeta) d\zeta \leq \sum_{\tau_j \leq \tau} W_O(j). \tag{B.4}$$

The left-hand inequality counts those customers who have already departed the system whereas the right-hand inequality also includes those who have arrived by time t but have not yet departed. Dividing the middle term by τ and taking the limit as $\tau \to \infty$ yields $\overline{N}_O$.

First look at the left hand inequality. Combining (B.1) and the fact that $D(\tau) \to \infty$ as $\tau \to \infty$ implies that

$$\lim_{\tau \to \infty} \frac{\sum_{j=1}^{D(\tau)} W_O(j)}{D(\tau)} = \overline{W}_O.$$

With this and (B.3), we divide by τ, take the limit and derive the following result:

$$\lim_{\tau \to \infty} \frac{\sum_{j=1}^{D(\tau)} W_O(j)}{\tau} = \lim_{\tau \to \infty} \frac{D(\tau)}{\tau} \frac{\sum_{j=1}^{D(\tau)} W_O(j)}{D(\tau)} = \frac{\overline{W}_O}{2\bar{s}}$$

If we can show that a similar bound holds for the right hand inequality, the proof is complete. Again, the existence of the limit in (B.1) implies

$$\lim_{J \to \infty} \frac{W_O(J)}{J} = \lim_{J \to \infty} \left[\frac{\sum_{j=1}^{J} W_O(j)}{J} - \left(\frac{\sum_{j=1}^{J-1} W_O(j)}{J-1} \right) \left(\frac{J-1}{J} \right) \right] = 0. \quad \text{(B.5)}$$

The Lemma 15 on the number in system process implies

$$\lim_{J \to \infty} \frac{J}{\tau_J} \leq \lim_{J \to \infty} \frac{D(\tau_J)}{\tau_J} + \frac{N(\tau_j)}{\tau_J} = \frac{1}{2\bar{s}(n)}. \quad \text{(B.6)}$$

Combining (B.5) and (B.6) yields the following limit:

$$\lim_{J \to \infty} \frac{W_O(J)}{\tau_J} = \lim_{J \to \infty} \frac{J}{\tau_J} \frac{W_O(j)}{J} = 0.$$

This limit implies that for any $\epsilon > 0$, there exists a K such that $W_O(j) \leq \tau_j \epsilon$ for all $j > K$. That is, a message arriving at time τ_j will have departed by time $\tau_j + W_j \leq (1+\epsilon)\tau_j$. We may use this to further upper bound the right hand inequality in (B.4).

$$\sum_{\tau_j \leq t} W_O(j) \leq \sum_{j=K+1}^{D(\tau(1+\epsilon))} W_O(j) + \sum_{j=1}^{K} W_O(j)$$

Dividing by τ and taking the limit, this upper bound converges to $\frac{\overline{W}_Q}{2\bar{s}}(1+\epsilon)$. Since ϵ may be arbitrarily small, the lemma is proven.

Appendix C

Proof of batch queuing time

In this appendix, we prove the following lemma, bounding T_Q, the time a batch spends in queue for the Source Only Policy given in Section 5.2.1.

Lemma 1: For the policy in Theorem 2(b) with batch time $T = \kappa \frac{4\beta^2 \lambda A}{v^2(1-\rho)^2 n}$ for some $\kappa > 1$, the delay of the batch in the queue is bounded by

$$T_Q = O(T).$$

Proof. T_Q may be bounded by using Kingman's bound for the delay in a G/G/1 queue. That is:

$$T_Q \leq \frac{\lambda_B \left(\sigma_A^2 + \sigma_S^2\right)}{2(1 - \rho_B)}. \tag{C.1}$$

where σ_A^2 is the variance of the interarrival times and σ_S^2 is the variance of the service times. In this context, the batch interarrival times are deterministic so $\sigma_A^2 = 0$ and $\lambda_B = 1/T$, the arrival rate of batches. With T fixed as in (5.27), $\rho_B \leq 1/\kappa$.

Bounding σ_S^2 requires some additional effort. First note that $\sigma_{T_B}^2 = E[T_B^2] - E[T_B]^2 \leq E[T_B^2]$. T_B has two parts: 1) $2L_{N_T}$ = the interdemand travel times for the pickup and delivery tours and 2) $2N_T \bar{s}(n)$ = the onsite service times for pickup and delivery.

Then

$$
\begin{aligned}
E[T_B^2] &= E[(2L_{N_T} + 2N_T\bar{s}(n))^2] && \text{(C.2)} \\
&= 4E[L_{N_T}^2] + 4\bar{s}(n)E[L_{N_T}N_T] + 4\bar{s}(n)^2E[N_T^2]. && \text{(C.3)}
\end{aligned}
$$

Compute the terms of (C.3) individually. First,

$$
\begin{aligned}
E[L_{N_T}^2] &= var(L_{N_T}) + E[L_{N_T}]^2 && \text{(C.4)} \\
&= var(E[L_{N_T}|N_T]) + E[var(L_{N_T}|N_T)] + E[E[L_{N_T}|N_T]]^2 && \text{(C.5)} \\
&\leq \beta^2 A var(\sqrt{N_T}) + \sum_{N_T} p(N_T)O(1) + \beta^2 AE[N_T] && \text{(C.6)} \\
&\leq 2\beta^2 AE[N_T] + O(1) && \text{(C.7)} \\
&= 2\beta^2 A\frac{\lambda T}{n} + O(1). && \text{(C.8)}
\end{aligned}
$$

where (C.5) uses the formulas for iterated variance and iterated expectation, (C.6) is by the BHH theorem, Theorem 1, and (C.7) is by concavity of $\sqrt{\cdot}$.

Next, the second term in (C.3) is:

$$
\begin{aligned}
E[L_{N_T}N_T] &= E[E[L_{N_T}N_T|N_T]] && \text{(C.9)} \\
&= \beta\sqrt{A}E[N_T^{3/2}] && \text{(C.10)} \\
&\leq \beta\sqrt{A}\left(E[N_T^2]\right)^{3/4} && \text{(C.11)} \\
&= \beta\sqrt{A}\left(\frac{\lambda T}{n} + \left(\frac{\lambda T}{n}\right)^2\right)^{3/4} && \text{(C.12)} \\
&\leq 2\beta\sqrt{A}\left(\frac{\lambda T}{n}\right)^{3/2}. && \text{(C.13)}
\end{aligned}
$$

where (C.11) is by concavity and (C.13) assumes that $\frac{\lambda T}{n} \geq 1 \Rightarrow \lambda = \Omega(n)$. Note that if $\frac{\lambda T}{n} < 1$, the system is very lightly loaded and a policy based on the worst case TSP may be used to again bound $T_Q = O(T)$ in a similar way, without the variance terms.

The last term in (C.3) is just the second moment of a Poisson variable:

$$E[N_T^2] = \frac{\lambda T}{n} + \left(\frac{\lambda T}{n}\right)^2 \tag{C.14}$$
$$\leq 2\left(\frac{\lambda T}{n}\right)^2. \tag{C.15}$$

Finally, put all of these terms together.

$$E[T_B^2] \leq 8\frac{\beta^2 A\lambda T}{n} + O(1) + 4\bar{s}(n)2\beta\sqrt{A}\left(\frac{\lambda T}{n}\right)^{3/2} + 4\bar{s}(n)^2 2\left(\frac{\lambda T}{n}\right)^2 \tag{C.16}$$
$$= 8\frac{\beta^2 A\lambda}{n}T + 4\rho\sqrt{\frac{\beta^2 A\lambda}{n}}T^{3/2} + 4\rho^2 T^2 + O(1) \tag{C.17}$$

Substituting this into Equation (C.1) above,

$$T_Q \leq \frac{8}{(1-\frac{1}{\kappa})^2}\frac{\beta^2\lambda A}{n} + \frac{4\rho\sqrt{\kappa}}{(1-\frac{1}{\kappa})^2}\frac{\beta^2\lambda A}{n} + \frac{4\rho^2\kappa}{(1-\frac{1}{\kappa})^2}\frac{\beta^2\lambda A}{n} + O(1) \tag{C.18}$$
$$= \left(\frac{8}{(1-\frac{1}{\kappa})^2} + \frac{4\rho\sqrt{\kappa}}{(1-\frac{1}{\kappa})^2} + \frac{4\rho^2\kappa}{(1-\frac{1}{\kappa})^2}\right)\frac{\beta^2\lambda A}{n} + O(1) \tag{C.19}$$
$$\leq \left(\frac{8}{(1-\frac{1}{\kappa})^2} + \frac{4\sqrt{\kappa}}{(1-\frac{1}{\kappa})^2} + \frac{4\kappa}{(1-\frac{1}{\kappa})^2}\right)\frac{\beta^2\lambda A}{n} + O(1) \tag{C.20}$$
$$= O\left(\frac{\beta^2\lambda A}{n}\right) \tag{C.21}$$

where (C.20) is given by $\rho \leq 1$.

Therefore, given the batch time $T = \kappa\frac{4\beta^2\lambda A}{v^2(1-\rho)^2 n}$, we see that $T_Q = O(T)$.

□

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