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Revenue in Truly Combinatorial Auctions and Adversarial Mechanism Design

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Abstract

Little is known about generating revenue in *unrestricted* combinatorial auctions. (In particular, the VCG mechanism has no revenue guarantees.)

In this paper we determine how much revenue can be guaranteed in such auctions.

Our analysis holds both in the standard model, when all players are independent and rational, as well as in a most adversarial model, where some players may bid collusively or even totally irrationally.

1 Our Goals

In a *combinatorial auction* there are multiple goods for sale, and each player i has a private *true valuation* for the goods —that is a function, denoted by TV_i, mapping each possible non-empty subset S of the goods to a non-negative number (representing i's value for S).

Combinatorial auctions are notoriously hard to work with, and thus researchers have considered several possible restrictions for them; in particular:

- *Sub-modularity.* Namely, $TV_i(S \cup T) \leq TV_i(S) + TV_i(T)$ for any subsets S and T of the goods;
- *Additive-Valuation.* Namely, $TV_i(S) = TV_i(g_1)+\ldots+TV_i(g_k)$ whenever $S = \{g_1, \ldots, g_k\}$;
- *Free-Disposal.* Namely, $TV_i(S) \leq TV_i(T)$ whenever $S \subset T$;
- *Single-mindedness.* Namely, for each i there is a subset of goods S and a value v such that $TV_i(T) = v$ if $T \supset S$, and 0 otherwise.
- *Unlimited supply.* Informally, an unbounded number of copies of each good are available, and each player values only sets of distinct goods.

In this paper, however, *we assume no restrictions whatsoever for combinatorial auctions:* whenever S and T are distinct subsets of goods, nothing can be inferred about $TV_i(S)$ from $TV_j(T)$. To emphasize that the players' true valuations can indeed be arbitrary, we may use the term *truly combinatorial*.

The classical goals of auction mechanisms are maximizing either *social welfare*, that is the sum of the values that each player has for the subset of goods he receives, or *revenue*, that is the sum of the prices paid by the players. For truly combinatorial auctions, however, essentially nothing is known about revenue. Accordingly, our goal is

Determining how much revenue is achievable in truly combinatorial auctions.

More generally, we want to determine how much revenue is achievable in truly combinatorial auctions in a broader and harder context, which we call *Adversarial Mechanism Design*. Essentially, this new and rapidly developing branch of game theory is concerned with the design of mechanisms when no information about the players is available, and some players act perfectly collusively or even irrationally. Accordingly, our expanded goal is

Putting forward notions and appropriate solution concepts for Adversarial Mechanism Design, and then providing such solutions for the case of truly combinatorial auctions.

(As revenue lowerbounds efficiency, we automatically derive efficiency bounds too. These are significant only if collusive and/or rational players are present, else the VCG mechanism already is perfectly efficient. Our bounds hold whether or not the resale of goods is allowed.)

2 Adversarial Mechanism Design

Adversaries are a fact of life, and several types of them have been considered in cryptography (the original field of the authors) and in game theory (see our Section 3 on related work). By "adversarial mechanism design" we do not mean the problem of designing some specific mechanisms resilient to some specific kind of adversarial players. Rather we mean to designate a *broad and coherent approach to mechanism design* comprising

(i) A *very general, simple, and adversarial "setting,"* describing the knowledge available to the designer and the possible behavior of the players, together with

(ii) A *matching notion of "mechanism performance."*

Given the focus of this paper, we shall explain this approach in terms of combinatorial auctions and revenue. The approach, however, is easily generalizable, and we believe and hope that it will be applied to many more areas.

The Adversarial Setting. We characterize the adversarial setting by two axioms:

1. *The Designer Has No Knowledge About the Players.*

   Axiom 1 is consistent with mechanism design in its purest form, where all knowledge about the players' true valuations resides with the players themselves. Of course, mechanism design could become easier by assuming that some special knowledge about the profile TV is available to the designer. In particular, a lot of theoretical work has been done in the Bayesian setting, where the designer is assumed to be aware of the probability distribution that generated the actual TV. Notably, in auctions of a single good, the celebrated result of Myerson provides optimal-revenue mechanisms in a very general Bayesian setting [19].

   Even Bayesian information, however, may be insufficiently helpful in many design problems. (In particular, nothing is known about extending Myerson's results to truly combinatorial auctions.) Moreover —and perhaps more importantly— precise Bayesian knowledge is rarely available. Both limitations of course apply to any other kind of special knowledge. Accordingly, relying on the availability to the designer of some special knowledge about TV cannot be the only battle plan in mechanism design. It is crucial to develop design tools that are helpful even when the designer knows nothing about the players. Because we can opt to ignore any knowledge we have (whether useful or not), if we can find reasonable solutions when we know nothing, we *a fortiori* find reasonable solutions no matter what knowledge is available to us.

2. *Players Can Unrestrictedly, Secretly, and Perfectly Collude.*

   Mechanism design traditionally guarantees a desired property "at equilibrium." But equilibria are very fragile: they are defined in terms of the deviation of a single player, assuming that all players are independent and rational. All bets are off when two or more players deviate from their equilibrium strategies, and such deviations should indeed be expected if there are *collusive* players. Here the term collusive is generically used to refer to "any multiplicity of players who coordinate their strategies." There are of course many models of collusion. For concreteness sake, Axiom 2 addresses one of them. (We only sketch this model informally because all our theorems are actually proven relative to the subsequent Axiom 2′, which substantially generalizes and subsumes Axiom 2, and any other collusive model we can think of.)

   Axiom 2 states that the members of a collusive set C have a "joint utility," an *unrestricted* function of their own true valuations, of the prices they pay, and of which subset of the goods each one of them receives. The mechanism designer may know what the utility function may be for the collusive sets, if any. But Axiom 2 states that collusion remains *secret*: namely, the designer does not know who the collusive players are, nor how many there are, if any. Finally, Axiom 2 states that members of a collusive set C enjoy *perfect* coordination. That is, they actually play the strategies maximizing C's utility, based on the information available to all of them —which may include information about the other players. (In particular, C's members cannot be tempted to act independently. Specifically, they might make side payments to each other, and enter binding agreements with each other on how to —in our case— bid.)

   Collusion —even in milder forms!— is a real problem in mechanism design. For instance, consider the famous VCG mechanism [5, 13, 21]. This mechanism is dominant-strategy truthful and, in truly combinatorial auctions, achieves perfect economic efficiency (but makes no claims about revenue —*even when all players are independent*). Yet, the VCG mechanism is totally vulnerable to collusion [1]. Indeed, its economic efficiency —let alone its revenue— may totally vanish when just two sufficiently knowledgeable players collude.[1]

[1]Let there be two goods for sale, a and b, and three players. Player 1 values only good a for x, Player 2 values only good b for y, and Player 3 values only the pair of goods $\{a, b\}$ for z, where z is much greater than both x and y. Assume that Players 1 and 2 known that Player 3 values only the pair $\{a, b\}$ for at most w. Then, under the VCG mechanism, Players 1 and 2 may collude so that each of them gets for free the goods he values. To make this happen, Player 1 bids w for a, and Player 2 bids w for b. Such collusive bids thus destroy both revenue and efficiency.

Much effort has been devoted to mitigating the problem of collusion. On the practical side, colluding has been made an offence punishable by law, and monitoring systems in auction rooms have made it harder for collusive players to coordinate themselves. Yet, some players have always succeeded in secretly colluding, and will likely continue to do so [7, 10, 12]. On the theoretical side, several "collusion-resilient" mechanisms have been developed (see Section 3), but only for auctions and/or collusion models of a restricted type.

It is thus crucial to provide solutions to the problem of collusion for truly combinatorial auctions and when the players are capable of unrestrictedly, secretly, and perfectly colluding. Of course, little or nothing can be guaranteed when all players are collusive. Accordingly,

> *We interpret Axiom 2 as a call to develop mechanisms that are capable of withstanding perfectly collusive players, so long as "independent" ones are also present.*

We study combinatorial auctions also in a much harder setting. We refer to it as **the Worst Setting**, and characterize it by our Axiom 1 and the following modification of Axiom 2:

$2'$. *Players Can Be Irrational.*

The traditional game theoretic assumption that all players are perfectly rational strikes us (and we are far from alone) as quite unrealistic —at least when the game at hand involves more than a handful of players and actions. It is thus crucial to develop combinatorial auctions whose performance is guaranteed even when some of the players are irrational, and thus may bid in a truly arbitrary manner, without any predictability and without any relation to their utilities.

Again, nothing can be guaranteed when all players are irrational. Accordingly,

> *We interpret Axiom $2'$ as a call to develop mechanisms that are capable of withstanding irrational players, so long as rational ones are present too.*

From now on "irrational" is a technical term, and subsumes "collusive". Indeed, by saying that the bids of irrational players are arbitrary, we mean that they are universally quantified. Therefore, whatever bid sub-profile the players of a collusive set C might choose, the same players, by virtue of being irrational, could equally well choose.

Accordingly: in the Worst Setting, we dispense with collusive players altogether and partition the set of all players into: the set of independent and rational players, I, and the set of irrational players, denoted by $-I$. (From now on, "independent" implies "rational".)

A Matching Notion of Mechanism Performance. A general notion of performance, appropriate to the adversarial (and the worst) setting, naturally follows from our two axioms.

The Influence of Axiom 1

When a distribution over the players' true valuations is known, the performance of a mechanism M can be defined as its expected performance under this distribution. But when Axiom 1 rules out this distributional knowledge, and any other knowledge about the players' true valuations, the meaningful (and in fact the only) object left to consider is *the actual true-valuation profile* TV. Accordingly:

What revenue would you be satisfied with if the players started with TV*?*

Answering this natural question for all possible TVs, one obtains a *benchmark:* a function $\mathbb{B}$ from true-valuation profiles to non-negative numbers. It is thus natural to define M's performance as "the fraction of $\mathbb{B}$" that M returns as revenue. Let us be a bit more precise.

Since Axioms 2 or $2'$ have not yet come into play, we have the momentary luxury of assuming that all players are independent and rational. Accordingly, any rational play of M will result in an equilibrium. Denoting by Σ_{TV} the set of possible equilibria (under mechanism M) when the players' true-valuation profile is TV, and disregarding for now the problem of equilibrium selection, a reasonable definition for M's revenue performance might be as follows:

We say that M*'s revenue achieves a fraction* c *of benchmark* $\mathbb{B}$ *if* $\forall TV$ *and* $\forall \sigma \in \Sigma_{TV}$*, the revenue of* $M(\sigma)$ *is at least*

$$c \cdot \mathbb{B}(TV).$$[2]

We note that the use of benchmarks is not only common in computer science, but also in game theory, although implicitly. For instance, saying "The VCG mechanism achieves economic efficiency", is equivalent to saying "The VCG mechanism returns a fraction 1 of the *maximum social welfare* benchmark." That is, we interpret the maximum social welfare as a benchmark, MSW, indeed mapping a valuation profile TV to the maximum over all possible allocations A of the social welfare of the allocation A, namely $\sum_i TV_i(A_i)$. It is just frosting on the cake that the VCG mechanism returns 100% of its benchmark in dominant strategies. Had it returned half of it, it would still be an impressive mechanism —and it would have obliged one to state its performance explicitly in terms of a benchmark.

[2] When M and the strategies of σ are probabilistic, the revenue of $M(\sigma)$ is taken to be the expected revenue computed over all possible coin tosses of M and the probabilistic strategies σ_i.

The Influence of Axioms 2 and 2'

Let us now explain how the presence of collusive and/or irrational players influences the above notion of mechanism performance. We derive our final notion from two conceptual points —the first more subjective than the second.

- *We do not count on collusive or irrational players for generating revenue.*
  That is, subjectively but perhaps realistically, we take the position that the seller should consider himself very lucky if —by a miracle— all collusive and irrational players kindly walk away (i.e., bid the null valuation), leaving only the *independent* players to bid. Accordingly: under Axioms 2 or $2'$, the performance of an auction mechanism M relative to a benchmark $\mathbb{B}$ is measured by comparing M's revenue not to $\mathbb{B}(TV)$ but to $\mathbb{B}(TV_I)$, where I denotes the set of the independent players. That is, the chosen benchmark is applied to just the true valuations of the independent players.
- *The Worst Setting calls for dominant-strategy truthful (DST) mechanisms.*
  Indeed, since irrational players bid arbitrarily, it is natural to demand equilibria σ such that, whenever i is an independent player, σ_i is i's best response to all possible bids of the other players. Which is exactly the definition of a dominant-strategy equilibrium.

The above points naturally yield the following definition of mechanism performance in the Worst Setting.

Definition 1. *We say that the revenue of a mechanism M achieves a fraction c of a benchmark $\mathbb{B}$, in the Worst Setting and in truly combinatorial auctions, if M is DST and*
$\forall$ true-valuation profiles TV, $\forall$ subset I of independent players, and $\forall$ bid sub-profile BID_{-I}:
The (expected) revenue of $M(TV_I \sqcup BID_{-I})$ is at least $c \cdot \mathbb{B}(TV_I)$.

The Significance of Our Notion. Definition 1 puts forward an incredibly strong notion. Putting it in general terms, the notion demands that a mechanism achieves its characteristic property not only in dominant strategies, but by means of a dominant-strategy equilibrium such that the desired property continues to be guaranteed even when —say— half of the players deviate from their equilibrium strategies. And the fact that the desired property is expressed in terms of the true valuations of the independent and rational players is both natural and necessary. Indeed, irrational players could act so as to hurt themselves and others. The question therefore is not whether our notion is meaningful, but whether such a meaningful notion can be achieved for a non-trivial benchmark. (Indeed, the identically-0 benchmark can always be achieved!)

We shall soon prove that a very significant benchmark can in fact be significantly achieved.

3 Related Work

As collusion is a real problem, several notions of "collusion-resiliency" appear in the literature. One such notion is that of a *group strategyproof* mechanism. Essentially, such a mechanism discourages collusion in that any gain for a collusive player is accompanied by a loss for another collusive player. Notable examples of group startegyproof mechanisms are those of [15, 18, 9]. Such a notion of collusion resiliency, however, is only meaningful when collusive players cannot make side payments to one another. No such restrictions occur in our model.

A stronger notion of collusion resiliency is that of a c-truthful mechanism [11]. Essentially, such a mechanism guarantees that fewer than c collusive players cannot "collectively gain more than they could by bidding individually." This notion, however, has very limited applicability. The authors prove that the only mechanisms satisfying it must work in a specific manner: for each subset S of the goods and for each player i these mechanisms must fix a price $p_{S,i}$ and offer S to i for that price. Thus, without any special knowledge about the players, such mechanisms cannot be designed to offer any revenue guarantee. The authors also put forward a weaker variant of their notion —*c-truthful with high probability*— for which they can approximate maximum revenue, but only for a very special type of auction: unlimited supply of a single good. Such auctions are much simpler that truly combinatorial ones (and their notion does not apply even to traditional single-good auctions).

Another class of revenue mechanisms have been developed for various restricted combinatorial auctions, sharing a similar algorithmic approach [3, 14, 17]. Let illustrate this approach following the specific incarnation of [3] for auctions of multiple goods in the unlimited supply model:

> The designer is assumed to have the following special knowledge: two integers L (for "lowerbound") and U (for "upperbound") such that, for any subset S of distinct goods and any player i, either $TV_i(S) = 0$ or $L \leq TV_i(S) \leq U$. Accordingly, the mechanism (1) randomly select a power of 2 between L and U, without loss of generality 2^k, and then (2) offer any subset of the goods for fixed price 2^k to any player who wants it.

These authors do not discuss collusion, but their approach is collusion-resilient: there is little for collusive players to do when every bundle of goods is offered at the same take-it-or-leave-it price. By contrast, we do not assume any special knowledge about the players, nor any restrictions on the type of auctions (in particular unlimited supply removes competition for the goods). We adapt, however, exponentially-distributed prices to our own use.

Another relevant mechanism is that of [8] for auctions of a single good, both in the limited and unlimited supply model. (In essence there are a number of lithographs from the same etching, and each player wants at most one lithograph.) Their mechanism achieves, within a constant factor, the following benchmark: *the maximum revenue that can be generated by fixing a price p lower than the second highest player's value for a copy of the good, and then offering a copy to any player willing to pay p for it.* The revenue guaranteed by their mechanism is again robust against collusion, but their auctions are far from combinatorial.

4 Our Benchmark

What revenue benchmarks should we choose for truly combinatorial auctions?

An obvious temptation is to consider MSW. After all, MSW upperbounds the revenue achievable by any DST auction mechanism. The problem is, however, that, even in the *absence* of collusive and irrational players, no DST mechanism can achieve a positive fraction of MSW.[3] We thus need to choose a less demanding benchmark.

Our chosen benchmark, $\mathbb{MSW}_{-\star}$, is defined to be the maximum social welfare after disregarding the valuation of the "star" player, that is, the player whose value for some subset of the goods is higher than the value attributed by any other player to any other subset. Let us now be a bit more precise and establish some useful notation along the way.

Definition 2. *Relative to a true-valuation profile V for a set of goods G, we say that player i is the* star player *if there exists $S \subset G$ such that, for any player j and any $T \subset G$:*

$$TV_i(S) \geq TV_j(T).$$

We denote the star player by "$\star$".

Because the maximum social welfare function, MSW, can be evaluated also on single valuations, the following is an alternative definition of the star player: $\star = \arg\max_i MSW(V_i)$. (Recall that our combinatorial auctions are not restricted to free disposal. Accordingly, $MSW(V_i)$ need not coincide with $V_i(G)$. Rather, $MSW(V_i) = V_i(S_i)$, where S_i is i's *favorite subset* of the goods: that is, $V_i(S_i) \geq V_i(T)$ for all $T \subset G$.)

[3] That is, *For any DST mechanism M, any truly combinatorial auction with n players and m goods, and positive constant $g_{n,m}$, there exists a valuation profile V such that the revenue of $M(V)$ is less than $g_{m,n} \cdot MSW(V)$.* This statement is actually trivial for $m = 1$ or deterministic DST mechanisms, and not difficult to prove in any case.

Definition 3. *We define the benchmark* $\mathbb{MSW}_{-\star}$ *as follows: for any valuation profile* V*,*

$$\mathbb{MSW}_{-\star}(V) = MSW(V_{-\star}).$$

That is, $\mathbb{MSW}_{-\star}$ is computed by first removing the valuation of the star player, and then computing the maximum social welfare of the remaining valuations. In other words: if $V = (V_1, \ldots, V_{\star-1}, V_\star, V_{\star+1}, \ldots, V_n)$, then $\mathbb{MSW}_{-\star}(V) = MSW(V_1, \ldots, V_{\star-1}, V_{\star+1}, \ldots, V_n)$.

Accordingly, in auctions of just one good, $\mathbb{MSW}_{-\star}$ coincides with the second-highest valuation. Indeed, when only a single good g is for sale, a player's valuation coincides with a single number: the value that the player has for g. Thus, the star player is the one who values g the most. And, after the star player is removed, the maximum social welfare of the remaining players is just the highest of the remaining valuations, and thus the second-highest of the original valuations.

The Significance of Our Benchmark. Four reasons make the $\mathbb{MSW}_{-\star}$ benchmark quite significant:

1. *It is quite achievable.*
   Quite differently from MSW, a reasonable fraction of it can be achieved in dominant strategies. Moreover, this is true not only when all players are independent, but also *in the Worst Setting.*

2. *It is quite large.*
   For a large variety of distributions D we expect $\mathbb{MSW}_{-\star}$ to be close to MSW when the true-valuation profile TV is drawn from D. (For instance, distributions D in which "no player is too special.") This statement *should not be confused* with working in the Bayesian setting. Indeed, in the Bayesian setting the designer *knows* the distribution D, while in the Adversarial Setting and in the Worst Setting D may exist but is not known by the designer.

   (In other words, to choose $\mathbb{MSW}_{-\star}$ as the benchmark of his mechanism, a designer need not have accurate knowledge of D. For instance, it suffices for him to know that, whatever the underlying distribution may be, it is one in which no player is "too special." This is indeed a much weaker, and thus much more realistic, requirement.[4])

[4]Consider a cattle rancher who has discovered half a dozen oil fields in his land and wishes to sell them in a combinatorial auction to Chevron, Shell, Exxon, BP, Total, and Mobil. Then, he may very well know nothing about the distribution of the values that these companies have for every subset of his oil fields, but he might know —for instance— that the maximum social welfare of all 6 companies is in the same ballpark as that of just 5 of the companies.

3. *It is quite natural.*

   As already observed, $\mathbb{MSW}_{-\star}$ generalizes the second-price benchmark in single-good auctions, and its reasonableness in combinatorial auctions can be "argued" along similar lines. Assume that the star player has an absolutely astronomical valuation for some subset of the goods, way out of scale with anyone else's valuation of any other goods. Then, because a DST mechanism M cannot charge the star player based on his bid, and because it "cannot charge anyone else more than their bids," then it is hard for M to produce revenue that is a good fraction of the star player's social welfare. It is thus somewhat natural to eliminate the star player from consideration when choosing a benchmark for DST mechanisms. Which is exactly what $\mathbb{MSW}_{-\star}$ does. But, having valid reasons for dismissing the star player, if we are interested in guaranteeing as much revenue as possible we should not "lower the bar" and dismiss some other player too. Nor should we lower our benchmark in any other way. Of course, by lowering our benchmark in some clever way, we might be able to prove a "better-looking" theorem, such as achieving 99.99% of this other benchmark. While valuable from a PR point of view, this effort would not have much scientific significance.

   In a sense, benchmark $\mathbb{MSW}_{-\star}$ is "always there." No matter what other benchmark $\mathbb{B}$ we may choose, $\mathbb{MSW}_{-\star}(TV)$ implicitly exists and any revenue could (and perhaps should) be compared to it. In sum, when choosing $\mathbb{MSW}_{-\star}$, we made a honest effort to select the "the highest possible reasonable benchmark" for revenue.

4. *It is quite "robust."*

   Although we have argued against dismissing players beyond the star one, it is natural to wonder what happens to our theorems in these other cases. As it turns out, *Theorems 1 and 2 are entirely unaltered by such a change of benchmark.* That is, relative to the "MSW minus k star players" benchmark, for any k, the mechanism we propose continues to be asymptotically optimal. While we have already justified our specific choice of $\mathbb{MSW}_{-\star}$ by means of its particular merits, here we note further that this benchmark characterizes a large class of related ones; in some sense, it is "robust."

(In principle, rather than considering benchmarks that remove star players, one might consider benchmarks that remove "star items." For instance, the Louvre might be happy benchmarking the value of its collection as the value of all its artwork except the Mona Lisa. Unfortunately no positive fraction of such benchmarks is achievable in DST mechanisms, for much the same reason that no positive fraction of MSW itself is DST-achievable.)

5 The Statements of Our Theorems

How much revenue can DST mechanisms return in a truly combinatorial auction?

Answering this question is precisely the goal of our theorems.

Our first theorem provides a very general lowerbound on the revenue achievable in truly combinatorial auctions. Making this lowerbound as high as possible requires defining the constants $c_{n,m}$, which will be closely approximated by $\log \min\{n, m\}$.

Definition 4. *For any positive integers n and m, we define $c_{n,m}$ to be the constant > 2 solving the equation $e^{x-2} = x \cdot \min\{n, m\}$.*

We prove that, *even in the worst case,* we can *always* achieve a fraction $\frac{1}{c_{n,m}}$ of $\mathbb{MSW}_{-\star}$.

Theorem 1. *There exists a DST mechanism M for truly combinatorial auctions such that, $\forall$ true-valuation profile TV, $\forall$ subset I of independent players, and $\forall$ bid sub-profiles BID_{-I}: The (expected) revenue of $M(TV_I \sqcup BID_{-I})$ is at least*

$$c_{n,m} \cdot \mathbb{MSW}_{-\star}(TV_I).$$

Setting aside small constants, we note that Theorem 1 states that, in *any* combinatorial auction with n players and m goods, *even in the Worst Setting,* the expected revenue of DST mechanism M exceeds a fraction $\frac{1}{\log \min\{n,m\}}$ of $\mathbb{MSW}_{-\star}$.

Is this the best revenue one can guarantee? For sufficiently large n or m, *YES!*

Definition 5. *Let $opt(n, m)$ denote the smallest $x \in \mathbb{R}^+$ for which there exists a DST mechanism $M^{n,m}$ such that, for all true-valuation profiles TV in a truly combinatorial auction with n players and m goods: the (expected) revenue of $M^{n,m}(TV)$ is at least $\frac{1}{x} \cdot \mathbb{MSW}_{-\star}(TV)$.*

Theorem 2.

$$\lim_{\min\{n,m\}\to\infty} \frac{c_{n,m}}{opt(n, m)} = 1.$$

Note that, in the Worst Setting, finding good mechanisms for truly combinatorial auctions becomes more and more difficult as the number of players and/or goods increases. Thus the asymptotic optimality of our revenue bound is significant.

Now that we know the precise answer to our revenue question when n and m become large, the only question remaining is what happens for auctions when both n and m are small. Accordingly, we prove a very general *upperbound* for the revenue obtainable in dominant strategies. Indeed, this bound applies to any truly combinatorial auction with more than one player and more than one good. The bound is related to harmonic numbers.

Definition 6. *The ith harmonic number, H_i, is $\sum_{j=1}^{i} 1/j$.*

Recall that H_i is essentially $\log(i)$.

Theorem 3. (Harmonic Revenue Bound) *For any $n, m > 1$, and any DST mechanism M, there exists a valuation profile BID for a truly combinatorial auction with n players and m goods such that the (expected) revenue of $M(BID)$ is at most*

$$\frac{\mathbb{MSW}_{-\star}(BID)}{H_{\min\{n,m\}} - 1}.$$

Finally, we prove that probabilism is essential to our endeavor. That is, while the mechanism we construct to prove Theorem 1 is probabilistic, we prove than any *deterministic* DST mechanism will perform exponentially worse than ours, in at least some auctions. Namely,

Theorem 4. *For any $n, m > 1$, and for any deterministic DST auction mechanism M, there exists a valuation profile BID for truly combinatorial auctions with n players and m goods such that the revenue of $M(BID)$ is at most*

$$\frac{\mathbb{MSW}_{-\star}(BID)}{\min\{n,m\} - 1}.$$

We warn the reader not to confuse MSW with $\mathbb{MSW}_{-\star}$ in the above statement. Indeed, relative to MSW and auctions of a single good, Theorem 4 would be trivial. And so it would also be relative to MSW and truly combinatorial auctions. (This is so because one could always consider valuation profiles BID in which all players only value the subset of the goods consisting of just the first good.)

6 The Positive News of Our Theorems

People hate inconvenient truths, and we fear that economists are no exception. Constantly seeking higher and higher revenue, one may easily hate the fact that no DST mechanism can guarantee revenue higher than a logarithmic (in the number of goods/players) fraction of $\mathbb{MSW}_{-\star}$. And lumping the message with the messenger, one may easily hate this paper too. Accordingly, we feel it necessary to argue that our theorems, beyond advancing our knowledge about DST mechanisms and combinatorial auctions, actually have some *positive* implications. Some of these are listed below.

1. *The Declining Intractability of Truly Combinatorial Auctions.*

   Let us recall that the only revenue guarantee known up to now for *truly* combinatorial auctions was precisely **0** —even assuming the rationality and independence of all players, and a sufficiently general Bayesian setting. As testified by the myriads of famous specialized subcases, truly combinatorial auctions were *totally untamed.*

   By forcing a revenue transition from 0 to $\mathbb{MSW}_{-\star}(TV_I)/c_{n,m}$, Theorem 1 finally brings some "domestication" to truly combinatorial auctions.

2. *The Rising Tractability of the Worst Setting.*

   It should not be lost to the reader that our theorems hold also in the Worst Setting. The irrationality of multiple players is perhaps the most severe threat to the very notion of an equilibrium, never mind to mechanism design. Thus, the ability to guarantee $\mathbb{MSW}_{-\star}(TV_I)/c_{n,m}$ revenue in the presence of irrational players is actually *excellent* news. (Indeed, guaranteeing any *positive* revenue might have been good news.)

3. *The function $c_{n,m}^{-1}$ is slowly decreasing.*

   Consider a totalitarian and corrupt country that suddenly decides to reform itself and wishes to place into private hands, through a giant combinatorial auction, several of its national resources: timber, oil, gas, diamonds, coal, etc. In such a sale, it is realistic to assume that the number of goods is < 299, and that the auction is *rife with collusion* —even if its players include several and reputable foreign firms. With these premises, guaranteeing (as per Theorem 1) revenue greater than 10% of the social welfare of the independent players is not a bad option.

   In any case, knowing that such expected revenue is available enables the new-and-improved government to "raise the bar" for whichever designer it chooses for its auction. Without knowing what can be actually guaranteed —and thus "what rights we have"— we cannot but smile and be thankful for whatever "solution" is offered to us. (Like that character of Molière who was ecstatic to learn that he always spoke prose.)

4. *Design Guidance.*

   Consider the problem of designing a DST mechanism for auctions with n players and m goods. If n and m are large, then a designer can simply use the mechanism $\mathbb{M}$ of the proof of Theorem 1. But if $n = 5$ and $m = 7$ and he wants to generate revenue greater than a fraction $1/c_{5,7}$ of $\mathbb{MSW}_{-\star}$, then Theorem 3 warns him not to investigate an approach that, if successful, would generate a constant fraction of $\mathbb{MSW}_{-\star}$ for all n and m, as this is impossible. (Indeed, when we strive to solve a specific problem, consciously or not we often seek solutions that are applicable to the general case, and we often succeed. It is thus useful to know when this approach is doomed to fail.)

5. *Clear Alternatives.*

   Theorem 3 tells us that only two alternatives are open to us if we seek to generate more revenue: either (1) capitalize on the presence of some special knowledge, or (2) resign to work with a weaker notion of equilibrium.

6. *New Techniques.*

   What suffices to prove that a given property can be guaranteed in dominant strategies is quite clear: namely, specifying a mechanism and analyzing it. But proving that no DST mechanism can guarantee a given property is much harder, since it involves "defeating a universal quantifier." In principle, one must analyze all mechanisms. Accordingly, such proofs tend to be quite rare (unless we are dealing with trivial properties). It is therefore crucial to develop the largest possible set of tools to establish the limitations of a given class of mechanisms. And we believe and hope that the techniques developed for proving Theorems 2 and 3 will be helpful in this regard. (We have indeed recently used them to solve other problems in mechanism-design.)

7. *A New Advantage of Probabilism.*

   It is clear that exogenous randomization is a very powerful tool, but it is much less clear exactly how much it can help us in a given setting. It is thus always significant to understand what additional power this crucial tool offers us as mechanism designers. We believe (also) this truth to be self-evident.

7 Preliminaries

Let us establish our notation and recall some basic terminology, concepts, and facts assuming that there are n players and m goods.

An allocation is a sequence $A = A_0, A_1, \ldots, A_n$, where A_i is the subset of goods allocated to player i, and A_0 the set of unallocated goods. The set of *winners* in an allocation A, denoted by Win_A, is the subsets of all players i such that A_i is non-empty. An outcome is a pair $\Omega = (A, P)$, where A is an allocation, and P a profile of prices (non-negative numbers).

The utility function u_i of player i maps i's true valuation and an outcome $\Omega = (A, P)$ to i's utility as follows: $u_i(TV_i, \Omega) = TV_i(A_i) - P_i$.

A bid is a valuation of the goods, that is a function mapping each of the 2^m subsets of the goods to a non-negative number, such that the empty subset is mapped to 0. If V_S and V_T are two valuation sub-profiles such that the subsets of the players S and T are disjoint, then by $V_S \sqcup V_T$ be denote the sub-profile mapping each player $i \in S \cup T$ to $(V_S)_i$ if $i \in S$, and to $(V_T)_i$ otherwise.

A mechanism M is a (possibly probabilistic) function mapping a profile of bids BID to an outcome (A, P) satisfying the *opt-out condition*: $P_i = 0$ whenever BID_i is the null valuation. We view each mechanism M as two separate functions: an *allocation function* M_a and a *price function* M_p. That is, for all bid profiles BID: $M(BID) = (M_a(BID), M_p(BID))$. The *expected revenue* of mechanism M on bid profile BID is $E[\sum_{i \in N} M_p(BID)_i]$. We say that M is DST if for all players i and bid sub-profile BID_{-i}: $E[u_i(TV_i, M(TV_i \sqcup BID'_{-i}))] \geq E[u_i(TV_i, M(BID'))]$.

The social welfare, best-allocation, and maximum social welfare functions —SW, BA, and MSW— are so defined. For each valuation sub-profile V_C and allocation A:

- $SW(V_C, A) = \sum_{i \in C} V_i(A_i)$,
- $BA(V_C) = argmax_{A \in \mathbb{A}(G)} SW(V_C, A)$, where $\mathbb{A}(G)$ denotes the set of all possible allocations of G, and
- $MSW(V_C) = SW(V_C, BestA(V_C))$.

By convention, (1) $argmax$'s ties are broken lexicographically, and (2) $BestA(V_C)_i \neq X$ for any subset of the goods X such that $V_i(X) = 0$.

A valuation v of a finite set of goods G is *single-minded* if there exists a single subset of goods S and $x \in \mathbb{R}^+$ such that $v(T) = x$ whenever $S \subset T$ and 0 otherwise. We compactly represent such a single-minded valuation v by the pair (S, x).

Let us explicitly highlight two properties of DST mechanisms which we are going to use extensively. (The first is an immediate consequence of the *opt-out* condition —that is, that by submitting the null valuation a player can guarantee that he wins nothing and pays nothing.)

DST-1: $\forall$ (probabilistic or not) DST mechanisms M, players i, and bid profile BID, we have: $0 \leq E[M_p(BID)_i] \leq E[BID_i(M_a(BID)_i)]$.

DST-2: $\forall$ deterministic DST mechanisms M, players i, and bid profiles BID and BID' such that $BID_{-i} = BID'_{-i}$, we have: $M_a(BID)_i = M_a(BID')_a$ implies $M_p(BID)_i = M_p(BID')_i$.

8 Proof of Theorem 1

We constructively prove Theorem 1 by explicitly putting forward a simple and probabilistic mechanism $\mathbb{M}$. In so doing, some of our choices are dictated by our desire for $\mathbb{M}$ to be DST; others by our desire for $\mathbb{M}$ to generate revenue approximating $\mathbb{MSW}_{-\star}$.

8.1 The Battle Plan

At the highest level, the idea is that of *trading efficiency for revenue.* We obtain $\mathbb{M}$ by starting with an underlying, deterministic, DST, and high-efficiency mechanism $\mathbb{M}$. We then modify $\mathbb{M}$ so as convert some of its efficiency to revenue. The first approach to implement such a plan consists of three stages. In the first stage, we run $\mathbb{M}$ on the profile of bids provided by the players and obtain an allocation A' and a profile of prices P'. In the second stage, we raise all prices in P' by a fixed amount ρ. In the third stage we decide decide the final allocation and prices as follows. If player i wins a set of goods S in A', and if $P'_i + \rho$ is less than i's bid for S, then we finally allocate S to i for a price of $P'_i + \rho$. Else, S will go unallocated, and i pays 0. This modification of $\mathbb{M}$ may cause a loss of revenue from some players, but such loss may be compensated by additional revenue from other players. Thus: How should prices be raised, and by how much, to get better revuenue?

As we shall argue later on, the revenue of *any* deterministic mechanism can only poorly approximate our benchmark. Thus, we shall raise prices *probabilistically.* Further, in light of our benchmark, one natural choice for ρ is a fraction α of $\mathbb{MSW}_{-\star}(BID)$, where the scaling factor α is probabilistically chosen between 0 and 1. This approach, however, needs to be refined. To begin with, to ensure that $\mathbb{M}$ is DST, we do not want player i's price to depend on his bid, and $\mathbb{MSW}_{-\star}(BID)$ may indeed depend on BID_i. This problem is traversed by continuing to choose α probabilistically between 0 and 1, but then raising price P'_i by $\alpha MSW(BID_{-i})$ instead. Our analysis will support that this small change does not alter the ability to achieve the chosen benchmark. At the same time, such a modification of $\mathbb{M}$ is guaranteed to be DST.

Two choices now remain to fully specify $\mathbb{M}$: that of the underlying mechanism $\mathbb{M}$ and that of scaling factor α. For $\mathbb{M}$, as we plan to turn efficiency into revenue, it is natural to choose the VCG mechanism, since it has optimal efficiency. (However, any $\mathbb{M}$ whose efficiency is a "sufficiently high" fraction of MSW would work too, leaving room for computationally more tractable auction mechanisms and other desiderata.) To choose α, we are actually guided by Theorem 3, which was indeed discovered before $\mathbb{M}$. Recall that that theorem essentially states that, in truly combinatorial auction with n players and m goods, no DST mechanism can guarantee revenue greater than a *logarithmic* (in $\mu = min\{n, m\}$) fraction of $\mathbb{MSW}_{-\star}(BID)$. With this limitation in mind, we choose α by means of an exponential distribution. However, unlike the discrete ones cited in Section 3, our distribution must be *continuous*. Else, we would uselessly lose significant revenue. Our specific selection of constants is solely justified by our desire to optimize our revenue guarantee.

8.2 Our Mechanism $\mathbb{M}$.

On input BID, a profile of n bids for a set of m goods, compute an outcome (A, P) as follows:

1. Pick a *scaling factor* $\alpha \in [0, 1]$ as follows:
   - Let $\mu = \min\{n, m\}$ and $c_{n,m}$ be the constant > 2 that solves the equation $e^{x-2} = x\mu$.
   - Flip a coin whose probability of Heads is $\frac{1}{c_{n,m}-1}$. If Heads, choose $\alpha = 0$. If Tails, draw r uniformly from $[-(c_{n,m} - 2), 0]$ and choose $\alpha = e^r$.
2. Compute the *provisional allocation* A' and the profile of *provisional prices* $P' = VCG_p(BID)$ — respectively the allocation and the prices of the VCG mechanism for the bid profile BID— and then the set of *provisional winners* W' consisting of all players that obtain a non-empty subset of goods in A'.
3. For each $i \in W'$ compute i's *offer price* $P'_i + \alpha MSW(BID_{-i})$. If i's bid $BID_i(A'_i)$ exceeds i's offer price, set $A_i = A'_i$ and $P_i = P'_i + \alpha MSW(BID_{-i})$; otherwise set $A_i = \emptyset$ and $P_i = 0$.

Remarks

- We note that $c_{n,m}$ is uniquely defined: for $\mu \geq 1$ the function $f_{n,m}(x) = e^{x-2} - x\mu$ is negative at $x = 2$, goes to infinity with x, and has positive second derivative everywhere.
- Notice that although each price P_i is personalized, it is obtained via the same choice of scaling factor α. Were we in a Bayesian setting, where different players have different distributions for their valuations, then we would optimally choose a separate scaling factor α_i for each player i.

8.3 $\mathbb{M}$ Satisfies the Requirements of Theorem 1

To prove Theorem 1, it suffices to prove two properties: namely,

P1. $\mathbb{M}$ is DST and

P2. $\forall\, TV$, $\forall$ subset I of independent players, and $\forall$ bid sub-profile BID_{-I}:

The expected revenue of $\mathbb{M}(TV_I \sqcup BID_{-I})$ is at least $c_{n,m} \cdot \mathbb{MSW}_{-\star}(TV_I)$.

It should be appreciated that property P1 clearly holds, as $\mathbb{M}$ has been obtained from the VCG mechanism via modifications that are well known to preserve dominant-strategy truthfulness. Only the second property needs to be proven. Prior to doing so, note that our benchmark is "player-monotone." That is,

Lemma 1. *If V is a sub-profile of $\mathcal{V}$, then* $\mathbb{MSW}_{-\star}(\mathcal{V}) \geq \mathbb{MSW}_{-\star}(V)$.

Proof. Let $\mathcal{N}$ be the set of players relative to $\mathcal{V}$ and let C the set of players relative to V. Then, $C \subset \mathcal{N}$ and $V = \mathcal{V}_C$. Note that the star player in $\mathcal{V}$ is either the star player in V, or belongs to $\mathcal{N} \setminus C$. In either case (abusing notation) we deduce that $V_{-\star}$ is a sub-profile of $\mathcal{V}_{-\star}$. Thus, by the monotonicity of MSW and the definition of $\mathbb{MSW}_{-\star}$, we have: $\mathbb{MSW}_{-\star}(\mathcal{V}) = MSW(\mathcal{V}_{-\star}) \geq MSW(V_{-\star}) = \mathbb{MSW}_{-\star}(V)$. *Q.E.D.*

In virtue of the above trivial lemma, our property can be restated as follows:

P2′. $\forall\, TV$, $\forall$ subset I of independent players, and $\forall$ bid sub-profile BID_{-I}:

The expected revenue of $\mathbb{M}(TV_I \sqcup BID_{-I})$ is at least $c_{n,m} \cdot \mathbb{MSW}_{-\star}(TV_I \sqcup BID_{-I})$.

Finally, because both TV_I and BID_{-I} are universally quantified, to prove Theorem 1 it suffices to prove the following theorem.

Theorem 1′: $\forall$ n and m, and $\forall$ *bid profiles BID in a truly combinatorial auction with n players and m goods:*

$$E\left[\sum_{i \in N} \mathbb{M}_p(BID)_i\right] \geq \frac{\mathbb{MSW}_{-\star}(BID)}{c_{n,m}}. \tag{1}$$

Proof. For each player i, let S_i be the (possibly empty) set player i provisionally wins, and let P_i' be the provisional price $VCG_p(BID)_i$. We divide our proof into two cases: in the first case the star player bids large enough that we derive the revenue bound solely based on the revenue $\mathbb{M}$ extracts from the star player. In the second case we must sum up the revenue that $\mathbb{M}$ extracts from each set-winning player.

Case 1: $BID_\star(S_\star) > P'_\star + \mathbb{MSW}_{-\star}(BID)$**.**

Note that the right-hand side of the inequality of this case is always ≥ 0, thus $BID_\star(S_\star) > 0$ always. This implies that $S_\star \neq \emptyset$; namely that $\star$ is a provisional winner. As such, when mechanism $\mathbb{M}$ "makes $\star$ the offer" $P'_\star + \alpha \cdot \mathbb{MSW}_{-\star}(BID)$ where $\alpha \leq 1$, the offer price will always be at most player $\star$'s bid for $S_\star$, and hence player $\star$ will always pay his offer price. Thus the expected revenue from player $\star$ is just the expected offer price, namely

$$\begin{aligned}
E[\mathbb{M}_p(BID)_\star] &= \frac{1}{c_{n,m}-1}P'_\star + (1-\frac{1}{c_{n,m}-1})\int_{-(c_{n,m}-2)}^{0} \frac{1}{c_{n,m}-2}\left(P'_\star + e^r \mathbb{MSW}_{-\star}(BID)\right)dr \\
&= \left(\frac{1}{c_{n,m}-1} + (1-\frac{1}{c_{n,m}-1})\right)P'_\star + (1-\frac{1}{c_{n,m}-1})\frac{1}{c_{n,m}-2}\mathbb{MSW}_{-\star}(BID)\int_{-(c_{n,m}-2)}^{0} e^r dr \\
&= P'_\star + \frac{1}{c_{n,m}-1}\mathbb{MSW}_{-\star}(BID)\int_{-(c_{n,m}-2)}^{0} e^r dr \\
&= P'_\star + \mathbb{MSW}_{-\star}(BID)\frac{1-e^{-(c_{n,m}-2)}}{c_{n,m}-1} \\
&\geq \mathbb{MSW}_{-\star}(BID)\frac{1-\mu e^{-(c_{n,m}-2)}}{c_{n,m}-1} = \mathbb{MSW}_{-\star}(BID)\frac{1-\frac{1}{c_{n,m}}}{c_{n,m}-1} = \frac{\mathbb{MSW}_{-\star}(BID)}{c_{n,m}},
\end{aligned}$$

where the inequality follows because $P'_\star \geq 0$ and $\mu \geq 1$, and the second to last equality is by the definition of $c_{n,m}$, namely that $c_{n,m}\mu e^{-(c_{n,m}-2)} = 1$. Thus we have the desired result in this case.

Case 2: $BID_\star(S_\star) \leq P'_\star + MSW(BID_{-\star})$**.**

Consider a provisional winner i, and consider his offer price $P'_i + \alpha \cdot MSW(BID_{-i})$. We note that when $\alpha = 0$ the offer price for player i is just P'_i, which is less than or equal to $BID_i(S_i)$ since the VCG mechanism never charges players more than their bid; thus when $\alpha = 0$ player i will "accept the offer" and pay P'_i. Since player i will pay the offer price whenever it is less than $BID_i(S_i)$, we have that i will pay whenever $\alpha < \frac{BID_i(S_i)-P'_i}{MSW(BID_{-i})}$. Recall that, by the definition of $\mathbb{M}$, when $\alpha \neq 0$ we have $\alpha = e^r$. Thus this condition becomes $r < \log_e \frac{BID_i(S_i)-P'_i}{MSW(BID_{-i})}$. We note that r is also bounded to be at most 0, but the other condition takes precedence since $\log_e \frac{BID_i(S_i)-P'_i}{MSW(BID_{-i})} \leq 0$, as we show by an analysis of two cases: when $i = \star$ the claim is equivalent to the condition of Case 2, $BID_\star(S_\star) \leq P'_\star + MSW(BID_{-\star})$;

otherwise, when $i \neq \star$ we have $BID_i(S_i) - P'_i \leq BID_i(S_i) \leq MSW(BID_\star) \leq (BID_{-i})$ (where $MSW(BID_\star)$ denotes the highest bid of the star player, which is higher than any other bid, including $BID_i(S_i)$ by assumption) which implies the ≤ 0 bound we wanted to prove. Thus the expected price paid by player i is exactly expressed as the following integral:

$$\frac{P'_i}{c_{n,m}-1} + (1 - \frac{1}{c_{n,m}-1}) \int_{-(c_{n,m}-2)}^{\max\left\{-(c_{n,m}-2), \log_e \frac{BID_i(S_i)-P'_i}{MSW(BID_{-i})}\right\}} \frac{1}{c_{n,m}-2} \left(P'_i + e^r MSW(BID_{-i})\right) dr.$$

We lower-bound this expression using the following two observations: first, since $P'_i \geq 0$ we may remove this term from inside the integral; second, since the integrand is always positive, if we decrease the upper limit of the integral to $\log_e \frac{BID_i(S_i)-P'_i}{MSW(BID_{-i})}$ the integral can only decrease (where we use the standard convention that an integral with upper limit less than its lower limit is evaluated with limits reversed and negated). Thus we have

$$\begin{aligned} E\left[\mathbb{M}_p(BID)_i\right] &\geq \frac{P'_i}{c_{n,m}-1} + (1 - \frac{1}{c_{n,m}-1}) \int_{-(c_{n,m}-2)}^{\log_e \frac{BID_i(S_i)-P'_i}{MSW(BID_{-i})}} \frac{1}{c_{n,m}-2} \left(e^r MSW(BID_{-i})\right) dr \\ &= \frac{P'_i}{c_{n,m}-1} + MSW(BID_{-i}) \frac{1}{c_{n,m}-1} \left(e^{\log_e \frac{BID_i(S_i)-P'_i}{MSW(BID_{-i})}} - e^{-(c_{n,m}-2)} \right) \\ &= \frac{1}{c_{n,m}-1} \left(BID_i(S_i) - e^{-(c_{n,m}-2)} MSW(BID_{-i}) \right) \end{aligned}$$

Summing up this inequality over all provisional winners i, we get

$$E\left[\sum_{i \in W'} \mathbb{M}_p(BID)_i\right] \geq \frac{1}{c_{n,m}-1} \left(\sum_{i \in W'} BID_i(S_i) - e^{-(c_{n,m}-2)} \sum_{i \in W'} MSW(BID_{-i}) \right).$$

Now notice that $\sum_{i \in W'} BID_i(S_i) = MSW(BID)$. Further since $|W'| \leq \mu$ and $MSW(BID_{-i}) \leq MSW(BID)$ we have $\sum_{i \in W'} MSW(BID_{-i}) \leq \mu \cdot MSW(BID)$. Thus we have

$$\begin{aligned} E\left[\sum_{i \in W'} \mathbb{M}_p(BID)_i\right] &\geq MSW(BID) \frac{1 - \mu e^{-(c_{n,m}-2)}}{c_{n,m}-1} \\ &= MSW(BID) \frac{1 - \frac{1}{c_{n,m}}}{c_{n,m}-1} = \frac{MSW(BID)}{c_{n,m}} \geq \frac{\mathbb{MSW}_{-\star}(BID)}{c_{n,m}}, \end{aligned}$$

where we invoke the definition of $c_{n,m}$ to derive the first equality. Thus we have the desired conclusion. *Q.E.D.*

Remarks.

- Notice that our mechanism $\mathbb{M}$ requires that its underlying DST mechanism be reasonably efficient. Indeed, in the analysis of Case 2, we rely on the fact that the VCG algorithm is 100% efficient: namely, we rely $\sum_{i \in W'} BID_i(S_i) = MSW(BID)$. If another DST mechanism is used, one should make sure that, for its provisional allocation A, $\sum_i BID_i(A_i)$ is a sufficient fraction of $MSW(BID)$.
- Notice that, when lower-bounding the revenue generated by $\mathbb{M}$, the profile of prices returned by the underlying DST mechanism are essentially ignored. However, were we to "simplify" the definition of $\mathbb{M}$ by replacing the provisional prices with zeros, the resulting mechanism would not be DST.

9 Proof of Theorem 3

Although Theorem 2 is logically coupled with Theorem 1, we find it technically convenient to derive its proof from that of Theorem 3, which we therefore prove first.

Before proceeding any further, let us establish a very simple lemma. It is obvious from Property DST-1 that, for any bid profile BID, the revenue generated by any DST mechanism —probabilistic or not— cannot exceed $MSW(BID)$. Let us now minimally extend this upper-bound. Namely, let us extend it to $\mathbb{MSW}_{-\star}$.

Lemma 2. *For any $n > 1$, any m, and any DST mechanism M, there exists a bid profile BID for truly combinatorial auctions with n players and m goods such that*

$$E\left[\sum_{i \in N} \mathcal{M}_p(BID)_i\right] \leq \mathbb{MSW}_{-\star}(BID).$$

Proof. Let BID be such that all players bid single-mindedly for the first good, specifically $BID_i = (\{1\}, 1)$ for all i. Then, for any possible allocation A of the goods, $SW(BID, A) \leq 1$. Thus, no matter what the DST mechanism $\mathcal{M}$ might be, in expectation $SW(BID, \mathcal{M}_a(BID)) \leq 1$. By Property DST-1, this implies that the expected revenue generated by $\mathcal{M}$ also is ≤ 1. Since obviously $\mathbb{MSW}_{-\star}(BID) = 1$, profile BID satisfies our thesis. *Q.E.D.*

9.1 A Desperate Battle Plan

Proving Theorem 3 requires proving that, for each possible n, m, and DST auction mechanism M, there is a bid profile $BID^{n,m,M}$ for truly combinatorial auctions with n players and m goods, that is "unprofitable" for M —that is, such that the expected revenue of $M(BID^{n,m,M})$ is at most $\mathbb{MSW}_{-\star}(BID^{n,m,M})/(H_\mu - 1)$.

Since there are infinitely many DST mechanisms M, one cannot construct an *ad hoc* unprofitable bid profile for each M. One way to prove the Harmonic Revenue Bound consists of exhibiting a single and uniform algorithm that, on inputs n, m, and M, outputs the desired $BID^{n,m,M}$, and argue —again in a uniform way— that all such outputs "do the job." Having tried this approach for a while, we do not recommend it. We thus try a different approach.

Our approach is non-constructive, that is we argue that each M has an unprofitable bid profile without explicitly finding it. Such an approach is acceptable for the problem at hand. When, like in case of Theorem 1, we need to prove that there exists a mechanism $\mathbb{M}$ enjoying some useful properties, constructiveness is very desirable. (After all, one may indeed want to run $\mathbb{M}$ to guarantee some revenue in a worst-setting combinatorial auction.) But for proving that "no good mechanism exists" constructiveness is not necessary.

To establish non constructively the existence of all required unprofitable profiles, we use a *probabilistic method*. Specifically, we provide a uniform procedure that, on inputs n and m, specifies a distribution $\mathcal{BID}^{n,m}$ over the bid profiles for combinatorial auctions with n players and m goods. Then we prove that, for all n, m and DST mechanism M (probabilistic or deterministic), the ratio of the expected revenue of $M(\mathcal{BID}^{n,m})$ and the expected value of the $\mathbb{MSW}_{-\star}(\mathcal{BID}^{n,m})$ is at most $\frac{1}{H_{\min\{n,m\}}-1}$. Because this could not happen if M's revenue were greater than $\frac{\mathbb{MSW}_{-\star}(BID)}{H_{\min\{n,m\}}-1}$ for all bid profiles BID in the support of $\mathcal{BID}^{n,m}$, the existence of an unprofitable bid profile is established.

This battle plan for proving the Harmonic Revenue Bound was somewhat counterintuitive to us. In essence, it uses the Bayesian setting to *upperbound* revenue, while Bayesian knowledge traditionally enables us to *increase* revenue. Accordingly, we adopted the plan when everything else failed. Indeed, an act of "desperation."

But, once refined, the plan got more risky. In fact, each chosen distribution $\mathcal{BID}^{n,m}$ has *finite support*. That is, we constrain ourselves to find an unprofitable bid profile —for each of the infinitely many DST mechanisms— from just a "handful" of possible candidates. In military terms, we choose to stand against infinitely many enemies with finitely many troops.

But the advantage of this risky plan is its simplicity. Indeed, after *guessing* the right distribution $\mathcal{BID}^{n,m}$, arguing that the expected revenue of $M(\mathcal{BID}^{n,m})$ is low for each DST mechanism boils down to just one insight, Lemma 3, and a few calculations.

9.2 Proof of the Harmonic Revenue Bound

We define the distribution $\mathcal{BID}^{n,m}$ in two steps. We start by defining a distribution, h_S^k, over the single-minded bids of a single player.

Definition 7. (Bounded-Harmonic Distributions) *For any subset of goods S and positive integer k, we denote by h_S^k the distribution assigning, for each integer $j \in [1, k]$, probability $\frac{1}{k}$ to the single-minded valuation $(S, \frac{1}{j})$.*

Definition 8. *In a combinatorial auction with n players and m goods, denoting the set of goods by $\{1, \ldots, m\}$ and letting $\mu = \min\{n, m\}$, we define the distribution $\mathcal{BID}^{n,m}$ as follows. For each player i: if $i \in \{1, \ldots, \mu\}$, then $\mathcal{BID}_i^{n,m} = h_{\{i\}}^{\mu}$; else $\mathcal{BID}_i^{n,m}$ is the null valuation.*

That is, in $\mathcal{BID}^{n,m}$ we essentially have $n = m = \mu$ and μ separate auctions, each with a single good and a single player. Indeed, each player i bids only for the subset $\{i\}$, and the amount of his bid is the inverse of an integer uniformly and independently chosen in $\{1, \ldots, \mu\}$.

We now prove a property of DST mechanisms that may be of independent interest.

Lemma 3. (Harmonic-Pricing) *For all probabilistic DST mechanisms M, all players i, all valuation sub-profiles BID_{-i}, all positive integers k, and all subsets of goods S,*

$$\underset{BID_i \leftarrow h_S^k}{E}[E[M_p(BID_{-i} \sqcup BID_i)_i] \leq \frac{1}{k}.$$

Proof. For each $j \leq k$ define α_j as the expected price paid by player i relative to the bid profile $BID_{-i} \sqcup (S, \frac{1}{j})$ under mechanism M, and let β_j be the probability that player i is allocated some set containing S. For notational convenience, let $\alpha_{k+1} = \beta_{k+1} = 0$. Expressed in this notation, the lemma states that

$$\frac{1}{k}\sum_{j=1}^{k} \alpha_j \leq \frac{1}{k} \quad \text{or, equivalently,} \quad \sum_{j=1}^{k} \alpha_j \leq 1.$$

We start by noting that, since M is DST, assuming that i's true valuation is $(S, \frac{1}{j})$ and that all other players bid according to the sub-profile BID_{-i}, i's utility is at least as large when he bids $(S, \frac{1}{j})$ as when he bids $(S, \frac{1}{j+1})$. That is,

$$\frac{1}{j}\beta_j - \alpha_j \geq \frac{1}{j}\beta_{j+1} - \alpha_{j+1}. \tag{2}$$

Suppose for the sake of contradiction that $\sum_{j=1}^{k} \alpha_j > 1$. Thus $1 < \sum_{j=1}^{k} \alpha_j = \sum_{j=1}^{k} j(\alpha_j - \alpha_{j+1})$ and further since for each j, $0 \le \beta_j \le 1$, we have $\beta_1 = \sum_{j=1}^{k}(\beta_j - \beta_{j+1}) \le 1$. Comparing these two sums term by term we note that there must exist a j such that the corresponding term from the first sum exceeds the term from the second sum, namely $j(\alpha_j - \alpha_{j+1}) > (\beta_j - \beta_{j+1})$. Dividing by j and rearranging terms yields $\frac{1}{j}\beta_j - \alpha_j < \frac{1}{j}\beta_{j+1} - \alpha_{j+1}$, which contradicts Equation 2. Thus $\sum \alpha_j \le 1$, as desired. *Q.E.D.*

Finally, let us restate and prove the Harmonic Revenue Bound.

Theorem 3. *For any $n, m > 1$, and any DST mechanism M, there exists a valuation profile BID for a truly combinatorial auction with n players and m goods such that, letting $\mu = \min\{n, m\}$ we have*

$$E\left[\sum_i M_p(BID)_i\right] \le \frac{\mathbb{MSW}_{-\star}(BID)}{H_\mu - 1}.$$

Proof. Let $\mathcal{BID}^{n,m}$ be the distribution of Definition 7 and fix arbitrarily a DST mechanism M. Invoking μ times (i.e., for each player $\le \mu$) Lemma 3 with $k = \mu$ we have

$$E\left[\sum_i M_p(\mathcal{BID}^{n,m})_i\right] \le 1. \tag{3}$$

That is, M's expected revenue (taken over $\mathcal{BID}^{n,m}$ and M's random choices, if any) is ≤ 1. At the same time,

$$E\left[\mathbb{MSW}_{-\star}(\mathcal{BID}^{n,m})\right] \le H_\mu - 1. \tag{4}$$

In fact,

(a) the expected value of MSW over $\mathcal{BID}^{n,m}$ is just $\sum_{j=1}^{\mu} 1/j = H_\mu$;

(b) there are at least two players by hypothesis; and

(c) the star player —whoever he may be— values his item for at most 1.

Inequalities 3 and 4 thus imply that the ratio between M's expected revenue and the expected $\mathbb{MSW}_{-\star}$ is at most $\frac{1}{H_\mu - 1}$. In turn, this implies the existence of a bid profile BID as per our thesis. *Q.E.D.*

10 Proof of Theorem 2

Theorem 2.

$$\lim_{\min\{n,m\}\to\infty} \frac{c_{n,m}}{opt(n,m)} = 1.$$

Proof. Let $\mu = \min\{n,m\}$. Theorem 3 and Lemma 2 imply that $opt(n,m) \geq \max\{1, H_\mu - 1\}$. Thus, because $H_\mu \geq \log_e \mu$ for all μ,

$$\frac{c_{n,m}}{opt(n,m)} \leq \frac{c_{n,m}}{\max\{1, H_{\min\{n,m\}} - 1\}} \leq \frac{c_{n,m}}{\max\{1, \log_e \mu - 1\}}. \tag{5}$$

Let us now upper-bound $c_{n,m}$. Recall that $c_{n,m}$ is the unique solution > 2 to the equation $e^{x-2} = x\mu$.

Rewriting our equation as $\frac{e^{x-2}}{x} = \mu$, we note that the left hand side is an increasing function of x for $x \geq 2$. Thus, if we find a value v for which $\frac{e^{v-2}}{v} > \mu$ then we will know that $v > c_\mu$.

Consider now the value $v = \log_e \mu + \log_e \log_e \mu + 4$. We have $e^{v-2} = e^2 \mu \log_e \mu$, which is easily seen to be greater than μv for $\mu \geq 2$. Thus we have upperbounded $c_{n,m}$ by $\log_e \mu + \log_e \log_e \mu + 4$, which yields

$$\frac{c_{n,m}}{opt(n,m)} \leq \frac{\log_e \mu + \log_e \log_e \mu + 4}{\max\{1, \log_e \mu - 1\}}, \tag{6}$$

which implies that $\lim_{n,m\to\infty} \frac{c_{n,m}}{opt(n,m)} \leq 1$, as desired.

Q.E.D.

11 Proof of Theorem 4

Let us restate and then prove Theorem 4. Namely,

Theorem 4: *For any $n, m > 1$, and for any deterministic auction mechanism M, there exists a valuation profile BID for truly combinatorial auctions with n players and m goods such, letting $\mu = \min\{n,m\}$, the revenue of $M(BID)$ is at most*

$$\frac{\mathbb{MSW}_{-\star}(BID)}{\mu - 1}.$$

Proof. We construct the desired bid profile within three steps.

Step 1. Define the single-minded valuation profile BID^0 as follows: BID^0_i equals $(\{i\}, 1)$ if player $i \leq \mu$, and the null valuation otherwise. It is thus clear that $\mathbb{MSW}_{-\star}(BID^0) = \mu - 1$, so that $\frac{\mathbb{MSW}_{-\star}(BID^0)}{\mu-1} = 1$. We distinguish two cases: namely, (1) $M_p(BID^0)_i > 0$ for no i and (2) $M_p(BID^0)_i > 0$ for some i. In the first case, the revenue of M on bid profile BID^0 is 0, and thus BID^0 is the profile required by the theorem. Otherwise, we proceed to Step 2.

Step 2. Let j be a player such that $M_p(BID^0)_j > 0$, and define for each integer $\alpha \geq 2$ the valuation profile $BID^\alpha = BID^0_{-j} \sqcup (\{j\}, \mu^\alpha)$. It is thus evident that, for all $\alpha \geq 2$, $\mathbb{MSW}_{-\star}(BID^\alpha) = \mu - 1$ and thus $\frac{\mathbb{MSW}_{-\star}(BID^\alpha)}{\mu-1} = 1$.

Let us now analyze the price side. Notice three facts: by construction, $BID^0_{-j} = BID^\alpha_{-j}$; by Property DST-1, j is allocated the subset of goods $\{j\}$ under bid profile BID^0; and, for all $\alpha \geq 2$, j's bid value for $\{j\}$ is higher in BID^α than in BID^0. Thus, because M is deterministic, Property DST-2 implies that, for all $\alpha \geq 2$, j continues to win the set $\{j\}$ in BID^α and to pay the same price he pays in BID^0, which is at most 1 —because of Property DST-1 and because $BID^0_j(\{j\}) = 1$.

We now distinguish two cases: (a) there is some integer $\bar{\alpha} \geq 2$ such that $M_p(BID^{\bar{\alpha}})_i = 0$ for all $i \neq j$, or (b) for each integer $\alpha \geq 2$ there is a player k_α, $k_\alpha \neq j$, such that $M_p(BID^\alpha)_{k_\alpha} > 0$. In the first case, the total revenue for M under bid profile $BID^{\bar{\alpha}}$ is at most 1, and thus $BID^{\bar{\alpha}}$ is the profile required by the theorem. Otherwise, we proceed to Step 3.

Step 3. By the opt-out condition, for each integer $\alpha > 2$, we have $k_\alpha \in \{1, \ldots, \mu\} \setminus \{j\}$. Thus, the pigeonhole principle implies the existence of $\beta, \gamma \in \{2, \ldots, \mu+1\}$ such that $k_\beta = k_\gamma$. Without loss of generality, let $\beta < \gamma$. Define now

$$k = k_\beta (= k_\gamma) \quad \text{and} \quad BID' = BID^\gamma_{-k} \sqcup (\{k\}, \mu^{\gamma+1}).$$

Since the star player in BID' is k, we have $\mathbb{MSW}_{-\star}(BID') = \mu^\gamma + \mu - 2 \geq \mu^\gamma$. Further, because γ and β are integers, $\gamma > \beta$, and $\beta \geq 2$, we have $\mu^\gamma > (\mu^\beta + \mu)(\mu - 1)$ and thus

$$\mu^\beta + \mu < \frac{\mathbb{MSW}_{-\star}(BID')}{\mu - 1}. \tag{7}$$

Let us now analyze the price situation. We consider the following two mutually exclusive cases.

Case 1: $M_p(BID')_j \leq \mu^\beta$**.**

The definition of BID' and Property DST-1 clearly imply that $\sum_{i\in-\{j,k\}} M_p(BID')_i \leq \mu - 2$. As for player k, we note that under bid profile BID^γ, he wins his set $\{k\}$, without paying more than 1, his bid for $\{k\}$. Further, the bid profile BID' is identical to BID^γ except for k's bid for $\{k\}$, which is higher in BID' than in BID^γ. Thus, k will continue to win $\{k\}$ in BID', without paying more than 1. Thus the revenue in this case is at most the sum of μ^β (from player j), 1 (from player k), and $\mu - 2$ (from all other players), totalling less than $\mu^\beta + \mu$.

Thus Inequality 7 implies that the valuation profile BID' satisfies our thesis.

Case 2: $M_p(BID')_j > \mu^\beta$**.**

Define

$$BID'' = BID^\beta_{-k} \sqcup (\{k\}, \mu^{\gamma+1}).$$

It is clear that $\mathbb{MSW}_{-\star}(BID'') = \mu^\beta + \mu - 2 \geq \mu^\beta$ and thus, since $\beta \geq 2$, we have

$$\mu < \frac{\mathbb{MSW}_{-\star}(BID'')}{\mu - 1}. \tag{8}$$

Turning our attention to prices, as for BID', it is clear that $\sum_{i\in-\{j,k\}} M_p(BID'')_i \leq \mu - 2$.

Let us now analyze the price paid by player j. Notice that BID' and BID'' differ only in the bid of player j, and that j bids higher for $\{j\}$ in BID' than in BID''. Thus, if j won $\{j\}$ in BID'', then he would win it too in BID' *for the same price*. However, by the assumption of this case, j's price is greater than μ^β, namely, greater than his valuation for $\{j\}$ under BID'', which implies that he cannot win it under bid profile BID''.

Finally, let us analyze the price of player k. Notice that BID'' is identical to BID^β except for k's bid for $\{k\}$, which is higher in BID^β than in BID''. Since k wins his set $\{k\}$ under BID^β paying at most 1, he continues to win $\{k\}$ in BID'', for at most 1.

Thus the revenue in this case is at most the sum of 0 (from player j), 1 (from player k), and $\mu - 2$ (from all other players), totalling less than μ.

Thus Inequality 8 implies that the valuation profile BID'' satisfies our thesis.

And thus, in all cases, we have exhibited a valuation profile that satisfies the theorem.

Q.E.D.

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