Velocity–Ion Temperature Gradient Driven Modes and Angular Momentum Transport in Magnetically Confined Plasmas

by

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Abstract

Plasma confinement experiments continue to uncover fascinating phenomena that motivate theoretical discussion and exploration. In this thesis, we consider the phenomenon of angular momentum transport in magnetically confined plasmas. Relevant experiments and theoretical developments are presented in order to motivate the derivation of a modified version of the three-field nonlinear Hamaguchi-Horton equations[1]. The equations are altered to include a zeroth-order parallel velocity inhomogeneity along the radially-analagous coordinate, resulting in a nonlinear system that describes the evolution of the velocity–ion temperature gradient-driven modes (VITGs). The equations are used to analyze VITG modes in the local approximation of a magnetized plasma, as well as in an inhomogeneous slab model. Applying quasilinear methods, we find a turbulent angular momentum flux in agreement with the accretion theory of the spontaneous rotation phenomenon[2]. More advanced applications are considered for future analysis.

Thesis Supervisor: Bruno Coppi
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Chapter 1

Introduction

Magnetically confined toroidal plasmas play a leading role in the pursuit of controlled thermonuclear fusion. The high energy densities required to yield reaction cross-sections large enough to consistently fuse light elements are so great that the fusile atoms are ionized, creating a plasma. In order to maintain the steep density and temperature gradients necessary to maintain such a high enery density in a modestly sized system, it is necessary to confine the plasma, which is most readily done by surrounding it with a closed magnetic flux surface; the simplest topological class allowable for such a surface under the constraint $\nabla \cdot \mathbf{B}$ is a toroid.

The strong influence of electromagnetic interactions in such a system, as well as its non-trivial geometry, mean that these laboratory plasmas are massively nonlinear systems with many degrees of freedom. Velocity, temperature, and density gradients for both ions and electrons induce different oscillatory modes within the plasma as well as drive particle transport, which in turn affect the magnetic confinement. Because of the significant complexity involved in studying plasmas, then, it has been helpful to characterize phenomena individually in order to better understand and manipulate plasma as a whole.

In this thesis we follow this trend by exploring the phenomenon of velocity–ion temperature gradient (VITG) driven modes and their associated turbulent transport of angular momentum in magnetically confined plasmas, which has been observed in most major tokamak experiments in recent years. Angular momentum transport is of unique interest due to its relevance to the observed "spontaneous" generation of angular momentum at the plasma's edge. This angular momentum is then transported to the center of the plasma column via the process we explore in this thesis. This phenomenon occurs in the absence of any measureable net contribution of external angular momentum (i.e., via such means as asymmetric neutral beam injection or ion cyclotron radio-frequency heating (ICRH)), and is transported with a behavior that cannot be explained by neoclassical theories. In view of these difficulties, the Accretion Theory[2] has been developed as a theoretical basis for this process and has been shown to explain experimental findings.

In this chapter we present the empirical findings and theoretical background that motivate our consideration of this problem. In chapter two, we develop an ordering of parameters that allows us to develop the modified Hamaguchi-Horton equations by using rigorous approximation techniques to simplify the fluid equations. Chapter three gives an analysis of VITG-driven modes in the local homogeneous approximation and a derivation via quasilinear analysis of the turbulent angular momentum flux associated with these modes. The basic theory for the inhomogeneous case is also developed in chapter three. Chapter four draws from the topics covered in the first three chapters to discuss the implications and potential future directions of this analysis.

1.1 Experimental Developments

There have been nearly two decades of experimental observations of the spontaneous rotation of toroidal plasma, as well as analyses of the associated anomolous transport mechanisms, including that of angular momentum (other quantities are also found to follow exhibit turbulent transport). Initial observations were made in conjunction with neutral-beam injection (NBI)[3, 4], whith the beam presenting a plausible source for angular momentum, although the anomolous, non-diffusive nature of the transport mechanism began to present itself[5, 6] with the use refined observational methods that simulataneously measured velocity and ion temperature gradients, suggesting the VITG-driven mode as an underlying mechanism.

Later investigations conducted with ICRH in the absence of NBI also observed the spontatneous rotation phenomenon and anomolous momentum transport attributable to the ITG-driven mode. While some observations seemed to implicate ICRH as the initiating mechanism[7], subsequent measurement showing approximate symnmetry of the rotation profile between the high- and low-field sides of the plasma column contradicted these conclusions[8].

Recent experiments have found that no auxilary heating or injection of angular momentum is necessary for spontaneous rotation and angular momentum transport to be initiated[9, 10, 11]. It has also been shown that there is an inversion of the rotation velocity during the transition between the low-confinement regime (L-regime) and the high-confinement regime (H-regime)[12].

1.2 Theoretical Investigations

The phenomenon of spontateous rotation is of theoretical interest for a number of reasons. Certainly, its study is pursued in part because of its novel "spontaneous nature", but rotation also plays an important role in understanding the transition between a Low-confinement (L) and a High-confinement (H) regime, due to the sudden rotational velocity disruptions which accompany it [12, 13]. Additionally, the anomolous transport mechanisms of angular momentum itself are turbulent in nature, alowing study of the topic to yield insight into turbulent particle and energy transport.

In addition to the study of angular momentum transport, the study of angular momentum inflow, or "pinch", at the plasma column's edge has received significant attention as well. Mechanisms described by collisional neoclassical theory have been proposed[14], while other analyses attribute the process to finite Larmor radius corrections[15]. More recently, resistive ballooning instabilities have been proposed as a mechanism by which angular momentum could be ejected from the plasma column[16].

Early theoretical investigations of the spontaneous rotation phenomenon focused largely on rotation influenced by NBI and ICRH[17, 18]. Although these auxiliary heating techniques did contribute to the toroidal angular momentum in some cases, to a large extent their dominant influence is due to their associated power flux into the plasma, feeding the instability and thereby driving the anomolous transport.

More recently, there has been a growing body of work that applies the gyrofluid approximation (similar to the drift approximation, but assuming larger variation of perturbed quantities). Analysis and simulation[19, 20] have yielded promising results, although simulating in the gyrofluid approximation requires quite sophisticated computational techniques, especially for the nonlinear or inhomogeneous cases.

In the analysis presented here, we consider only Ohmic heating within the scope of the quasilinear Accretion Theory [2, 21], which is a drift-ordered approximation derived from fluid theory. We will show that an accurate transport equation is readily obtainable from our simple modified system using quasilinear analysis.

Chapter 2

The Modified Hamaguchi-Horton Equations: A Drift Model with Finite Parallel Velocity Gradient

Due to the complex nature of plasmas, they are exceedingly difficult to model accurately over all length and time scales. The full mathematical description of a plasma lies within a $6N$-dimensional phase-space, for a plasma composed of N particles. Fluid descriptions of a plasma, which rely on successive moments of the particle distribution function (which exists in a 6-dimensional phase-space), are similarly complex, as the equations that govern the distribution function are structured such that each moment of the equation is coupled with the next higher-order moment. This necessitates an approximate closure of the system of equations that includes only the first several moments. This is generally done by ordering the relevant plasma parameters such that zeroth- and first-order terms in a small-parameter expansion capture the length and time scales relevant to the regime of interest.

In this chapter, we introduce the relevant equations that govern plasma evolution and then close the system of equations within the regime relevant to our discussion of angular momentum transport.

2.1 Kinetic Equation

A typical fusion plasma is considered to be be wholly described by its distribution in phase space, to which classical mechanics and electrodynamics can be applied (quantum and relativistic effects are negligible at common energy and length scales encountered in the laboratory). That is to say, if a distribution function $f_s(\mathbf{x}, \mathbf{v}, t)$ is defined for each plasma species s, and boundary values are known, all relevant parameters of the system are determined for all time t. The velocities and positions of all the constituent particles are defined by $\{f_s\}$ and $\mathbf{E}$ and $\mathbf{B}$ can be determined using Maxwell's equations from the first and second moments of the distribution function, which give the charge density and current density, respectively.

Enforcing conservation of the particles described by the distribution function by requiring $\frac{d}{dt} \iint_V f_s(\mathbf{x}, \mathbf{v}, t)\, d^3x\, d^3v = 0$, we obtain the microscopic kinetic equation for f_s,

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} f_s + \mathbf{a} \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0\,, \tag{2.1}$$

where the acceleration $\mathbf{a}$ is due to the Lorentz force.

Eq. (2.1) includes interactions on all length scales and in fact necessitates that f_s be a discontinuous sum of Dirac delta functions, due to the discrete nature of microscopic interactions. In order to obtain a smooth function which we can approximate analytically, we write the same equation for the ensemble averages of the distribution function and the acceleration, denoted by $\bar{f}_s$ and $\bar{\mathbf{a}}$, respectively.

Unfortunately for the plasma physicist, equality between $\bar{\mathbf{a}} \cdot \frac{\partial \bar{f}_s}{\partial \mathbf{v}}$ and $\langle \mathbf{a} \cdot \frac{\partial f_s}{\partial \mathbf{v}} \rangle$ only holds if $\mathbf{a}$ and f_s are uncorrelated (that is, if there are no Coulombic interactions between particles), which in many cases is not a valid approximation to make. We therefore add a correction term (or collision operator), $C(f_s)$, to the ensemble-averaged equation which captures the interaction between particles. This yields the kinetic equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} f + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = C(f)\,, \tag{2.2}$$

where we have dropped the s subscript and taken f and $\mathbf{a}$ to be ensemble average values. Although there is no closed form for $C(f)$, it can be expressed order by order depending on the regime of interest.

2.2 The Drift Approximation

The drift approximation is a common approximation used in plasma physics in which quantities are ordered according to the to the ratio of the Larmor radius to macroscopic length scales of the system. Since the Larmor radius varies inversely with B, the drift approximation holds best for a strongly-magnetized plasma. As we will show, this ordering allows us to greatly simplify our equations by averaging particle trajectories over their periodic orbits.

2.2.1 Ordering of Parameters

In the case of a magnetized plasma, which we may assume for all cases considered here, the microscopic length scale of species s is determined by the thermal gyroradius, $\rho_s \equiv v_{ts}/\Omega_s$[1], which is the approximate orbital radius of particles about the magnetic field lines. Since ρ_s is much smaller than any characteristic scale length, L, of the plasma (such as the plasma minor radius or density scale length), we define an order parameter

$$\delta \equiv \frac{\rho_s}{L} \ll 1 \,. \tag{2.3}$$

This parameter defines a scale which we shall use to compare other quantities of interest in our system and develop an ordering which captures the important aspects of the regime we wish to consider.

We will consider first the time scales of the two mechanisms by which particles are accelerated. The first of these is the is due to the collision operator $C(f)$ and is typified by the collision frequency ν. In order to continue consideration of the plasma as magnetized, we must restrict ν such that $\nu/\Omega_s \sim \delta$. Otherwise, collisions would

[1] $v_{ts} \equiv \sqrt{\frac{2T_s}{m_s}}$ is the thermal velocity of particles of species s and $\Omega_s \equiv \frac{q_s B}{m_s c}$ is the gyrofrequency. q_s is the particle charge of the species, T_s its temperature, m_s its mass. $B = |\mathbf{B}|$ is the magnitude of the magnetic field, and c is the speed of light in vacuum.

dominate over gyration, invalidating the average over the gyroangle which we shall undertake shortly.

The other mechanism for acceleration is due to the electric field parallel to $\mathbf{B}$, denoted $\mathbf{E}_{\|}$. The acceleration term due to to $\mathbf{E}_{\|}$ is given by

$$\mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{e}{m} \mathbf{E}_{\|} \cdot \frac{\partial f}{\partial \mathbf{v}} \sim \frac{e}{m} \frac{E_{\|}}{v_t} f \equiv \nu_E f \,, \tag{2.4}$$

where $1/\nu_E$ is the characteristic time-scale of acceleration due to $\mathbf{E}_{\|}$. We must treat ν_E in our ordering as $\frac{\nu_E}{\Omega} \sim \delta$. Otherwise, to lowest order, $\mathbf{E}_{\|}$ acceleration would be unbalanced by any collisional force, since $\frac{\nu}{\Omega} \sim \delta$. As such, there would be no equilibrium $\mathbf{E}_{\|}$ to zeroth order.

Similar to the treatment of the collisional interactions, we wish to ensure that the phase space distribution function doesn't vary on a time scale faster than that of gyration. Thus, we order the partial time derivative of f as

$$\frac{\partial f}{\partial t} \sim \delta \Omega f \,. \tag{2.5}$$

This allows to meaningfully average over the gyroangle, which varies on a timescale $\sim 1/\Omega$.

The final consideration of our ordering is the particle interaction with the electric field perpendicular to $\mathbf{B}$, $\mathbf{E}_{\perp}$. This is dominated by the $\mathbf{E} \times \mathbf{B}$ drift, which has velocity

$$\mathbf{V}_E \equiv c \frac{\mathbf{E} \times \mathbf{B}}{B^2} \,. \tag{2.6}$$

Within the regime we shall consider, we treat the $\mathbf{E}_{\perp}$ interaction as being of order δ compared to the particle gyration according to the relation

$$\frac{V_E}{v_t} \sim \delta \,. \tag{2.7}$$

This last relation characterizes the drift kinetic approximation. It is a useful approximation to use when considering fairly small perturbations to the distribution function ($\sim \delta f$) on length scales no shorter than the gyroradius. It is important to note that only slow variations of the magnetic field are included in the drift kinetic approximation. Since $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, and $\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$, the non-electrostatic component of the $\mathbf{E} \times \mathbf{B}$ velocity,

$$\mathbf{V}_E - \mathbf{V}_E^{Electrostatic} = \frac{1}{B^2}\frac{\partial \mathbf{A}}{\partial t} \times \mathbf{B}, \tag{2.8}$$

is of order δV_E, due to the time-derivative term, and is thus of order $\delta^2 v_t$ by Eq. (2.7).

To summarize, we have developed an ordering that reflects the importance of the parameters in our regime of interest. This drift kinetic ordering can be expressed as

$$\frac{\nu}{\Omega} \sim \frac{\nu_E}{\Omega} \sim \frac{1}{\Omega}\frac{\partial}{\partial t} \sim \frac{V_E}{v_t} \sim \delta \equiv \frac{\rho}{L} \ll 1\,. \tag{2.9}$$

Taken together, these relations will allow us to to find a closed system of equations in the next subsection which approximate the first few moments of the kinetic equation.

2.2.2 Approximations of the Modified Hamaguchi-Horton Equations

As noted earlier, the kinetic equation is not tractable due to the infinitely nested coupling between its successive moments. In fact, even if we could find an explicit solution for the kinetic equation, it would be difficult to immediately draw meaningful conclusions about the actual underlying processes of the plasma's behavior. An approximate model of the plasma can be very helpful to this end, and a good model can give quite accurate predictions while still being simple enough to allow one to build intuition. We shall attempt to derive such a model by examining the zeroth-, first-, and second- order moments of the kinetic equation (that is, the convolution of powers of $\mathbf{v}_s$ with the distribution function, taken within velocity space, so that the result is in terms of the average flow velocity, $\mathbf{V}_s$), and then rewrite the higher-order dependencies in terms of lower-order quantities. The equations derived in this manner are referred to as the Hamaguchi-Horton equations[1], after their developers, but we shall modify them slightly by assuming a zeroth-order equilibrium toroidal velocity inhomogeneity, $\mathbf{V}_{\|}(x)$.

We begin by finding the moments of the kinetic equations:

$$\frac{\partial n_s}{\partial t} = -\boldsymbol{\nabla} \cdot (n_s \mathbf{V}_s) \, , \tag{2.10}$$

$$\frac{\partial}{\partial t}(m_s n_s \mathbf{V}_s) = -\boldsymbol{\nabla} \cdot \mathbf{p}_s + q_s n_s \left(\mathbf{E} + \frac{1}{c} \mathbf{V}_s \times \mathbf{B} \right) + \mathbf{F}_s \, , \tag{2.11}$$

and

$$\frac{3}{2} \frac{\partial p_s}{\partial t} = -\boldsymbol{\nabla} \cdot \left(\frac{3}{2} p_s \mathbf{V}_s + \mathbf{q}_s \right) - \mathbf{p}_s \colon \boldsymbol{\nabla} \mathbf{V}_s + W_s \, . \tag{2.12}$$

These are the well-known *fluid balance equations* and shall be our starting point for deriving the modified Hamaguchi-Horton equations.

Here we have defined $\mathbf{q}_s$ as the heat flux vector, p_s as the species s scalar pressure, and $\mathbf{p}_s$ as the rest-frame pressure tensor, given by

$$\mathbf{p}_s \equiv p_s \mathbf{I}_3 + \boldsymbol{\pi}_s \tag{2.13}$$

where $\mathbf{I}_3$ is the 3×3 identity tensor and $\boldsymbol{\pi}_s$ is the generalized viscosity tensor. $\mathbf{F}_s$ and W_s are the first and second moments of the colision operator. $\mathbf{p}_s \colon \boldsymbol{\nabla} \boldsymbol{V_s}$ denotes the colon product between the tensors $\mathbf{p}_s$ and $\boldsymbol{\nabla} \boldsymbol{V_s}$ such that the resultant scalar is given by

$$\mathbf{p}_s \colon \boldsymbol{\nabla} \boldsymbol{V_s} = \sum_i \sum_j (p_s)_{ij} \frac{\partial (V_s)_i}{\partial x_j} \, . \tag{2.14}$$

For our derivation, we will also be assuming that our quantities of interest (density, ion temperature, electrostatic potential, and parallel flow velocity) are separable into unperturbed equilibrium components and smaller perturbed components (denoted with a tilde):

$$n_i(\mathbf{x}, t) = n_{i0}(\mathbf{x}) + \tilde{n}_i(\mathbf{x}, t) \, , \tag{2.15}$$

$$T_i(\mathbf{x}, t) = T_{i0}(\mathbf{x}) + \tilde{T}_i(\mathbf{x}, t) \, , \tag{2.16}$$

$$\phi(\mathbf{x}, t) = \phi_0(\mathbf{x}) + \tilde{\phi}(\mathbf{x}, t) \, , \tag{2.17}$$

$$V_{\|}(\mathbf{x},t) = V_{\|0}(\mathbf{x}) + \tilde{V}_{\|}(\mathbf{x},t)\,. \tag{2.18}$$

We assume the electron temperature to be constant due to the small contribution of electrons to the total angular momentum and extremely weak coupling between species. We also adopt a generalization of our ordering from Sec. 2.2.1 so that

$$\rho\nabla_{\perp} \sim 1\,,\quad \rho\nabla_{\|} \sim \delta\,,\quad \frac{1}{\Omega}\frac{\partial}{\partial t} \sim \delta \;\text{ for perturbed components, and} \tag{2.19}$$

$$\rho\nabla_{\perp} \sim \delta\,,\quad \rho\nabla_{\|} \sim \delta^2\,,\quad \frac{1}{\Omega}\frac{\partial}{\partial t} \sim \delta^3 \;\text{ for unperturbed components} \tag{2.20}$$

in order to reflect the relative stability of the unperturbed quantities in comparison to the perturbed quantities.

2.3 Deriving the Modified Three-Field System

We shall now derive a modified version of the Hamaguchi-Horton equation using the first three moments of the ion-specific kinetic equation. These give the ion continuity equation, the ion equation of motion (or momentum equation), and the ion pressure equation, respectively. We shall examine each equation component-wise, applying our ordering definition to arrive at a set of three approximate homogeneous equations which describe the evolution of the perturbed portion of three plasma quantities: the ion pressure, the electrostatic potential, and the parallel plasma velocity.

If we consider the ordering definition from (2.3), we can reasonably assume that an ensemble average is applicable locally within the plasma, and propose a local equation of state. This rather strong approximation, which must be verified *a posteriori* is that plasma constituents are in local thermodynamic equilibrium, with a Maxwellian velocity distribution, and can therefore be assigned a temperature. In fact, due to the large mass difference between the electron and ion species within the plasma, each species has its own local temperature, as well as an equation of state which we will assume to be analogous to that of an ideal gas;

$$p_s = n_s T_s \,, \tag{2.21}$$

where for simplicity, the Boltzmann factor has been absorbed into T_s.

We make the fairly reasonable assumption of local charge neutrality ($n_i(\mathbf{x},t) \approx n_e(\mathbf{x},t) \approx n(\mathbf{x},t)$) which, along with our ordering from Sec. 2.2.1, implies that the unperturbed electrostatic potential is zero ($\phi(\mathbf{x},t) = \tilde{\phi}(\mathbf{x},t)$). We also make a few stronger assumptions. The first is that the generalized viscosity tensor $\boldsymbol{\pi}_s$, the electron–ion collisional heat exchange W_s, and the heat flux vector $\mathbf{q}_s$ are all negligible[1]. Also, we assume that the collisional friction force $\mathbf{F}_s$ only has a parrallel component. Finally, we neglect the electron mass, since $m_e/m_i \approx 1/3700$. With these assumptions, along with the drift ordering, we can now derive a symplified version of the system (2.10–2.10).

Defining the unit vector $\hat{\mathbf{b}} \equiv \mathbf{B}/|B|$, we first take the cross product of $\hat{\mathbf{b}}$ with Eq. (2.11), keeping in mind the definition of $\mathbf{V}_{i\perp} \equiv \hat{\mathbf{b}} \times (\mathbf{V}_i \times \hat{\mathbf{b}})$ and keeping terms only through the frist order in δ. Doing this we find

$$\mathbf{V}_{i\perp} = \frac{c}{B^2}(\mathbf{B} \times \boldsymbol{\nabla}\tilde{\phi}) + \frac{c}{q_i n_0 B^2}(\mathbf{B} \times \boldsymbol{\nabla} p_i) + \frac{1}{\Omega_i}\hat{\mathbf{b}} \times \left(\frac{\partial}{\partial t} + \mathbf{V}_i \cdot \boldsymbol{\nabla}\right)\mathbf{V}_i \,. \tag{2.22}$$

The first term in (2.22) is the electrostatic (or $\mathbf{E} \times \mathbf{B}$) drift velocity, $\mathbf{V}_E$. It is typically the largest component of the perpendicular ion velocity. The second term is the ion diamagnetic drift velocity, $\mathbf{v}_{id}$, which causes a magnetic field opposite to the direction of the prevailing field, similar to material diamagnetism. The final term is the ion polarization drift velocity , $\mathbf{v}_{ip}$, which can be expanded by components of $\mathbf{V}_i$ so that to leading order

$$\mathbf{v}_{ip} = \frac{1}{\Omega_i}\hat{\mathbf{b}} \times \left(\frac{\partial}{\partial t} + \mathbf{V}_i \cdot \boldsymbol{\nabla}\right)\mathbf{V}_i \approx \frac{1}{\Omega_i}\hat{\mathbf{b}} \times \left(\frac{\partial}{\partial t} + \mathbf{V}_E \cdot \boldsymbol{\nabla}\right)\mathbf{V}_E \,. \tag{2.23}$$

We neglect this term due to the ordering $V_E/v_t \sim \delta$.

We now turn our attention to Eq. (2.10), the ion continuity equation. Neglecting the ion diamagnetic drift velocity and taking $\nabla \cdot \mathbf{V}_E = 0$, $\nabla_\| V_\| = 0$ per Eq. (2.20) we find

$$\frac{\partial \tilde{n}}{\partial t} = -(\mathbf{V}_\| + \mathbf{V}_E) \cdot \nabla (n_0 + \tilde{n}) - n_0 \nabla_\| \tilde{u}_\| \,. \tag{2.24}$$

Additionally, our ordering allows us to simplify this expression by writing

$$-\mathbf{V}_E \cdot \nabla n_0 = -\frac{c}{B}\left(\hat{\mathbf{b}} \times \nabla \tilde{\phi}\right)_x \frac{\partial n_0}{\partial x} = \frac{c}{B}\frac{\partial \tilde{\phi}}{\partial y} n_0 L_n^{-1} \,, \tag{2.25}$$

$$-\mathbf{V}_E \cdot \nabla \tilde{n} = -\frac{c}{B}\left(\frac{\partial \tilde{\phi}}{\partial x}\frac{\partial \tilde{n}}{\partial y} - \frac{\partial \tilde{\phi}}{\partial y}\frac{\partial \tilde{n}}{\partial x}\right) = \frac{c}{B}\left[\tilde{\phi}, \tilde{n}\right] \,, \tag{2.26}$$

$$-\mathbf{V}_\| \cdot \nabla n_0 = 0 \,, \text{ and} \tag{2.27}$$

$$-\mathbf{V}_\| \cdot \nabla \tilde{n} = -V_\| \nabla_\| \tilde{n} \,, \tag{2.28}$$

where $[A, B]$ is a simplifying notation called a Poisson bracket. Finally, defining the dimensionless quantities $\bar{n} \equiv \tilde{n}/n_0$ and $\bar{\phi} \equiv e\tilde{\phi}/T_e$, as well as the ion-acoustic diamagnetic drift velocity $V_{de} = c_s T_e/qBL_n$ we rewrite Eq. (2.24) as

$$\frac{\partial \bar{n}}{\partial t} = -V_{ds}\frac{\partial \bar{\phi}}{\partial y} - V_\| \nabla_\| \bar{n} - \nabla_\| \bar{V}_\| - \frac{c^2}{\Omega_i}\left[\bar{\phi}, \bar{n}\right] \,. \tag{2.29}$$

From Eq. (2.11) we find the perpendicular ion momentum conservation equation:

$$m_i n \left(\frac{\partial}{\partial t} + \mathbf{V}_i \cdot \nabla\right) \mathbf{V}_{i\perp} = -\nabla_\perp p_i - q_i n \nabla_\perp \phi + \frac{q_i n}{c}\left(\mathbf{V}_i \times \mathbf{B}\right) \,. \tag{2.30}$$

Multiplying (2.30) vectorially by $\hat{\mathbf{b}}$ and then taking the divergence of the resulting equation, we find

$$\begin{aligned} &m_i n \nabla \cdot \left\{\hat{\mathbf{b}} \times \left[\frac{\partial}{\partial t} + (\mathbf{V}_E + \mathbf{v}_{id}) \cdot \nabla\right] \mathbf{V}_E\right\} \\ &= -\nabla \cdot \left(\hat{\mathbf{b}} \times \nabla p_i\right) - q_i n \nabla \cdot \left(\hat{\mathbf{b}} \times \nabla \tilde{\phi}\right) \end{aligned} \tag{2.31}$$

to lowest order in δ. Noting the relation

$$\frac{c}{B}\nabla \cdot \left[\hat{\mathbf{b}} \times \left(\hat{\mathbf{b}} \times \nabla \tilde{\phi}\right)\right] = -\frac{c}{B}\nabla_\perp^2 \tilde{\phi} \,, \tag{2.32}$$

where $\nabla_\perp^2 = \nabla_x^2 + \nabla_y^2$ is the perpendicular Laplacian, we reduce Eq. (2.31) to

$$\frac{\partial}{\partial t}\left(\nabla_\perp^2 \tilde{\phi}\right) = -\frac{1}{\Omega_i m_i n_0}\frac{dp_{i0}}{dx}\frac{\partial}{\partial y}\left(\nabla_\perp^2 \tilde{\phi}\right) - \frac{c}{B}\left[\tilde{\phi}, \nabla_\perp^2 \tilde{\phi}\right] . \tag{2.33}$$

We define the quantities $\eta_i \equiv L_n/L_{T_i}$, $\theta_i \equiv T_{i0}/T_e$, and $\kappa_i \equiv \theta_i(1+\eta_i)$ so that from our equation of state (2.21), we find

$$\begin{aligned} \frac{dp_{i0}}{dx} &= T_{i0}\frac{dn_0}{dx} + n_0\frac{dT_{i0}}{dx} \\ &= T_{i0}\frac{dn_0}{dx}\left[1 + \left(\frac{d}{dx}\ln T_{i0}\right)\left(\frac{d}{dx}\ln n_0\right)^{-1}\right] \\ &= -\theta_i(1+\eta_i)/L_n = \kappa_i/L_n\,, \end{aligned} \tag{2.34}$$

enabling us to rewrite Eq. (2.33) as

$$\frac{\partial}{\partial t}\left(\nabla_\perp^2 \bar{\phi}\right) = -V_{ds}\kappa_i\frac{\partial}{\partial y}\nabla_\perp^2 \bar{\phi} - \frac{c^2}{\Omega_i}\left[\bar{\phi}, \nabla_\perp^2 \bar{\phi}\right] . \tag{2.35}$$

Multiplying through by ρ_i^2 and subtracting from Eq. (2.29), our final result is

$$\begin{aligned} \frac{\partial}{\partial t}\left(\bar{n} - \rho_i^2\nabla_\perp^2 \bar{\phi}\right) = &\ -V_{ds}\tfrac{\partial}{\partial y}\left(1 + \kappa_i\rho_i^2\nabla_\perp^2\right)\bar{\phi} - \nabla_\| \tilde{V}_\| \\ &-V_\|\nabla_\|\bar{n} - \tfrac{c^2}{\Omega_i}\left[\bar{\phi}, \left(\bar{n} - \rho_i^2\nabla_\perp^2\bar{\phi}\right)\right] . \end{aligned} \tag{2.36}$$

We also examine the parallel ion momentum:

$$m_i n\left(\frac{\partial}{\partial t} + \mathbf{V}_i \cdot \boldsymbol{\nabla}\right)V_\| = -\nabla_\| p_i - q_i n \nabla_\| \phi\,, \tag{2.37}$$

where we have discarded the $\mathbf{V}_i \times \mathbf{B}$ force, since it is perpendicular to $\mathbf{B}$. Neglecting the ion diamagnetic term $(\mathbf{v}_{id}\cdot\boldsymbol{\nabla})\,V_\|$, and using Eq. (2.26) with the substitution of $V_\|$ for $\tilde{n}$, we find

$$\frac{\partial \tilde{V}_\|}{\partial t} = -\frac{1}{m_i n_0}\nabla_\|\tilde{p}_i - \frac{q_i}{m_i}\nabla_\|\tilde{\phi} - V_\|\nabla_\|\tilde{V}_\| + \frac{c}{B}\frac{\partial\tilde{\phi}}{\partial y}\frac{\partial V_\|}{\partial x} - \frac{c}{B}\left[\tilde{\phi}, \tilde{V}_\|\right] . \tag{2.38}$$

Noting that $T_e/m_i = c^2$, we write (2.38) using dimensionless parameters as

$$\frac{\partial \tilde{V}_\|}{\partial t} = -\frac{1}{m_i n_0}\nabla_\| \tilde{p}_i - c^2 \nabla_\| \bar{\phi} - V_\| \nabla_\| \tilde{V}_\| + \frac{c^2}{\Omega_i}\frac{\partial \bar{\phi}}{\partial y}\frac{\partial V_\|}{\partial x} - \frac{c^2}{\Omega_i}\left[\bar{\phi}, \tilde{V}_\|\right]. \tag{2.39}$$

Finally, we consider the ion energy equation (2.12). Keeping in mind our ordering thus far, we can write it as

$$\frac{\partial \tilde{p}_i}{\partial t} = -V_\| \nabla_\| \tilde{p}_i - \mathbf{V}_E \cdot \boldsymbol{\nabla}_\perp \left(p_{i0} + \tilde{p}_i\right) - \gamma p_{i0} \nabla_\| \tilde{V}_\| . \tag{2.40}$$

Recalling our expression of $\boldsymbol{\nabla}_\perp p_{i0}$ from Eq. (2.34) and our usage of the Poisson bracket (2.26), we write (2.40) as

$$\frac{\partial \tilde{p}_i}{\partial t} = -V_\| \nabla_\| \tilde{p}_i - \kappa_i m c_s^2 n V_{ds} \frac{\partial \bar{\phi}}{\partial y} - \gamma p_{i0} \nabla_\| \tilde{V}_\| - \frac{c^2}{\Omega_i}\left[\bar{\phi}, \tilde{p}_i\right]. \tag{2.41}$$

Together, Eqs. (2.36), (2.39), and (2.41) comprise the Hamaguchi-Horton equations, modified to include zeroth-order parallel velocity. We collect the equations here for convenience:

$$\begin{aligned}
\frac{\partial}{\partial t}(\bar{n} - \rho_i^2 \nabla_\perp^2 \bar{\phi}) = -V_{ds}\frac{\partial}{\partial y}\left(1 + \kappa_i \rho_i^2 \nabla_\perp^2\right)\bar{\phi} - \nabla_\| \tilde{V}_\| & \\
-V_\| \nabla_\| \bar{n} - \frac{c_s^2}{\Omega_i}\left[\bar{\phi}, \left(\bar{n} - \rho_i^2 \nabla_\perp^2 \bar{\phi}\right)\right], & \qquad (2.42) \\
\frac{\partial \tilde{V}_\|}{\partial t} = -\frac{1}{m_i n_0}\nabla_\| \tilde{p}_i - c_s^2 \nabla_\| \bar{\phi} - V_\| \nabla_\| \tilde{V}_\| + \frac{c_s^2}{\Omega_i}\frac{\partial \bar{\phi}}{\partial y}\frac{\partial V_\|}{\partial x} - \frac{c_s^2}{\Omega_i}\left[\bar{\phi}, \tilde{V}_\|\right], & \qquad (2.43) \\
\frac{\partial \tilde{p}_i}{\partial t} = -V_\| \nabla_\| \tilde{p}_i - \kappa_i m c_s^2 n V_{ds}\frac{\partial \bar{\phi}}{\partial y} - \gamma p_{i0} \nabla_\| \tilde{V}_\| - \frac{c_s^2}{\Omega_i}\left[\bar{\phi}, \tilde{p}_i\right]. & \qquad (2.44)
\end{aligned}$$

We have substituted c here for the ion-acoustic velocity $c_s = (T_e/m_i)^(1/2)$, which better approximates the speed of light in plasma due lights interaction with the free-charge medium.

Chapter 3

Physics of Relevant Modes and Angular Momentum Transport

3.1 The Hamaguchi-Horton Equations Applied to Velocity–Ion Temperature Gradient Driven Modes

Eqs. (2.42–2.43) are interesting in that they include both the gradients of the parallel velocity and of the ion temperature (the term η_i in the definition of κ_i is a temperature-gradient term). The novelty of the inclusion of these terms comes from the fact that their inclusion has a significant affect on first order affects since, unlike equilibrium quantities, gradients can be either positive or negative. We will now see how these two quantities effect the resulting behavior of the collective modes described by our model.

3.1.1 Locally Approximate Dispersion Relation for the VITG-Driven Modes

We now seek solutions of the modified Hamaguchi-Horton equations of the form $\tilde{Q}\exp\left(ik_\perp y + ik_z z - i\Omega t\right)$ for an unknown perturbed quantity $\tilde{Q}$, where $k_\perp$ and k_z

are the perpendicular and parralel wavenumbers, and $\Omega(\mathbf{k}) = \omega(\mathbf{k}) + i\gamma(\mathbf{k})$ is the complex frequency; $\omega(\mathbf{k})$ is the ordinary frequency, and $\gamma(\mathbf{k})$ is the growth/damping factor (depending on the sign of $\gamma(\mathbf{k})$. Adopting this form for our solutions, we see that

$$\frac{\partial \tilde{Q}}{\partial t} = -i\Omega\tilde{Q}\,, \;\; \frac{\partial \tilde{Q}}{\partial y} = ik_\perp\tilde{Q}\,, \;\; \nabla_\parallel\tilde{Q} = ik_z\tilde{Q}\,. \tag{3.1}$$

This situation is analogous to a model in which we only consider the local plasma parameters, which are all constant within that region. We insert this *ansatz* into Eqs. (2.42)-(2.44), and neglecting the nonlinear terms contained in the Poisson brackets (which are relatively small), we find

$$\Omega\left(\bar{n} + \rho_i^2 k_\perp^2 \bar{\phi}\right) - k_\perp V_{ds}\left(1 - \kappa_i \rho_i^2 k_\perp^2\right)\bar{\phi} - k_z\tilde{V}_\parallel - k_z V_\parallel \bar{n} = 0\,, \tag{3.2}$$

$$\Omega\tilde{V}_\parallel - \frac{k_z}{m_i n_0}\tilde{p}_i - k_z c^2\bar{\phi} - k_z V_\parallel \tilde{V}_\parallel + \frac{k_\perp c^2}{\Omega_i}\frac{dV_\parallel}{dx}\bar{\phi} = 0\,, \tag{3.3}$$

$$\Omega\tilde{p}_i - k_z V_\parallel \tilde{p}_i - k_\perp V_{ds}\kappa_i\bar{\phi} - k_z\Gamma p_{i0}\tilde{V}_\parallel = 0\,, \tag{3.4}$$

so that we now have a homogeneous system of algebraic equations relating four variables. In order to close the system, we make the adiabatic approximation for the electron population,

$$\tilde{n}T_e \simeq en\tilde{\phi} \tag{3.5}$$

which allows us to take $\bar{\phi} = \bar{n}$.

We now have a homogeneous system of three equations, relating three independent variables. We enforce the criterion for a non-trivial solution, that the determinant of the characteristic matrix be zero. This yeilds an equation relating Ω to $\mathbf{k}$, which is the dispersion relation for the ITG driven mode:

$$\Omega'^2 - \omega_{d,T}\Omega' - \omega_{s,T}^2 = 0\,, \tag{3.6}$$

where $\Omega' = \Omega - k_z V_\parallel$ is the Doppler-shifted frequency. We have simplified the expression by defining the quantities

$$\omega_{d,T} \equiv \frac{k_\perp V_{ds,T}}{1+\rho_s^2 k_\perp^2(1+k_z V_\|)}\,, \quad \omega_{s,T}^2 \equiv \frac{k_z^2 c_{s,T}^2}{1+\rho_s^2 k_\perp^2(1+k_z V_\|)}\,, \tag{3.7}$$

where

$$V_{ds,T} \equiv \left(1-\kappa_i \rho_s^2 k_\perp^2\right)\,, \quad c_{s,T}^2 \equiv c_s^2 \left(\frac{\alpha+\kappa_i \frac{k_\perp V_{ds}}{\Omega'}}{1-\Gamma\theta_i \frac{k_z^2 c_s^2}{\Omega'^2}}\right)\,, \tag{3.8}$$

and

$$\alpha \equiv 1 - \frac{k_\perp}{k_z \Omega_i}\frac{dV_\|}{dx}\,. \tag{3.9}$$

For sufficiently large $|\Omega^2|$, such that

$$\Gamma\theta_i \frac{k_z^2 c_s^2}{|\Omega'^2|} \ll 1\,, \tag{3.10}$$

we can neglect this term, allowing us to write the dispersion relation as

$$\left[1+\rho_s^2 k_\perp^2(1+k_z V_\|)\right]\Omega'^2 - k_\perp V_{ds}\left(1-\kappa_i\rho_s^2 k_\perp^2\right)\Omega' - k_z^2 c_s^2\left(\alpha - \kappa_i \frac{k_\perp V_{ds}}{\Omega'}\right) = 0\,, \tag{3.11}$$

which is a cubic polynomial in Ω'.

We can make a number of further assumptions to simplify the dispersion relation so that we may analyze the complex and real parts of Ω. The first of these, is that the perpendicular wave number is sufficiently large that

$$\rho_s k_\perp \sim \kappa_i^{-1/2}\,, \quad |\Omega| \ll |k_\perp| V_{de}\,, \tag{3.12}$$

allowing us to drop the second term and write a simplified relation

$$\Omega' = \left[\frac{k_z k_\perp c_s^2}{(1+\rho_s^2 k_\perp^2)}\left(\frac{1}{\Omega'} k_z \kappa_i V_{de} - \frac{1}{\Omega_i}\frac{dV_\|}{dx}\right)\right]^{1/2} \tag{3.13}$$

We see that the instability of the mode (the existence of complex-valued Ω') depends upon the sign and magnitude of the parallel velocity gradient. If it is sufficiently large and positive, there will be two complex roots that are a complex conjugate pair, plus one real root (to ensure that the cubic equation has real coefficients). The parallel velocity gradient therefore is a determining factor in the mode's instability.

We solve this equation, assuming a positive and large velocity gradient. We then expand $Im(\Omega_+)$ about 0 to fifth order in $\frac{\Omega_i}{k_\perp V_{de}}$ to find the approximate growth rate,

$$\gamma_+(k_\perp, k_z) = c_s \left[\frac{k_\perp k_z}{\Omega_i(1+\kappa_i)}\frac{dV_\parallel}{dx}\right]^{1/2} + \frac{3k_z\Omega_i^2 V_{de}^2\kappa_i}{8c_s(\frac{dV_\parallel}{dx})^3}\left[\frac{k_z\Omega_i\kappa_i(1+\kappa_i)}{k_\perp}\frac{dV_\parallel}{dx}\right]^{1/2} - \frac{105k_z\Omega_i^4 V_{de}^4\kappa_i}{256c_s^3\left(\frac{dV_\parallel}{dx}\right)^6}\left[\frac{k_z\Omega_i\kappa_i(1+\kappa_i)}{k_\perp}\frac{dV_\parallel}{dx}\right]^{1/2} . \qquad (3.14)$$

Figs. A-1 and A-2 in Appendix A illustrate the shape of the growth rate in the region of our interest.

We find that the growth rate increases without bound as $k_z \to \infty$, since we have neglected higher-order damping terms both in our model and in the taylor-series expansion of $\gamma_+(k_\perp, k_z)$. However, these first several terms allow us some idea of the ITG-driven mode's behavior in the regime of interest, and are in fact quite accurate within the limit $\Omega/k_\perp V_{de} \ll 1$, which begins to become valid at around $k_\perp = 1000$ cm^{-1}. Perhaps most interestingly, we see that the instability increases as the product $\frac{k_z}{k_\perp}\frac{dV_\parallel}{dx}$ becomes more negative, implying that the instability will be dominated by shorter and shorter parallel length scales with a phase velocity along the direction determined by $k_\perp$ and k_z that yield $\frac{k_z}{k_\perp}\frac{dV_\parallel}{dx} < 0$.

3.2 A Model for Angular Momentum Transport

Conservation of the canonical toroidal angular momentum within a plasma can be expressed through the usual conservation equation

$$\sum_s \frac{\partial J_s}{\partial t} + \sum_s \nabla \cdot \mathbf{\Gamma}_{J_s} = S_J , \qquad (3.15)$$

where J_s is the canonical toroidal angular momentum density of species s, $\mathbf{\Gamma}_{J_s}$ is the flux vector of J_s, and S_J is a source of toroidal angular momentum which we shall assume to be localized at the edge of the plasma in the absence of any direct injection of angular momentum to the plasma through either ICRH or neutral beam injection.

The canonical toroidal angular momentum density for a species s can be expressed as

$$J_s = m_s n_s v_T^s R - \frac{n_s q_s}{c}\mathbf{A}\,, \tag{3.16}$$

where v_T^s is the toroidal particle velocity and R is the torus major radius. If we invoke the quasineutrality approximation, however, we see that for a sum over all species, the second term of (3.16) cancels out due to the opposing signs of q_s, so that the total toroidal angular momentum density is

$$J = \sum_s m_s n_s v_T^s R\,. \tag{3.17}$$

The form of (3.15) remains the same.

The main challenge of determining the behavior of the toroidal angular momentum profile lies in finding an accurate model for the transport term $\mathbf{\Gamma}_J$. The accretion theory[2] proposes a tramsport model given by

$$\mathbf{\Gamma}_{J_s} \simeq -\left(J_{0_s} v_J^s + D_J^s \frac{\partial J_s}{\partial x}\right)\hat{\mathbf{e}}_x \tag{3.18}$$

3.3 Theoretical Considerations of the Model

Of the quantities included in Eq. (3.18), $J(r)$ is the dependent variable, a function of radius. J_0 is an equilibrium angular momentum parameter, which can be seen by setting the flux equal to zero. This yields a first-order integrable ODE that we can solve by assuming that V_J/D_J is an increasing function of x, as has been shown to be the case in the quasilinear analysis of anomolous particle transport. In particular, [2] assumes that

$$\frac{v_J}{D_J} \simeq 2\alpha_J \frac{x}{a^2}\,, \tag{3.19}$$

so that we find that in equilbrium

$$J(x) = J_0 \left(1 - \alpha_J \frac{x^2}{a^2}\right) . \tag{3.20}$$

Thus $J(x = 0) = J_0$, and $J(x = \pm a) = J_0(1 - \alpha_J)$, and α_J determines the degree to which the angular momentum density profile is peaked. Γ_J is given by the second vector-product moment of the distribution function, remembering our drift ordering in which the dominant drift velocity is $\mathbf{V}_E$ and the time-varying parallel velocity is $\tilde{\mathbf{V}}_\parallel$:

$$\Gamma_J = \int \mathbf{V}_E \times (m_i \tilde{\mathbf{V}}_\parallel) f_s(\mathbf{x}, \mathbf{V}, t)\, dV^3 = m_i n_i \langle V_e \tilde{V}_\parallel \rangle . \tag{3.21}$$

This leave us with the task of finding D_J.

Recalling Eq. (2.29), we write

$$\Omega' \bar{n} = k_\perp V_{ds} \bar{\phi} + k_z n \tilde{V}_\parallel , \tag{3.22}$$

and solve for $\tilde{V}_\parallel$ to find

$$\tilde{V}_\parallel = \frac{\Omega'}{k_z} \bar{n} - \frac{k_\perp}{k_z} V_{ds} \bar{\phi} . \tag{3.23}$$

In order to find the parallel momentum flux, given by eq. (3.21), we multiply through by $m_i n \mathbf{V}_{E_x} = -i m_i n \frac{cT_e k_\perp}{eB} \bar{\phi}$, yielding

$$\Gamma_J = -i m_i n D_B \frac{k_\perp}{k_z} \left(\Omega' - k_\perp V_{ds}\right) \langle |\bar{\phi}|^2 \rangle , \tag{3.24}$$

where $D_B = cT_e/eB$. Having seen from the previous section that $\Omega \gg k_\perp V_{ds}$, we neglect the second term in parentheses. We may additionally may write $-i\Omega$ as $-(\gamma + i\omega)$. We therefore find

$$\Gamma_J = -m_i n D_B \frac{k_\perp}{k_z} \gamma \langle |\bar{\phi}|^2 \rangle , \tag{3.25}$$

since Γ_J is real.

3.4 Modified Hamaguch-Horton Equations in a Inhomogeneous Plasma

We will now consider the case of an inhomogeneous plasma, with zeroth-order equilibrium parameters $n_0(x)$, $T_{i0}(x)$, and $V_{\parallel}(x)$ assumed to be symmetric across the $y-z$ plane and vary along the x-axis. The derivatives of these quantities are assumed to have sign $-x/|x|$; that is to say, the profiles for the quantities are peaked, with a maximum at $x = 0$. As functions of these three quantities, $p_{i0}(x)$, $\kappa_i(x)$, $V_{ds}(x)$, $L_n(x)$, and $L_{T_i}(x)$ also vary along the x-axis.

The inhomogeneity in x prevents the reduction of Eqs. (2.42—2.43) to an algebraic equation, as for the local approximation. The inhomogeneity necessitates that we Instead, we are left with the linear homogeneous system of ordinary differential equations

$$\begin{aligned} \left(\Omega' + k_z V_{\parallel}\right)\left[1 + \rho_i^2\left(k_\perp^2 - \frac{d^2}{dx^2}\right)\right]\bar{\phi}(x) - k_z V_{\parallel}(x) & \\ -k_\perp V_{ds}(x)\left(1 - \kappa_i(x)\rho_i^2\left(k_\perp^2 - \frac{d^2}{dx^2}\right)\right)\bar{\phi}(x) - k_z\tilde{V}_{\parallel}(x) &= 0\,, \quad (3.26) \\ \Omega\tilde{V}_{\parallel}(x) - \frac{k_z}{m_i n_0(x)}\tilde{p}_i(x) - k_z c^2\bar{\phi}(x) - k_z V_{\parallel}(x)\tilde{V}_{\parallel}(x) + \frac{k_\perp c^2}{\Omega_i}\frac{dV_{\parallel}(x)}{dx}\bar{\phi}(x) &= 0\,, \quad (3.27) \\ \Omega\tilde{p}_i(x) - k_z V_{\parallel}\tilde{p}_i(x) - k_\perp V_{ds}(x)\kappa_i(x)\bar{\phi}(x) - k_z\Gamma p_{i0}(x)\tilde{V}_{\parallel}(x) &= 0\,. \quad (3.28) \end{aligned}$$

We solve Eq. (3.28) for $\tilde{p}_i(x)$ and insert this into Eq. (3.27) to find $\tilde{V}_{\parallel}(x)$. Substituting our result into Eq. (3.26), we find a homogeneous second-order linear ODE describing $\bar{\phi}(x)$:

$$g(x)\frac{d^2\bar{\phi}}{dx^2} + h(x)\bar{\phi} = 0\,, \tag{3.29}$$

which, surprisingly, has no dependence on x-directed component of the magnetic field, $E_x = -d\bar{\phi}/dx$. The functions $g(x)$ and $h(x)$ are given by

$$\begin{aligned} h(x) &= \Omega_i\left[\Omega'\left(1 + k_\perp^2\rho_i^2\right) + k_\perp^2\rho_i^2\left(k_z V_{\parallel}(x) - \kappa_i(x)k_\perp V_d(x)\right) - k_\perp V_d(x)\right]\left(3m\Omega'^2 - 5k_z^2 T_i(x)\right) \\ &- 3k_z m c_s^2\left[k_z k_\perp \Omega_i \kappa_i(x) V_d(x) + c_s^2 m n(x)\Omega'\left(k_z\Omega_i - k_\perp\frac{dV_{\parallel}}{dx}\right)\right]\,, \text{ and} \end{aligned} \tag{3.30}$$

$$g(x) = \Omega_i \rho^2 \left[\Omega' + k_z V_\parallel(x) - k_\perp \kappa_i(x) V_d(x) \left(3m\Omega'^2 - 5k_z^2 T_i(x)\right)\right] . \tag{3.31}$$

Thus, $\bar{\phi}(x)$ should be obtainable by integrating. However, the x-dependencies of $g(x)$ and $h(x)$ are quite complicated, even if we choose simple quadratic profiles for the plasma parameters such that for an equilibrium quantity $Q_0(x)$,

$$Q_0(x) = (Q_0 - Q_e)\left(1 - \left(\frac{x}{a}\right)^{\alpha_Q}\right)^{\beta_Q} + Q_e , \tag{3.32}$$

where a is the distant from the origin to the plasma edge, Q_0 is the on-axis value of $Q_0(x)$, and Q_e is its value at the edge, the equation is insoluble by any practical analytical means. To make matters worse, the dispersion relation arises from a quite complicated eigenvalue problem for which a tecnique such as the shooting method may be helpful.

Chapter 4

Conclusion and Remarks

In this thesis we have derived a three-field nonlinear plasma fluid model, based on the Hamaguchi-Horton equations, that are capable of describing the time-evolution of the unstable VITG mode, even in relatively simple geometry, such as the local homogeneous slab model considered in Sec. 3.1.1. Our model incorporates a zeroth order inhomogeneous parallel velocity which, approximate drift-fluid model for a magnetized plasma, based on the Hamaguchi-Horton fluid equations and which incorporates a zeroth-order equilibrium parallel velocity inhomogeneity along the radial-analagous perpendicular axis. This model can be used to investigate a local-plasma linear approximation and associated unstable VITG-Driven modes, or to study the inhomogeneous plasma problem, which yields a second-order linear ODE for the electrostatic potential, which does not allow a tractable analytical solution for realistic plasma profiles.

4.1 Conclusions from the Model

Solving the dispersion relation given by the model in the local approximation, we were able to determine not only that there was an instability, but that it depends significantly on the size and magnitude of the parallel velocity gradient and its interaction with $k_z/k_\perp$. Specifically, we found that short parallel length-scales dominate the stability, due to the positive dependence of the growth rate $\gamma(k_\perp, k_z)$ on k_z. These dominant modes have a preferred phase velocity component in direction direction as determined by the sign of $\frac{k_z}{k_\perp}\frac{dV_\parallel}{dx}$.

In addition to determining the behavior of the dominant unstable VITG-driven modes, we derived a quasilinear angular momentum flux which depends on the growth rate, which we have shown to be dependent largely on the sign and magnitude of the parallel velocity gradient.

4.2 Questions that Remain Unanswered—Directions for Future Study

4.2.1 Analyzing the Modified Hamaguchi-Horton Equations

Although the model we derived, which incorporates a zeroth-order velocity inhomogeneity into the well-known Hamaguchi-Horton equation, shows promise in providing meaningful and physically-motivated solutions within a relatively simple framework, it is important that we study the equations further to determine whether they are appropriate for future analyses, and, if so, what kind. There are a number of considerations that must be made, but chief among these is whether the equations abide by all relevant conservation laws.

With a better understanding of the behavior of the equation, we would be well-equipped to model it numerically in order to determine the dispersion relation and growth rates of the dominant modes. A numerical solution to the inhomogeneous case would be very interesting, especially with the introduction of an x-dependence in the dispersion relation. However, as a numerical solution, it will not be transparent to the simple analyses we performed on the local dispersion relation.

4.2.2 Understanding the Local Dispersion Relation

Despite our analysis of the approximate local-plasma dispersion relation, we were unable to find a tractable solution to even the total locally-approximate dispersion relation without making quite substantial further approximations. In searching for a more appropriate expression, we would hope to find that there is a cut-off in the growth rate for some value of k_z, which was not observed in our analysis (see Figs. A-1 and A-2). Instead, we find the somewhat unphysical situation in which the growth rate is unbounded in k_z for some values of $k_\perp$. This is due in part to the dropping of the nonlinear components of the equations. If we were to re-incorporate these, we would likely be able to observe a saturation of the growth rate for sufficiently high k_z. This value should be tested against experiment to determine the overall accuracy of the model.

4.2.3 Comparison of Computed Behavior to Experiment

Further work also needs to be done to compare the behavior observed in our model to a physical plasma to see which regimes find the two in agreement and at what points they depart from each other. Most telling would be a comparison between the calculated and observed growth rate of these modes. A solution to the 2nd-order equation would be especially valuable in this endeavor, as the length scales over which the homogeneous dispersion relation is valid are actually quite small.

4.3 Conclusion

In conclusion, the VITG-driven modes are a very promising candidate in determining the mechanism underlying the turbulent transport of toroidal angular momentum. Our preliminary results have shown the role that parallel velocity inhomogeneity plays in the instability of these modes and have derived a plausible quasilinear toroidal angular momentum flux. However, future investigation, involving principally the solution of the inhomogeneous and nonlinear equations, is of utmost importance in drawing any further conclusions.

Appendix A

Figures

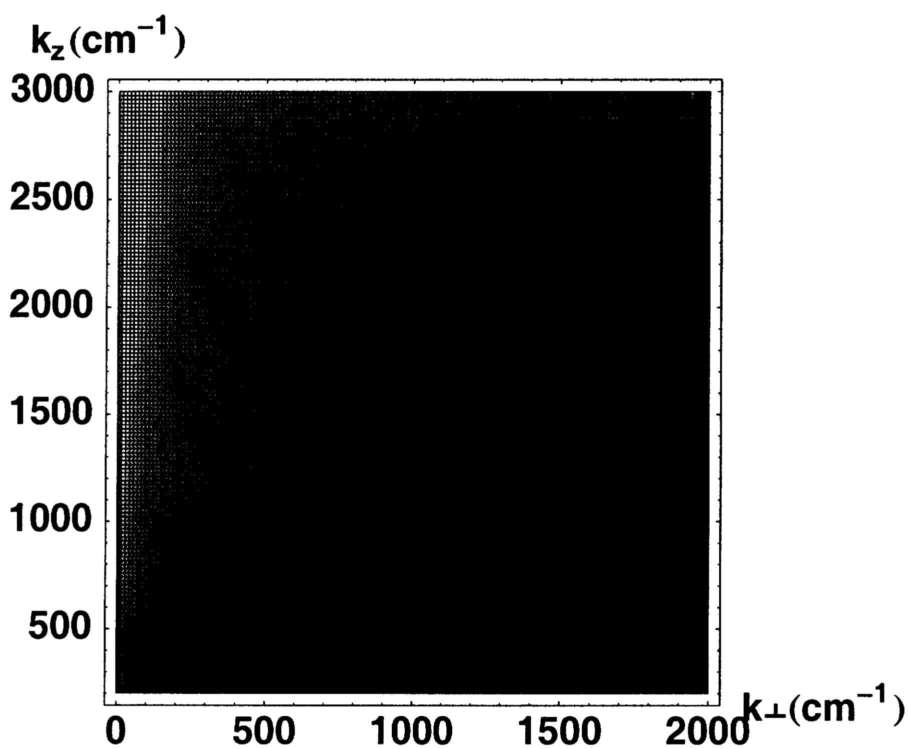

Figure A-1: Density-contour plot of growth rate $\gamma_+(k_z, k_\perp)$ in the $k_z - k_\perp$ plane in large $k_\perp$ approximation. Lighter shading indicates higher values of $\gamma_+(k_z, k_\perp)$.

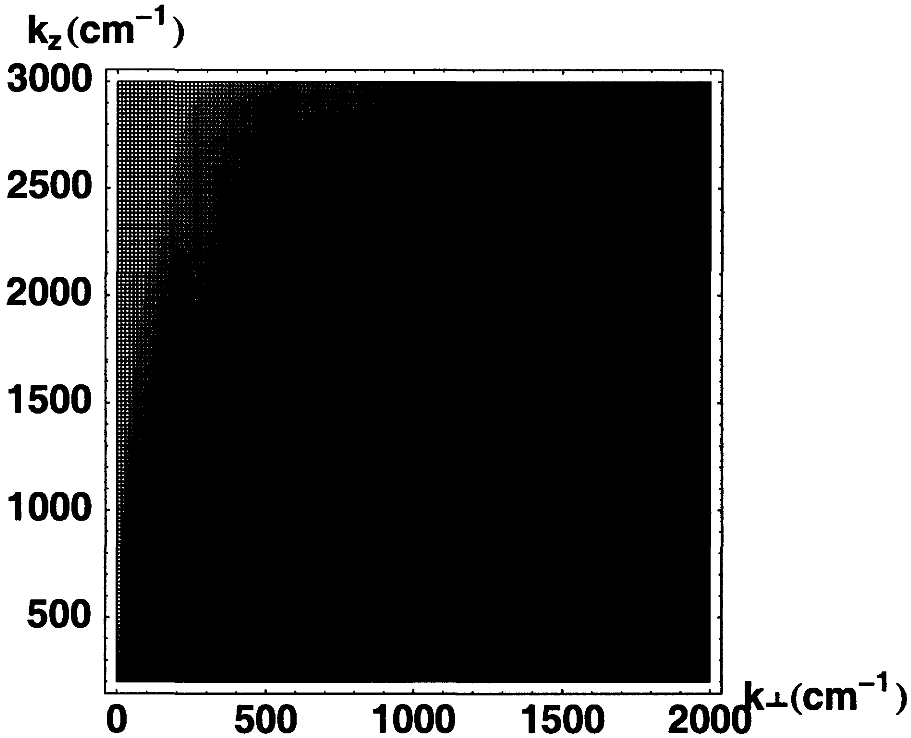

Figure A-2: Density-contour plot of growth rate $\gamma_+(k_z, k_\perp)$ expanded by powers of $\Omega_i/k_\perp V_{ds}$ in the $k_z - k_\perp$ plane for the large $k_\perp$ approximation. Lighter shading indicates higher values of $\gamma_+(k_z, k_\perp)$

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