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# Computer Visualization of Julia Sets for Maps beyond Complex Analyticity

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**Abstract.** Using the computer program creating Julia sets for two-dimensional maps we have made computer animation showing how Julia sets for the Peckham map alters when the parameter of the map is changing. The Peckham map is a one-parameter map which includes the complex map z=z^2+c, and is non-analytical for other values of the parameter. Computer animation of Julia fractal sets allows seeing how patterns typical for complex maps gradually destroy.

## 1 Introduction

Iterations of complex analytic maps appears to be a wonderful source of fractal structures, interesting from both mathematical and aesthetic viewpoints. Already the simplest map

$$z=z^2+c, \quad (1)$$

where c is a constant complex number contains very complicated dynamics, and Julia sets for certain values of the parameter c represent a true computer art (see examples in [1]).

There are several methods to generalize the map (1). One way is to retain the complex analyticity and consider polynomial or transcendent maps. There is a well developed mathematical theory for such kind of maps (see, for example, Milnor's lectures [2] and references therein). Julia sets for such maps has roughly the same structure, though can get additional symmetries in the polynomial case. Particular case of Zhukovski map have been considered in [3], for more general forms of maps within complex analyticity see, for example, [4]. Among recent results we can indicate the paper [5] where it was shown that the Julia set of a transcendental meromorphic map with at most finitely many poles cannot be contained in any finite set of straight lines and the paper [6] which describes the conditions of existence of Sierpiński carpet Julia sets and estimation of it's Hausdorff dimension.

We should also note that in some cases Julia set can bee smooth, see [7] which gives the examples of smooth Julia sets, among them: a circle, a segment, an infinite interval, a straight line, and the complex plane. It is shown that the functions studied in [7] are chaotic on their Julia sets. An interesting generalization of Mandelbrot and Julia sets to hyperbolic dynamics can be found in the paper [8].

Abandoning complex analyticity gives rise to new events, and the corresponding mathematical theory is developing intensively at the present time [9]. Visual patterns of analogs of the Mandelbrot set usually differ qualitatively from the fractal structure of the "true" Mandelbrot set, which can be seen clearly for the "burning ship" map [10]. This map differs from (1) only by the presence of moduli, and it is enough to convert the Mandelbrot set into amorphous structure with thin linear bands. In a broader contexts, both complex analytic maps and the "burning ship" map represent particular cases of two-dimensional real maps, which can be derived from the complex form by extracting real and imaginary parts. In principle, one-dimensional real maps also can have a complicated chaotic nature (see, for example, [11]), though they are less interesting from a viewpoint of a computer art.

In this form the complex analytic map (1) has the form

$$x=x^2-y^2+R(c), \quad (2)$$

$$y=2xy+I(c) \quad (3)$$

and in the "burning ship" map the equation (2) remains the same, while (3) becomes

$$y=2|x||y|+I(c) \quad (4)$$

## 2 Results

For Mandelbrot and Julia sets for two-dimensional real maps these long quasi-parallel bands and other linear structures appear to be rather typical (see examples in [1]). This means that complex analytic maps have some special properties. Formally, they demand Cauchy-Riemann condition to be satisfied, which leads to absence of saddle points and absence of appearance or disappearance of fixed points with change of the

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parameter of the map (1) [9]. The letter follows directly from the fundamental theorem of algebra, the former follows from the fact that a complex multiplication is a combination of stretching and rotation, while a map in a saddle point stretches in one dimension and contracts in another one.

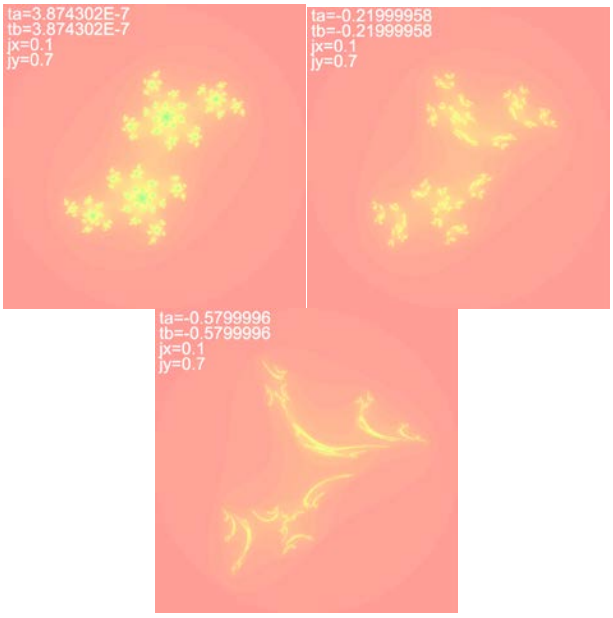


**Fig.1.** Three frame from the animation showing gradual change of Julia set with the deviation of complex analyticity. Parameters of the map is shown in the pictures. Here jx is real, and jy is the imagery parts of c, ta and tb represents a and b (they are equal for the Packham map). Left panel is the analytic map

In the paper [9] B. Peckham described how this properties have been lost when a map becomes non-analytical. For this purpose the map

$$z = z^2 + c + a \qquad (5)$$

have been used. It coincides with (1) for a=0, and is not analytic for any nonzero a. The goal of our paper is to describe typical visual forms of the Peckham map (gradual developing of linear structures through the classical Mandelbrot set for the Peckham map have been considered in [12].

Strictly speaking, the results presented below are related to filled Julia sets, i.e. sets of initial points such that iterations of the map (5) remake bounded from above. They are shown in white. Initial points from other regions leave the vicinity of the coordinate origins (we set it to a radius 10 disc in our numerical studies), the color encodes the escape time needed.

Keeping in mind possible generalizations of the Packham map, we realize more general map

$$x = x^2 - y^2 + R(c) + ax, \qquad (6)$$

$$y = 2xy + I(c) - by. \qquad (7)$$

However, for the animations we have chosen the original Packham map, with a=b. Animations show gradual change of Julia sets with changing parameter a.

Visual patterns of Julia sets rapidly loose features known from the complex analytic dynamics, becoming rather prolonged (and, usually, less aesthetically attractive). Figure 1 represent three frames from our animation, showing the transition from complex analytic pattern (left figure) to quasi-linear structures (right figure).

Some examples of Julia sets for the map with a not equal to be are presented in Fig.2. To our mind, current results indicate that "aesthetic maximum" is realized usually for the complex analytic map with a=b =0 in comparison with non-analytic maps.

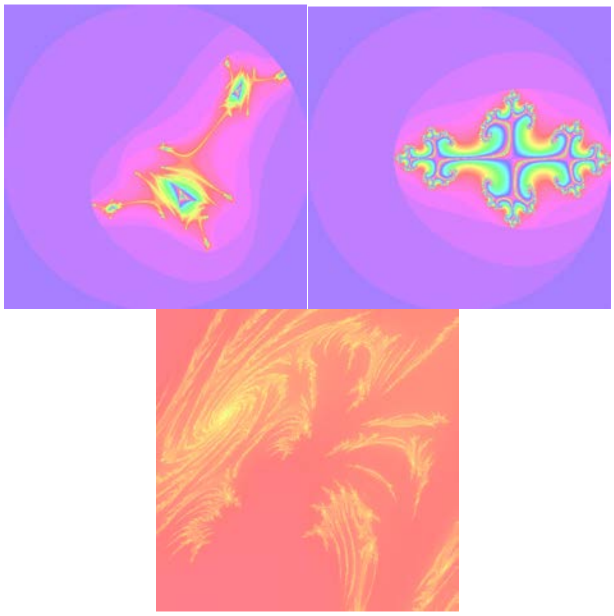

**Fig.2.** Examples of diagrams for general case of different parameters a and b. From left to right: a=- 0,75, b=1,6, x=0,74, y=0,97; a=1,14, b=1,39, x=0,28, y=0,01; a=- 0,6, b=0,3, x=-0,7, y=0,3

## 3 Discussion and Conclusions

We should however note that complex analytic dynamics is not unique in producing visually attractive Julia sets among two-dimensional real maps. Another example is given by the "burning ship" map. It's Mandelbrot set is rather typical for real maps, however, Julia sets represent unexpectedly interesting constructions, though probably less attractive than for analytic maps.

Fig.3 represents three frame from our visualization of Julia sets of the "burning ship" map. Adding the term proportional to $\bar{z}$ into the right hand side of the equations for "burning ship" map should destroy such structures. This is illustrated in Fig.4 created for the map

$$x = x^2 - y^2 + R(c) + ax, \qquad (8)$$

$$y = 2(x)(y) + I(c) - by. \qquad (9)$$

The Fig.4 shows four frames from the corresponding animation.

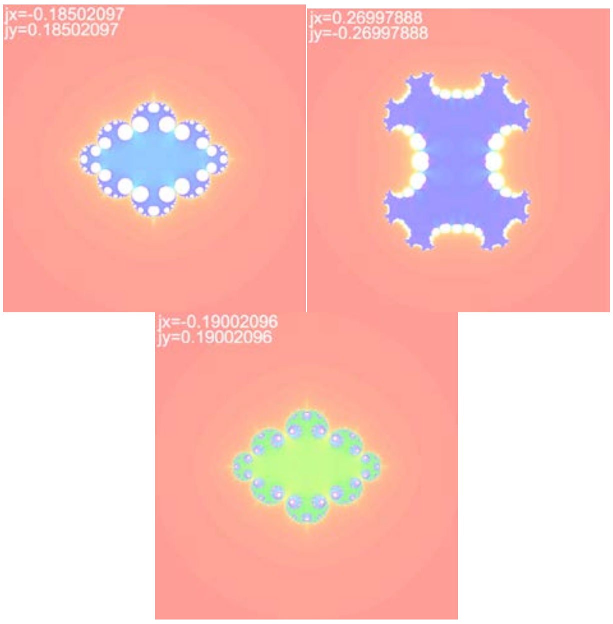


**Fig.3.** Examples of Julia sets for the "burning ship" map

We have considered the influence of complex non-analyticity up on visual forms of Julia sets of corresponding maps. For the Packham map the Julia sets rapidly evolve from known "classical" forms of the map (1) to forms typical for two-dimensional real maps with increasing of modulus of the parameter a. Already for |a| about 0.2 traces of complex analyticity have become hardly detectable. We can even claim that a general aesthetic attraction falls rapidly with deviating from complex analyticity (recognizing that such statement is inevitably subjective and nonformalizable) confirming a special role of complex analytical maps in computer graphics.

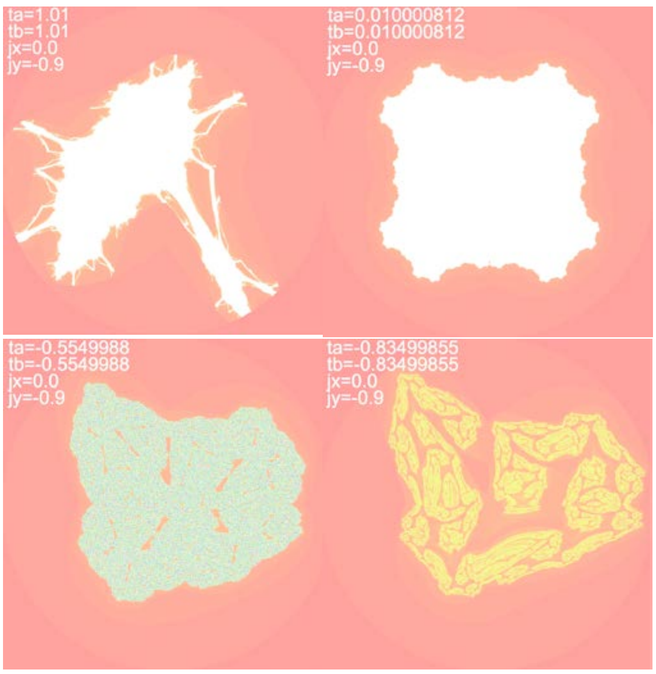


**Fig.4.** Diagrams for "deformed burning ship" map (8)-(9)

Nevertheless, examples of Julia sets for non-analytic map (2,4) show that the Cauchy - Riemann conditions are not strictly necessary for Julia sets to be aesthetically attractive (which is also interesting since the Mandelbrot set for the map (2,4) has totally different visual forms). This means that even for rather simple case of two-dimensional maps the connection between mathematical struture and aesthetic attraction is not so tight, and the search for aesthetically attractive Julia sets should not be bound only to complex analytic maps, despite their special role in computer art.

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