Effective Contracts in Supply Chains

by

Wanhang (Stephen) Shum

Submitted to the Sloan School of Management
in partial fulfillment of the requirements for the degree of

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Abstract

In the past decade, we have seen significant increase in the level of outsourcing in many industries. This increase in the level of outsourcing increases the importance of implementing effective contracts in supply chains. In this thesis, we study several issues in supply chain contracts. In the first part of the thesis, we study the impact of effort in a supply chain with multiple retailers. The costly effort engaged by a retailer may increase or decrease the demands of other retailers. However, effort is usually not verifiable and hence not contractible. Based on the impact of a retailer's effort on its own and other retailers' revenue, we classify each retailer into different categories. According to the corresponding categories of all retailers, we identify coordinating contracts and general classes of contracts that cannot coordinate.

Second, we study the stability of coordinating contracts in supply chains. We illustrate that, due to competition, not all coordinating contracts are achievable. Thus, we introduce the notion of rational contracts, which reflects the agents "bargaining power". We propose a general framework for coordinating and rational contracts. Using this framework, we analyze two supply chains, a supply chain with multiple suppliers and single retailer, and a supply chain with a single supplier and price-competing retailers. We identify coordinating contracts for each case and characterize the bounds on profit shares for the agents in any rational contracts.

Finally, we study the robustness of coordinating contracts to renegotiation. Applying the concept of contract equilibrium, we show that many coordinating contracts are not robust to bilateral renegotiation if the relationship between the supplier and the retailers is a one-shot game. If the supplier and retailers engage in long-term relationship, then many coordinating contracts are robust to bilateral renegotiation. We also extend concept of contract equilibrium to the concept of strong contract equilibrium to study the robustness of contracts to multilateral renegotiation. We show that, in repeated game setting, the concept of strong contract equilibrium is related to the concept of rational contracts.

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Chapter 1

Introduction

In the past decade, we have seen significant increase in the level of outsourcing in many industries. In 2004, 90% of all U.S. businesses were outsourcing some work, as noted in an article in Foust (2004). Indeed, brand-named technology companies now outsource not only the production but also the design of their products to companies called original design manufacturers (ODM), most of which located in Taiwan. According to another article in Engardio and Einhorn (2005), it was estimated that 20% of the 700 million mobile phones produced in 2005 are expected to be designed and produced by Taiwanese ODMs, in addition to 30% of digital cameras, 65% of MP3 players and 70% of the PDAs.

This increase in the level of outsourcing significantly increases the role and importance of supply contracts. Indeed, in addition to acting as a binding agreement of the terms of trade, supply contracts can also serve as a mechanism to induce different parties in the supply chain to achieve global efficiency. That is, supply contracts can be used to "coordinate", or align the interest of the different parties with the objective of the supply chain. In addition to the successful implementation of revenue sharing contracts in the video rental industry, academicians and practitioners have witnessed the use of different forms of rebates, returns and other incentives in many industries.

Since contracts now play a more significant role in the industry, the literature on supply contracts has been growing in the last few years (see Cachon (2003) and Lariviere (1999) for recent reviews). Existing literature on supply contracts has the following three emphasis:

- **Coordinating Contracts:** Motivated by double marginalization, see Spengler (1950), significant work has been done on identifying coordinating contracts, i.e., contracts under which the supply chain optimal decisions is a equilibrium. Different contracts, such as buyback contracts (Pasternack (1985)) and revenue sharing contracts (Pasternack (2002); Cachon and Lariviere (2005)) have been proposed, assuming that different parties in the supply chain will agree to the proposed contract.

- **Contractible Actions:** Many papers focus on the relationship between contracts and the agents' verifiable decisions such as pricing, inventory and capacity. This type of verifiable decisions are contractible, i.e., payment can be expressed as a function of these decisions. Some literature (Taylor (2002); Plambeck and Taylor (2006)) study non-contractible decisions, but they focus on supply chains with only two parties.

- **Specific Types of Contracts:** Many different types of contracts, including buyback contracts (Pasternack (1985)), revenue sharing contracts (Pasternack (2002); Cachon and Lariviere (2005)) and PDS scheme (Bernstein and Federgruen (2005)), have been studied. The focus is on studying specific types of contracts with no attention to the analysis of general classes of contracts.

Our objective is to extend existing literature in supply contracts beyond these three emphasis.

First, we analyze contractible as well as non-contractible decisions, because not all decisions in a supply chain are verifiable. For example, many retailers can influence demand by investing in product advertisement, improved product display or customer service representatives. This costly effort, in many cases, is not verifiable, and hence not contractible. To coordinate effort of a retailer, incentives have to be given using verifiable measures that depend on effort. One goal of this thesis is to study the impact of non-contractible effort on the design of supply contracts.

Second we study general classes of contracts in addition to specific types of contracts. Each class of contracts may include many different types of contracts. Understanding the performance of different classes of contracts in different cases can help us design effective supply contracts. One objective here is to identify certain classes of contracts that can or cannot coordinate the supply chain in different cases.

Third, we try to identify contracts that are not just coordinating but also stable. A coordinating contract may not be stable in the sense that some rational decision makers may have incentives to deviate and not agree to the contracts. For instance, a coordinating contract that generates negative expected profits for one or more agents will never be agreed by all agents, and hence not stable. In many cases, a coordinating contract which generates non-negative expected profit for every agent in the supply chain may also not be stable. The thesis develops and applies different notions of stability to identify effective contracts.

In Chapter 2, we study a supply chain with one supplier selling to multiple retailers. Each retailer has to decide its inventory level, and in addition, its effort level, which is non-contractible. The effort of each retailer affects its own demand and the demands of other retailers. According to the impact of the effort of the retailers, we classify the supply chain into different cases, and identify coordinating contracts for each case. We also analyze general classes of contracts to identify cases in which certain class of contracts do not coordinate.

In Chapter 3, we introduce a notion called *rational contracts*, using the concept of the core, to eliminate contracts that are not stable. We apply this notion to study a supply chain with multiple retailers and a supply chain with multiple suppliers. In each case, we identify contracts that are both coordinating and rational contracts. With the notion of rational contracts, we endogenously characterize the bounds of profits for different parties in the supply chain which reflect their relative bargaining powers.

In Chapter 4, we apply a notion called *contract equilibrium* (Crémer and Riordan (1987); O'Brien and Shaffer (1992)) to study a supply chain with one supplier selling to multiple retailers. The notion of contract equilibrium studies the robustness of bilateral renegotiation when a supplier signs bilateral contracts with multiple retailers. We show that in a one-stage game setting, many contracts are not robust to bilateral renegotiations. On the other hand, if the supplier engages in repeated relationship with the retailers, then many contracts are robust to bilateral renegotiations. We also extend this concept of contract equilibrium to study the robustness to multilateral renegotiations, and relate it to the notion of rational contracts.

Chapter 2

Coordinating Efforts of Multiple Retailers

2.1 Introduction

In many supply chains, retailers invest in product advertisement, improved product display or customer service representatives to influence customer demand for certain products. For example, in the automobile industry, local car dealers may buy TV commercials, place advertisements in newspapers, and hire many sales representatives to increase their demands and sales. In other industries, the quality of service provided by the retailer may directly affect demand. For example, according to Agnese (2006), *"As grocery chains focus on quality, selection and convenience – not just pricing – Wal-Mart may become less of a threat"*. All these efforts, which come at a cost to the retailer, benefit the supplier and either help or hurt other retailers offering the same product.

Of course, suppliers would like retailers to invest in as much effort as possible to promote their products, because they are not paying for the effort that increases their demand. On the other hand, retailers may not be willing to invest resources in this costly effort. This conflict of interest has been observed in the partnership between multinational manufacturers and local distributors. According to Arnold (2000), as multinational manufacturer expand into a new market, they typically partner with local distributors. However, manufacturers typically complain that "*the distributors didn't invest in business growth*". A possible solution, according to Arnold (2000), is "*to create an agreement with strong incentives for appropriate goals*".

The examples above have illustrated the importance of identifying coordinating contracts that motivate retailers to promote the suppliers' products. However, achieving coordination in an effort-dependent demand environment is not simple even for a supply chain with a single supplier and a single retailer. Indeed, it cannot be achieved by revenue sharing (Cachon and Lariviere (2005)) or buyback contract (Taylor (2002)). Thus, the academic community has proposed more sophisticated contracts to coordinate retailer's effort in a supply chain with a single supplier and a single retailer. For example, Taylor (2002) shows that buyback contracts with a per unit target sales rebate is coordinating when (i) the demand is the product of the effort and a uniformly distributed random perturbation, and (ii) the cost of effort is quadratic in the level of effort invested by the retailer.

The existence of multiple retailers adds even more complexity to the problem. One expects that the effort of a specific retailer increases its own demand; however, the impact on the demand faced by other retailers may vary. In the automobile industry case, for example, the TV commercial of one dealer may have a "brand-name promotion effect", and hence may increase the demands faced by other dealers for the same cars. In the grocery industry, however, the improvement of the quality of service of one grocery store may decrease the demands of other stores because customers may switch to the one with better service.

In this chapter, we develop a general framework to model a supply chain with a single supplier and multiple retailers where retailers' efforts have an impact on their own demand as well as the demand faced by others. Demand of every retailer is a function of the effort levels of all retailers and a *random perturbation*. We assume a retailer's demand is stochastically increasing in its effort, but we make no assumption on how a retailer's demand react to other retailers' efforts. Our first objective in this chapter is to identify coordinating contracts, i.e., contracts that achieve global optimization as a Nash equilibrium. Three issues need to be discussed when proposing these contracts.

First, the effort level and cost of effort is not verifiable in many cases. For example, when effort means quality of service provided by the retailers, it is hard to measure and verify the quality of service. Even when effort means advertising, a retailer can invest more or less time/effort in designing the advertising campaign and this time/effort is hard to measure and verify. Therefore, terms of payment or terms of the contract should not depend on the effort level or the cost of effort.

Second, sales and demand may be verifiable, but the realized random perturbation is not verifiable. Therefore, terms of payment or terms of the contract should not depend on the realized random perturbation.

Third, a retailer may not be willing to share its own sales and demand information with other retailers. Hence, terms of payment between the supplier and one retailer should not depend on the demand or sales of other retailers.

We start with revenue sharing contract and show that it is coordinating under some very restrictive conditions. Unfortunately, even under these conditions, revenue sharing is not flexible, that is, it does not allow any allocation of profit between the supplier and the retailers. Hence, we propose a fixed target rebate to go with the revenue sharing contracts. Under a fixed target rebate, the retailer receives a fixed rebate amount from the supplier if the performance reaches certain level. Interestingly, this fixed target rebate is similar to a type of "promotional allowances" commonly used in the food industry (Calvin and Cook (2001); Stecklow et al. (2003)).

Two common forms of promotional allowances are applied in the food industry. In the first type, the supplier promises to rebate a percentage of a product's cost if the the retailer achieves a certain level of sales performance. However, this type of rebate often results in large retailers negotiating huge rebates for aggressive targets, purchasing huge volume and then selling at very low prices, or even below costs. According to a report by the U.S. Department of Agriculture (Calvin and Cook (2001)), many fresh food suppliers regard this type of allowance *"as harmful or neutral rather than beneficial"*.

The second type of promotional allowances is a fixed fee paid by the supplier to the retailer. Sixty two percent of the fresh fruit and vegetables shippers have offered or received requests for this type of allowance (Calvin and Cook (2001)). This allowance is usually paid upfront, and there may be an associated sales commitment by the retailer. If this type of allowance is tied to a commitment, the suppliers *"may gain, for example, if an advertisement for a product stimulates demand". Our fixed target rebate is exactly equivalent to this type of allowance with commitment*, and we will show that revenue sharing with fixed target rebate is coordinating and flexible under certain conditions.

Our second objective in this chapter is to identify classes of contracts that cannot coordinate the supply chain under certain conditions. For this purpose, we introduce two classes of contracts, namely the monotone contracts, and the quantity and sales only contracts. We identify conditions under which certain contracts are coordinating and characterize classes of contracts that will not coordinate if some or all of these conditions are violated.

The chapter is organized as follows. Section 2.2 provides a review of related literature. Section 4.2 presents our model and introduces the notions of *facilitator* and *competitor*. In Section 2.4, we study three types of contracts and identify conditions under which they are coordinating. Section 2.5 provides an analysis of these conditions and identify classes of contracts which cannot coordinate when these conditions are violated. Section 4.5 provides a summary together with some concluding remarks.

2.2 Literature Review

The literature on supply contracts has been growing in the last few years, see Cachon (2003) and Lariviere (1999) for recent reviews. Significant work has been done on identifying coordinating contracts in decentralized supply chains with a single supplier and a single retailer. Examples include buyback contracts (Pasternack (1985)), revenue sharing contracts (Pasternack (2002); Cachon and Lariviere (2005)) and quantity flexibility contracts (Tsay (1999)). Most of these contracts are motivated by double marginalization, see Spengler (1950).

The impact of retailer's effort on supply contracts has also received some attention in the literature. For example, Netessine and Rudi (2004) studies an on-line retailer responsible for advertising and promoting products while the supplier manages inventory and delivers to the consumer. They show that coordination is achieved when the supplier shares a portion of the retailer's advertising cost. This implies, that in this model, the retailer's effort cost can be verified by the supplier.

Krishnan et al. (2004) studies a similar model where demand faced by the retailer is a function of its effort and a random perturbation, and the effort is decided after the random perturbation is realized. They analyze a contract similar to the one in Netessine and Rudi (2004) as well as two other coordinating contracts, namely markdown allowance and constrained buyback. The markdown allowance contract requires the random perturbation to be observable by both the retailer and the supplier, while the constrained buyback requires the random perturbation to be verifiable by the supplier, or by an outside third party.

Taylor (2002) analyzes a single retailer single supplier model and shows that a buyback contract with a per unit target sales rebate is coordinating when the demand is the product of the effort and a uniformly distributed random perturbation, and the effort cost is quadratic in the effort level. Interestingly, this contract does not require the effort (as in Netessine and Rudi (2004) and Krishnan et al. (2004)) or the random perturbation (as in Krishnan et al. (2004)) to be verifiable.

In addition to contract agreements between firms, the effort level is also an important issue in wage contracts involving managerial efforts and salesforce incentives. Research in this area typically assumes a principal agent model in which the principal decides the agent's wage contract. A number of economists have studied this problem to identify optimal contract structure for the principal (Mirrless (1976); Holmstrom (1979); Grossman and Hart (1983); Rogerson (1985); Bolton and Dewatripont (2005)). Chen (2000, 2005) extends the model and incorporates the inventory decisions of the principal.

Another line of research in supply contracts important to our study is the impact of multiple, sometimes competing, retailers on the contract structure and the retailers strategy. Here, the focus is typically on the impact of the retailers pricing strategy, not their effort level, on the behavior of other retailers as well as on the structure of the contract. For example, Bernstein and Federgruen (2005) studies price competition among multiple retailers and shows that buyback contracts with price dependent wholesale and buyback prices are coordinating.

Chen et al. (2006) analyze a model similar to the one in Bernstein and Federgruen (2005) and observe that not all coordinating contracts are implementable. Thus, they introduce the notion of rational contracts, which reflects the agents bargaining power. This bargaining power is directly related to the level of loyalty customers demonstrate for each retailer. They show that revenue sharing with price rebate is coordinating and allows any distribution of the system profit among the players, as long as this distribution satisfies certain requirement specified by the agents relative bargaining powers.

A different line of research that combines contracts and multiple retailers focuses on inventory-based competition or cooperation among retailers. For example, in Anupindi et al. (2001) retailers are tied together not through price competition, but rather through transhipment of inventory after demand is realized. The authors identify allocation mechanisms under which the system optimal procurement strategy is a Nash equilibrium, and the optimal system transhipment strategy lies in the core.

Anupindi and Bassok (1999) and Netessine and Rudi (2002) both study a model in which during retails stock-out period, customers may search for the same or similar product at a different retailer. In Anupindi and Bassok (1999), the authors compare the equilibrium inventory strategy of a decentralized system with two retailers, i.e., a system in which each retailer decides on its inventory level independently of the other retailer, to a centralized system, i.e., one in which retailers inventory is shared by both. They show that the manufacturer prefers a decentralized system when, during stock-out, the search level is high, while the retailers always prefer a centralized system. Netessine and Rudi (2002) perform a similar comparison between centralized and decentralized inventory strategy for a supply chain with any number of retailers, and show that the inventory levels in a centralized system can be higher or lower than that in a decentralized system depending on the demand structure.

Finally, Wang and Gerchak (2001) analyzed a model where the retailer's demand depends on its inventory level. They find coordination mechanisms for the single retailer case, and show that coordination is impossible when two retailers co-exist.

2.3 The Model

Consider a single supplier selling to n retailers, $i = 1, \ldots, n$, each facing an uncertain demand D_i. At the beginning of the period, after finalizing the contract with the supplier but before demand is realized, every retailer decides the level of effort, e_i, in addition to the order quantity q_i. The supplier will produce and deliver after receiving orders from the retailers. We assume the contracts are binding, i.e., the delivery quantity to retailer i equals its order quantity, q_i. The demand of each retailer is realized after all deliveries are made.

We model the impact of the retailers effort on the demand of each retailer by a function, $D_i(\boldsymbol{e}, \epsilon_i)$, of the effort profile $\boldsymbol{e} = \{e_1, \ldots, e_n\}$ and a random variable ϵ_i which is independent of $\boldsymbol{e}$. $D_i(\cdot)$ has a distribution function $F_i(\cdot|\boldsymbol{e})$ and density function $f_i(\cdot|\boldsymbol{e})$, both differentiable in $\boldsymbol{e}$. We also assume the following:

Assumption 2.3.1 *(a) $D_i(\boldsymbol{e}, \epsilon_i)$ is non-decreasing and concave in e_i.*

(b) $F_i(\cdot|\boldsymbol{e})$ is non-increasing in e_i.

(c) $F_i(\cdot|\boldsymbol{e})$ is convex in e_i.

Assumption 2.3.1 implies that the demand of each retailer is stochastically non-decreasing in its own level of effort, but the marginal payoff of effort is stochastically decreasing. Assumption 2.3.1 (a) is very general. For example, the multiplicative variable model, $D_i(\boldsymbol{e}, \epsilon_i) = d_i(\boldsymbol{e})\epsilon_i$, and additive variable model, $D_i(\boldsymbol{e}, \epsilon_i) = d_i(\boldsymbol{e}) + \epsilon_i$ both satisfy this condition when $d_i(\boldsymbol{e})$ is non-decreasing and concave in e_i.

Assumption 2.3.1(b) is the stochastic dominance condition (SDC) while assumption 2.3.1(c) is the convexity of the distribution function condition (CDFC). SDC and CDFC have been always assumed in the economics literature studying effort and moral hazard effects (Mirrless (1976); Holmstrom (1979); Rogerson (1985); Bolton and Dewatripont (2005)). SDC is very general, and it is implied by Assumption 2.3.1(a) in the multiplicative variable and the additive variable models. Although CDFC is quite restrictive, it is necessary for analytical results even for simple principal-agent models (Rogerson (1985), Bolton and Dewatripont (2005)). When demand follows additive variable model or multiplicative model, then CDFC is satisfied if $d_i(\cdot)$ is non-decreasing in e_i, and ϵ_i follows a uniform distribution. Hence, the CDFC assumption is less restrictive than the assumptions made in Taylor (2002), which assumes $d_i(\boldsymbol{e}) = e_i\epsilon_i$ where ϵ_i is uniformly distributed. In fact, in Appendix A.1 we show that Assumption 2.3.1, including CDFC, is satisfied with a multiplicative variable model when the distribution of ϵ_i follows truncated version of some commonly used distributions, e.g., Exponential, Gamma and Normal.

We make no assumption on the impact of a retailer's level of effort on other retailers' demand other than differentiability. Therefore, the model allows one retailer's effort to increase or decrease the demand of other retailers.

The supplier produces and delivers to each retailer at a per-unit cost c_i, while each retailer sells at a price of p_i per unit. Moreover, each retailer has a cost $g_i(e_i)$ for exerting an effort level of e_i, where $g_i(e_i)$ is differentiable, increasing and convex in e_i. Hence, it is increasingly more expensive to exert every unit of effort.

We assume that the supplier and all retailers are risk neutral, i.e., the objective of every agent is to maximize its expected profit. Hence, coordination is achieved by maximizing the total expected profit, given by

$$\Pi(\boldsymbol{q}, \boldsymbol{e}) = \sum_{i=1}^{n} [p_i S_i(q_i, \boldsymbol{e}) - c_i q_i - g_i(e_i)],$$

where $S_i(q_i, \boldsymbol{e}) = q_i - \int_0^{q_i} F_i(d_i|\boldsymbol{e})\partial d_i$ is the expected sales of retailer i.

It will be unreasonable for system optimality to be achieved when one retailer does not order anything or orders an infinite quantity. It would also be unrealistic to expect that any retailer will exert infinite effort level at system optimal. Hence, we assume that any $(\boldsymbol{q}^*, \boldsymbol{e}^*)$ which maximizes $\Pi(\boldsymbol{q}, \boldsymbol{e})$ has a finite and strictly positive $\boldsymbol{e}^*$ and $\boldsymbol{q}^*$. Then, any optimal $(\boldsymbol{q}^*, \boldsymbol{e}^*)$ must satisfy the first-order optimality conditions, given by

$$\frac{\partial \Pi(\boldsymbol{q}^*, \boldsymbol{e}^*)}{\partial e_i} = \sum_{j=1}^{n} p_j \frac{\partial S_j(q_j^*, \boldsymbol{e}^*)}{\partial e_i} - g_i'(e_i^*) = 0, \; i = 1, \ldots, n, \tag{2.1}$$

and

$$\frac{\partial \Pi(\boldsymbol{q}^*, \boldsymbol{e}^*)}{\partial q_i} = p_i \frac{\partial S_i(q_j^*, \boldsymbol{e}^*)}{\partial q_i} - c_i = 0, \; i = 1, \ldots, n. \tag{2.2}$$

Observe that whether some given $(\boldsymbol{q}, \boldsymbol{e})$ satisfies the first-order conditions depends on $\frac{\partial S_j(\boldsymbol{q},\boldsymbol{e})}{\partial e_i}$, which is the rate of change of the expected sales of retailer j when retailer i increases its effort. We therefore introduce the notions of facilitator and competitor, depending on the values of $\frac{\partial S_j(\boldsymbol{q},\boldsymbol{e})}{\partial e_i}$.

Definition 2.3.2 *(1) Retailer i is a* competitor to the rest of the market *at $(\boldsymbol{q}', \boldsymbol{e}')$ if $\sum_{j \neq i} p_j \frac{\partial S_j(q_j', \boldsymbol{e}')}{\partial e_i} < 0$, i.e. the total expected sales revenue of other retailers will decrease if this retailer increases its effort at the current quantity and effort level.*

(2) Retailer i is a facilitator to the rest of the market *at $(\boldsymbol{q}', \boldsymbol{e}')$ if $\sum_{j \neq i} p_j \frac{\partial S_j(q_j', \boldsymbol{e}')}{\partial e_i} > 0$, i.e. the total expected sales revenue of other retailers will increase if this retailer increases its effort at the current quantity and effort level.*

(3) Retailer i is a self-facilitator *at $(\boldsymbol{q}', \boldsymbol{e}')$ if $\frac{\partial S_i(q_i', \boldsymbol{e}')}{\partial e_i} > 0$, i.e. the retailer's own expected sales volume will strictly increase if its effort increase at the current quantity and effort level.*

(4) Retailer i is a partial self-facilitator *at $\boldsymbol{e}'$ if it is a self facilitator at $(\boldsymbol{q}, \boldsymbol{e}')$ for some $\boldsymbol{q}$.*

Hence, a retailer is a competitor(facilitator) to the rest of the market at certain effort and quantity level if its effort has a strictly negative(positive) impact to the aggregated expected sales revenue of all other retailers at that effort and quantity level. A retailer is a self-facilitator at certain effort and quantity level if its effort has a strictly positive impact on its expected sales revenue at that effort and quantity level. A retailer is a partial self-facilitator at certain effort and quantity level if its effort has a strictly positive impact on its expected sales revenue at the same effort but a different quantity level.

Observe that a retailer can be a partial self-facilitator but not a facilitator. For example, if the retailer's inventory level is too low, so that there will always be a shortage, then effort has no impact on the current expected sales revenue. However, effort will have an impact if the retailer orders more.

Observe also that these notions apply to a specific retailer at a specific strategy $(\boldsymbol{q}, \boldsymbol{e})$. At any $(\boldsymbol{q}, \boldsymbol{e})$, it is possible that some retailers are facilitators to the rest of the supply chain, while some other retailers are competitors. Furthermore, a retailer may be a competitor to the rest of the market at some $(\boldsymbol{q}, \boldsymbol{e})$ and a facilitator to the rest of the market at some $(\boldsymbol{q}', \boldsymbol{e}')$. The next lemma shows that there is a relationship between being a self-facilitator, or a facilitator to the rest of the supply chain, at a system optimal strategy.

Lemma 2.3.3 *Let $(\boldsymbol{q}^*, \boldsymbol{e}^*)$ be a system optimal strategy. If retailer i is not a facilitator to the rest of the market at $(\boldsymbol{q}^*, \boldsymbol{e}^*)$, then it must be a self facilitator at $(\boldsymbol{q}^*, \boldsymbol{e}^*)$.*

Proof. Since $(\boldsymbol{q}^*, \boldsymbol{e}^*)$ is a system optimal strategy, it must satisfy the first order condition given by (2.1), or equivalently,

$$\frac{\partial S_i(q_i^*, \boldsymbol{e}^*)}{\partial e_i} = -\sum_{j \neq i}^{n} p_j \frac{\partial S_j(q_j^*, \boldsymbol{e}^*)}{\partial e_i} + g_i'(e_i^*) = 0, \; i = 1, \ldots, n.$$

Suppose retailer i is not a facilitator at $(\boldsymbol{q}^*, \boldsymbol{e}^*)$. Then $-\sum_{j \neq i}^{n} p_j \frac{\partial S_j(q_j^*, \boldsymbol{e}^*)}{\partial e_i} + g_i'(e_i^*) > 0$ because $\sum_{j \neq i}^{n} p_j \frac{\partial S_j(q_j^*, \boldsymbol{e}^*)}{\partial e_i} \leq 0$ and $g_i'(e_i^*) > 0$. Hence, $\frac{\partial S_i(q_i^*, \boldsymbol{e}^*)}{\partial e_i} > 0$ and so retailer i is self-facilitator. ■

Lemma 2.3.3 implies that a retailer has to be either a facilitator to the rest of the market, or a self-facilitator, or both, at any system optimal strategy. This is quite intuitive. If a retailer is neither a facilitator to the rest of the market nor a

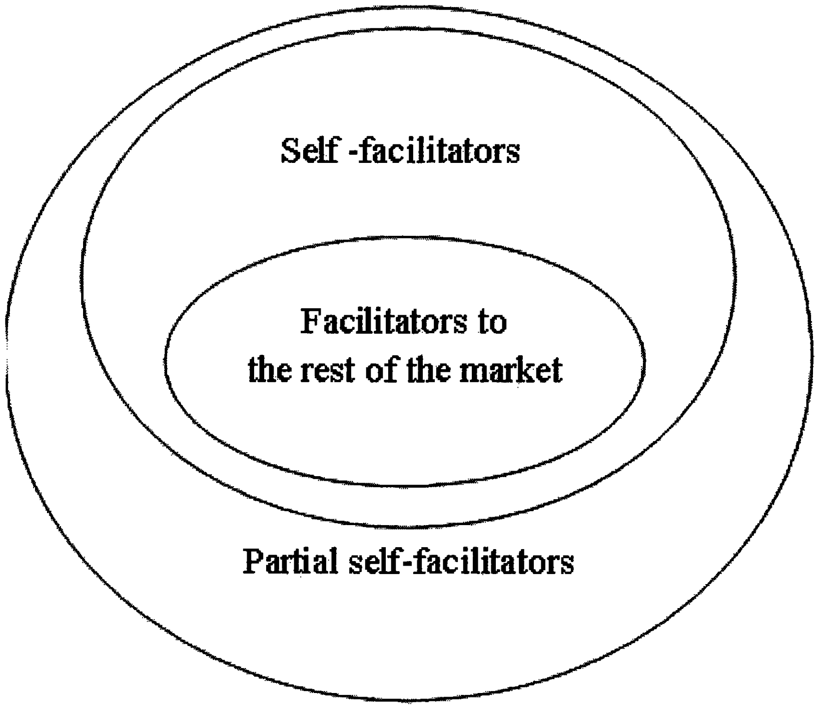

Figure 2-1: Different types of retailers at every system optimal $(\boldsymbol{q}^*, \boldsymbol{e}^*)$

self-facilitator at some strategy $(\boldsymbol{q}, \boldsymbol{e})$, then decreasing its effort by a tiny amount will not hurt its expected sales revenue or the expected sales revenue of the rest of the supply chain, but it will decrease the effort cost. Hence, decreasing the retailer's effort by a tiny amount will increase the total system expected profit, which implies that the given strategy is not system optimal.

Lemma 2.3.3 together with Definition 2.3.2 implies a relationship among different types of retailers for every system optimal $(\boldsymbol{q}^*, \boldsymbol{e}^*)$, as shown in Figure 2-1. If a retailer is not a facilitator to the rest of the market, it must be a self-facilitator and hence also a partial self-facilitator. If a retailer is a self-facilitator, it must also be a partial self-facilitator, and maybe a facilitator to the rest of the market. Finally, if a retailer is a partial self-facilitator, it may be a facilitator to the rest of the market and not a self-facilitator.

The following lemma identifies a property of the demand distribution function of a self-facilitator.

Lemma 2.3.4 *Retailer i is a self-facilitator at $(\boldsymbol{q}', \boldsymbol{e}')$ if and only if there exists*

$q_i'' \leq q_i'$ *such that* $F_i(q_i''|\boldsymbol{e})$ *is strictly decreasing at* $\boldsymbol{e}'$, *i.e.*, $\frac{\partial F_i(q_i''|\boldsymbol{e}')}{\partial e_i} < 0$.

Proof. For one direction, suppose on the contrary that $\frac{\partial F_i(q_i''|\boldsymbol{e}')}{\partial e_i} = 0$ for all $q_i'' \leq q_i'$. Then,

$$\frac{\partial S_i(q_i', \boldsymbol{e}')}{\partial e_i} = -\int_0^{q_i} \frac{\partial F_i(d_i|\boldsymbol{e})}{\partial e_i} \partial d_i = 0,$$

and since this is not strictly positive, retailer i cannot be a self-facilitator. For the other direction, suppose there exists $q_i'' \leq q_i'$ such that $\frac{\partial F_i(q_i''|\boldsymbol{e}')}{\partial e_i} < 0$. Since $F_i(x|\boldsymbol{e})$ is twice differentiable in $(x, \boldsymbol{e})$, there exists $\tilde{q}_i$ such that $\frac{\partial F_i(q_i|\boldsymbol{e}')}{\partial e_i} < 0$ for all $\tilde{q}_i \leq q_i \leq q_i''$. Then,

$$\frac{\partial S_i(q_i', \boldsymbol{e}')}{\partial e_i} = -\int_0^{\tilde{q}_i} \frac{\partial F_i(d_i|\boldsymbol{e})}{\partial e_i} \partial d_i - \int_{\tilde{q}_i}^{q_i''} \frac{\partial F_i(d_i|\boldsymbol{e})}{\partial e_i} \partial d_i - \int_{q_i''}^{q_i'} \frac{\partial F_i(d_i|\boldsymbol{e})}{\partial e_i} \partial d_i > 0,$$

■

2.4 Coordinating Contracts

In this section we study three types of contracts and identify conditions under which each of these contracts is coordinating.

2.4.1 Revenue Sharing Contracts

First, we study the case when the supplier and each retailer are engaged in a revenue sharing contract. In this case, every retailer pays the supplier a wholesale price, w_i, for each unit of product ordered, and a proportion, α_i, of its sales revenue. Hence, the expected profit of retailer i is

$$\pi_i^r(\boldsymbol{e}, q_i) = (1 - \alpha_i) p_i S_i(q_i, \boldsymbol{e}) - g_i(e_i) - w_i q_i$$

The expected profit of retailer i is continuous and concave in e_i and q_i. Hence, there exists a pure strategy which is Nash equilibrium, and the following first-order conditions are necessary and sufficient.

$$(1-\alpha_i)p_i\frac{\partial S_i(q_i,\boldsymbol{e})}{\partial e_i} - g_i'(e_i) = 0, i = 1,\ldots,n, \tag{2.3}$$

$$(1-\alpha_i)p_i\frac{\partial S_i(q_i,\boldsymbol{e})}{\partial q_i} - w_i = 0, i = 1,\ldots,n. \tag{2.4}$$

Theorem 2.4.1 *For any system optimal strategy $(\boldsymbol{q}^*,\boldsymbol{e}^*)$ where every retailer is not a facilitator to the rest of the market (and hence is a self-facilitator by Lemma 2.3.3), there exists exactly one revenue sharing contract such that $(\boldsymbol{q}^*,\boldsymbol{e}^*)$ is a Nash equilibrium.*

The Theorem thus implies that revenue sharing contracts are coordinating when there exists a system optimal strategy where every retailer is not a facilitator to the rest of the market.

Proof. Consider a revenue sharing contract (α_i, w_i) between the supplier and each retailer that satisfies

$$1-\alpha_i = \frac{w_i}{c_i} \equiv \lambda_i, \tag{2.5}$$

and

$$-\alpha_i p_i \frac{\partial S_i(q_i^*,\boldsymbol{e}^*)}{\partial e_i} = \sum_{j\neq i} p_j \frac{\partial S_j(q_j^*,\boldsymbol{e}^*)}{\partial e_i}. \tag{2.6}$$

It is easy to verify that $1 > \alpha_i \geq 0$ and $w_i > 0$, and hence all (α_i, w_i) are valid revenue sharing contracts.

Now, since $(\boldsymbol{q}^*,\boldsymbol{e}^*)$ is system optimal, it satisfies the first order conditions given by (2.1) and (2.2). Substituting (2.5) and (2.6) will give (2.3) and (2.4), implying that $(\boldsymbol{q}^*,\boldsymbol{e}^*)$ is a Nash equilibrium under the given revenue sharing contract.

To show that only one such contract exists, suppose $(\boldsymbol{q}^*,\boldsymbol{e}^*)$ is a Nash equilibrium under revenue sharing contracts (α_i, w_i). Then, the system optimal strategy $(\boldsymbol{q}^*,\boldsymbol{e}^*)$ satisfies the conditions given by (2.3) and (2.4). Comparing with the system optimal necessary conditions (2.1) and (2.2), it is easy to show that equations (2.5) and (2.6) must be satisfied. Equation (2.6) uniquely determines α_i, and then combined with (2.5), w_i is also uniquely determined. ∎

Under the coordinating revenue sharing contract given by (2.5) and (2.6), the expected profit of retailer i is:

$$\pi_i(\mathbf{e}^*, q_i^*) = \lambda_i[p_i S_i(q_i^*, \mathbf{e}^*) - c_i q_i^*] - g_i(e_i^*),$$

where

$$\lambda_i = 1 - \alpha_i = 1 + \frac{1}{p_i \frac{\partial S_i(q_i^*, \mathbf{e}^*)}{\partial e_i}} \sum_{j \neq i} p_j \frac{\partial S_j(q_j^*, \mathbf{e}^*)}{\partial e_i}$$

This implies,

Corollary 2.4.2 *Under a coordinating revenue sharing contract, the more competitive retailer i is to the rest of the supply chain at the optimal strategy, or equivalently the smaller $\sum_{j \neq i} p_j \frac{\partial S_j(q_j^*, \mathbf{e}^*)}{\partial e_i}$ is, the smaller the expected profit of retailer i.*

To understand the intuition behind this property, observe that when a retailer is too competitive, additional effort by this retailer may hurt the system total profit. The only way to motivate the retailer to decrease its effort is by reducing the retailer's marginal return of effort. In revenue sharing contracts, this can only be done through decreasing the retailer's proportion of revenue share, i.e., decreasing $1 - \alpha_i$, or equivalently, decreasing λ_i. Hence, this implies a decrease in the retailer's expected profit.

In the special case when retailer i is not a competitor to the rest of the market, retailer i receives all the expected system profit due to its own sales, that is, $\alpha_i = 0$ and $w_i = c_i$. This is true, because when i is neither a facilitator nor a competitor, the right hand side of (2.6) equal to zero. Thus, in this case, the coordinating contract reduces to a wholesale price contract where the wholesale price is the same as the production cost. This is exactly the situation in the case of a supply chain with a single retailer and a single supplier. This observation agrees with the results of Cachon and Lariviere (2005) that revenue sharing contracts do not coordinate supply chains with one retailer and one supplier with effort effect, since wholesale price contract with $w_i = c_i$ is not regarded as a valid revenue sharing contract. Yet, it is important to observe that the coordinating revenue sharing contract reduces to a wholesale price contract only when a retailer i is not a competitor to the rest of the market. In all other cases, the coordinating revenue sharing contracts generate positive profits for the supplier.

Unfortunately, Theorem 2.4.1 has two limitations. First, it assumes that every retailer is not a facilitator to the rest of the market at the system optimal strategy. In section 2.5.1, we show that many commonly used contracts, including revenue sharing and buy back, do not coordinate the supply chain if this condition is not satisfied, i.e., if at least one of the retailers is a facilitator to the rest of the market. Specifically, in Section 2.5.1, we show that even if one of the retailers is a facilitator to the rest of the market, then revenue sharing and buyback contracts do not coordinate.

A different important limitation is that the set of coordinating revenue sharing contracts does not allow any flexibility in the distribution of profit. Hence, a side payment may be needed to guarantee a win-win situation.

These two limitations of revenue sharing contracts motivate us to propose other contracts that address these issues.

2.4.2 Revenue Sharing Contract with Fixed Target Sales Rebate

Under a revenue sharing contract with fixed target sales rebate, every retailer i pays the supplier a wholesale price w_i for every unit of product and a fixed cost K_i for the entire order, and shares a proportion α_i of its sales revenue with the supplier. In return, the retailer receives a fixed rebate of R_i if the sales reaches a target level T_i.

If retailer i orders no less than the target quantity, i.e., $q_i \geq T_i$, then its expected profit is given by

$$\overline{\pi}_i^T(q_i, \boldsymbol{e}) = (1 - \alpha_i)p_i S_i(q_i, \boldsymbol{e}) - w_i q_i + R_i[1 - F_i(T_i|\boldsymbol{e})] - g_i(e_i) - K_i. \tag{2.7}$$

Otherwise, if retailer i orders less than the target sales quantity, i.e, $q_i < T_i$, then the contract is just the same as a revenue sharing contract with a fixed ordering cost, and the expected profit of the retailer is given by

$$\underline{\pi}_i^T(q_i, \boldsymbol{e}) = (1 - \alpha_i)p_i S_i(q_i, \boldsymbol{e}) - w_i q_i - g_i(e_i) - K_i. \tag{2.8}$$

It is obvious that $\overline{\pi}_i^T(q_i, \boldsymbol{e}) \geq \underline{\pi}_i^T(q_i, \boldsymbol{e})$ for all $\boldsymbol{q}$ and all $\boldsymbol{e}$. Therefore, if some (q_i, e_i) maximizes $\overline{\pi}_i^T(q_i, \boldsymbol{e})$ given $\boldsymbol{e}_{-i}$ and $q_i \geq T_i$, then (q_i, e_i) is the best response for $\boldsymbol{e}_{-i}$.

Since $\overline{\pi}^T(q_i, \boldsymbol{e})$ is concave in (q_i, e_i) for all $(q_i, \boldsymbol{e})$, the following first-order conditions of equation (2.7) are sufficient for Nash equilibrium for $(\boldsymbol{q}, \boldsymbol{e})$ where $q_i \geq T_i$ for all i:

$$(1 - \alpha_i)p_i \frac{\partial S_i(q_i, \boldsymbol{e})}{\partial e_i} - R_i \frac{\partial F_i(T_i|\boldsymbol{e})}{\partial e_i} - g_i'(e_i) = 0 \tag{2.9}$$

$$(1 - \alpha_i)p_i \frac{\partial S_i(q_i, \boldsymbol{e})}{\partial q_i} - w_i = 0. \tag{2.10}$$

Theorem 2.4.3 *For every system optimal strategy $(\boldsymbol{q}^*, \boldsymbol{e}^*)$ where every retailer is a self-facilitator, there exists a revenue sharing contract with fixed target sales rebate such that $(\boldsymbol{q}^*, \boldsymbol{e}^*)$ is a Nash equilibrium. Hence, revenue sharing contract with fixed target sales rebate is coordinating if there exists a system optimal strategy where every retailer is a self-facilitator.*

Proof. Consider a revenue sharing contract with fixed target sales rebate such that for every retailer i, $(\alpha_i, w_i, R_i, T_i, K_i)$, we have:

$$T_i \leq q_i^*, \tag{2.11}$$

$$R_i \frac{\partial F_i(T_i|\boldsymbol{e}^*)}{\partial e_i} = -\sum_{j \neq i} p_j \frac{\partial S_j(q_j^*, \boldsymbol{e}^*)}{\partial e_i} - \alpha_i p_i \frac{\partial S_i(q_i^*, \boldsymbol{e}^*)}{\partial e_i}, \tag{2.12}$$

and

$$1 - \alpha_i = \frac{w_i}{c_i} = \lambda_i. \tag{2.13}$$

It is easy to verify using Lemma 2.3.4 that there exits $(\alpha_i, w_i, R_i, T_i, K_i)$ that satisfies the above conditions and $0 < \alpha_i < 1$, $w_i > 0$ and $R_i \geq 0$.

Since $(\boldsymbol{q}^*, \boldsymbol{e}^*)$ is system optimal, it must satisfy (2.1) and (2.2). Together with the contracts defined by (2.11), (2.12) and (2.13), it is easy to show that $(\boldsymbol{q}^*, \boldsymbol{e}^*)$ satisfies

(2.9) and (2.10) and hence is a Nash equilibrium under the given contract. ■

In Section 2.4.1, we discussed two disadvantages of standard revenue sharing contracts. First, revenue sharing contracts are coordinating only when there exists a system optimal strategy where every retailer is not a facilitator to the rest of the market. Adding a fixed target sales rebate and a fixed ordering cost removes this disadvantage, because coordination can now be achieved at the system optimal strategy even if one or more retailers are facilitators to the rest of the supply chain. The only condition required for coordination is that every retailer is a self-facilitator at a system optimal.

Another disadvantage of revenue sharing contracts is that it allows only one allocation of profit. The next theorem shows that the fixed target sales rebate provides total flexibility to the contract, i.e., the expected system profit due to the selling through one retailer can be arbitrarily allocated between this retailer and the supplier.

Theorem 2.4.4 *Let $(\boldsymbol{q}^*, \boldsymbol{e}^*)$ be a system optimal strategy where every retailer is a self-facilitator, and $\pi^{i*} = p_i S_i(q_i^*, \boldsymbol{e}^*) - c_i q_i^* - g_i(e_i^*)$ be the expected system profit due to selling through retailer i at this optimal $(\boldsymbol{q}^*, \boldsymbol{e}^*)$. For all $0 \leq \kappa_i \leq \pi^{i*}$, there exists a revenue sharing contract with fixed target sales rebate such that $(\boldsymbol{q}^*, \boldsymbol{e}^*)$ is a Nash equilibrium and the expected profit of retailer i at $(\boldsymbol{q}^*, \boldsymbol{e}^*)$ is exactly κ_i at $(\boldsymbol{q}^*, \boldsymbol{e}^*)$.*

Proof. From the proof of Theorem 2.4.3, we know that the contract is coordinating if (2.11), (2.12) and (2.13) are satisfied. For every retailer i, we consider two cases:

Case 1: Retailer i is a competitor to the rest of the market at $(\boldsymbol{q}^*, \boldsymbol{e}^)$. By the proof of Theorem 2.4.1, we can see that there exists $0 < \alpha_i < 1$ and $w_i > 0$ such that (2.13) is satisfied and $-\sum_{j \neq i} p_j \frac{\partial S_j(q_j, e)}{\partial e_i} - \alpha_i p_i \frac{\partial S_i(q_i, e)}{\partial e_i} = 0$. Pick $T_i \leq q_i$ such that $F_i(T_i|\boldsymbol{e}^*) = 0$ and $\frac{\partial F_i(T_i|e^*)}{\partial e_i} = 0$. Then, (2.11), (2.12) and (2.13) are satisfied for any R_i and K_i. The profit of retailer i can then be any arbitrary number by picking the correct R_i and K_i.

Case 2: Retailer i is not a competitor to the rest of the market at $(\boldsymbol{q}^*, \boldsymbol{e}^)$. First pick any $0 \leq \alpha_i \leq 1$. If $-\sum_{j \neq i} p_j \frac{\partial S_j(q_j^*, e^*)}{\partial e_i} - \alpha_i p_i \frac{\partial S_i(q_i^*, e^*)}{\partial e_i} = 0$, pick $T_i \leq q_i^*$ such that $\frac{\partial F_i(T_i|\boldsymbol{e}^*)}{\partial e_i} = 0$ and any R_i. Otherwise, pick $T_i \leq q_i^*$ such that $\frac{\partial F_i(T_i|\boldsymbol{e}^*)}{\partial e_i} < 0$ and let $R_i = \frac{1}{\frac{\partial F_i(T_i|\boldsymbol{e}^*)}{\partial e_i}}[-\sum_{j\neq i} p_j \frac{\partial S_j(q_j^*,\boldsymbol{e}^*)}{\partial e_i} - \alpha_i p_i \frac{\partial S_i(q_i^*,\boldsymbol{e}^*)}{\partial e_i}]$. Then, (2.11), (2.12) and (2.13) are satisfied. For $K_i = 0$, retailer i's profit can be anything between $\pi_i' = \pi^{i*} - \frac{1-F_i(T_i|\boldsymbol{e}^*)}{\frac{\partial F_i(T_i|\boldsymbol{e}^*)}{\partial e_i}} \sum_{j\neq i} p_j \frac{\partial S_j(q_j^*,\boldsymbol{e}^*)}{\partial e_i}$ and $\pi_i'' = \frac{1-F_i(T_i|\boldsymbol{e}^*)}{\frac{\partial F_i(T_i|\boldsymbol{e}^*)}{\partial e_i}}[-\sum_{j\neq i} p_j \frac{\partial S_j(q_j^*,\boldsymbol{e}^*)}{\partial e_i} - p_i \frac{\partial S_i(q_i^*,\boldsymbol{e}^*)}{\partial e_i}] - g_i(e_i^*)$ by picking different α_i. Profit of retailer i can also be anything below this range by picking the correct $K_i > 0$. ■

A few important points about revenue sharing contract with fixed target sales rebate should be mentioned. First, it is sometimes possible to pick T_i such that $F_i(T_i|\boldsymbol{e}^*) = 0$, i.e., the probability that the demand is below the target sales level at the system optimal effort is zero. This does not mean a rebate for the retailer regardless of its effort, because $F_i(T_i|\boldsymbol{e})$ may be positive at other effort profiles. In this case, the retailer is guaranteed the rebate when every retailer is at this optimal effort level, $\boldsymbol{e}^*$.

Second, when the retailer is a facilitator to the rest of the supply chain, R_i is increasing in $\sum_{j\neq i} p_j \frac{\partial S_j(q_j^*,\boldsymbol{e}^*)}{\partial e_i}$, i.e., increasing in the marginal impact of retailer i's effort on other retailers' sales revenue. That is, the more facilitative retailer i is, the higher its target sales rebate. This is intuitive, since the more facilitative retailer i is, the more this retailer helps others. To motivate retailer i to help other retailers, the supplier should give retailer i a higher rebate. Of course, increasing the rebate also increases the retailer's expected profit. Hence, a fixed ordering cost is needed to allow a lower retailer expected profit, and to achieve any allocation of profit. This is similar to the retailer paying a deposit or sales guarantee beforehand.

Interestingly, the contract analyzed in this section is similar to the *promotional allowance with commitment* used extensively in the fresh food industry, see Section 2.1. Indeed, given a desired allocation of profit between each retailer and the supplier, one can set α_i in our contract, and achieve coordination with $R_i > K_i$. This is equivalent to the supplier paying retailer i an allowance of $R_i - K_i$ at the beginning of the period and then charging the retailer with a penalty K_i, if the retailer does not meet the target sales performance. Thus, our fixed target sales rebate is exactly equivalent to the promotional allowance with commitment.

2.4.3 Revenue Sharing Contract with Fixed Target Demand Rebate

We now introduce a revenue sharing contract with fixed target demand rebate. Fixed target demand rebate is similar to fixed target sales rebate, except that retailer i receives the rebate R_i as long as demand is above T_i, regardless of the order and sales quantity. The expected profit of retailer i under this contract is given by

$$\pi_i^{TD}(q_i, \boldsymbol{e}) = (1-\alpha_i)p_i S_i(q_i, \boldsymbol{e}) - w_i q_i + R_i[1 - F_i(T_i|\boldsymbol{e})] - g_i(e_i) - K_i,$$

Since $\pi_i^{TD}(q_i, \boldsymbol{e})$ is concave, first-order conditions, which are the same as the first-order conditions for $\overline{\pi}_i^T(q_i, \boldsymbol{e})$ in (2.9) and (2.10), are necessary and sufficient for Nash equilibrium. The next theorem shows that revenue sharing contract with fixed target demand rebate is coordinating under certain conditions.

Theorem 2.4.5 *Let $(\boldsymbol{q}^*, \boldsymbol{e}^*)$ be a system optimal strategy such that every retailer is a partial self-facilitator at $\boldsymbol{e}^*$, and let $\pi^{i*} = p_i S_i(q_i^*, \boldsymbol{e}^*) - c_i q_i^* - g_i(e_i^*)$ be the expected profit through retailer i at this optimal $(\boldsymbol{q}^*, \boldsymbol{e}^*)$. Suppose $0 \leq \kappa_i \leq \pi^{i*}$. Then, there exists a coordinating revenue sharing contract with fixed target demand rebate such that $(\boldsymbol{q}^*, \boldsymbol{e}^*)$ is a Nash equilibrium and the expected profit of retailer i at $(\boldsymbol{q}^*, \boldsymbol{e}^*)$ is exactly κ_i.*

Proof. Since every retailer is a partial self-facilitator at $\boldsymbol{e}^*$, by Definition 2.3.2 and Lemma 2.3.4 there exists d_i such that $\frac{\partial F_i(d_i|\boldsymbol{e}^*)}{\partial e_i} < 0$ for all i. The rest of the proof is similar to proof of Theorem 2.4.3 and Theorem 2.4.5 except that (2.11) is not required in the coordinating contract, i.e., T_i can be larger than q_i^*. ∎

Theorem 2.4.4 implies that revenue sharing contract with fixed demand rebate is coordinating and flexible if there exists a system optimal strategy where every retailer is a partial self-facilitator. This implies that, unlike revenue sharing contract with fixed target sales rebate, where every retailer must be a self-facilitator at an optimal effort level and quantity. Here, every retailer needs only to be a self-facilitator at some quantity, not necessarily the optimal one. Interestingly, in the next section we show that there does not exist any coordinating contract if this condition is not satisfied.

Finally, it is appropriate to point out that unlike revenue sharing contract with fixed target sales rebate, the implementation of revenue sharing contract with fixed target demand rebate is more difficult. Indeed, this contract is probably applicable for online stores, but much more difficult to implement in conventional stores.

2.5 Conditions for Coordination

In section 2.4, we show that revenue sharing contracts are coordinating when there exists an optimal $(\boldsymbol{q}^*, \boldsymbol{e}^*)$ where every retailer is not a facilitator to the rest of the market. Revenue sharing contracts with fixed target sales rebate coordinate under a less restrictive condition, i.e., when there exists an optimal $(\boldsymbol{q}^*, \boldsymbol{e}^*)$ such that every retailer is a self-facilitator. By using a fixed target demand rebate instead of a fixed target sales rebate, the supply chain can be coordinated under the least restrictive condition, i.e., when there exists an optimal $\boldsymbol{e}^*$ where every retailer is a partial self-facilitator.

In this section, we do not restrict ourselves to any specific type of contract. Instead, we study each of these conditions, i.e., facilitator to the rest of the market, self facilitator, and partial self-facilitator, and identify a certain class of contracts that cannot coordinate when one of these conditions is violated. To achieve this, we model the payment from retailer i to the supplier as $t_i(\boldsymbol{q}, D_i)$, which is a function of the order quantities of all retailers and the demand of retailer i. We limit our analysis to types of contracts which satisfy the following assumption.

Assumption 2.5.1 *For all i, all $\boldsymbol{q}$ and $\boldsymbol{e}$, $\int t_i(\boldsymbol{q}, d_i) f(d_i|\boldsymbol{e}) \partial d_i$ exists.*

This assumption is quite general. For example, one sufficient condition for the assumption to be true is that $t_i(\boldsymbol{q}, d_i) f(d_i|\boldsymbol{e})$ is bounded and has finitely many points of discontinuity in d_i (Rudin (1976)). Many well known contracts, such as revenue sharing contracts, buyback contracts, wholesale price contracts and quantity flexibility contracts, when combined with many commonly used distributions such as uniform, Normal or exponential, satisfy this assumption.

Hence, the expected profit of the retailer i is given by:

$$\pi_i^t(q_i, \boldsymbol{e}) = p_i S_i(q_i, \boldsymbol{e}) - \int t_i(\boldsymbol{q}, d_i) f(d_i|\boldsymbol{e}) \partial d_i - g_i(e_i)$$

The existence of a pure-strategy Nash equilibrium is not guaranteed for any arbitrary contract $t_i(q_i, D_i)$. However, if there exists a pure-strategy Nash equilibrium which is finite and strictly positive, then the pure-strategy Nash equilibrium must satisfy the following first-order condition:

$$p_i \frac{\partial S_i(q_i, \boldsymbol{e})}{\partial e_i} - \frac{\partial}{\partial e_i} \int t_i(q_i, d_i) f(d_i|\boldsymbol{e}) \partial d_i - g_i'(e_i) = 0, i = 1, \ldots, n. \tag{2.14}$$

Finally, it is important to emphasize that in this chapter we do not consider mixed strategies. First, it will be difficult to convince managers that decisions should be made by choosing randomly from a menu of alternatives. Second, if there is a unique system optimal effort and quantity profile, then any mixed strategy is suboptimal. Therefore, contracts that do not achieve system optimum as pure-strategy Nash equilibrium are considered to be not coordinating.

2.5.1 Monotone Contracts

In this subsection, we study a special class of contracts called monotone contracts.

Definition 2.5.2 *A contract $(t_i(\boldsymbol{q}, D_i))$ is monotone if $(t_i(\boldsymbol{q}, D_i))$ is non-decreasing in D_i for all D_i and $\boldsymbol{q}$.*

The definition implies that many commonly used contracts, such as revenue sharing contracts, buyback contracts and quantity flexibility contracts, are monotone contracts. The following lemma shows that under monotone contracts, the expected payment to the supplier is non-decreasing with effort.

Lemma 2.5.3 *A monotone contract $t_i(\boldsymbol{q}, D_i)$ must satisfy*

$$\frac{\partial}{\partial e_i}\int t_i(\boldsymbol{q}, d_i) f_i(d_i|\boldsymbol{e})\partial d_i \geq 0$$

for all $\boldsymbol{q}$ and $\boldsymbol{e}$.

Proof.

$$\frac{\partial}{\partial e_i}\int t_i(\boldsymbol{q}, d_i) f(d_i|\boldsymbol{e})\partial d_i = \frac{\partial}{\partial e_i}\int t_i(\boldsymbol{q}, D_i(\epsilon_i, \boldsymbol{e}))\tilde{f}_i(\epsilon_i)\partial\epsilon_i,$$

where $\tilde{f}_i(\cdot)$ is the density function of ϵ_i.

By Assumption 2.3.1, $D_i(\epsilon_i, \boldsymbol{e})$ is increasing in e_i. Since the contract is monotone, $t_i(\boldsymbol{q}, D_i(\epsilon_i, \boldsymbol{e}))$ is increasing in D_i. Hence, $t_i(\boldsymbol{q}, D_i(\epsilon_i, \boldsymbol{e}))\tilde{f}_i(\epsilon_i)$ is increasing in e_i, and this property is kept by integrating over ϵ_i, implying $\frac{\partial}{\partial e_i}\int t_i(\boldsymbol{q}, d_i) f(d_i|\boldsymbol{e})\partial d_i \geq 0$. ■

The lemma thus implies that the expected payment from a retailer to the supplier under a monotone contract must be non-decreasing in the retailer's effort. This is intuitive, because under a monotone contract, payment is non-decreasing in demand which is non-decreasing in the retailer's effort.

The next theorem characterizes cases in which monotone contracts are not coordinating.

Theorem 2.5.4 *Suppose a retailer is a facilitator to the rest of the supply chain at a system optimal strategy $(\boldsymbol{q}^*, \boldsymbol{e}^*)$, then $(\boldsymbol{q}^*, \boldsymbol{e}^*)$ will not be a Nash equilibrium under any monotone contract.*

Proof. Suppose $(\boldsymbol{q}^*, \boldsymbol{e}^*)$ is a Nash equilibrium under some monotone contracts. Then $(\boldsymbol{q}^*, \boldsymbol{e}^*)$ must satisfy the first-order condition in (2.14), i.e.,

$$p_i\frac{\partial S_i(q_i^*, \boldsymbol{e}^*)}{\partial e_i} - \frac{\partial}{\partial e_i}\int t_i(\boldsymbol{q}^*, d_i) f(d_i|\boldsymbol{e}^*)\partial d_i - g_i'(e_i^*) = 0.$$

Since $(\boldsymbol{q}^*, \boldsymbol{e}^*)$ is system optimal, it must also satisfy (2.1). Comparing the two conditions, we have

$$-\frac{\partial}{\partial e_i}\int t_i(\boldsymbol{q}^*, d_i) f(d_i|\boldsymbol{e}^*)\partial d_i = \sum_{j\neq i} p_j \frac{\partial S_j(q_j^*, \boldsymbol{e})}{\partial e_i}.$$

If retailer i is a facilitator to the rest of the market, then $\sum_{j\neq i} p_j \frac{\partial S_j(q_j^*, \boldsymbol{e})}{\partial e_i} > 0$. Hence $\frac{\partial}{\partial e_i}\int t_i(\boldsymbol{q}^*, d_i) f(d_i|\boldsymbol{e}^*)\partial d_i < 0$ which violates the monotone condition according to Lemma 2.5.3.

■

Theorem 2.5.4 implies that buyback contracts, revenue sharing contracts, quantity discount contracts and quantity flexibility contracts are not coordinating when some (even one) retailers are facilitators to the rest of the market at the system optimal strategy. Since revenue sharing contracts are coordinating when there is a system optimal where no retailer is a facilitator to the rest of the supply chain, a further implication of Theorem 2.5.4 is that no monotone contract can coordinate when a revenue sharing contract is not coordinating. By contrast, revenue sharing contracts with fixed target sales rebate is coordinating when one or more retailer is a facilitator to the rest of the market. This is, of course, consistent with Theorem 2.5.4, since in this case the contract is not monotone.

Before we move on to another class of contracts, let us introduce a sub-class of monotone contracts, called the quantity only contracts.

Definition 2.5.5 *A contract $(t_i(\boldsymbol{q}, D_i))$ is a quantity only contract if terms of payment depends on quantity only, i.e., $t_i(\boldsymbol{q}, D_i)$ can be written as $t_i(\boldsymbol{q})$.*

Since all quantity only contracts are monotone contracts, Theorem 2.5.4 implies that quantity only contracts do not coordinate when some retailers are facilitators to the rest of the market at all system optimal $(\boldsymbol{q}^*, \boldsymbol{e}^*)$. Hence, if we want to achieve a system optimal where at least one retailer is facilitator to the rest of the market, we need more parameters in the contract payment, e.g., revenue sharing contracts with fixed target sales rebate.

2.5.2 Quantity and Sales Only Contract

In this subsection we study a different class of contracts, the quantity and sales only contract defined below.

Definition 2.5.6 *A* quantity and sales only contract *is a contract such that the terms of payment depends on the quantity and sales only, i.e., the payment from retailer i to the supplier can be written as $t_i(\boldsymbol{q}, \min(q_i, D_i))$.*

Definition 2.5.6 implies that almost all contracts used are quantity and sales only contracts. It is usually difficult to implement if the terms of payment in the contract depend on other parameters such as the retailer's demand (and not sales) or other retailers' sales. Indeed, most contracts proposed in literature fall under this class.

Observe that the class of monotone contracts and the class of quantity and sales only contracts maybe quite distinct. For example, the revenue sharing contract with fixed target sale rebate described in Section 2.4.2 belongs to the second class but not the first. The following lemma identifies a property for quantity and sales only contracts.

Lemma 2.5.7 *A quantity and sales only contract $t_i(\boldsymbol{q}', \min(q_i', D_i))$ must satisfy*

$$\frac{\partial}{\partial e_i}\int t_i(\boldsymbol{q}', d_i) f(d_i|\boldsymbol{e}')\partial d_i = 0$$

for all $(\boldsymbol{q}', \boldsymbol{e}')$ where retailer i is not a self-facilitator.

Proof. By Lemma 2.3.4, since retailer i is not a self-facilitator at $(\boldsymbol{q}', \boldsymbol{e}')$, $\frac{\partial F_i(q_i''|e')}{\partial e_i'} = 0$ for all $q'' \leq q'$, it is obvious that $\frac{\partial f_i(q_i''|e')}{\partial e_i} = \frac{\partial}{\partial e_i}\frac{\partial}{\partial q_i}F_i(q_i''|\boldsymbol{e}') = \frac{\partial}{\partial q_i}\frac{\partial}{\partial e_i}F_i(q_i''|\boldsymbol{e}') = 0$ for all $q'' \leq q'$.

Hence,

$$\begin{aligned}
\frac{\partial}{\partial e_i}\int t_i(\boldsymbol{q}', \min(q_i', d_i)) f(d_i|\boldsymbol{e}')\partial d_i &= \frac{\partial}{\partial e_i'}\left[\int_0^{q_i'} t_i(\boldsymbol{q}', d_i) f(d_i|\boldsymbol{e}')\partial d_i + (1 - F_i(q_i'))t_i(\boldsymbol{q}', q_i')\right] \\
&= \int_0^{q_i'} t_i(\boldsymbol{q}', d_i)\frac{\partial f(d_i|\boldsymbol{e}')}{\partial e_i'}\partial d_i - \frac{\partial F_i(q_i')}{\partial e_i'} t_i(\boldsymbol{q}', q_i') \\
&= 0.
\end{aligned}$$

■

Lemma 2.5.7 implies that under a quantity and sales only contract, the expected payment from a retailer to the supplier is not affected by small changes in effort when that retailer is not a self-facilitator. The intuition is that when a retailer is not a self-retailer, small variation in its effort has no impact on its distribution function of its sales, and hence has no impact on the contract payment.

The following theorem identifies conditions under which quantity and sales only contracts cannot coordinate.

Theorem 2.5.8 *Suppose a retailer i is not a self-facilitator at some strategy $(\boldsymbol{q}', \boldsymbol{e}')$ which is finite and strictly positive, then $(\boldsymbol{q}', \boldsymbol{e}')$ cannot be a Nash equilibrium strategy under any quantity and sales only contract.*

Proof. Suppose on the contrary that retailer i is a not a self-facilitator at some Nash equilibrium strategy $(\boldsymbol{q}', \boldsymbol{e}')$ under some contract $t_i(\cdot)$, i.e., $\frac{\partial S_i(q_i', \boldsymbol{e}')}{\partial e_i} = 0$. Then according to the first-order condition in (2.14),

$$\frac{\partial}{\partial e_i} \int t_i(\boldsymbol{q}', d_i) f(d_i|\boldsymbol{e}') \partial d_i = -g_i'(e_i') < 0,$$

implying $t_i(\cdot)$ is not a quantity and sales contract according to Lemma 2.5.7. ■

Theorem 2.5.8 implies that there does not exist any coordinating quantity and sales only contract if there does not exist any system optimal strategy $(\boldsymbol{q}, \boldsymbol{e})$ where every retailer is a self-facilitator. Hence, by Theorem 2.4.3, when revenue sharing contract with fixed target sales rebate is not coordinating, then there does not exist any quantity and sales only contract that can coordinate.

2.5.3 All Contracts

So far we have studied two categories of contracts, namely the monotone contracts and quantity and sales only contracts, and identified conditions under which contracts in these classes cannot coordinate. Here, we expand our focus and study the grand class of all contracts. The next lemma shows a property that every contract must satisfy.

Lemma 2.5.9 *Any contract $t_i(\boldsymbol{q}, D_i)$ must satisfy*

$$\frac{\partial}{\partial e_i}\int t_i(\boldsymbol{q}', d_i) f(d_i|\boldsymbol{e}')\partial d_i = 0$$

for all q_i' and $\boldsymbol{e}'$ such that retailer i is not a partial self-facilitator.

Proof. Suppose retailer i is not a partial self-facilitator. By Definition 2.3.2 and Lemma 2.3.4, $\frac{\partial F_i(q_i''|e')}{\partial e_i'} = 0$ for all q''. Hence,

$$\frac{\partial f_i(q_i''|e')}{\partial e_i} = \frac{\partial}{\partial e_i}\frac{\partial}{\partial q_i}F_i(q_i''|e') = \frac{\partial}{\partial q_i}\frac{\partial}{\partial e_i}F_i(q_i''|\boldsymbol{e}') = 0.$$

Therefore,

$$\frac{\partial}{\partial e_i}\int t_i(\boldsymbol{q}', d_i) f(d_i|\boldsymbol{e}')\partial d_i = \int t_i(\boldsymbol{q}', d_i)\frac{\partial f(d_i|\boldsymbol{e}')}{\partial e_i'}\partial d_i = 0.$$

■

Lemma 2.5.9 implies that when a retailer is not a partial-self facilitator, small variations in a its effort has no impact on its expected payment to the supplier under any contract. This is intuitive. When a retailer is not a partial self-facilitator, small change in its effort has no impact on its demand distribution, and hence no impact on the expected payment which depends only on the order quantity and demand distribution.

The next theorem characterizes conditions under which every contract is not coordinating.

Theorem 2.5.10 *Let $(\boldsymbol{q}', \boldsymbol{e}')$ be a finite and strictly positive system strategy where not all retailers are partial self-facilitators. Then, $(\boldsymbol{q}', \boldsymbol{e}')$ cannot be a Nash equilibrium under any contract $t_i(\boldsymbol{q}, D_i)$.*

Proof. Suppose on the contrary that $(\boldsymbol{q}', \boldsymbol{e}')$ is a Nash equilibrium under some contract $t_i(\cdot)$. Then, the first-order condition in (2.14) must be satisfied, i.e., $p_i \frac{\partial S_i(q_i', \boldsymbol{e}')}{\partial e_i} - \frac{\partial}{\partial e_i} \int t_i(\boldsymbol{q}', d_i) f(d_i|\boldsymbol{e}') \partial d_i - g_i'(e_i') = 0$.

By Definition 2.3.2, we know that retailer i is not a self-facilitator due to the assumption that $\frac{\partial F_i(d_i|\boldsymbol{e}')}{\partial e_i} = 0$ for all d_i. Hence, we have

$$0 = \frac{\partial}{\partial e_i} \int t_i(\boldsymbol{q}', d_i) f(d_i|\boldsymbol{e}') \partial d_i = -g_i'(e_i') < 0,$$

which is a contradiction. ∎

Theorem 2.5.10 implies that if there does not exists an optimal strategy $(\boldsymbol{q}^*, \boldsymbol{e}^*)$ where every retailer is a partial self-facilitator, then there does not exist any coordinating contract. Therefore, by Theorem 2.4.5, when revenue sharing contract with fixed target demand rebate does not coordinate, there does not exist any coordinating contracts.

2.6 Conclusions

In this chapter, we have developed a model to study a decentralized supply chain with a single supplier and multiple retailers where demands depend on the efforts of the retailers. We have introduced the notion of self-facilitator, partial self-facilitator, facilitator to the rest of the supply chain, and competitor to the rest of the supply chain. We show that if a retailer is not a facilitator at a system optimal, then it must be a self facilitator at this optimal, which in turn implies that the retailer is a partial self-facilitator at this optimal. We also introduce two categories of contracts, namely the monotone contracts, and quantity and sales only contracts.

Our analysis show that, when there exists an optimal strategy where every retailer is not a facilitator to the rest of the supply chain, and hence is a self-facilitator, there exists exactly one revenue sharing contract where this optimal is Nash equilibrium. Hence, revenue sharing contract is coordinating. On the other hand, when this condition is violated, i.e. when some retailers are facilitators to the rest of the market, all monotone contracts such as buyback contracts and quantity discount contracts cannot coordinate. Therefore, there exists a coordinating monotone contract if and only if there exits a coordinating revenue sharing contract.

Hence, we propose a new type of contract called the revenue sharing contract with fixed target sales rebate. Interestingly, similar contracts have been used in the fresh food industry. We show that, when there exists an optimal strategy where every retailer is a self-facilitator, then this contract is coordinating and allows arbitrary allocation of profits. We also show that when this condition is violated, then all quantity and sales only contracts, including revenue sharing contracts with fixed target sales rebate, are not coordinating. Therefore, there exists a coordinating quantity and sales only contract if and only if a revenue sharing contract with fixed target sales rebate is coordinating.

The last type of contracts we propose is the revenue sharing contract with fixed target demand rebate. When there exists a system optimal where every retailer is a partial self-facilitator, then revenue sharing contract with fixed target demand rebate is coordinating and allows arbitrary allocation of profits. We also show that there does not exist coordinating contracts when this condition is violated.

All results can be extended into the situation when revenue sharing contracts are substituted by buyback contracts. A summary of our results and the extended results can be found in Table 2.1.

Interestingly, the model analyzed here is quite simple but relatively general. It does not assume any specific function for the demand and the effort cost, nor does it assume any relationship between the effort of one retailer and the demands of other retailers. Although the assumption of CDFC in Assumption 2.3.1(c) is restrictive, it is only required for the coordination and flexibility results in Section 2.4.2 and Section 2.4.3. Furthermore, it is possible that CDFC can be relaxed and substituted by other general assumptions if more demand information is known. For example, in Appendix A.2, we show that CDFC can be relaxed when demand follows a multiplicative variable model.

Condition for optimal $(\boldsymbol{q}^*, \boldsymbol{e}^*)$	Every retailer is not a facilitator to the rest of the market	Every retailer is a self-facilitator	Every retailer is a partial self-facilitator
Condition Satisfied	Standard Revenue sharing & buy-back contracts are coordinating.	Revenue sharing & buyback contracts with fixed target sales rebate are coordinating and flexible	Revenue sharing & buyback contracts with fixed target demand rebate are coordinating and flexible
Condition Violated	No monotone contract is coordinating.	No quantity and sales only contract is coordinating.	No contract is coordinating

Table 2.1: Summary of Results.

One limitation is that our proposed contracts require that sales or demand of each retailer to be verifiable by the supplier. Hence they may not be implementable in supply chains where retailers do not have adequate technology to verify their sales. Indeed, when the supplier and a retailer do not agree over the sales level, there will be disagreement over the payment amount and hence expensive lawsuit may occur. An alternative to sales based contracts is quantity only contracts, which is a sub-class of monotone contracts. Unfortunately, these contracts are not coordinating in many cases. In particular, quantity only contracts do not coordinate when some retailers are facilitators to the rest of the supply chain at all system optimal strategies. Of course, even quantity and sales contracts do not coordinate in all cases.

Another limitation is that our model focuses on the impact of retailers' efforts on the demand and ignores other factors which may also affect the demands, such as supplier's effort, pricing strategies and inventory on the shelves. The analysis here and in related literature (Bernstein and Federgruen (2005); Chen et al. (2006)) suggests that it may be possible to extend our model to these cases as well.

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Chapter 3

Coordinating and Rational Contracts

3.1 Introduction

The literature on supply contracts has been growing in the last few years. Significant work has been done on identifying coordinating contracts in decentralized supply chains with risk neutral decision makers. In that case, a contract is called coordinating if the equilibrium decisions made by the different players maximize the expected total profit of the entire supply chain.

Unfortunately, *a coordinating contract may not be stable,* in the sense that *some rational decision maker(s) may have incentive to deviate from this contract and hence will not agree to it.* For instance, consider a supply chain with a single supplier and a single retailer. A coordinating contract may specify that for any realization of the demand, the supplier will always take 50% of the supply chain's profit plus a fixed payment and the retailer will take 50% of the system's profit minus the fixed payment. Evidently, in this contract the retailer's order quantity is such that the system's expected profit is maximized. Therefore, for any fixed payment, this contract is a coordinating contract. However, if the fixed payment is larger than 50% of the system's expected profit, the retailers' expected profit is negative and it will not partner with the supplier. Therefore, in this case, this coordinating contract is not stable.

In the above example, it is obvious that any stable coordinating contract must generate non-negative expected profits for both the supplier and the retailer. However, identifying stable coordinating contracts may not be so simple in other cases, especially when there are competing agents. In Section 3.2, we present an example and show that a coordinating contract which generates non-negative expected profit for every agent in the supply chain may actually not be stable, since some or all of the agents may have other options that generate higher expected profit.

Our objective in this chapter is to identify coordinating contracts that are stable. To address this challenge, we introduce the notion of *rational contracts*. To define this notion, we analyze the system as a cooperative game. We consider a game where every agent in the supply chain is a player, and the payoff of signing a coordinating contract for each player is his/her objective value given the contract. A rational contract is defined as a coordinating contract which belongs to the core of the cooperative game. In other words, under a rational contract, every subset of agents receives no less than its predefined value, which reflects the *bargaining power* of the agents. This notion allows us to eliminate certain coordinating contracts that are not stable. More importantly, it allows us to capture the relative bargaining powers and to evaluate the profit shares of the agents endogenously.

Recently, analyzing bargaining powers and predicting stable outcomes in a supply chain have received some attention in the operations literature. A first line of research models the bargaining process explicitly as a Stackelberg type game and examines the equilibrium. For example, Ertogral and Wu (2001) study a bargaining game close to Rubinstein's model in which the supplier and the retailer make alternating offers. Bernstein and Marx (2006) study a supply chain with one supplier and multiple retailers where each retailer can set its reservation profit level. This type of literature helps us understand the dynamics among different agents in the supply chain, but their results depend heavily on details of the bargaining process and usually restrict the types of contracts proposed during negotiations. A second line of research applies the Nash bargaining solution to examine the outcomes. Examples include Gurnani and Shi (2006) and Nagarajan and Bassok (2002). Nash bargaining solution allows prediction of outcomes while abstracting out the details of the bargaining process. However, when it is applied to a situation when an agent has to bargain with multiple agents independently, e.g., an assembler bargaining with its component suppliers (Nagarajan and Bassok (2002)), then the sequence of negotiations has huge impact on the outcomes.

This chapter has similar flavor to Nagarajan and Sosic (2007), as both apply cooperative game theory concept to study cooperative elements in supply chains to predict stable outcomes, while abstracting out the details of the bargaining process. However, the objective and methodology are quite different. Nagarajan and Sosic (2007) analyze stable pricing cartel structures in a competitive market, using the concept of farsighted stability. In this chapter, we apply the traditional concept of the core to introduce the notion of rational contracts. This allows us to analyze stable coordinating contracts and examine the share of profits according to the relative endogenous bargaining power of the agents.

To illustrate the key idea of the rational contract concept, we apply it to analyze two supply chain models. First, we analyze a supply chain with a single risk neutral supplier and multiple risk neutral retailers competing on inventory and pricing. Second, we study a supply chain with a single risk averse retailer and multiple risk averse suppliers. In both cases, we identify coordinating contracts, and more importantly, we apply the concept of rational contracts to illustrate how the system profits can be distributed among the players according to their relative bargaining powers.

The organization of this chapter is as follows. In Section 3.2, we propose a general model which allows us to study coordination among a set of agents. We define rational contracts for such a general system. We then apply the concepts of coordinating contracts and rational contracts to analyze two supply chains. Specifically, in Section 3.3, we analyze a supply chain with a single supplier and multiple competing retailers where all agents are risk-neutral. In Section 3.4, we study a supply chain consisting of multiple competing suppliers and a single retailer where the agents are risk-averse. Finally, we conclude in Section 3.5.

3.2 The Model

In this section, we introduce the concept of coordinating contracts defined by Gan et al. (2004), which allows to model both risk-neutral and risk-averse players. We propose a new concept, rational contracts, in a general system with multiple agents. This concept, which is based on the definition of a core in a cooperative game (see Osborne and Rubinstein (1994)), allows us to eliminate certain coordinating contracts that are not stable.

Consider a system consisting of a set $N = \{1, \ldots, n\}$ of agents who face some uncertainty with a probability space (Ω, F, P). Every agent has to choose its strategy s_i, which may be a single stage static action or a sequence of dynamic actions (see Fudenberg and Tirole (1991) for definition of strategy in dynamic setting), from the strategy space S_i. Before the agents choose their strategies, they have to agree, either through negotiation or any other means, on the sharing rule $\theta(\cdot) \in \Theta$, referred to as the *contract*, which is a function of action vector $\boldsymbol{a}$ and uncertainty ω.

Let $\boldsymbol{s} = (s_1, \ldots, s_n)$ be the strategy profile of the agents. Assume that every agent in the system has an objective which is a real-valued function of the strategy profile and the sharing rule, $U_i(\boldsymbol{s}, \theta(\cdot))$. Examples of an objective function would be the agent's expected profit, mean-variance tradeoff, expected utility, or CVaR (to be defined later). Let $\boldsymbol{U}(\boldsymbol{s}, \theta(\cdot)) = (U_1(\boldsymbol{s}, \theta(\cdot)), \ldots, U_n(\boldsymbol{s}, \theta(\cdot)))$ be the objective vector of the agents.

Definition 3.2.1 *A strategy profile $\boldsymbol{s} \in S_1 \times \ldots \times S_n$ is optimal under the sharing rule $\theta(\cdot)$ if the joint objective $\boldsymbol{U}(\boldsymbol{s}, \theta(\cdot))$ is a Pareto-optimal point of the set*

$$G = \{\boldsymbol{U}(\boldsymbol{s}, \theta(\cdot)) \mid \boldsymbol{s} \in S_1 \times \ldots \times S_n, \theta \in \Theta\},$$

i.e., it is not possible to improve the objective of one agent by changing the strategy profile or sharing rule without decreasing the objective of another agent.

The definition implies that the optimality of a strategy profile depends on the sharing rule. Unfortunately, there may not exist optimal strategy profile under some sharing rules. When the objectives of the agents are transferrable such as expected profit, CVaR and mean-variance tradeoff, the Pareto-optimal frontier is the plane where sum of objectives of all agents is maximized. In this case, additional side payments do not affect the existence of optimal strategy profiles under any sharing rules. If, in addition, the objectives of the agents are expected profits, then there exists optimal strategy profiles under any sharing rules. Next, we are going to define coordination.

Definition 3.2.2 *A sharing rule $\theta(\cdot)$ is said to coordinate the system if it satisfies the following conditions:*

(a) There exists a subgame perfect equilibrium strategy profile $\mathbf{s}^*$ under this sharing rule $\theta(\cdot)$.

(b) There exists an optimal strategy profile s^θ under this sharing rule θ.

(c) At least one optimal strategy profile in (b), $\mathbf{s}^\theta$, is a subgame perfect equilibrium under the sharing rule $\theta(\cdot)$.

A subset of sharing rules (or a type of contracts), Θ_0, is said to coordinate if there exists $\theta(\cdot) \in \Theta_0$ where $\theta(\cdot)$ is coordinating.

So far, we have focused on a sharing rule that coordinates the system. However, it is not clear that a given coordinating sharing rule is stable. In Section 3.1, we discussed an example of a supply chain with a single supplier and a single retailer, and show that a coordinating contract may not be stable, for example, when the expected profit of one of the agents is negative. The following example shows that a coordinating contract may not be stable even when the expected profits of all agents are non-negative.

Example 3.2.3 *Consider two risk-neutral supplers selling the same product to a risk-neutral retailer. The per unit production costs of supplier 1 and supplier 2 are \$8 and \$9 respectively. The retailer sells at \$20 per unit and faces an uncertain demand uniformly distributed between 800 and 1800. Since supplier 1 has lower production* *cost, system optimality is achieved when only supplier 1 produces, and the system optimal production quantity is 1400 units. Now consider the following contract profile, call it* Contract Profile $\mathcal{A}$*.*

- *Between the retailer and supplier 1: buyback contract with a wholesale price of \$18.8 and a buyback price of \$18.*
- *Between the retailer and supplier 2: wholesale price of \$19*

Under this contract, the retailer's optimal decision is to order 1400 units from supplier 1 and order nothing from supplier 2. Hence, Contract Profile $\mathcal{A}$ *is coordinating. However, we argue that* Contract Profile $\mathcal{A}$ *is not stable. Under* Contract Profile $\mathcal{A}$*, the expected profits of supplier 1, supplier 2 and the retailer are \$11880, \$0 and \$1320 respectively. Suppose supplier 2 proposes that the retailer will only be charged \$10 per unit if the retailer orders at least 1400 units from supplier 2. Observe that if the retailer orders 1400 units from supplier 2 instead of supplier 1, it can increase its expected profit to \$10400, which is much higher than the expected profit of \$1320 it gets under* Contract Profile $\mathcal{A}$*. Hence, the retailer will accept the new deal. Supplier 2 also gains from the new deal as it now enjoys a positive expected profit. It is important to observe that both supplier 2 and the retailer can come to an agreement which gives both of them a higher expected profit than* Contract Profile $\mathcal{A}$*. This higher expected profit is guaranteed regardless of the action of supplier 1, i.e., even when supplier 1 refuses to supply to the retailer anymore. Therefore, we say* Contract Profile $\mathcal{A}$ *is not stable.*

The example thus implies that a given coordinating sharing rule is not stable if there is a subset of the agents who can do better by collaborating on their own and not participating in the contract. We thus introduce the notion of a *rational sharing rule.*

We assume that the agents' objectives are transferrable; this is the case when the agent objectives are expected profit, CVaR, or mean-variance tradeoff. Given a coordinating sharing rule $\theta(\cdot)$, and a strategy profile $\boldsymbol{s}^*$ which is both optimal and subgame perfect equilibrium , the payoff of each agent is simply its objective $U_i(s^*, \theta(s^*, \cdot))$. We assume that there exists a value $v(C)$ associated with every coalition C of agents. The definition of $v(C)$ will be discussed later.

Consider a cooperative game in which every agent is a player. Define a rational contract as one such that the objective value vector, $U_i(s^*, \theta(s^*, \cdot))$ associated with an optimal and subgame perfect equilibrium strategy is in the core of the cooperative game. Formally, rational contracts are defined as follows.

Definition 3.2.4 *Let $\theta(\cdot) \in \Theta$ be a coordinating contract. The contract $\theta(\cdot)$ is called rational if there exits an optimal and subgame perfect equilibrium strategy profile, s^*, such that*

$$\sum_{i \in C} U_i(s^*, \theta(s^*, \cdot)) \geq v(C) \textit{ for all } C \subset N.$$

Hence, if a type of sharing rules is coordinating and flexible, i.e. it can achieves any point on the Pareto-optimal frontier as equilibrium, then there exists a coordinating and rational contract if an only if the core is non-empty. If a type of sharing rule is coordinating, but can only achieves a portion of the Pareto-optimal frontier as equilibrium, then showing the existence of coordinating and rational contract is equivalent to showing that one of the equilibria that is on the Pareto-optimal frontier also lies in the core.

A question that remains is how to define the value of each coalition $v(C)$, which has been studied ever since the pioneering work of von Neumann and Morgenstern (1944). The two most popular and widely accepted ones are the $\alpha-$ and $\beta-$ definitions (see Aumann (1967)). Briefly, $v_\alpha(C)$ is defined as the payoff that the coalition C can guarantee, while $v_\beta(C)$ is the payoff of coalition C that agents outside the coalition cannot prevent C from getting. The two definitions are equivalent when the agents payoffs are transferrable (Aumann (1967)).

Other definitions have been proposed, such as the $\gamma-$core in Chander and Tulkens (1997). However, these definitions depend on assumptions on the behaviors of agents outside the coalition, which are usually hard to justify. In particular, in a supply chain, agents outside the coalition may act cooperatively, non-cooperatively, or split into a number of coalitions.

Other approaches of cooperative games study the stability conditions of overall coalition structure (Ray and Vohra (1997, 1999)), or by studying games of coalition formation (Block (1997); Hart and Kurz (1983); Yi (1997)). These approaches are too complicated in studying contracts in a general system where agents have continuous strategy space. One possible approach which has been used in the operations literature is to study farsighted coalition structures (Chwe (1994)). This approach requires not only the value of each coalition, but in addition, the value of each agent in every coalition structure. Hence we follow the traditional cooperative game approach and use the $\alpha-$core and $\beta-$core definitions as the values of the coalitions.

3.3 Risk Neutral Single Supplier-Multiple Retailer Supply Chain

In this section, we identify coordinating and rational contracts in a supply chain with a single risk-neutral supplier and n competing risk-neutral retailers. Our model is similar to the one in Bernstein and Federgruen (2005). One difference is that they assume the supplier will deliver the exact amount ordered by each retailer, while we allow the supplier to ship less than the amount ordered by a retailer.

At the beginning of the period, after the contracts between the supplier and the retailers are finalized, each retailer decides its own selling price p_i and order quantity, q_{r_i}. After observing the decisions of the retailers, the supplier decides the production and delivery quantity to each retailer q_{s_i}, where $q_{s_i} \leq q_{r_i}$ for all i. Hence, the production quantities, q_{s_i}'s, have a direct impact on the agents' profits. On the other hand, the order quantities, q_{r_i}'s, have an indirect impact on the agents' profits since they limit the possible choices of q_{s_i}'s for the supplier. Let $\boldsymbol{q_s} = (q_{s_1}, \ldots, q_{s_n})$ be the production and delivery quantity vector.

The supplier pays a cost of c_i per unit produced and delivered to retailer i. Demand is realized once all the decisions have been made. Specifically, each retailer faces an uncertain demand $D_i(\boldsymbol{p}) = d_i(\boldsymbol{p})\epsilon_i$, where ϵ_i is a nonnegative random variable independent of $\boldsymbol{p}$ with a distribution function $F_i(\cdot)$. For each i, $d_i(\boldsymbol{p})$ is a deterministic function of the price decision vector $\boldsymbol{p} = (p_1, \ldots, p_n)$ and is assumed to satisfy the following conditions.

Assumption 3.3.1 *For each i, $d_i(\boldsymbol{p})$ satisfies:*

(a) For all i, $\frac{\partial d_i(\boldsymbol{p})}{\partial p_i} \leq 0$.

(b) For all $i \neq j$, $\frac{\partial d_i}{\partial p_j} \geq 0$.

Assumption 3.3.1 (a) implies that retailer i's demand is non-increasing in its own price, while Assumption 3.3.1 (b) implies that retailer i's demand is nondecreasing in its competitors' prices. These assumptions are commonly used in the literature, see Bernstein and Federgruen (2005).

Since we assume that all agents are risk neutral, the objective of every agent is to maximize its expected profit. Hence coordination is achieved when the total expected profit of the system, given by

$$E[\pi^*(\boldsymbol{p}, \boldsymbol{q_s})] = \sum_{i=1}^{n} p_i E[\min(d_i(\boldsymbol{p})\epsilon_i, q_{s_i})] - \sum_{i=1}^{n} c_i q_{s_i},$$

is maximized. To avoid unrealistic situation, we assume that any $(\boldsymbol{p}, \boldsymbol{q_s})$ that maximizes $E[\pi^*(\boldsymbol{p}, \boldsymbol{q_s})]$ is positive and finite.

3.3.1 Coordinating Contracts

Consider a revenue sharing contract with price rebate between the supplier and each retailer i. In this contract, retailer i pays a per unit cost w_i for ordering and receives a price rebate of $f_i(p_i)$ if it sets the selling price to p_i. After demand is realized, the supplier receives a proportion α_i of retailer i's sales revenue. Since $q_{s_i} \leq q_{r_i}$, retailer i's expected profit is thus

$$E[\pi_{r_i}] = (1 - \alpha_i)p_i E[min(d_i(\boldsymbol{p})\epsilon_i, q_{s_i})] - w_i q_{s_i} + f_i(p_i),$$

while the expected profit of the supplier is

$$E[\pi_s] = \sum_{i=1}^{n} (\alpha_i p_i E[min(d_i(\boldsymbol{p})\epsilon_i, q_{s_i})] - (c_i - w_i)q_{s_i} - f_i(p_i) +) .$$

Let $(\boldsymbol{p}^*, \boldsymbol{q_s}^*)$ be the supply chain optimal price and quantity vectors. In the next theorem we show that there exists a revenue sharing contract with price rebate such that $(\boldsymbol{p}, \boldsymbol{q_r}, \boldsymbol{q_s}) = (\boldsymbol{p}^*, \boldsymbol{q_s}^*, \boldsymbol{q_s}^*)$ is a subgame perfect equilibrium and hence this type of contracts is coordinating.

Theorem 3.3.2 *Revenue sharing contracts with price rebate are coordinating and flexible.*

Proof. First, for $1 \le i \le n$, pick $0 \le \beta_i \le 1$ such that $\sum_{i=1}^{n} \beta_i < 1$. Let $w_i = \beta_i c_i$, $\alpha_i = 1 - \beta_i$ and $f_i(p_i) = \beta_i \sum_{j \neq i} p_j^* E[min(q_{s_j}^*, d_j(\boldsymbol{p}_{-i}^*, p_i)\epsilon_j)]$.

If other retailers set their prices at $\boldsymbol{p}_{-i}^*$, the expected profit of retailer i becomes

$$E[\pi_{r_i}(p_i, q_{s_i})] = \beta_i E[\pi^*(\boldsymbol{p}_{-i}^*, p_i, \boldsymbol{q_s}_{-i}^*, q_{s_i})] + \beta_i \sum_{j \neq i} c_j q_{s_j}^* ,$$

which is maximized when $(p_i, q_{s_i}) = (p_i^*, q_{s_i}^*)$. Suppose the supplier produces and delivers $\boldsymbol{q_s}^*$ if the price and order decisions of the retailers are $(\boldsymbol{p}, \boldsymbol{q_r}) = (\boldsymbol{p}^*, \boldsymbol{q_s}^*)$. Then, retailer i cannot achieve a higher objective by deviating from $(p_i, q_{r_i}) = (p_i^*, q_{s_i}^*)$.

Suppose every retailer i sets the price at p_i^* and orders $q_{s_i}^*$, the expected profit of the supplier becomes

$$E[\pi_s(\boldsymbol{p}^*, \boldsymbol{q_s})] = E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s})] - \sum_{i=1}^{n} \beta_i E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s}_{-i}^*, q_{s_i})] - \sum_{i=1}^{n} \beta_i \sum_{j \neq i} c_j q_{s_j}^* ,$$

which is maximized when $\boldsymbol{q_s} = \boldsymbol{q_s}^*$. Hence the supplier cannot achieve a higher objective by deviating from $\boldsymbol{q_s} = \boldsymbol{q_s}^*$

Therefore, using the *one-stage-deviation principle* Fudenberg and Tirole (1991), $(\boldsymbol{p}, \boldsymbol{q_r}, \boldsymbol{q_s}) = (\boldsymbol{p}^*, \boldsymbol{q_s}^*, \boldsymbol{q_s}^*)$ is a subgame perfect equilibrium. Observe that the expected profit of the system can be arbitrarily distributed among the supplier and the retailers by changing β_i's. Hence, revenue sharing contracts with price rebate but are coordinating and flexible. ∎

The proof of the theorem implies that β_i is the fraction of the system profit obtained by retailer i. It is important to note that in the proposed price rebate function,

$$\frac{df_i(p_i)}{dp_i} = \beta_i \sum_{j \neq i} p_j^* \frac{\partial d_j(p_{-i}^*, p_i)}{\partial p_i} \left[F_j(z_j) z_j - \int_0^{z_j} F_j(x) dx \right] \geq 0$$

where the above inequalities follow from Assumption 3.3.1, and $z_j = \frac{q_{s_j}^*}{d_j(p_{-i}^*, p_i)}$.

The above formula implies that the retailer receives higher rebate when charging the customers a higher price. Thus, the price rebates can effectively prevent the retailers from engaging in price competition.

We point out that the contract remains coordinating if the price rebate is implemented in other different forms. One possible implementation is that retailer i receives the rebate $f_i(p_i^*)$ if $p_i \geq p_i^*$ but no rebate otherwise. This implementation of the price rebate is similar to the minimum advertised price policy (see Charness and Chen (2002)), which is commonly used in the electronics industry (Patterson (1999)). Under minimum advertised price policy,a retailer receives a rebate from the supplier if it does not advertise the product below a certain price.

As mentioned at the beginning of this section, Bernstein and Federgruen (2005) study a similar model in which the supplier will satisfy any order placed by the retailer. They assume that customer demand satisfies Assumption 3.3.1 and show that buyback contract is coordinating and flexible when the wholesale price and buyback price both depend on the selling price of the retailer. It is not hard to verify that this contract is also coordinating in the case when the supplier can ship less than the amount ordered by a retailer. Unfortunately, such retail-price dependent wholesale prices i.e., wholesale prices that depend on retail prices, are not viewed favorably by managers as they are seen to be "eroding the power of the brand" (see, Ailawadi et al. (1999)). On the other hand, in our proposed contract, only the rebate depends on retail price;

the wholesale price and revenue sharing proportion are all independent of the retail price.

Bernstein and Federgruen (2005) also show that if the demand distribution satisfies, in addition to Assumption 3.3.1, a certain set of assumptions, e.g., when customer demand has increasing differences in prices, then there exists a coordinating buyback contract with fixed wholesale price and fixed buyback price. Unfortunately, as the authors pointed out, there may only be a single such coordinating contracts. Hence, the share of profit between the supplier and the retailers may not be flexible. This implies the contract may not satisfy the restrictions on the distribution of expected profit for rational contract discussed later in Section 3.3.2.

3.3.2 Rational Contracts

In this subsection we show that the coordinating contracts described in Section 3.3.1, i.e., revenue sharing with price rebate, are rational. More importantly, we characterize bounds on the expected profit shared by each agent and indeed show that rational contracts resrtics the agent expected profits to certain intervals. We point out that Bernstein and Federgruen (2005) also observe the existence of such bounds, but do not evaluate them or propose any method to study them explicitly. In particular, they mention "Participation constraints for the supplier and the retailer, e.g., ensuring that their expected profits are in excess of those achieved prior to coordination, result in a lower bound $\underline{\alpha}$ and upper bound $\overline{\alpha}$, respectively. The exact choice of $[\underline{\alpha}, \overline{\alpha}]$ depends on the chain members' bargaining powers...".

For this purpose, we need to specify the value of any coalition. Suppose that a subset of the retailers form a coalition C. Since the supplier is outside the coalition, the coalition cannot guarantee the supplier will sell them any products. Hence the coalition cannot guarantee any profit and therefore $v(C) = 0$ for any coalition excluding the supplier.

Now consider a coalition consisting of the supplier only and no retailers. The supplier cannot guarantee the sales of any products, and hence cannot guarantee any positive profit. Therefore, $v(\text{supplier}) = 0$.

Finally, if a coalition is formed by the supplier and a subset of the retailers, i.e. $C = \{supplier\} \cup \mathcal{R}$ where $\mathcal{R}$ is a subset of retailers, the coalition has two choices. First, the coalition may sell the products to each retailer i, $i \notin \mathcal{R}$, at a wholesale price w_i. Unfortunately, the coalition cannot guarantee non-zero sales revenue from selling to the retailers outside the coalition (since retailers may decide not to order anything from the coalition). Second, the coalition may decide not to sell to the retailers outside the coalition. In this case, the retailers outside the coalition will be driven out of the market and their demand is redistributed. Specifically, the demand for each retailer j in the coalition would be a new function $d_j^{\mathcal{R}}(\boldsymbol{p}_{\mathcal{R}})$ which is a function of the price profile of the retailers in the coalition. We make two important assumptions regarding the relation between $d_j^{\mathcal{R}}(\boldsymbol{p}_{\mathcal{R}})$ and $d_j(\boldsymbol{p})$.

Assumption 3.3.3

(a) $d_j^{\mathcal{R}}(\boldsymbol{p}_{\mathcal{R}}) \geq d_j(\boldsymbol{p}_{\mathcal{R}}, \boldsymbol{p}_{-\mathcal{R}})$ *for all* $\boldsymbol{p}_{\mathcal{R}}$ *and* $\boldsymbol{p}_{-\mathcal{R}}$.

(b) $\max_{\boldsymbol{p}_{\mathcal{R}}, \boldsymbol{q}_{s\mathcal{R}}} \sum_{j\in\mathcal{R}} \big(p_j E[\min(d_j^{\mathcal{R}}(\boldsymbol{p}_{\mathcal{R}})\epsilon_j, q_{s_j})] - c_j q_{s_j}\big) < \sum_{i=1}^{n} \big(p_i^* E[\min(d_i(\boldsymbol{p}^*)\epsilon_i, q_{s_i}^*)] - c_i q_{s_i}^*\big)$.

The first assumption implies that when some retailers are driven out of the market, the demand of each remaining retailer does not decrease, given the price profile unchanged. The second assumption requires that the maximum possible profit of a system consisting of only a proper subset of retailers is less than the optimal system profit with all retailers.

Given the two choices of the coalition $C = \{supplier\} \cup \mathcal{R}$, the highest profit guaranteed is obtained when the coalition picks the second option, i.e., they do not sell to retailers outside the coalition. Hence, the value of the coalition becomes

$$v(C) = \max_{\boldsymbol{p}_{\mathcal{R}}, \boldsymbol{q}_{s\mathcal{R}}} \sum_{i\in\mathcal{R}} \big(p_i E[min(d_i^{\mathcal{R}}(\boldsymbol{p}_{\mathcal{R}})\epsilon_i, q_{s_i})] - c_i q_{s_i}\big).$$

Our evaluation for the values of coalitions is consistent with the $\alpha-$core approach. As discussed in Section 3.2, we believe this approach is appropriate since the value of a coalition is defined as the expected profit the coalition can *guarantee*, independent of the behaviour of other agents, rather than the profit the coalition can *expect* assuming certain type of behaviour by agents outside the coalition. Indeed, a similar approach is used by other researchers. For example, when Plambeck and Taylor (2004) study capacity trading in a supply chain with a single manufacturer and multiple retailers, they "focus on coalitions that include the manufacturer because buyers cannot trade capacity without the manufacturer's cooperation". Milgrom (2004) also use exactly the same approach to define the values of coalitions when studying the core of a single-buyer-multiple-seller auction.

We are ready to show the existence of a rational contract and how this concept can help eliminate certain coordinating contracts that are not stable. First, according to Section 3.3.1, there exists a coordinating contract such that under the system optimal strategy, the expected profit of retailer i, $i = 1, 2, \ldots, n$, is

$$E[\pi_{r_i}] = \beta_i E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s}^*)],$$

and the expected profit of the supplier is

$$E[\pi_s] = (1 - \sum_{i=1}^{n} \beta_i) E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s}^*)].$$

The following theorem illustrates that when we choose β_i, $i = 1, \ldots, n$, appropriately, every coalition can get no less than its guaranteed value.

Theorem 3.3.4 *There exists β_i's such that $\sum_{i \in C} \beta_i E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s}^*)] \geq v(C)$ for all $C \subseteq \mathcal{A}$, where $\mathcal{A}$ is the union of the supplier and the set of all retailers.*

Proof. Let Γ be the set of possible coalitions consisting of the supplier and a proper subset of the retailers. Let $C \in \Gamma$. Then, by Assumption 3.3.3 (b), $v(C) < E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s}^*)]$. Let $v^* = max_{C \in \Gamma} v(C)$. Since Γ is a finite set, $v^* < E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s}^*)]$.

Now, choose β_i's such that $(1 - \sum_{i=1}^{n} \beta_i) E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s}^*)] \geq v^*$. Then $v(C) = 0$ for all coalitions C excluding the supplier, and $v(C) \leq E[\pi_s(\boldsymbol{p}^*, \boldsymbol{q_s}^*)]$ for all coalitions including the supplier. ■

The theorem implies that revenue sharing contracts with price rebate (our proposed contract) are rational. In the next two lemmas, we show that there exist constraints on the values of β_i, which in some sense capture the bargaining power of the agents.

Define $m_{ij}(\boldsymbol{p}) = d_j^{-i}(\boldsymbol{p}_{-i}) - d_j(\boldsymbol{p})$, referred to as the *customer demand mobility* from retailer i to retailer j at price $\boldsymbol{p}$. That is, $m_{ij}(\boldsymbol{p})$ represents the increase in retailer j demand due to the departure of retailer i from the system. If m_{ij} is relatively small (large), then customers are loyal (not loyal) to retailer i and they leave (stay in) the system when retailer i departs the system.

Let

$$\tilde{\beta}_i = \frac{p_i^* E[\min(d_i(\boldsymbol{p}^*)\epsilon_i, q_{s_i}^*)] - c_i q_{s_i}^*}{E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s}^*)]},$$

and for $i = 1, \ldots, n$,

$$\begin{aligned} \overline{\beta}_i &= \tilde{\beta}_i - \frac{\sum_{j\neq i}(p_j^*\mu_j - c_j\eta_j)m_{ij}(\boldsymbol{p}^*)}{E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s}^*)]} \\ &= \frac{E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s}^*)] - \sum_{j\neq i}(p_j^*\mu_j - c_j\eta_j)d_j^{-i}(\boldsymbol{p}_{-i}^*)}{E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s}^*)]} \end{aligned} \tag{3.1}$$

where $\eta_j = \frac{q_{s_j}^*}{d_j(\boldsymbol{p}^*)}$, or equivalently $F_j(\eta_j) = \frac{p_j^* - c_j}{p_j^*}$, and $\mu_j = E[\min(\epsilon_j, \eta_j)]$.

Then, $\tilde{\beta}_i$ is the proportion of total supply chain profit due to selling through retailer i. To understand $\overline{\beta_i}$, observe that the numerator in the second equation of (3.1) represents an upper bound on the system profit loss due to the departure of retailer i while the denominator is the maximum system profit with all retailers included. Thus, $\overline{\beta}_i$ is an upper bound on the fraction of profit lost to the system due to the departure of the ith retailer. Evidently, $\overline{\beta}_i$ is directly related to the concept of customer mobility, $m_{ij}(\boldsymbol{p})$. That is, the lower the customer mobility from i to other retailers, the higher $\overline{\beta}_i$ and hence the higher the potential loss to the system due to the departure of retailer i.

In the next lemma we show that the distribution of system profit among the different retailers is limited to a set defined by the $\overline{\beta}_i$, $i = 1, \ldots, n$.

Lemma 3.3.5 *Let $\theta(\cdot)$ be a rational contract such that under the system optimal* *strategy, the expected profit of each retailer is*

$$E[\pi_{r_i}] = \beta_i E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s}^*)], i = 1, \ldots, n,$$

with $0 < \beta_i < 1$ *and* $0 < \sum_{i=1}^n \beta_i < 1$. *Then, for all* $1 \le i \le n$, $\beta_i \le \overline{\beta}_i$.

Proof. Consider a coalition C including every agent but retailer i. Let $d_j^{-i}(\cdot)$ be the demand function of each retailer $j \in C$. The value of this coalition is

$$v(C) = \max_{\boldsymbol{p}_{-i}, \boldsymbol{q_s}_{-i}} \sum_{j \neq i} \big(p_j E[\min(d_j^{-i}(\boldsymbol{p}_{-i})\epsilon_j, q_{s_j})] - c_j q_{s_j}\big).$$

For the contract to be rational, the expected profit of this coalition has to be no less than its value, or,

$$(1-\beta_i)E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s}^*)] \ge v(C) = \max_{\boldsymbol{p}_{-i}, \boldsymbol{q_s}_{-i}} \sum_{j \neq i} \big(p_j E[\min(d_j^{-i}(\boldsymbol{p}_{-i})\epsilon_j, q_{s_j})] - c_j q_{s_j}\big).$$

Hence, for all $\boldsymbol{q_s}_{-i}$,

$$\begin{aligned}\beta_i &\le 1 - \frac{\sum_{j\neq i}\big(p_j^* E[\min(d_j^{-i}(\boldsymbol{p}_{-i}^*)\epsilon_j, q_{s_j})] - c_j q_{s_j}\big)}{E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s}^*)]} \\ &= \tilde{\beta}_i + \frac{\sum_{j\neq i}\big(p_j^* E[\min(d_j(\boldsymbol{p}^*)\epsilon_j, q_{s_j}^*)] - p_j^* E[\min(d_j^{-i}(\boldsymbol{p}_{-i}^*)\epsilon_j, q_{s_j})] - c_j(q_{s_j}^* - q_{s_j})\big)}{E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s}^*)]}.\end{aligned}$$

In particular, choose $q_{s_j} = q_{s_j}^* \frac{d_j^{-i}(\boldsymbol{p}_{-i}^*)}{d_j(\boldsymbol{p}^*)}$. Let $\eta_j = \frac{q_{s_j}}{d_j^{-i}(\boldsymbol{p}_{-i}^*)} = \frac{q_{s_j}^*}{d_j(\boldsymbol{p}^*)}$ and $\mu_j = E[\min(\epsilon_j, \eta_j)]$. Then,

$$p_j^* E[\min(d_j(\boldsymbol{p}^*)\epsilon_j, q_{s_j}^*)] - p_j^* E[\min(d_j^{-i}(\boldsymbol{p}_{-i}^*)\epsilon_j, q_{s_j})] - c_j(q_{s_j}^* - q_{s_j}) = (p_j^*\mu_j - c_j\eta_j)m_{ij}(\boldsymbol{p}^*).$$

Therefore,

$$\beta_i \le \tilde{\beta}_i - \frac{\sum_{j\neq i}(p_j^*\mu_j - c_j\eta_j)m_{ij}(\boldsymbol{p}^*)}{E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s}^*)]} = \overline{\beta}_i.$$

■

The upper bounds on the retailers' profit imply a lower bound on the supplier profit. To characterize this bound on the supplier's profit, let

$$\underline{\beta}_0 = \sum_{i=1}^{n} \frac{\sum_{j \neq i} (p_j^* \mu_j - c_j \eta_j) m_{ij}(\boldsymbol{p}^*)}{E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s}^*)]}. \tag{3.2}$$

Lemma 3.3.6 *Let $\theta(\cdot)$ be a rational contract such that under the system optimal strategy, the expected profit of the supplier is $E[\pi_s] = \beta_0 E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s}^*)]$, $0 < \beta_0 < 1$. Then, $\beta_0 \geq \underline{\beta}_0$.*

Proof. Lemma 3.3.5 shows that $\beta_i \leq \overline{\beta}_i$ for all $i = 1, \ldots, n$. Hence,

$$\begin{aligned}
\beta_0 &= 1 - \sum_{i=1}^{n} \beta_i \\
&\geq 1 - \sum_{i=1}^{n} \overline{\beta}_i \\
&\geq 1 - \sum_{i=1}^{n} (\tilde{\beta}_i - \frac{\sum_{j \neq i} (p_j^* \mu_j - c_j \eta_j) m_{ij}(\boldsymbol{p}^*)}{E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s}^*)]}) \\
&= 1 - \sum_{i=1}^{n} \tilde{\beta}_i + \sum_{i=1}^{n} \frac{\sum_{j \neq i} (p_j^* \mu_j - c_j \eta_j) m_{ij}(\boldsymbol{p}^*)}{E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s}^*)]} \\
&= \sum_{i=1}^{n} \frac{\sum_{j \neq i} (p_j^* \mu_j - c_j \eta_j) m_{ij}(\boldsymbol{p}^*)}{E[\pi^*(\boldsymbol{p}^*, \boldsymbol{q_s}^*)]}.
\end{aligned}$$

■

We conclude:

Observation 3.3.7 $\beta_i \leq \overline{\beta}_i \leq \tilde{\beta}_i, \forall i \geq 1$. *Thus, a retailer cannot receive more than its expected contribution to the system profit, i.e., the expected profit of the system generated by the demand of this retailer.*

Observation 3.3.8 *The higher the mobility of retailer i's customers, i.e., the lower retailer i's customers' loyalty, the lower this retailer's upper bound on its share of profit.*

By definition, $\overline{\beta}_i$ is an upper bound on the fraction of profit lost to the system due to the departure of retailer i. Lemma 3.3.5 implies that this value also represents an upper bound on the fraction of system profit that can be claimed by retailer i. Thus, we refer to $\overline{\beta}_i$ as retailer i's bargaining power. The observation thus implies that a retailer's *bargaining power* increases with the loyalty of its customers.

Observation 3.3.9 *The upper bound of each retailer's share of profit does not depend on the customers mobility of other retailers.*

Observation 3.3.10 *The lower bound on supplier's share of profit increases with the customer mobility of every retailer.*

This is intuitive, since as customer mobility from say retailer i to other retailers increases, the upper bound on retailer i's share of profit, $\overline{\beta}_i$, or alternatively, retailer i's bargaining power, decreases. Indeed, the mobile customers of every retailer lead to a potential increase in the profits of other retailers. Thus, the definition of $\underline{\beta}_0$, (3.2), implies that these potential increase in retailers profit is claimed by the supplier in any rational contract.

We finally consider two special yet interesting cases. In the first case, we assume all retailers are homogenous, i.e., they face the same demand functions, and hence the system optimal price is identical for all the retailers, i.e., $p^*_{s_i} = p^*_{s_j}$ for all i and j. Let

$$l_i(\boldsymbol{p}^*) = \frac{d_i(\boldsymbol{p}^*) - \sum_{j \neq i}[d_j^{-i}(\boldsymbol{p}^*_{-i}) - d_j(\boldsymbol{p}^*)]}{d_i(\boldsymbol{p}^*)} = l(\boldsymbol{p}^*)$$

be the customer loyalty of retailer i at price $\boldsymbol{p}$. Then, $l_i(p^*)$ is exactly the fraction of retailer i's customers the system will lose when retailer i departs. Since every retailer is identical, $l_i(p^*)$ is also the fraction of customers in the system who will only buy from one single retailer at the current system optimal price level. By Assumption 3.3.3, $l(\boldsymbol{p}^*) > 0$. Since all retailers are homogenous, we have that

$$\overline{\beta}_i = \frac{1}{n} l(\boldsymbol{p}^*) \text{ for } 1 \leq i \leq n,$$

and

$$\underline{\beta}_0 = 1 - l(\boldsymbol{p}^*),$$

which is the fraction of all customers in the system who are willing to switch to another retailer if their serving retailer departs. Therefore, the higher the number of customers willing to switch retailers, i.e., as $l(\boldsymbol{p}^*)$ decreases, the higher the lower bound on the supplier's profit under a rational contract. This implies a higher bargaining power for the supplier.

In the second case. we assume that each retailer's demand is independent of the pricing strategy of other retailers. This implies $\overline{\beta}_i = \tilde{\beta}_i$ for $1 \leq i \leq n$, and, it is easy to show that in this case every coordinating contract is rational if and only if $\beta_i \leq \tilde{\beta}_i$ for all retailers. Observe that if retailer i gains some influence on other retailers' demand through its pricing strategy, then retailer i's upper bound on profit share, $\overline{\beta}_i$, will be smaller than $\tilde{\beta}_i$. Therefore, the destructive power of a retailer does not earn it a higher possible profit when there is a monopolistic supplier who can control the impact of the destruction caused by this retailer.

Finally, we hope to point out that most of our results in this section carry over to the case with additive demand random variable ($D_i(\boldsymbol{p}) = d_i(\boldsymbol{p}) + \epsilon_i$). It is easy to show that results for coordinating and rational contracts can be carried over. The bounds on the profit share of the agents can be found following the same approach and lead to similar observations.

3.4 Single Retailer Multiple Suppliers System Under Risk Aversion

In this section, we identify coordinating and rational contracts in a supply chain with a single retailer and multiple suppliers where the agents are risk-averse. The retailer faces an uncertain demand D with a continuous distribution function $F(\cdot)$. Before the realization of the demand, the retailer orders a quantity q_{r_i} from each supplier i. Each supplier i then produces and delivers a quantity $q_{s_i} \leq q_{r_i}$, also before demand is realized. Let $\boldsymbol{q_r} = (q_{r_1}, \ldots, q_{r_n})$ and $\boldsymbol{q_s} = (q_{s_1}, \ldots, q_{s_n})$. Notice that $\boldsymbol{q_s} \leq \boldsymbol{q_r}$. The selling price of the retailer is p per unit and the production cost of supplier i is c_i

per unit. Without loss of generality we assume that $c_1 \leq c_1 \leq \ldots \leq c_n$. Finally, before making the ordering decisions, the retailer and each of the suppliers agree to a payment scheme (or contract) denoted as $T_i(\boldsymbol{q_s}, D)$, which is the payment from the retailer to supplier i. We denote $\boldsymbol{T} = (T_1(\boldsymbol{q_s}, D), \ldots, T_n(\boldsymbol{q_s}, D))$ as a payment vector, i.e. the vector of all payment schemes from the retailer to the suppliers. We focus on payment vectors $\boldsymbol{T}$ such that

1. $\int T_i(\boldsymbol{q_s}, D)dF(D)$ exists for all $\boldsymbol{q_s}$ and i.

2. $T_i(\boldsymbol{q_s}, D)$ is either left-continuous or right-continuous for all $\boldsymbol{q_s}$ and i.

Let $\mathcal{T}$ be the set of all possible payment vectors satisfying these assumptions.

The decisions of the agents depend not only on the payment vector $\boldsymbol{T}$ but also on the objective functions of the agents. The most commonly used risk-averse objective function in economics and operations literature is the expected concave utility function. However, firms usually do not have such an explicit form of utility function. Instead, risk-averse decision makers may consider Value at Risk (VaR), Conditional Value-at-Risk (CVaR), or mean-variance. The problem with mean-variance objective measure is that it penalizes upside risk as well as downside risk (March and Shapira (1987)). Indeed, a study of managerial perspective towards risk (Shapira (1994)) points out that managers "claimed to be primarily concerned with the downside of the distribution of outcomes". The study also reveals that "most managers referred to the 'worst possible outcome' in defining risk. Such a definition is reflected by the tail of the distribution...". In view of this study, Value-at-Risk(VaR) and Conditional Value-at-Risk(CVaR) may be the risk measures that appropriately capture this managerial perspective towards risk. Unfortunately, VaR measure faces some fundamental challenges such as not preserving the property of subadditivity (see Rockafellar and Uryasev (2000)). Hence, we focus on a model where risk aversion is measured by CVaR.

CVaR is a risk measure commonly used in finance and insurance literature, and it is closely related to Value-at-Risk(VaR) (see Jorion (1997); Dowd (1998); Duffie and Pan (1997)). It was first used by Chen et al. (2003) to study inventory management of a risk averse agent. To define CVaR, we follow Rockafellar and Uryasev (2002). First, given a random variable $\tilde{z}$, VaR is defined as the $1-\eta$-percentile of the random variable $\tilde{z}$, i.e., $VaR_\eta(\tilde{z}) = \sup\{z | 1 - \Pr(\tilde{z} \le z) \ge \eta\}$. Then, we introduce the η-head distribution of the random variable $\tilde{z}$, $\Psi_\eta(z)$, as follows.

$$\Psi_\eta(z) = \begin{cases} 1, & \text{if } z \ge VaR_\eta(\tilde{z}); \\ \frac{\Pr(\tilde{z} \le z)}{1-\eta}, & \text{if } z < VaR_\eta(\tilde{z}). \end{cases}$$

The η-CVaR of the random variable $\tilde{z}$ is then defined as the mean of the η-head distribution of $\tilde{z}$, i.e.,

$$CVaR_\eta(\tilde{z}) = \frac{1}{1-\eta} \int_{-\infty}^{z_\eta -} z d \Pr(\tilde{z} \le z) + \Big(1 - \frac{1-\eta'}{1-\eta}\Big) z_\eta,$$

where for simplicity, $z_\eta = VaR_\eta(\tilde{z})$ and $\eta \le \eta' = 1 - \Pr(\tilde{z} < z_\eta)$. It is clear that when $\eta = 0$, $CVaR_\eta(\tilde{z})$ reduces to the expectation of $\tilde{z}$. Further, as η increases, the decision maker based on $CVaR_\eta$ risk measure becomes more risk averse.

Now assume that $\tilde{z}$ is a function of a continuous random variable $\tilde{v} \in V$, i.e., $\tilde{z} = g(\tilde{v})$. It is easy to show that

$$CVaR_\eta(\tilde{z}) = E[g(\tilde{v}) | \tilde{v} \in \mathcal{V}],$$

where $\mathcal{V}$ is any subset of V satisfying (a)$Pr(\tilde{v} \in \mathcal{V}) = 1 - \eta$, and (b)$g(\tilde{v}') \le g(\tilde{v}'')$ for all $v' \in \mathcal{V}$ and $v'' \notin \mathcal{V}$.

Given the delivery quantity vector $\boldsymbol{q_s}$ and the payment vector $\boldsymbol{T}(\boldsymbol{q_s}, D)$, the profit of the retailer π_0 is given by $\pi_0(\boldsymbol{q_s}, D, \boldsymbol{T}) = p \min(q_0, D) - \sum_{i=1}^n T_i(\boldsymbol{q_s}, D)$ where $q_0 = \sum_{i=1}^n q_{s_i}$. The CVaR objective of the retailer, is given by

$$U_0(\boldsymbol{q_s}, \boldsymbol{T}) = CVaR_{\eta_0}(\pi_0(\boldsymbol{q_s}, D, \boldsymbol{T})) = E[\pi_0(\boldsymbol{q_s}, D, \boldsymbol{T}) | D \in \mathcal{D}_0],$$

where $\mathcal{D}_0$ is any subsect of $\mathbf{R}^+$ satisfying $Pr(D \in \mathcal{D}_0) = 1 - \eta_0$ and $\pi_0(\boldsymbol{q_s}, D', \boldsymbol{T}) \le \pi_0(\boldsymbol{q_s}, D'', \boldsymbol{T})$ for all $D' \in \mathcal{D}_0$ and $D'' \notin \mathcal{D}_0$. Similar, supplier i's profit is given by

$\pi_i(\boldsymbol{q_s}, D, \boldsymbol{T}) = T_i(\boldsymbol{q_s}, D) - c_i q_{s_i}$, and its CVaR objective, given by

$$U_i(\boldsymbol{q_s}, \boldsymbol{T}) = CVaR_{\eta_i}(\pi_i(\boldsymbol{q_s}, D, \boldsymbol{T})) = E[\pi_i(\boldsymbol{q_s}, D, \boldsymbol{T}) | D \in \mathcal{D}_i],$$

where $\mathcal{D}_i$ is any subsect of $\mathbf{R}^+$ satisfying $Pr(D \in \mathcal{D}_i) = 1 - \eta_i$ and $\pi_i(\boldsymbol{q_s}, D', \boldsymbol{T}) \leq \pi_i(\boldsymbol{q_s}, D'', \boldsymbol{T})$ for all $D' \in \mathcal{D}_i$ and $D'' \notin \mathcal{D}_i$.

For the simplicity of presentation, we assume that no two suppliers have the same risk-aversion level for the rest of the section, i.e., $\eta_i \neq \eta_j$ whenever $i \neq j$ and $i, j \neq 0$. Most of the results in this section can be extended to the case when two or more suppliers have the same risk-aversion level. Define m as the index of the least risk averse supplier, i.e., $\eta_m \leq \eta_i$ for all $i > 0$. Let $\underline{\eta} = \min(\eta_0, \eta_m)$ be the percentile used for the CVaR objective of the least risk averse agent.

3.4.1 Coordinating Payment Vectors

It is easy to show that CVaR is a transferrable objective. Thus, finding the Pareto-optimal frontier is equivalent to optimizing $\sum_{i=0}^{n} U_i(\boldsymbol{q_s}, \boldsymbol{T})$. The following lemma shows that only the least cost suppliers will produce in a coordinated supply chain.

Lemma 3.4.1 *If $\boldsymbol{q_s}^*$ is a supply chain optimal production vector, then $q^*_{s_i} = 0$ for all $i > l$, where l is the largest index of suppliers having the lowest production cost.*

Proof. Suppose $\boldsymbol{q_s}^*$ is the optimal production vector under $\boldsymbol{T}$ and $q^*_{s_i} \neq 0$ for some $i > l$. Now, define $\boldsymbol{q_s}'$ such that $q'_{s_1} = q^*_{s_1} + q^*_{s_i}$ and $q'_{s_i} = 0$. Let $\boldsymbol{T}'(\boldsymbol{q_s}', D) = \boldsymbol{T}(\boldsymbol{q_s}^*, D)$ for all $\boldsymbol{q_s}$. Then $\boldsymbol{T}' \in \mathcal{T}$ and the total objective given by $(\boldsymbol{T}', \boldsymbol{q_s}')$ is greater than the total utility given by $(\boldsymbol{T}, \boldsymbol{q_s}^*)$, which contradicts the optimality of $\boldsymbol{q_s}^*$. ∎

The lemma thus suggests that Pareto-optimality can be achieved only when all products are produced by the suppliers with the lowest production cost, which is intuitive. The following lemma identifies conditions for optimality of payment vectors.

Lemma 3.4.2 *The following conditions are necessary and sufficient for the existence of optimal quantity vector $\boldsymbol{q_s}^*$ under payment vector $\boldsymbol{T}(\cdot)$.*

(a) For an agent a who is more risk averse than the least risk-averse agent, its $1-\eta$ percentile profit is constant as long as $\eta \geq \underline{\eta}$, i.e., $VaR_{\eta}(\pi_a(\boldsymbol{q_s}^*, D, \boldsymbol{T})) = VaR_{\underline{\eta}}(\pi_a(\boldsymbol{q_s}^*, D, \boldsymbol{T}))$ if $\eta \geq \underline{\eta}$ and $\eta_a > \underline{\eta}$.

(b) For any agent a, $\pi_a(\boldsymbol{q_s}^*, \boldsymbol{T}(\cdot), D') \leq \pi_a(\boldsymbol{q_s}^*, \boldsymbol{T}(\cdot), D'')$ for all $D' \leq VaR_{\underline{\eta}}(D) \leq D''$ with probability 1. In other words, when the demand falls below $1-\underline{\eta}$ percentile of demand, the profit of every agent is below its $1-\underline{\eta}$ percentile of profit.

In addition, the probability that the demand falls below optimal production quantity $q_0^* = \sum_{i=1}^{n} q_{s_i}^*$ is no more than $1-\underline{\eta}$, i.e., $P[D \leq q_0^*] \leq 1-\underline{\eta}$. Hence, the profit for any agent who is more risk-averse than the least risk averse agent is constant when demand is less than the total production quantity.

Proof. Let $\underline{m}$ be the least risk-averse agent, i.e. $\eta_{\underline{m}} = \underline{\eta}$. We first show by contradiction that condition (a) is necessary. Suppose that $VaR_{\eta}(\pi_a(\boldsymbol{q_s}^*, D, \boldsymbol{T})) \neq VaR_{\underline{\eta}}(\pi_a(\boldsymbol{q_s}^*, D, \boldsymbol{T}))$ for agent a who is more risk averse than the least risk averse agent for some $\eta > \underline{\eta}$. Obviously, there exists $\boldsymbol{T}'$ such that for all D, $\pi_a(\boldsymbol{q_s}^*, D, \boldsymbol{T}') = 0$, $\pi_{\underline{m}}(\boldsymbol{q_s}^*, D, \boldsymbol{T}')) = \pi_a(\boldsymbol{q_s}^*, D, \boldsymbol{T}) + \pi_{\underline{m}}(\boldsymbol{q_s}^*, D, \boldsymbol{T})$ and $\pi_{a'}(\boldsymbol{q_s}^*, D, \boldsymbol{T}') = \pi_{a'}(\boldsymbol{q_s}^*, D, \boldsymbol{T})$ for $a' \notin \{a, \underline{m}\}$. Then, $CVaR_{\eta_{a'}}(\pi_{a'}(\boldsymbol{q_s}^*, D, \boldsymbol{T}')) = CVaR_{\eta_{a'}}(\pi_{a'}(\boldsymbol{q_s}^*, D, \boldsymbol{T}))$ for $a' \notin \{a, \underline{m}\}$ and $CVaR_{\eta_a}(\pi_a(\boldsymbol{q_s}^*, D, \boldsymbol{T}')) = 0$. Let $\mathcal{D}$ be any subsect of $\mathbf{R}^+$ satisfying $Pr(D \in \mathcal{D}) = 1-\underline{\eta}$ and $\pi_{\underline{m}}(\boldsymbol{q_s}^*, D', \boldsymbol{T}') \leq \pi_{\underline{m}}(\boldsymbol{q_s}^*, D'', \boldsymbol{T}')$ for all $D' \in \mathcal{D}$ and $D'' \notin \mathcal{D}$. Then $CVaR_{\eta_{\underline{m}}}(\pi_{\underline{m}}(\boldsymbol{q_s}^*, D, \boldsymbol{T}')) = E[\pi_{\underline{m}}(\boldsymbol{q_s}^*, D, \boldsymbol{T}')|D \in \mathcal{D}] = E[\pi_a(\boldsymbol{q_s}^*, D, \boldsymbol{T}) + \pi_{\underline{m}}(\boldsymbol{q_s}^*, D, \boldsymbol{T})|D \in \mathcal{D}]$.

First, let $\mathcal{D}'$ be any subset of $\mathbf{R}^+$ satisfying $Pr(D \in \mathcal{D}') = 1-\underline{\eta}$ and $\pi_{\underline{m}}(\boldsymbol{q_s}^*, D', \boldsymbol{T}) \leq \pi_{\underline{m}}(\boldsymbol{q_s}^*, D'', \boldsymbol{T})$ for all $D' \in \mathcal{D}'$ and $D'' \notin \mathcal{D}'$. Then, $Pr(D \in \mathcal{D}'') = Pr(D \in \mathcal{D}) = 1-\underline{\eta}$, and $\pi_{\underline{m}}(\boldsymbol{q_s}^*, D', \boldsymbol{T}) \leq \pi_{\underline{m}}(\boldsymbol{q_s}^*, D'', \boldsymbol{T})$ for all $D' \in \mathcal{D}'$ and $D'' \in \mathcal{D}$ but $D'' \notin \mathcal{D}'$. Hence, $E[\pi_{\underline{m}}(\boldsymbol{q_s}^*, D, \boldsymbol{T})|D \in \mathcal{D}] \geq CVaR_{\eta_{\underline{m}}}(\pi_{\underline{m}}(\boldsymbol{q_s}^*, D, \boldsymbol{T}))$.

Second, let $\mathcal{D}''$ be any subset of $\mathbf{R}^+$ satisfying $Pr(D \in \mathcal{D}'') = 1-\eta_a$ and $\pi_a(\boldsymbol{q_s}, D', \boldsymbol{T}) \leq \pi_a(\boldsymbol{q_s}, D'', \boldsymbol{T})$ for all $D' \in \mathcal{D}''$ and $D'' \notin \mathcal{D}''$. Then, $Pr(D \in \mathcal{D}'') = 1-\eta_a < 1-\underline{\eta} = Pr(D \in \mathcal{D})$, and $\pi_a(\boldsymbol{q_s}, D', \boldsymbol{T}) \leq \pi_a(\boldsymbol{q_s}, D'', \boldsymbol{T})$ for all $D' \in \mathcal{D}''$ and $D'' \in \mathcal{D}$ but $D'' \notin \mathcal{D}''$. Also, since (a) does not hold, $Pr(\pi_a(\boldsymbol{q_s}, D', \boldsymbol{T}) <$

$\pi_a(\boldsymbol{q_s}, D'', \boldsymbol{T}), D' \in \mathcal{D}'', D'' \in \mathcal{D}, D'' \notin \mathcal{D}'') > 0$. Hence,

$$E[\pi_a(\boldsymbol{q_s}^*, D, \boldsymbol{T}) | D \in \mathcal{D}] > CVaR_{\eta_a}(\pi_a(\boldsymbol{q_s}^*, D, \boldsymbol{T})).$$

Therefore,

$$CVaR_{\eta_{\underline{m}}}(\pi_{\underline{m}}(\boldsymbol{q_s}^*, D, \boldsymbol{T}')) > CVaR_{\eta_{\underline{m}}}(\pi_{\underline{m}}(\boldsymbol{q_s}^*, D, \boldsymbol{T})) + CVaR_{\eta_a}(\pi_a(\boldsymbol{q_s}^*, D, \boldsymbol{T})),$$

violating the optimality of $\boldsymbol{T}$. Thus, (a) holds.

To show that (b) is necessary, first notice that

$$CVaR_{\eta_a}(\pi_a(\boldsymbol{q_s}^*, \boldsymbol{T}(\cdot), D)) \leq E[\pi_a(\boldsymbol{q_s}^*, \boldsymbol{T}(\cdot), D) | D \leq VaR_{\underline{\eta}}]$$

for all agent a, because $Pr(D \leq VaR_{\underline{\eta}}) \geq 1 - \eta_a$. Now assume to the contrary that for some agent a', $Pr(\pi_{a'}(\boldsymbol{q_s}^*, \boldsymbol{T}(\cdot), D') > \pi_{a'}(\boldsymbol{q_s}^*, \boldsymbol{T}(\cdot), D''), D' \leq VaR_{\underline{\eta}}, D'' > VaR_{\underline{\eta}}) > 0$. Then, $CVaR_{\eta_{a'}}(\pi_{a'}(\boldsymbol{q_s}^*, \boldsymbol{T}(\cdot), D)) < E[\pi_{a'}(\boldsymbol{q_s}^*, \boldsymbol{T}(\cdot), D) | D \leq VaR_{\underline{\eta}}]$. Hence, $\sum_a CVaR_{\eta_a}(\pi_a(\boldsymbol{q_s}^*, \boldsymbol{T}(\cdot), D)) < \sum_a E[\pi_a(\boldsymbol{q_s}^*, \boldsymbol{T}(\cdot), D) | D \leq VaR_{\underline{\eta}}] = E[p \min(q_0^*, D) - \sum_{i=1}^n c_i q_{s_i}^* | D \leq VaR_{\underline{\eta}}]$. Now, define $\boldsymbol{T}'$ such that the least risk risk-averse agent gets all system profit while other agents get nothing. Then sum of objectives is exactly $E[p \min(q_0^*, D) - \sum_{i=1}^n c_i q_{s_i}^* | D \leq VaR_{\underline{\eta}}]$, contradicting the optimality of $\boldsymbol{T}$. Thus, (b) holds.

Given (a) and (b), the sum of objectives can be given by $CVaR_{\eta_0}(p \min(q_0^*, D) - T_m(\boldsymbol{q_s}^*, D)) + CVaR_m(T_m(\boldsymbol{q_s}^*, D)) - c_1 q_0^*$. We consider two cases. For the first case, let $\underline{\eta} = \eta_0 \leq \eta_m$. Then, because of (a) and (b), the sum of objectives is given by $CVaR_{\underline{\eta}}(p \min(q_0^*, D)) - c_1 q_0^* = pE[\min(q_0^*, D) | D \leq VaR_{\underline{\eta}}(D)] - c_1 q_0^*$, where the optimal $q_0^* < VaR_{\underline{\eta}}(D)$. Also, the sum of objectives is the same for all payment vectors satisfying (a) and (b). For the second case, let $\eta_0 > \eta_m = \underline{\eta}$. In this case, because of (a) and (b), the retailer's profit is constant for $D \leq VaR_{\underline{\eta}}(D)$, meaning $T_m(\boldsymbol{q_s}, D) = p \min(\boldsymbol{q_s}, D) + K(\boldsymbol{q_s})$ for $D \leq VaR_{\underline{\eta}}(D)$. Hence, the sum of objectives is also given by $pE[\min(q_0^*, D) | D \leq VaR_{\underline{\eta}}(D)] - c_1 q_0^*$, which is strictly decreasing when $q_0^* \geq VaR_{\underline{\eta}}(D)$. Thus, the optimal $q_0^* < VaR_{\underline{\eta}}(D)$, implying $P[D \leq q_0^*] \leq 1 - \underline{\eta}$.

Again, the sum of objectives is the same for all payment vectors satisfying (a) and (b). Hence, (a) and (b) are sufficient for the existence of the optimal $\boldsymbol{q_s}^*$. ■

Observe that under any practical contract, the profit of every agent should remain constant when demand is higher than the total production quantity. Hence, Lemma 3.4.2 implies the following.

Observation 3.4.3 *In any practical coordinating contract, the least risk-averse agent will take all the risk while all other agents will receive a payment independent of the realization of the demand.*

This observation is very different from the results of similar studies that consider expected concave utility and mean-variance tradeoff. For example, Agrawal and Seshadri (2000) study a supply chain with one supplier and multiple independent non-competing retailers where every agent maximizes the expectation of a concave utility function. They show that it will be optimal if there exists a risk neutral intermediate agent taking all the risks. Spulber (1985) study a similar model and show that global optimality is achieved when all agents share the risk, unless when there is a risk-neutral agent, in which case it is optimal when the risk neutral agent takes all the risks. Gan et al. (2004) study a supply chain with a single supplier and a single retailer, and model risk aversion by expected exponential utility objective and the mean-variance objective. They show that the agents share risk under coordinating contracts in both cases. All these studies show that coordination is achieved when the agents share the risks. On the other hand, we find that when the agents consider CVaR, then the system is coordinated only when the least risk-averse agent(s) take all the risks.

The case where the least risk-averse agent is a supplier that does not have the lowest production cost deserves special attention. In this case, the suppliers with the lowest production cost produce everything, while the least risk-averse supplier takes all the risk and the retailer's profit is independent of the demand. It is clear that this case is equivalent to the following scenario: the least risk-averse supplier purchases products from the lowest-cost suppliers, while selling the products at the retailer under a consignment contract.

Now that we have studied the necessary conditions for coordination, we shall propose payment vectors that coordinate the supply chain. For this purpose, let M be a large number and consider a payment scheme between supplier i and the retailer in which the retailer pays Mq_i to supplier i. This payment scheme implies that the retailer will never procure from this supplier.

We start by focusing on the case in which the retailer has the same risk-averse level as the least risk averse supplier, which also has the lowest production cost.

Theorem 3.4.4 *Suppose the retailer has the same level of risk aversion as the least risk-averse supplier and this supplier also has the lowest production cost, i.e.,* $\eta_0 = \eta_1 = \underline{\eta}$. *Then* $\boldsymbol{T} \in \mathcal{T}$ *with*

(a) $T_i(\boldsymbol{q_s}, D) = Mq_{s_i}$ *for all* $i \neq 1$, *and*

(b) $T_1(q_{s_1}, D)$ *being any contract that coordinates a supply chain with a single risk-neutral supplier and a single risk-neutral retailer*

is coordinating.

Proof. Notice that when M is large enough (such as larger than p), the retailer will not order from suppliers other than supplier 1. Now the objectives of the remaining two agents, namely the retailer and supplier 1, are

$$U_0(\cdot) = E_D[\pi_r(q_{s_1}, D)|D \leq VaR_{\underline{\eta}}(D)] \text{ and } U_1(\cdot) = E_D[\pi_{s_1}(q_{s_1}, D)|D \leq VaR_{\underline{\eta}}(D)],$$

which can be reduced to an equivalent risk-neutral model. Indeed, let $\tilde{D}$ be a random variable with cdf $\tilde{F}_r$ with $\tilde{F}_r(d) = \frac{F(d)}{(1-\underline{\eta})}$ for $d \leq VaR_{\underline{\eta}}(D)$ and $\tilde{F}_r(d) = 1$ for $d \geq VaR_{\underline{\eta}}(D)$. It is easy to show that

$$U_r(q_{s_1}) = E_{\tilde{D}}[\pi_r(q_{s_1}, \tilde{D})] \text{ and } U_{s_1}(q_{s_1}) = E_{\tilde{D}}[\pi_{s_1}(q_{s_1}, \tilde{D})].$$

Hence, as long as T_1 is coordinating in the risk-neutral setting with a single supplier and a single retailer, then $\boldsymbol{T}$ is coordinating. $\blacksquare$

Given the extensive research on coordinating contracts in the risk-neutral single supplier, single retailer setting, there is no need to propose any specific contract for this case. Indeed, consider the special case with a risk-averse single supplier and a risk-averse single retailer when the two agents have the same risk averse level. Theorem 3.4.4 implies that any coordinating contract for the risk-neutral single supplier single retailer setting is also coordinating in this case.

We now propose coordinating contracts for other cases.

Theorem 3.4.5 *We consider three cases:*

(a) The risk-averse level of the retailer is less than or equal to the risk-averse levels of all suppliers, i.e., $\eta_0 \leq \eta_i$ *for all* i. *Then* $\boldsymbol{T} \in \mathcal{T}$ *where*

$$T_i(\boldsymbol{q_s}, D) = \begin{cases} \alpha pCVaR_{\underline{\eta}}(\min(q_{s_1}, D)) + (1-\alpha)c_1 q_{s_1}, & \text{if } i = 1; \\ Mq_{s_i}, & \text{if } i \neq 1; \end{cases}$$

satisfying $0 \leq \alpha \leq 1$ *is a coordinating payment vector. Moreover,* T_1 *is a quantity discount contract.*

(b) Suppose the risk-averse level of the retailer is higher than or equal to the level of the least risk-averse supplier, which does not have the least production cost, i.e., $\eta_m \leq \eta_0$ *but* $c_m \neq c_1$. *Then* $\boldsymbol{T} \in \mathcal{T}$ *where*

$$T_i(\boldsymbol{q_s}, D) = \begin{cases} \alpha pCVaR_{\underline{\eta}}(q_{s_1}, D) + (1-\alpha)c_1 q_{s_1}, & \text{if } i = 1; \\ p\min(q_0, D) - [\alpha + \gamma(1-\alpha)]pCVaR_{\underline{\eta}}(\min(q_0, D)) & \\ \quad -(1-\gamma)(1-\alpha)c_1 q_0 + Mq_{s_m}, & \text{if } i = m; \\ Mq_{s_i}, & \text{otherwise}; \end{cases}$$

satisfying $0 \leq \alpha \leq 1$, $0 \leq \gamma \leq 1$ *is a coordinating payment vector. Note that here* T_1 *is a quantity discount contract and* T_m *is a consignment contract.*

(c) *Suppose the supplier with the lowest production cost has the least risk-averse level among all agents, i.e.,* $\underline{\eta} = \eta_1 \leq d_r$. *Then* $\boldsymbol{T} \in \mathcal{T}$ *where*

$$T_i(\boldsymbol{q_s}, D) = \begin{cases} p\min(q_{s_1}, D) - (1-\alpha)pCVaR_{\underline{\eta}}(min(q_{s_1}, D)) & \\ +(1-\alpha)c_1 q_{s_1}, & \text{if } i = 1; \\ Mq_{s_i}, & \text{if } i \neq 1; \end{cases}$$

satisfying $0 \leq \alpha \leq 1$ *is a coordinating payment vector. Note that here* T_1 *is a consignment contract.*

Proof. In all cases, when M is large enough the retailer will only order from supplier 1. Let a be the least risk-averse agent in each case. Then, under the proposed contracts, the objective of supplier 1 is $U_1(\cdot) = \beta_1 \sum_{i=0}^{n} U_i(\cdot)$ and the objective of the retailer is $U_0(\cdot) = \beta_0 \sum_{i=1}^{n} U_i(\cdot)$ where $\beta_1 = \alpha$ in all cases, and $\beta_0 = 1 - \alpha$ in (a) and (c) while $\beta_0 = (1-\alpha)\gamma$ in (b). Hence it is easy to show by the *one-stage-deviation principle* that $(\boldsymbol{q_r}, \boldsymbol{q_s}) = (\boldsymbol{q_s}^*, \boldsymbol{q_s}^*)$, where $q^*_{s_i} = 0$ for $i \neq 1$ and $q^*_{s_1}$ is the system optimal quantity, is a subgame perfect equilibrium. Therefore the proposed contracts are coordinating contracts. It is easy to show by calculus that in both (a) and (b), $\frac{\partial T_1(\cdot)}{\partial q_{s_1}} \leq 0$, and hence the contract between the retailer and supplier 1 is a quantity discount contract. ■

It is important to notice that the contracts proposed in Theorem 3.4.5 are flexible, i.e. they allow arbitrary division of objectives among the retailer and all suppliers who have a role (either production or risk-taking) in the supply chain. We now discuss the results of the theorem. In the first case, the retailer is the least risk-averse. Then, it is globally optimal for the retailer to take all the risks. In this case, a properly designed quantity discount contract with supplier 1 is coordinating, if other suppliers charge high enough so that it is optimal for the retailer not to order from them.

In the second case, there exists a least risk-averse supplier (supplier m) who is the least risk-averse agent in the supply chain, but it does not have the lowest production cost. As discussed earlier, it would be globally optimal in this case for this supplier to take all the risk, while the lowest cost supplier is responsible for production. Then, a properly designed quantity discount contract between the retailer and supplier 1, and a properly designed consignment contract between the retailer and supplier m is coordinating, given that other suppliers charge high enough so that the retailer will no order from them. It is important to observe that in the consignment contract between the retailer and supplier m, the latter charges the former for a very high price per unit of item it produces. Hence, functionally, this supplier is an intermediate agent who is only responsible for the risk and the ownership of the inventories at the retailer.

In the last case, the lowest cost supplier is also the least risk-averse agent. Then, it is optimal for this supplier to take the risk and be responsible for the production at the same time. In this case, a properly designed consignment contract between the retailer and this supplier is coordinating, again if other suppliers charge high enough.

Observe that the three cases in the statement of Theorem 3.4.5 may overlap. For example, when supplier 1 is the least risk-averse supplier and has the same level of risk aversion as the retailer. Then, besides fitting the situation in Theorem 3.4.4, it also satisfies the conditions in all cases in the above theorem. Hence in this case, these contracts proposed in all (a), (b) and (c) coordinate as well.

3.4.2 Rational Payment Vectors

We have identified coordinating payment vectors which depend on the level of risk aversion of the suppliers and the retailer. The contracts we propose in Theorem 3.4.5 require suppliers who should not be responsible for anything under global optimal to charge very high wholesale price so that the supplier will not order from it. However, since these suppliers are getting nothing under the proposed contracts, they would try to propose other deals to the retailer so that the retailer will order from them. Hence, the question which remains is whether the proposed contracts can prevent this from happening. In other words, are the proposed contracts stable and therefore rational.

For this purpose, we need to find the value of each coalition C of agents. Any coalition has to include the retailer and at least one supplier in order to build a subsystem with non-zero value. Hence, we have the following.

(a) Any coalition C with a single agent has a value $v(C) = 0$.

(b) For any coalition C without the retailer, the value of the coalition $v(C) = 0$.

Now consider a coalition C of agents including the retailer and at least one supplier. If C is a separate supply chain, then the sum of objectives of agents in C is maximized when the supply chain is coordinated. From previous analysis, this is achieved when the least risk-averse agent takes all the risk while the supplier with the lowest production cost does all the production. The sum of objectives of all agents in C is the one that the coalition can guarantee. Hence, the value of the coalition is given by

$$v(C) = \max_q (pCVaR_{\eta_a}(\min(q, D)) - c_i q),$$

where a is the least risk-averse agent in the coalition and supplier i is the supplier with the lowest production cost in this coalition.

Now that we have identified conditions for coordinating and the value of each coalition, we proceed to analyze coordinating payment vectors that are also rational. To do this, we introduce a few notions. Let

$$S_i(c_j) = \max_q (pCVaR_{\eta_i}(min(q, D)) - c_j q),$$

which is the maximum system objective function value when supplier j is responsible for all production and agent i (where $i = 0$ denote the retailer and $i > 1$ denote supplier i) is the risk taker.

Let

$$S^* = max_{i,j} S_i(c_j)$$

be the optimal system total objective. Given $\boldsymbol{T}$ and the corresponding optimal production quantity vector $\boldsymbol{q_s}^*$, let $\beta_i = \frac{U_{s_i}(\boldsymbol{T}, \boldsymbol{q_s}^*)}{S^*}$ be the fraction of supplier i's objective over the system total objective and $\beta_0 = \frac{U_r(\boldsymbol{T}, \boldsymbol{q_s}^*)}{S^*}$ be the fraction of retailer's objective over the system total objective.

The next three theorems identify conditions for coordinating contracts to be rational. The three cases correspond to the cases in Theorem 3.4.5. The bounds on β_i reflect the *bargaining powers* of the agents.

Theorem 3.4.6 *(a) Suppose the retailer is the least risk-averse agent (i.e., $\eta_0 \leq \eta_i$ for all i). Then any coordinating $\boldsymbol{T} \in \mathcal{T}$ is rational if and only if*

(1) $0 \leq \beta_1 \leq \overline{\beta}_1 = 1 - \frac{S_0(c_2)}{S^*}$.

(2) $1 \geq \beta_0 = \frac{U_r(\boldsymbol{q_s}^*, \boldsymbol{T})}{S^*} \geq \underline{\beta}_0 = \frac{S_0(c_2)}{S^*}$.

(3) $\beta_i = 0$ *for* $i \neq 0$ *or 1.*

(b) Suppose the least risk-averse supplier is also the least risk-averse agent (i.e., $\underline{\eta} = \eta_m \leq \eta_0$) and this supplier is not supplier 1 (i.e., $m \neq 1$). Then any coordinating payment vector $\boldsymbol{T} \in \mathcal{T}$ is rational if and only if

(1) $0 \leq \beta_1 \leq \overline{\beta}_1 = 1 - \frac{S_m(c_2)}{S^*}$.

(2) $0 \leq \beta_m \leq \overline{\beta}_m = 1 - max_{j \neq m} \frac{S_j(c_1)}{S^*}$.

(3) $1 \geq \beta_0 \geq \underline{\beta}_0 = max_{j \neq 1,m} \frac{S_j(c_2)}{S^*}$

(4) $\beta_i = 0$ *for* $i \neq 0$ *or 1 or* m.

(c) Suppose supplier 1 is the least risk-averse agent (i.e., $\underline{\eta} = \eta_1 \leq \eta_0$). Then, any coordinating payment vector $\boldsymbol{T} \in \mathcal{T}$ is rational if and only if

(1) $\beta_1 \leq \overline{\beta}_1 = 1 - \max_{j \neq 1} \frac{S_j(c_2)}{S^*}$.

(2) $\beta_0 \geq \underline{\beta}_0 = \max_{j \neq 1} \frac{S_j(c_2)}{S^*}$.

(3) $\beta_i = 0$ *for* $i \neq 0$ *or 1.*

Proof.

(a) For one direction, suppose the conditions are satisfied. $v(C) = S^* = U_r(\boldsymbol{q_s}^*, \boldsymbol{T}) + U_{s_1}(\boldsymbol{q_s}^*, \boldsymbol{T})$ if the coalition C includes the retailer and supplier 1. For a coalition C without supplier 1, $v(C) \leq S_0(c_2) \leq U_r(\boldsymbol{q_s}^*, \boldsymbol{T})$, where the second inequality follows from (2). Hence $v(C) \leq \sum_{a \in S} U_a(\boldsymbol{q_s}^*, \boldsymbol{T})$ and $\boldsymbol{T}$ is rational.

For the opposite direction, suppose that $\boldsymbol{T}(\cdot)$ is rational. Let $i \geq 2$. Consider a coalition with everyone but supplier 1. The value of the coalition is given by $v(C) = S_0(c_2)$, meaning that this coalition should get no less than $S_0(c_2)$ under any rational contract. Hence, $U_{s_1}(\boldsymbol{q_s}^*, \boldsymbol{T}) \leq S^* - S_0(c_2)$, implying $\beta_1 \leq \overline{\beta}_1$. For $i \geq 2$, consider a coalition with everyone but supplier i. The value of the coalition is given by $v(C) = S^*$. Hence, the sum of objectives of agents other than supplier i should be no less than S^* under any rational contract, implying $U_{s_i}(\boldsymbol{q_s}^*, \boldsymbol{T}) = 0$, i.e., $\beta_i = 0$. Thus, $\beta_0 = 1 - \sum_{i=1}^n \beta_i \geq 1 - \overline{\beta}_1 = \underline{\beta}_0$.

(b) For one direction, suppose conditions (1),(2),(3) and (4) are all satisfied. For any coalitions C including the retailer, supplier 1 and supplier m, $v(C) = S^* = U_r(\boldsymbol{q_s}^*, \boldsymbol{T}) + U_{s_1}(\boldsymbol{q_s}^*, \boldsymbol{T}) + U_{s_m}(\boldsymbol{q_s}^*, \boldsymbol{T})$. For any coalition C including the retailer and supplier m but excluding supplier 1, $v(C) \leq S_m(c_2) \leq S^* - U_1(\boldsymbol{q_s}^*, \boldsymbol{T}) = U_r(\boldsymbol{q_s}^*, \boldsymbol{T}) + U_{s_m}(\boldsymbol{q_s}^*, \boldsymbol{T})$, where the second inequality follows from (1). For any coalition C including the retailer but excluding supplier m, $v(C) \leq \max_{j \neq m} S_j(c_1) \leq S^* - U_m(\boldsymbol{q_s}^*, \boldsymbol{T}) = U_r(\boldsymbol{q_s}^*, \boldsymbol{T}) + U_{s_1}(\boldsymbol{q_s}^*, \boldsymbol{T})$, where the second inequality follows from (2). For coalition C including the retailer but excluding both suppliers 1 and m, $v(C) \leq \max_{j \neq 1,m} S_j(c_2) \leq U_r(\boldsymbol{q_s}^*, \boldsymbol{T})$, where the second inequality follows from (3). Hence the payment vector $\boldsymbol{T}$ is rational.

For the opposite direction, suppose $\boldsymbol{T}$ is rational. Consider a coalition C with every agent but supplier 1. This coalition should get no less than $v(C) = S_m(c_2)$ under any rational contract. Hence, $U_1(\boldsymbol{q_s}^*, \boldsymbol{T}) \leq S^* - v(C) = S^* - S_m(c_2)$, giving the bound $\overline{\beta}_1$. Consider a coalition C with every agent but supplier m. This coalition should get no less than $v(C) = \max_{j \neq m} S_j(c_1)$. Hence, $U_m(\boldsymbol{q_s}^*, \boldsymbol{T}) \leq S^* - v(C) = S^* - \max_{j \neq m} S_j(c_1)$, giving the bound $\overline{\beta}_m$. Consider a coalition with every agent but supplier i where $i \neq 1$ or m. This coalition should not get less than its value $v(C) = S^*$. Then, $U_i(\boldsymbol{q_s}^*, \boldsymbol{T}) \leq S^* - v(C) = S^* - S^* = 0$, and hence $\beta_i = 0$. Finally, consider a coalition C with every agent but suppliers 1 and m. For this coalition, $v(C) = \max_{j \neq 1,m} S_j(c_2)$. For $\boldsymbol{T}$ to be rational, coalition C has to get no less than $v(C)$, while every agent in this coalition other than the retailer has to get 0 objective. Hence, $U_r(\boldsymbol{q_s}^*, \boldsymbol{T}) \geq v(C) - \sum_{i\neq 1,m} U_{s_i}(\boldsymbol{q_s}^*, \boldsymbol{T}) = v(C)$, giving the bound $\underline{\beta}_0$.

(c) For one direction, suppose conditions (1), (2) and (3) are all satisfied. For any coalition C including both the retailer and supplier 1, $v(C) = S^* = U_r(\boldsymbol{T}, \boldsymbol{q_s}^*) + U_{s_1}(\boldsymbol{T}, \boldsymbol{q_s}^*)$. For any coalition C including the retailer but excluding supplier 1, $v(C) \leq \max_{j\neq 1} S_j(c_2) \leq U_r(\boldsymbol{T}, \boldsymbol{q_s}^*)$. Hence $\boldsymbol{T}$ is rational.

For the other direction, suppose $\boldsymbol{T}$ is rational. Consider a coalition C consisting of every agent but supplier i, $i > 1$. $v(C) = S^*$, which implies $\beta_i = 0$. Consider a coalition including every agent but supplier 1. $v(C) = \max_{j\neq 1} S_j(c_2)$, which gives the bound on β_1. Finally, $\beta_0 = 1 - \sum_{i=1}^{n} \beta_i \geq 1 - \overline{\beta}_1 = \underline{\beta}_0$.

■

Theorem 3.4.6 implies that any coordinating contracts are rational as long as the agents' objectives satisfy certain conditions, which may be different depending on different cases. It can be verified that these conditions can be satisfied by any coordinating contracts that are flexible. As discussed before, the coordinating contracts proposed in Theorem 3.4.5 are flexible. Hence, Theorem 3.4.6 implies that the contracts proposed in Theorem 3.4.5 are also rational. Now, consider the special case where supplier 1 has the same level of risk aversion as the retailer, in addition to the assumptions in Theorem 3.4.6. By Theorem 3.4.4, any contract that coordinates a supply chain with a risk-neutral single supplier and a risk-neutral single retailer also coordinates in this case. It is well known that revenue sharing and buyback contracts coordinate such systems (see Cachon (2003)), and these contracts allow any arbitrary allocation of system profit (objective). Hence, there exist coordinating buyback and revenue sharing contracts satisfying the conditions in Theorem 3.4.6 and are, therefore, rational.

It is important to discuss the bounds on the agents' share of the system objective. First, in all cases, the objective, and hence the profit, of an agent who is responsible for neither the production nor risk taking should be zero. This is reasonable, because this agent is not making any contribution to the supply chain.

In cases (a) and (by) of Theorem 3.4.6, supplier 1 is responsible for production only and it is not taking any risk of the suppl chain. Observe that the bound on this supplier's share of system objective $\overline{\beta}_1$ is smaller when its production cost is close to the cost of supplier 2, the second lowest cost supplier. Indeed as the difference between the production costs of these two suppliers decreases, supplier 1's bargaining power decreases and hence the fraction of system objective this supplier can claim.

In case (b), supplier m is taking all the risks of the supply chain, but it is not producing anything. It is important to observe that this supplier is competing on the level of risk aversion with all other suppliers, and even with the retailer. As the level of risk of the second least risk averse agent decreases, supplier m's bargaining power decreases, and hence its maximum claim on the fraction of system objective also decreases. In particular, this can be due to a decrease in the retailer's risk aversion level.

Consider the special case when supplier m has the same level of risk aversion as the retailer. Then, its objective value is zero under any rational coordinating contract. This is intuitive since this supplier will not produce anything and its only potential contribution is to take some of the risk. However, because the retailer has the same level of risk aversion, the system total objective is the same with or without this supplier. Hence, this supplier does not have any bargaining power.

Finally, in case (c), since supplier 1 is responsible for production and risk-taking at the same time, its bargaining power is affected by both the production costs and levels of risk aversion of other agents in the supply chain.

3.5 Conclusions

In this chapter, we have developed a model to study a decentralized supply chain with multiple agents facing demand uncertainties where the agents may be either risk neutral or risk averse. We have defined coordination and coordinating contracts in such a system and introduced the notion of rational contracts. These concepts are applied to analyze two supply chains: (i) a supply chain with a single risk neutral supplier and multiple risk neutral retailers; and (ii) a supply chain with a single risk averse retailer and multiple risk averse suppliers.

The coordinating and rational contract concepts defined here are quite general and can be applied to analyze other complex supply chains as well. The rational contract concept, however, has a few limitations. First, as mentioned before, we use the $\alpha-$core concept as the definition of rational contract, which may be conservative in calculating the value of each coalition of agents. This rational contract concept allows us to exclude some "non-stable" coordinating contracts, but the set of rational contracts may still be very broad. One possible remedy is to use other core concepts from cooperative game theory when finding the set of rational contract.

Finally, we discuss some possible extensions of our results. Notice that in the first model, retailers compete on price while in the second model, suppliers compete on production costs and the risk-aversion levels. There are many other dimensions to differentiate the agents in the supply chain. For instance, the availability of information and different capacities may differentiate different agents. It may be possible to apply our approach to study supply chains in which the agents compete on these dimensions.

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Chapter 4

Robustness to Renegotiation

4.1 Introduction

The efficiency of a decentralized supply chain depends on the actions of the firms in the supply chain, and the actions of the firms depend on the contracts among these firms. Many actions, whether contractible or non-contractible, can be coordinated with formal contracts (see Lariviere (1999); Cachon (2003); Shum and Simchi-Levi (2006)) and relational contracts (see Levin (2003); Plambeck and Taylor (2006); Taylor and Plambeck (2007a,b)). However, even after coordinating contracts are signed, the parties may act selfishly which may affect the efficiency of the supply chain.

Consider a typical scenario with a monopolistic supplier selling to multiple competing retailers. This may be the case when a manufacturer sells to multiple distributors, or when a franchisor servicing several franchisees. Suppose the supplier has signed coordinating contracts with the retailers. Unfortunately, global efficiency is not guaranteed, if one or more firms are looking for opportunities to sign new contracts or to renegotiate existing contracts.

First, the supplier may sign contracts with new potential retailers. In this case, there may exist contracts between the supplier and the new retailers, such that these new contracts, together with the contracts between the supplier and the existing retailers, coordinate the supply chain. However, the goal of the supplier and the new retailers is to maximize their joint profit, as the existing contracts have been signed.

Therefore in this case, it is not clear whether the new contracts between the supplier and the new retailers will coordinate the supply chain.

Second, the supplier may renegotiate existing contracts with one or more retailers. For example, the supplier can offer a lower marginal price to one retailer such that this retailer will order a larger quantity and sell at a lower price. This way, this retailer and the supplier can increase their joint profits at the expense of other retailers. The new quantity and price of this retailer may not be optimal for the supply chain. Hence, the renegotiation between the supplier and a subset of retailers affect the efficiency of the supply chain.

In this chapter, we study the second type of opportunism, i.e., the incentive of the supplier to renegotiate signed contracts with one or more retailers and the robustness of different contracts to renegotiation. This incentive depends on a number of factors. First, it depends on structure of the signed contracts. For any given non-coordinating contract, there is potential for the supplier and a subset of retailers to renegotiate and jointly gain from new contracts. This implies that non-coordinating contracts are not robust to renegotiation. Hence, we focus on coordinating contracts. Our objective is to identify coordinating contracts that do not provide incentive for the supplier to renegotiate with one or more retailers.

The incentive to renegotiate also depends on the prospect of future relationship. If the relationship between the supplier and the retailers is a one-shot game, then there may be more incentive for the supplier to renegotiate with a one or more retailers to exploit the immediate gain. However, in case of repeated interaction, there is less incentive for the supplier to exploit an immediate gain at the expense of some retailers, because it is important for the supplier to keep a trusting relationship with all retailers. Therefore, it is interesting to study the robustness of contracts to renegotiation when the relationship between the supplier and the retailers is short term or long term.

We first apply the concept of contract equilibrium (Crémer and Riordan (1987); O'Brien and Shaffer (1992)) to study the incentive of the supplier to renegotiate with any single retailer in a one shot-game. Then we study the case when the supplier and the retailers engage in a long-term relationship. To do this, we analyze a repeated game and employ the concept of relational contracts. Our objective is to identify coordinating contracts that are robust to bilateral renegotiation in a long-term relationship.

Contracts that are robust to bilateral renegotiation may not be robust to multilateral renegotiation. This means that, the supplier can renegotiate with more than one retailer and jointly gain from the renegotiation. Hence, we extend the concept of contract equilibrium to the concept of strong contract equilibrium. With this new concept, we study the incentive of the supplier to renegotiate with any proper subset of retailers in both the one-shot game and repeated game setting.

In Section 4.2, we introduce a general model with a supplier serving multiple retailers. In Section 4.3, we apply the concept of contract equilibrium to study the robustness to bilateral renegotiation both when the supplier-retailers relationship is a one shot-game and when the supplier interact with the retailers repeatedly. In Section 4.4, we introduce the concept of strong contract equilibrium to study the robustness of coordinating contracts to multilateral renegotiation. Finally, Section 4.5 provides a summary with some concluding remarks.

4.2 Model

Consider a two echelon supply chain with a supplier serving n multiple retailers, who face some uncertainty ω. At the beginning of the period, after finalizing the contracts with the supplier but before the uncertainty is realized, each retailer has to decide its action, σ_i, which may be vector. Examples of retailers' actions include price, inventory and effort. At the same time, the supplier has to decide its action, σ_0, which may include capacity, quality, etc. We denote $\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_n)$ as the action profile of the supplier and all retailers.

The demand of each retailer, $d_i(\boldsymbol{\sigma}, \omega)$, which is a function of the actions of the supplier and all retailers, has a distribution function $F_i(d|\boldsymbol{\sigma})$. We make the following assumptions on the demand of each retailer.

Assumption 4.2.1 *$F_i(d_i|\boldsymbol{\sigma})$ is continuous in d_i for all d_i, $\boldsymbol{\sigma}$ and i.*

Assumption 4.2.2 *$\frac{\partial F_i(d_i|\boldsymbol{\sigma})}{\sigma_j^k} \neq 0$ for $inf\{d|F_i(d|\boldsymbol{\sigma}) > 0\} < d_i < sup\{d|F_i(d|\boldsymbol{\sigma}) < 1\}$ for all $j \neq i$ and k.*

Assumption 4.2.3 *$F_i(d_i|\boldsymbol{\sigma})$ is differentiable in $\boldsymbol{\sigma}$ for all d_i, $\boldsymbol{\sigma}$ and i.*

Assumption 4.2.1 implies that the demand of each retailer has a continuous distribution. Assumption 4.2.2 implies that every point between the lowest possible demand and the highest possible demand has a position probability density. Many commonly used distributions satisfy Assumptions 4.2.1 and 4.2.2. Assumption 4.2.3 implies that demand of each retailer is differentiable in the action profile, and it is satisfied by many commonly used demand models.

The payoff of the supplier, $u_0(\boldsymbol{\sigma})$, depends on the its action and the actions of all retailers. This payoff may be the production cost of the supplier, in which case the revenue is negative. The payoff of each retailer, $u_i(\sigma_0, \sigma_i, d_i(\cdot))$, is a function of its demand, its action and the supplier's actions. A retailer's payoff can include only the sales revenue and other components, such as the effort cost if effort is a decision of the retailer. We assume the following.

Assumption 4.2.4 *$u_i(\sigma_0, \sigma_i, d_i)$ is increasing in d_i for all σ_0, σ_i and i.*

Assumption 4.2.4 is intuitive. In all supply chains, the payoff of a retailer is increasing in its demand.

At the beginning of the period, before everyone decides its action, the supplier signs a contract with each retailer, regarding the terms of payment. Each contract is denoted as $t_i(\sigma_0, \sigma_i, d_i)$, which is the payment function from retailer i to the supplier. Let $\boldsymbol{T} = (t_1, \ldots, t_n)$ be the profile of contracts between the supplier and all retailers.

We assume that every firm, either a retailer or the supplier, is risk-neutral and maximizes its expected profit when making decisions. Each retailer i receives a payoff of $u_i(\cdot)$ and pays the supplier $t_i(\cdot)$. Hence, its profit function is given by

$$\pi_i(\sigma_0, \sigma_i, d_i) = u_i(\sigma_0, \sigma_i, d_i) - t_i(\sigma_0, \sigma_i, d_i)$$

and its expected profit of retailer is given by

$$U_i(\boldsymbol{\sigma}) = E[u_i(\sigma_0, \sigma_i, d_i(\boldsymbol{\sigma}, \omega)) - t_i(\sigma_0, \sigma_i, d_i(\boldsymbol{\sigma}, \omega))].$$

The supplier gets a revenue of $u_0(\cdot)$ and receives a payment of $t_i(\cdot)$ from each retailer. Hence, its expected profit is given by

$$U_0(\boldsymbol{\sigma}) = E[\sum_{i=1}^{n} t_i(\sigma_0, \sigma_i, d_i(\boldsymbol{\sigma}, \omega)) + u_0(\boldsymbol{\sigma})].$$

The total expected profit of the supply chain, which is the sum of the expected profits of the supplier and all retailers, is given by

$$\Pi(\boldsymbol{\sigma}) = \sum_{i=0}^{n}(\boldsymbol{\sigma}) = u_0(\boldsymbol{\sigma}) + \sum_{i=1}^{n} E[u_i(\sigma_0, \sigma_i, d_i(\boldsymbol{\sigma}, \omega))].$$

We denote $\boldsymbol{\sigma}^* = argmax_{\boldsymbol{\sigma}}\Pi(\boldsymbol{\sigma})$ as the supply chain optimal action profile.

4.3 Bilateral Renegotiation

In this section, we study robustness of contracts to bilateral renegotiation. In particular, we study whether there is joint incentive for the supplier and any one retailer to change their bilateral contract, after the supplier has signed contracts with all the retailers. We first start with the case when the supplier-retailers relationship is a one-shot game, and then move to the case when the supplier engages in long term relationship with the retailers.

4.3.1 One-shot game

We first apply the concept of contract equilibrium (Crémer and Riordan (1987); O'Brien and Shaffer (1992)) to study the case when the supplier-retailers relationship is a one-shot game. The concept of contract equilibrium studies whether there is joint incentive for the supplier and any one retailer to unilaterally change their contract terms, given the contracts between the supplier and other retailers unchanged. A contract profile and a corresponding Nash equilibrium action profile constitutes a contract equilibrium if the supplier and any one retailer do not have a joint profitable deviation.

Definition 4.3.1 *A contract profile* $\boldsymbol{T}$ *and a corresponding Nash equilibrium decision profile* $\boldsymbol{\sigma}^*$ *is a contract equilibrium if* $U_0(\boldsymbol{\sigma}^*) + U_i(\boldsymbol{\sigma}^*) \geq U_0(\sigma_0, \sigma_i, \boldsymbol{\sigma}^*_{-\{0,1\}}) + U_i(\sigma_0, \sigma_i, \boldsymbol{\sigma}^*_{-\{0,1\}})$ *for all* σ_0 *and* σ_i *for all* i.

A contract equilibrium is formed if no supplier-retailer pair can cooperate to increase their total profits, given the contract with other retailers and other retailers' decisions unchanged. Other retailers' decisions are fixed because there may exist no or multiple equilibria after the supplier changes its contract with one retailer. Another reason is that the retailer can change its action late enough so that other retailers do not have time to respond. The contracts between the supplier and other retailers are fixed because these contracts are already signed. Hence, the concept of contract equilibrium studies whether the supplier and any one single retailer can make a jointly profitable change to their contract after the supplier finalizes its contract with all other retailers. This is similar to the market-by-market bargaining restriction in Hart and Tirole (1990) and the pair-wise-proof concept in McAfee and Schwartz (1994).

There exist coordinating contracts that can constitute contract equilibrium. For example, if the supplier takes all the supply chain revenue while each retailer only receives a fixed payment regardless of the demand, then the supply chain optimal action profile is a Nash equilibrium. Under this contract, there is no jointly profitable deviation of the supplier and any one retailer from the supply chain optimal action, if other retailers carry out the supply chain optimal actions. This type of contract together with the system optimal action profile constitute a contract equilibrium

However, there are also many coordinating contracts that are not robust to bilateral renegotiation. As we show later in this section, there exists a large class of contracts, which include revenue sharing, wholesale price and buyback contract, which cannot constitute a contract equilibrium in many cases. To do this, we first introduce an important class of contracts.

Definition 4.3.2 *A contract profile* $\boldsymbol{T}(\cdot)$ *is* profit-demand monotone *if the profit of retailer* i, $\pi_i(\sigma_0, \sigma_i, d_i) = u_i(\sigma_0, \sigma_i, d_i) - t_i(\sigma_0, \sigma_i, d_i)$, *is non-decreasing in the demand* d_i *for all* $\boldsymbol{\sigma}$ *for all* i.

Many contracts, including wholesale price, buyback, revenue sharing, and consignment contracts, belong to the class of profit-demand monotone contracts. In fact, all contracts that have been studied in literature belong to this class. Any contract that is not profit-demand monotone is not practical because it is not reasonable to penalize the retailer for a higher demand. Next, we will introduce a general property of profit-demand monotone contracts.

Definition 4.3.3 *A contract profile* $\boldsymbol{T}(\cdot)$ *is* strictly profit-demand monotone *if*

(1) $\boldsymbol{T}(\cdot)$ *is monotone.*

(2) For all i, *there exists* $inf\{d|F(d|\boldsymbol{\sigma}^*) > 0\} < d_i < \{d|F(d|\boldsymbol{\sigma}^*) < 1\}$ *such that the profit of retailer* i, $\pi_i(\boldsymbol{\sigma}^*, d_i)$, *is strictly increasing in* d_i.

Definition 4.3.3 implies that the profit of every retailer is strictly increasing in demand at some point if the supplier and all retailers carry out the supply chain optimal actions. Hence, if $\boldsymbol{T}$ is strictly profit-demand monotone, then for each retailer i there exists a demand point d_i such that $\pi_i(\sigma_0, \sigma_i, d_i + \epsilon) > \pi_i(\sigma_0, \sigma_i, d_i)$ for all $\epsilon > 0$. Many contracts, including wholesale price, buyback, revenue sharing and sales rebate contracts are strictly profit-demand monotone. The only contracts that are profit-demand monotone but not strictly profit-demand monotone are contracts under which the retailers' profits do not change in the demand. As mentioned before, this type of contracts always constitute contract equilibrium with the system optimal action profile. Therefore, our focus is to study whether strictly profit-demand monotone contracts can constitute a contract equilibrium with the system optimal action profile. To do this, we introduce a common property of retailers' decisions.

Definition 4.3.4 *A component of a retailer's decision σ_i^k has homogenous impact on other retailers at $\boldsymbol{\sigma}$ if*

(a) $\frac{\partial F_j(d_j|\boldsymbol{\sigma})}{\sigma_i^k} \neq 0$ *and* $\frac{\partial F_{j'}(d_{j'}|\boldsymbol{\sigma})}{\sigma_i^k} \neq 0$

(b) $sgn(\frac{\partial F_j(d_j|\boldsymbol{\sigma})}{\sigma_i^k}) = sgn(\frac{\partial F_{j'}(d_{j'}|\boldsymbol{\sigma})}{\sigma_i^k})$

for all $j, j' \neq i$, $inf\{d|F_j(d|\boldsymbol{\sigma}) > 0\} < d_j < sup\{d|F_j(d|\boldsymbol{\sigma}) < 1\}$ *and* $inf\{d|F_{j'}(d|\boldsymbol{\sigma}) > 0\} < d_{j'} < sup\{d|F_{j'}(d|\boldsymbol{\sigma}) < 1\}$.

Definition 4.3.4 implies that a retailer's decision has a homogenous impact on other retailers if it affects other retailers' demands in the same direction. For example, in a supply chain with multiple retailers selling substitutable products, the demand of every retailer is increasing in another retailer's price. In this case, the price of every retailer has homogenous impact on other retailers.

A retailer's inventory decision may also have homogenous impact on other retailers. For example, a retailer's inventory display may increase its demand and decrease the demand of other retailers (Wang and Gerchak (2001)). In this case, a retailer's inventory affects other retailers' demand in the same direction.

A retailer's effort may increase or decrease other retailers' demand (Shum and Simchi-Levi (2006)), depending on the type of effort. For example, if effort means customer service for every retailer, then every retailer's effort hurts any other retailer's demand. Hence, in this case, a retailer's effort has homogenous impact on other retailers.

Therefore, many types of retailers' decision have homogenous impact on other retailers. As we show in the next theorem, when a component of a retailer's decision has homogenous impact on other retailers, it is hard to find a coordinating contract profile that form a contract equilibrium with the supply chain optimal decision profile.

Theorem 4.3.5 *Suppose $\boldsymbol{T}(\cdot)$ is coordinating, i.e. $\boldsymbol{\sigma}^*$ is a Nash equilibrium under contract profile $\boldsymbol{T}(\cdot)$. If $\boldsymbol{T}(\cdot)$ is strictly profit-demand monotone and a component of retailer i's decision, σ_i^k, has homogeneous impact on other retailers at $\boldsymbol{\sigma}^*$, then $\boldsymbol{T}(\cdot)$ is not contract equilibrium at $\boldsymbol{\sigma}^*$.*

Proof. Since $\boldsymbol{\sigma}^*$ is system optimal, then $\frac{\Pi(\boldsymbol{\sigma}^*)}{\partial \sigma_i^k} = 0$. Now, consider $U_0(\boldsymbol{\sigma}) + U_i(\boldsymbol{\sigma}) = \Pi(\boldsymbol{\sigma}) - \sum_{j \notin \{0,i\}} U_j(\boldsymbol{\sigma})$. Let $G_j(\cdot|\boldsymbol{\sigma}^*)$ be the conditional distribution function of π_j given $\boldsymbol{\sigma}^*$. Since $\boldsymbol{T}(\cdot)$ is monotone, $G_j(x|\boldsymbol{\sigma}) = F_j(d_j(x, \boldsymbol{\sigma})|\boldsymbol{\sigma})$ where $d_j(x, \boldsymbol{\sigma}) = sup(d|\pi_j(\boldsymbol{\sigma}, d) \leq x)$ for all $\boldsymbol{\sigma}$ and $j = 1, \ldots, n$ such that $j \neq i$. Hence,

$$\begin{aligned} U_j(\boldsymbol{\sigma}) &= E[\pi_j(\boldsymbol{\sigma}, d_j(\cdot))] \\ &= \int_0^\infty [1 - G_j(x|\boldsymbol{\sigma})]dx - \int_{-\infty}^0 G_j(x|\boldsymbol{\sigma})dx \\ &= \int_0^\infty [1 - F_j(d_j(x, \boldsymbol{\sigma})|\boldsymbol{\sigma})]dx - \int_{-\infty}^0 F_j(d_j(x, \boldsymbol{\sigma})|\boldsymbol{\sigma})dx. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial U_j(\boldsymbol{\sigma})}{\partial \sigma_i^k}|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}^*} &= \int_{-\infty}^\infty \frac{F_j(d_j(x, \boldsymbol{\sigma}))}{\partial \sigma_i^k}|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}^*} dx \\ &= \int_{-\infty}^\infty |\frac{F_j(d_j(x, \boldsymbol{\sigma}))}{\partial \sigma_i^k}|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}^*} dx, \end{aligned}$$

where the second equality follows from the fact that σ_i^k has homogenous impact on other retailers.

Since $\boldsymbol{T}$ is strictly monotone at $\boldsymbol{\sigma}*$, there exists $x' < x"$ such that $0 < F(d_j(x', \boldsymbol{\sigma}^*)) < F(d_j(x", \boldsymbol{\sigma}*)) < 1$. Hence, $\frac{\partial F(d_j(x,\boldsymbol{\sigma}))}{\partial \sigma_i^k}|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}^*} \neq 0$ for $\tilde{x}' \leq x \leq \tilde{x}"$ for some $x' \leq \tilde{x}' \leq \tilde{x}" \leq x"$, implying $\frac{\partial U_j(\boldsymbol{\sigma})}{\partial \sigma_i^k}|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}^*} \neq 0$.

Therefore,

$$\begin{aligned} \frac{\partial}{\partial \sigma_i^k}[U_0(\boldsymbol{\sigma}) + U_i(\boldsymbol{\sigma})]|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}^*} &= \frac{\partial \Pi(\boldsymbol{\sigma})}{\partial \sigma_i^k}|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}^*} - \sum_{j \notin \{0,i\}} \frac{\partial U_j(\boldsymbol{\sigma})}{\partial \sigma_i^k}|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}^*} \\ &= - \sum_{j \notin \{0,i\}} |\frac{\partial U_j(\boldsymbol{\sigma})}{\partial \sigma_i^k}|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}^*} \\ &\neq 0 \end{aligned}$$

where the second equality follows from the fact that $\frac{\partial \Pi(\boldsymbol{\sigma})}{\partial \sigma_i^k}|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}^*} = 0$ and σ_i^k has homogeneous impact on the output of other retailers. ■

Theorem 4.3.5 implies that if retailers compete in price, inventory, or effort, or more than one or even all of them, then a large class of contracts, including buyback, revenue sharing and sales rebate contracts cannot coordinate the supply chain and constitute a contract equilibrium with the supply chain optimal decision profile at the same time. In other words, many coordinating contracts are not robust to renegotiation when the supplier-retailers relationship is only a one-shot game. Next, we will study the case when the supplier and the retailers interact repeatedly to see if more coordinating contracts are robust to renegotiation under long-term relationship.

4.3.2 Long-term relationship

Now suppose that the game in 4.3.1 repeats in periods $\tau = 1, 2, \ldots$, with the supplier selling to the retailers in each period. We assume that no inventory is carried over from one period to another. This is the case when, for example, the products sold by the supplier are perishable, or the supplier sells distinctive products in each period.

To study the long-term relationship between the supplier and the retailers, we employ the concept of relational contracts. Relational contracts have been studied in economics literature (Baker et al. (2002); Levin (2003); Bolton and Dewatripont (2005)) to coordinate non-contractible decisions. For example, in a employer-employee relationship, it is hard to include detailed criteria of promotion and compensation in a formal employment contract. In this case, relational contracts ensure both the employer and the employee keep their promise.

Relational contracts have also been studied in different supply chains. Relational contracts can be used to ensure that all firms invest the optimal effort in a production partnership (Plambeck and Taylor (2006)), or to provide incentive to the supplier to invest in a higher capacity in the early product development stage (Taylor and Plambeck (2007a,b)).

In all these examples, relational contracts are used to create incentives. Here, we use relational contracts to eliminate the incentive of any supplier-retailer pair to deviate from a coordinating contract. Relational contracts allow contracts that are otherwise not coordinating to coordinate the supply chain. However, our objective here is to study the incentive of the firms to deviate from coordinating contracts. Hence, we assume that $\boldsymbol{T}(\cdot)$ is a coordinating contract profile. Furthermore, we assume that there is one unique supply chain system optimal profile, and all firms pick the Pareto-efficient equilibrium when there exists multiple equilibria. Then, $\boldsymbol{\sigma}^*$ is the resulting action if all firms do not deviate from $\boldsymbol{T}(\cdot)$.

We assume that the supplier and all retailers employ trigger strategies that are standard in economics literature (Baker et al. (2002); Levin (2003); Bolton and Dewatripont (2005)). This means that, the supplier and each retailer will renew the same contract and transact every period, until one firm refuses to do so, and then refuse to perform any transaction between them in all subsequent periods.

Under this trigger strategy, the supplier renews the same contract $t_i(\cdot)$ with each retailer every time period. In this case, $\boldsymbol{\sigma}^*$ is the resulting action in every period. The profit-to-go for the supplier is given by

$$U_0^\infty = \frac{1}{1-\delta} U_0(\sigma^*).$$

The profit-to-go for each retailer is given by

$$U_i^\infty = \frac{1}{1-\delta} U_i(\sigma^*), i = 1, \ldots, n.$$

On the other hand, if the supplier and retailer i change their bilateral contract and deviate from $\boldsymbol{\sigma}^*$ after the supplier signs contracts with all retailers, then other retailers $j \neq i$ will refuse to renew contracts and refuse to transact with the supplier for all subsequent periods. In this case, we consider the profits of retailer i and the supplier for the current period and all subsequent periods.

If the supplier and retailer i change their terms of contract, their maximum total expected profits for the current period is

$$U^{i*} = max_{\sigma_0,\sigma_i}(U_0(\sigma_0, \sigma_i, \boldsymbol{\sigma}^*_{\{-0,i\}}) + U_i(\sigma_0, \sigma_i, \boldsymbol{\sigma}^*_{\{-0,i\}}))$$

Here, we assume that other retailers still play $\boldsymbol{\sigma}^*_{\{-0,i\}}$, i.e., the supplier and retailer i

change late enough so that other retailers do not have time to respond. Suppose the supplier and retailer i achieve their maximized total profit U^{i*} in the renegotiated contract. Then, the supplier is allocated an expected profit of U_0^i and retailer i is allocated an expected profit of U_i^i, where $U_i^i + U_0^i = U^{i*}$

For subsequent periods, since retailers other than retailer i will refuse to renew contract and transact with the supplier, retailer i and the supplier will face new demand function, $\tilde{d}_i(\sigma_0, \sigma_i, \omega)$, and new payoff functions, $\tilde{u}_0^i(\sigma_0, \sigma_i, d_i)$ and $\tilde{u}_i(\sigma_0, \sigma_i, d_i)$. We assume that the supplier and retailer i follows the Nash bargaining solution in subsequent periods. Hence, there exists α_i such that the per-period expected profits in each subsequent period for retailer i and the supplier are $\alpha_i \tilde{U}^{i*}$ and $(1-\alpha_i)\tilde{U}^{i*}$ respectively, where $\tilde{U}^{i*} = max_{\sigma_0,\sigma_i} E[\tilde{u}_0^i(\sigma_0, \sigma_i, d_i) + \tilde{u}_i(\sigma_0, \sigma_i, d_i)]$

Combining the expected profits for current period and for all subsequent periods, the profit-to go of the supplier if it changes its terms of contract with retailer i is given by

$$\tilde{U}_0^{i,\infty} = \frac{\delta}{1-\delta}(1-\alpha_i)\tilde{U}^{i*} + U_0^i.$$

The profit-to go of retailer i is given by

$$\tilde{U}_i^{\infty} = \frac{\delta}{1-\delta}\alpha_i \tilde{U}^{i*} + U_i^i,$$

where δ is the discount factor and it is assumed to be the same for the supplier and all retailers.

The supplier and retailer i will both agree to change the terms of contracts if they cannot increase their profits-to-go simultaneously. We call a coordinating contract profile $\boldsymbol{T}(\cdot)$ and the supply chain optimal action profile $\boldsymbol{\sigma}^*$ a *contract equilibrium in the repeated game* if there is no incentive for the supplier and any single retailer to change the terms of contracts. Formally

Definition 4.3.6 *A coordinating contract profile* $\boldsymbol{T}(\cdot)$ *and the supply chain optimal action profile* $\boldsymbol{\sigma}^*$ *form a* contract equilibrium in the repeated game *if there does not exist* U_i^i *and* U_0^i *where* $U_i^i + U_0^i = U^{i*}$ *such that* $\tilde{U}_i^{\infty} > U_i^{\infty}$ *and* $\tilde{U}_0^{i,\infty} > U_0^{i,\infty}$.

Hence, $\boldsymbol{T}(\cdot)$ and $\boldsymbol{\sigma}^*$ form a contract equilibrium in the repeated game if and only if there does not exist U_i^i and U_0^i where $U_i^i + U_0^i = U^{i*}$ such that

$$\frac{\delta}{1-\delta}(U_0(\sigma^*) - (1-\alpha_i)\tilde{U}^{i*}) < U_0^i - U_0(\sigma^*), \tag{4.1}$$

and

$$\frac{\delta}{1-\delta}(U_i(\sigma^*) - \alpha_i\tilde{U}^{i*}) < U_i^i - U_i(\sigma^*). \tag{4.2}$$

Combining equations (4.1) and (4.2), we show the following lemma.

Lemma 4.3.7 *A coordinating contract profile $\boldsymbol{T}(\cdot)$ and the supply chain optimal action profile $\boldsymbol{\sigma}^*$ form a contract equilibrium in the repeated game if and only if*

$$\frac{\delta}{1-\delta}(U_0(\sigma^*) + U_i(\sigma^*) - \tilde{U}^{i*}) \geq U^{i*} - (U_0(\sigma^*) + U_i(\sigma^*)) \tag{4.3}$$

for all $i \in \{1, \ldots, n\}$

Proof. For one direction, if $\boldsymbol{T}(\cdot)$ and $\boldsymbol{\sigma}^*$ do not form a contract equilibrium in the repeated game, there exists U_i^i and U_0^i where $U_i^i + U_0^i = U^{i*}$ such that (4.1) and (4.2) hold. Combining the two equations, we have $\frac{\delta}{1-\delta}(U_0(\sigma^*) + U_i(\sigma^*) - \tilde{U}^{i*}) < U^{i*} - (U_0(\sigma^*) + U_i(\sigma^*))$, violating equation (4.3).

For the other direction, if equation (4.3) is violated, i.e., $\frac{\delta}{1-\delta}(U_0(\sigma^*) + U_i(\sigma^*) - \tilde{U}^{i*}) < U^{i*} - (U_0(\sigma^*) + U_i(\sigma^*))$. Then we can find U_i^i and U_0^i such that (4.1) and (4.2) hold, and hence $\boldsymbol{T}(\cdot)$ and $\boldsymbol{\sigma}^*$ do not form a contract equilibrium in the repeated game. ■

Observe that $\frac{\delta}{1-\delta}(U_0(\sigma^*) + U_i(\sigma^*) - \tilde{U}^{i*})$ is the loss of the supplier and retailer i due to the loss of transactions with other retailers in subsequent periods. $U^{i*} - (U_0(\sigma^*) + U_i(\sigma^*))$ represent the maximum amount that the supplier and retailer i can gain additionally in the current period by changing the terms of their contract. Hence, a contract equilibrium is formed if loss in future periods exceed the gain in the current period. The next proposition identifies contracts which cannot form a contract equilibrium in the repeated game.

Lemma 4.3.8 *For any contract profile $\boldsymbol{T}$ and its Nash equilibrium action profile $\boldsymbol{\sigma}$, if*

$$U^{i*} > U_0(\sigma^*) + U_i(\sigma^*) \tag{4.4}$$

for some i, then $\boldsymbol{T}$ and $\boldsymbol{\sigma}$ is not a contract equilibrium for all δ.

Proof. First, $U^{i*} - (U_0(\sigma^*) + U_i(\sigma^*)) \geq 0$ because $U^{i*} = max_{\sigma_0, \sigma_i}(U_0(\sigma_0, \sigma_i, \boldsymbol{\sigma}^*_{\{-0,i\}}) + U_i(\sigma_0, \sigma_i, \boldsymbol{\sigma}^*_{\{-0,i\}}))$. Hence, if $U_0(\sigma^*) + U_i(\sigma^*) - U^{i*} < 0$, then equation (4.3) does not hold for all δ. ■

Equation 4.4 implies that the supplier and retailer i receive an expected profit which is less than the maximum expected profit they can get even if they don't transact with other retailers. Hence, according to Proposition 4.3.8, if a contract profile that allocate to any supplier-retailer pair an expected profit which is less than the maximum profit they can get even when they do not transact with other retailers, then this contract profile cannot form a contract equilibrium in the repeated game.

Proposition 4.3.8 implies that certain $\boldsymbol{T}(\cdot)$ and $\boldsymbol{\sigma}$ may form a contract equilibrium in the one-shot game, but not a contract equilibrium in the repeated game. For example, as discussed before, if all retailers receive a fixed profit regardless of demand under $\boldsymbol{T}(\cdot)$, then $\boldsymbol{T}(\cdot)$ and $\boldsymbol{\sigma}^*$ form a contract equilibrium. However, if the total expected profit of the supplier and some retailer i is less than U^{i*}, i.e., the maximum expected profit they can get without transacting with other retailers, then $\boldsymbol{T}(\cdot)$ and $\boldsymbol{\sigma}^*$ do not form a contract equilibrium in the repeated game. In fact, if the supplier and some retailer i receive a total expected profit of less than U^{i*} under $\boldsymbol{T}(\cdot)$, they should not agree to $\boldsymbol{T}(\cdot)$ because $\boldsymbol{T}(\cdot)$ is not a rational contract profile (as discussed in Chapter 3).

The concept of rational contracts was developed to better understand whether some rational decision maker(s) may have incentive to deviate from any particular coordinating contract and hence will not agree to it. A coordinating contract profile $\boldsymbol{T}(\cdot)$ is rational if every subset of agents receive an expected profit no less than what they can guarantee on their own.

To study whether $\boldsymbol{T}(\cdot)$ is rational, we have to consider the case when only retailers $i \in \mathcal{J}$ remains in the supplier chain and the supplier does not transact with retailers $i \notin \mathcal{J}$. Let $d_i(\sigma_0, \boldsymbol{\sigma}_{\mathcal{J}}, \omega), i \in \mathcal{J}$ be the new demand functions, and let $\tilde{u}_0^{\mathcal{J}}(\sigma_0, \sigma_{\mathcal{J}}, d_i)$ and $\tilde{u}_i^{\mathcal{J}}(\sigma_0, \sigma_{\mathcal{J}}, d_i)$ be the new payoff functions. Then, the maximum expected profit the supplier and this subset of retailers $i \in \mathcal{J}$ can get on their own is $\tilde{U}^{\mathcal{J}*} = max_{\sigma_0, \boldsymbol{\sigma}_{\mathcal{J}}} E[u_0^{\mathcal{J}}(\sigma_0, \sigma_{\mathcal{J}}, d_i) + \sum_{i \in \mathcal{J}} u_i^{\mathcal{J}}(\sigma_0, \sigma_{\mathcal{J}}, d_i)]$. A contract profile $\boldsymbol{T}$ is rational if and only if the total expected profit of the supplier and all retailers in the subset $\imath \in \mathcal{J}$, $U_0(\sigma^*) + \sum_{i \in \mathcal{J}} U_i(\sigma^*)$ is no less than $\tilde{U}^{\mathcal{J}*}$ for all $\mathcal{J}$.

Hence, if $\boldsymbol{T}(\cdot)$ is rational, then $\tilde{U}^{i*} \leq U_0(\sigma^*) + U_i(\sigma^*)$ for all i. This implies that, if $\tilde{U}^{i*} > U_0(\sigma^*) + U_i(\sigma^*)$ for some i, then $\boldsymbol{T}(\cdot)$ is not rational. This explains why a contract profile $\boldsymbol{T}(\cdot)$ may form a contract equilibrium in the one-shot game but not in the repeated game. After the supplier signs $\boldsymbol{T}(\cdot)$ with the retailers, even if the supplier and retailer i receive an expected profit less than what they can get without transacting with other retailers, there may be no incentive for the supplier and any single retailer to change the terms of contracts because the supplier can no longer choose not to transact with other retailers. However, the supplier and retailer i may choose not to transact with other retailers or not to renew the same contract profile $\boldsymbol{T}(\cdot)$ in subsequent periods. Of course, it is questionable why the supplier and retailer i agree to $\boldsymbol{T}(\cdot)$ in the first place. The next theorem shows that any coordinating contract profile, $\boldsymbol{T}(\cdot)$, which forms a contract equilibrium in the one-shot game with $\boldsymbol{\sigma}^*$ also forms a contract equilibrium in the repeated game if $\boldsymbol{T}(\cdot)$ is rational.

Theorem 4.3.9 *Let $\boldsymbol{T}(\cdot)$ be a coordinating and rational contract profile. If $\boldsymbol{T}(\cdot)$ and $\boldsymbol{\sigma}^*$ form a contract equilibrium in the one-shot game, then for all δ such that for all $\delta \geq \underline{\delta}$, $\boldsymbol{T}(\cdot)$ and $\boldsymbol{\sigma}^*$ also form a contract equilibrium in the repeated game.*

Proof. Since $\boldsymbol{T}(\cdot)$ is rational, $U_0(\sigma^*)+U_i(\sigma^*)-\tilde{U}^{i*} \geq 0$. In addition, $\boldsymbol{T}(\cdot)$ and $\boldsymbol{\sigma}^*$ form a contract equilibrium in the one-shot game, implying that $U^{i*}-(U_0(\sigma^*)+U_i(\sigma^*)) = 0$. Hence, equation (4.3) holds for all δ. ∎

Theorem 4.3.9 shows that any coordinating and rational contract profile forms a contract equilibrium in the repeated game with the supply chain optimal action profile if they form a contract equilibrium in the one shot game. However, as Theorem 4.3.5 shows, many coordinating contracts do not form a contract equilibrium in the one shot game with the supply chain optimal action profile. On the other hand, the supplier may have less incentive to change terms of contracts with a retailer in the current period in the fear of loss of transactions with other retailers in subsequent periods. Hence, the questions is whether some coordinating contracts that do not form a contract equilibrium in the one-shot game can form a contract equilibrium in the repeated game. The next theorem shows that it is possible.

Theorem 4.3.10 *Let $\boldsymbol{T}(\cdot)$ be a coordinating and rational contract profile. If*

$$U^{i*} < U_0(\sigma^*) + U_i(\sigma^*), \tag{4.5}$$

then there exists $\underline{\delta} > 0$ such that $\boldsymbol{T}(\cdot)$ and $\boldsymbol{\sigma}^*$ form a contract equilibrium in the repeated game for all $\delta \geq \underline{\delta}$.

Proof. Let

$$\underline{\delta}_i = \frac{U^{i*} - (U_0(\boldsymbol{\sigma}^*) + U_i(\boldsymbol{\sigma}^*))}{U^{i*} - \tilde{U}^{i*}}. \tag{4.6}$$

Let $\underline{\delta} = \max_i \underline{\delta}_i$. Then for all $\delta \geq \underline{\delta}$ and $i = 1, \ldots, n$,

$$\begin{aligned}
\frac{\delta}{1-\delta}(U_0(\boldsymbol{\sigma}^*) + U_i(\boldsymbol{\sigma}^*) - \tilde{U}^{i*}) &\geq \frac{\underline{\delta}}{1-\underline{\delta}}(U_0(\boldsymbol{\sigma}^*) + U_i(\boldsymbol{\sigma}^*) - \tilde{U}^{i*}) \\
&\geq \frac{\underline{\delta}_i}{1-\underline{\delta}_i}(U_0(\boldsymbol{\sigma}^*) + U_i(\boldsymbol{\sigma}^*) - \tilde{U}^{i*}) \\
&= U^{i*} - (U_0(\boldsymbol{\sigma}^*) + U_i(\boldsymbol{\sigma}^*)).
\end{aligned}$$

Hence (4.3) holds for all i for all $\delta \geq \underline{\delta}$. ■

The proof of Theorem 4.3.10 implies that the minimum discount factor, $\underline{\delta}$, is the maximum of all $\underline{\delta}_i$, where $\underline{\delta}_i$ is the minimum discount factor such that there is no incentive for the supplier and retailer i to change the terms of their contract. From the proof, we observe the following.

Observation 4.3.11 *For all i,*

(1) $\underline{\delta}_i$ is strictly increasing in $U^{i*} - (U_0(\boldsymbol{\sigma}^*) + U_i(\boldsymbol{\sigma}^*))$.

(2) $\underline{\delta}_i$ is strictly decreasing in $U_0(\boldsymbol{\sigma}^*) + U_i(\boldsymbol{\sigma}^*) - \tilde{U}^{i*}$

$U^{i*} - (U_0(\boldsymbol{\sigma}^*) + U_i(\boldsymbol{\sigma}^*))$ represents the potential gain in current period by changing the terms of contract. Hence, the higher the potential gain in current period by changing the terms of contract, the higher the incentive for changing the terms of contract, and hence a higher discount factor is needed for the value of future relationship to offset the potential gain in current period. On the other hand, $U_0(\boldsymbol{\sigma}^*) + U_i(\boldsymbol{\sigma}^*) - \tilde{U}^{i*}$ represents the per-period loss in profit due to the loss of relationship with other retailers. The higher this loss is, the more valuable transactions with other retailers in future periods are, and hence a lower discount factor is needed to counteract the incentive to change the terms of contract for a gain in the current period.

Theorem 4.3.10 implies that a coordinating contract profile can form a contract equilibrium in the repeated game if $U^{i*} < U_0(\sigma^*) + U_i(\sigma^*)$, i.e., if supplier and retailer i are getting under $\boldsymbol{T}(\cdot)$ an expected profit that is strictly more than that they can guarantee on their own without transacting with other retailers. This condition is similar to but not exactly the same as the conditions for rational contracts. If a contract profile is rational, then the supplier and any single retailer get more than or equal to what they can guarantee without transacting with other retailers. However, if the supplier and a single retailer only receive a total expected profit equal to what they can guarantee without transacting with other retailers, there is no loss to them if other retailers refuse to transact with them in future periods. Hence, if there is a gain in current period by changing the terms of contract, there is still incentive to change. In this case, the contract does not form a contract equilibrium in the repeated game.

Although rational contracts do not guarantee the condition in equation (4.5), many coordinating contracts satisfy this condition. In fact, any coordinating and flexible contracts satisfy this condition. Hence, Theorem 4.3.10 implies that any coordinating and flexible contracts form a contract equilibrium in the repeated game with the supply chain optimal action profile as long as the discount factor is significantly high.

4.4 Multilateral Renegotiation

Besides changing terms of contract with one retailer, the supplier can also change the terms of contracts with several retailers. That is to say, even in cases when there is no incentive the supplier and any single retailer to change their terms of contracts, it is possible that the supplier and a subset of retailers can jointly gain by changing their terms of contracts. In this section, we study the robustness of contracts to multilateral renegotiation, i.e., we study whether there is incentive for the supplier and a subset of retailers to modify the terms of contracts.

4.4.1 One-Stage Game

We start with analyzing the case when the interaction between the supplier and the retailers is a one-shot game. In Section 4.3.1, we apply the concept of contract equilibrium to study the robustness of bilateral renegotiation. In this section, we extend the concept of contract equilibrium to study robustness of contracts to multilateral renegotiation.

To extend the concept of contract equilibrium, we introduce the notion of *strong contract equilibrium*, which has similar flavor to the concept of strong equilibrium in Aumann (1959).A strategy profile is a strong equilibrium if no subset can increase the payoffs of all its members by using a different strategy, given the strategies of players outside the subset unchanged. Similarly, we define *strong contract equilibrium* as follows.

Definition 4.4.1 *A contract profile $\boldsymbol{T}$ and a corresponding Nash equilibrium decision profile $\boldsymbol{\sigma}^*$ is a strong contract equilibrium if*

$$U_0(\boldsymbol{\sigma}^*) + \sum_{i \in \mathcal{J}} U_i(\boldsymbol{\sigma}^*) \geq U_0(\sigma_0, \boldsymbol{\sigma}_{\mathcal{J}}, \boldsymbol{\sigma}^*_{-\{0\}\cup\mathcal{J}}) + \sum_{i \in \mathcal{J}} U_i(\sigma_0, \boldsymbol{\sigma}_{\mathcal{J}}, \boldsymbol{\sigma}^*_{-\{0\}\cup\mathcal{J}})$$

for all σ_0 and $\boldsymbol{\sigma}_{\mathcal{J}}$ for all subset of retailers $\mathcal{J} \subseteq \{1, \ldots, n\}$.

A strong contract equilibrium is formed in the one-shot game, if the supplier and any subset of retailers cannot increase their profits by changing the terms of their contracts, holding the supplier's contracts with other retailers and the actions of other retailers constant. Hence, in strong contract equilibrium, the objective is to find whether there is incentive for the supplier to change its contracts with a subset of retailers after it has signed contracts with all retailers. For example, the contract discussed in Section 4.3.1, where each retailer receives a fixed payment independent of the demand and the supplier receives all the sales revenue, forms a strong contract equilibrium in the one-shot game with the supply chain optimal action profile.

However, many coordinating contracts do not form a strong contract equilibrium with the supply chain optimal action profile in the one-shot game. In fact, Definition 4.3.2 and Definition 4.4.1 together imply that, any coordinating contract that does not a form contract equilibrium in the one-shot game also does not form a strong contract equilibrium in the one-shot game. Hence, together with Theorem 4.3.5, we have the following theorem.

Theorem 4.4.2 *Suppose $\boldsymbol{\sigma}^*$ is a Nash equilibrium under contract profile $\boldsymbol{T}(\cdot)$ and $\boldsymbol{\sigma}^*$ is system optimal. If $\boldsymbol{T}(\cdot)$ is strictly monotone at $\boldsymbol{\sigma}^*$ and a component of retailer i's decision, σ_i^k, has homogeneous impact on other retailers at $\boldsymbol{\sigma}$, then $\boldsymbol{T}(\cdot)$ is not strong contract equilibrium at $\boldsymbol{\sigma}^*$.*

Theorem 4.4.2 implies that many coordinating contracts, such as buyback contracts, revenue sharing contracts, sales rebate and PDS schemes do not form strong contract equilibria with the supply chain optimal action profile in the one-shot game. In section 4.3.2, we show that long-term relationship can reduce the incentive for bilateral renegotiation between the supplier and any single retailer. Similarly, we want to know whether long-term relationship can reduce the incentive for multilateral renegotiation between the supplier and a subset of retailers. In particular, we will study whether more coordinating contracts form strong contract equilibria in the repeated game with the supply chain optimal action profile.

4.4.2 Long-term Relationship

To study robustness to multilateral renegotiations in long-term relationship, we analyze a repeated game similar to the one in Section 4.3.2. The supplier sells to all retailers in periods $\tau = 1, 2, \ldots$. We assume that no inventory is carried over from one period to another, i.s., the products are perishable or the products in different periods are distinctive.

We employ the same rational contracts and trigger strategies as in Section 4.3.2. The supplier renews the same contract $t_i(\cdot)$ with each retailer i, until one firm refuses to do so, and then refuse to perform any transactions between them in all subsequent periods.

Unless the supplier or one retailer refuses to transact in a period, the contract between the supplier and each retailer i remains $t(\cdot)$ for all periods $\tau = 1, 2, \ldots$.

In this infinite horizon model, the game is repeated every period. If nothing changes, the supplier renews the same contract with each retailer. In this case, the profit-to-go of the supplier is given by

$$U_0^\infty = \frac{1}{1-\delta} U_0(\sigma^*),$$

and profits-to-go for the retailers are given by

$$U_i^\infty = \frac{1}{1-\delta} U_i(\sigma^*), i = 1, \ldots, n.$$

On the other hand, if the supplier renegotiates its contract with a subset of retailers $\mathcal{J}$, then other retailers $j \notin \mathcal{J}$ will refuse to renew contracts and will end relationship forever with this supplier in future periods. In this case, we have to consider the profits of retailers $i \in \mathcal{J}$ and the supplier for this period and future periods.

if the supplier and a subset of retailers $\mathcal{J}$ change their terms of contract, their maximum total expected profit for the current period is given by

$$U^{\mathcal{J}*} = max_{\sigma_0, \boldsymbol{\sigma}_{\mathcal{J}}}(U_0(\sigma_0, \boldsymbol{\sigma}_{\mathcal{J}}, \boldsymbol{\sigma}^*_{-\{0\}\cup\mathcal{J}}) + U_i(\sigma_0, \boldsymbol{\sigma}_{\mathcal{J}}, \boldsymbol{\sigma}^*_{-\{0\}\cup\mathcal{J}})).$$

We assume that the supplier and the subset of retailers $\mathcal{J}$ achieve their maximum total expected profit in the renegotiated contracts. In addition, The supplier is allocated an expected profit of $U_0^{\mathcal{J}}$, and each retailer $i \in \mathcal{J}$ is allocated an expected profit of $U_i^{\mathcal{J}}$, where $\sum_{i \in \mathcal{J}} U_i^{\mathcal{J}} + U_0^{\mathcal{J}} = U^{\mathcal{J}*}$.

In subsequent periods, the supplier will lose $n - |\mathcal{J}|$ retailers since retailers $i \notin \mathcal{J}$ will refuse to transact with the supplier. In this case, the demand and payoff functions for retailers $i \in \mathcal{J}$ and the supplier will be different. Each retailer $i \in \mathcal{J}$ will face a new demand function $\tilde{d}_i(\sigma_0, \boldsymbol{\sigma}_{\mathcal{J}}, \omega)$ and a new payoff function $\tilde{u}_i^{\mathcal{J}}(\sigma_0, \boldsymbol{\sigma}_{\mathcal{J}}, d_i)$. The supplier will face a new payoff function $\tilde{u}_0^{\mathcal{J}}(\sigma_0, \boldsymbol{\sigma}_{\mathcal{J}}, \mathbf{y}_{\mathcal{J}})$.

We assume that profits in future periods are allocated according to the Nash bargaining solution. Hence, there exists α_i, $0 < \alpha_i < 1$, $i \in \mathcal{J}$, such that the per-period expected profits in future periods for each retailer $i \in \mathcal{J}$ and the supplier will be $\alpha_i \tilde{U}^{\mathcal{J}*}$ and $(1 - \sum_{i \in \mathcal{J}} \alpha_i)\tilde{U}^{\mathcal{J}*}$ respectively, where

$$\tilde{U}^{\mathcal{J}*} = max_{\sigma_0, \boldsymbol{\sigma}_{\mathcal{J}}} E[\tilde{u}_0^{\mathcal{J}}(\sigma_0, \boldsymbol{\sigma}_{\mathcal{J}}, \mathbf{y}_{\mathcal{J}}) + \tilde{\sum}_{i \in \mathcal{J}} u_i^{\mathcal{J}}(\sigma_0, \boldsymbol{\sigma}_{\mathcal{J}}, d_i)]$$

is the maximum total expected profit for the supplier and all retailers $i \in \mathcal{J}$ in each subsequent period.

Hence, if the supplier renegotiates contracts with a subset of retailers $\mathcal{J}$, the profit-to-go of the supplier is given by

$$\tilde{U}_0^{\mathcal{J},\infty} = \frac{\delta}{1-\delta}(1 - \sum_{i \in \mathcal{J}} \alpha_i)\tilde{U}^{\mathcal{J}*} + U_0^{\mathcal{J}},$$

and the profit-to-go of each retailer $i \in \mathcal{J}$ is given by

$$\tilde{U}_i^{\mathcal{J},\infty} = \frac{\delta}{1-\delta}\alpha_i \tilde{U}^{\mathcal{J}*} + U_i^{\mathcal{J}}$$

The supplier and a subset of retailers $\mathcal{J}$ will all agree to change the terms of contracts if they can all increase their profits simultaneously. We call a coordinating contract profile $\boldsymbol{T}(\cdot)$ and the supply chain optimal action profile $\boldsymbol{\sigma}^*$ a *strong contract equilibrium in the repeated game* if there is no incentive for the supplier and any subset of retailers $\mathcal{J}$ to change the terms of contracts. Formally,

Definition 4.4.3 *a coordinating contract profile* $\boldsymbol{T}(\cdot)$ *and the supply chain optimal action profile* $\boldsymbol{\sigma}^*$ *a* strong contract equilibrium in the repeated game *if there does not exist a subset of retailers* $\mathcal{J} \subseteq \{1, \ldots, n\}$, $U_i^{\mathcal{J}}, i \in \mathcal{J}$ *and* $U_0^{\mathcal{J}}$ *where* $\sum_{i \in \mathcal{J}} U_i^{\mathcal{J}} + U_0^{\mathcal{J}} = U^{\mathcal{J}*}$ *such that* $\tilde{U}_i^{\mathcal{J},\infty} > U_i^{\infty}$ *for all* $i \in \mathcal{J}$ *and* $\tilde{U}_0^{\mathcal{J},\infty} > U_0^{\infty}$.

Equivalently, $\boldsymbol{T}(\cdot)$ and $\boldsymbol{\sigma}^*$ form a strong contract equilibrium in the repeated game if there does not exist $\mathcal{J} \subseteq \{1, \ldots, n\}$, $U_i^{\mathcal{J}}, i \in \mathcal{J}$ and $U_0^{\mathcal{J}}$ where $\sum_{i \in \mathcal{J}} U_i^{\mathcal{J}} + U_0^{\mathcal{J}} = U^{\mathcal{J}*}$ such that

$$\frac{\delta}{1-\delta}(U_0(\sigma^*) - (1-\alpha_i)\tilde{U}^{\mathcal{J}*}) < U_0^{\mathcal{J}} - U_0(\sigma^*), \tag{4.7}$$

and

$$\frac{\delta}{1-\delta}(U_i(\sigma^*) - \alpha_i \tilde{U}^{\mathcal{J}*}) < U_i^{\mathcal{J}} - U_i(\sigma^*) \tag{4.8}$$

for all $i \in \mathcal{J}$. Combining (4.7) and (4.8), we have the following lemma.

Lemma 4.4.4 *A coordinating contract profile,* $\boldsymbol{T}(\cdot)$, *and the supply chain optimal action profile,* $\boldsymbol{\sigma}^*$, *form a strong contract equilibrium in the repeated game if and only if*

$$\frac{\delta}{1-\delta}(U_0(\sigma^*) + \sum_{i \in \mathcal{J}} U_i(\sigma^*) - \tilde{U}^{\mathcal{J}*}) \geq U^{\mathcal{J}*} - (U_0(\sigma^*) + \sum_{i \in \mathcal{J}} U_i(\sigma^*)) \tag{4.9}$$

for all $\mathcal{J} \subseteq \{1, \ldots, n\}$.

Proof. The proof is similar to the proof of Lemma 4.3.7 and hence we omit the proof. ∎

The left hand side of (4.9) represents the loss in future periods of the supplier and retailers $i \in \mathcal{J}$ due to the loss of transactions with retailers $i \notin \mathcal{J}$. The right hand side of (4.9) represents their gain in current period by changing the terms of their contracts. Hence, a coordinating contract profile and the supply chain optimal action profile form a strong contract equilibrium in the repeated game if the supplier and any subset of retailers lose more in future periods than their gain in current period by changing their terms of contracts.

Lemma 4.4.4 implies that a contract profile and the supply chain optimal action profile do not form a contract equilibrium if for some subset of retailers $\mathcal{J}$

$$U_0(\sigma^*) + \sum_{i \in \mathcal{J}} U_i(\sigma^*) - \tilde{U}^{\mathcal{J}*} < 0, \tag{4.10}$$

because the right hand side of Equation (4.9) is always non-negative. Since the condition in (4.10) is equivalent to the contract profile $\boldsymbol{T}(\cdot)$ not being rational, we have the following theorem.

Theorem 4.4.5 *A coordinating contract profile $\boldsymbol{T}(\cdot)$ does not form a strong contract equilibrium in the repeated game with the supply chain optimal action profile $\boldsymbol{\sigma}^*$ if it is not rational.*

Theorem 4.4.5 implies that any coordinating contract that is not rational cannot form a strong contract equilibrium in the repeated game with the supply chain optimal action profile, even if it forms a strong contract equilibrium in the one-shot game. If a contract profile is not rational, then the supplier and a subset of retailers receive an expected profit less than what they can get without transacting with other retailers. This does not mean that the contract profile does not form a strong contract equilibrium with the supply chain optimal action profile in the one-shot game, because it may not be possible for the supplier and the same subset of retailers to increase their profit by changing their terms of contracts when contracts with other retailers are already signed. However, the supplier and this subset of retailers may choose not to transact or renew the same contracts with other retailers. Hence, this contract profile does not form a strong contract equilibrium with the supply chain optimal action profile in the repeated game.

Now we know that any coordinating contract that is not rational cannot form a strong contract equilibrium with the supply chain optimal action profile in the repeated game. The next question is whether the reverse is true, i.e., whether any coordinating contract that is rational forms a strong contract equilibrium in the repeated game with the supply chain optimal action profile. The following theorem shows that this is true if the contract profile and the supply chain optimal action profile is a strong contract equilibrium in the one-shot game.

Theorem 4.4.6 *Suppose a coordinating contract profile $\boldsymbol{T}(\cdot)$ forms a strong contract equilibrium in the one-shot game with the supply chain optimal action profile $\boldsymbol{\sigma}^*$. Then, for all δ, $\boldsymbol{T}(\cdot)$ and $\boldsymbol{\sigma}^*$ form a strong contract equilibrium in the repeated game if and only if $T(\cdot)$ is rational.*

Proof. Theorem 4.4.5 implies that $\boldsymbol{T}(\cdot)$ and $\boldsymbol{\sigma}^*$ is not a strong contract equilibrium in the repeated game if $\boldsymbol{T}(\cdot)$ is not rational. Hence, one direction is shown.

For the other direction, if $\boldsymbol{T}(\cdot)$ is rational, $U_0(\sigma^*) + \sum_{i\in\mathcal{J}} U_i(\sigma^*) - \tilde{U}^{\mathcal{J}*} \geq 0$. In addition $\boldsymbol{T}(\cdot)$ and $\boldsymbol{\sigma}^*$ is a strong equilibrium in the one-shot game, meaning $U^{\mathcal{J}*} - (U_0(\sigma^*) + \sum_{i\in\mathcal{J}} U_i(\sigma^*)) = 0$. Hence, equation (4.9) holds for all δ. ■

Theorem 4.4.6 implies that a coordinating contract and the supply chain optimal action profile that form a strong contract equilibrium in the one-shot game also form a strong contract equilibrium in the repeated game if this contract profile is rational. However, as discussed before, many coordinating contracts do not form strong contract equilibria with the supply chain optimal action profile in the one-shot game. Hence, the question is whether some coordinating contracts that do not form strong contract equilibria in the one-shot game can form strong contract equilibria in the repeated game with the supply chain optimal action profile. In particular, we want to know if all coordinating and rational contracts form strong contract equilibria with the supply chain optimal action profile in the repeated game.

Observe that Definition 4.3.6 and Definition 4.4.3 imply that a coordinating contract profile and the supply chain optimal action profile do not form a strong contract equilibrium in the repeated game if they do not form a contract equilibrium in the repeated game. As discussed in Section 4.3.2, a coordinating and rational contract profile and the supply chain optimal action profile may not form a contract equilibrium in the repeated game. For example, suppose the supplier and a retailer receive an expected profit that is exactly same as the expected profit they get without transacting with other retailers. In this case, future relationship with other retailers does not increase the profit of the supplier and this retailer in future periods. Hence, they will change the terms of contract if they can gain in the current period, even if other retailers will refuse to transact with them in this case.

To solve this issue, we introduce the notion of *strictly rational contracts*, which is similar to the concept of rational contracts. A coordinating contract profile is strictly rational if every proper subset receive an expected profit that is strictly more than that they can guarantee on their own. Formally,

Definition 4.4.7 *A coordinating contract profile $\boldsymbol{T}(\cdot)$ is* strictly rational *if*

$$U_0(\sigma^*) + \sum_{i \in \mathcal{J}} U_i(\sigma^*) > \tilde{U}^{\mathcal{J}*} \tag{4.11}$$

for all $\mathcal{J}$ where $\mathcal{J}$ is proper subset of the set of all retailers $\{1, \ldots, n\}$.

Definition 4.4.7 implies that all strictly rational contracts are rational. On the other hand, although not all rational contracts are strictly rational, many of them are. In fact, all coordinating contracts that are flexible are also strictly rational. Hence, in this case, many coordinating contracts (such as the PDS schemes in Bernstein and Federgruen (2005), revenue sharing contracts with price rebates in Chapter 3, and revenue sharing with fixed target rebates in Chapter 2 are strictly rational. In the next theorem, we show that all coordinating and strictly rational contracts form contract equilibria with the supply chain optimal action profile if the discount factor is high enough.

Theorem 4.4.8 *Suppose $\boldsymbol{T}(\cdot)$ is a coordinating and strictly rational contract profile. Then, there exists $\underline{\delta}$ such that $\boldsymbol{T}(\cdot)$ and $\boldsymbol{\sigma}^*$ form a strong contract equilibrium in the repeated game for all $\delta \geq \underline{\delta}$.*

Proof. Since $\boldsymbol{T}(\cdot)$ is strictly rational, $U_0(\sigma^*) + \sum_{i \in \mathcal{J}} U_i(\sigma^*) - \tilde{U}^{\mathcal{J}*} > 0$ for all $\mathcal{J}$. Let

$$\underline{\delta}_{\mathcal{J}} = \frac{U^{\mathcal{J}*} - (U_0(\boldsymbol{\sigma}^*) + \sum_{i \in \mathcal{J}} U_i(\boldsymbol{\sigma}^*))}{U^{\mathcal{J}*} - \tilde{U}^{\mathcal{J}*}}. \tag{4.12}$$

Let $\underline{\delta} = \max_{\mathcal{J}} \underline{\delta}_{\mathcal{J}}$. Then, equation (4.9) holds for all $\delta \geq \underline{\delta}$ and all $\mathcal{J}$. ∎

Since many coordinating contracts are strictly rational, Theorem 4.4.8 implies that many coordinating contracts form strong contract equilibria with the supply chain optimal action profile for sufficiently large discount factor. In addition, the proof of Theorem 4.4.8 implies the following about $\underline{\delta}$, the minimum discount factor for any specific coordinating and strictly rational contract to form a strong contract equilibrium with the supply chain optimal action profile.

Observation 4.4.9 *(1) $\underline{\delta}$ is increasing with $U^{\mathcal{J}*} - (U_0(\boldsymbol{\sigma}^*) + \sum_{i\in\mathcal{J}} U_i(\boldsymbol{\sigma}^*))$ for all $\mathcal{J}$.*

(2) $\underline{\delta}$ is decreasing with $U_0(\boldsymbol{\sigma}^*) + \sum_{i\in\mathcal{J}} U_i(\boldsymbol{\sigma}^*) - \tilde{U}^{\mathcal{J}*}$ for all $\mathcal{J}$.

Hence, the higher the supplier and a subset of retailers can gain in the current period by changing the terms of contract, the higher the discount factor needed for the value of future relationship with other retailers to offset this gain so that there is no incentive for the supplier and this subset of retailers to change the terms of contracts. On the other hand, the higher the loss in future periods when transactions with other retailers are lost, the more valuable future relationship with the other retailers. In this case, a lower discount factor is required for future loss to offset current gain when the supplier and a subset of retailers change the terms of their contracts.

4.5 Conclusions

In this chapter, we study the robustness of coordinating contracts to bilateral and multilateral renegotiation. We analyze a very general model with a supplier serving multiple retailers, where the retailers may compete in one or more than one dimensions including price and inventory.

We apply the concept of contract equilibrium to study the robustness of coordinating contracts to bilateral renegotiation. In addition, we extend the concept of contract equilibrium to introduce the concept of strong contract equilibrium, which studies the incentive of the supplier to renegotiate with any subset of retailers. When a component of a retailer's decision affect all other retailers' demands in the same direction, then there does not exist any coordinating profit-demand monotone contract that is robust to renegotiation, either bilateral or multilateral or both, in the one-shot game.

If the supplier and the retailers engage in long-term relationship, the future value of a trusting relationship reduce the incentive the supplier to renegotiate with one or more retailers. In particular, any coordinating and strictly rational contracts are robust to both bilateral and multilateral renegotiation.

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Appendix A

CDFC and Multiplicative Variable Demand Model

We will study the convexity of distribution function condition (CDFC) assumption when the demand follows a multiplicative variable model. Suppose $D(\boldsymbol{e}, \epsilon_i) = d_i(\boldsymbol{e})\epsilon_i$ where $d_i(\boldsymbol{e})$ is non-decreasing and concave in e_i. Then, Assumption 2.3.1 (a) and (b) are satisfied. Hence, the only concern is the CDFC of Assumption 2.3.1(c).

Let $\tilde{F}(\cdot)$ and $\tilde{f}(\cdot)$ be the distribution and density functions of ϵ_i. Then, $F(x|\boldsymbol{e}) = \tilde{F}(\frac{x}{d_i(\boldsymbol{e})})$, and CDFC condition is equivalent to $\frac{\partial^2}{\partial e_i^2}\tilde{F}(\frac{x}{d_i(\boldsymbol{e})}) \geq 0$ for all x.

A.1 CDFC and Common Distributions

The first and second derivatives of $F(x|\boldsymbol{e}) = \tilde{F}(\frac{x}{d_i(\boldsymbol{e})})$ with respect to e_i are given by

$$\frac{\partial}{\partial e_i}\tilde{F}_i(\frac{x}{d_i(\boldsymbol{e})}) = -\tilde{f}_i(\frac{x}{d_i(\boldsymbol{e})})\frac{x}{[d_i(\boldsymbol{e})]^2}\frac{\partial d_i(\boldsymbol{e})}{\partial e_i}$$

and

$$\frac{\partial^2}{\partial e_i^2}\tilde{F}(\frac{x}{d_i(\boldsymbol{e})}) = \frac{x}{[d_i(\boldsymbol{e})]^3}(\frac{\partial d_i(\boldsymbol{e})}{\partial e_i})^2[\frac{x}{d_i(\boldsymbol{e})}\tilde{f}_i{}'(\frac{x}{d_i(\boldsymbol{e})})+2\tilde{f}_i(\frac{x}{d_i(\boldsymbol{e})})]-\tilde{f}_i(\frac{x}{d_i(\boldsymbol{e})})\frac{x}{[d_i(\boldsymbol{e})]^2}\frac{\partial^2 d_i(\boldsymbol{e})}{\partial e_i^2}$$

Hence, CDFC is satisfied if $\frac{x}{d_i(\boldsymbol{e})}\tilde{f}_i{}'(\frac{x}{d_i(\boldsymbol{e})})+2\tilde{f}_i(\frac{x}{d_i(\boldsymbol{e})}) \geq 0$ for all x and $\boldsymbol{e}$, or equivalently,

$$\epsilon_i\tilde{f}_i{}'(\epsilon_i) + 2\tilde{f}_i(\epsilon_i) \geq 0 \tag{A.1}$$

for all ϵ_i where $f(\epsilon_i) > 0$.

Normal Distribution

Proposition A.1.1 *CDFC can be satisfied by a truncated Normal distribution.*

Proof. For a normal distribution, $f(x) = \frac{e^{-\frac{(x-m)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$ and $f'(x) = -\frac{x-m}{\sigma^2}\frac{e^{-\frac{(x-m)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} = -\frac{x-m}{\sigma^2}f(x)$. Hence, $xf'(x)+2f(x) = [2-\frac{x(x-m)}{\sigma^2}]f(x)$, which is positive in $[0, \frac{m+\sqrt{m^2+8\sigma^2}}{2}]$. Therefore, if the normal distribution is truncated for this region $[0, \frac{m+\sqrt{m^2+8\sigma^2}}{2}]$, then (A.1) is satisfied and CDFC is also satisfied. ■

Exponential Distribution

Proposition A.1.2 *CDFC can be satisfied by a truncated exponential distribution.*

Proof. For an exponential distribution, $f(x) = \lambda e^{-\lambda x}$ and $f'(x) = -\lambda^2 e^{-\lambda x} = -\lambda x$. Hence if the exponential distribution is truncated at $x = \frac{2}{\lambda} = 2E[X]$, then (A.1) is satisfied and CDFC is also satisfied. ■

Other Distributions

Similar analysis can be used to show that CDFC is satisfied in the multiplicative variable demand model with other distributions, such as truncated Gamma distribution and truncated Rayleigh distribution.

A.2 Relaxation of CDFC

Now, we try to relax CDFC for the case when demand follows a multiplicative variable model, i.e., $D_i(\boldsymbol{e}, \epsilon_i) = d_i(\boldsymbol{e})\epsilon_i$ where ϵ_i has distribution and density functions $\tilde{F}_i(\cdot)$ and $\tilde{f}_i(\cdot)$. We assume that $\boldsymbol{e} \geq 0$. Furthermore, we substitute CDFC with the following set of assumptions.

Assumption A.2.1 *(a) For all i, $d_i(\mathbf{e})$ is strictly increasing in e_i for all $\mathbf{e}$, i.e., $\frac{\partial d_i(\mathbf{e})}{\partial e_i} > 0$.*

(b) For all i, there exists $\tilde{\epsilon}_i > 0$ such that $\tilde{f}_i(\frac{x}{d_i(\mathbf{e})}) > 0$ and $\frac{\partial^2}{\partial e_i^2}\tilde{F}_i(\frac{x}{d_i(\mathbf{e})}) \geq 0$ for all $0 < \frac{x}{d_i(\mathbf{e})} \leq \tilde{\epsilon}_i$.

Assumption A.2.1(a) implies that demand of a retailer is stochastically strictly increasing in its effort for all possible demand realizations. Assumption A.2.1(b) is more general than CDFC. Using similar analysis in Appendix A.1, it can be shown that Assumption A.2.1(b) is satisfied under Assumption A.2(a) and many common distributions such as Normal distribution, exponential distribution and gamma distribution.

Since CDFC is required only for Theorems 2.4.3, 2.4.4 and 2.4.5, it is only necessary to show that revenue sharing contract with fixed target sales rebate (and hence fixed target demand rebate) is coordinating and flexible under Assumption A.2.1.

Theorem A.2.2 *Suppose Assumption 2.3.1(a)(b) and Assumption A.2.1 is satisfied. Then revenue sharing contract with fixed target sales rebate is coordinating and flexible.*

Proof. Let $(\boldsymbol{q_r}^*, \mathbf{e}^*)$ be a system optimal strategy. Let $\tilde{T}_i = d_i(\mathbf{e}^*_{-i}, 0)\tilde{\epsilon}_i$. Let $T_i = \tilde{T}_i$ if $\tilde{T}_i < q_i^*$ and T_i be something below q_i^* otherwise. Then $\frac{T_i}{d_i(\mathbf{e}^*_{-i}, e_i)} \leq \epsilon_i$ for all $\tilde{e}_i$, implying $\frac{\partial}{\partial e_i}\tilde{F}_i(\frac{T_i}{d_i(\mathbf{e}^*_{-i}, e_i)}) < 0$ and $\frac{\partial^2}{\partial e_i^2}\tilde{F}_i(\frac{T_i}{d_i(\mathbf{e}^*_{-i}, e_i)}) > 0$ for all e_i. Hence, the expected profit of retailer i is concave in (q_i, e_i) for all $q_i \geq q_i^*$ given $\mathbf{e}^*_{-i}$ and a revenue sharing contract with fixed target sales rebate with a target sales of T_i or smaller. Hence, first-order-condition is sufficient for Nash equilibrium, and coordination and flexibility can be shown using the same analysis in the proofs of Theorems 2.4.3 and 2.4.4. ∎

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