Optimal Reduced Order Modeling of Power Systems Based on Synchronic Modal Equivalencing

by

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Abstract

In this thesis we develop optimal reduced-order models of linear time invariant dynamic systems using *Synchronic Modal Equivalencing* (SME). The first part of this thesis consists of a method for solving the finite-time grammians problem. This method is used to obtain optimal finite-time reduced-order models based on SME. A special case of zero eigenvalue decoupling pertaining to Power System models for infinite time is examined.

The second part of the thesis is the development of a fast and efficient algorithm to identify basis variables for modal reduction. The concept of *Multiple Synchrony* is used to obtain a synchronic basis to build the transient response of the nonrelevant variables of the system for the inter-area modes.

The previous methods are applied to a 111 state variable power system model to obtain a reduced-order equivalent.

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DEDICATED TO MY SISTER MARIA EVELYN TALEI CASTRILLON CANDAS

AND TO

THE MEMORY OF MY FATHER SAMUEL CASTRILLON LEON

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Chapter 1

Introduction

Due to the size and the complexity of interconnected power systems, model reduction is a necessary tool for analysis and design. A recent contribution to the field was developed by Ramaswamy *et al* [20], known as *Selective Modal Equivalencing (SME)* based on the idea of *Synchrony.* The theory was developed for interconnected power systems but it applies to any *Linear Time Invariant* (L.T.I.) system which exhibits dynamics that are essentially localized to a subset of the variables; that is, there is a subset of variables that interact with each other if certain modes are excited. This occurs in systems which have strong dynamic links between groups of variables which, in turn, are connected to other groups of variables through weak connections. If one is interested in studying a specific group of variables, then a reduced order model can be achieved by excluding modes that are local to other groups of variables. The challenge, then, is to identify the modes that correspond to local dynamics, to partition the system into groups which exhibit local dynamics, and to apply reduced order model techniques to preserve the local modes associated with the variables of interest, which we term the *study group*, and the extensive modes that correspond to interaction with other groups. The details of the method upon which this thesis is based may be found in [21] [22] [23] and [24].

1.1 Previous Work

Model reduction is a difficult problem. No general technique exists to solve every class of problems accurately. Such methods as balanced truncation and optimal Hankel model reduction are well known schemes given any generic model but do not take advantage of the structure of the model to minimize the error. For a comprehensive review of model reduction methods see [14].

The model reduction in this thesis involves the partitioning of the system into different groups or *areas*. One *study* area is chosen and it is assumed that all the disturbances are localized to that area. At this point several approaches to model reduction have been examined including removing the nearly uncontrollable and/or unobservable dynamics from the study area [8], and modal reduction [27].

We concentrate on the approach of partitioning the system into *coherent* generators. Two generators are exactly or approximately coherent given a particular disturbance if the transient angle responses are exactly or approximately equal. For analytical conditions for coherency see [7], [11], [12], [13] and [28]. Approximate coherency has been studied in [18] and [19]. Slow coherency, or coherency in the slowest modes of the system, is discussed in detail in [10].

The research in this thesis is based on synchrony in which relevant generator angles move in constant proportion. The details can be found in [20] [21] [22] [23] and [24]. An overview of this theory will be presented in Chapter 2.

The S.M.E. method proposed in [22] gives a good *relative* angle performance (see Figure 1.1 top part) however the *absolute* angles are not always well preserved (lower part). The transient responses are simulated on a linearized swing equation model, see Chapter 2 and subsection 3.3.5 for a brief description of the model . This motivates the development of methods to optimize the framework.

The main objective in this thesis is to optimize the S.M.E. framework. Numerical techniques such as a Quasi-Newton method are used to obtain the optimal equivalent.

In Chapter 2, the theoretical background necessary to understand the concepts discussed in this thesis is presented. An overview of the S.M.E. method is explained

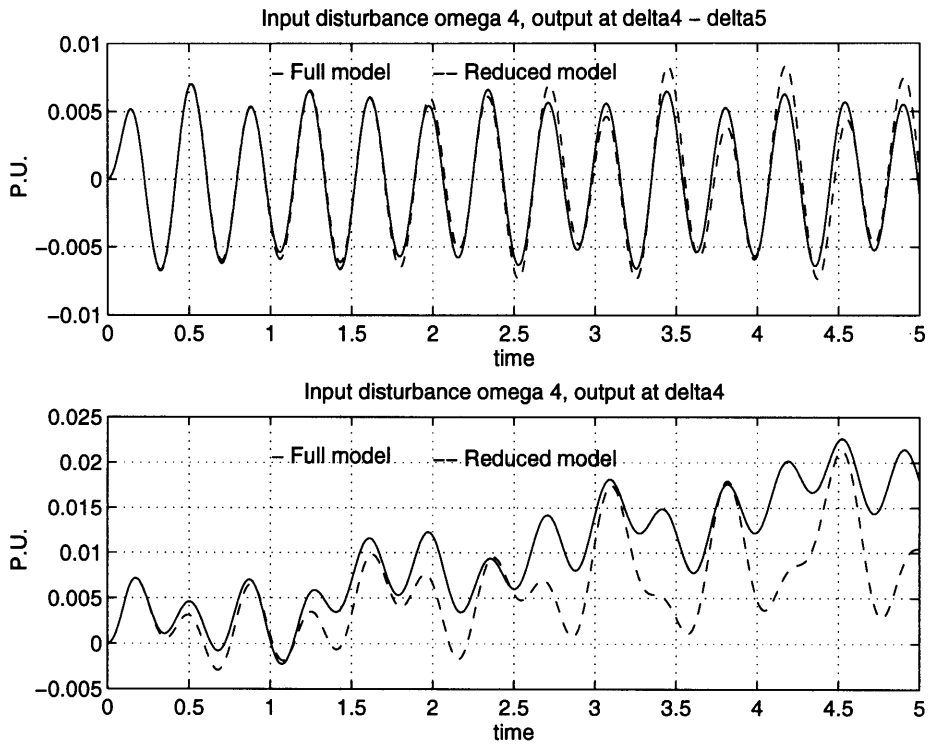

Figure 1-1: Comparison of the full model and the reduced-order model using S.M.E. reduced order dynamic equivalent. Input disturbance into machine 4 of 1 p.u. for 0.1s

together with the simplest model of a power system. Other preliminary materials such as system projections are introduced.

In Chapter 3, the necessary tools to solve optimization problems involving the solution to finite-time grammians under most conditions of stability are developed. This method is applied in conjunction with a Quasi-Newton method to find the minimum of the finite-time impulse response of the error system (Full minus Reduced model). A special case of infinite-time optimization for power systems is presented and solved through zero-eigenvalue decoupling.

In Chapter 4, a new algorithm is proposed for selecting the basis variables used in the model reduction. Assuming that only the modes between areas, referred to as the *interarea* modes, and the study area modes are excited from a disturbance in study area, the algorithm selects basis variables in such a way that their solutions span as much as possible the solution subspace corresponding to the non-relevant variables.

In Chapter 5, the optimization methods developed in the previous chapters are

applied to a realistic 111 state variable linearized model produced by the simulation and analysis software for power systems called *Eurostag*. This is an especially troublesome model for which an attempt is made to find an optimal equivalent.

In Chapter 6 conclusions and suggestions for future work are presented.

Chapter 2

Background Theory

2.1 Swing Equation Model

The theory developed in this thesis is based on the swing equation models. Consider a lossless network with n generators (see [3] for more details). The electrical power is injected into the network at each generator is given by

$$P_{Gi} = \sum_{j=1}^{n} \mid V_i \mid\mid V_j \mid sin(\delta_i - \delta_j) B_{ij} \tag{2.1}$$

where the operands of the summation are the power from bus i transmitted to bus j through a line with admittance B_{ij} and voltage V_i at the buses. These are the algebraic constraints of the power system. The simplest dynamic constraints at generator i are given by:

$$M_i \ddot{\delta}_i + D_i \dot{\delta}_i + P_{Gi} = P_{Mi} \tag{2.2}$$

where M_i is the moment of inertia of generator i, D_i is the damping factor and P_{Mi} denotes the mechanical power applied to generator i. Equations (2.1) and (2.2) constitute the *swing equation* model with $P_{Gi} = P_{Mi}$ at the operating point.

In matrix form the *swing equation* model can be represented by

$$\begin{bmatrix} \dot{\delta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & -\mathbf{M^{-1}D} \end{bmatrix} \begin{bmatrix} \delta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{M^{-1}} \end{bmatrix} \Delta\mathbf{P}(\delta) \tag{2.3}$$

where

$$\delta = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{bmatrix} \qquad \omega = \begin{bmatrix} \dot{\delta_1} \\ \dot{\delta_2} \\ \vdots \\ \dot{\delta_n} \end{bmatrix}$$

$\mathbf{M}$ = $n \times n$ diagonal matrix containing the inertia constants of the generators

$\mathbf{D}$ = $n \times n$ diagonal matrix containing the damping factors of the generators

$\Delta\mathbf{P}(\delta)$ = $n \times n$ diagonal matrix containing the excess power injected at the buses

In this thesis we use the linearized version of the swing equation model

$$\begin{bmatrix} \Delta\dot{\delta} \\ \Delta\dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mathbf{M^{-1}F} & -\mathbf{M^{-1}D} \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta\omega \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{M^{-1}} \end{bmatrix} \Delta u \tag{2.4}$$

where Δu is a small perturbation in the power vector, δ is the operating point, and

$$\mathbf{F} = -\frac{\partial}{\partial\delta}(\Delta\mathbf{P}(\delta))|_{\delta_o} = \begin{bmatrix} \frac{\partial P_1}{\partial\delta_1} & \cdots & \frac{\partial P_1}{\partial\delta_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial P_n}{\partial\delta_1} & \cdots & \frac{\partial P_1}{\partial\delta_n} \end{bmatrix} \tag{2.5}$$

where

$$\begin{aligned} \frac{\partial P_i}{\partial\delta_k} &= \sum_{j=1, j\neq i}^{n} \mid V_i \mid\mid V_j \mid cos(\delta_i - \delta_j)B_{ij} && \text{if } i = k \\ \frac{\partial P_i}{\partial\delta_k} &= -\mid V_i \mid\mid V_j \mid cos(\delta_i - \delta_j)B_{ij} && \text{if } i \neq k \end{aligned} \tag{2.6}$$

2.2 Synchrony

The first step in this method of reduced-order modeling is to identify the areas which are weakly coupled (see [10]) to each other but strongly interconnected internally. In the limit of vanishing weak interconnections in a swing-equation model of the power system, the generator's angles in each area can be shown to move identically, provided only a certain number (number of areas -1) of the slowest oscillatory mode-pairs of the system are excited. This is termed slow-coherency [10].

Slow-coherency is an ideal setting. To deal with more realistic models the notion of synchrony was developed in [21]. Given that only a subset of all the modes of the system are excited (termed a *chord* of the system) we present the following definition from [20].

Definition 2.2.1 *Machine i is synchronic with machine j in the chord v, or is v-synchronic with j, if there exists a scalar constant k_{ij} such that for any initial condition and for all time t the angles $\triangle\delta_i(t)$ and $\triangle\delta_j(t)$ satisfy*

$$\triangle\delta_i(t) - k_{ij}\triangle\delta_j(t) = \gamma_{ij}(t) \tag{2.7}$$

where $\gamma_{ij}(t)$ contains none of the modes in v.

Areas can arise in which the machine's angles move v - *synchronically* with respect to each other. An algorithm was developed in [21] to identify such areas and assign a reference machine to each area. Thus the reference machine, represents the dynamical modes of the rest of the machines in that area.

The eigenvalues of the system are used to identify synchronic variables. Let the columns of the $n \times p$ full column rank matrix $\mathbf{V}$ (assuming distinct eigenvalues) be compromised of the eigenvectors corresponding to the modes in v, and let a_i be the i_{th} row of $\mathbf{V}$. Note that n is the size of system matrix and p is the number of modes in the chord. The following lemma from [20] is important for identifying the reference machines.

Lemma 2.2.1 *State variable x_i exhibits one dimensional synchrony with respect to state variable x_j in the chord v if and only if the corresponding rows of* $\mathbf{V}$ *are related by a scalar as follows:*

$$\mathbf{a_i} = k_{ij}\mathbf{a_j} \tag{2.8}$$

This lemma forms the basis to identify such *synchronic* areas, by identifying each group of variables whose eigenvector matrix rows are approximately equal in the modes of the chord. Once these areas are identified, one area is retained in full detail (which we shall call the *study area variables*), along with a reference machine in each external area. The dynamics of the rest of the machines are equivalenced using

$$x_i = k_{ij}x_j. \tag{2.9}$$

The undamped swing equation model (explained further in this thesis) is used to identify these areas.

2.3 Multi-Dimensional Synchrony

In practice, the *one-dimensional* relation described in Equation (2.9) is approximate. Accuracy can be improved by expressing the dynamics of a variable outside the study area as a linear combination of the dynamics of a set of basis variables in the chord v. Thus, if we can find a relation of the form

$$\delta_1(t) = k_{12}\delta_2(t) + k_{13}\delta_3(t) \tag{2.10}$$

for all initial conditions that excite only the modes in v, then we can say that δ_1 exhibits two dimensional synchrony with respect to the *basis variables* δ_2 and δ_3 in the chord v. This extension preserves the notion of *synchrony* and is appropriately called *Multi-dimensional Synchrony.*

In general, we partition our system into study area variables, basis variables and nonrelevant variables denoted by x_s, x_b and x_z respectively and describe the motion

of the non-relevant machines in terms of the basis variables using

$$x_z = K x_b \tag{2.11}$$

which is called the equivalent model, or equivalent for short.
Methods have been developed in various papers to approximate K in Equation (2.11) implicitly and explicitly. In this thesis we use the notion of *Multi-dimensional Synchrony* to define the form of a reduced order model, and formulate the calculation of the Grouping matrix K as an optimization problem.

Consider the partitioned system:

$$\begin{bmatrix} \dot{x}_s \\ \dot{x}_b \\ \dot{x}_z \end{bmatrix} = \begin{bmatrix} A_{ss} & A_{sb} & A_{sz} \\ A_{bs} & A_{bb} & A_{bz} \\ A_{zs} & A_{zb} & A_{zz} \end{bmatrix} \begin{bmatrix} x_s \\ x_b \\ x_z \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix} u \tag{2.12}$$

where x_s x_b x_z are, respectively, the study, basis and non-relevant state variables. Using the Equivalent described by (2.11) we substitute it in Equation (2.12) and renaming our state variables to $\tilde{x}_s$ and $\tilde{x}_b$ we obtain the **reduced model**

$$\begin{bmatrix} \dot{\tilde{x}}_s \\ \dot{\tilde{x}}_b \end{bmatrix} = \begin{bmatrix} A_{ss} & A_{sb} + A_{sz}K \\ A_{bs} & A_{bb} + A_{bz}K \end{bmatrix} \begin{bmatrix} \tilde{x}_s \\ \tilde{x}_b \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u \tag{2.13}$$

Three explicit equivalents were developed in [20] and are used in this thesis

- K = 0: No equivalent

- K = $V_z V_b^{-1}$: In this equivalent we retain the eigenvalues and right eigenvectors in the chord v exactly and all the left eigenvectors except for those associated with the study group reference machine are preserved. The equivalent will perform very well as long as the perturbation is introduced to nonreference study group variables only.

- K = $V_z V_b^{\dagger}$, where † is the pseudoinverse. In this case we use fewer reference variables or modes in the chord. than areas. This equivalent is the least square solution and does not preserve the eigenvalues and eigenvectors exactly.

2.4 Reference Variable Selection

In this section we describe the basis selection algorithm developed in [20]. The reference variables are then used as basis variables for the reduced-order model. Later in Chapter 4 we develop an improved algorithm for *variable* and *machine* basis selection. The structure of the undamped swing equation model is

$$\ddot{\delta} = A\delta. \tag{2.14}$$

For machine selection we will use the right eigenvector $n \times p$ matrix $\mathbf{V}$ corresponding to the matrix A and the modes excited in the chord v of the undamped swing equation model.

Recall that the $n \times p$ matrix is comprised of the p eigenvectors corresponding to the chosen chord v, and the i^{th} row of $\mathbf{V}$ is $\mathbf{a_i}$. Now if the one dimensional synchrony between variables i and j were exact, the angle between $\mathbf{a_i}$ and $\mathbf{a_j}$ would be 0 or 180^o and the cosine of this angle will have magnitude 1. This motivates the following definitions cited from [20].

Definition 2.4.1 *Let the columns of* $\mathbf{V}$ *comprise the eigenvectors in the chosen chord* v. *Let* $\mathbf{a_i}$ *be the ith row of* $\mathbf{V}$, *corresponding to the variable* j. *Let* $\mathcal{D}_i$ *be the diagonal weighting matrix with non-negative entries. Then*

- *The* **angle of synchrony** *of variable* i *with variable* j, *when* $\mathbf{a}_j \neq 0$, *is given by*

$$\begin{aligned} \phi_{ij} &= cos^{-1}\left(\frac{(\mathbf{a_i}\mathcal{D}_i)(\mathbf{a_j}\mathcal{D}_i)^T}{\|\mathbf{a_i}\mathcal{D}_i\|\|\mathbf{a_j}\mathcal{D}_i\|}\right) && if\ \mathbf{a}_i \neq 0 \\ &= 0 && otherwise \end{aligned} \tag{2.15}$$

where $\|.\|$ *is the vector 2-norm*

- The **degree of synchrony** of variable i with variable j, when $\mathbf{a}_j \neq 0$, is given by

$$\mu_{ij} = \cos \phi_{ij} \tag{2.16}$$

• The **synchronic distance** of variable i from variable j, when ϕ_{ij} exists, is given by

$$d_{ij} = \|\mathbf{a}_i\| \sin \phi_{ij} \tag{2.17}$$

The choice for $\mathcal{D}_i$ which will make the right eigenvectors invariant to scaling is given by

$$\mathcal{D}_i = Diag_k[|w_{ik}|] \tag{2.18}$$

where w_{ik} is the left eigenvector component of the state variable i in the k mode.

2.4.1 Variable Selection Algorithm

Algorithm 2.4.1 *(Selecting Reference Variables) The inputs are the matrix* $\mathbf{V}$ *containing the right eigenvector component for* v*, the matrices* $\{\mathcal{D}_i\}$ *for* v*, and the desired number of groups,* q*. The outputs are the rows of* $\mathbf{a}_{i^\star}$ *of* $\mathbf{V}$ *that correspond to the selected reference variables, or equivalently , a vector* $\mathbf{b}$ *containing the indices* $i^\star$ *of the chosen reference variables.*

- *Set* $\mathbf{y_1} = \mathbf{a}_{i^\star}$*, where* $i^\star = \arg\max \|\mathbf{a_i}\mathcal{D}_i\|$.
- *For* r = *2,...,q, set* $\mathbf{y_r} = \mathbf{a}_{i^\star}$ *where*

$$i^* = \arg\max_i [\min_j(d_{ij})],\ 1 \leq j \leq r\text{-}1$$

The following paragraph is quoted from [20]:

> The first reference picked is the variable with the largest net participation in the chosen chord, as measured by the norm of $\mathbf{a_i}\mathcal{D}_i$. The second reference variable picked is the variable with the largest synchronic distance to the first reference machine, and the remaining references are picked similarly, so that the references are mutually as "far away" as pos-

sible in terms of the synchronic distance. This provides a good basis for representing the remaining non-reference variables.

Note that in general $\phi_{ij} \neq \phi_{ji}, \mu_{ij} \neq \mu_{ji}$, and $d_{ij} \neq d_{ji}$. The anti-symmetric properties of the previous metrics preclude the identification of orthogonal subspaces by the reference variable selection algorithm. This motivates us to define new metrics and develop a new algorithm in Chapter 4.

2.5 System Projections and Matrix Decomposition

One of the main parts of this thesis involves the solution of finite-time grammians. We project the full system onto various subsystems and solve them in the decomposed spaces. Such system decomposition can be achieved by the following method. Suppose we have a system $\dot{x} = Ax + Bu,\ y = Cx$ where $x \in \mathbf{R^n}, y \in \mathbf{R^p}, u \in \mathbf{R^m}, A \in \mathbf{R^{n \times n}}, B \in \mathbf{R^{n \times m}}, C \in \mathbf{R^{p \times n}}$ and we desire to decompose it into smaller subsystems. The first step is to find a unitary matrix V^T, through a Schur decomposition, such that

$$\hat{A} = V^T A V = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} \tag{2.19}$$

where $\hat{A}_{11}$ and $\hat{A}_{22}$ contain the eigenvalues of the original system. Through the transformation

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = V^T x \tag{2.20}$$

Thus we can assign the various modes of the full system into either of the subsystems $\hat{A}_{11}$ and $\hat{A}_{22}$. The next steps are to solve the following Sylvester equation

$$\hat{A}_{11} X - X \hat{A}_{22} + \hat{A}_{12} = 0 \tag{2.21}$$

and to calculate the following matrices:

$$\begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} VB \tag{2.22}$$

and

$$\begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix} = CV^T \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}. \tag{2.23}$$

This yields the two state space projections: $\dot{\hat{x}}_1 = \hat{A}_{11}\hat{x}_1 + \hat{B}_1 u$, $y = \hat{C}_1\hat{x}_1$ and $\dot{\hat{x}}_2 = \hat{A}_{12}\hat{x}_2 + \hat{B}_2 u$, $y = \hat{C}_2\hat{x}_2$. If the previous two systems are added in parallel the original system will be obtained. Note that this method will only work if the decomposition is well posed. Attempting to break up a complex pair or a set of non distinct eigenvalues will not be successful.

Chapter 3

Optimal Synchronic Modal Equivalencing

This is the main chapter of the thesis we shall optimize the *SME* framework developed in [20] through numerical optimization techniques given a set of basis variables. A new method to solve finite-time grammians which is the key to an efficient numerical optimization is presented. Also a zero eigenvalue decoupling method is presented for the case where we desire to preserve the zero eigenvalue exactly but optimize the rest of the modes.

3.1 Solution to Finite-time Grammians

In this section we introduce a new method to compute finite-time grammians fast and efficiently for stable and unstable linear time invariant systems. We shall rely heavily on the theory developed in this chapter to solve the model reduction problem. The solution for finite time grammians for *stable* systems is solved in [5] finding the steady state solution to the time varying Lyapunov equation and going backward in time to find the finite-time solution. For *unstable* systems the steady state solution does not exist therefore the method in [5] is not applicable. The method proposed here is fundamentally different since the solution is obtained over a finite time interval. It applies independent of the stability of the system. The following theorem is essential

for the optimization scheme.

Theorem 3.1.1 *Given $A_1 \in \mathcal{R}^n$, $A_2 \in \mathcal{R}^m$, $C_1 \in \mathcal{R}^{p\times n}$ and $C_2 \in \mathcal{R}^{p\times m}$, the solution to the integral*

$$X = \int_0^{t_f} e^{A_1^T t} C_1^T C_2 e^{A_2 t} dt \tag{3.1}$$

satisfies:

$$[A_1]^T X + X[A_2] = -C_1^T C_2 + e^{A_1^T t_f} C_1^T C_2 e^{A_2 t_f} \tag{3.2}$$

Proof: Substituting (3.1) into left side of (3.2) we obtain

$$A_1^T \int_0^{t_f} e^{A_1^T t} C_1^T C_2 e^{A_2 t} dt + \int_0^{t_f} e^{A_1^T t} C_1^T C_2 e^{A_2 t} A_2 dt \tag{3.3}$$

$$= \int_0^{t_f} \frac{d}{dt}[e^{A_1^T t} C_1^T C_2 e^{A_2 t}] dt \tag{3.4}$$

$$= e^{A_1^T t} C_1^T C_2 e^{A_2 t} \mid_0^{t_f} \tag{3.5}$$

$$= -C_1^T C_2 + e^{A_1^T t_f} C_1^T C_2 e^{A_2 t_f} \tag{3.6}$$

□

No assumptions are made concerning the locations of the eigenvalues of matrix A_1 and A_2, however the uniqueness of solutions of Equation (3.2) does depend on the eigenvalues. If (3.2) has multiple solutions for X, only one corresponds to the integral given by (3.1). The following theorem (see [4]) which describes exactly the necessary and sufficient conditions for uniqueness is stated.

Theorem 3.1.2 *The solution to the Sylvester equation $AX + XB = C$ is unique* **if and only if** *the eigenvalues $a_1, a_2, ..., a_m$ of A and $b_1, b_2, ..., b_n$ of B satisfy*

$$a_i + b_j \neq 0 \ \ (i = 1, 2, ..., m; j = 1, 2, ..., n) \tag{3.7}$$

A characteristic of typical power systems models is the presence of a zero eigenvalue. Under certain conditions, (3.1) and (3.2) can be solved to obtain a unique solution. In the next subsection a particular case of a single zero eigenvalue is solved, which is enough for most swing equation models of power systems. Then the complete solution for the more general case involving distinct symmetrical eigenvalues will be solved.

3.1.1 Space Decomposition Method

Suppose we have a system $\dot{x} = Ax + u, y = Cx$, represented by $[A, I, C, 0]$ as show in Figure 3-1, where A contains a zero eigenvalue, **no** oscillatory modes and no poles symmetric around the origin. The truncated two norm to an impulse of size u for this system is

$$\|(y)_T\| = \int_0^{t_f} y^T y \, dt = u^T \left(\int_0^{t_f} e^{A^T t} C^T C e^{At} dt \right) u. \tag{3.8}$$

Now, let us decompose $[A, I, C, 0]$ into two subsystems connected in parallel (see [25], implemented under the *slowfast* command in matlab , see [9]) denoted $[A_1, B_1, C_1, 0]$ and $[A_2, B_2, C_2, 0]$ as shown in 3-1 where A_1 contains the zero eigenvalue (scalar), and A_2 contains the rest of the modes. The truncated two norm to an impulse of size u for this system is

$$\begin{aligned}
\|(y)_T\| &= \int_0^{t_f} y^T y \, dt = \int_0^{t_f} u^T [C_1 e^{A_1 t} B_1 + C_2 e^{A_2 t} B_2]^T [C_1 e^{A_1 t} B_1 + C_2 e^{A_2 t} B_2] u \, dt \\
&= u^T [\int_0^{t_f} B_1^T e^{A_1^T t} C_1^T C_1 e^{A_1 t} B_1 dt + \int_0^{t_f} (B_1^T e^{A_1^T t} C_1^T C_2 e^{A_2 t} B_2 \\
&+ B_2^T e^{A_2^T t} C_2^T C_1 e^{A_1 t} B_1) dt + \int_0^{t_f} B_2^T e^{A_2^T t} C_2^T C_2 e^{A_2 t} B_2 dt] u.
\end{aligned} \tag{3.9}$$

Since the *combined* system and the original system are the same and A_1 is a zero matrix, then

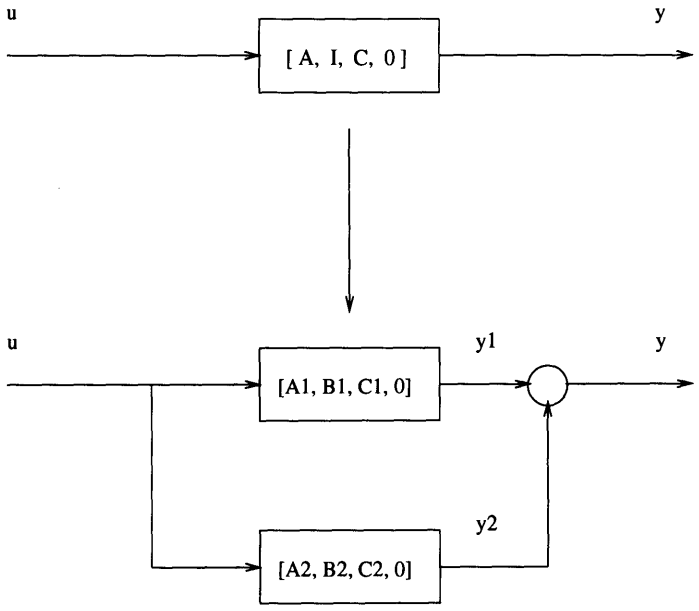

Figure 3-1: Transformation of [A,I,C,0] system

$$\begin{aligned} \|(y)_T\| &= u^T(\int_0^{t_f} e^{A^T t} C^T C e^{At} dt)u \\ &= u^T[B_1^T C_1^T C_1 B_1 * t_f + B_1^T C_1^T C_2 A_2^{-1}[e^{A_2 t_f} - I]B_2 \\ &+ B_2^T[e^{A_2 t_f} - I]^T A_2^{-T} C_2^T C_1 B_1 + \int_0^{t_f} B_2^T e^{A_2^T t} C_2^T C_2 e^{A_2 t} B_2 dt]u. \quad (3.10) \end{aligned}$$

Now, the input u is arbitrary and $\|(y)_T\|$ has the form $u^T N^T N u$ for both equations (3.9) and (3.10), therefore

$$\begin{aligned} \int_0^{t_f} e^{A^T t} C^T C e^{At} dt &= B_1^T C_1^T C_1 B_1 * t_f + B_2^T[e^{A_2 t_f} - I]^T A_2^{-T} C_2^T C_1 B_1 \\ &+ B_1^T C_1^T C_2 A_2^{-1}[e^{A_2 t_f} - I]B_2 \\ &+ \int_0^{t_f} B_2^T e^{A_2^T t} C_2^T C_2 e^{A_2 t} B_2 dt \quad (3.11) \end{aligned}$$

The integral term in the right side of equation (3.10) is readily solved by a Lya-

punov equation if A_2 satisfies Theorem 3.1.2. Notice that A_2 has no zero eigenvalues and is thus invertible.

□

Example 3.1.1 *Calculate the finite time observability grammian for the system* $\dot{x} = Ax + Bu$, $y = Cx + Du$ *where*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \; B = 0, \; C = I, \; D = 0, \; t_f = 0.1\,. \tag{3.12}$$

This system has eigenvalues $\lambda = 16.1168, -1.1168$ *and 0. Using numerical integration, and a reasonably small step size, the answer is*

$$X = \int_0^{0.1} e^{A^T t} C^T C e^{At} dt = \begin{bmatrix} 0.2104 & 0.1593 & 0.2083 \\ 0.1593 & 0.3169 & 0.2745 \\ 0.2083 & 0.2745 & 0.4406 \end{bmatrix}, \; \textit{time elapsed} = 79.1515s\,. \tag{3.13}$$

Using zero eigenvalue decomposition and Lyapunov equation the answer is

$$X = \int_0^{0.1} e^{A^T t} C^T C e^{At} dt = \begin{bmatrix} 0.2104 & 0.1593 & 0.2083 \\ 0.1593 & 0.3169 & 0.2744 \\ 0.2083 & 0.2744 & 0.4406 \end{bmatrix} \; \textit{time elapsed} = 0.92s\,. \tag{3.14}$$

Both answers are nearly equal but the eigenvalue decomposition method is clearly more efficient and precise. Note that the computational effort for the second method does not depend on the length of time (t_f)*; however if the system is unstable, overflow problems arise for* t_f *too large.*

3.2 Space Decomposition

In this section the solution to the integral

$$\|(y)_T\| = u^T(\int_0^{t_f} e^{A^T t} C^T C e^{At} dt)u \tag{3.15}$$

is determined for all the cases where the eigenvalues do not satisfy Theorem 3.1.2 except for the non distinct symmetrical eigenvalues case. It starts with the zero eigenvalue decoupling and then in the following subsection the method is extended to include all types of distinct eigenvalue combinations.

3.2.1 Zero Eigenvalue Decomposition

As in the previous section, we decompose the system $[A, I, C, 0]$ into two subsystems $[A_1, B_1, C_1, 0]$ and $[A_2, B_2, C_2, 0]$. Where A_1 is composed of all the zero eigenvalues and is in *Jordan form* with all the zero eigenvalues and A_2 is invertible.

Then,

$$\begin{aligned}\|(y)_T\| = \int_0^{t_f} B_1^T e^{A_1^T t} C_1^T C_1 e^{A_1 t} B_1 dt + \int_0^{t_f} (B_1^T e^{A_1^T t} C_1^T C_2 e^{A_2 t} B_2 \\ + B_2^T e^{A_2^T t} C_2^T C_1 e^{A_1 t} B_1) dt + \int_0^{t_f} B_2^T e^{A_2^T t} C_2^T C_2 e^{A_2 t} B_2 dt\end{aligned} \tag{3.16}$$

Now we solve each integral separately. The first integral is solved by expansion,

$$\begin{aligned}\int_0^{t_f} e^{A_1^T t} C_1^T C_1 e^{A_1 t} dt &= \int_0^{t_f} C_1^T C_1 e^{A_1 t} dt \\ &+ \int_0^{t_f} A_1^T t C_1^T C_1 e^{A_1 t} dt \\ &+ \int_0^{t_f} \frac{(A_1^T)^2 t^2}{2!} C_1^T C_1 e^{A_1 t} dt \\ &\vdots \\ &+ \int_0^{t_f} \frac{(A_1^T)^m t^m}{m!} C_1^T C_1 e^{A_1 t} dt,\end{aligned}$$

and integration by parts:

$$
\begin{aligned}
&= C_1^T C_1 L(t_f) \\
&+ A_1^T t_f C_1^T C_1 L(t_f) - \int_0^{t_f} A_1^T C_1^T C_1 L(t) dt \\
&+ \frac{(A_1^T)^2 t_f^2}{2!} C_1^T C_1 L(t_f) - \int_0^{t_f} (A_1^T)^2 t C_1^T C_1 L(t) dt \\
&\vdots \\
&+ \frac{(A_1^T)^m t_f^m}{m!} C_1^T C_1 L(t_f) - \int_0^{t_f} \frac{(A_1^T)^m t^{m-1}}{(m-1)!} C_1^T C_1 L(t) dt
\end{aligned}
$$

$$
\begin{aligned}
&= C_1^T C_1 L(t_f) \\
&+ A_1^T t_f C_1^T C_1 \left[L(t_f) - I\frac{t_f^2}{2!} - A_1 \frac{t_f^3}{3!} - \ldots - A_1^m \frac{t_f^{m+2}}{(m+2)!} \right] \\
&+ \frac{(A_1^T)^2 t_f^2}{2!} C_1^T C_1 [L(t_f) - I\frac{t_f^3}{1!3} - A_1 \frac{t_f^4}{2!4} \\
&- \ldots - A_1^m \frac{t_f^{m+3}}{(m+1)!(m+3)}] \\
&\vdots \\
&+ \frac{(A_1^T)^m t_f^m}{m!} C_1^T C_1 [L(t_f) - I\frac{t_f^{m+1}}{1!(m+1)} - A_1 \frac{t_f^{m+2}}{2!(m+2)} \\
&- A_1^2 \frac{t_f^{m+3}}{3!(m+3)} - \ldots - A_1^m \frac{t_f^{2m+1}}{(m+1)!(2m+1)}] + \ldots \quad (3.17)
\end{aligned}
$$

where

$$
L(t) = It + \frac{A_1 t^2}{2!} + \frac{A_1^2 t^3}{3!} + \ldots + \frac{A_1^m t^{m+1}}{(m+1)!} \tag{3.18}
$$

Since a Jordan decomposition can be written as $J = D + N$, where D is a diagonal matrix and N is a nilpotent matrix, there exists a positive integer k such that $N^k = 0$; moreover, k is the order of the largest Jordan block (see [16] p. 139)). In the previous series there are $k(k+2)$ terms to calculate and A_1 is a nilpotent matrix.

In practice we obtain a schur decomposition for A_1 which has this nilpotent structure, and since A_1 is highly sparse (at most $(n-1)$ non-zero elements), the calculation of the previous series is fast.

The second integral $\int_0^{t_f} B_1^T e^{A_1^T t} C_1^T C_2 e^{A_2 t} B_2 dt$ and its transpose are readily solved

by Theorem 3.1.1 since then eigenvalues of A_1 and A_2 satisfy the uniqueness conditions of Theorem 3.1.2. The third integral is solved in the last subsection.

3.2.2 Distinct Eigenvalue Symmetrical Decomposition

If $[A_2, B_2, C_2, 0]$ has eigenvalues symmetrically placed around the origin of the complex plane then the conditions for uniqueness of Theorem 3.1.2 fails. Now if these symmetrical eigenvalues are distinct then we can solve the problem numerically and analytically. As in the previous subsection we decompose the system $[A_2, I, C_2, 0]$ into two subsystems $[A_3, B_3, C_3, 0]$ and $[A_4, B_4, C_4, 0]$ where A_3 is a diagonal matrix with all the distinct symmetrical eigenvalues (including the jw axis) and A_4 the rest. Then,

$$\begin{aligned}\int_0^{t_f} e^{A_2^T t} C_2^T C_2 e^{A_2 t} dt &= \int_0^{t_f} B_3^T e^{A_3^T t} C_3^T C_3 e^{A_3 t} B_3 dt + \int_0^{t_f} B_3^T e^{A_3^T t} C_3^T C_4 e^{A_4 t} B_4 dt \\ &+ \int_0^{t_f} B_4^T e^{A_4^T t} C_4^T C_3 e^{A_3 t} B_3 dt + \int_0^{t_f} B_4^T e^{A_4^T t} C_4^T C_4 e^{A_4 t} B_4 dt\end{aligned} \tag{3.19}$$

The integral $\int_0^{t_f} e^{A_3^T t} C_3^T C_3 e^{A_3 t}$ dt becomes

$$\int_0^{t_f} \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_m t} \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1m} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{m1} & q_{m2} & \cdots & q_{mm} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_m t} \end{bmatrix} dt \tag{3.20}$$

where $\lambda_1, \lambda_2, ..., \lambda_m$ are the eigenvalues of A_3 and q_{ij} are the elements of $C_3^T C_3$. Applying matrix multiplication equation (3.20) becomes

$$\int_0^{t_f} \begin{bmatrix} e^{(\lambda_1+\lambda_1)t}q_{11} & e^{(\lambda_1+\lambda_2)t}q_{12} & \cdots & e^{(\lambda_1+\lambda_m)t}q_{1m} \\ e^{(\lambda_2+\lambda_1)t}q_{21} & e^{(\lambda_2+\lambda_2)t}q_{22} & \cdots & e^{(\lambda_2+\lambda_m)t}q_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ e^{(\lambda_m+\lambda_1)t}q_{m1} & e^{(\lambda_m+\lambda_2)t}q_{m2} & \cdots & e^{(\lambda_m+\lambda_m)t}q_{mm} \end{bmatrix} dt. \tag{3.21}$$

Integrating the above matrix we have

$$\begin{bmatrix} c_{11}e^{(\lambda_1+\lambda_1)t}q_{11} & c_{12}e^{(\lambda_1+\lambda_2)t}q_{12} & \cdots & c_{1m}e^{(\lambda_1+\lambda_m)t}q_{1m} \\ c_{21}e^{(\lambda_2+\lambda_1)t}q_{21} & c_{22}e^{(\lambda_2+\lambda_2)t}q_{22} & \cdots & c_{2m}e^{(\lambda_2+\lambda_m)t}q_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1}e^{(\lambda_m+\lambda_1)t}q_{m1} & c_{m2}e^{(\lambda_m+\lambda_2)t}q_{m2} & \cdots & c_{mm}e^{(\lambda_m+\lambda_m)t}q_{mm} \end{bmatrix}_0^{t_f}. \tag{3.22}$$

Where $c_{ij} = t$ if $\lambda_i = -\lambda_j$ and $c_{ij} = \frac{1}{\lambda_i+\lambda_j}$ otherwise.

The solution of the rest of equation (3.20) is easily solved with Theorem 3.1.1.

3.3 Optimization

In this section we apply the result from the previous section to optimize the dynamic equivalent described in Chapter 2. A suitable cost function involving the difference between the full and reduced systems variables is introduced and minimized over finite time.

3.3.1 Problem Statement

The objective of this section is to find an optimal reduced-order model for any given area partitioned L.T.I. system. Specifically we want to calculate the grouping matrix K such that the equivalent $x_z = Kx_b$ minimizes the difference between *study* area variables of the full model and the reduced-order model. Mathematically we state the problem as follows:

Given a partitioned linear model

$$\begin{bmatrix} \dot{x}_s \\ \dot{x}_b \\ \dot{x}_z \end{bmatrix} = \begin{bmatrix} A_{ss} & A_{sb} & A_{sz} \\ A_{bs} & A_{bb} & A_{bz} \\ A_{zs} & A_{zb} & A_{zz} \end{bmatrix} \begin{bmatrix} x_s \\ x_b \\ x_z \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix} u \tag{3.23}$$

where x_s x_b x_z are, respectively, the study, basis and non-relevant state variables. We want to find a **reduced model**

$$\begin{bmatrix} \dot{\tilde{x}}_s \\ \dot{\tilde{x}}_b \end{bmatrix} = \begin{bmatrix} A_{ss} & A_{sb} + A_{sz}K \\ A_{bs} & A_{bb} + A_{bz}K \end{bmatrix} \begin{bmatrix} \tilde{x}_s \\ \tilde{x}_b \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u \tag{3.24}$$

such that the output $\| x_s - \tilde{x}_s \|$ is minimized for a suitable norm and all norm bounded inputs u. The **error system** is defined to be

$$\begin{bmatrix} \dot{x}_s \\ \dot{x}_b \\ \dot{x}_z \\ \dot{\tilde{x}}_s \\ \dot{\tilde{x}}_b \end{bmatrix} = \begin{bmatrix} A_{ss} & A_{sb} & A_{sz} & 0 & 0 \\ A_{bs} & A_{bb} & A_{bz} & 0 & 0 \\ A_{zs} & A_{zb} & A_{zz} & 0 & 0 \\ 0 & 0 & 0 & A_{ss} & A_{sb} + A_{sz}K \\ 0 & 0 & 0 & A_{bs} & A_{bb} + A_{bz}K \end{bmatrix} \begin{bmatrix} x_s \\ x_b \\ x_z \\ \tilde{x}_s \\ \tilde{x}_b \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \\ 0 \\ B_1 \\ 0 \end{bmatrix} u \tag{3.25}$$

$$e(t) = \begin{bmatrix} I & 0 & 0 & -I & 0 \end{bmatrix} \begin{bmatrix} x_s \\ x_b \\ x_z \\ \tilde{x}_s \\ \tilde{x}_b \end{bmatrix}. \tag{3.26}$$

We optimize K such that for a suitable set of inputs, the error is minimized

$$\min_{\forall K} \| \mathbf{e(t)} \| . \tag{3.27}$$

While the study area variables may be specified or determined a priori, the basis variables are not. Determining the appropriate basis variables will be also studied in Chapter 4.

3.3.2 Cost function

The selection of a good performance index is essential to our optimization. A natural measure of error is given by the $\mathcal{H}_\infty$ norm since it is the least upper bound for the maximum $l_2 - l_2$ gain. There are several problems associated with this norm:

1) Linearized models of Interconnected Power Systems have a zero mode, thus $\| \mathbf{e(s)} \|_{\mathcal{H}_\infty}$ does not exist. We tried to avoid this problem by filtering out the zero eigenvalue at the output, but the pole -zero cancellation introduced a numerical instability in the optimization routine.

2) We have to introduce stability constraints on $\mathbf{e(s)}$ which increases the number of iterations.

3) There might not exist a reduced order model such that $\mathbf{e(s)}$ is stable in the *synchrony* framework.

4) Unstable modes must be retained exactly otherwise the error vector will tend to infinity .

5) Calculation of the $\mathcal{H}_\infty$ norm through frequency sampling or D-K iteration is numerically inefficient for large systems.

This leads us to look for another approach. A finite-time time-domain optimization is suggested since it will allow us to examine a system with unstable and zero modes.

3.3.3 Finite-time Optimization

Consider two L.T.I. systems

$$\begin{aligned} \dot{z_1} &= A_1 z_1 + B_1 u \\ y_1 &= C_1 z_1 \\ \dot{z_2} &= (A_2 + F) z_2 + B_2 u \\ y_2 &= C_2 z_2 \end{aligned}$$

where $y_1, y_2 \in \mathcal{R}^s$, $A_1 \in \mathcal{R}^n$, $A_2 \in \mathcal{R}^{s+b}$, $B_1 \in \mathcal{R}^{n \times m}$, $B_2 \in \mathcal{R}^{(s+b) \times m}$, $C_1 \in \mathcal{R}^{s \times n}$ and

$C_2 \in \mathcal{R}^{s\times(s+b)}$

The impulse responses are given by

$$y_1 = C_1 e^{A_1 t} B_1 u_1 \tag{3.28}$$

$$y_2 = C_2 e^{(A_2+F)t} B_2 u_2 \tag{3.29}$$

where u_1 and u_2 are the areas of the impulses and F is a static feedback term. To find the optimal finite time response with respect to F, we minimize the error

$$\begin{aligned} \| (y_1 - y_2)_T \|^2_{lpe_2} &= \int_0^{t_f} u_1^T B_1^T e^{A_1^T t} C_1^T C_1 e^{A_1 t} B_1 u_1 dt \\ &- 2\int_0^{t_f} u_1^T B_1^T e^{A_1^T t} C_1^T C_2 e^{(A_2+F)t} B_2 u_2 dt \\ &+ \int_0^{t_f} u_2^T B_2^T e^{(A_2+F)^T t} C_2^T C_2 e^{(A_2+F)t} B_2 u_2 dt \end{aligned} \tag{3.30}$$

where t_f is the final time. The solution to these quadratic equations were solved in the first section of this chapter. Thus we

$$\min_F [u_1^T B_1^T X B_1 u_1 - 2u_1^T B_1^T U_1 B_2 u_2 + u_2^T B_2^T U_2 B_2 u_2] \tag{3.31}$$

where

$$[A_1]^T X + X[A_1] = -C_1^T C_1 + e^{A_1^T t_f} C_1^T C_1 e^{A_1 t_f} \tag{3.32}$$

$$A_1^T X_1 + X_1[A_2 + F] = -C_1^T C_2 + e^{A_1^T t_f} C_1^T C_2 e^{(A_2+F)t_f} \tag{3.33}$$

$$[A_2 + F]^T X_2 + X_2[A_2 + F] = -C_2^T C_2 + e^{(A_2+F)^T t_f} C_2^T C_2 e^{(A_2+F)t_f} \tag{3.34}$$

It is assumed that the solutions to Equations (3.32)- (3.34) have unique solutions,

otherwise we decompose them as described in Section 3.1 to obtain their unique solutions. We can solve this optimization problem by applying an unconstrained optimization method (Quasi-Newton method) defining F as a matrix of the optimizing variables.

3.3.4 Implementation

In our context, A_1 is the full model and A_2 is the reduced model with no equivalent, i.e. $K = 0$, and the matrix of the optimizing variables is given by

$$F = \begin{bmatrix} 0_{ss} & A_{sz}K \\ 0_{bs} & A_{bz}K \end{bmatrix}. \tag{3.35}$$

By making u_1=u_2 a vector of ones and optimizing over infinite time we minimize the $\mathcal{H}_2$ norm, in other words the least upper bound for the l_2 - *power*. Although it is a well defined norm, we suggest minimizing an averaging of the inputs:

$$\min_{\forall K} \sum_{i}^{m} u_i^T B_1^T X B_1 u_i - 2u_i^T B_1^T X_i B_2 u_i + u_i^T B_2^T X_2 B_2 u_i \tag{3.36}$$

but still finite-time so we retain numerical stability.

As an initial condition we can use any finite K although it makes sense to use the *psuedoinverse* or inverse equivalent. The space decomposition method described in Chapter 2 is used to solve this.

3.3.5 Simulations

In this section we apply the previous optimization method to the 39 bus, 10 machines New England power system as described in [24], and compare the results with other equivalents. In this case the system has damping and we use the 20^{th}-order linearized swing equation model. The partitioning algorithm described in [20] assigned machines {4 5 7 9} as the study area , {1 6 10} the basis, and {2 3 8} the non-relevant machines.

The following equivalents are examined:

- K = 0

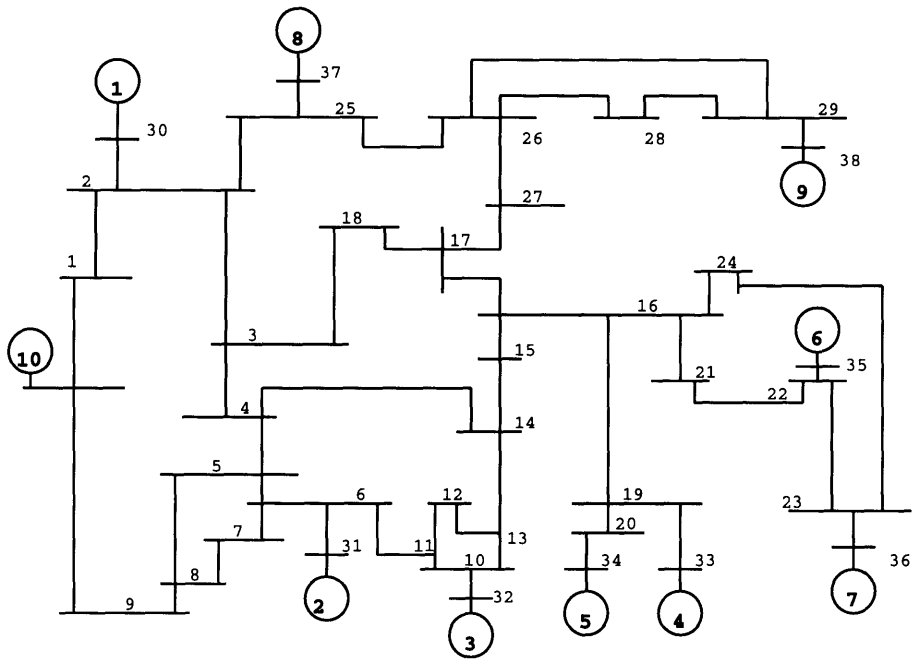

Figure 3-2: New England System Line Diagram. Machines {4 5 6 7 9} as the study area, {1 6 10} the basis, and {2 3 8} the non-relevant machines.

- K=$V_z V_b^{-1}$
- K=$Kopt_{5s}$: corresponds to the optimization method for 5 seconds with initial condition chosen to be K=$V_z V_b^{-1}$. The algorithm converged in around 10 minutes using a Sun Sparcstation 5.

We introduce an input disturbance of a pulse of 1 p.u. for 0.1 seconds into machine angular speed ω_4 and the outputs (which we will be the vector $e(t)$) are sampled every 0.01 seconds for 5 seconds and consist of the study area machines without the reference machine. In Table 3.1 we can appreciate the *quantitative* error reduction at the output $e(t)$. The peak error is reduced by a fourth and the sum of squared errors is one-thirtieth of the best equivalent of $K = 0$ and $K = V_z V_b^{-1}$.

In Figures 3-3 and 3-4 we can appreciate the *qualitative* differences between $K = V_z V_b^{-1}$ and the Optimal equivalent. The $K = V_z V_b^{-1}$ equivalent for *relative* angles is good but for the absolute angles the dynamic equivalent is poor as shown in Figure 3-3. In Figure 3-4 the transient for the relative angle is nearly indistinguishable from the full model and the absolute angle shows a very good match.

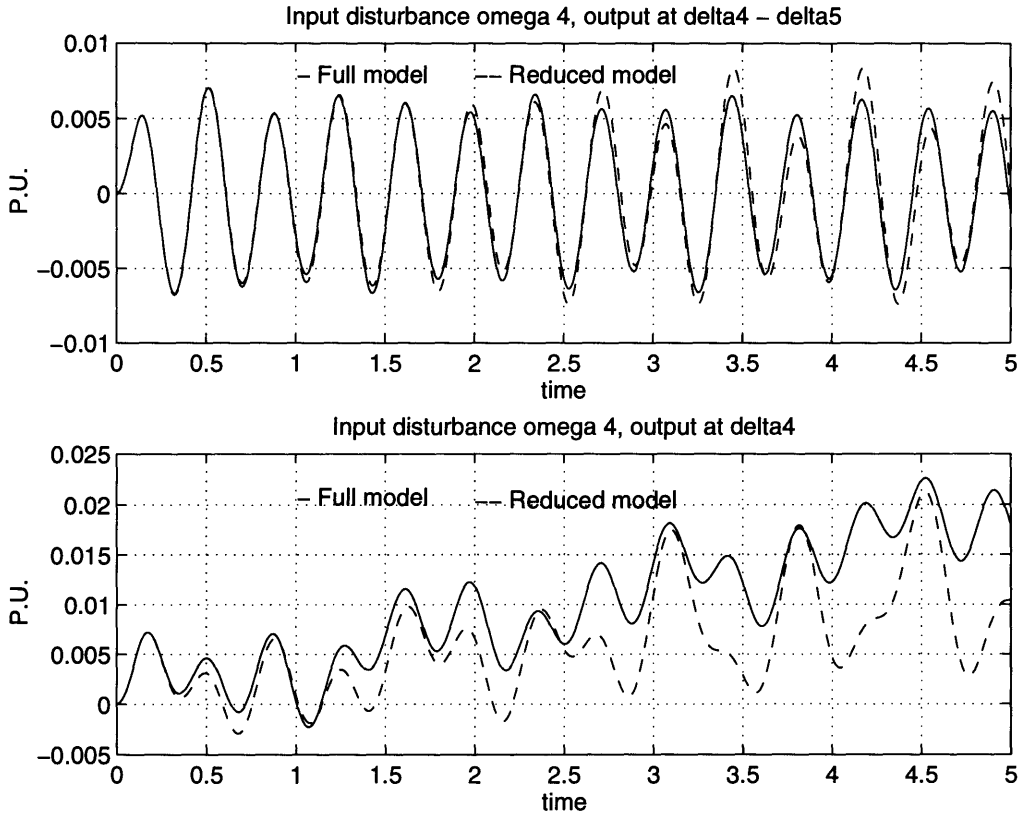

Figure 3-3: Comparison of the full model and the reduced-order model using the $K = V_z V_b^{-1}$ equivalent. Input disturbance into machine 4.

Table 3.1: Output vector $\mathbf{e(t)}$ to an input disturbance of one p.u. for 0.1s on machine angular speed ω_4. Output sampled every 0.01s for 5s.

Equivalent:	K = 0	K = $V_z V_b^{-1}$	K=$Kopt_{5s}$
$\| e(t) \|_\infty$	4.78×10^{-2}	7.96×10^{-2}	1.13×10^{-2}
$\sum$ of squared errors	0.9611	1.7115	0.0349

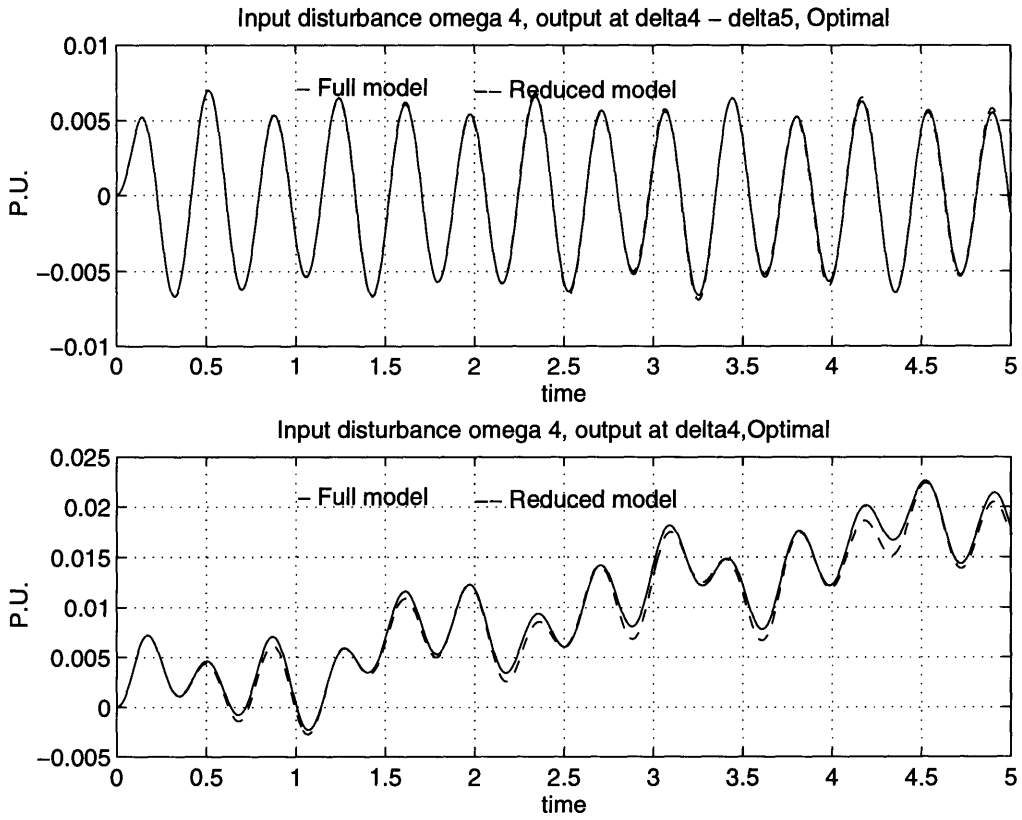

Figure 3-4: Comparison of the full model and the reduced-order model for the Optimized equivalent. Input disturbance into machine 4.

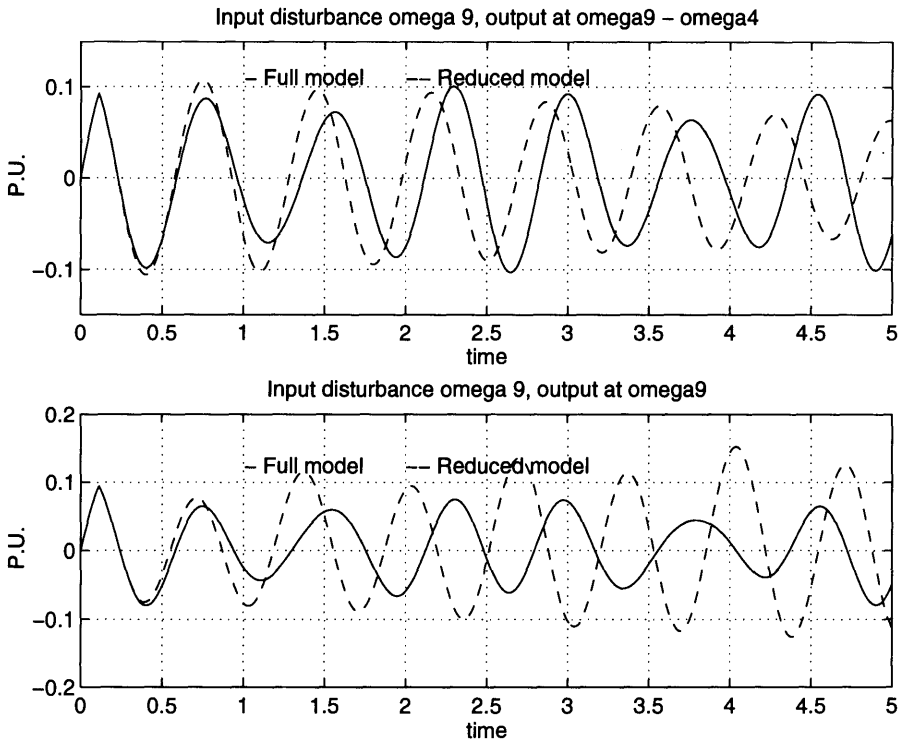

Figure 3-5: Comparison of the full model and the reduced-order model using the $K = V_z V_b^{-1}$ equivalent. Input disturbance into machine 9.

Table 3.2: Output vector $\mathbf{e(t)}$ to an input disturbance of one p.u. for 0.1s on machine angular speed ω_9. Output sampled every 0.01s for 5s.

Equivalent:	K = 0	K = $V_z V_b^{-1}$	K=$Kopt_{5s}$
$\| e(t) \|_\infty$	0.1363	0.2474	3.15×10^{-2}
$\sum$ of squared errors	2.8975	16.7093	0.1599

In Figure 3-5 we show that the relative angle speed is not very good and the absolute is even worse for $K = V_z V_b^{-1}$ equivalent given a disturbance into machine 9. In Figure 3-6 the optimized model reduces substantially the error as reported in Table 3.1, however it is not as good as in 3-4. This might be due to a improper selection of the chord modes, which will be addressed further in Chapter 5.

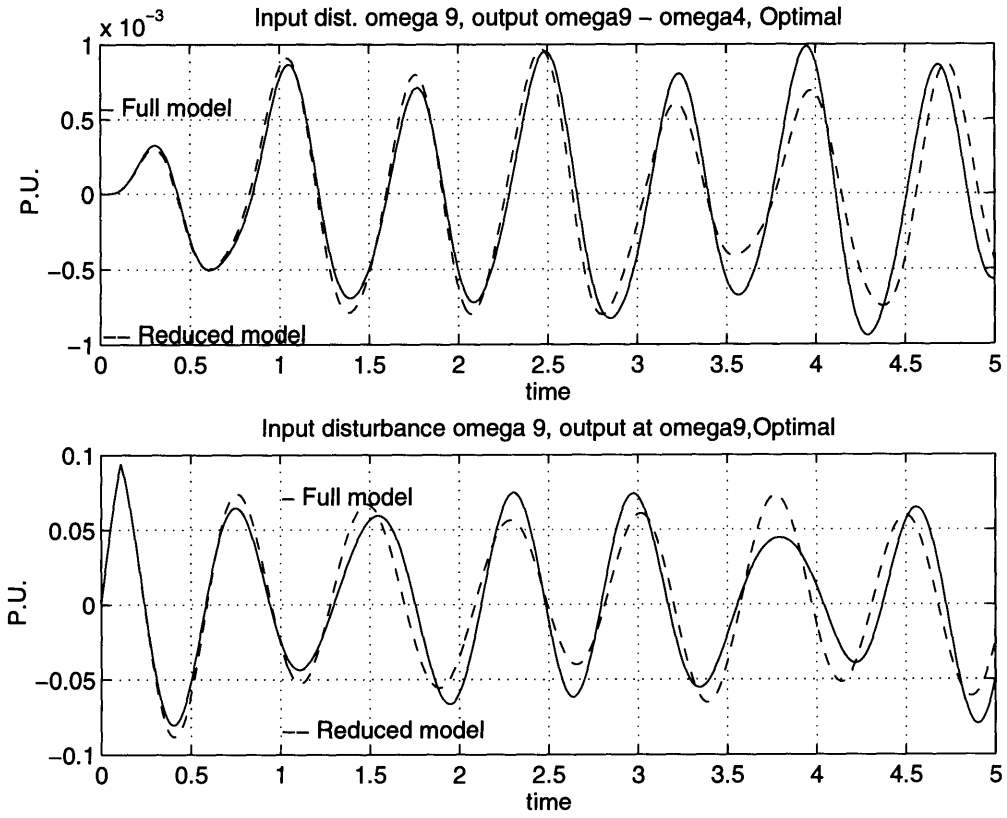

Figure 3-6: Comparison of the full model and the reduced-order model for the Optimized equivalent. Input disturbance into machine 9.

3.4 Zero Eigenvalue Decoupling

In many power system applications the relative angles are sufficient to determine stability. Under such conditions it may be favorable to decouple the mode zero (angle reference) and examine the *synchrony* that may arise naturally in the system (this is left as future work). The following section describes a well known method to decouple the zero eigenvalue of a swing equation model.

Suppose we have a system $\dot{x} = Ax + Bu$ partitioned as in (3.23), then the transformation

$$\tilde{\delta}_i = \delta_i - \delta_j, \ \forall i = 1, 2, 3, ...j - 1, j + 1, ... \tag{3.37}$$

and

$$\tilde{\omega}_i = \omega_i - \omega_j, \ \forall i = 1, 2, 3, ...j - 1, j + 1, ... \tag{3.38}$$

where j is the reference machine, decouples the reference machine from the rest of the system. It is recommended to use a non-relevant machine angle since the similarity transformation will not affect the inputs of the system and the decoupled system will retain physical sense. We reorder the system

$$\begin{bmatrix} \dot{x}_s \\ \dot{x}_b \\ \dot{x}_z \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} A_{ss} & A_{sb} & A_{sz} & A_{s(n-1)} & A_{sn} \\ A_{bs} & A_{bb} & A_{bz} & A_{b(n-1)} & A_{bn} \\ A_{zs} & A_{zb} & A_{zz} & A_{z(n-1)} & A_{zn} \\ A_{(n-1)s} & A_{(n-1)b} & A_{(n-1)z} & A_{(n-1)(n-1)} & A_{(n-1)n} \\ A_{ns} & A_{nb} & A_{nz} & A_{n(n-1)} & A_{nn} \end{bmatrix} \begin{bmatrix} x_s \\ x_b \\ x_z \\ x_{n-1} \\ x_n \end{bmatrix} \tag{3.39}$$

where x_{n-1} corresponds to δ_j, x_n corresponds to ω_j and the non-relevant machines have two less state variables.

Now we construct the similarity transformation. Let

$$\tilde{x_s} = x_s - \begin{bmatrix} T_{s\delta} & T_{s\omega} \end{bmatrix} \begin{bmatrix} x_{n-1} \\ x_n \end{bmatrix} \tag{3.40}$$

$$\tilde{x_b} = x_b - \begin{bmatrix} T_{b\delta} & T_{b\omega} \end{bmatrix} \begin{bmatrix} x_{n-1} \\ x_n \end{bmatrix} \tag{3.41}$$

$$\tilde{x_z} = x_z - \begin{bmatrix} T_{z\delta} & T_{s\omega} \end{bmatrix} \begin{bmatrix} x_{n-1} \\ x_n \end{bmatrix} \tag{3.42}$$

where

$$T_{s\delta} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}_{s\times 1}, \; T_{s\omega} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{s\times 1}, \tag{3.43}$$

and $T_{b\delta}, T_{b\omega}, T_{z\delta}$ and $T_{z\omega}$ are similarly defined. Then the similarity transformation is given by

$$T = \begin{bmatrix} I & 0 & 0 & -T_{s\delta} & -T_{s\omega} \\ 0 & I & 0 & -T_{b\delta} & -T_{b\omega} \\ 0 & 0 & I & -T_{z\delta} & -T_{z\omega} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.44}$$

It is a simple task now to form the new decoupled system. By a similarity transformation the system is

$$\tilde{x} = Tx \tag{3.45}$$

Thus the new decoupled system is

$$\dot{\tilde{x}} = TAT^{-1}\tilde{x} + TBu \tag{3.46}$$

$$y = CT^{-1}\tilde{x} \tag{3.47}$$

The system matrix will have the same modes as the original system but the new transformed system will have zeros in its last column, effectively decoupling x_{n-1} from the rest of the system. The previous decomposition retains the physical meaning between states and machines, and the system can be repartitioned under a new chord.

After finding the optimal reduced order system corresponding to the zero eigenvalue system it is possible to reintroduce the zero eigenvalue and obtain a system in terms of the absolute variables.

Suppose we find a Grouping matrix K where

$$\tilde{x}_z = K\tilde{x}_b \tag{3.48}$$

is as best approximated. Replacing $\tilde{x}_z$ and $\tilde{x}_b$ by their absolute definitions, we obtain

$$x_z - T_{z\delta}x_{n-1} - T_{z\omega}x_n = K[x_b - T_{b\delta}x_{n-1} - T_{b\omega}x_n]. \tag{3.49}$$

Solving explicitly for x_z gives the equivalent

$$x_z = Kx_b + (T_{z\omega} - KT_{b\omega})x_n + (T_{z\delta} - KT_{b\delta})x_{n-1} \tag{3.50}$$

and substituting (3.50) into (3.39) gives the reduced-order model

$$\begin{bmatrix} \dot{x}_s \\ \dot{x}_b \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} A_{ss} & A_{sb} + A_{sz}K & A_{sz}(T_{z\delta} - KT_{b\delta}) + A_{s(n-1)} \\ A_{bs} & A_{bb} + A_{bz}K & A_{bz}(T_{z\delta} - KT_{b\delta}) + A_{b(n-1)} \\ A_{(n-1)s} & A_{(n-1)b} + A_{(n-1)z}K & A_{(n-1)z}(T_{z\delta} - KT_{b\delta}) + A_{(n-1)(n-1)} \\ A_{ns} & A_{nb} + A_{nz}K & A_{nz}(T_{z\delta} - KT_{b\delta}) + A_{n(n-1)} \end{bmatrix}$$

$$\begin{bmatrix} A_{sz}(T_{zw} - KT_{bw}) + A_{sn} \\ A_{sz}(T_{zw} - KT_{bw}) + A_{bn} \\ A_{(n-1)z}(T_{zw} - KT_{bw}) + A_{(n-1)n} \\ A_{nz}(T_{zw} - KT_{bw}) + A_{nn} \end{bmatrix} \begin{bmatrix} x_s \\ x_b \\ x_{n-1} \\ x_n \end{bmatrix} \tag{3.51}$$

Since the transformation back can be put in the form of a similarity transformation the zero eigenvalue is retained for any finite K. The offset can be adjusted by calculating the steady state solution of the original model and adding a constant vector to the output of the reduced model to compensate for any steady state error.

Example 3.4.1 *In this example we optimize the swing equation model as described in Section 3.3.5. We denote the optimized system (for 5 seconds) as the coupled system and the decoupled system corresponds to the zero eigenvalue decoupling, optimizing it for 5 seconds and then transformation back. Comparisons are made in the following figures and table. For both figures the error vector $e(t)$ corresponds to the outputs in all the study area machines. The rising responses correspond to the δ variables and the oscillatory responses correspond to the ω variables.*

As we can see from the results the decoupled system shows an improvement in the δ group over the coupled system, although the latter better represents the ω group. In Table 3.3 we notice that the delta variables have lower errors for the decoupled system over the coupled system, however the faster dynamics are not well equivalenced. At longer time periods the coupled system degrades significantly due to lack of the presence of the zero eigenvalue, thus the decoupled system is a better equivalent for long time scales. Notice, however that we can repartition the system with the zero eigenvalue decoupled since the transformation retains physical meaning and then assigning the inputs into the new study area. This is left as future work.

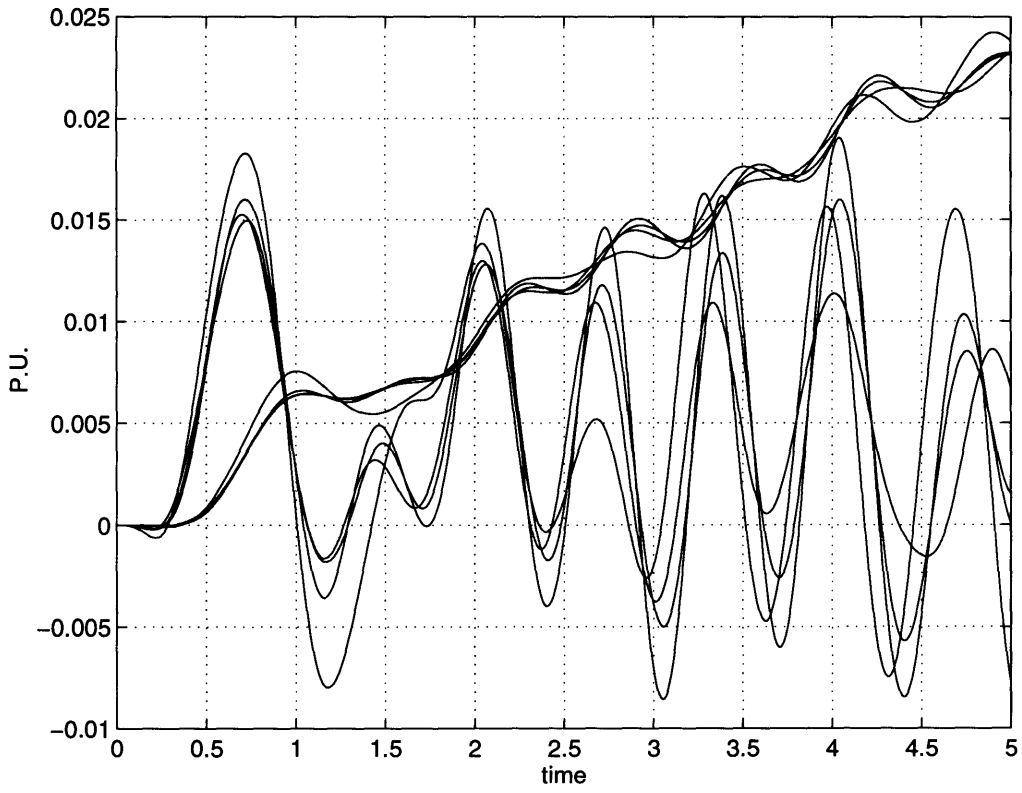

Figure 3-7: $e(t)$ for coupled system. Disturbance of 1 p.u. for 0.1s into machine ω_9.

Table 3.3: Output vector $\mathbf{e(t)}$ to an input disturbance of one p.u. on machine speed ω_9 for 0.1 s. Output sampled every 0.01s for 5s.

Method:	Coupled	Decoupled zero eigenvalue
$\| e(t)_{\delta_4} \|_\infty$	0.0232	0.0192
$\| e(t)_{\omega_4} \|_\infty$	0.0160	0.0395
$\| e(t)_{\delta_5} \|_\infty$	0.0232	0.0200
$\| e(t)_{\omega_5} \|_\infty$	0.0191	0.0484
$\| e(t)_{\delta_7} \|_\infty$	0.0233	0.0184
$\| e(t)_{\omega_7} \|_\infty$	0.0153	0.0462
$\| e(t)_{\delta_9} \|_\infty$	0.0242	0.0206
$\| e(t)_{\omega_9} \|_\infty$	0.0183	0.0595
$\sum$ of squared errors $e(t)_{\delta_4}$	0.0934	0.0480
$\sum$ of squared errors $e(t)_{\omega_4}$	0.0266	0.1415
$\sum$ of squared errors $e(t)_{\delta_5}$	0.0936	0.0492
$\sum$ of squared errors $e(t)_{\omega_5}$	0.0360	0.2128
$\sum$ of squared errors $e(t)_{\delta_6}$	0.0931	0.0480
$\sum$ of squared errors $e(t)_{\omega_6}$	0.0226	0.1437
$\sum$ of squared errors $e(t)_{\delta_9}$	0.0946	0.0488
$\sum$ of squared errors $e(t)_{\omega_9}$	0.0363	0.2702

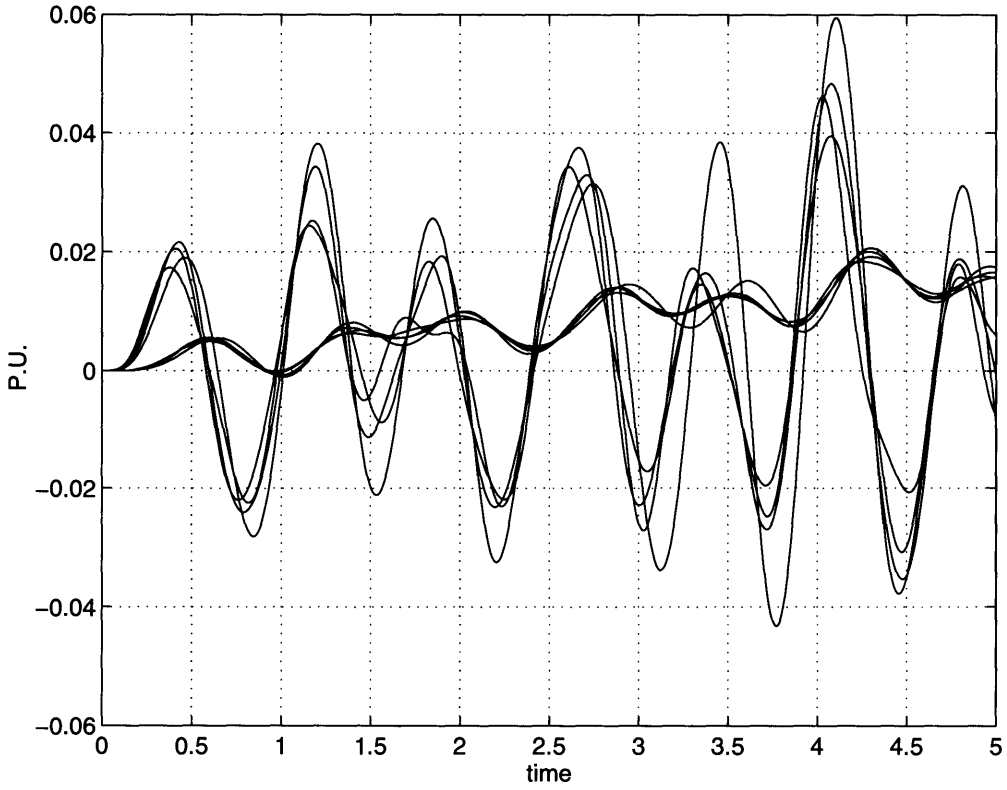

Figure 3-8: $e(t)$ for decoupled system. Disturbance of 1 p.u. for 0.1s into machine ω_9.

Chapter 4

Basis Variable Selection

To achieve a "good" reduced-order model the choice of basis variables is critical. Assuming that we have information on which modes are excited we can pick such a set in an intelligent manner. The purpose of this chapter is to develop an algorithm which is both fast and numerically stable. The alternative is to try every possible combination of basis variables; this would be computationally prohibitive even for small power system models.

In [20] the selection of reference machines is based on **one dimensional synchrony**. The algorithm and associated definitions are described in Chapter 2 of this thesis. In this chapter we propose a new method of selecting basis variables using the concept of *multi-dimensional synchrony.*

4.1 Motivation

The purpose of this chapter is to develop a method for selecting basis variables which is both numerically efficient and precise. We rely heavily on the theory developed in [20].

For reference we state the definition of **one dimensional synchrony**.

Definition 4.1.1 *Machine i is synchronic with machine j in the chord v, or is v-synchronic with j, if there exists a scalar constant k_{ij} such that for any initial condition and for all t the angles $\triangle\delta_i(t)$ and $\triangle\delta_j(t)$ satisfy*

$$\triangle\delta_i(t) - k_{ij}\triangle\delta_j(t) = \gamma_{ij}(t) \tag{4.1}$$

where $\gamma_{ij}(t)$ contains none of the modes in v.

The metric of **synchronic distance** between the variables formulated in [20] is based on the previous definition. The initial reference machine is picked with the largest net participation in the chord v. This machine may have extensive participation in all the modes of the chord, thus it may be difficult to choose the second reference machine such that the **synchronic angle** is orthogonal, while a highly localized initial choice would make it easy to find any other state variable which is nearly orthogonal to it. **Synchronic distance** does not convey information on the extensiveness of the modes in any particular state variable, only the angle and net participation in that state variable. This motivates to look for a different approach which captures the greater degree of freedom implicit in **multiple synchrony**.

4.2 Formulation

Suppose we can find n_e basis variables x_b which participate in a single mode (or two modes if we have a complex conjugate pair) but different modes in the chord v. Under such conditions we can build the solution to x_z completely given that only the chord v is excited by taking a linear combination of the solutions of the entries of x_b to a given input in a single area; i.e. x_b would form a nearly orthonogal basis which would span the subspace corresponding to x_z in the chord x_z. Thus we have a relation of the form of

$$x_z = Kx_b \tag{4.2}$$

In practice we do not have such a clear separation of basis variables but we can still identify those variables which have a high participation in single modes. We shall introduce two metrics which will help us identify the basis variables but first we will discuss a suitable scale-invariant matrix to identify the mode shapes of the system

for each variable in the model.

The matrix that captures the information of how the various states interact with respect to each other is the right eigenvector matrix since if two variables exhibit synchrony with respect to each other then the rows of the eigenvector matrix are linearly dependent, as shown in lemma 2.1.1. However the right eigenvector matrix is subject to scaling. If we scale the various columns of the right eigenvector matrix then the **localizability index** and **synchronic angle** (defined in the next page) would be altered for every different type of scaling. Thus we weigh this matrix to make it more robust to scaling. Let $\mathbf{V}$ the right eigenvector of the matrix and let $\mathbf{W^T}$ be the left eigenvector matrix, where

$$\mathbf{V} = [v_1, v_2, ..., v_{n_e}], \mathbf{W} = [w_1, w_2, ..., w_{n_e}]. \tag{4.3}$$

The new weighted matrix $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \|w_1\|_2 & 0 & \cdots & 0 \\ 0 & \|w_2\|_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \|w_n\|_2 \end{bmatrix} \tag{4.4}$$

As we can see, we scale the right eigenvector by the two norm of the left eigenvector once. Suppose we scale the first column of $\mathbf{V}$ by a positive real variable α, thus the first diagonal entry of the weighting matrix is multiplied by α^{-1}, and as a result the magnitude of the original scaling is retained. The right eigenvector matrix is not completely invariant to scaling. The magnitude is invariant but the phase of the entries can be affected.

The objective then is to identify which row vectors are as localized as much as possible and thus forming a nearly orthogonal *synchronic* basis.

4.2.1 Basis Variable Selection Algorithm

Suppose we have constructed a **weighted right eigenvector** $\mathbf{P}$ matrix corresponding to the modes in v. Let a_i for i = 1,2,...n be the rows of the matrix and calculate $\mathbf{L}$ = [Re($\mathbf{P}$) Im($\mathbf{P}$)] . We define two metrics:

Definition 4.2.1 *Let μ be called the* **localizability index** *and a_{ij} correspond to the j^{th} component of the row vector a_i where j = 1,2, ... n_e then*

$$\mu_i = variance(a_{i1}, a_{i2}, ...a_{in_e})2n \tag{4.5}$$

Definition 4.2.2 *Let ϕ be called the* **synchronic angle** *between the row vector a_j and the projected row vector y on the subspace Φ. Let $\Phi = [a'_1, a'_2, a'_3, ..., a'_i]$ (where a'_i is transpose(a_i)) correspond to the matrix which spans this space. Then*

$$\phi_{j\Phi} = \cos^{-1}\left(\frac{ya'_j}{\|y\|\|a'_j\|}\right) \tag{4.6}$$

$$y = \Phi(\Phi'\Phi)^{-1}\Phi' a'_j \tag{4.7}$$

Thus the first definition gives us a measure of the localizability of the variable in the modes corresponding to the chord v weighted by the participation in that chord. For a high μ the variable is localized in a single mode and participates heavily in that chord and for μ = 0 the participation is evenly spread among all the modes. The second definition indicates the angle between the vector j and the projection of that vector upon the subspace Φ . With our metrics defined it is a simple task to formulate the following algorithm for basis variable selection which is a modification of the *reference selection algorithm* contained in [20].

Algorithm 4.2.1 *(Basis Variable Selection) Inputs: Real matrix* $\mathbf{L}$ *which corresponds to real and imaginary parts of* $\mathbf{P}$ *for the chord v. Let n_e be the number of modes contained in v.*

- *Set* $\mathbf{y_1} = \mathbf{a}_{i^*}$, *where* $i^* = \arg\max(\mu_i)$
- *For r = 2,...,n_e, set* $\mathbf{y_r} = \mathbf{a}_{i^*}$ *where*

$$i^* = \arg\max_i [\mu_i \sin \phi_{i\Phi}]$$
$$\Phi = [y'_1, y'_2, y'_3, ..., y'_j] \; 1 \leq j \leq r\text{-}1$$

The algorithm works as follows: First we pick a row which has the largest variance, in other words the most localized modes and greatest participation on the modes of the chord v. The second one is picked such that is has the highest angle with respect to first row and the highest localizability and participation. For the successive variables the row vectors are projected on the subspace spanned by the previous picks and the angle is calculated. The variance is also calculated and a decision is made based upon the combinations of these two. Thus we have selected a set of basis variables which will span the subspace corresponding to x_z in the modes of the chord. Note that if we wish to include more basis variables we simply run the algorithm as many times as the number of user-defined basis variables are desired and/or the chord can be extended to include more modes.

We note that any choice of basis variables will enable us to calculate a Grouping Matrix K, however, if the system exhibits local and extensive behavior, and if the chord is chosen properly, then we would expect that the Grouping Matrix K will be minimized (using a suitable matrix norm) with the proper choice of basis variables.

4.2.2 Example

Consider the 20^{th} order swing equation model of the 10 machine 38 bus New England model with damping described in [20]. The synchronic areas were determined in [24]. Corresponding to the study area are machines {4, 5, 6, 7, 9} (state variables: {11 12 7 8 9 10 13 14 17 18 }) and two non relevant areas: {1, 2, 3, 8} and {10} (which are respectively state variables {1 2 3 4 5 6 15 16} and {19 20}). The basis machines picked by the fuzzy algorithm in [20] are {1,6,10} which corresponds to state variables {1 2 11 12 19 20} and the chord is v = [0 1 3] corresponding to the modes [0,0, $\pm$ 4.750i, $\pm$ 8.4601i] in the undamped swing equation model.

The basis variables picked by the algorithm proposed in Section 4.2.1 are {5 6 9 10 19 20} (corresponding to machines { 3 5 10 }) which we denote as the new or

Table 4.1: Comparison of errors between old and new basis selection. Output vector e(t) to an input disturbance of one p.u. on machine speed ω_4. Output sampled every 0.01s for 30s.

Equivalent:	K = 0 (old basis)	K = $V_z V_b^{-1}$ (old basis)	$K = V_z V_b^{-1}$ (new basis)
$\| e(t)_{\delta_4} \|_\infty$	0.1134	0.0498	0.0486
$\| e(t)_{\omega_4} \|_\infty$	0.0443	0.0932	0.0647
$\| e(t)_{\delta_7} \|_\infty$	0.1139	0.0485	0.0453
$\| e(t)_{\omega_7} \|_\infty$	0.0463	0.0671	0.0144
$\| e(t)_{\delta_9} \|_\infty$	0.1128	0.0485	0.0478
$\| e(t)_{\omega_9} \|_\infty$	0.0586	0.0594	0.0451
$\sum$ of squared errors $e(t)_{\delta_4}$	12.950	2.0134	2.0877
$\sum$ of squared errors $e(t)_{\omega_4}$	0.8745	3.8428	1.9763
$\sum$ of squared errors $e(t)_{\delta_7}$	12.953	2.0044	2.0819
$\sum$ of squared errors $e(t)_{\omega_7}$	0.8838	2.2102	0.0953
$\sum$ of squared errors $e(t)_{\delta_9}$	12.9432	1.9999	2.0966
$\sum$ of squared errors $e(t)_{\omega_9}$	1.3212	1.8946	1.0193

refined basis, with {4 5 6 7 9} being the study area machines and {1 2 8} the non relevant machines. Note that the total computational time for selecting these basis variables was half a second.

A perturbation is introduced at machine ω_4 and the output *error* was sampled every 0.01s for 30 seconds. The results are shown in Table 4.1. As we can see, the $K = V_z V_b^{-1}$ equivalent with the new basis is overall a better reduced order model. Table 4.2 presents the results for a perturbation into machine ω_9. We observe there is a significant reduction in the error using the new basis selection demonstrating the effectiveness of the algorithm.

Additional justification for the new basis variable selection algorithm is obtained by calculating the Frobenius norm of the Grouping Matrix K. For the old basis this is 7.9845; for the new basis this is 1.6920. In fact, a combinatorial search showed that the basis variable selected by the new algorithm resulted in the Grouping Matrix K with the lowest Frobenius norm of all possible basis variable combination. In this sense, the algorithm picked the optimal solution.

The optimized results for both cases are nearly the same, they reduce to about the

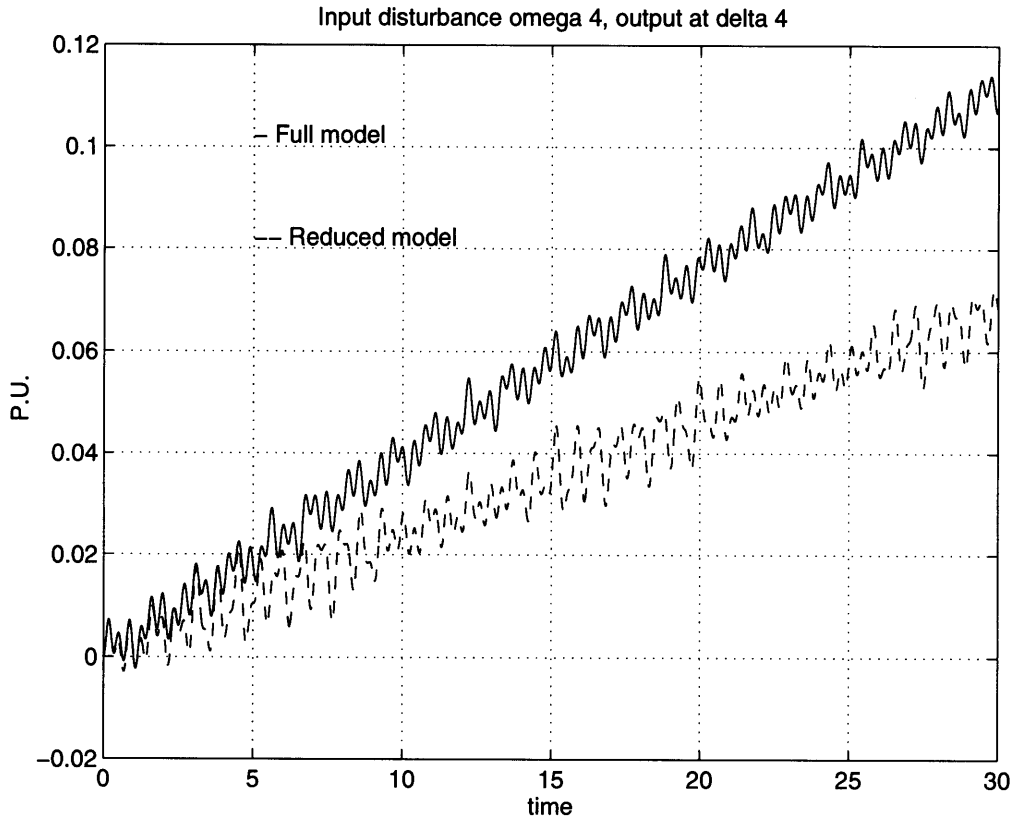

Figure 4-1: Comparison of full and reduced order model. K = $V_z V_b^{-1}$ equivalent old basis.

Table 4.2: Comparison of errors between old and new basis selection. Output vector $\mathbf{e(t)}$ to an input disturbance of one p.u. on machine speed ω_9. Output sampled every 0.01s for 30s.

Equivalent:	K = 0 (old basis)	K = $V_z V_b^{-1}$ (old basis)	$K = V_z V_b^{-1}$ (new basis)
$\| e(t)_{\delta_4} \|_\infty$	0.1361	0.0971	0.0537
$\| e(t)_{\omega_4} \|_\infty$	0.0707	0.2128	0.0413
$\| e(t)_{\delta_7} \|_\infty$	0.1359	0.0959	0.0535
$\| e(t)_{\omega_7} \|_\infty$	0.0696	0.1950	0.0403
$\| e(t)_{\delta_9} \|_\infty$	0.1445	0.1001	0.0637
$\| e(t)_{\omega_9} \|_\infty$	0.1699	0.1902	0.1791
$\sum$ of squared errors $e(t)_{\delta_4}$	18.834	7.2686	2.6387
$\sum$ of squared errors $e(t)_{\omega_4}$	1.923	26.318	0.9151
$\sum$ of squared errors $e(t)_{\delta_7}$	18.834	7.1996	2.6379
$\sum$ of squared errors $e(t)_{\omega_7}$	1.8243	20.429	0.8341
$\sum$ of squared errors $e(t)_{\delta_9}$	19.063	7.1885	2.8749
$\sum$ of squared errors $e(t)_{\omega_9}$	16.958	20.678	18.411

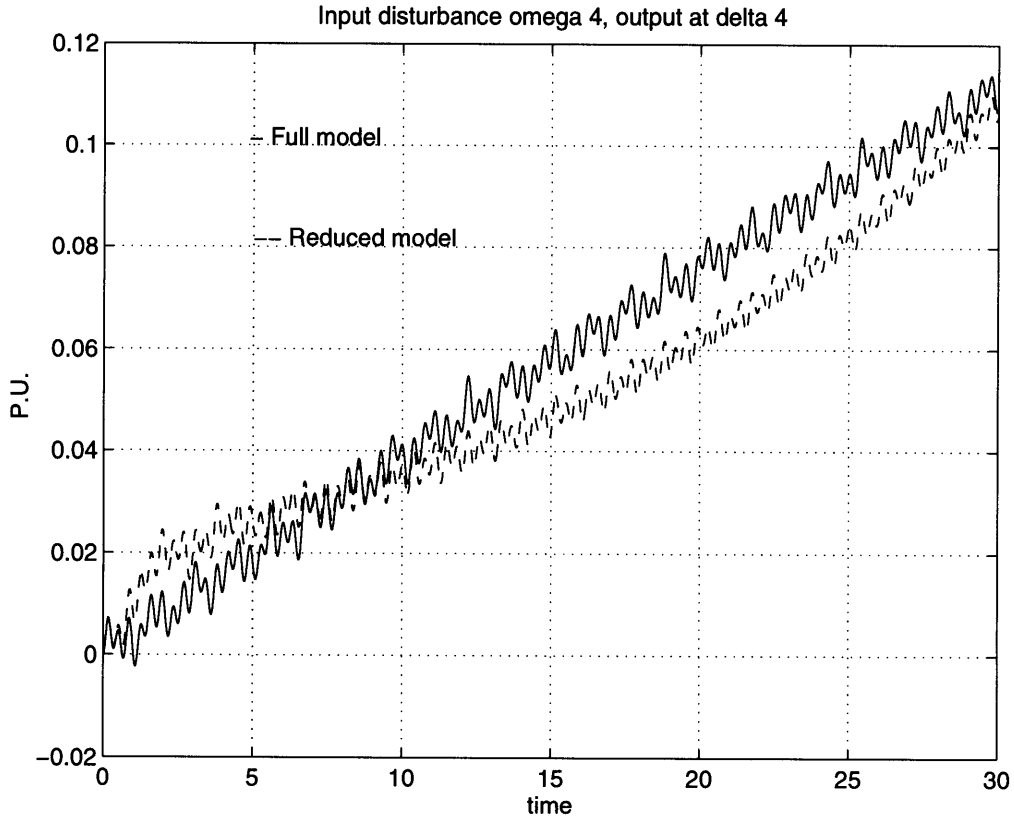

Figure 4-2: K = Optimal for 30 seconds, K = $V_z V_b^{-1}$ initial condition, old basis.

same cost function although the new basis variables (initial condition K = $V_z V_b^{-1}$) is much closer to optimal than the old basis. In Figures 4-1 and 4-2 we compare the K = $V_z V_b^{-1}$ and optimized equivalent for the old basis. At this time scale of optimization the overall error increases, this leads us to conclude the necessity for the addition of other modes in the chord to compensate for the error. This is left for future research.

Chapter 5

Detailed Model

In this chapter we implement the previously developed methods on the New England 20 machines, 38 bus and 111 state variable power system L.T.I. model produced from a nonlinear model in *Eurostag*. The model presents certain challenges including the presence of multiple zero eigenvalues and increased size.

5.1 Zero Eigenvalues Decoupling

The presence of uncontrollable zero eigenvalues in the system give rise to multiple solutions for the Lyapunov equations. We cannot produce a minimal realization without destroying the physical meaning of the partitioning. We describe a simple method to decouple these eigenvalues similar to the space decomposition method in Chapter 3 . In this case we break up the Lyapunov equation in Sylvester equations to speed up the algorithm.

Suppose we have the *full* and *reduced* system with no equivalent ($K = 0$)

$$\dot{x_{full}} = A_{full}x_{full} + B_{full}u$$
$$y_1 = C_{full}x_{full}$$
$$\dot{x_{red}} = (A_{red} + F)x_{red} + B_{red}u$$
$$y_2 = C_{red}x_{red}$$

(5.1)

where

$$F = \begin{bmatrix} 0_{ss} & A_{sz}K \\ 0_{bs} & A_{bz}K \end{bmatrix}. \tag{5.2}$$

We further calculate the rank of both A_{full} and A_{red} to indicate the number of zero eigenvalues in the system. Now decompose the system impulse response solutions into zero and full rank parts (as described in [25]). Then

$$\begin{aligned} y_1 = C_{full}e^{A_{full}t}B_{full}u &= (C_1e^{A_1t}B_1 + C_2e^{A_2t}B_2)u = (C_1L_1B_1 + C_2e^{A_2t}B_2)u \\ y_2 = C_{red}e^{(A_{red}+F)t}B_{red}u &= (C_3e^{A_3t}B_3 + C_4e^{A_4t}B_4)u = (C_3L_3B_3 + C_4e^{A_4t}B_4)u \end{aligned} \tag{5.3}$$

where L_1 and L_3 correspond to the solutions of e^{A_1t} and e^{A_3t} respectively and are obtained by the method described in Section 3.2.1.

The finite time lpe_2 response is:

$$\begin{aligned} \| (y_1 - y_2)_T \|^2_{lpe_2} = \int_0^{t_f} & u[B_2^T e^{A_2^T t}C_2^T C_2 e^{A_2 t}B_2 + 2B_2^T e^{A_2^T t}C_2^T C_1 L_1 B_1 \\ & -2B_2^T e^{A_2^T t}C_2^T C_4 e^{A_4 t}B_4 - 2B_2^T e^{A_2^T t}C_2^T C_3 L_3 B_3 + B_1^T L_1^T C_1^T C_1 L_1 B_1 \\ & -2B_1^T L_1^T C_1^T C_4 e^{A_4 t}B_4 - 2B_1^T L_1^T C_1^T C_3 L_3 B_3 + B_4^T e^{A_4^T t}C_4^T C_4 e^{A_4 t}B_4 \\ & +2B_4^T e^{A_4^T t}C_4^T C_3 L_3 B_3 + B_3^T L_3^T C_3^T C_3 L_3 B_3]u \; dt \\ = u[& B_2^T X_1 B_2 + B_4^T X_2 B_4 - 2B_2^T X_3 B_4 + B_1^T L_1^T C_1^T C_1 L_1 B_1 t_f \\ & +B_3^T L_3^T C_3^T C_3 L_3 B_3 t_f + 2B_2^T (e^{A_2 t_f} - I)^T A_2^{-T} C_2^T C_1 L_1 B_1 \\ & +2B_4^T (e^{A_4 t_f} - I)^T A_4^{-T} C_4^T C_3 L_3 B_3 - 2B_2^T (e^{A_2 t_f} - I)^T A_2^{-T} C_2^T C_3 L_3 B_3 \\ & -2B_1^T L_1^T C_1^T C_4 A_4^{-1} (e^{A_4 t_f} - I) B_4 - 2B_1^T L_1^T C_1^T C_3 L_3 B_3 t_f]u \end{aligned} \tag{5.4}$$

where

$$[A_2]^T X_1 + X_1[A_2] = -C_2^T C_2 + e^{A_2^T t_f} C_2^T C_2 e^{A_2 t_f} \tag{5.5}$$

$$[A_4^T] X_2 + X_2[A_4] = -C_4^T C_4 + e^{A_4^T t_f} C_4^T C_4 e^{A_4 t_f} \tag{5.6}$$

$$[A_2^T] X_3 + X_3[A_4] = -C_2^T C_4 + e^{A_2^T t_f} C_2^T C_4 e^{A_4 t_f} \tag{5.7}$$

Notice that we only have to calculate the space decomposition for A_{full} only once, but $A_{red} + F$ has to be decomposed at every step. Fortunately the matrix is smaller and thus the computational effort is smaller. The introduction of the space decomposition does not greatly affect much the computational burden on the algorithm.

5.2 Basis Selection

Since the SME framework consists of equivalencing machines, we use the algorithm selection program on the damped New England swing equation model. The basis variables previously identified belonged to machines {3 5 10}. In this case the equivalent with the new basis variables is $K = V_z V_{b*}^{\dagger}$ where † is the *psuedoinverse* and V_{b*} is constructed from the detailed model corresponding to the modes [0, ± 4.750i, ± 8.4601] of the undamped swing equation model. Notice that one of the zero modes is lacking since we cannot clearly identify this mode with the modes of the detailed model. Note that the output transients for $K = V_z V_{b*}^{\dagger}$ and $K = V_z V_b^{-1}$ for the old basis are practically the same. As shown in Table 5.1 and Figure 5.3, there is a marked improvement in time domain simulations, and the zero mode gain nearly matches the full model zero mode gain. The relative angle is not affected much as we can see in Figure 5.3.

Table 5.1: Output vector $\mathbf{e(t)}$ to an input disturbance of one p.u. on machine angle ω_4.

Equivalent:	K = $V_z V_b^{-1}$ (Old Basis)	$K = V_z V_{b*}^{\dagger}$(New Basis)
$\| e(t)_{\delta_6} \|_\infty$	22.999	18.600
$\| e(t)_{\delta_4} \|_\infty$	22.992	18.629
$\sum$ of squared errors $e(t)_{\delta_6}$	6432	2006
$\sum$ of squared errors $e(t)_{\delta_4}$	6437	2009

5.3 Optimization

We optimized this model with the new basis variable selection for 100 seconds. The results for 10 seconds are shown in Figures 5-3 and 5-4. The full line is the full model and the dashed is the reduced order model. As we can see from the figures the fast modes are relatively well preserved but the slow modes are not retained. We conclude that a critical or critical combination of modes have not been well identified in the chord selection process. The next logical step is to develop a new algorithm for chord selection which takes into account the detailed model modes.

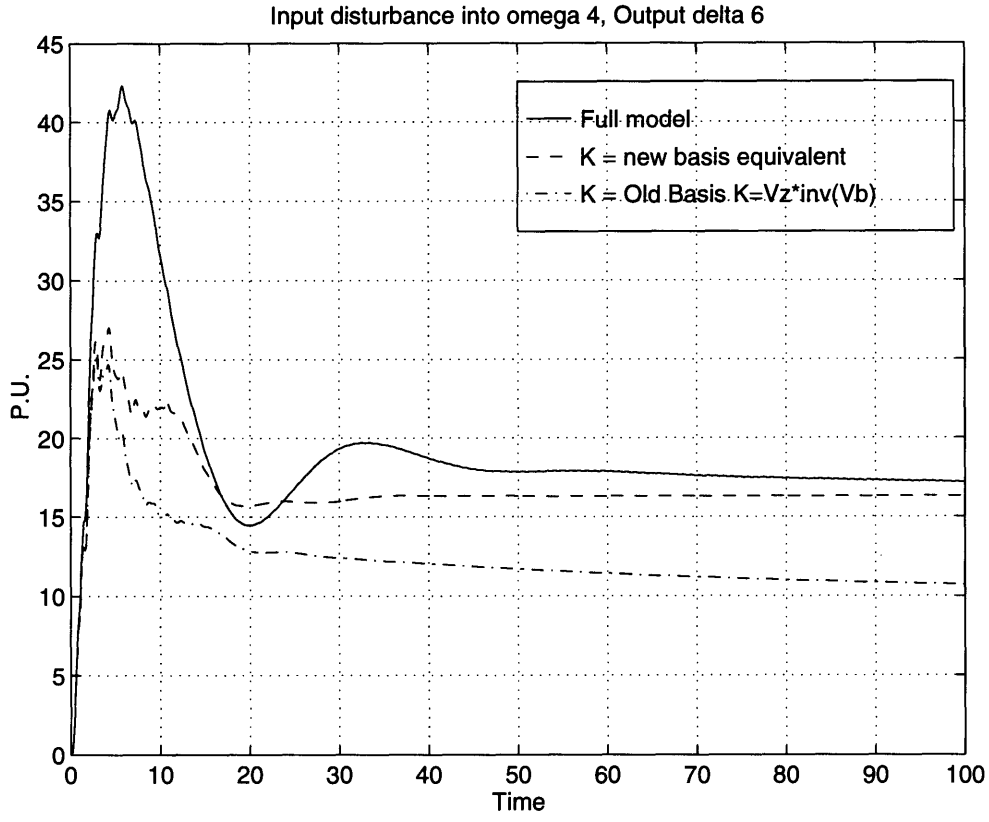

Figure 5-1: Detailed model simulation. Input disturbance of 1 p.u. for 1s. Angle δ_6 output.

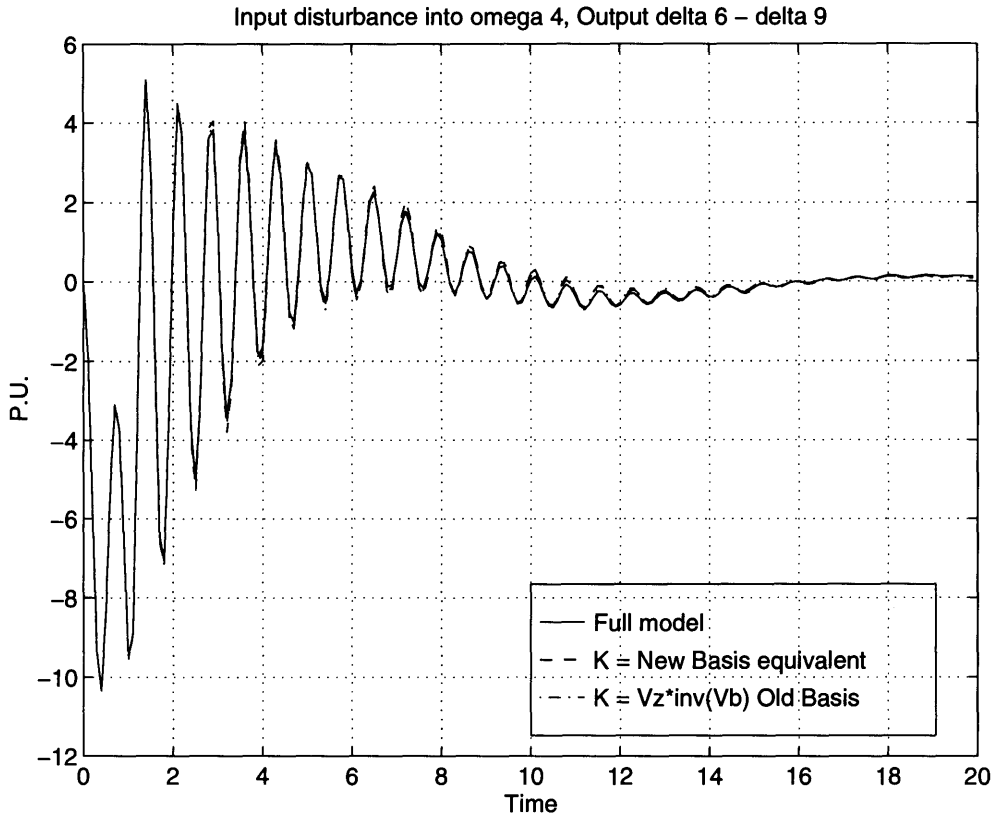

Figure 5-2: Relative angle detailed model simulation. Input disturbance of 1 p.u. for 1s. Angle δ_6 output.

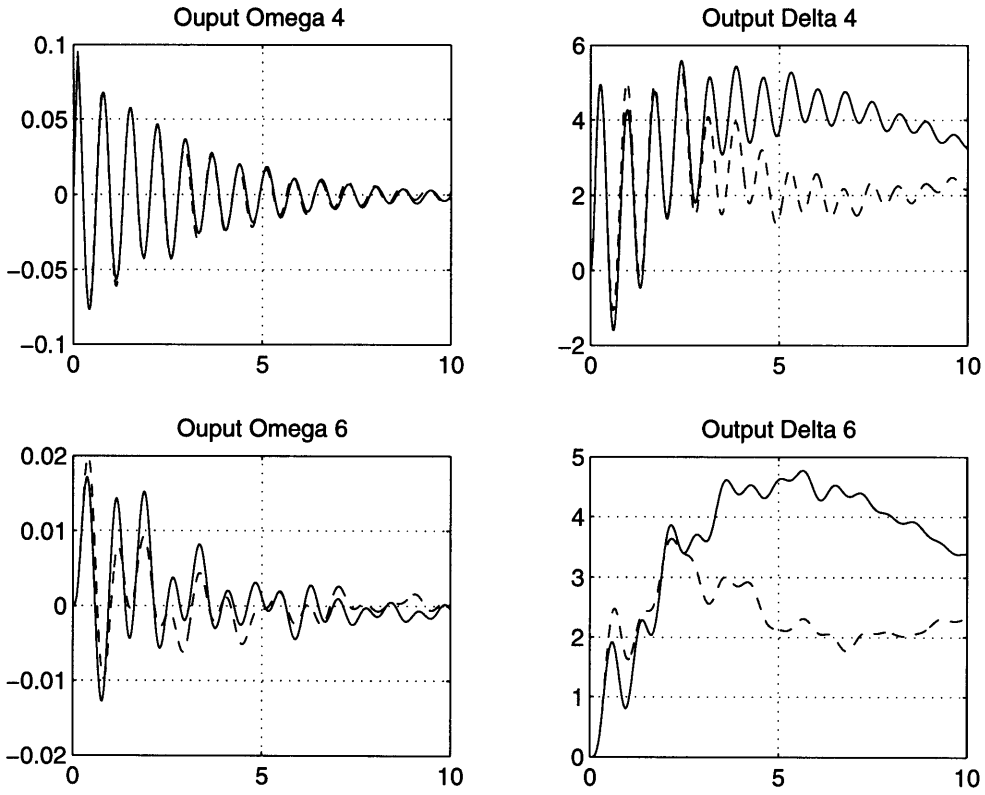

Figure 5-3: Detailed model simulation, Input ω_4, Output machines 4 and 6.

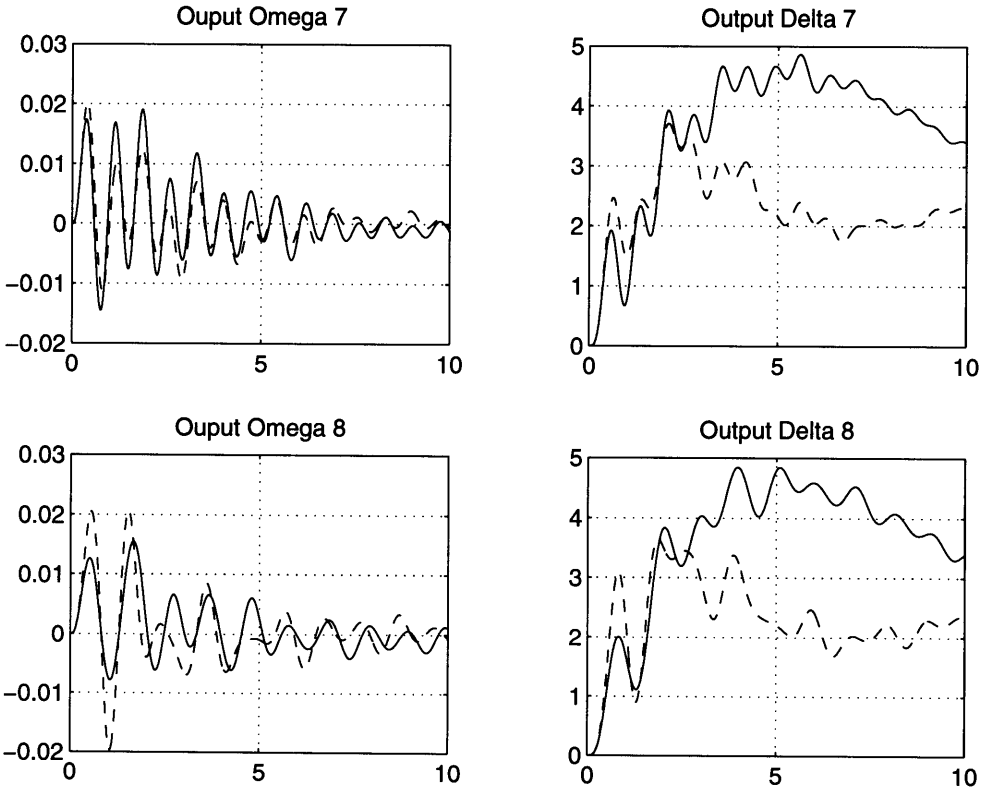

Figure 5-4: Detailed model simulation, Input ω_4, Output machines 7 and 8.

Chapter 6

Conclusions

6.1 Summary

In Chapter 3, optimization methods were used to find the optimal equivalent given a set of basis variables. A finite-time lpe_2 norm of the output was introduced to minimize the error of the "energy". The solution of the cost function was solved by a method developed in that chapter involving the solution to finite-time grammians of the system.

In Chapter 4 we developed a new basis variable selection algorithm based on the concept of *Multiple Synchrony*. Assuming that the left and right eigenvectors corresponding to the chord do not change significantly (see Section 2.2), the algorithm attempts to find the best set of basis variables such that their solution (with respect to an impulse or an initial condition) span as much as possible the subspace spanned by the non relevant variables at every point in time given that only modes in the chord v are excited. The set of basis variables thus selected have high participation in the v modes, high localizability to a single mode and are as mutually orthogonal as possible from each other. The simulations presented in Chapter 4 and Chapter 5 clearly show improvements in transient response in the basis selection algorithm.

In Chapter 5 we applied the previous methods to a detailed 111 state variable New England power system. The optimized and non-optimized S.M.E. methods were compared. The results showed that the basis selection is critical for obtaining a

good dynamic equivalent. The basis specified by the multi-dimensional synchrony algorithm reduced the error by a third over that of the basis identified by the one-dimensional synchrony algorithm.

6.2 Future work

Throughout this thesis we examine and optimize various aspects of the S.M.E. framework for reduced order modeling. However we have not looked at all the possible avenues for optimization. We present several suggestions to further improve the framework.

The issue that was least addressed in this thesis is the chord selection process. Although the results from the small swing equation model for the chord given in [20] are very good given that the chord selection was done on that smaller model, it is not clear we can justify such a selection for the full detailed system especially if other modes besides the swing modes are heavily excited. We have to reformulate the chord selection processes in an optimal manner with emphasis on detailed models.

An issue which was not investigated is the size of the chord and the number of basis variables. It is possible to define a minimum tolerance between the different eigenvector matrix rows and form a new partitioning thus increasing the number of basis variables.

For large scale systems a frequent problem associated with the gradient optimization method is the presence of several minima, i.e. we have a non-convex optimization problem. It would be interesting to study *Homotopy* methods to correct this problem although these methods are very slow (see [1] [2] [15] [17] [26])

To smooth the optimization it is helpful to obtain the exact gradients with respect to the optimizing variables $\mathbf{K}$. In Appendix A the solution is derived although the zero-eigenvalue decomposition is lacking. It involves solving k Lyapunov equations where k is the number of elements in $\mathbf{K}$. Reducing the number of calculations would certainly make the optimizing algorithm more numerically attractive.

The finite time grammian solution solved in Chapter 5 is not limited to the S.M.E.

model reduction framework. One can envision other methods such as the Balanced Truncation or Optimal Hankel model reduction for finite time where this result would be useful. In summary it is applicable to any method involving the computation of grammians.

Lastly we did not solve the case for non distinct symmetric eigenvalues. This is a minor issue since in practice this case is unlikely to be encountered. For completeness we would like to have the entire general solution.

Appendix A

Gradients

A.1 Gradients

Sometimes perturbation techniques are not accurate enough to obtain the gradients for the optimization method used in Chapter 3. The optimization is usually more stable and accurate if we can calculate the gradients directly. The following section describes a method to solve for the gradients although has not been implemented in the thesis.

Suppose we wish to find the gradients of

$$X = \int_0^{t_f} e^{A^T t} C^T C e^{At} dt \tag{A.1}$$

with respect to the entries in A. For simplicity we assume that A satisfies the conditions of Theorem (3.1.2). Now, X satisfies

$$[A]^T X + X[A] = -C^T C + e^{A^T t_f} C^T C e^{At_f} \tag{A.2}$$

and differentiating with respect to a_{ij}, where a_{ij} are the entries of A, yields

$$[A]^T X_d + X_d[A] = -(Z^T X + XZ) + \frac{d}{da_{ij}} e^{A^T t_f} L e^{At_f} \tag{A.3}$$

where

$$\begin{aligned} X_d &= \frac{d}{da_{ij}} \int_0^{t_f} e^{A^T t} C^T C e^{At} dt \\ Z &= \frac{dA}{da_{ij}} \\ L &= C^T C \end{aligned}$$

Now

$$\begin{aligned} [A]^T X_d + X_d [A] &= -(Z^T X + XZ) + \frac{d}{da_{ij}} e^{A^T t_f} L e^{At_f} \\ &= -(Z^T X + XZ) + \frac{d}{da_{ij}} \frac{d}{dt_f} \int_0^{t_f} e^{A^T \sigma} L e^{A\sigma} d\sigma \\ &= -(Z^T X + XZ) + \dot{X}_d \end{aligned} \tag{A.4}$$

and we have a dynamical Lyapunov equation. The solution to equation (A.4) is given by ([6], see pp. 58-59):

$$\begin{aligned} X_d(t_f) &= e^{A^T t_f} X_d(0) e^{At_f} + \int_0^{t_f} e^{A^T (t_f - \sigma)} Q(\sigma) e^{A(t_f - \sigma)} d\sigma \\ &= \int_0^{t_f} e^{A^T (t_f - \sigma)} Q(\sigma) e^{A(t_f - \sigma)} d\sigma \end{aligned} \tag{A.5}$$

where

$$Q(\sigma) = -(Z^T X(\sigma) + X(\sigma) Z). \tag{A.6}$$

Now we solving for $X_d(t_f)$ yields,

$$\begin{aligned} \frac{d}{d\sigma} e^{A^T (t_f - \sigma)} Q(\sigma) e^{A(t_f - \sigma)} &= -A^T (e^{A^T (t_f - \sigma)} Q(\sigma) e^{A(t_f - \sigma)}) + e^{A^T (t_f - \sigma)} Q(\sigma) e^{A(t_f - \sigma)} A \\ &+ e^{A^T (t_f - \sigma)} \frac{dQ(\sigma)}{d\sigma} e^{A(t_f - \sigma)} \end{aligned} \tag{A.7}$$

$\Rightarrow$

$$\int_0^{t_f} \frac{d}{d\sigma} e^{A^T(t_f-\sigma)} Q(\sigma) e^{A(t_f-\sigma)} d\sigma = -A^T \int_0^{t_f} (e^{A^T(t_f-\sigma)} Q(\sigma) e^{A(t_f-\sigma)}) d\sigma$$
$$+ \int_0^{t_f} e^{A^T(t_f-\sigma)} Q(\sigma) e^{A(t_f-\sigma)} d\sigma A + \int_0^{t_f} e^{A^T(t_f-\sigma)} \frac{dQ(\sigma)}{d\sigma} e^{A(t_f-\sigma)} d\sigma \qquad \text{(A.8)}$$

$\Rightarrow$

$$e^0 Q(t_f) e^0 - e^{A^T t_f} Q(0) e^{A t_f} = -A^T X_d(t_f) - X_d(t_f) A$$
$$+ \int_0^{t_f} e^{A^T(t_f-\sigma)} [Z^T \frac{dX(\sigma)}{d\sigma} + \frac{dX(\sigma)}{d\sigma} Z] e^{A(t_f-\sigma)} d\sigma \qquad \text{(A.9)}$$

$\Rightarrow$

$$\begin{aligned} Q(t_f) &= -A^T X_d(t_f) - X_d(t_f) A \\ &+ \int_0^{t_f} e^{A^T(t_f-\sigma)} [Z^T e^{A^T\sigma} L e^{A\sigma} + e^{A^T\sigma} L e^{A\sigma} Z] e^{A(t_f-\sigma)} d\sigma \end{aligned} \qquad \text{(A.10)}$$

$\Rightarrow$

$$\begin{aligned} Q(t_f) &= -A^T X_d(t_f) - X_d(t_f) A \\ &+ \int_0^{t_f} e^{A^T(t_f-\sigma)} [Z^T e^{A^T\sigma} L e^{A\sigma} + e^{A^T\sigma} L e^{A\sigma} Z] e^{A(t_f-\sigma)} d\sigma \end{aligned} \qquad \text{(A.11)}$$

$\Rightarrow$

$$\begin{aligned} 0 &= -Q(t_f) - A^T X_d(t_f) - X_d(t_f) A + e^{A^T t_f} (\int_0^{t_f} e^{-A^T\sigma} Z^T e^{A^T\sigma} d\sigma) L e^{A t_f} \\ &+ e^{A^T t_f} L (\int_0^{t_f} e^{A\sigma} Z e^{-A\sigma} d\sigma) e^{A t_f} \end{aligned} \qquad \text{(A.12)}$$

We now have a stationary Lyapunov equation, which can be easily solved.

The solution of the integral

$$X = \int_0^{t_f} e^{-A^T t} Z e^{At} dt \tag{A.13}$$

is given by the integral of the *Baker Hausdorff* formula(see [5]):

$$X = \int_0^{t_f} Q + [At, Z] + \frac{1}{2!}[At, [A, Z]] + ...dt \tag{A.14}$$

where [.,.] is called the **Lie Bracket** and is defined as follows

Definition A.1.1 *Suppose* $A \in R^n$ *and* $B \in R^n$ *then the* **Lie Bracket** *of [A,B] is equal to*

$$[A, B] = AB - BA \tag{A.15}$$

The above series is infinite but due to the factorial nature of this series in practice it converges fast. Notice also that Z has only one non zero element thus it is diagonal or nilpotent for the off diagonal case.

This method fails if the above equation has multiple solutions. It is suggested to use the space decomposition method developed in Chapter 3 to separate the eigenvalues which fail Theorem 3.1.2 in order to obtain the gradients for the decomposed system. Although for a power system model we would only have to worry about the zero eigenvalues. This is left as future work.

Appendix B

Matlab Basis Selection Program

```
function [b,Y,L,L1,scale] = selectbasis(Aedf, chord)

% Syntax: [b,Y,L,L1,scale] = selectbasis(A, chord)
% Multi-Synchrony Basis Variables algorithm
% Last Revised: September 20, 1995
% Written by: Julio E. Castrillon

[V,D]=eig(Aedf);
W=inv(V);
Part = [];
chord = chord*2 + ones(1,length(chord));
[dummy,Dord] = sort(diag(D));

%reorder the chord to match eigenvalues
newchord = [];
V_chord =[];
W_chord = [];
for I=1:length(chord),
                newchord=[newchord, Dord(chord(I)), Dord(chord(I)+1)];
                end;
for I=1:length(newchord),
                 V_chord=[V_chord, V(:,newchord(I))];
                 end;
```

```
for I=1:length(newchord),
                W_chord=[W_chord; W(newchord(I),:)];
                end;
Part_chord = V_chord*diag(sum(abs(W_chord)'));
[n,p] = size(Part_chord);


                                                                    30
L1 = [real(Part_chord) imag(Part_chord)];
[usless,p]=size(L1);
scale = sum(L1')';
[u1,a] = size(L1);
L = L1;

%pick highest participation in a single mode
[u1, a] = max(L);
[u2, b] = max(u1);
                                                                    40
L(a(b),b); %maximum participation in a single mode
q=2*length(chord);
%find the rest of variables
Y=L(a(b),:);
b = a(b);

for k = 1:q-1,
           ci = 1;
           vc = L(1,:);
           for m = 2:n,                                             50
                       one = partdegB(Y,vc,2*n);
                       two = partdegB(Y,L(m,:),2*n);
                       if two > one vc = L(m,:); ci = m; end
           end
           Y = [Y;vc];
           b = [b ci];
end

%============================== End of Module ==============================
```

```
                                                                        60
function [Mu] = partdegB(Y, V, mx)

% Syntax:  [Mu] = partdegB(Y, V, mx)
% Last Revised Sept.  10, 1995
% Written by:  Julio E. Castrillon.


[n, p] = size(V);
[nr, p] = size(Y);                                                      70
Mu = zeros(n,nr);
              A  = Y';
              sub_space  = A*inv(A'*A)*A'*V';
              if (norm(V)*norm(sub_space))==0 m=0; else
              m = acos((V*sub_space)/(norm(V)*norm(sub_space))); end;
              m = m*mx*var(V);
              Mu = m;

%============================== End of Module ================================
                                                                        80
```

Bibliography

[1] V. AJJARAPU and C. CHRISTY. The continuation power flow: a tool for steady state voltage stability analysis. *2nd international workshop on bulk power system voltage phenomena, voltage and security, Maryland*, 1991.

[2] E.E. ALLGOWER. A survey of homotopy methods for smooth mappings. *Lecture notes in mathematics, No. 878, springer verlag, pp. 1-29*, 1980.

[3] A.R. BERGEN. *Power Systems Analysis.* Prentice Hall, 1986.

[4] W.G. BOCKLEY and J. McNAMEE. Matrix and other direct methods for the solution of systems of linear difference equations. *Philos. Trans. Roy. Soc.(London) Ser. A, 252 (1960), p. 69-131*, 1960.

[5] R. W. BROCKETT. System theory on group manifolds and coset spaces. *SIAM*, May 1972.

[6] R.W. BROCKETT. *Finite Dimensional Linear Systems.* John Wiley and Sons,Inc., 1970.

[7] M. BRUCOLI, R. SBRIZZAI, F. TORELLI, and M. TROVATO. Structural coherency in interconnected power systems. *Large Scale Systems, Vol 5. pp. 169-188*, 1983.

[8] M. BRUCOLI, R. SBRIZZAI, F. TORELLI, and M. TROVATO. Structural approach for the identification of power network dynamic equivalents. *International Journal of System Science, Vol. 23, No.1 pp. 1-16*, 1992.

[9] R.Y. CHIANG and M.G. SAFONOV. *Robust Control Toolbox.* The Mathworks Inc., 1992.

[10] J.H. CHOW. *Time-Scale Modeling of Dynamic Networks with Applications to Power Systems, Lecture Notes in Control and Information Sciences. Vol. 46.* Springer Verlag, 1982.

[11] U. DI CAPRIO. Conditions for theoretical coherency in multimachine power systems. *Automatica, Vol. 17, No.5, pp. 687-701*, 1981.

[12] U. DI CAPRIO. Theoretical and practical dynamic equivalents in multimachine power systems. *Electrical Power and Energy Systems, Vol. 4, No.4 pp.224-232*, October 1982.

[13] J. DORSEY and R.A. SCHLUETER. Structural archetypes for coherency: A framework for comparing power systems equivalents. *Automatica, Vol. 20, No.3, pp. 349-352*, October 1984.

[14] L. FORTUNA, G. NUNNARI, and A. GALLO. *Model Order Reduction Techniques with Applications in Electrical Engineering.* Spinger-Verlag, 1992.

[15] C.B. GARCIA and W.I. ZANGWILL. *Pathways to solutions, Fixed points and equilibria,.* Englewood Cliffs, NJ: Prentice-hall, Inc. pp. 1-23, 1981.

[16] R. A. HORN and C.R. JOHNSON. *Matrix Analysis.* Cambridge University Press, 1993.

[17] K. OKUMURA, TERAI K., and A. KISHIMA. Solution of ill-conditioned load flow equation by homotopy continuation method. *Proc. IEEE Int, Symp. on circuits and Systems, singapore pp. 2897-2899.*, 1991.

[18] M.A. PAI and R.P. ADGAONKAR. Identification of coherent generators using weighted eigenvectors. *IEEE Power Engineering Society Winter Meeting, New York, Paper No. A 79 022-5*, February 1979.

[19] R. PODMORE. Identification of coherent generators for dynamic equivalents. *IEEE Transactions on Power Apparatus and Systems, Vol. PAS-97, No.4, pp. 1344-1354*, July 1978.

[20] G.N. RAMASWAMY. *MODAL STRUCTURES AND MODEL REDUCTION with Application to Power System Equivalencing.* Ph.D. Thesis, Department of Electrical Engineering and Computer Science, MIT, Cambridge MA, 1995.

[21] G.N. RAMASWAMY, VERGHESE G.C., VIALAS C., and DEMARCO C.L. Going beyond coherency: Synchrony as a basis for dynamic equivalencing in power systems models. *North American Power Symposium*, October 1993.

[22] G.N. RAMASWAMY, VERGHESE G.C., ROUCO L., LESIEUTRE B.C., and FILLÂTRE O. Synchronic modal equivalencing (sme): A new framework for constructing dynamic equivalents in power systems. *North American Power Symposium*, September 1994.

[23] G.N. RAMASWAMY, VERGHESE G.C., ROUCO L., VIALAS C., and DEMARCO C. L. Synchrony, aggregation and multi-area eigenanalysis. *IEEE PES Winter Meeting*, January 1995.

[24] G.N. RAMASWAMY, ROUCO L., FILLÂTRE O., VERGHESE G.C., PANCIATICI P., LESIEUTRE B.C., and PELTIER D. Synchronic modal equivalencing (sme) for structure-preserving dynamic equivalents. *IEEE PES Winter Meeting*, January 1995.

[25] M.G. SAFONOV, E.A. JONCKHEERE, M. VERMA, and D.J.N. LIMEBEER. Synthesis of positive real multivariable feedback systems. *Int. J. Control, Vol. 45, No. 3, pp. 817-842*, 1987.

[26] M. TOLIKAS, Lj. TRAJKOVIC, and M.D. ILIC. Homotopy methods for solving decoupled power flow equations. *IEEE Int. Symp. on Circuits and Systems, San Diego CA*, May 1992.

[27] J.M. UNDRILL and A.E. TURNER. Construction of power systems electromechanical equivalents by modal analysis. *IEEE Transactions on Power Apparatus and Systems, Vol. PAS-90, pp. 2049-2059*, September 1971.

[28] F. WU and N. NARASIMHAMURTHI. Coherency identification for power systems dynamic equivalents. *IEEE Transactions on Circuits and Systems, Vol. CAS-30, No.3, pp. 140-147*, March 1983.