Differential Operators for Edge Detection

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Abstract:

We present several results characterizing two differential operators used for edge detection:
the Laplacian and the second directional derivative along the gradient. In particular,

- (a) we give conditions for coincidence of the zeros of the two operators, and
- (b) we show that the second derivative along the gradient has the same zeros of the
  normal curvature in the gradient direction.

Biological implications are also discussed. An experiment is suggested to test which of the
two operators may be used by the human visual system.

Introduction

Edges are significant properties of the physical surface. Physical edges are often correlated with intensity changes in an image. A classical problem in computer vision is how these changes can be detected, extracted and best represented. We will not consider here the problem of how to detect and classify edges on the basis of their physical and geometrical origin. The structure of many existing edge detection schemes includes a first step in which the image is appropriately filtered in order (a) to set the spatial scale at which sharp changes in intensity must be detected; (b) to eliminate noise; (c) to interpolate the discrete array of sampled values into a smooth, differentiable surface. The second step consists of a derivative operation, since sharp intensity changes in 1-D correspond to extrema of the first derivative, equivalently locations at which the second derivative crosses zero and changes sign. In 2-D, various types of derivatives are possible and many local differential operators have indeed been proposed.

In this note, the question of the appropriate filter will not concern us. We will consider the derivative operation only, with the goal of characterizing two of the most interesting differential operators which have been proposed: the Laplacian and the second directional derivative along the gradient (Canny, 1983; Haralick, 1982; Havens & Strickwerda, 1982).

Differential operators

Two-dimensional differential operators which can be used for detecting sharp changes in intensity can be classified according to whether they are (a) linear or nonlinear, (b) directional or rotationally symmetric.

Rotationally symmetric operators have several attractive features. Two of the most interesting operators of this class are the Laplacian (∇^2 which is linear) and the second directional derivative along the gradient ($\frac{\partial^2}{\partial n^2}$) which is nonlinear.

Cartesian and polar form

We give the explicit representation of the two operators in cartesian and polar coordinates:

$$\nabla^2 f = f_{xx} + f_{yy} = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \theta^2} \tag{1}$$

$$\frac{\partial^2 f}{\partial n^2} = \frac{f_x^2 f_{xx} + 2 f_x f_y f_{xy} + f_y^2 f_{yy}}{f_x^2 + f_y^2} \longleftrightarrow \{ \frac{2}{\rho^2} \frac{\partial f}{\partial \rho} \frac{\partial f}{\partial \theta} \frac{\partial^2 f}{\partial \rho \partial \theta} + \frac{1}{\rho^4} \left(\frac{\partial^2 f}{\partial \theta} \right)^2$$

$$\frac{\partial^2 f}{\partial \theta^2} - \frac{1}{\rho^3} \frac{\partial f}{\partial \rho} \left(\frac{\partial f}{\partial \theta} \right)^2 + \left(\frac{\partial f}{\partial \rho} \right)^2 \frac{\partial^2 f}{\partial \rho^2} \} \frac{1}{\left(\frac{\partial f}{\partial \rho} \right)^4 + \frac{1}{\rho^2} \left(\frac{\partial f}{\partial \theta} \right)^2} \qquad (2)$$

We also give the explicit representation for the second directional derivative in the direction orthogonal to the gradient.

$$\frac{\partial^2 f}{\partial n_\perp^2} = \frac{f_y^2 f_{xx} - 2 f_x f_y f_{xy} + f_x^2 f_{yy}}{f_x^2 + f_y^2} \qquad (3)$$

Remark:

The representation in polar coordinates shows clearly that the two operators are rotationally symmetric, since their form does not change for a rotation of the coordinate system ϕ. We can state

•Characteristic Property of Rotationally Symmetric Operators: A sufficient condition for an operator to be rotationally invariant is that θ appears only as derivative in the polar representation of the operator.

Simple properties of ∇^2 and $\frac{\partial^2}{\partial n^2}$

We state here three obvious properties.

(I)If the image $f(x,y)$ can be represented as a function of only one variable, i.e., $f(x, y_o)$, the two operators ∇^2 and $\frac{\partial^2}{\partial n^2}$ are then equivalent, i.e., $\frac{\partial^2}{\partial n^2} f = \nabla^2 f$.

Corollary

For $f(x, y_o)$, the zeros of $\frac{\partial^2}{\partial n^2}$ and of $\nabla^2 f$ coincide with the zeros of the normal curvature along the gradient.

Remark

Property (I) is not equivalent to the "linear variation" condition of Marr & Hildreth (1980), which states that if f chanages at most linearly along the edge direction ℓ, then $\nabla^2 f = \frac{\partial^2 f}{\partial n^2}$.

(II)If $f(x,y) = f(\rho)$ is rotationally symmetrical, $\nabla^2 f$ and $\frac{\partial^2}{\partial n^2}$ differ by the additive term $\frac{1}{\rho} \frac{\partial f}{\partial \rho}$.

Remark

For circularly symmetric functions, the zeros of $\nabla^2 f$ are more separated than the zeros of $\frac{\partial^2}{\partial n^2} f$. This lack of localization by ∇^2 (for circularly symmetric patterns) can also be seen in the fact that zeros of ∇^2 (but not of $\frac{\partial^2}{\partial n^2}$) "swing wide" of corners.

(III).(a) $\frac{\partial^2}{\partial n^2}$ is nonlinear.

(b) $\frac{\partial^2}{\partial n^2}$ neither commutes nor associates with the convolution, i.e.,

$$\frac{\partial^2}{\partial n^2}(g * f) \neq \left(\frac{\partial^2}{\partial n^2}g\right) * f$$
$$\left(\frac{\partial^2}{\partial n^2}g\right) * f \neq g * \frac{\partial^2}{\partial n^2}f$$

(4)

(c) $\frac{\partial^2}{\partial n^2}$ is a linear operator on f, if $f = f(\rho)$.

(d) ∇^2 corresponds in polar Fourier coordinates to $-w^2$, where w is $w^2 = \omega_x^2 + \omega_y^2$.

(e) $\frac{\partial^2}{\partial n^2}$ gives, in general, a function with nonzero mean when applied to a zero-mean function (it gives zero if applied to $f(x, y) = const$). The reason, of course, is that $\frac{\partial^2}{\partial n^2}$ is a nonlinear operator.

Geometric characterization of zeros of ∇^2 and $\frac{\partial^2}{\partial n^2}$

Let us consider the intensity surface represented as $X = (x, y, z)$, where $z = f(x, y)$ with $f \epsilon C^r(D)$, $D < R^2$. The coefficients of the first fundamental form $I(dx, dy)$ of x, called the first fundamental coefficients, are

$$E = 1 + f_x^2 \quad F = f_x f_y \quad G = 1 + f_y^2$$

(6)

The normal to the surface **N** is given by

$$\mathbf{N} = \frac{\mathbf{X}_x \times \mathbf{X}_y}{|\mathbf{X}_x \times \mathbf{X}_y|} = \frac{(-f_x, -f_y, 1)}{g}$$

(7)

with $g^2 = 1 + f_x^2 + f_y^2$.

Similarly, the coefficients of the second fundamental form $II(dx, dy)$ are

$$L = \frac{f_{xx}}{g} \quad M = \frac{f_{xy}}{g} \quad N = \frac{f_{yy}}{g}$$

(8)

The mean curvature of the surface X is

$$H = \frac{EN + GL - 2FM}{2g^2} = \frac{(1 + f_x^2)f_{yy} + (1 + f_y^2)f_{xx} - 2f_x f_y f_{xy}}{2g^3} \tag{9}$$

which is the average value of the two principal curvatures k_1 and k_2, i.e.,

$$H = \frac{k_1 + k_2}{2} \tag{10}$$

In particular, H can be computed in terms of the surface normal, i.e., $H = -\nabla \cdot \mathbf{N}$.

The gaussian curvature is given by

$$K = \frac{LN - M^2}{EG - F^2} = \frac{LN - M^2}{g^2} = \frac{f_{xx}f_{yy} - f_{xy}^2}{g^2} = \frac{|Hess f|}{g^4} \tag{11b}$$

which is the product of the principal curvatures, i.e.,

$$K = k_1 \cdot k_2 \tag{12}$$

We use equations (2) and (3) and the property

$$\nabla^2 f = \left(\frac{\partial^2}{\partial n^2} + \frac{\partial^2}{\partial n_\perp^2} \right) f \tag{13}$$

for writing H in terms of ∇^2 and $\frac{\partial^2}{\partial n^2}$:

$$H = \frac{1}{2g^3} \left(g^2 \nabla^2 f - (\nabla f)^2 \frac{\partial^2 f}{\partial n^2} \right) \tag{14}$$

We can now characterize the connection between the zeros of ∇^2 and the zeros of $\frac{\partial^2}{\partial n^2} f$ as

Property (1): The zeros of $\frac{\partial^2}{\partial n^2} f$ are different from zeros of $\nabla^2 f$ if $(\nabla f)^2 \leq 0$ and $|Hess(f)| > 0$ (in this case H is not zero).

Property (2): If $\nabla f \neq 0$, the zeros of $\frac{\partial^2}{\partial n^2} f$ coincide with the zeros of ∇^2 *iff* the mean curvature H is zero.

Property (3): For surfaces with minimal curvature ($H = 0$), the zeros of $\frac{\partial^2 f}{\partial n^2}$ coincide with the zeros of $\nabla^2 f$ (where the gradient of f is different from zero).

Note that if f and its derivatives are small, $H \approx \frac{1}{2}\nabla^2 f$, $K = f_{xx}f_{yy} - f_{xy}^2$. In this case, zero crossings of the Laplacian correspond to zeros of the mean curvature.

5

The normal curvature

The second directional derivative along the gradient has a simple interpretation in terms of the normal curvature along the gradient. The normal curvature K_n in the direction of the gradient is

$$K_n = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2} \tag{15}$$

where du and dv are the direction numbers of the gradient. Assuming $dn^2 + dv^2 = 1$, we obtain

$$\begin{aligned} du &= \frac{f_x}{|\nabla f|} \\ dn &= \frac{f_y}{|\nabla f|} \end{aligned} \tag{16}$$

Thus, equations (6) and (8) lead, together with equations (15), (16), to

$$K_n = \frac{1}{g^3} \frac{\partial^2}{\partial n^2} f \tag{17}$$

In particular, it follows

Property 4

The second directional derivative along the gradient and the normal curvature in the direction of the gradient have the same zeros.

Invariance properties

Since $\frac{\partial^2}{\partial n^2}$ and ∇^2 are rotationally symmetric operators, their zeros are also rotationally invariant, i.e., their geometry remains invariant for rotations of the image. In addition, the zeros of $\frac{\partial^2}{\partial n^2}$ and ∇^2 are invariant to photometric scaling (i.e., change in contrast in the image $I(x, y) \mapsto cI(x, y)$.

Remark

The operator $\frac{\partial^2}{\partial n^2}$ and the normal curvature in the direction of the gradient K_n are not defined when $\nabla f = 0$. In this case, the direction of the gradient is undertermined, although

the hessian can of course be diagonalized (determining the principal directions). Thus $\frac{\partial^2}{\partial n^2}$ has the disadvantage with respect to ∇^2 that it is not defined everywhere.

Potential biological consequences

A natural question arising from these comparisons is, which derivative operators are used by the human visual system? Zero-crossings in the output of directional second derivatives approximated by the difference of one-dimensional gaussians (DOG) were suggested by Marr & Poggio (1977) in their theory of stereo matching. Marr & Hildreth (1979) later proposed the rotationally symmetric Laplacian (approximated by $\nabla^2 G$, i.e., a rotationally symmetric DOG). Psychophysical evidence does not rule out either of these schemes. Physiology shows that a class of retinal ganglion cells is performing a roughly linear operation quite similar to the convolution of the image with the laplacian of a gaussian. Data on cortical cells are still somewhat contradicting on whether some simple cells may perform the equivalent of a linear directional derivative operation or instead signal the presence (and perhaps the slope) of a zero-crossing of the rotationally symmetric $\nabla^2 G$.

On physiological grounds, it seems unlikely that retinal cells could perform the rotationally symmetric nonlinear $\frac{\partial^2}{\partial n^2}$ operation, although not all classes of ganglion cells have been tested properly to allow a firm conclusion. In particular, one-dimensional and rotationally symmetric patterns are customarily used: in the first case $\frac{\partial^2}{\partial n^2}$ and ∇^2 are equivalent, whereas in the second case, they may be distinguishable only quantitatively.

(a) An interesting possibility to distinguish the laplacian from the directional second derivative on the basis of physiological experiments is suggested by the observation that the zero-crossings of the laplacian "swing wide" of grey-level corners. In particular, the zero-crossings associated with an elongated black bar, say, coincide for ∇^2 and for $\frac{\partial^2}{\partial n^2}$, whereas they differ for a circular black disk.

Notice that in this case, both operators are linear. They associate therefore with gaussian convolution. The corresponding point-spread function are

(a) for the one-dimensional, $f(x)$:

$$\frac{\partial^2}{\partial x^2} G = \frac{2}{2\sigma^2}\left(\frac{\partial x^2}{2\sigma^2} - 1\right)e^{\frac{-x^2}{2\sigma^2}} \tag{18}$$

(b) for the two-dimensional $f(\rho)$

7

$$\nabla^2 G \mapsto \frac{4}{2\sigma^2}\left(\frac{\rho^2}{2\sigma^2} - 1\right)e^{\frac{-\rho^2}{2\sigma^2}}$$
$$\frac{\partial^2}{\partial n^2} \mapsto \frac{2}{2\sigma^2}\left(\frac{2\rho^2}{2\sigma^2} - 1\right)e^{\frac{-\rho^2}{2\sigma^2}}$$

(19)

where σ is the standard derivation of the gaussian function.

Let us call w the diameter of the central region of these masks, i.e., the distance between the central zeros. w_{1D} denotes the diameter for the one-dimensional case and w_{2D} for the two-dimensional case. It is easy to see that the second directional derivative has $w_{1D}^d = w_{2D}^d$ whereas this is not true for the laplacian, $w_{1D}^\ell \neq w_{2D}^\ell (w_{1D}^\ell = w_{1D}^d)$. From (a) and (b) we get

$$w_{2D}^d = w_{1D}^d = \sigma w_{1D}^\ell = 2\sigma$$
$$w_{2D}^\ell = 2\sqrt{2}\sigma$$

(20)

Notice that the difference is less for patterns of finite size. Furthermore, convolution with a gaussian will tend to displace the inflexion point of the intensity distribution toward the inside, possibly correcting for the effect of the Laplacian.

In any case, a possible psychophysical test is

Test: If zero crossing in the laplacian are used by our visual system to estimate position of edges, the apparent width of a 1-D bar and of a circle (with equal physical widths) should be different. The bar should appear smaller than the circle. This is not expected if the second directional derivative is used. Appropriate vernier acuity experiments are planned to answer this question.

(b) There are classes of intensity edges that generate zeros in $\frac{\partial^2}{\partial n^2}$ but not in ∇^2. An example is given by

$$I(x,y) = \left(1 + e^{\beta y}\right)\frac{e^{\alpha x}}{1 + e^{\alpha x}}$$

(21)

which, with appropriate values of β does not satisy $\nabla^2 I = 0$ for any $y \geq 0$. It is possible, however, to find solutions to $\frac{\partial^2}{\partial n^2}I = 0$. Thus, the edge I could again be used to discriminate psychophysically between ∇^2 and $\frac{\partial^2}{\partial n^2}$.

(c) Functions $h\epsilon C^2$ in a certain region D are subharmonic *iff* $\nabla^2 h \geq 0$ in D. This subharmonic functions do not have zero crossings of the laplacian; in general zero-crossings of $\frac{\partial^2}{\partial n^2}$ are present. There are special cases, however, in which both $\frac{\partial^2}{\partial n^2}$ and ∇^2 do not have any zero. An example is given by $f = \cos\ x + bx^2$ with $\nabla^2 f = \frac{\partial^2}{\partial n^2}f = -\cos^2 x + 2bx$, which does not have any zero crossings if $b > \frac{1}{2}$. It would be interesting to test psychophysically and physiologically this kind of pattern.

8

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