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Dynamics of Manipulators with Less Than One Degree of Freedom

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ABSTRACT. We have developed an efficient Lagrangian formulation of manipulators with small numbers of degrees of freedom. The efficiency derives from the lack of velocities, accelerations, and generalized forces. The number of additions and multiplications remains constant, independent of the number of joints, as long as the number of joints remains less than one. While this is a restricted class of manipulators, we believe that it is important to understand it fully before studying of more complex systems. Manipulators with less than one degree of freedom are by far the most common manipulators used by industry. We have also noticed that many of the multiple-degree-of-freedom manipulators in our laboratory tend to be used in a zero-degree-of-freedom mode. With this formulation of the dynamics it should be possible in principle to compute the Lagrangian dynamics of manipulators with less than one degree of freedom in real time.

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## Introduction

The problem of computing the joint torques required to produce given joint positions, velocities, and accelerations, for a manipulator can be greatly simplified by eliminating all joints. Until recently the solutions to these equations required the use of more general programs which required seconds or even minutes of computer time for each trajectory point. They could not be computed in real time.

Two different types of schemes were proposed to render the Lagrangian dynamics computationally feasible: (1) simplifying the dynamics by ignoring some terms and correcting errors with feedback, and (2) simplifying the manipulator by removing degrees of freedom. We have chosen the second since it allows the inclusion of the fun-to-compute Coriolis and centrifugal forces.

### The Uicker/Kahn Lagrangian Formulation

The standard formulation for manipulator dynamics may be found in Uicker [1], who represents Lagrangian based dynamics using four by four matrices. The formulations we use taken directly from Hollerbach [2], who uses the notation of Kahn [3]. Kahn numbers the links of a manipulator from 1 to  $n$  starting from the base to the tip, with the reference frame numbered as link 0. Since we are concerned only with manipulators with no joints the zero link is of utmost importance. The coordinate system is fixed in each link as follows:

$z_i$  is along the axis of joint  $i + 1$ ,

$x_i$  is along the common normal from  $z_{i-1}$  to  $z_i$ , and

$y_i$  goes off in some completely different direction.

The symbols  $s_i$ ,  $\alpha_i$ ,  $\theta_i$ , and  $a_i$  are of no concern to us since they represent relative position of two adjacent links. Since this would make the equations go away entirely, we shall complicate the situation by modeling our single-link manipulator as a two-link system with a fixed connecting joint. In other words,  $\theta_0 = 0$ .

Following Hollerbach [2], we let  ${}^i v_j = (1 \ x \ y \ z)^T$  be a vector from coordinate system  $i$  to a point

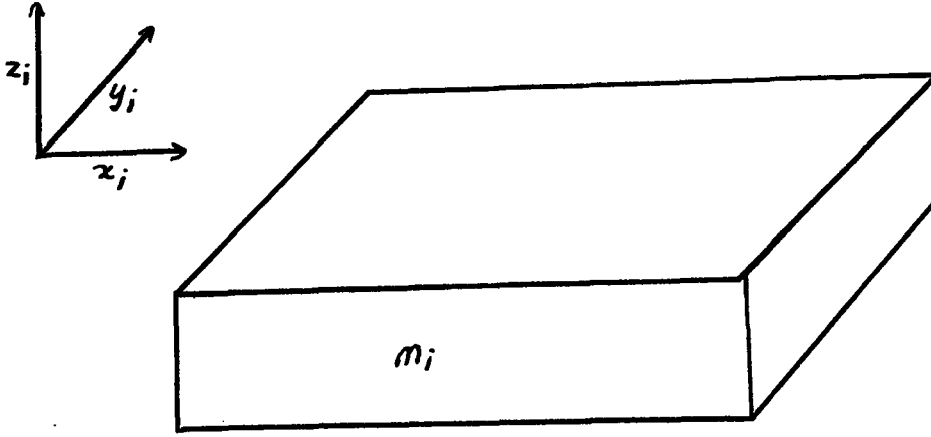


Figure 1. Standard coordinate axes of a manipulator with no degrees of freedom

fixed in link  $j$  expressed in link  $i$  coordinates. Then adjacent coordinate systems are related by the relation:

$${}^{i-1}v_j = A_i {}^i v_j \quad (1)$$

where  $A_i$  is the 4x4 transformation matrix between coordinate systems  $i-1$  and  $i$ :

$$A_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_i \cos \theta_i & \cos \theta_i & -\sin \theta_i \cos \alpha_i & \sin \theta_i \sin \alpha_i \\ a_i \sin \theta_i & \sin \theta_i & \cos \theta_i \cos \alpha_i & -\cos \theta_i \sin \alpha_i \\ s_i & 0 & \sin \alpha_i & \cos \alpha_i \end{bmatrix} \quad (2)$$

We define  $A_0 = I$ , the identity matrix. Any two coordinate systems  $i$  and  $j$  can be related by cascading the transformations:

$${}^i v_k = {}^i W_j {}^j v_k \quad (3)$$

where  ${}^i W_j = A_{i+1} A_{i+2} \dots A_j$  for  $i < j$ . We define  ${}^j W_j = I$ . Like Hollerbach [3], we omit the 0

superscript when referring to the base coordinate system, so that  $v_k = {}^0v_k$  and  $W_j = {}^0W_j$ .

Using this notation, the generalized forces  $F_i$  can be derived from the Lagrange equations [1,2,3]:

$$F_i = \sum_{j=i}^n \left[ \sum_{k=1}^j \left( \text{tr} \left( \frac{\partial W_j}{\partial q_i} J_j \frac{\partial W_j^T}{\partial q_k} \right) \right) \ddot{q}_k + \sum_{k=1}^j \sum_{l=1}^j \left( \text{tr} \left( \frac{\partial W_j}{\partial q_i} J_j \frac{\partial^2 W_j^T}{\partial q_k \partial q_l} \right) q_k q_l \right) - m_j g^T \frac{\partial W_j}{\partial q_i} {}^j r_j \right] \quad (4)$$

where

$J_j$  is the inertia tensor with respect to the proximal joint of link  $j$  expressed in  $j$ 's body coordinates,

$m_j$  is the mass of link  $j$ ,

$g$  is the gravity vector, and

${}^j r_j$  is the coordinate of the center of mass of link  $j$  expressed in link  $j$ 's body coordinates.

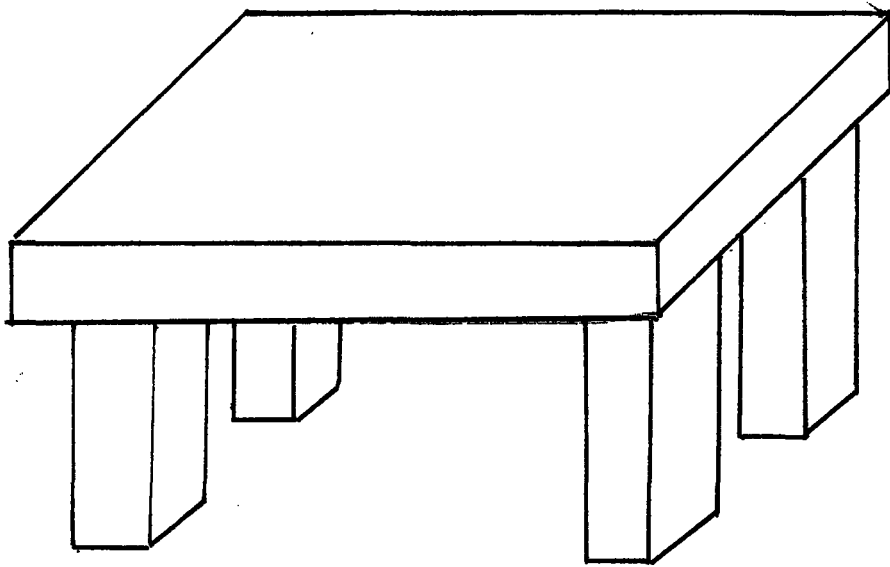
To make the paper look more impressive we include here some addition equations:

$$\sum_{k=0}^{n+1} \sum_{i=0}^{n-k} \binom{n+1}{k} \binom{n-k}{i} \frac{(-1)^{n-k-i} z^i}{(\lambda n)^{n-k}}$$

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n E(|x_i - E(x_i)|^k)}{(\sum_{i=1}^n E(|x_i - E(x_i)|^2))^{\frac{k}{2}}} = 0$$

for  $k > 2$

Some researchers [4] have considered this too slow for real time computation. But notice that, in (4), if for all  $i$  all  $\theta_i = 0$  and  $\alpha_i = 0$ , then all of the matrices  $A_i = 0$  will be identity matrices. Thus,  $F_0 = m_0 a_0$ . Given the proper initial conditions, it is possible to make the manipulator stand still by applying no force. This is fortunate since there is no way to apply force anyway. We believe that this computation is simple enough to be computed in real time.



**Figure 2.** Industrial use of four zero-degree-of-freedom manipulators to support table top.

## References

- [1] J. J. Uicker, "On the Dynamic Analysis of Spatial Linkages Using 4x4 Matrices," Ph. D. Thesis, Northwestern University, August 1965.
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