DEFINITE INTEGRALS

by

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Submitted in Partial Fulfillment of
the Requirements for the Degree of
Bachelor of Science in Mathematics

Massachusetts Institute of Technology
June 1937

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INTRODUCTION

The first two chapters consist of the evaluation and extension of a number of definite integrals; whereas, in the third chapter, an attempt is made to show a possible method for obtaining the sum of a finite series by means of definite integrals somewhat of the form used in Chapters I and II.

It is perhaps advisable to outline the useful notions and formulae which are used repeatedly in the evaluation of the definite integrals. This will enable the reader to follow the steps more readily without an undue repetition of the same theorems, etc., between equations. The notions and formulae, the proofs of which may be found in practically any text on Functions of a Complex Variable, will merely be stated as it is not deemed necessary to repeat such proofs since this would merely take up space.

Methods of finding residues:

(1) Simple pole as of function $\frac{1}{z - a} f(z) = \phi(z)$

Then $\lim_{z \to a} (z - a)\, \phi(z) = f(a)$ is the residue at a

if the Limit is a definite number.

(2) It follows from L'Hospital's rule that, if $\phi(z)$ and $\psi(z)$ are holomorphic at a and if $z - a$ is a non-repeated factor of $\psi(z)$, the residue of $\frac{\phi(z)}{\psi(z)}$ at a is $\frac{\phi(a)}{\psi'(a)}$.

(3) If we have poles of higher multiplicity, then we use Taylor's expansion $f(z) = \frac{1}{(z-a)^n}\phi(z)$ where a is real or complex.

$$\frac{1}{(z-a)^n}\left[\phi(a) + (z-a)\phi'(a) + (z-a)^2\frac{\phi''(a)}{2!} + \ldots\ldots\ldots\right]$$

If $n = 1$ then $\phi(a)$ is the residue at a

If $n = 2$ " $\phi'(a)$ " " " " a

If $n = 3$ " $\frac{\phi''(a)}{2!}$ " " " " a

If $n = n$ " $\frac{\phi^{n-1}(a)}{(n-1)!}$ " " " " a

Two useful theorems for showing that $\int$ around R or r (where $R \to \infty$ and $r \to 0$) = 0 are the following:

$$\lim_{R\to\infty}\int_R f(z)\,dz = i(\theta_2 - \theta_1)K \quad \text{around large circle}$$

$$\text{and} \lim_{z\to\infty} z f(z) = K$$

If $\lim_{z\to a}(z-a)f(z) = k$ where k is a constant, then

$$\lim_{r\to 0}\int f(z)\,dz = i(\theta_2 - \theta_1)k$$

The following useful formulae are given:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \cosh z = \frac{e^{z} + e^{-z}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \sinh z = \frac{e^{z} - e^{-z}}{2}$$

$$\sin i z = i \sinh z \qquad e^{2n\pi i} = 1$$

$$\cos i z = \cosh z \qquad e^{(2n-1)\pi i} = -1$$

$$\sin \theta \geqq \frac{2\theta}{\pi} \quad \text{where} \quad 0 \leqq \theta \leqq \frac{\pi}{2}$$

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Chapter I

EVALUATION OF DEFINITE INTEGRALS

$\int_0 \frac{\cos m x}{x^4 + a^4} dx \qquad z^4 = - a^4$

$$z = a \left[\cos \frac{(\pi + 2k\pi)}{4} + i \sin \frac{(\pi + 2k\pi)}{4}\right]$$

where $k = 0, 1, 2, 3$

Consider

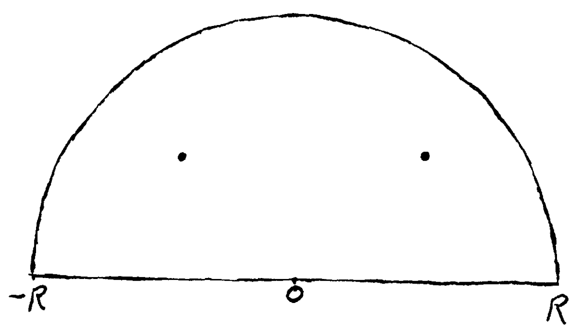
$$\int \frac{e^{imz}}{z^4 + a^4} dz$$

$z_1 = a e^{i \frac{\pi}{4}}$

$z_2 = a e^{i \frac{3\pi}{4}}$

$z_3 = a e^{i \frac{5\pi}{4}}$

$z_4 = a e^{i \frac{7\pi}{4}}$



Integral around $R = 0$ for

$$\int_0^{\pi} \frac{e^{im(R \cos \theta + i \sin \theta)}}{R^4 e^{4i\theta} + a^4} Re^{i\theta} d\theta$$

$$< \int_0^{\pi} \frac{R e^{-mR \sin \theta}}{R^4 - a^4} d\theta < \frac{2 R}{R^4 - a^4} \int_0^{\frac{\pi}{2}} e^{-mR \sin \theta} d\theta$$

$$< \frac{2R}{R^4 - a^4} \int_0^{\frac{\pi}{2}} e^{-\frac{2mR\theta}{\pi}} d\theta < \frac{- 2 R \pi}{2 m R (R^4 - a^4)} = 0 \text{ as } R \to \infty$$

$$\text{Residue at } \alpha = \frac{e^{im\alpha}}{4\alpha^3}$$

$$\text{Residue at } ae^{i\frac{\pi}{4}} = \frac{e^{ima(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})}}{4\,a^3\,e^{i\frac{3\pi}{4}}} = \frac{e^{i\frac{ma}{\sqrt{2}}}\cdot e^{-\frac{ma}{\sqrt{2}}}}{4a^3\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)}$$

$$\text{Residue at } ae^{i\frac{3\pi}{4}} = \frac{e^{ima(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})}}{4\,a^3\,e^{i\frac{9\pi}{4}}}$$

$$= \frac{e^{-i\frac{ma}{\sqrt{2}}}\cdot e^{-\frac{ma}{\sqrt{2}}}}{4\,a^3\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)}$$

Adding residues, we have

$$\frac{e^{-\frac{ma}{\sqrt{2}}}}{4\,a^3}\left[\frac{e^{i\frac{ma}{\sqrt{2}}}}{-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}} - \frac{e^{-i\frac{ma}{\sqrt{2}}}}{-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}}\right]$$

$$= \frac{\sqrt{2}\,e^{-\frac{ma}{\sqrt{2}}}\left[e^{i\frac{ma}{\sqrt{2}}}(-1-i) - e^{-i\frac{ma}{\sqrt{2}}}(-1+i)\right]}{4\,a^3\left(+\frac{1}{2} + \frac{1}{2} - \frac{1}{2}i + \frac{1}{2}i\right)}$$

$$= \frac{\sqrt{2}\,e^{-\frac{ma}{\sqrt{2}}}}{4\,a^3}\left[\left(e^{i\frac{ma}{\sqrt{2}}}\right)-(\sqrt{2})e^{i\frac{\pi}{4}} - \left(e^{-i\frac{ma}{\sqrt{2}}}\right)(-\sqrt{2})\,e^{-i\frac{\pi}{4}}\right]$$

$$= -\frac{ie^{-\frac{ma}{\sqrt{2}}}}{a^3}\sin\left(\frac{\pi}{4} + \frac{ma}{\sqrt{2}}\right)$$

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{x^4 + a^4}\, dx = \frac{2\, e^{-\frac{ma}{\sqrt{2}}} \sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)}{a^3}$$

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^4 + a^4}\, dx = \frac{2\, e^{-\frac{ma}{\sqrt{2}}} \sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)}{a^3}$$

$$\int_{0}^{\infty} \frac{\cos mx}{x^4 + a^4}\, dx = \frac{e^{-\frac{ma}{\sqrt{2}}} \sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)}{2a^3}$$

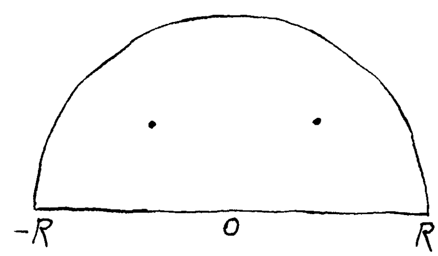
$$\int_{-\infty}^{\infty} \frac{\sin mx\, dx}{x^4 + a^4} = 0$$

$$\int_{0}^{\infty} \frac{x \sin mx}{x^4 + a^4}\, dx$$

m and a positive

Consider

$$\int \frac{z\, e^{imz}}{z^4 + a^4}\, dz$$



Integral around $R = 0$ for

$$\int_{0}^{\pi} \frac{R\, e^{i\theta}\, e^{imR(\cos\theta + i\sin\theta)}\, i R e^{i\theta}\, d\theta}{R^4 e^{4i\theta} + a^4}$$

$$< \int_{0}^{\pi} \frac{R^2\, e^{-mR\sin\theta}}{R^4 - a^4}\, d\theta < \frac{2R^2}{R^4 - a^4} \int_{0}^{\frac{\pi}{2}} e^{-mR\sin\theta}\, d\theta$$

$$< \frac{2R^2}{R^4 - a^4} \int_{0}^{\frac{\pi}{2}} e^{-\frac{2mR\theta}{\pi}}\, d\theta = -\frac{2R^2\pi}{2mR(R^4 - a^4)} \left[e^{-\frac{2R\theta}{\pi}} \right]_{0}^{\frac{\pi}{2}}$$

$$= \frac{\pi R}{R^4 - a^4} \left[1 - e^{-R}\right] \longrightarrow 0 \text{ as } R \longrightarrow \infty$$

$$\text{Residue at } \alpha = \frac{e^{im\alpha}}{4\alpha^2}$$

$$\text{Residue at } ae^{i\frac{\pi}{4}} = \frac{e^{i\,ma\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)}}{4\,a^2\,e^{i\frac{\pi}{2}}} = \frac{e^{i\frac{ma}{\sqrt{2}}}\,e^{-\frac{ma}{\sqrt{2}}}}{4\,a^2\,i}$$

$$\text{Residue at } ae^{i\frac{3\pi}{4}} = \frac{e^{i\,ma\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)}}{4\,a^2\,e^{i\frac{3\pi}{2}}} = \frac{e^{-i\frac{ma}{\sqrt{2}}}\,e^{-\frac{ma}{\sqrt{2}}}}{-4\,a^2\,i}$$

Adding residues

$$\frac{e^{-\frac{ma}{\sqrt{2}}}}{2\,a^2}\left[\frac{e^{i\frac{ma}{\sqrt{2}}} - e^{-i\frac{ma}{\sqrt{2}}}}{2\,i}\right] = \frac{e^{-\frac{ma}{\sqrt{2}}}}{2\,a^2}\sin\frac{ma}{\sqrt{2}}$$

$$\int_{-\infty}^{\infty}\frac{x\,e^{imx}}{x^4 + a^4}\,dx = \frac{\pi\,i\,e^{-\frac{ma}{\sqrt{2}}}}{a^2}\sin\frac{ma}{\sqrt{2}}$$

$$\int_{-\infty}^{\infty}\frac{x\cos mx}{x^4 + a^4}\,dx + i\int_{-\infty}^{\infty}\frac{x\sin mx}{x^4 + a^4}\,dx = \frac{i\pi\,e^{-\frac{ma}{\sqrt{2}}}}{a^2}\sin\frac{ma}{\sqrt{2}}$$

$$\int_{-\infty}^{\infty}\frac{x\cos mx}{x^4 + a^4}\,dx = 0$$

$$\int_{0}^{\infty}\frac{x\sin mx}{x^4 + a^4}\,dx = \frac{\pi\,e^{-\frac{ma}{\sqrt{2}}}}{2a^2}\sin\frac{ma}{\sqrt{2}}$$

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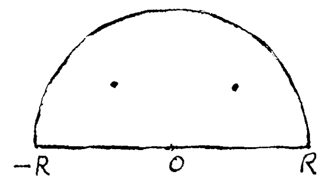
$a > 0, \quad m > 0$

$$\int_0^{\infty} \frac{x^3 \sin mx}{x^4 + a^4}\, dx$$



$$\int \frac{z^3 e^{imz}}{z^4 + a^4}\, dz$$

$$\int_0^{\pi} \frac{R^3 e^{3i\theta} e^{imR(\cos\theta + i \sin\theta)}}{R^4 e^{4i\theta} + a^4}\, R\, i\, e^{i\theta}\, d\theta$$

$$< \int_0^{\pi} \frac{R^4 e^{-mR\sin\theta}}{R^4 - a^4}\, d\theta < \frac{2R^4}{R^4 - a^4} \int_0^{\frac{\pi}{2}} e^{-\frac{2mR\theta}{\pi}}\, d\theta$$

$$\simeq \frac{2R^4 \pi}{(R^4 - a^4)\, 2mR}\, [1 - e^{-mR}] \to 0 \text{ as } R \to \infty$$

$$\text{Residue at } \alpha = \frac{\alpha^3 e^{im\alpha}}{4\alpha^3}$$

$$\text{Residue at } ae^{i\frac{\pi}{4}} = \frac{e^{ima\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)}}{4} = \frac{e^{i\frac{ma}{\sqrt{2}}}\, e^{-\frac{ma}{\sqrt{2}}}}{4}$$

$$\text{Residue at } ae^{i\frac{3\pi}{4}} = \frac{e^{ima\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)}}{4} = \frac{e^{-i\frac{ma}{\sqrt{2}}}\, e^{-\frac{ma}{\sqrt{2}}}}{4}$$

Adding residues

$$\frac{e^{-\frac{ma}{\sqrt{2}}}}{2}\left[\frac{e^{i\frac{ma}{\sqrt{2}}} + e^{-i\frac{ma}{\sqrt{2}}}}{2}\right] = \frac{e^{-\frac{ma}{\sqrt{2}}}}{2} \cos\frac{ma}{\sqrt{2}}$$

$$\int_{-\infty}^{\infty} \frac{x^3 e^{imx}}{x^4 + a^4}\, dx = \pi i\, e^{-\frac{ma}{\sqrt{2}}} \cos\frac{ma}{\sqrt{2}}$$

$$\int_{-\infty}^{\infty} \frac{x^3 \cos mx}{x^4 + a^4}\, dx + i \int_{-\infty}^{\infty} \frac{x^3 \sin mx}{x^4 + a^4}\, dx = i \pi e^{-\frac{ma}{\sqrt{2}}} \cos \frac{ma}{\sqrt{2}}$$

$$\int_{-\infty}^{\infty} \frac{x^3 \cos mx}{x^4 + a^4}\, dx = 0$$

$$\int_{0}^{\infty} \frac{x^3 \sin mx}{x^4 + a^4}\, dx = \frac{\pi}{2} e^{-\frac{ma}{\sqrt{2}}} \cos \frac{ma}{\sqrt{2}}$$

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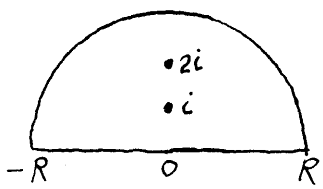
$$\int_{0}^{\infty} \frac{x^2\, dx}{x^4 + 5x^2 + 4}$$

Consider

$$\int \frac{z^2\, dz}{z^4 + 5a^2 + 4}$$



$$z^2 = \frac{-5 \pm \sqrt{25 - 16}}{2}$$

$$z^2 = -1 \text{ or } -4$$

$$z = \pm i \text{ or } \pm 2i$$

residue at pole α $\left.\frac{z^2}{4z^3 + 10z}\right]_\alpha = \frac{\alpha^2}{4\alpha^3 + 10\alpha}$

$$= \frac{\alpha}{4\alpha^2 + 10}$$

residue at $i = \frac{i}{4(-1) + 10} = \frac{i}{6}$

" " $2i = \frac{2i}{4(-4) + 10} = -\frac{i}{3}$

Since the exponent of the numerator is 2 less than that of the biggest exponent in denominator

(i.e., $\frac{z^2}{z^4}$ the $\int_R = 0$ as $R \to \infty$)

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{x^4 + 5x^2 + 4} = 2\pi i\left[\frac{i}{6} - \frac{i}{3}\right]$$

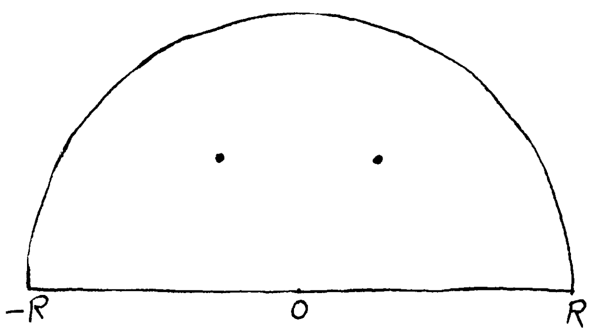
since we have an even function

$$2\int_{0}^{\infty} \frac{x^2\,dx}{x^4 + 5x^2 + 4} = 2\pi i\left[-\frac{i}{6}\right]$$

$$\int_{0}^{\infty} \frac{x^2\,dx}{x^4 + 5x^2 + 4} = \frac{\pi}{6}$$

If $M > 0$ evaluate

$$\int_{0}^{\infty} \frac{\cos mx\,dx}{1 + x^2 + x^4}$$



consider

$$\int \frac{e^{imz}\,dz}{1 + z^2 + z^4} =$$

$$1 + z^2 + z^4 = \frac{z^6 - 1}{z^2 - 1} \qquad \therefore \quad z^6 - 1 = 0$$

wherever $1 + z^2 + z^4 = 0$

$$z = \cos\left(\frac{0 + 2k\pi}{6}\right) + i\sin\left(\frac{0 + 2k\pi}{6}\right) \qquad (k = 0, 1, 2, 3, 4, 5)$$

$\therefore \dfrac{e^{imz}}{1 + z^2 + z^4}$ has poles at $e^{i\frac{\pi}{3}}$, $e^{i\frac{2\pi}{3}}$, above the x axis, the residue at $e^{i\frac{\pi}{3}}$ + the residue at $e^{i\frac{2\pi}{3}}$

$$\lim_{z \to e^{i\frac{\pi}{3}}} \frac{e^{imz}}{2z + 4z^3} + \lim_{z \to e^{2i\frac{\pi}{3}}} \frac{e^{imz}}{2z + 4z^3} =$$

$$\frac{e^{im\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}}{2\, e^{i\frac{\pi}{3}}\left(1 + 2\, e^{2i\frac{\pi}{3}}\right)} + \frac{e^{im\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}}{2\, e^{i\frac{2\pi}{3}}\left(1 + 2\, e^{4i\frac{\pi}{3}}\right)} =$$

$$\frac{e^{i\frac{m}{2}}\, e^{-m\frac{\sqrt{3}}{2}}}{2\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)\left[1 + 2\left(+\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^2\right]} + \frac{e^{-i\frac{m}{2}}\, e^{-m\frac{\sqrt{3}}{2}}}{2\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^2\left[1 + 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^4\right]} =$$

$$\frac{e^{-m\frac{\sqrt{3}}{2}}\, e^{i\frac{m}{2}}}{(1 + \sqrt{3}\, i)\, \sqrt{3}\, i} + \frac{e^{-m\frac{\sqrt{3}}{2}}\, e^{-i\frac{m}{2}}}{(-1 + \sqrt{3}\, i)(-\sqrt{3})} =$$

$$\frac{e^{-m\frac{\sqrt{3}}{2}}}{i\sqrt{3}} \; \frac{e^{i\frac{m}{2}}(1 - i\sqrt{3}) + e^{-i\frac{m}{2}}(1 + i\sqrt{3})}{(1 + i\sqrt{3})\,(1 - i\sqrt{3})} =$$

$$\frac{e^{-m\frac{\sqrt{3}}{2}}}{i\sqrt{3}}\left[\frac{2e^{i\frac{m}{2}}\, e^{-i\frac{\pi}{3}} + 2e^{-i\frac{m}{2}}\, e^{i\frac{\pi}{3}}}{(1 + i\sqrt{3})\,(1 - i\sqrt{3})}\right] = \frac{e^{-m\frac{\sqrt{3}}{2}}}{\sqrt{3}\, i} \cos\left(\frac{m}{2} - \frac{\pi}{3}\right)$$

on the circle

$$z = R\, e^{i\theta}$$

$$dz = R\, i\, e^{i\theta}\, d\theta$$

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{1 + x^2 + x^4}\, dx + \int_0^{\pi} \frac{e^{-m R \sin\theta}\, e^{im R \cos\theta}}{1 + R^2\, e^{2i\theta} + R^4\, e^{4i\theta}}\, R\, i\, e^{i\theta}\, d\theta$$

$$= \frac{2\; i\; e^{-m\frac{\sqrt{3}}{2}}}{i\sqrt{3}} \cos\left(\frac{\pi}{3} - \frac{m}{2}\right)$$

the second $\int \to 0$ when $R \to \infty$

$$\int_{-\infty}^{\infty} \frac{\cos mx}{1 + x^2 + x^4}\, dx + i \int_{-\infty}^{\infty} \frac{\sin mx}{1 + x^2 + x^4}\, dx = \frac{2\pi}{\sqrt{3}} e^{-m\frac{\sqrt{3}}{2}} \cos\left(\frac{\pi}{3} - \frac{m}{2}\right)$$

$$\therefore \quad \int_{0}^{\infty} \frac{\cos mx}{1 + x^2 + x^4}\, dx = \frac{\pi}{\sqrt{3}} e^{-m\frac{\sqrt{3}}{2}} \cos\left(\frac{\pi}{3} - \frac{m}{2}\right)$$

$$\text{and} \quad \int_{-\infty}^{\infty} \frac{\sin mx}{1 + x^2 + x^4}\, dx = 0$$

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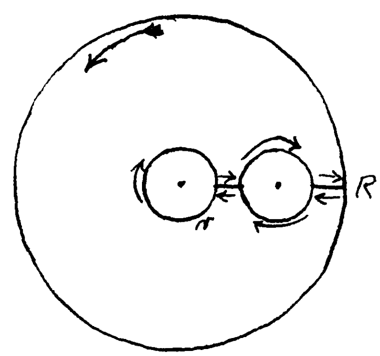
$0 < a < 1$



$$P \int_0 \frac{x^{a-1}}{1 - x}\, dx$$

$$\int \frac{z^{a-1}}{z - 1}\, dz$$

$$z = r e^{i\theta}$$

$$dz = r i e^{i\theta}\, d\theta$$

$$\lim_{r \to 0} \int_{2\pi}^{0} \frac{r^{a-1} e^{i(a-1)\theta}\, r\, i\, e^{i\theta}\, d\theta}{r e^{i\theta} - 1}$$

$$\leqq \int_{2\pi}^{0} \frac{|r^a|\, d\theta}{|r + 1|} \leqq \frac{r^a}{r + 1} \int_{2\pi}^{0} d\theta \to 0$$

$$\lim_{R \to \infty} \int_{0}^{2\pi} \frac{R^{a-1} e^{i(a-1)\theta}}{R e^{i\theta} - 1} R\, i\, e^{i\theta}\, d\theta$$

$$\leqq \int_{0}^{2\pi} \frac{R^a}{R + 1}\, d\theta = \frac{R^a}{R + 1} \int_{0}^{2\pi} d\theta \to 0$$

$$\int_0^{1-\varepsilon} \frac{x^{a-1}}{x-1}\,dx + \int_{1+\varepsilon}^{\infty} \frac{x^{a-1}}{x-1}\,dx$$

$$- e^{2a\pi i} \int_0^{1-\varepsilon} \frac{x^{a-1}}{x-1}\,dx + \int_{1+\varepsilon}^{\infty} \frac{x^{a-1}}{x-1}\,dx =$$

$$\pi i \left[e^{2\pi(a-1)i} + e^{0(a-1)i} \right]$$

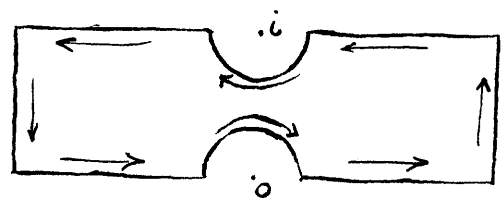
$$(e^{2a\pi i} - 1)\, P \int_0^{\infty} \frac{x^{a-1}}{1-x}\,dx = \pi i\,(e^{2\pi ai} + 1)$$

$$P \int_0^{\infty} \frac{x^{a-1}}{1-x}\,dx = \frac{\pi i\,(e^{2a\pi i} + 1)}{e^{2a\pi i} - 1} = \frac{\frac{\pi i}{2i}\,(e^{a\pi i} + e^{-a\pi i})}{\frac{e^{a\pi i} - e^{-a\pi i}}{2i}}$$

$$= \frac{\pi \cos a\pi}{\sin a\pi} = \pi \operatorname{ctn} a\pi$$

$-\pi < a < \pi$

$$\int_0^{\infty} \frac{\sinh ax}{\sinh \pi x}\,dx$$



Consider $\int \frac{e^{az}\,dz}{\sinh \pi z}$

since $\sinh z = \frac{e^{\pi z} - e^{-\pi z}}{2}$

we have poles when $e^{\pi z} = e^{-\pi z}$

$e^{2\pi z} = 1$ i.e., when $z = 0$ and $z = ki$

when $z = i$ need only concern us, using above contour.

$$z = R + i\,y$$

$$dz = i\,dy$$

$$\int_0^1 \frac{2\,e^{aR}\,e^{iay}\,i\,dy}{e^{\pi R}\,e^{i\pi y} - e^{-\pi R}\,e^{-i\pi y}} = \int_0^1 \left|\frac{2\,e^{aR}\,dy}{e^{\pi R} - e^{-\pi R}}\right| \leq \frac{2}{e^{(\pi-a)R} - \frac{1}{e^{\pi R}}} \int_0^1 dy$$

tends to 0 as $R \to \infty$

$$z = -R + i\,y$$

$$dz = i\,dy$$

$$\frac{2}{e^{-(\pi-a)R} - \frac{1}{e^{-\pi R}}}$$ tends to 0 as $R \to \infty$

$$\int_0^\infty \frac{e^{ax}}{\sinh \pi x}\,dx + \int_\infty^0 \frac{e^{a(x+i)}}{\sinh \pi(x+i)}\,dx$$

$$\int_0^{-\infty} \frac{e^{a(x+i)}}{\sinh \pi(x+i)}\,dx + \int_{-\infty}^0 \frac{e^{ax}\,dx}{\sinh \pi x}$$

$$\lim_{z \to 0} \frac{2z\,e^{az}}{e^{\pi z} - e^{-\pi z}} = \frac{2az\,e^{az} + 2e^{az}}{\pi e^{\pi z} + \pi e^{-\pi z}}$$

$$= \lim_{z \to i} \frac{1}{\pi}(\pi i) \frac{2(z-i)\,e^{az}}{e^{\pi z} - e^{-\pi z}} = -i\,e^{ia}$$

residue about little semicircle $\frac{1}{\pi}\pi i = i$

Combining

$$(1 + e^{ia}) \int_0^\infty \frac{e^{ax}}{\sinh \pi x}\,dx + (1 + e^{ia}) \int_{-\infty}^0 \frac{e^{ax}}{\sinh \pi x}\,dx + i - i\,e^{ia} = 0$$

$$(1 + e^{ia}) \int_0^{\infty} \frac{e^{ax} - e^{-ax}}{2\,(\sinh \pi x)}\, dx = -\frac{i}{2}\,[1 - e^{ia}]$$

$$\int_0^{\infty} \frac{\sinh a x}{\sinh \pi x}\, dx = \frac{i}{2}\,\frac{[1 - e^{ia}]}{[1 + e^{ia}]} = -\frac{ii}{2}\;\frac{\dfrac{e^{-i\frac{a}{2}} - e^{i\frac{a}{2}}}{2 - i}}{\dfrac{e^{-i\frac{a}{2}} + e^{i\frac{a}{2}}}{2}}$$

$$= \frac{1}{2}\,\frac{\sin \frac{a}{2}}{\cos \frac{a}{2}} = \frac{1}{2} \tan \frac{a}{2}$$

- -

$$\int_0^{2\pi} \cot \frac{\theta - a - bi}{2}\, d\theta$$

$$z = e^{i\theta}$$

$$\int_0^{2\pi} \frac{\cos \frac{\theta - a - bi}{2}}{\sin \frac{\theta - a - bi}{2}}\, d\theta =$$

$$dz = i\, e^{i\theta}\, d\theta$$

$$\frac{dz}{iz} = d\theta$$

$$\int_0^{2\pi} i\, \frac{e^{i\frac{(\theta-a-bi)}{2}} + e^{-i\frac{(\theta-a-bi)}{2}}}{e^{i\frac{(\theta-a-bi)}{2}} - e^{-i\frac{(\theta-a-bi)}{2}}}\, d\theta =$$

$$\int_0^{2\pi} i\, \frac{e^{i\theta - ia + b} + 1}{e^{i\theta - ia + b} - 1}\, d\theta =$$

$$\int_c \frac{z + e^{ia-b}}{z - e^{ia-b}}\, \frac{dz}{z} = -2\pi i \qquad \text{If } b < 0$$

$$= -2\pi i + 4\pi i \qquad e^{ia-b} \text{ is outside unit circle}$$

$$= 2\pi i \qquad e^{ia-b} \text{ is inside unit circle}$$

If $b > 0$

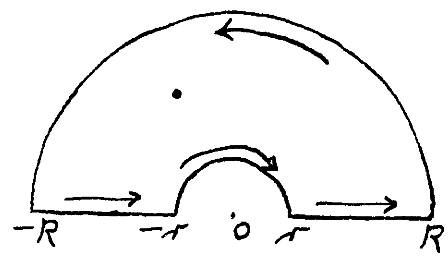
$$\lim_{z \to 0} \frac{z\,(z - e^{ia-b})}{(z - e^{ia-b})\,z} = -2\pi i$$

$$\int_0^{2\pi} \cot \frac{\theta - a - bi}{2}\, d\theta = 2\pi i \quad \text{when } b > 0$$

$$= -2\pi i \quad \text{when } b < 0$$

$0 < a < 2$

$$\int_0^{\infty} \frac{x^{a-1}}{1 + x + x^2}\, dx$$



Consider

$$\int \frac{z^{a-1}}{1 + z + z^2}\, dz = \int \frac{z^a\, dz}{z\,(1 + z + z^2)}$$

$$z^2 + z + 1 = 0$$

$$z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$z = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$

inside figure

$$\lim_{z \to -\frac{1}{2} + \frac{\sqrt{3}}{2} i} \frac{[z - (-\frac{1}{2} + \frac{\sqrt{3}}{2} i)]\, z^a}{z\,[z - (-\frac{1}{2} + \frac{\sqrt{3}}{2} i)][z - (-\frac{1}{2} - \frac{\sqrt{3}}{2} i)]}$$

$$= \frac{e^{i\frac{2\pi a}{3}}}{e^{i\frac{2\pi}{3}}\ \sqrt{3}\, i}$$

Integrated around r =

$$\pi i \lim_{z \to 0} \frac{z\, z^a}{z\,(1 + z + z^2)} = 0$$

$$\therefore \int_{-\infty}^{0} \frac{x^{a-1}}{1 + x + x^2}\, dx + \int_0^{\infty} \frac{x^{a-1}}{1 + x + x^2}\, dx = 2\pi i \cdot \text{Residue}$$

$$\int_0^{\infty} \frac{x^{a-1}\, e^{\pi i(a-1)}}{1 - x + x^2}\, dx + \int_0^{\infty} \frac{x^{a-1}}{1 + x + x^2}\, dx$$

$$= \frac{2\pi}{\sqrt{3}} \quad \frac{\cos \frac{2\pi a}{3} + i \sin \frac{2\pi a}{3}}{\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}}$$

$$= \frac{2}{\sqrt{3}} \quad \frac{(\cos \frac{2\pi a}{3} + i \sin \frac{2\pi a}{3})(\cos \frac{2\pi}{3} - i \sin \frac{2\pi}{3})}{(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3})(\cos \frac{2\pi}{3} - i \sin \frac{2\pi}{3})}$$

$$= \frac{2}{\sqrt{3}} \left[\cos \left(\frac{2\pi a}{3} - \frac{2\pi}{3}\right) + i \sin \left(\frac{2\pi a}{3} - \frac{2\pi}{3}\right)\right]$$

$$\int_0^{\infty} \frac{x^{a-1}}{1 - x + x^2} \cos \pi a\, dx + \int_0^{\infty} \frac{x^{a-1}}{1 + x + x^2}\, dx = \frac{2\pi}{\sqrt{3}} \cos \left(\frac{2\pi a}{3} - \frac{2\pi}{3}\right)$$

$$i \int_0^{\infty} \frac{x^{a-1}}{1 - x + x^2} \sin \pi a\, dx = i \sin \left(\frac{2\pi a}{3} - \frac{2\pi}{3}\right)$$

$$\int_0 \frac{x^{a-1}}{1 - x + x^2}\, dx = \frac{2}{\sqrt{3}} \quad \frac{\sin \frac{2\pi a - 2\pi}{3}}{\sin \pi a}$$

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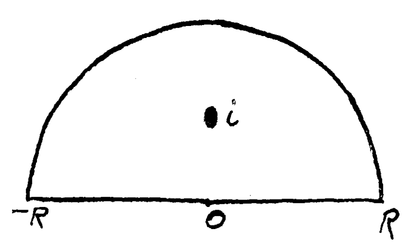
$$\int_0^1 \frac{\log (x + \frac{1}{x})}{1 + x^2}\, dx = \frac{\pi}{2} \log 2$$



Consider $\int \frac{\log (z + i)}{z^2 + 1}\, dz$

$\int$ around R = 0, for $\lim_{z \to \infty} z\, f(z) = 0$

$$\frac{z \log (z + i)}{z^2 + 1} = \frac{\frac{z}{z + i} + \log (z + i)}{2 z}$$

$$= \frac{\frac{1}{z + i} + \frac{\log z + i}{z}}{2} = 0$$

$$\text{Residue at } i = \lim_{z \to i} \frac{z - i \log (z + i)}{(z - i)(z + i)} = \frac{\log 2i}{2i}$$

$$\int_{-\infty}^{\infty} \frac{\log (x + i)}{x^2 + 1} dx = \pi \log 2i$$

R) $$\int_{-\infty}^{\infty} \frac{\frac{1}{2} \log (x^2 + 1)}{x^2 + 1} dx = \pi \log 2$$

$$\int_{-\infty}^{\infty} \frac{\log x (x + \frac{1}{x})}{x^2 + 1} dx =$$

$$\frac{1}{2} \int_0^{\infty} \frac{\log x}{x^2 + 1} dx + \frac{1}{2} \int_0^{\infty} \frac{\log (x + \frac{1}{x})}{x^2 + 1} dx$$

or

$$\int_0^1 \frac{\log (x + \frac{1}{x})}{x^2 + 1} dx = \frac{\pi}{2} \log 2$$

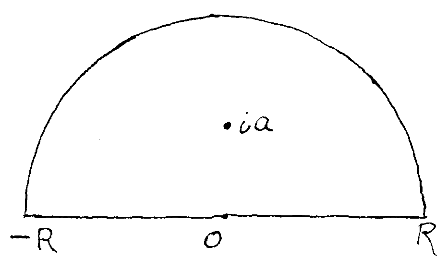
Chapter II

EXTENSION OF THE EVALUATION OF ONE DEFINITE INTEGRAL TO THAT OF MANY OTHERS

The following example will show from the evaluation of one definite integral how we can obtain a great number of other integrals:

$a > 0$

$$\int_{-\infty}^{\infty} \frac{a\cos x + x\sin x}{x^2 + a^2}\,dx$$



Consider

$$\int \frac{e^{iz}}{z - ai}\,dz$$

$$\int_0^{\pi} \frac{e^{iR(\cos\theta + i\sin\theta)}\,R\,i\,e^{i\theta}\,d\theta}{R\,e^{i\theta} - ai}$$

$$< \int_0^{\pi} \frac{Re^{-R\sin\theta}}{R + a} < \frac{2R}{R+a}\int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\,d\theta < \frac{2R}{R+a}\int_0^{\frac{\pi}{2}} e^{-2\frac{R\theta}{\pi}}\,d\theta$$

$$< + \frac{2R}{(R+a)\,2R}\left[1 - e^{-R}\right] \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty$$

$$\text{Residue at } ia = \lim_{z \to ia} \frac{(z - ai)\,e^{iz}}{z - ai} = e^{-a}$$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x - ai}\,dx = 2\pi i\,e^{-a}$$

$$\int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{x - ai}\,dx = 2\pi i\,e^{-a}$$

$$\int_{-\infty}^{\infty} \frac{(\cos x + i\sin x)(x + ai)}{(x - ai)(x + ai)}\,dx = 2\pi i\,e^{-a}$$

$$\int_{-\infty}^{\infty} \frac{x \cos x - a \sin x}{x^2 + a^2} dx + i \int_{-\infty}^{\infty} \frac{a \cos x + x \sin x}{x^2 + a^2} dx = 2\pi i \; e^{-a}$$

(1) $\therefore \quad \int_{-\infty}^{\infty} \frac{a \cos x + x \sin x}{x^2 + a^2} dx = 2\pi \; e^{-a}$

(2) $\int_{-\infty}^{\infty} \frac{x \cos x - a \sin x}{x^2 + a^2} dx = 0$

= 0 because odd function

$$\int_{-\infty}^{\infty} \frac{x \cos x}{x^2 + a^2} dx - a \int_{-\infty}^{\infty} \frac{\sin x}{x^2 + a^2} dx = 0$$

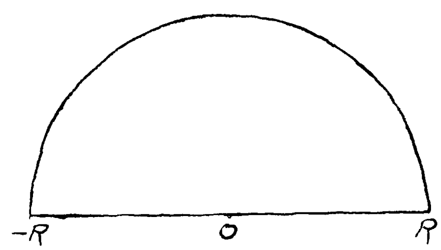
(3) $\therefore \quad \int_{-\infty}^{\infty} \frac{\sin x}{x^2 + a^2} dx = 0$

- -

$a > 0$

$$\int_{-\infty}^{+\infty} \frac{-a \cos x + x \sin x}{x^2 + a^2} dx = 0$$

Consider $\int \frac{e^{iz}}{z + ai} dz$



$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x + ia} dx = 0$$

$$\int_{-\infty}^{\infty} \frac{\cos x + i \sin x \; (x - ia)}{(x + ia)(x - ia)} dx = 0$$

$$\int_{-\infty}^{\infty} \frac{x \cos x + a \sin x}{x^2 + a^2} dx + i \int_{-\infty}^{\infty} \frac{-a \cos x + x \sin x}{x^2 + a^2} dx = 0$$

(4) $\int_{-\infty}^{\infty} \frac{-a \cos x + x \sin x}{x^2 + a^2} dx = 0$

(5) $\int_{-\infty}^{\infty} \frac{x \cos x + a \sin x}{x^2 + a^2} dx = 0$

Adding (1) and (4)

(6) $$\int_{-\infty}^{\infty} \frac{2\, x \sin x}{x^2 + a^2}\, dx = 2\pi e^{-a}$$

(7) $$\int_{0}^{\infty} \frac{x \sin x}{x^2 + a^2}\, dx = \frac{\pi}{2} e^{-a}$$

Adding (3) and (6)

(8) $$\int_{-\infty}^{\infty} \frac{(x + 1) \sin x}{x^2 + a^2}\, dx = \pi e^{-a}$$

Adding (6) and (8)

$$\int_{-\infty}^{\infty} \frac{(2x + 1) \sin x}{x^2 + a^2}\, dx = 2\pi e^{-a}$$

Generalizing,

(9) $$\int_{-\infty}^{\infty} \frac{(nx + 1) \sin x}{x^2 + a^2}\, dx = n\pi e^{-a}$$ where n is an integer

Chapter III

SUMMATION OF A FINITE SERIES BY MEANS OF DEFINITE INTEGRALS

The scheme consists of evaluating $\int_0^\infty \frac{x^{2m}}{x^{2n} + 1}\, dx$ by a standard method; then, knowing the answer, we compare this with the finite summations in the real and imaginary parts of an equation gotten by taking a number (n) of poles on the unit circle, where we may let these poles approach arbitrarily close by increasing n sufficiently.

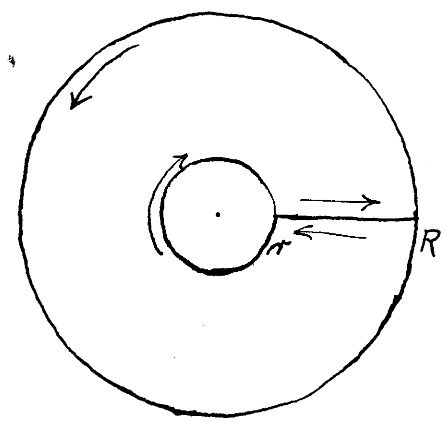
$$0 < p < 1$$

$$\int_0^\infty \frac{x^{p-1}}{1 + x}\, dx$$

$$\int \frac{z^{p-1}}{1 + z}\, dz$$

$$\lim_{R \to \infty} z\, f(z) = 0$$

$$\lim_{z \to 0} z\, f(z) = 0$$



$$\int_0^\infty \frac{x^{p-1}}{1 + x}\, dx + \int_\infty^0 \frac{(x\, e^{2\ \ i})^{p-1}}{1 + x\, e^{2\pi i}}\, dx = 2\pi i\, e^{(p-1)\pi i}$$

since
$$\lim_{z \to -1} \frac{(1 + z)\, z^{p-1}}{(z + 1)} = (e^{\pi i})^{p-1}$$

$$\int_0^\infty \frac{x^{p-1}}{1 + x}\, dx + \int_\infty^0 \frac{e^{2p\pi i}\, x^{p-1}}{1 + x}\, dx = 2\pi i\, e^{(p-1)\pi i}$$

$$(1 - e^{2p\pi i}) \int_0^\infty \frac{x^{p-1}}{1 + x}\,dx = -2\pi i\, e^{p\pi i}$$

$$\int_0^\infty \frac{x^{p-1}}{1 + x}\,dx = \frac{+2\pi i}{(-e^{-p\pi i} + e^{p\pi i})} = \frac{\pi}{\sin p\pi}$$

$$0 < p < 1 \quad \text{then} \quad \int_0^\infty \frac{x^{p-1}}{1 + x}\,dx = \frac{\pi}{\sin p\pi}$$

If m and n are positive integers, and $m < n$

$$\int_0^\infty \frac{x^{2m}}{x^{2n} + 1}\,dx \qquad\qquad x^{2n} = y$$

$$2nx^{2n-1}dx = dy$$

$$\frac{1}{2n}\int_0^\infty \frac{y^{\frac{m}{n}}\, y^{\frac{1-2n}{2n}}}{1 + y}\,dy \qquad\qquad dx = \frac{x^{1-2n}}{2n}\,dy$$

$$\frac{1}{2n}\int_0^\infty \frac{y^{\left(\frac{2m+1}{2n} - 1\right)}}{1 + y}\,dy \qquad\qquad dx = \frac{y^{\frac{1-2n}{2n}}}{2n}\,dy$$

$$x = y^{\frac{1}{2n}}$$

Since $\frac{2m + 1}{2n} < 1$ $\qquad\qquad x^{2m} = y^{\frac{m}{n}}$

$$\frac{1}{2n}\int_0^\infty \frac{y^{\left(\frac{2m+1}{2n} - 1\right)}}{1 + y}\,dy = \frac{\pi}{2n \sin \frac{2m + 1}{2n}}$$

$$\frac{\pi}{2n \sin \frac{2m + 1}{2n}} = \int_0^\infty \frac{x^{2m}}{x^{2n} + 1}\,dx$$

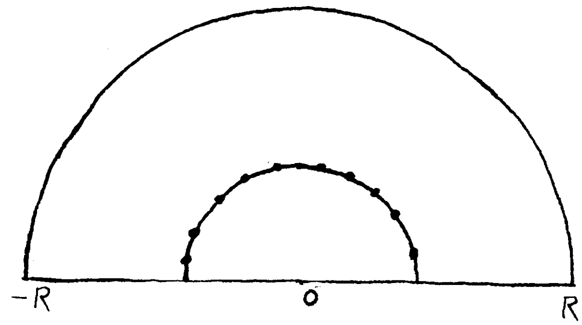
We now attack the problem in a different manner which will permit us to get the sum of a finite series by means of definite integrals. We have the advantage now of knowing the value of the definite integral $\int_0^\infty \frac{x^{2m}}{x^{2n} + 1}\,dx$

m and n positive integers

with $m < n$

$$\int_0^{\infty} \frac{x^{2m}}{x^{2n}+1}\,dx$$

$$\int \frac{z^{2m}}{z^{2n}+1}\,dz$$



$$z^{2n} = -1$$

$$z = 1\left(\cos\frac{\pi + 2k\pi}{2n} + i\sin\frac{\pi + 2k\pi}{2n}\right) \qquad (k = 0, 1, 2 \ldots 2n-1)$$

$\therefore$ we have poles at angles $\frac{\pi}{2n}, \frac{3\pi}{2n} \ldots\ldots \frac{(4n-1)\pi}{2n}$

on unit circle

The poles from $\frac{\pi}{2n} \ldots\ldots \frac{(2n-1)\pi}{2n}$ on unit circle are n poles in upper half of the plane and as long as n is finite there are no poles on the x-axis.

$$\lim_{R\to\infty} \int_{AB} f(z)\,dz = i(\theta_2 - \theta_1)\frac{a}{b}$$

$$\text{If} \quad f(z) = \frac{\phi(z)}{\psi(z)}$$

where $\psi(z)$ is a polynomial of degree n and $\phi(z)$ is a polynomial of degree less than n; where a and b are the coefficients of z^{n-1} and z^n in $\phi(z)$ and $\psi(z)$ respectively. Since the degree of $\phi(z)$ is $\leqq n-2$, $a = 0$ and $\therefore$ the integral of $f(z)$ round semicircle of radius R is 0.

$$\int_{-R}^{R} \frac{x^{2m}}{x^{2n}+1}\,dx = 2\pi i \sum \text{Residues}$$

Taking pole at $e^{\frac{p\pi}{2n}}$

$$\frac{[e^{(\frac{p\pi}{2n})}]^{2m}}{2n\,(e^{\frac{p\pi}{2n}})^{2n-1}} = \frac{e^{\frac{pm\pi}{n}}}{2\,n\,e^{p\pi}\,e^{-\frac{p\pi}{2n}}} = -\frac{e^{\frac{(2m+1)p\pi}{2n}}}{2\,n\,e^{p\pi}}$$

$p = 1, 3, \ldots\ldots\ 2n - 1.$ Substitute $p = 2p - 1$

$$\sum_{p=1}^{n} -\frac{e^{\frac{(2m+1)(2p-1)\pi}{2n}}}{2n} = -\frac{1}{2n}\sum_{p=1}^{n} e^{\frac{2m+1}{2n}(2p-1)\pi}$$

$$-\frac{1}{2n}\sum_{p=1}^{n}\left(\cos\frac{2m+1}{2n}(2p-1)\pi + i\sin\frac{2m+1}{2n}(2p-1)\pi\right)$$

(A) $$\int_{-\infty}^{\infty}\frac{x^{2m}}{x^{2n}+1}\,dx = -\frac{2\pi i}{2n}\sum_{p=1}^{n}\cos\frac{2m+1}{2n}(2p-1)\pi + \frac{2\pi}{2n}\sum_{p=1}^{n}\sin\frac{2m+1}{2n}(2p-1)\pi$$

since we have an even function

$$\int_{0}^{\infty}\frac{x^{2m}}{x^{2n}+1}\,dx = \frac{\pi}{2n}\sum_{p=1}^{n}\sin\frac{2m+1}{2n}(2p-1)\pi$$

$$\int_0^\infty \frac{x^{2m}}{x^{2n}+1}\,dx = \frac{\pi}{2n}\sum_{p=1}^{n} \sin\frac{2m+1}{2n}(2p-1)\pi$$

$$= \frac{\pi}{2n\sin\frac{2m+1}{2n}\pi}$$

from comparison with the answer obtained in the first method.

Since $\sum_{p=1}^{n} \cos\frac{2m+1}{2n}(2p-1)\pi = 0$

for the imaginary must vanish in equation (A) as we know from the answer obtained in the first way.

Checking the results of this specific finite series:

$$\sum_{p=1}^{n} \cos(ap+b) = \sum_{p=1}^{n} e^{i(ap+b)} \quad \text{(only real part of it)}$$

$$= e^{ib}\sum_{p=1}^{n} e^{iap}$$

Using geom. series

$$\sum_{p=1}^{n} e^{iap} = \frac{e^{ia(n+1)} - e^{ia}}{e^{ia}-1} = \frac{e^{ia}\left(e^{i\pi(2m+1)}-1\right)}{e^{ia}-1}$$

$$= i \cdot \frac{i\,2\,e^{i\frac{a}{2}}}{e^{i\frac{a}{2}} - e^{-i\frac{a}{2}}} = \frac{e^{i\frac{a}{2}}}{\sin\frac{a}{2}}$$

$$\sum_{p=1}^{n} \cos\,(ap + b) \;=\; \frac{e^{ib}\;\;i\;e^{ia}}{\sin\frac{a}{2}} \;=\; -\,\frac{\sin\,(b + \frac{a}{2})}{\sin\frac{a}{2}}$$

$$= \;-\,\frac{\sin 0}{\sin\frac{a}{2}} \;=\; 0$$

$$\sum_{p=1}^{n} \sin\frac{2m + 1}{2n}\,(2p - 1)\pi \;=\; \frac{1}{\sin\frac{2m + 1}{2n}\,\pi}$$

$$\sum_{p=1}^{n} \sin\,(ap + b) \;=\; \sum_{p=1}^{n} e^{i(ap+b)} \qquad \text{for (only imag. coeff.)}$$

$$= \;e^{ib}\sum_{p=1}^{n} e^{iap} \;=\; \frac{e^{ib}\;\;i\;e^{ia}}{\sin\frac{a}{2}}$$

$$= \;\frac{\cos\,(b + \frac{a}{2})}{\sin\frac{a}{2}} \;=\; \frac{\cos 0}{\sin\frac{(2m + 1)}{2n}\,\pi}$$

$$= \;\frac{1}{\sin\frac{(2m + 1)}{2n}\,\pi}$$

We note the peculiarity of the finite series

$$\sum_{p=1}^{n} \cos\frac{2m + 1}{2n}\,(2p - 1)\pi \;=\; 0$$

that is, no matter how many terms we have in the series given by n the sum is always zero (i.e., if n is a finite integer no matter how large n is (with $m < n$) then wherever the series is broken off (at $n = 5, 1000, 10^{10}$, or any such integral number) the sum is zero.

Checking the result in special cases

Suppose for simplicity $m = 1$

For n = 2

$$\sum_{p=1}^{2} \cos \frac{3}{4} (2p - 1)\pi = \cos \frac{3\pi}{4} + \cos \frac{9\pi}{4}$$

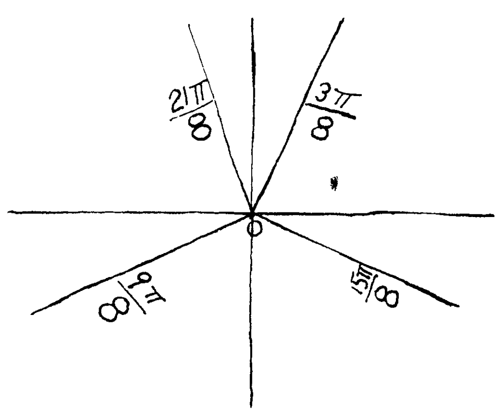
$$= -\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2} = 0$$

For n = 3

$$\sum_{p=1}^{3} \cos \frac{3}{6} (2p - 1)\pi = \cos \frac{\pi}{2} + \cos \frac{3\pi}{2} + \cos \frac{5\pi}{2} = 0$$

For n = 4

$$\sum_{p=1}^{4} \cos \frac{3}{8} (2p - 1)\pi = \cos \frac{3\pi}{8} + \cos \frac{9\pi}{8} + \cos \frac{15\pi}{8} + \frac{21\pi}{8} = 0$$



Since from the diagram and knowledge of sign of cosine being positive in the first and third quadrants and negative in the second and fourth quadrants.

$$\cos \frac{3\pi}{8} + \cos \frac{21\pi}{8} = 0$$

$$\cos \frac{9\pi}{8} + \cos \frac{15\pi}{8} = 0$$

etc., for $n > 4$ (continuing the same process).