Mathematical Modeling and Analysis of Flexible Production Lines

by

Young Jae Jang

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Abstract

We present a model and analysis of a production line that processes different part types on unreliable machines which operate according to a priority rule. The production line consists of machines separated by storage areas in which parts flow in a fixed sequence. A machine operates on the highest priority part whenever possible, and only operates on lower priority parts when unable to produce the higher priority parts. Part priorities are static and are a function only of part type. The purpose of this thesis is to present mathematical formulations and algorithms for estimating production rates and average inventory levels for each part type in a flexible production line. The qualitative behavior of the multiple-part-type line under different supply and demand scenarios is described.

Thesis Supervisor: Stanley B. Gershwin
Title: Senior Research Scientist, Department of Mechanical Engineering

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Chapter 1

Introduction

This thesis presents a model and analysis of a synchronous tandem production line that produces different part types on unreliable machines. Inventory is stored between machines in finite buffers. It is assumed that machines in the processing line are flexible in that they can operate on different part types, and there are no set-up penalties incurred when machines switch production from one part type to another. The machines operate according to a static priority rule, working on the highest priority part whenever possible, and on lower priority parts only when unable to process those with higher priorities due to either blockage or starvation. In this thesis, we investigate the flow behavior of a production system for a synchronous tandem flow line consisting of machines separated by storage areas in which parts flow in a fixed sequence. Such a tandem flow line can be observed in automotive fabrication, electronic components assembly and consumer products manufacturing industries.

The purpose of this research is to develop mathematical formulations and algorithms to analyze a manufacturing system processing more than one part type. That is, the goal of this thesis is to answer the question, "What average throughput rate and *work-in-progress* (WIP) for each part type can be expected from a given flexible production line design?" An accurate and logical answer to this question will help in understanding the best way to allocate buffer capacity in order to achieve the maximum throughput rate for each part type for a given process.

Analyzing the system behavior and evaluating the performance of manufacturing systems is challenging due to the complexity and randomness of the systems. Systematic approaches to the problem include real world experimentation, simulation and analytical methods. Real-world experimentation is not feasible when analysis is required to design a new production system — a production line in which experiments need to be performed does not exist. Even if a production line is available for actual experimentation, doing so is often too expensive and too time-consuming. Simulation is widely used in industry to predict the behavior of production lines. It can be useful for the detailed evaluation of a single final design. However, simulation requires a significant amount of computation and modeling time to generate analytically meaningful results. Analytical models are by far the fastest method for evaluating system performance, but very hard to construct. The analytical approach is usually quicker, and cheaper than simulation, but subject to errors associated with the abstracting of reality. This thesis focuses on developing an analytical model to evaluate real production lines by accurately accounting for key system phenomena.

1.1 Motivation

Since the idea of *Flexible Manufacturing Systems* (FMS) — in which installed machines and operations provide the production line with the ability to build different kinds of products on the same production facility — was proposed in the 1950s, various industries have implemented it in actual production lines. In the automotive industry, for instance, with the development of the robot technology and advanced automated material handling systems, machines are able to switch operations from one product type to another flexibly. This system flexibility has been adopted as a way to produce several different types of products on the same production line. For instance, an article in the *New York Times*[11] describes the current trends of FMS in the automotive industry:

"Detroit automakers are striving to embrace this practice, which is a hallmark of Japanese manufacturers. It allows companies to quickly change the mix of vehicles that their plants make to reflect demand. In 2004, Ford said, only 38 percent of its plants were flexible. By 2008, the company expects that 82 percent of its plants will be flexible, an increase from its previous goal of 75 percent."

Producing a diversity of products in a single production line has become a common practice in high-tech industries, such as semiconductor memory chips and liquid-crystal-display (LCD) panel productions. For example, the author observed that an LCD panel production line typically produces several different sizes of panels. A flexible and rapid response to the customers needs of different product types have recently become the primary determinant of competitiveness. Considering the enormous capital investment of production lines (well in excess of 2 billion dollars), it is crucial to be able to understand and predict the overall behavior of the FMS production line. However, little analysis of flexible production has been done. Due to the lack of tools or methods, line designers in industry cannot consider the dynamic behavior of the production line processing multiple part types. Analysis and intuition about production lines as well as methods to predict their systems behavior and estimate their performance are needed.

1.2 General Modeling Assumptions

A *manufacturing system* (also called *transfer line, production line* and within the queueing literature, *tandem queueing system with finite buffers* consists of parts (material), machines (work stages), and finite storage areas or *buffers*. In modeling the production lines in this research, we applied the concept of independence of events, which says that events in the future are only contingent on the present state of the system and are otherwise independent of each other and of past events [8]. In the production line model, this means that failure times and repair times on different machines are independent of one another. It also implies that the time between failures of a given machine are independent of previous failure times, repair times and the amount of material in the buffer, assuming the machine is not blocked or starved. This independence assumption allows the system to be modeled as a *Markov process*. Markovian models are frequently employed in analyzing production systems because of their tractability.

The *cycle time* is the time required for a single operation on an isolated machine. Cycle times are considered *fixed and deterministic* when they do not vary from one part to the next on a specific process. *Stochastic* cycle times vary randomly from part to part. Since production lines are usually designed to produce similar or identical mature products in large quantities, they usually perform their tasks with a low level of variability when they are operational. Therefore, in our model it is assumed that there is no variation in the processing time.

A *machine failure* occurs when a machine in the production line ceases to run due to a malfunction. There are two important categories of failures. *Operation dependent failure*, or ODFs, occur as a function of the amount of material processed since the last time a machine was repaired. *Time dependent failures*, or TDFs, occur as a function of the time a machine runs since its last failure [8]. There is a significant impact on flow line model estimate based on whether one assumes TDFs or ODFs. However, casual observation have shown that TDFs are not very prevalent. It is assumed that the machines in the proposed model operate with ODFs.

Quality issues, rework and yield, customer satisfaction, environmental impact, machine maintenance polices, and other management issues are not addressed in this thesis.

1.3 Literature Review

Buzacott [5] modeled a two-machine one-finite-buffer production system as a Markov process. The model assumed that machines are unreliable and subject to fail under operations dependent failures (ODFs). Also the model assumed that operations times of the two machines were identical and deterministic. Failure and repair times were geometrically distributed. As a justification for using the ODF assumption, Buzacott and Hanifin [4] provided data from Chrysler Corporation that showed that the majority of failures are ODFs. Production lines longer than two machines are not in general analytically tractable. However, that most factories have production lines with many machines. One method which has proved practically robust for production line with finite buffers and unreliable machines is *decomposition* which was introduced by Gershwin [7]. This model evaluates the throughput and average buffer levels of the line with tandem configurations. Later this model was extended to assembly/disassembly network by Gershwin [1]. Under the same assumption, a decomposition method for lines with a loop configuration was introduced by researchers [10]. However, all the models described so far were limited to a single part type. Nemec [12] formulated a deterministic single failure multi-part type transfer line processing two-part type. However, this formulation worked only for small two-part type lines, and there is no clear way of generalizing his equations for longer lines and more than two-part types. Tolio [13] proposed a way of analyzing two-part type lines with multiple failure modes. However, the Markov model of the two-machine line, the basic building block of the decomposition, for this model was complex. Moreover, in Tolio's model, the number of equations to solve increase exponentially as the production line becomes longer.

1.4 Thesis Outline

This thesis is organized as follows. Chapter 2 reviews the analysis of a single-part type production line using decomposition. Since the modeling and analysis of the multiple-part type production line follows the assumptions made for the single-part type line, it is critical to review this system first. Chapter 3 introduces a Markov model of a processing line with multiple-part types. The decomposition of the long line into smaller, tractable two-machine lines is also discussed. Chapter 4 presents the analysis of the Markov chain for two-machine lines. As a first step in modeling a production line with more than one part type, we restrict ourselves to the case where there are three part types. Type 1, Type 2, and Type 3 decompositions are introduced in Chapter 5, Chapter 6, and Chapter 7. Then in Chapter 8, we propose a way to expand the three-part-type line to general line processing more then three part types. An algorithm to solve the decomposition is presented in Chapter 9, as are numerical results concerning the accuracy of the decomposition. The system behavior of the three-part type processing line under different supply and demand scenarios is also investigated in Chapter 10. In Chapter 11, we present a decomposition for a re-entrant production line. In developing equations for the building blocks for the re-entrant production line, we modify the decomposition model that has been created for the multiple-part-type line. Semiconductor device and liquid crystal display (LCD) fabrication processes are characterized as a re-entrant process, in which a similar sequence of processing step is repeated several times. Chapter 12 summarizes this work and proposes future research.

Chapter 2

Review of Production Lines Processing a Single Part Type

2.1 Single-Part Type Production Line

This section reviews the analysis of a production line processing a single part type particularly for deterministic processing model introduced by Gershwin [7]. The basic model of a production line consists of *machines* and *buffers*. Figure 2.1.1 shows a diagram of a production line processing a single part type. In the figure, machines and buffers are depicted by squares and circles, respectively. A machine, denoted by the letter M, is a processing unit where parts are processed. Between the machines, there are buffers, denoted by the letter B, where *in-process inventory* or *in-process WIP* are stored. In the model, it is assumed that the buffer space is limited. A part flows though the machines and buffers linearly in that it begins at the first machine and then is sent to the next downstream buffer immediately after the completion of the process in the first machine. Then the part is loaded to the second machine and so on. In the model, it is assumed that a part always enters to the line through the

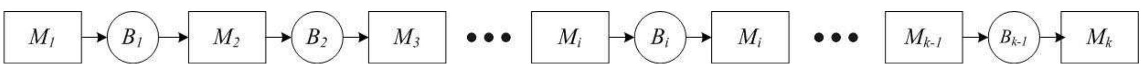

Figure 2.1.1: A production line processing a single-part type

first machine and travels all the way down to the last machine. There is no re-entrant flow in the line and parts are never created no destroyed. Therefore, once a part enters into the machine M_1, it moves on to B_1, M_2, B_2, M_3 and so on to M_K and then it exits the line.

We assume that the production times of each machine are deterministic and identical and this time is unity. However, machines are subject to failures and the failure quantities and repair quantities are not necessarily the same. The model also assumes that the machines have operation dependent failures and therefore, they can only fail while working on a part. It is further assumed that the machines have geometrically distributed up and down times. That is, in any given time step, each machine, M_i has a fixed probability p_i of failing if it is working on a part and has a fixed probability r_i of being repaired, if it is down.

The size of buffer B_i is denoted by N_i and the level of buffer B_i at time t is denoted by $n_i(t)$. We assume that the buffer is finite $0 \leq n_i(t) \leq N_i$ for all time t. Note that the buffer sizes are not required to be identical. In this configuration of the line, a machine can be idle due to *blockage* (i.e. $n_i = N_i$) or *starvation* (i.e. $n_i = 0$). Note that a machine not processing a part due to failure is different from machine being idle due to starvation or blockage. If a machine is starved or blocked, it is forced to stop processing although it is up and operational.

2.2 Two-Machine One-Buffer Building Block

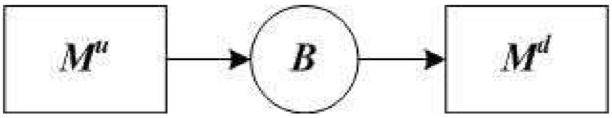

Figure 2.2.1: Two-machine one-buffer building block

In the production line analysis, typical questions that need to be answered are "What is the throughput rate?" and "Which machine is the bottleneck?", and "How much inventory is stored in the line?" Answering these questions is not trivial and providing the exact answers for the general case may not be possible. However, the exact analysis for a small system is possible. Let us consider a system consisting of two processing machines and one buffer as shown in Figure 2.2.1. In the line, a finite buffer of size N is placed between the machines. The first machine, denoted by M^u, is called the *upstream machine*, and the second machine, denoted by M^d, is called the *downstream machine*. All the assumptions described for the long production line also apply to this line. That is, the machines operate at the same rate and they can be down only when they are operating on a part. The machine M^u can fail with probability p^u while it is processing a part and it can be repaired with probability of r^u when it is down. Likewise, M^d has machine parameters of p^d and r^d. Let α^u denote the state of the machine M^u. Let us define α^u to be 1 if M^u is up and α^u to be 0 if M^u is down. Similarly, α^d represents the state of machine M^d. The size of the buffer is N and therefore, the level of buffer n satisfies $0 \leq n \leq N$. Then the state of the two-machine one-buffer line can be completely characterized as $s = (n, \alpha^u, \alpha^d)$. Let E^u be the probability that M^u makes a part in a given time step and E^d be the probability that M^d makes a part in a given time step. These quantities are determined by r^u, p^u, r^d, p^d, and N. Again, it is assumed that no part is created nor discarded in the middle of the line, that is, once a part enters M^u , the part always leaves from M^d. Therefore, $E^u = E^d = E$, where E is called the *production rate* or *throughput rate* of the line. That is, E is the probability that the line produces a part in any given time step. Since the operation time is 1, E is also the production rate or throughput rate.

This two-machine one-buffer system can be modeled as a discrete-time, discrete-state Markov process and exact values of the production rate and average buffer level can be calculated. Also other interesting performance measures such as the probability of blockage of M^u, denoted by P_b, and the probability of starvation of M^d, denoted by P_s, can be evaluated.

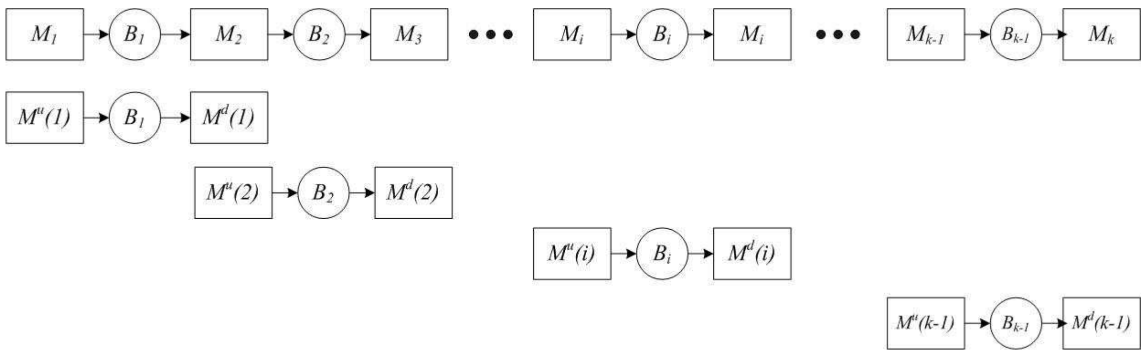

Figure 2.3.1: A decomposed production line processing a single-part type

2.3 Decomposition

Unfortunately, production lines longer than two machines are not analytically tractable. Since the most factories have production lines with many machines, a method that can analyze a longer production line is needed. Developing a method that evaluates exact solutions for longer lines may not be possible, therefore a method that approximates the solution is useful. One method, which has proved be robust, accurately approximating performance measures for longer production lines with unreliable machines and finite buffers is *decomposition*. This method decomposes the long production line into tractable two-machine one-buffer production lines, called the *building blocks*. Each individual building block is analyzed and then the building blocks are patched back together in a way that the performance measures of the longer line are approximated by those of the building blocks.

The idea of the decomposition can be better explained with the decomposed line shown in Figure 2.3.1. Suppose that there is an observer in each buffer in the line. Each observer watches the flow of parts entering and leaving the buffer that she is in. Although the observer watches the flow of the actually line, she is informed and believes that the buffer she is watching is not part of the long production line, but it is the buffer of a small line consisting of two machines and one buffer.

For any buffer in the line, the inflow may stop for two reasons: the machine directly upstream of the buffer may have failed, or some machine further upstream may have failed, starving the line downstream of it. Likewise, the outflow may stop for two reasons: the machine directly downstream of the buffer may have failed, or a machine further downstream may have failed, blocking the line upstream of it. However, the observer is unable to determine the reason that inflow or out flow has ceased. The parameters of the machines of the building block are found by the decomposition so that judging by the behavior of the inflow and outflow, the observer in the buffer is unable to tell whether of not she is in a buffer of the real line or building block. The decomposition is based on equations that relate the parameters of the building blocks to those of the neighboring building blocks and their performance. The derivations for the equations for the building blocks and detailed analysis of the decomposition method is described in Gershwin [8].

Chapter 3

Three-Part-Type Processing Line

3.1 Notation

Figure 3.1.1 represents a production line processing three different part types. The line consists of two kinds of components: machines M_i, denoted by the squares; and finite-capacity storage buffers $B_{i,j}$ for work in process inventory, denoted by the circles. Let us define K to be the number of machines that are processing two different part types in the line. At the beginning and end of the line, there are supply machines $M_{0,1}$, $M_{0,2}$, and $M_{0,3}$, and demand machines, $M_{K+1,1}$, $M_{K+1,2}$, and $M_{K+1,3}$.

Machines $M_{0,j}$ and $M_{K+1,j}$ process only Type j parts. Each machine, other than the supply and demand machines, processes all part types. We assume that there is no set-up time incurred when the machines switch production from one part type to another. When M_i completes work on a part, it sends the part to a buffer downstream of the machine. Each part type has a distinct buffer after each machine. For instance, a Type 1 part processed at M_i would be sent to $B_{i,1}$ and a Type 2 part processed at the same machine would be sent to $B_{i,2}$.

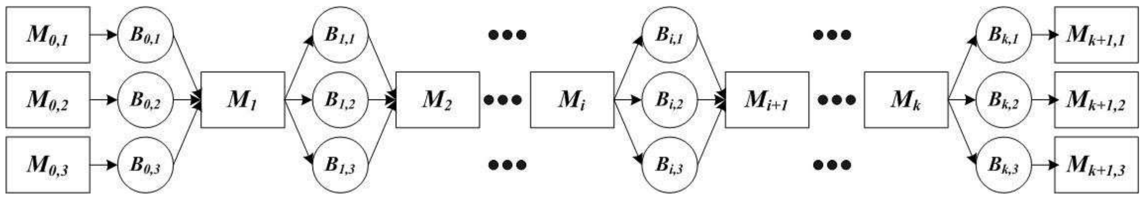

Figure 3.1.1: A two-part type production line

3.2 Assumptions and Approximations

It is assumed that all the machines in the line, including supply and demand machines, are unreliable. Let $\alpha(t)$ denote the state of a machine at time t. If $\alpha(t) = 1$, the machine is said to be *up* or *working*. If $\alpha(t) = 0$, the machine is said to be *down* or *failed*. We let $\alpha_{0,j}(t)$ denote the state of supply machine $M_{0,j}$ at the end of time t. For the demand machine, $M_{K+1,j}$, we let the corresponding state variables be $\alpha_{K+1,j}(t)$. For processing machine M_i, the state variable representing the state of the machine at the end of time t is written $\alpha_i(t)$. We make the assumption that all the machines in the line, including the supply and demand machines, have *homogeneous processing times*. That is, the lengths of time that parts spend in machines are fixed, known in advance, and the same for all the machines. For convenience, the processing times are assumed to be scaled to unity. Furthermore, we assume that the yield of all machines is 100%. That is, we do not model the scrapping or rework of parts.

We assume that all buffers, including the supply and demand buffers, have finite size. The size of buffer $B_{i,j}$ is denoted $N_{i,j}$, where i indicates the production stage, and j, represents the part type. We let buffers $B_{0,j}$ denote the supply buffers for Type j. Likewise, buffers $B_{K,j}$ denote the demand buffers for Type j. We denote the level of $B_{i,j}$ at the end of time t by $n_{i,j}(t)$. Therefore, $0 \leq n_{i,j}(t) \leq N_{i,j}$, for all (i, j), and for all $t \geq 0$. A machine is said to be *starved* for a given part type if the upstream buffer corresponding to that part type is empty. It is *blocked* for a given part type if the corresponding downstream buffer is full. We make the assumptions that the supply machines are never starved and the demand machines are never blocked.

As mentioned earlier, all machines in the line are assumed to be unreliable. We further assume that machines cannot fail if they are idle. This is called *operation dependent failures*. It means that the supply machines cannot fail if they are blocked and the demand machines cannot fail if they are starved. A processing machine cannot fail if it is either starved or blocked for the all the part types at the same time unit.

All machines are assumed to have geometrically distributed up and down times. We assume that the probability that M_i fails is the same, regardless of the part type the processing machine is working on. We let r_i represent the probability that M_i is up at time $t+1$, given it was down at time t. Likewise, p_i represents the probability that M_i is down at time $t+1$, given it was up and not blocked or starved at time t. For the supply machines, we let $r_{0,j}$ represent the probability that $M_{0,j}$ is up at time $t+1$, given it was down at time t. Also, $p_{0,j}$ represents the probability that $M_{0,j}$ is down at time $t+1$, given it was up *and* not blocked at time t. For the demand machines $M_{K+1,j}$, the corresponding parameters are written $r_{K+1,j}$ and $p_{K+1,j}$. For M_i, the machine parameters defines the state transitions as follows:

$$\begin{aligned} r_i &= Pr\left[\alpha_i(t+1) = 1 | \alpha_i(t) = 0\right] \\ p_i &= Pr\left[\alpha_i(t+1) = 0 | \right. \\ &\qquad\qquad \{\alpha_i(t) = 1 \cap n_{i-1,1}(t) > 0 \cap n_{i,1}(t) < N_{i,1}\} \cup \\ &\qquad\qquad \{\alpha_i(t) = 1 \cap n_{i-1,2}(t) > 0 \cap n_{i,2}(t) < N_{i,2}\} \cup \\ &\qquad\qquad \left.\{\alpha_i(t) = 1 \cap n_{i-1,3}(t) > 0 \cap n_{i,3}(t) < N_{i,3}\}\right] \\ &\text{for } i = 1, \ldots, K \end{aligned} \tag{3.2.1}$$

Likewise, for the supply and demand machines, the machine parameters are defined as:

$$
\begin{aligned}
r_{0,j} &= Pr\left[\alpha_{0,j}(t+1) = 1 | \alpha_{0,j}(t) = 0\right] \\
p_{0,j} &= Pr\left[\alpha_{0,j}(t+1) = 0 | \alpha_{0,j}(t) = 1 \cap n_{0,j}(t) < N_j\right] \\
&\quad \text{for } j = 1, 2, 3
\end{aligned}
$$

$$
\begin{aligned}
r_{K+1,j} &= Pr\left[\alpha_{K+1,j}(t+1) = 1 | \alpha_{K+1,j}(t) = 0\right] \\
p_{K+1,j} &= Pr\left[\alpha_{K+1,j}(t+1) = 0 | \right. \\
&\qquad\quad \left. \alpha_{K+1,j}(t) = 1 \cap n_{K,j}(t) > 0\right] \\
&\quad \text{for } j = 1, 2, 3
\end{aligned}
$$

We define e_i to be *isolated production rate* of machine M_i. Then e_i is the production rate of M_i if the machine were never impeded by blockages or starvations. It is given by

$$
e_i = \frac{r_i}{r_i + p_i} \tag{3.2.2}
$$

It is the same s the isolated efficiency because the operation time is 1. In order to derive the decomposition, we need to make a crucial approximation. We will assume that the probability that a machine M_i is simultaneously starved and blocked for a given part type is negligible. That is, we assume that

$$
\begin{aligned}
&Pr[n_{i,j}(t) = 0 \cap n_{i+1,j}(t) = N_{2,j}] \approx 0 \\
&\quad i = 1 \ldots K \\
&\quad j = 1, 2, 3
\end{aligned} \tag{3.2.3}
$$

We justify this approximation with the following argument. In order for the machine to be both starved and blocked for the same part type simultaneously, it is necessary that at some point, the machine had exactly one part in the upstream buffer, and exactly one space in the downstream buffer. At the same time, the upstream machine must be unable to process parts to place in its downstream buffer, and the downstream machine must be unable to process parts, depleting the stores of its upstream buffer. Since the probability of a machine failing is usually small — on the order of 0.01 in a time step — the probability that all three of these occurrences happen at the same time is likely to be quite low. In fact, testing this hypothesis using discrete event simulation has shown that the approximation holds in many systems with moderate sized buffers. Also, this approximation works successfully with the single-part-type decomposition method of Gershwin [7].

3.3 Part Type Priority Policy

Since each machine in the production line must choose which part to work on when it has a choice, we are required to state a policy by which that choice is made. Our assumption is that each machine will work on Type 1 parts whenever the machine is up, the upstream buffer for Type 1 parts is not empty, and the downstream buffer for Type 1 parts is not full. Each machine will only work on Type 2 parts if it is up, and either blocked or starved for Type 1 parts, and not blocked or starved for Type 2 parts. If a machine is blocked or starved for Type 1 and Type 2, it seeks to work for Type 3. In other words, when the machine has a choice, it always chooses the highest priority part type possible.

3.4 Production Rate

Let us denote the production rate of Type j parts at M_i by $E_{i,1}$. This is the fraction of time that M_i is working on Type 1 parts. We know that M_i will make a Type 1 part at the end of time step $t+1$ if M_i is not starved for Type 1 parts at time t, M_i is not blocked for Type 1 parts at time t, and M_i is up at the end of time step $t+1$. This probability is expressed as follows:

$$E_{i,1} = Pr\left[\alpha_i(t+1) = 1 \cap n_{i-1,1}(t) > 0 \cap n_{i,1}(t) < N_{i,1}\right] \tag{3.4.1}$$

Let the quantity $E_{i,2}$ denote the production rate of Type 2 parts. This is the fraction of time that M_i is working on Type 2 parts. From our assumptions, we know that M_i will make a Type 2 part at time $t+1$, if M_i is either blocked or starved for Type 1 at time t; M_i is not starved or blocked for Type 2; and M_i is up at the end of time $t+1$. This is:

$$\begin{aligned} E_{i,2} \quad = \quad & Pr[\alpha_i(t+1) = 1 \cap \\ & (n_{i-1,1}(t) = 0 \cup n_{i,1}(t) = N_{i,1}) \cap \\ & (n_{i-1,2}(t) > 0 \cap n_{i,2}(t) < N_{i,2})] \end{aligned} \tag{3.4.2}$$

Similarly, for the efficiency for Type 3,

$$\begin{aligned} E_{i,2} \quad = \quad & Pr[\alpha_i(t+1) = 1 \cap \\ & (n_{i-1,1}(t) = 0 \cup n_{i,1}(t) = N_{i,1}) \cap \\ & (n_{i-1,2}(t) = 0 \cup n_{i,2}(t) = N_{i,2}) \cap \\ & (n_{i-1,3}(t) > 0 \cap n_{i,3}(t) < N_{i,3})] \end{aligned} \tag{3.4.3}$$

In steady state, because of conservation of flow, we require that each machine in the line makes the same number of parts — Type 1, Type 2, and Type 3 combined. If we denote the throughput of the demand machine for Type j parts by $E_{K+1,j}$, and the supply machine of Type j parts by $E_{0,j}$, then we must have

$$E_{0,j} = E_{1,j} = E_{2,j} = \ldots = E_{i,j} = E_{K+1,j},$$
$$\text{for } j = 1, 2, 3$$

3.5 Basic Idea of Decomposition

We intend to break down the larger system into analytically tractable two-machine lines, which are called *building blocks.* The building blocks capture the local behavior of the long line, as seen by an observer in a buffer, by choosing appropriate parameters of machines This decomposition procedure is represented in Figure 3.5.1. As discussed earlier, the idea is to fool an observer in a buffer in the long, multi-part type processing line into thinking he is in a two-machine line. In the figure, the inflow and outflow behavior of material that an observer in buffer $B_{i,1}$ could see is modeled by the two-machine, one-part line $L(i, 1)$.

Close observation of the dynamics of the long line, however, shows the necessity for a new two-machine line model. The reason is as follows. The building block used in the single-part-type line decomposition is not adequate here. Suppose that we take the point of view of an observer in $B_{i,1}$. We misinform this observer: we lead him to believe that he is watching the flow in the only buffer in a two-machine, one-buffer, one-part type system. Let us assume that the observer sees that the outflow from his buffer has ceased, but the inflow has not. Eventually, unless the outflow resumes or the inflow ceases, $B_{i,1}$ will fill up. According to our scheduling rule, M_i will immediately begin making Type 2 parts, if it is able to. Suppose it does, and that M_i fails while making a Type 2 part. Now suppose that while M_i is down, the outflow from $B_{i,1}$ begins again. Then the sequence of events that the observer will see are that the outflow ceased, the buffer filled up, but when the outflow began again and the buffer was not full, the inflow did not begin. As far as the observer in the buffer is concerned, the machine upstream of him failed while it was blocked.

There is subtlety here. While this apparent idleness failure is behavior that an observer in a buffer sees, the real machines do not fail when they are idle. It only appears to the observer that the machine has failed during an idle period because the observer believes that he is in a two-machine, one-part type line. Therefore, while in the previous model in [7] it is assumed that the pseudo-machines in the two-machine sub-lines had operation dependent failures, we must relax that assumption for the two-machine sub-lines in the multiple-part type case. Thus, a new two-machine line model is in order. We present a discrete-time, discrete-state Markov model of precisely such a line in Chapter 4.

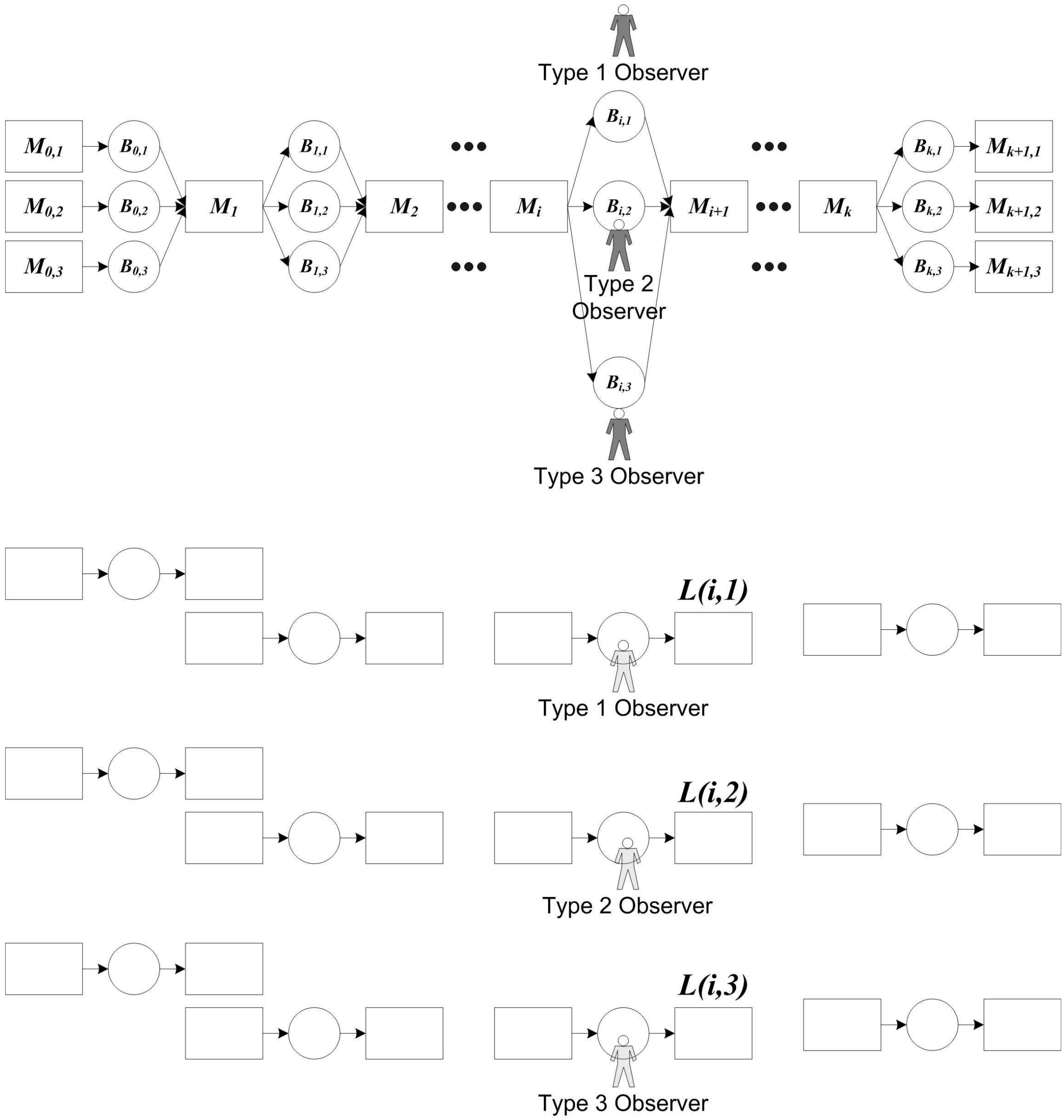

Figure 3.5.1: The decomposition of a line into building blocks

Chapter 4

Two-Machine Lines with Idleness Failures

4.1 Idleness Failure and Failure-Mode Change

As discussed in Chapter 3, in order to decompose the Markov chain model of the two-part-type processing line, we need a new two-machine line building block. The two-machine-line presented here is similar to the deterministic processing time with multiple-failure model described by Tolio [13]. However, the model presented here differs in that the machines are allowed to fail while they are idle.

4.1.1 Idleness Failure

We use the concept of multiple failure modes in constructing the building block. As in the Tolio decomposition, the machines of the building block can fail in *local failure modes* and *remote failure modes*. A local failure mode is a failure of a machine adjacent to the buffer in the real line. A remote failure mode is a failure introduced to account for the effect of a failure of a machine not adjacent to the buffer. In this model, the machines in the building block are no longer restricted to failing only if they are not blocked and not starved. Since a machine in the two-machine line can fail while it is idle — starved or blocked — we call the line *a building block with idleness failure.*

4.1.2 Failure Mode Change

In order to explain the failure mode change, let us consider the following events.

1. $M_{i'}$ is failed.

2. M_i is blocked. $i < i'$

3. M_i begins to process Type 2 parts.

4. M_i gets failed.

Suppose that M_i is blocked for Type 1 due to a failure of a machine in the downstream, for example, $M_{i'}, i < i'$. In this case, the downstream machine, $M^d(i-1,1)$, is failed in a remote failure mode. Now, M_i begins working on Type 2 parts. However, while M_i is working on Type 2, it gets failed. At this moment, $M^d(i-1,1)$ will switch from the original failure mode to a new failure mode. We call this shifting mode change *a failure mode change*. There are two important observations about failure-mode changes. The first is that a failure mode can only change to a mode corresponding to a machine which is closer to the observer. The reason for this is that the initiating failure corresponds to a real failure of some machines, which has propagated by means of starvation or blockage to the observer's location.

4.2 Two-Machine-Line Notation and Parameters

The building blocks are illustrated in Figure 4.1.1. The Markov chain transition graphs of each machine are shown. As is our convention, the machines are denoted by squares, and the buffer by circles. We denote the upstream machine by M^u, and the downstream machine by M^d. We denote the size of the intermediate buffer by N, and the current level of the intermediate buffer by n. It follows that $0 \leq n \leq N$. We define the state of the two-machine line to be $s = (n, \alpha^u, \alpha^d)$. α^u is Δ_i if M^u is down at mode i, and Υ^u if M^u is up. α^d is Δ_j^d if M^d is down at mode j, and Υ^d if M^d is

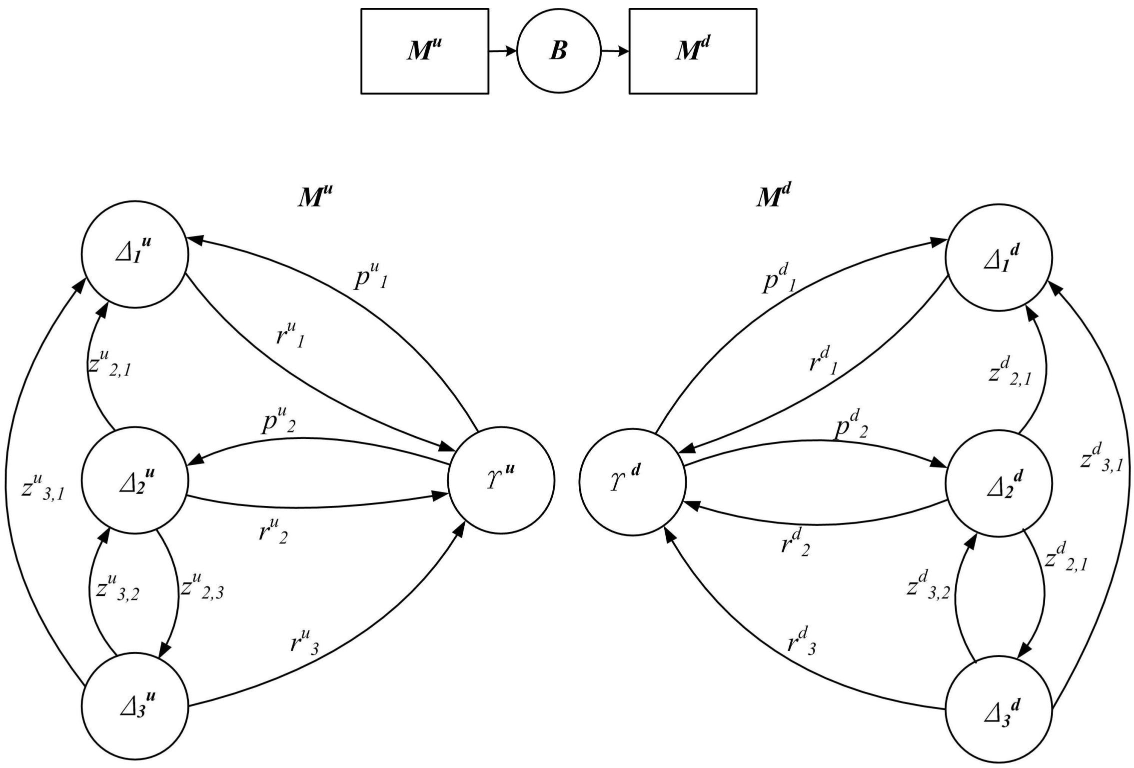

Figure 4.1.1: Example of Markov states with three failure modes

up.

Material flows into the upstream machine from an infinite supply, is processed by the machine, and when processing is complete, the material is placed into the buffer until it is processed by the downstream machine. Upon finishing being processed by the downstream machine, the part leaves the line. We assume that there is always room for the downstream machine to unload a part it has just completed processing. We make the assumption that there is only one class of parts produced by the line, and that the production time at each of the machines is identical, and equal to one.

The machines are unreliable and can fail in multiple failure modes. We assume that the machines can fail while they are either operating on a part or idle, but we do not assume that the probabilities of failure are identical. In particular, we assume that the probability that M^u fails into mode i while it is working on a part, given it is not blocked, is p_i^u, and the probability that it fails into mode j while it is blocked is q_j^u. Note that we do *not* assume that there is a new failure mode, but only that there is a new way of reaching the failure mode. We define the quantities p_i^d and q_j^d for M^d similarly. Finally, we denote the probabilities that M^u and M^d are repaired while they are down at failure mode j by r_j^u and r_j^d, respectively.

The quantity $z_{i,i'}^u$ is the probability of the upstream machine changing from down mode i to down mode i'. The expression $z_{j,j'}^d$ represents the probability that the downstream machine changes from down mode j to down mode j'. Defining $\alpha^\dagger(t)$ as the state (up state or down state) of a machine $\dagger$ at time t (where $\dagger$ is either u or d for upstream or downstream), then we can define r, p, q, and z as

$$\begin{aligned}
r_{j,i}^\dagger &= Pr[\alpha^\dagger(t+1) = \Upsilon_i^\dagger \mid \alpha^\dagger(t) = \Delta_j^\dagger] \\
p_{i,j}^\dagger &= Pr[\alpha^\dagger(t+1) = \Delta_j^\dagger \mid \alpha^\dagger(t) = \Upsilon_i^\dagger \text{ and } n(t) < N] \\
q_{i,j}^\dagger &= Pr[\alpha^\dagger(t+1) = \Delta_j^\dagger \mid \alpha^\dagger(t) = \Upsilon_i^\dagger \text{ and } n(t) = N] \\
z_{j,j'}^\dagger &= Pr[\alpha^\dagger(t+1) = \Delta_{j'}^\dagger \mid \alpha^\dagger(t) = \Delta_j^\dagger] \\
& \text{for } \dagger = u \text{ and } d
\end{aligned}$$

We also define P^u and P^d such that

$$P^u = \sum_{j=1}^{J} p_j^u \qquad \text{and} \qquad P^d = \sum_{l=1}^{L} p_l^d$$

where J and L are the numbers of failure modes for the upstream machine and downstream machine respectively. The set of parameters p_j^u and p_l^d must be such that $P^u < 1$ and $P^d < 1$. We define

$$Q^u = \sum_{j=1}^{J} q_j^u \qquad \text{and} \qquad Q^d = \sum_{l=1}^{L} q_l^d$$

and again, the set of parameters q_j^u and q_l^d must be such that $Q^u < 1$ and $Q^d < 1$.

4.3 Transition Equations

4.3.1 Transient states

In the steady-state, the probability of a two-machine line being on a transient state is zero. Since idleness failure is considered, there are fewer transient states than in Gershwin and Tolio's model. The following are transient states:

$P(0, \Upsilon^u, \Delta_l^d)$ **and** $P(0, \Upsilon^u, \Upsilon^d)$: Since the upstream machine is up, it puts a part at the beginning of a time step and therefore the buffer will have the part at the end of the time step. There is no transition to this state.

$P(N, \Delta_j^u, \Upsilon^d)$ **and** $P(N, \Upsilon^u, \Upsilon^d)$: Since the downstream machine is up it takes out a part at the beginning of a time step and therefore, buffer will have a part always less then N at the end of the time step. There is no transition to these states.

4.3.2 Non-transient states

A *internal state* is the state where buffer level satisfies $2 \leq n \leq N-2$. The system can get from $P(n, \Delta_j^u, \Upsilon^d)$ to $P(n, \Delta^u, \Delta^d)$ if the upstream machine is repaired from failure mode j but the downstream machine remains down. The probability of this is $r_j^u(1-P^d)$. It can move from $P(n, \Upsilon^u, \Delta_l^d)$ to $P(n, \Delta^u, \Delta^d)$ if the upstream machine is not repaired but the downstream machine is repaired from mode l. Therefore the transition probability is $(1-Pu)r_l^d$. The transition probability from $P(n, \Delta_j^u, \Delta_l^d)$ to $P(n, \Upsilon^u, \Upsilon^d)$ is $r_j^u r_l^d$ since the upstream and downstream machines are repaired from the down mode j and l, respectively. Likewise, the transition probability from $P(n, \Upsilon_j^u, \Upsilon_l^d)$ to $P(n, \Upsilon^u, \Upsilon^d)$ is $(1-P^u)(1-P^d)$ since the upstream and downstream machines remain up. No other transitions from non-transient states are possible.

Consequently, the first equation is:

$$\begin{aligned} P(n, \Upsilon^u, \Upsilon^d) \quad &= \sum_{j=1}^{J}\sum_{l=1}^{L} P(n, \Delta_j^u, \Delta_l^d) r_j^u r_l^d \\ &+ \sum_{j=1}^{J} P(n, \Delta_j^u, \Upsilon^d) r_j^u (1 - P^d) \\ &+ \sum_{l=1}^{L} P(n, \Upsilon^u \Delta_l^d)(1 - P^u) r_l^d \\ &+ P(n, \Upsilon^u \Upsilon^d)(1 - P^u)(1 - P^d) \end{aligned} \tag{4.3.1}$$

The other equations are similar and they are represented in the Appendix. The lower boundary states ($n \leq 1$) and upper boundary ($n \geq N - 1$) are also shown in the Appendix.

4.3.3 State Classification

For our convenience and later use, we define the following quantities. For the upstream machine,

- $W^u \quad P(\alpha^u = \Upsilon^u, n < N)$
- $X_j^u \quad P(\alpha^u = \Delta_j^u, n < N)$, (*remote failure $j = 1 \ldots J - 1$, real failure $j = J$).
- $D_b^u \quad P(\alpha^u = \Delta_J^u, n = N, \alpha^d = \Delta_l^d)$, (*only real failure Δ_J^d).
- $P_b \quad P(\alpha^u = \Upsilon^u, n = N, \alpha^d = \Delta_l^d)$

These states are mutually exclusive and collectively exhaustive. Therefore, we can state

$$W^u + \sum_{j=1}^{J} X_j^u + \sum_{l=1}^{L} Db_l^u + \sum_{l=1}^{L} Pb_l = 1 \tag{4.3.2}$$

For the downstream machine,

- $W^d \quad P[\alpha^d = \Upsilon^d, n > 0q]$

- X_l^d $P[\alpha^d = \Delta_l^d, n > 0]$, (*remote failure $l = 2 \ldots L$, real failure $l = 1$).
- D_s^d $P[\alpha^u = \Delta_j^u, \alpha^d = \Delta_1^d, n = 0]$, (*only real failure Δ_1^d).
- P_s $P[\alpha^u = \Delta_j^u, \alpha^d = \Upsilon^d, n = 0]$

These states are mutually exclusive and collectively exhaustive. Therefore, we can states such that

$$W^d + \sum_{l=1}^{L} X_l^d + \sum_{j=1}^{J} Ds_l^d + \sum_{j=1}^{J} Ps_j = 1 \tag{4.3.3}$$

4.4 Performance Measures

The main performance measures of interest in the two-machine line are the *throughput, average buffer level, probability of starvation*, and *probability of blockage.*

4.4.1 Efficiency

The throughput of the upstream machine is the probability that the upstream machine is working in time step $t+1$, and not blocked at time t. This is,

$$E^u = Pr[\alpha^u(t+1) = \Upsilon^u \cap n(t) < N] \tag{4.4.1}$$

Similarly, the throughput of the downstream machine is the probability that the downstream machine is working at time $t+1$ and not starved at time t. This is,

$$E^d = Pr[\alpha^d(t+1) = \Upsilon^u \cap n(t) > 0] \tag{4.4.2}$$

Observe the equation in (4.4.1) that this expression has both time step $t+1$ and time step t in it. We proceed by conditioning on events occurring time step t to write (4.4.1) in terms of events occurring entirely in time step t. By doing so, we will be able to express the production rate of the upstream machine entirely in terms of the state probabilities, which are defined only for one time step.

Before we derive a single-time-step expression for the efficiency, notice that for every machine failure, there is also a repair. That is, for both the up-and downstream machines in the building block, the probability that a repair occurs at any time (the repair frequency) equals the probability that a failure occurs at any time(the failure frequency), consequently. For the upstream machine, that is expressed as

$$\begin{aligned} r_j^u(Pr[\alpha^u = \Delta_j^u \cap n < N] + Pr[\alpha^u = \Delta_j^u \cap n = N]) \\ = p_j^u Pr[\alpha^u = \Upsilon_j^u \cap n < N] + q_j^u Pr[\alpha^u = \Upsilon_j^u \cap n = N] \end{aligned} \tag{4.4.3}$$

Likewise, for the downstream machine,

$$\begin{aligned} r_l^d(Pr[\alpha^d = \Delta_l^d \cap n > 0] + Pr[\alpha^d = \Delta_l^d \cap n = 0]) \\ = p_l^d Pr[\alpha^d = \Upsilon_l^d \cap n > 0] + q_l^d Pr[\alpha^d = \Upsilon_l^d \cap n = 0] \end{aligned} \tag{4.4.4}$$

We can use (4.4.3) and (4.4.4) to derive expressions for the efficiencies for upstream and downstream machines defined in (4.4.1) and (4.4.2):

$$\begin{aligned} E^u &= Pr[\alpha^u(t+1) = \Upsilon^u \cap n(t) < N] \\ &= Pr[\alpha^u(t+1) = \Upsilon^u | \alpha^u(t) = \Upsilon^u \cap n(t) < N] \\ &\quad \times Pr[\alpha^u(t) = \Upsilon^u \cap n(t) < N] \\ &\quad + \sum_{j=1}^{J} (Pr[\alpha^u(t+1) = \Upsilon^u | \alpha^u(t) = \Delta_j^u \cap n(t) < N] \\ &\quad \times Pr[\alpha^u(t) = \Delta_j^u \cap n(t) < N]) \\ &= (1 - P^u) Pr[\alpha^u(t) = \Upsilon^u \cap n(t) < N] \\ &\quad + \sum_{j=1}^{J} r_j^u Pr[\alpha^u(t) = \Delta_j^u \cap n(t) < N] \end{aligned}$$

If we apply the fact that the repair frequency equals failure frequency (4.4.3), then E^u is

$$\begin{aligned}
E^u &= Pr[\{\alpha^u(t) = \Upsilon^u\} \cap \{n(t) < N\}] \\
&+ Q^u Pr[\{\alpha^u(t) = \Upsilon^u\} \cap \{n(t) = N\}] \\
&- \sum_{j=1}^{J} r_j^u Pr[\{\alpha^u(t) = \Delta_j^u\} \cap \{n(t) = N\}]
\end{aligned}$$

Since

$$\begin{aligned}
Pr[\{\alpha^u(t) = \Upsilon^u\} \cap \{n(t) = N\}] &= \sum_{l=1}^{L} Pr(N, \Upsilon^u, \Delta_l^d) \\
Pr[\{\alpha^u(t) = \Delta_j^u\} \cap \{n(t) = N\}] &= \sum_{l=1}^{L} Pr(N, \Delta_j^u, \Delta_l^d)
\end{aligned}$$

E^u can be

$$\begin{aligned}
E^u &= \sum_{n=0}^{N-1} Pr(n, \Upsilon^u, \Upsilon^d) + \sum_{n=0}^{N-1} \sum_{l=1}^{L} Pr(n, \Upsilon^u, \Delta_j^d) \\
&+ Q^u \sum_{l=1}^{L} Pr(N, \Upsilon^u, \Delta_l^d) - \sum_{j=1}^{J} r_j^u \sum_{l=1}^{L} Pr(N, \Delta_j^u, \Delta_l^d)
\end{aligned} \tag{4.4.5}$$

Note that if we were not considering idleness failure, Q^u would be zero and the last term of (4.4.5) also would be zero, since there will be no transition to the state $(N, \Delta_j^u, \Delta_l^d)$. Therefore, only the first two terms of (4.4.5) would be left. In this case, the efficiency is the probability that the upstream machine is up and buffer is not full. This expression is the same as the one described by Gershwin [8] in which the model does not consider idleness failure. Similarly for the downstream machine:

$$\begin{aligned} E^d &= Pr[\{\alpha^d(t) = \Upsilon^d\} \cap \{n(t) > 0\}] \\ &+ Q^d Pr[\{\alpha^d(t) = \Upsilon^d\} \cap \{n(t) = 0\}] \\ &- \sum_{1=1}^{L} r_l^d Pr[\{\alpha^d(t) = \Delta_j^d\} \cap \{n(t) = 0\}] \end{aligned}$$

Since,

$$\begin{aligned} Pr[\{\alpha^d(t) = \Upsilon^d\} \cap \{n(t) = 0\}] &= \sum_{j=1}^{J} Pr(0, \Delta_j^u, \Upsilon^d) \\ Pr[\{\alpha^d(t) = \Delta_j^d\} \cap \{n(t) = 0\}] &= \sum_{j=1}^{J} Pr(0, \Delta_j^u, \Delta_l^d) \end{aligned}$$

, we can rewrite E^d as

$$\begin{aligned} E^d &= \sum_{n=1}^{N} Pr(n, \Upsilon^u, \Upsilon^d) + \sum_{n=1}^{N}\sum_{j=1}^{J} Pr(n, \Delta_j^u, \Upsilon^d) \\ &+ Q^d \sum_{j=1}^{J} Pr(0, \Delta_j^u, \Upsilon^d) - \sum_{l=1}^{L} r_l^d \sum_{j=1}^{J} Pr(0, \Delta_j^u, \Delta_l^d) \end{aligned} \tag{4.4.6}$$

Note that in the expressions (4.4.6), if we were not consider the idleness failure, only the first two terms would be left. In this case, the efficiency is the probability that the downstream machine is up and there is at least one part in the buffer.

4.4.2 Average buffer level, probability of blockage, and probability of starvation

The average buffer level is denoted $\bar{n}$, and is given by

$$\bar{n} = \sum_{n=0}^{N} nP(n, \Upsilon^u, \Upsilon^d) + \sum_{j=1}^{J}\sum_{n=0}^{N} nP(n, \Delta_j^u, \Upsilon^d) \\ + \sum_{l=1}^{L}\sum_{n=0}^{N} nP(n, \Upsilon^u, \Delta_l^d) + \sum_{j=1}^{J}\sum_{l=1}^{L}\sum_{n=0}^{N} nP(n, \Delta_j^u, \Delta_l^d) \quad (4.4.7)$$

The probability that the downstream machine is starved, denoted P_s, is the probability that the machine is up, and there is no part in the buffer. This is given by

$$P_s = P(\alpha^u = \Upsilon^u, \alpha^d = \Delta_l^d, n = N) \quad (4.4.8)$$

The probability that the upstream is blocked, denoted P_b, is the probability that the machine is up, and that the intermediate buffer is full. This is expressed as

$$P_b = P(\alpha^u = \Delta_j^u, \alpha^d = \Upsilon^d, n = 0) \quad (4.4.9)$$

4.5 Solution Algorithm

In this section, we propose a numerical solution algorithm to solve the transition equations expressed in Section 4.3. We denote the number of states by η and it is

$$\eta = (N+1)(J+1)(L+1)$$

,where J and L are the total number of the down states for M^u and M^d, respectively.

The steady state transition equations of the system has a form

$$p = Ap \quad (4.5.1)$$

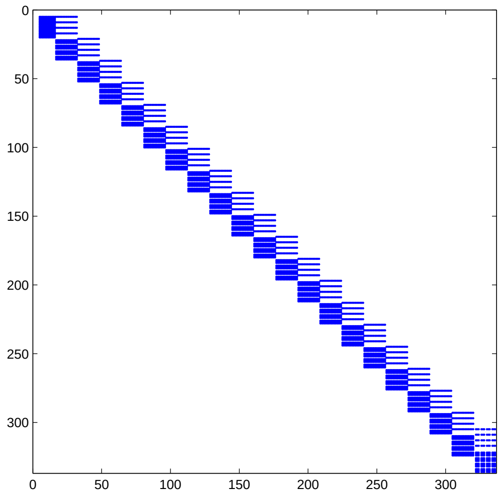

Figure 4.5.1: Example of a sparsity pattern $J = 3$ and $L = 3$

$$v^T p = 1 \tag{4.5.2}$$

where

- p is an unknown n-vector.
- A is an $\eta \times \eta$ matrix. The rank if $A - I$ is $\eta - 1$.
- v is an η-vector, each of whose elements is 1.

Note that the matrix, A, comes from the transition equations in the Appendix.

We know that p is the steady-state probability distribution of a Markov process and A is the transition matrix. The basic method of solving the linear equation is iterating the equation

$$p(k+1) = Ap(k) \tag{4.5.3}$$

where $p(0)$ is chosen to satisfy (4.5.2), until $p(k)$ converges. Practical convergence is defined, in this case, according to a criterion like

$$\delta_i(k+1) = |p_i(k+1) - p_i(k)| < \epsilon \qquad \text{for all } i \tag{4.5.4}$$

for some suitable ϵ.

Note that most elements in the matrix A are zero. Figure 4.5.1 shows an example of matrix A. In this matrix, a nonzero block is indicated with 1 and zero element is left blank. As the figure shows, the majority of elements are zero. We can take the advantage of the sparsity when we numerically solve the equation.

Chapter 5

Decomposition Analysis for Type 1

In this chapter, we introduce the derivation of the decomposition equations for Type 1. The building block that imitates the flow in $B_{i,j}$ denoted by $L(i,j)$. For example, $L(1,2)$ represents the first building block imitating the behavior of Type 2. The upstream and downstream machines in $L(i,j)$ are denoted by $M^u(i,j)$ and $M^d(i,j)$. We consider the perspective of an observer in a buffer for Type 1 part, $B_{i,1}$. Then we seek to capture the upstream and downstream behavior, as seen by the observer.

5.1 State and Parameter Definitions

The downstream machine, $M^d(i,1)$ $(i = 0, \ldots K-1)$, represents the Type 1 flow behavior as it leaves $B_{i,1}$. The downstream machine $M^d(i,1)$ is up when M_{i+1} is up and is not blocked for a Type 1 part. That is,

$$\Upsilon^d(i,1) = \{\alpha_{i+1} = 1 \cap n_{i+1,1} < N_{i+1,1}\}$$

That is, the observer in $B_{i,1}$ will see a part moving out of the buffer, when M_{i+1} is up is not blocked for Type 1. There are three cases in which the observer does not see a part moving out of the buffer and therefore believes that $M^d(i,1)$ is down. These cases are:

- $\Delta_1^d(i,1) = \{M_{i+1} \text{ is down}\}$

- $\Delta_1^d(i,1) = \{M_{i+1}$ is up but blocked due to a failure of a machine downstream$\}$

- $\Delta_1^d(i,1) = \{M_{i+1}$ is down and also blocked due to a failure of a machine down stream$\}$

Note that outflow from the buffer may stop for two reasons. First, the machine directly downstream of the buffer may have failed. This case is called the *local failure*. For the the other case, a machine further downstream have failed, blocking the upstream of it. The first down state, $\Delta_1^d(i,1)$ represents the failure of the machine immediate downstream of the buffer the observer is watching. The second down state, $\Delta_2^d(i,1)$, indicates a failure in a machine further downstream. This case is called the *remote failure*. Therefore, the first down state, $\Delta_1^d(i,1)$, is caused by the local failure, while the second down state,$\Delta_2^d(i,1)$, is caused by the remote failure.

The first and second down states can be seen in a building block for the single-part-type line decomposition. The last down state, $\Delta_3^d(i,1)$, exists only when machines are processing multiple part types. This down state is a mixture of local and remote failures. This failure occurs when the following sequence of failures occurs: Suppose M_{i+2} is down. If that failure persists long enough, it will make $B_{i+1,1}$ full, causing the blockage of M_{i+1}. Now, M_{i+1} is blocked for Type 1 parts, and $M^d(i,1)$ will be down in the second down state described above. While M_{i+1} is blocked for Type 1, it may work on a Type 2 part. Now suppose that M_{i+1} fails while it is working on a Type 2 part. At this moment, M_{i+2} is down and $B_{i+1,1}$ is full and M_{i+1} is also down. The observer in $B_{i,1}$ sees that its downstream machine is not only down but also blocked for Type 1 parts. In order to get into this down state, the remote failure must occur first before the local failure takes place. This is because the blockage cannot occur when the machine is down already. Therefore, from the observer's view point, the third down state, $\Delta_3^d(i,1)$, can be reached only from the second down state, $\Delta_2^d(i,1)$.

The states of $M_d(i,1)$ defined in terms of line L are

$$
\begin{aligned}
\Upsilon^d(i,1) &= \{\alpha_{i+1} = 1 \cap n_{i+1,1} < N_{i+1,1}\} \\
\Delta_1^d(i,1) &= \{\alpha_{i+1} = 0 \cap n_{i+1,1} < N_{i+1,1}\} \\
\Delta_2^d(i,1) &= \{\alpha_{i+1} = 1 \cap n_{i+1,1} = N_{i+1,1}\} \\
\Delta_3^d(i,1) &= \{\alpha_{i+1} = 0 \cap n_{i+1,1} = N_{i+1,1}\} \\
& \quad i = 0 \dots K-1
\end{aligned} \tag{5.1.1}
$$

Similarly, the state definitions for $M^u(i,1)$ are

$$
\begin{aligned}
\Upsilon^u(i,1) &= \{\alpha_i = 1 \cap n_{i-1,1} > 0\} \\
\Delta_1^u(i,1) &= \{\alpha_i = 0 \cap n_{i-1,1} > 0\} \\
\Delta_2^u(i,1) &= \{\alpha_i = 1 \cap n_{i-1,1} = 0\} \\
\Delta_3^u(i,1) &= \{\alpha_i = 0 \cap n_{i-1,1} = 0\} \\
& \quad i = 1 \dots K
\end{aligned} \tag{5.1.2}
$$

5.2 Equalities

For convenience, we define the following probabilities for a building block:

$$
\begin{aligned}
W^u(i,1) &= Pr[\Upsilon^u(i,1) \cap n_{i,1} > 0] \\
W^d(i,1) &= Pr[\Upsilon^d(i,1) \cap n_{i,1} > 0] \\
X^u_m(i,1) &= Pr[\Delta^u_m(i,1) \cap n_{i,1} < N_{i,1}], m = 1,2,3 \\
X^d_m(i,1) &= Pr[\Delta^d_n(i,1) \cap n_{i,1} > 0], n = 1,2,3 \\
P_b(i,1) &= Pr[\Upsilon^u(i,1) \cap n_{i,1} = N_{i,1}] \\
P_s(i,1) &= Pr[\Upsilon^d(i,1) \cap n_{i,1} = 0] \\
D_b(i,1) &= Pr[\Delta^u_1(i,1) \cap n_{i,1} = N_{i,1}] \\
D_s(i,1) &= Pr[\Delta^d_1(i,1) \cap n_{i,1} = 0] \\
& \quad i = 0 \ldots K
\end{aligned}
\tag{5.2.1}
$$

The quantity, $W^u(i,1)$, is the probability that $M^u(i,1)$ is up and not starved. The quantity $X^u_m(i,1)$ is the probability that $M^u(i,1)$ is in the down state $\Delta^u_m(i,1), m = 1,2,3$. $P_b(i,1)$ is the probability that $M^u(i,1)$ is up but idle because of blockage. $D_b(i,1)$ is the probability that $M^u(i,1)$ is down and also blocked for a part. Likewise, the quantity, $W^d(i,1)$, is the probability that $M^d(i,1)$ is up and not blocked. The quantity $X^d_m(i,1), m = 1,2,3$ is the probability that $M^d(i,1)$ is down at the down state, $\Delta^d_m(i,1)$. $P_s(i,1)$ is the probability that $M^d(i,1)$ is up but idle because of starvation. $D_s(i,1)$ is the probability that M_{i+1} is down and also starved for a part.

$W^d(i,1)$ is the probability that that M_{i+1} is up and neither starved nor blocked for Type 1. If we relate this quantity in the building block with events the real line, then

$$\begin{aligned} W^d(i,1) &= Pr[\Upsilon^d(i,1) \cap n_{i,1} > 0] \\ &= Pr[\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1}] \end{aligned}$$

Notice that this expression can be written as

$$\begin{aligned} &Pr[\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1}] \\ &\qquad = Pr[\Upsilon^u(i+1,1) \cap n_{i+1,1} < N_{i+1,1}] \\ &\qquad = W^u(i+1,1) \end{aligned}$$

Therefore,

$$W^d(i,1) = W^u(i+1,1) \tag{5.2.2}$$

This equality is a statement of the conservation of flow.

Next, $X_1^d(i,1)$ is the probability that the downstream machine of $L(i,1)$ is down in mode 1 and not starved. From definitions (5.1.1) and (5.2.1), it is

$$\begin{aligned} X_1^d(i,1) &= Pr[\Delta_1^d(i,1) \cap n_{i,1} > 0] \\ &= Pr[\alpha_{i+1} = 0 \cap n_{i+1,1} < N_{i+1,1} \cap n_{i,1} > 0] \\ &= Pr[\Delta^u(i+1,1) \cap n_{i+1,1} < N_{i+1,1}] \\ &= X_1^u(i+1,1) \end{aligned}$$

Therefore,

$$X_1^d(i,1) = X_1^u(i+1,1) \tag{5.2.3}$$

Next, $X_2^d(i,1)$ is the probability that the downstream machine of $L(i,1)$ is down in mode 2 and not starved. From (5.1.1), it is

$$\begin{aligned} X_2^d(i,1) &= Pr[\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cap n_{i+1,1} = N_{i+1,1}] \\ &= Pr[\Upsilon^u(i+1,1) \cap n_{2,1} = N_{i+1,1}] \\ &= P_b(i+1,1) \end{aligned}$$

A similar equality can be derived for $X_2^u(i+1,1)$. Therefore,

$$X_2^d(i,1) = P_b(i+1,1) \tag{5.2.4}$$

$$X_2^u(i+1,1) = P_s(i,1) \tag{5.2.5}$$

Last,

$$\begin{aligned} X_3^d(i,1) &= Pr[\alpha_{i+1} = 0 \cap n_{i,1} > 0 \cap n_{i+1,1} = N_{i+1,1}] \\ &= Pr[\Delta_1^u(i+1,1) \cap n_{i+1,1} = N_{i+1,1}] \\ &= D_b(i+1,1) \end{aligned}$$

Again, $X_3^u(i+1,1)$ can be derived in the similar way. Therefore,

$$X_3^d(i,1) = D_b(i+1,1) \tag{5.2.6}$$

$$X_3^u(i+1,1) = D_s(i,1) \tag{5.2.7}$$

5.3 Resumption of Flow

Resumption of flow is a transition from a down state to an up state. Before we formulate the resumption of flow equations, we first define the following quantity for our convenience. Let $R^d(i,1)$, the resumption of flow probability of $M^d(i,1)$ denote the probability that $M^d(i,1)$ is up at $t+1$ given that it was down at t. This quantity is actually the weighted sum of the transition probabilities from all the down states to the up state. Therefore,

$$R^d(i,1) = \frac{\sum_{n=1}^3 r_n^d(i,1)X_n^d(i,1)}{\sum_{n=1}^3 X_n^d(i,1)}$$

likewise,

$$R^u(i,1) = \frac{\sum_{m=1}^3 r_m^u(i,1)X_m^u(i,1)}{\sum_{m=1}^3 X_m^d(i,1)}$$

Now, let us formulate each resumption of flow equation. First, $r_1^d(i,1)$, the transition probability from the first down state to the up state is

$$\begin{aligned} r_1^d(i,1) &= Pr[\Upsilon^d(i,1) \text{ at } t+1 \,|\Delta_1^d(i,1) \text{ at } t] \\ &= Pr[\alpha_{i+1}(t+1) = 1 \cap \\ &\quad n_{i+1,1}(t+1) < N_{i+1,1}|\alpha_{i+1}(t) = 0 \cap n_{i+1,1}(t) < N_{i+1,1}] \\ &= r_{i+1} \end{aligned} \tag{5.3.1}$$

That ism, since $\Delta_1^d(i,1)$ represents a local failure, its repair probability is equal to the repair probability of the real machine M_{i+1}. Next, the repair probability of the remote down state is

$$\begin{aligned} r_2^d(i,1) &= Pr[\Upsilon^d(i,1) \text{ at } t+1 \,|\Delta_2^d(i,1) \text{ at } t] \\ &= Pr[\alpha_{i+1}(t+1) = 1 \cap n_{i+1,1}(t+1) < N_{i+1,1}| \\ &\quad \alpha_{i+1}(t) = 1 \cap n_{i+1,1}(t) = N_{i+1,1}] \\ &= \Big(1 - q_1^u(i+1,1)\Big)R^d(i+1,1) \end{aligned} \tag{5.3.2}$$

This is because the remote failure occurred due to the blockage of $B_{i+1,1}$ which was caused by the failure of M_{i+2} or some further downstream machine. Therefore, $M^d(i+1,1)$ must be repaired in order for flow of $M^d(i,1)$ to resume. There is also one more condition required for the resumption of the flow. Notice that at time t, M_{i+1} is blocked and therefore, it could work on a Type 2 or lower priority part. The resumption of the flow happens at the next time step only when M_{i+1} does not go down while it is working on Type 2. The probability that M_{i+1} goes down while it is blocked and processing a Type $j, j \geq 2$ part is $q_1^u(i,1)$ by the definition. Therefore, the transition probability from $\Delta_2^d(i,1)$ to $\Upsilon^d(i,1)$ is $\Big(1-q_1^u(i+1,1)\Big)R^d(i+1,1)$.

For the third repair probability, $r_3^d(i,1)$ is the transition probability from $\Delta_3^d(i,1)$ to $\Upsilon^d(i,1)$. We argue that this is

$$
\begin{aligned}
r_3^d(i,1) &= Pr[\Upsilon^d(i,1) \text{ at } t+1 \,|\Delta_3^d(i,1) \text{ at } t] \qquad (5.3.3)\\
&= Pr[\alpha_{i+1}(t+1) = 1 \cap n_{i+1,1}(t+1) < N_{i+1,1}| \\
&\qquad \alpha_{i+1}(t) = 0 \cap n_{i+1,1}(t) = N_{i+1,1}] \\
&= r_{i+1}R^d(i+1,1)
\end{aligned}
$$

As shown in the equation, M_{i+2} or some further downstream machine is down at t, causing $B_{i+1,1}$ to be full, and also M_2 is down at time t. Therefore, the both machines must be repaired for the flow of a Type 1 part to resume.

Similarly, the resumption of flow equations for $M^u(i+1,1)$ are:

$$
\begin{aligned}
r_1^u(i+1,1) &= r_{i+1} && (5.3.4)\\
r_2^u(i+1,1) &= \Big(1-q^d(i,1)\Big)R^u(i,1) && (5.3.5)\\
r_3^u(i+1,1) &= R^u(i,1)r_{i+1} && (5.3.6)
\end{aligned}
$$

5.4 Failure Mode Change

Failure mode changes are transitions taking place between down states. First, we consider transitions from $\Delta_1^d(i,1)$ to other down states. This down state is the failure of M_{i+1}. Since the local machine is down and is not blocked, any state change further downstream this will not change the state of $M^d(i,1)$. The only event that will change the state of $\Delta_1^d(i,1)$ is the resumption of flow. Therefore, there is no transition from $\Delta_1^d(i,1)$ to the rest of the down states. That is,

$$z_{1,2}^d(i,1) = 0 \tag{5.4.1}$$

$$z_{1,3}^d(i,1) = 0 \tag{5.4.2}$$

Next, we consider $\Delta_2^d(i,1)$. In this down state, M_{i+1} is up but is blocked for a Type 1 part. Both other down states od $M^d(i,1)$ can be reached from this one. First, consider the case in which M_{i+1} goes down, while processing Type 2 or lower priority part. In this case, M_{i+1} will be down and blocked at the same time. Therefore $M^d(i,1)$ will move to $\Delta_3^d(i,1)$. Since M_{i+1} goes down while it is blocked and $M^d(i+1,1)$ is not repaired, the transition probability from $\Delta_2^d(i,1)$ to $\Delta_3^d(i,1)$ is $q_1^u(i+1,1)\Big(1-R^d(i+1,1)\Big)$. The probability of this failure mode change is

$$\begin{aligned} z_{2,3}^d(i,1) &= Pr[\Delta_3^d(i,1) \text{ at } t+1 \ |\Delta_2^d(i,1) \text{ at } t\ ] \\ &= q_1^u(i+1,1)\Big(1-R^d(i+1,1)\Big) \end{aligned} \tag{5.4.3}$$

The second failure mode change from $\Delta_2^d(i,1)$ is the case that M_{i+1} goes down while it is blocked, but $M^d(i+1,1)$ gets repaired. In this case, M_{i+1} will be no longer blocked, but will move to the local failure mode. Therefore, with the transition probability of $q^u(i+1,1)R^d(i+1,1)$, $M^d(i,1)$ will move from $\Delta_2^d(i,1)$ to $\Delta_1^d(i,1)$. That is,

$$
\begin{aligned}
z^d_{2,1}(i,1) &= Pr[\Delta^d_1(i,1) \text{ at } t+1 \ |\Delta^d_2(i,1) \text{ at } t\ ] \\
&= q^u_1(i+1,1)R^d(i+1,1)
\end{aligned} \tag{5.4.4}
$$

The last down state to be considered is $\Delta^d_3(i,1)$. In this state, $M^d(i,1)$ is down because of the local failure of M_{i+1} and the blockage caused by the failure of M_{i+2} or some further downstream machine. The failure mode change to $\Delta^d_2(i,1)$ happens when M_{i+1} gets repaired, but $M^d(i+1,1)$ remains down. That is,

$$
\begin{aligned}
z^d_{3,2}(i,1) &= Pr[\Delta^d_2(i,1) \text{ at } t+1 \ |\Delta^d_3(i,1) \text{ at } t\ ] \\
&= r_{i+1}\Big(1 - R^d(i+1,1)\Big)
\end{aligned} \tag{5.4.5}
$$

On the other hand, when $M^d(i,1)$ is in $\Delta^d_3(i,1)$, if $M^d(i+1,1)$ gets fixed while M_{i+1} remains down, M_{i+1} will be no longer be blocked for a Type 1 part. Instead it will be in a local failure mode. Therefore,

$$
\begin{aligned}
z^d_{3,1}(i,1) &= Pr[\Delta^d_1(i,1) \text{ at } t+1 \ |\Delta^d_3(i,1) \text{ at } t\ ] \\
&= \Big(1 - r_{i+1}\Big)R^d(i+1,1)
\end{aligned} \tag{5.4.6}
$$

Following similar approaches, we also can derive the failure mode change probabilities for $M^u(i+1,1)$:

$$
\begin{aligned}
z^u_{1,2}(i+1,1) &= 0 && (5.4.7)\\
z^u_{1,3}(i+1,1) &= 0 && (5.4.8)\\
z^u_{2,1}(i+1,1) &= q^d_1(i,1)R^u(i,1) && (5.4.9)\\
z^u_{2,3}(i+1,1) &= \Big(1-R^u(i,1)\Big)q^d_1(i,1) && (5.4.10)\\
z^u_{2,1}(i+1,1) &= q^d_1(i,1)R^u(i,1) && (5.4.11)\\
z^u_{3,1}(i+1,1) &= R^u(i,1)\Big(1-r_{i+1}\Big) && (5.4.12)\\
z^u_{3,2}(i+1,1) &= \Big(1-R^u(i,1)\Big)r_{i+1} && (5.4.13)
\end{aligned}
$$

5.5 Interruption of Flow

The interruption of flows of out $B_{i,j}$ are failures of $M^d(i,1)$. The quantity $p^d_1(i,1)$ is the probability of from $\Upsilon^d(i,1)$ to $\Delta^d_1(i,1)$, which represents the failure of M_{i+1} when $B_{i,1}$ is not empty. This transition probability is

$$
\begin{aligned}
p^d_1(i,1) &= Pr[\Delta^d_1(i,1) \text{ at } t+1 \mid && (5.5.1)\\
&\qquad \Upsilon^d(i,1) \cap n_{i,1} > 0 \text{ at } t] &&\\
&= Pr[\alpha_2(t+1) = 0 \cap n_{i+1,1}(t+1) < N_{i+1,1}| &&\\
&\qquad \alpha_{i+1}(t) = 1 \cap n_{i,1}(t) > 0 \cap n_{i+1,1}(t) < N_{i+1,1}] &&\\
&= p_{i+1} &&
\end{aligned}
$$

Next, we derive the interruption of flow equation for $p^d_2(i,1)$, the transition probability from the up state to the state in which $M^d(i+1,1)$ is down and $B_{i+1,1}$ is full. This transition probability is

$$p^d_2(i,1) = Pr[\Delta^d_2(i,1) \text{ at } t+1 \,|\Upsilon^d(i,1) \cap n_{i,1} > 0 \text{ at } t]$$

We first start the derivation of this quantity by applying the fact that the probability of going out of a state is equal to the probability of going into that state. In this case,

$$\begin{aligned}
X_2^d(i,1) \quad & \left(\left(1-q^u(i+1,1)\right)R^d(i+1,1)+q^u(i+1,1)R^d(i+1,1)\right. \\
& \left. +q^u(i+1,1)\left(1-R^d(i+1,1)\right)\right) \\
& = W^d(i,1)p_2^d(i,1)+X_3^d(i+1,1)r_{i+1}(1-R^d(i+1,1))
\end{aligned}$$

If we simplify this equation, then

$$\begin{aligned}
& X_2^d(1,1)\left(R^d(i+1,1)+q^u(i+1,1)\left(1-R^d(i+1,1)\right)\right) \\
& \quad = W^d(i,1)p_2^d(i,1)+X_3^d(i,1)r_{i+1}\left(1-R^d(i+1,1)\right)
\end{aligned}$$

That is,

$$\begin{aligned}
p_2^d(i,1) = {} & \frac{1}{W^d(i,1)}\times \\
& \left[X_2^d(i,1)\left(R^d(i+1,1)+q^u(i+1,1)\left(1-R^d(i+1,1)\right)\right)\right. \\
& \left. -X_3^d(i,1)r_{i+1}(1-R^d(i+1,1))\right]
\end{aligned} \tag{5.5.2}$$

Next, we consider the interruption of flow equation for the third down state, $\Delta_3^d(i,1)$. This is the transition that goes to the state in which M_{i+1} is down and blocked for a Type 1 part. Note that when M_{i+1} is up and not blocked for a Type 1 part from $\Upsilon^d(i,1)$, it is impossible for M_{i+1} to be down and blocked at the next time step because once the machine is down, it will not move a part into the downstream buffer. Therefore,

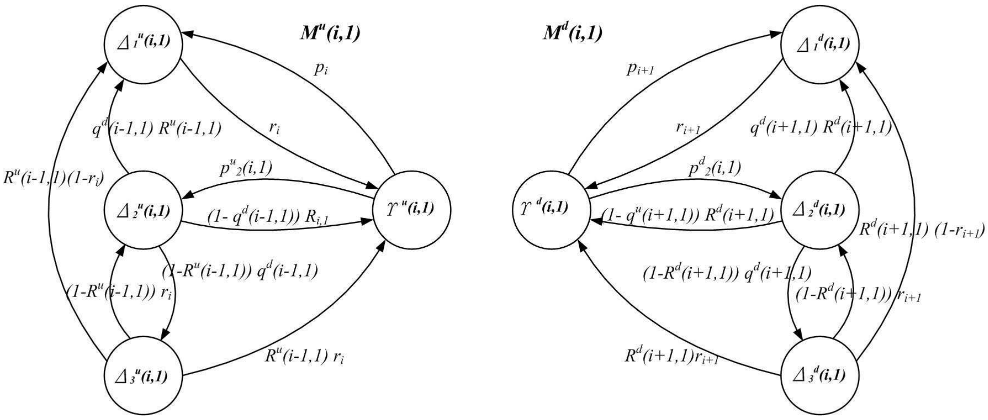

Figure 5.5.1: Markov Chain of $M^u(i,1)$ and $M^d(i,1)$

$$
\begin{aligned}
p_3^d(i,1) \quad &= \quad Pr[\Delta_3^d(i,1) \text{ at } t+1 \mid \\
&\qquad \Upsilon^d(i,1) \cap n_{i,1} > 0 \text{ at } t] = 0
\end{aligned} \tag{5.5.3}
$$

Similarly, the interruption of flow equations for $M^u(i+1,1)$ are

$$
\begin{aligned}
&p_1^u(i+1,1) = p_{i+1} \\
&p_2^u(i+1,1) = \frac{1}{W^u(i+1,1)} \times \\
&\qquad \Bigg[X_2^d(i+1,1)\Bigg(R^u(i,1) + \Big(1 - R^u(i,1)\Big)q^d(i,1)\Bigg) \\
&\qquad\qquad - X_2^d(i+1,1)\Big(1 - R^u(i,1)\Big)r_{i+1}\Bigg] \\
&p_3^u(i+1,1) = 0
\end{aligned} \tag{5.5.4}
$$

As a summary, Figure 5.5.1 shows all the possible transitions of $M^d(i,1)$ and $M^u(i+1,1)$.

5.6 Idleness Failure of Type 1

Now we need to derive expressions for the idleness failure of the building block. The idleness failure is the transition from an up state to a down state while it is idle. Note that there are three down states. However, the idleness failure occurs only to the Mode 1 down state, $\Delta_1^d(i,1)$. Suppose that $M^d(i,1)$ is up and starved. That is, M_{i+1} is up and $B_{i,1}$ is empty. The assumption in (3.2.3) implies that M_{i+1} cannot get blocked at this moment. Therefore, $M^d(i,1)$ cannot go to the second or third down state when it is starved. As a result,

$$\begin{aligned} q_2^d(i,1) &= 0 \\ q_3^d(i,1) &= 0 \end{aligned}$$

The probability $q_1^d(i,1)$ represents the probability that M^d(i,1) fails to the first down state at $t+1$ given that it was up and starved at t. That is,

$$\begin{aligned} q_1^d(i,1) &= Pr[\Delta_1^d(i,1) \text{ at } t+1 \,|\Upsilon^d(i,1) \cap n_{i,1} = 0 \text{ at } t] \\ &= Pr[\alpha_{i+1} = 0 \text{ at } t+1 | \\ &\qquad \alpha(t) = 1 \cap n_{i,1}(t) = 0 \cap n_{i+1,1} < N_{i+1,1} \text{ at } t] \end{aligned}$$

Let us consider the probability that M_{i+1} fails while it is either blocked or starved for Type 1. This quantity is related to the parameters of Type 2 building blocks since M_{i+1} fails while it is working Type 2. When M_{i+1} is either starved or blocked for a Type 1 part, and it is also starved for a Type 2 part, $M^d(i,2)$ can be one of the following three conditions:

- $M^d(i,2)$ is up and working ($\Upsilon^d(i,2) \cap n_{i,2} > 0$).
- $M^d(i,2)$ is starved ($\Upsilon^d(i,2) \cap n_{i,2} = 0$).
- $M^d(i,2)$ is down in Mode 2 ($\Delta_2^d(i,2) \cap n_{i,2} > 0$).

However, among these events, M_{i+1} can fail only when it is working on a Type 2 part. Therefore, $q_1^d(i,1)$ is a fraction of the failure probability, p_{i+1} and we state the idleness failure such that

$$q_1^d(i,1) = p_{i+1}\frac{W_d(i,2)}{W_d(i,2)+X_2(i,2)+P_s(i,2)} \tag{5.6.1}$$

Similarly,

$$q_1^u(i+1,1) = p_{i+1}\frac{W^u(i+1,2)}{W^u(i+1)+P_b(i+1,1)+X_2(i+1,1)} \tag{5.6.2}$$

5.7 Boundary Conditions

Let us consider the last building block for Type 1, $L(K,1)$. The only reason that the observer in the buffer $B_{K,1}$ does not find a part moving out of the buffer is that the machine, $M_{K+1,1}$, is down. Since $M_{K+1,1}$ is the last machine and only machine downstream from the buffer, no remote failure is possible for machine $M^d(K,1)$. Therefore the downstream machine has only one down state. Also $M_{K+1,1}$ processes only Type 1 parts, therefore no idleness failure occurs in $M^d(K,1)$. The parameters for $M^d(K,1)$ are

$$\begin{aligned} r^d(K,1) &= r_{K+1,1} \\ p^d(K,1) &= p_{K+1,1} \\ q^d(K,1) &= 0 \end{aligned} \tag{5.7.1}$$

Similarly, the parameters for $M^u(1,1)$ are

$$\begin{aligned} r^d(K,0) &= r_{0,1} \\ p^d(0,1) &= p_{0,1} \\ q^d(0,1) &= 0 \end{aligned} \tag{5.7.2}$$

5.8 Summary of the Decomposition Equations for Type 1

For $i = 0 \ldots K$

5.8.1 Resumption of flow

$$r^d(i,1) = r_{i+1} \tag{5.8.1}$$

$$r_2^d(i,1) = (1 - q^u(i+1,1))R^d(i+1,1)) \tag{5.8.2}$$

$$r_3^d(i,1) = r_{i+1}R^d(i+1,1) \tag{5.8.3}$$

$$r_1^u(i+1,1) = r_{i+1} \tag{5.8.4}$$

$$r_2^u(i+1,1) = \Big(1 - q^d(i,1)\Big)R^u(i,1) \tag{5.8.5}$$

$$r_3^u(i+1,1) = R^u(i,1)r_{i+1} \tag{5.8.6}$$

5.8.2 Failure mode change

$$
\begin{aligned}
z^d_{1,2}(i,1) &= 0 && (5.8.7)\\
z^d_{1,3}(i,1) &= 0 && (5.8.8)\\
z^d_{2,1}(i,1) &= q^u_1(i+1,1)R^d(i+1,1) && (5.8.9)\\
z^d_{2,3}(i,1) &= q^u_1(i+1,1)((1-R^d(i+1,1)) && (5.8.10)\\
z^d_{3,1}(i,1) &= \Big(1-r_{i+1}\Big)R^d(i+1,1) && (5.8.11)\\
z^d_{3,2}(i,1) &= r_{i+1}\Big(1-R^d(i+1,1)\Big) && (5.8.12)\\
z^u_{1,2}(i+1,1) &= 0 && (5.8.13)\\
z^u_{1,3}(i+1,1) &= 0 && (5.8.14)\\
z^u_{2,1}(i+1,1) &= q^d_1(i,1)R^u(i,1) && (5.8.15)\\
z^u_{2,3}(i+1,1) &= \Big(1-R^u(i,1)\Big)q^d_1(i,1) && (5.8.16)\\
z^u_{2,1}(i+1,1) &= q^d_1(i,1)R^u(i,1) && (5.8.17)\\
z^u_{3,1}(i+1,1) &= R^u(i,1)\Big(1-r_{i+1}\Big) && (5.8.18)\\
z^u_{3,2}(i+1,1) &= \Big(1-R^u(i,1)\Big)r_{i+1} && (5.8.19)
\end{aligned}
$$

5.8.3 Interruption of flow

$$
\begin{aligned}
p_1^d(i,1) &= p_{i+1} && (5.8.20)\\
p_2^d(i,1) &= \frac{1}{W^d(i,1)} \times && (5.8.21)\\
&\quad \Bigg[X_2^d(i,1)\Bigg(R^d(i+1,1) + q^u(i+1,1)\Big(1 - R^d(i+1,1)\Big)\Bigg)\\
&\qquad -X_3^d(i,1)r_{i+1}(1 - R^d(i+1,1))\Bigg]\\
p_3^d(i,1) &= 0 && (5.8.22)\\
p_1^u(i+1,1) &= p_{i+1} && (5.8.23)\\
p_2^u(i+1,1) &= \frac{1}{W^u(i+1,1)} \times && (5.8.24)\\
&\quad \Bigg[X_2^d(i+1,1)\Bigg(R^u(i,1) + \Big(1 - R^u(i,1)\Big)q^d(i,1)\Bigg)\\
&\qquad -X_2^d(i+1,1)\Big(1 - R^u(i,1)\Big)r_{i+1}\Bigg]\\
p_3^u(i+1,1) &= 0 && (5.8.25)
\end{aligned}
$$

5.8.4 Idleness failure

$$
\begin{aligned}
q_1^d(i,1) &= p_{i+1}\frac{W_d(i,2)}{W_d(i,2) + X_2(i,2) + P_s(i,2)} && (5.8.26)\\
q_2^d(i,1) &= 0 && (5.8.27)\\
q_3^d(i,1) &= 0 && (5.8.28)\\
q^u(i+1,1) &= p_{i+1}\frac{W^u(i+1,2)}{W^u(i+1,2) + P_b(i+1,2) + X_2(i+1,2)} && (5.8.29)\\
q_2^u(i+1,1) &= 0 && (5.8.30)\\
q_3^u(i+1,1) &= 0 && (5.8.31)
\end{aligned}
$$

Chapter 6

Decomposition Analysis for Type 2

The existence of Type 2 parts is made apparent to Type 1 parts only through the existence of idleness failures. However, the existence of Type 1 parts has a major influence on the production of Type 2 parts, and therefore the derivation for the Type 2 part decomposition is much more complicated. In particular, we now have to account for the possibility that an observer in a Type 2 buffer will see flow in and out of his buffer cease because the machines switched from making Type 2 to Type 1 parts.

6.1 State definitions

A downstream machine $M^d(i,2)$, $i = 1, ..., k-1$, represents all the Type 2 flow behavior from downstream $B_{i,2}$. Note that the machine $M^d(i,2)$ is up when M_{i+1} is up and is either blocked or starved for Type 1 but not blocked for Type 2. That is,

$$\begin{aligned} \Upsilon^d(i,2) \quad = \quad & \{\alpha_{i+1} = 1 \cap (n_{i,2} = 0 \cup n_{i+1,1} = N_{i+1,1}) \\ & \cap n_{i+1,2} < N_{i+1,2}\} \end{aligned} \tag{6.1.1}$$

The observer does not see a part moving out of the buffer $B_{i,2}$ when M_{i+1} is down; or M_{i+1} is up but processing Type 1; or M_{i+1} is up and it is either starved or blocked for Type 1, and it is blocked for Type 2. As we defined in Chapter 5, the first case is the local failure. The second case is a failure that cannot be observed in a Type 1 building block. We call this failure as the *type failure*. For our convenience, we combine the two failures — the local failure and the type failure – and create a new failure called the *local-type failure*. The down state caused by the local-type failure is stated such that,

$$\{\alpha_{i+1} = 0 \cup \{\alpha_{i+1} = 1 \cup (n_{i,1} > 0 \cup n_{i+1} < N_{i+1,1})\} \tag{6.1.2}$$

Note that the complement of the local-type down state is,

$$\begin{aligned} &\sim \{\alpha_{i+1} = 0 \cup \{\alpha_{i+1} = 1 \cup (n_{i,1} > 0 \cup n_{i+1} < N_{i+1,1})\} \\ &= \{\alpha_{i+1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1})\} \end{aligned}$$

With the local-type failure, we defined the following down states for $M^d(i,2)$:

- $M^d(i,2)$ is local-type down and not blocked for Type 2.
- $M^d(i,2)$ is up and blocked for Type 2.
- $M^d(i,2)$ is local-type down and blocked for Type 2.

These down sates are:

$$\begin{aligned} \Delta_1^d(i,2) &= \{\alpha_{i+1} = 0 \cup \{\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cup n_{i+1,1} < N_{i+1,1}\} \\ &\quad \cap n_{i+1,2} < N_{i+1,2}\} \\ \Delta_2^d(i,2) &= \{\alpha_{i+1} = 1 \cap \{n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}\} \\ &\quad \cap n_{i+1,2} = N_{i+1,2}\} \\ \Delta_3^d(i,2) &= \{\alpha_{i+1} = 0 \cup \{\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cup n_{i+1,1} < N_{i+1,1}\} \\ &\quad \cap n_{i+1,2} = N_{i+1,2}\} \end{aligned} \tag{6.1.3}$$

It is observed that the down state definitions for Type 2 are similar to those of the Type 1 described in (5.1.1).

6.2 Equalities

For our convenience, we define the following building block probabilities:

$$\begin{aligned}
W^u(i,2) &= Pr[\Upsilon^u(i,2) \cap n_{i,2} < N_{i,2}] \\
W^d(i,2) &= Pr[\Upsilon^d(i,2) \cap n_{i,2} > 0] \\
X^u_m(i,2) &= Pr[\Delta^u_m(i,2) \cap n_{i,2} < N_{i,2}], \qquad m = 1,2,3 \\
X^d_n(i,2) &= Pr[\Delta^d_n(i,2) \cap n_{i,2} > 0], \qquad n = 1,2,3 \\
Pb(i,2) &= Pr[\Upsilon^u(i,2) \cap n_{i,2} = N_{i,2}] \\
Ps(i,2) &= Pr[\Upsilon^d(i,2) \cap n_{i,1} = 0] \\
Db(i,2) &= Pr[\Delta_1(i,2) \cap n_{i,1} = N_{i,2}] \\
Ds(i,2) &= Pr[\Delta_1(i,2) \cap n_{i,1} = 0] \\
& \quad i = 1 \dots k-1
\end{aligned} \tag{6.2.1}$$

In(6.1.1), the following equality can be stated.

$$\Upsilon^d(i,2) = \{(\Delta^d_2(i,1) \cup \Delta^u_2(i+1,1)) \cap n_{i+1,2} < N_{i+1,2}\}$$

Then

$$\begin{aligned}
W^d(i,1) &= Pr[\Upsilon^d(i,2) \cap n_{i,2} < N_{i,2}] \\
&= Pr[(\Delta^d_2(i,1) \cup \Delta^u_2(i+1,1)) \cap n_{i+1,2} < N_{i+1,2} \cap n_{i,2} > 0] \\
&= W^u(i+1,2)
\end{aligned}$$

Next, $X^d_1(i,1)$ is the probability that the downstream machine of $L(i,1)$ is down in mode 1 and not starved. From the definition (6.1.3) and (7.2.1), it is

$$\begin{aligned}
X_1^d(i,2) &= Pr[\Delta_1^d(i,2) \cap n_{i,2} > 0] \\
&= Pr[\alpha_{i+1} = 0 \cup (\alpha_{i+1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1})) \\
&\qquad \cap (n_{i+1,2} < N_{i+1,2} \cap n_{i,2} > 0)] \\
&= Pr[\Delta^u(i+1,2) \cap n_{i+1,2} < N_{i+1,2}] \\
&= X_1^u(i+1,2)
\end{aligned}$$

Therefore,

$$X_1^d(i,2) = X_1^u(i+1,2) \tag{6.2.2}$$

Next, $X_2^d(i,2)$ is the probability that the downstream machine of $L(i,2)$ is down in mode 2 and not starved. From the definition (6.1.3), it is

$$\begin{aligned}
X_2^d(i,2) &= Pr[\{\alpha_{i+1} = 1 \cap \{n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}\} \\
&\qquad \cap n_{i+1,2} = N_{i+1,2} \cap n_{i,2} > 0] \\
&= Pr[\Upsilon^u(i+1,2) \cap n_{i+1,2} = N_{i+1,2}] \\
&= P_b(i+1,2)
\end{aligned}$$

A similar equality can be derived for $X_2^u(i+1,1)$. Therefore,

$$X_2^d(i,2) = P_b(i+1,2) \tag{6.2.3}$$

$$X_2^u(i+1,2) = P_s(i,2) \tag{6.2.4}$$

Last,

$$\begin{aligned}
X_3^d(i,1) &= Pr[\alpha_{i+1} = 0 \cup \{\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cup n_{i+1,1} < N_{i+1,1}\} \\
&\qquad \cap n_{i,2} > 0 \cap n_{i+1,2} = N_{i+1,2}] \\
&= Pr[\Delta_3^u(i+1,2) \cap n_{i+1,2} = N_{i+1,2}] \\
&= D_b(i+1,2)
\end{aligned}$$

Again, $X_3^u(i+1,1)$ can be derived in the similar way. Therefore,

$$\begin{aligned}
X_3^d(i,2) &= D_b(i+1,2) \qquad (6.2.5) \\
X_3^u(i+1,2) &= D_s(i,2) \qquad (6.2.6)
\end{aligned}$$

6.3 Resumption of Flow

Since there are three down states for $M^d(i,2)$, three separate repair probabilities need to be derived. First, we define the following quantities for the derivations of the resumption of flow equations.

$$R^d(i,2) = \frac{\sum_{n=1}^{3} r_n^d(i,2) X_n^d(i,2)}{\sum_{n=1}^{3} X_n^d(i,2)}$$

likewise,

$$R^u(i,2) = \frac{\sum_{m=1}^{3} r_m^u(i,2) X_m^u(i,2)}{\sum_{m=1}^{3} X_m^d(i,2)}$$

The definition of the repair probability from the down Mode 1 is,

$$\begin{aligned}
r_l^d(i,2) &= Pr[\Upsilon^d(i,2) \text{ at } t+1 | \Delta_l^d(i,1) \text{ at } t] \\
&\quad l = 1,2,3
\end{aligned}$$

The transition probability from $\Delta_1^d(i,2)$ to $\Upsilon^d(i,2)$ denoted by $r_1^d(i,2)$ is derived such that

$$\begin{aligned} r_1^d(i,2) \quad = \quad & Pr[\alpha_{i+1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \cap \\ & \qquad n_{i+1,2} < N_{i+1,2} \text{ at } t+1 | \\ & \qquad \alpha_{i+1} = 0 \cup (\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1}) \\ & \qquad \cap n_{i+1,2} < N_{i+1,2} \text{ at } t] \end{aligned}$$

Note that in this conditional probability, the event of $n_{i+1,2} < N_{i+1,2}$ is not affected during the transition. That is,

$$\begin{aligned} Pr[n_{i+1,2} = N_{i+1,2} \text{ at } t+1 | \alpha_{i+1} = 0 \cup (\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1}) \\ \cap n_{i+1,2} < N_{i+1,2} \text{ at } t] = 0 \end{aligned}$$

Therefore it can be stated that

$$\begin{aligned} r_1^d(i,2) \quad = \quad & Pr[\alpha_{i+1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \text{ at } t+1 \\ & \qquad \alpha_{i+1} = 0 \cup (\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1}) \text{ at } t] \end{aligned}$$

In other words, a buffer $n_{i+1,2}$ cannot get full if M_{i+1} was either down or working for a Type 1 part in the previous time step. Also we assume that a machine is not likely be starved and blocked simultaneously, and therefore it can be stated that

$$\begin{aligned} r_1^d(i,2) \quad = \quad & Pr[\alpha_{i+1} = 1 \cap n_{i,1} = 0 \text{ at } t+1 | \\ & \qquad \alpha_{i+1} = 0 \cup (\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1}) \text{ at } t] \\ & + Pr[\alpha_{i+1} = 1 \cap n_{i+1,1} = N_{i+1,1} \text{ at } t+1 | \\ & \qquad \alpha_{i+1} = 0 \cup (\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1}) \text{ at } t] \end{aligned}$$

The term $\{\alpha_{i+1} = 0\}$ is decomposed into the following mutually exclusive terms:

- $\{\alpha_{i+1} = 0 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1}\}$
- $\{\alpha_{i+1} = 0 \cap n_{i,1} = 0\}$
- $\{\alpha_{i+1} = 0 \cap n_{i+1,1} = N_{i+1,1}\}$

Using the law of probability,

$$
\begin{aligned}
& r_1^d(i,2) = \\
& \quad 1/\Big(Pr[\{\alpha_{i+1} = 0 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1}\} \cup \{\alpha_{i+1} = 0 \cap n_{i,1} = 0\} \\
& \qquad \cup \{\alpha_{i+1} = 0 \cap n_{i+1,1} = N_{i+1,1}\} \cup \{\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1}\}\Big) \times \\
& \qquad \Big(Pr[\alpha_{i+1} = 1 \cap n_{i,1} = 0 \text{ at } t+1 | \alpha_{i+1} = 0 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1} \text{ at } t] \times \\
& \qquad\quad Pr[\alpha_{i+1} = 0 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1}] \\
& \qquad + Pr[\alpha_{i+1} = 1 \cap n_{i,1} = 0 \text{ at } t+1 | \alpha_{i+1} = 0 \cap n_{i,1} = 0 \text{ at } t] \times \\
& \qquad\quad Pr[\alpha_{i+1} = 0 \cap n_{i,1} = 0] \\
& \qquad + Pr[\alpha_{i+1} = 1 \cap n_{i,1} = 0 \text{ at } t+1 | \alpha_{i+1} = 0 \cap n_{i+1,1} = N_{i+1,1} \text{ at } t] \times \\
& \qquad\quad Pr[\alpha_{i+1} = 0 \cap n_{i+1,1} = N_{i+1,1}] \\
& \qquad + Pr[\alpha_{i+1} = 1 \cap n_{i,1} = 0 \text{ at } t+1 | \alpha_{i+1} = 1 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1} \text{ at } t] \times \\
& \qquad\quad Pr[\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1}] \\
& \qquad + Pr[\alpha_{i+1} = 1 \cap n_{i+1,1} = N_{i+1,1} \text{ at } t+1 | \alpha_{i+1} = 0 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1} \text{ at } t] \times \\
& \qquad\quad Pr[\alpha_{i+1} = 0 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1}] \\
& \qquad + Pr[\alpha_{i+1} = 1 \cap n_{i+1,1} = N_{i+1,1} \text{ at } t+1 | \alpha_{i+1} = 0 \cap n_{i,1} = 0 \text{ at } t] \times \\
& \qquad\quad Pr[\alpha_{i+1} = 0 \cap n_{i,1} = 0] \\
& \qquad + Pr[\alpha_{i+1} = 1 \cap n_{i+1,1} = N_{i+1,1} \text{ at } t+1 | \alpha_{i+1} = 0 \cap n_{i+1,1} = N_{i+1,1} \text{ at } t] \times \\
& \qquad\quad Pr[\alpha_{i+1} = 0 \cap n_{i+1,1} = N_{i+1,1}] \\
& \qquad + Pr[\alpha_{i+1} = 1 \cap n_{i+1,1} = N_{i+1,1} \text{ at } t+1 | \alpha_{i+1} = 1 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1} \text{ at } t] \\
& \qquad\quad Pr[\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1}]\Big)
\end{aligned}
$$

Based on the assumption in (3.2.3) and the priority rules, the quantity of the following conditional probabilities are equal to zero.

- $Pr[\alpha_{i+1} = 1 \cap n_{i,1} = 0 \text{ at } t+1 | \alpha_{i+1} = 0 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1} \text{ at } t]$
- $Pr[\alpha_{i+1} = 1 \cap n_{i,1} = 0 \text{ at } t+1 | \alpha_{i+1} = 0 \cap n_{i+1,1} = N_{i+1,1} \text{ at } t]$
- $Pr[\alpha_{i+1} = 1 \cap n_{i+1,1} = N_{i+1,1} \text{ at } t+1 | \alpha_{i+1} = 0 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1} \text{ at } t]$
- $Pr[\alpha_{i+1} = 1 \cap n_{i+1,1} = N_{i+1,1} \text{ at } t+1 | \alpha_{i+1} = 0 \cap n_{i+1,1} = N_{i+1,1} \text{ at } t]$

The rest of the terms are evaluated such that

- $Pr[\alpha_{i+1} = 1 \cap n_{i,1} = 0 \text{ at } t+1 | \alpha_{i+1} = 0 \cap n_{i,1} = 0 \text{ at } t] = r_{i+1}(1 - R^u(i,1))$
  Note that this is a conditional probability that a machine M_{i+1} being down and starved for Type 1 is fixed but it is still starved for Type 1. Therefore, this quantity is the product of r_{i+1} and the probability that any failure causing the starvation is not fixed, which is $1 - R^u(i,1)$.

- $Pr[\alpha_{i+1} = 1 \cap n_{i,1} = 0 \text{ at } t+1 | \alpha_{i+1} = 1 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1} \text{ at } t] = p_2^u(i,1)$
  This is the quantity that $M_{i+1,1}$ gets starved for Type 1. We define this to be $p_2^u(i,1)$ and derivation of this quantity is introduce in Section 6.5

- $Pr[\alpha_{i+1} = 1 \cap n_{i+1,1} = N_{i+1,1} \text{ at } t+1 | \alpha_{i+1} = 0 \cap n_{i,1} = 0 \text{ at } t] = r_{i+1}(1 - R^d(i+1,1))$
  This is a case that machine M_{i+1} being down and blocked for Type 1 is fixed but it is still blocked for Type 1. Therefore, this quantity is the product of r_{i+1} and the probability that any failure causing the blockage is not fixed, which is $1 - R^d(i+1,1)$.

- $Pr[\alpha_{i+1} = 1 \cap n_{i+1,1} = N_{i+1,1} \text{ at } t+1 | \alpha_{i+1} = 1 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1} \text{ at } t] = p_2^d(i,1)$
  This is the quantity that $M_{i+1,1}$ gets blocked for Type 1, which is $p_2^d(i,1)$

From the notation described in Chapter 5, the first repair probability is therefore,

$$
\begin{aligned}
r_1^d(i,2) \;=\; & 1/\Big(W^d(i,1) + D_s(i,1) + D_b(i+1,1) + X_1^d(i,1)\Big) \times \\
& \Big(W^d(i,1)(p_2^u(i+1,1) + p_2^d(i,1)) \\
& \quad +D_s(i,1)(1 - R^u(i,1))r_{i+1} \\
& \quad +D_b(i+1,1)r_{i+1}(1 - R^d(i+1,1))\Big)
\end{aligned}
\tag{6.3.1}
$$

The next down repair probability, $r^d(i,2)$ is expressed as

$$
\begin{aligned}
r_2^d(i,2) &= Pr[\Upsilon^d(i,2) \text{ at } t+1 | \Delta_2^d(i,2) \text{ at } t] \qquad (6.3.2)\\
&= Pr[\alpha_{i+1,1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \cap n_{i+1,2} < N_{i+1,2} \text{ at } t+1 | \\
&\qquad \alpha_{i+1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \cap n_{i+1,2} = N_{i+1,2} \text{ at } t]
\end{aligned}
$$

This transition indicates that M_{i+1}, which was once blocked for Type 2, starts working on Type 2. Machine M_{i+1} was in the state that it was up, and either blocked or starved for Type 1, and blocked for Type 2. This transition probability is approximated such that,

$$
\begin{aligned}
r_2^d(i,2) \approx\; & Pr[\alpha_{i+1} = 1 \cap (n_{i+1,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \text{ at } t+1 | \\
& \quad \alpha_{i+1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \cap n_{i+1,2} = N_{i+1,2} \text{ at } t] \\
& \times Pr[n_{i+1,2} < N_{i+2,2} \text{ at } t+1 | \\
& \quad \alpha_{i+1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \cap n_{i+1,2} = N_{i+1,2} \text{ at } t]
\end{aligned}
\tag{6.3.3}
$$

In the first conditional probability in (6.3.3) indicates that M_{i+1} remains up during the transition. That is, M_{i+1} should not get failed while it is blocked for Type 2.

This transient probability is equivalent to

$$\begin{aligned} &Pr[\alpha_{i+1} = 1 \cap (n_{i+1,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \text{ at } t+1| \\ &\qquad \alpha_{i+1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \cap n_{i+1,2} = N_{i+1,2} \text{ at } t] \\ &\qquad = (1 - q^u(i+1,2)) \end{aligned}$$

The next conditional probability in (6.3.3) is the transition that M_{i+1} gets not blocked for Type 2. Also note that the state of machine M_{i+1} and the Type 1 buffers, $n_{i,1}$ and n_{i+1}, are independent of the state of the Type 2 buffer $n_{i+1,2}$. Therefore, the transition occurs when $M^d(i+1)$ gets repaired:

$$\begin{aligned} &Pr[n_{i+1,2} < N_{i+2,2} \text{ at } t+1| \\ &\qquad \alpha_{i+1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \cap n_{i+1,2} = N_{i+1,2} \text{ at } t] \\ &= Pr[n_{i+1,2} < N_{i+2,2} \text{ at } t+1|n_{i+1,2} = N_{i+1,2} \text{ at } t] \\ &= R^d(i+1,2) \end{aligned}$$

Then $r_2^d(i,2)$ is approximated such that

$$r_2^d(i,2) = (1 - q^u(i+1,2))R^d(i+1,2) \tag{6.3.4}$$

The third repair probability , $r_3^d(i,1)$, is the transition probability from $\Delta_3^d(i,1)$ to $\Upsilon^d(i,1)$. That is,

$$\begin{aligned} r_3^d(i,2) &= Pr[\Upsilon^d(i,2) \text{ at } t+1|\Delta_3^d(i,2) \text{ at } t] \qquad (6.3.5) \\ &= Pr[\alpha_{i+1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \cap n_{i+1,2} < N_{i+1,2} \text{ at } t+1| \\ &\qquad \alpha_{i+1} = 0 \cup (\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cup n_{i+1,1} < N_{i+1,1}) \\ &\qquad \cap n_{i+1,2} = N_{i+1,2} \text{ at } t] \end{aligned}$$

This transition probability is approximated such that

$$
\begin{aligned}
r^d(i,2) \;\approx\; & Pr[\alpha_{i+1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \text{ at } t+1| \\
& \qquad \alpha_{i+1} = 0 \cup (\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cup n_{i+1,1} < N_{i+1,1}) \\
& \qquad\qquad \cap n_{i+1,2} = N_{i+1,2} \text{ at } t] \\
& \times Pr[n_{i+1,2} = N_{i+1,2}) \text{ at } t+1| \\
& \qquad \alpha_{i+1} = 0 \cup (\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cup n_{i+1,1} < N_{i+1,1}) \\
& \qquad\qquad \cap n_{i+1,2} = N_{i+1,2} \text{ at } t]
\end{aligned}
\tag{6.3.6}
$$

Note that in (6.3.6), the machine state and buffer states for Type 1, $n_{i,1}$ and $n_{i+1,1}$, are independent of the the Type 2 buffer state $n_{i+1,2}$ and therefore,

$$
\begin{aligned}
r^d(i,2) \;\approx\; & Pr[\alpha_{i+1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \text{ at } t+1| \\
& \qquad \alpha_{i+1} = 0 \cup (\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cup n_{i+1,1} < N_{i+1,1}) \text{ at } t] \\
& \times Pr[n_{i+1,2} < N_{i+1,2} | n_{i+1,2} = N_{i+1,2}]
\end{aligned}
\tag{6.3.7}
$$

Observe that the first conditional probability is equal to $r_1^d(i,2)$ and the second term is $R^d(i+1,2)$;

$$r_3^d(i,2) = r_1^d(i,2) R^d(i+1,2) \tag{6.3.8}$$

As a summary, the resumption of flow equations for $M^d(i,1)$ are:

$$
\begin{aligned}
r_1^d(i,2) &= 1/\Big(W^d(i,1) + D_s(i,1) + D_b(i+1,1) + X_1^d(i,1)\Big) \times \\
&\quad \Big(W^d(i,1)(p_2^u(i+1,1) + p_2^d(i,1)) \qquad (6.3.9) \\
&\qquad +D_s(i,1)(1 - R^u(i,1))r_{i+1} \\
&\qquad +D_b(i+1,1)r_{i+1}(1 - R^d(i+1,1))\Big) \\
r_2^d(i,2) &= (1 - q^u(i+1,2))R^d(i+1,2) \qquad (6.3.10) \\
r_3^d(i,2) &= r_1^d(i,2)R^d(i+1,2) \qquad (6.3.11)
\end{aligned}
$$

Similarly, the resumption of flow equations for $M^u(i,1)$ are:

$$
\begin{aligned}
r_1^u(i,2) &= 1/\Big(W^d(i,1) + D_s(i,1) + D_b(i+1,1) + X_1^d(i,1)\Big) \times \\
&\quad \Big(W^d(i,1)(p_2^u(i+1,1) + p_2^d(i,1)) \qquad (6.3.12) \\
&\qquad +D_s(i,1)(1 - R^u(i,1))r_{i+1} \\
&\qquad +D_b(i+1,1)r_{i+1}(1 - R^d(i+1,1))\Big) \\
r_2^u(i,2) &= (1 - q^u(i+1,2))R^d(i+1,2) \qquad (6.3.13) \\
r_3^u(i,2) &= r_1^d(i,2)R^d(i+1,2) \qquad (6.3.14)
\end{aligned}
$$

6.4 Failure mode change

Failure mode changes are transitions between down states. First, the transition from $\Delta_1^d(i,2)$ to other down states is considered. This down state represents the state that either M_{i+1} is down or M_{i+1} is working on Type 1. From the observer in $B_{i,2}$, any state change further downstream of M_{i+1}, except the resumption of flow, will not change the state of $M^d(i,2)$. Therefore, there in no transition from $\Delta_1^d(i,1)$ to any other down state. Therefore,

$$z^d_{1,2}(i,2) = 0 \tag{6.4.1}$$
$$z^d_{1,2}(i,2) = 0 \tag{6.4.2}$$

Next, once $M^d(i,2)$ is in the down state $\Delta^d_2(i,2)$ — M_{i+1} is up and is able to work for Type 2 due to starvation or blockage of Type 1 but idle due to blockage for Type 2 — it can make a transition into two other down states. First, let us consider the transition to $\Delta^d_3(i,2)$

$$\begin{aligned} z^d_{2,3}(i,1) &= Pr[\Delta^d_3(i,2) \text{ at } t+1 | \Delta^d_2(i,2) \text{ at } t] \\ &= Pr[\alpha_{i+1} = 0 \cup (\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cup n_{i+1,1} < N_{i+1,1}) \\ &\qquad\qquad \cap n_{i+1,2} = N_{i+1,2} \text{ at } t+1 | \\ &\qquad\quad \alpha_{i+1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \cap n_{i+1,2} = N_{i+1,2} \text{ at } t] \end{aligned} \tag{6.4.3}$$

Then

$$\begin{aligned} z^d_{2,3}(i,1) &= Pr[\alpha_{i+1} = 0 \cup (\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cup n_{i+1,1} < N_{i+1,1}) \text{ at } t+1 | \\ &\qquad \alpha_{i+1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \cap n_{i+1,2} = N_{i+1,2} \text{ at } t] \\ &\quad \times Pr[n_{i+1,2} = N_{i+1,2} \text{ at } t+1 | \\ &\qquad \alpha_{i+1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \cap n_{i+1,2} = N_{i+1,2} \text{ at } t] \end{aligned}$$

Since the state $n_{i+1,2} = N_{i+1,2}$ and the state of M_{i+1},$n_{i,1}$, and $n_{i+1,1}$ are independent, it can be approximated as

$$\begin{aligned} z^d_{2,3}(i,1) &\approx Pr[\alpha_{i+1} = 0 \cup (\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1}) \text{ at } t+1 | \\ &\qquad \alpha_{i+1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \cap n_{i+1,2} = N_{i+1,2} \text{ at } t] \\ &\quad \times Pr[n_{i+1,2} = N_{i+1,2} \text{ at } t+1 | n_{i+1,2} = N_{i+1,2} \text{ at } t] \end{aligned}$$

The first term is $q^u(i+1,2)$ while the second term is $(1-R^d(i+1,2)$. Therefore,

$$z^d_{2,3} = q^u(i+1,2)(1-R^d(i+1,2)) \tag{6.4.4}$$

The next failure mode change in the second down state, $z^d_{3,2}(i,2)$ is derived with the same procedure.

$$\begin{aligned}
z^d_{2,1}(i,1) &= Pr[\Delta^d_1(i,2) \text{ at } t+1 | \Delta^d_2(i,2) \text{ at } t] \qquad (6.4.5)\\
&= Pr[\alpha_{i+1} = 0 \cup (\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cup n_{i+1,1} < N_{i+1,1})\\
&\qquad \cap n_{i+1,2} < N_{i+1,2} \text{ at } t+1|\\
&\qquad \alpha_{i+1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \cap n_{i+1,2} = N_{i+1,2} \text{ at } t]\\
&= Pr[\alpha_{i+1} = 0 \cup (\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cup n_{i+1,1} < N_{i+1,1}) \text{ at } t+1|\\
&\qquad \alpha_{i+1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \cap n_{i+1,2} = N_{i+1,2} \text{ at } t]\\
&\quad \times Pr[n_{i+1,2} < N_{i+1,2} \text{ at } t+1|\\
&\qquad \alpha_{i+1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \cap n_{i+1,2} = N_{i+1,2} \text{ at } t]\\
&\approx Pr[\alpha_{i+1} = 0 \cup (\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cup n_{i+1,1} < N_{i+1,1}) \text{ at } t+1|\\
&\qquad \alpha_{i+1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \text{ at } t]\\
&\quad \times Pr[n_{i+1,2} < N_{i+1,2} \text{ at } t+1 | n_{i+1,2} = N_{i+1,2} \text{ at } t]\\
&= q^u(i+1,2)R^d(i+1,2)
\end{aligned}$$

Likewise, the failure mode changes from the last down state $\Delta_3^d(i,2)$ are

$$\begin{aligned}
z_{3,2}^d(i,1) &= Pr[\Delta_2^d(i,2) \text{ at } t+1|\Delta_3^d(i,2) \text{ at } t] \qquad (6.4.6)\\
&= Pr[\alpha_{i+1}=1\cap(n_{i,1}=0\cup n_{i+1,1}=N_{i+1,1})\cap n_{i+1,2}=N_{i+1,2} \text{ at } t+1|\\
&\qquad \alpha_{i+1}=0\cup(\alpha_{i+1}=1\cap n_{i,1}>0\cup n_{i+1,1}<N_{i+1,1})\cap n_{i+1,2}=N_{i+1,2} \text{ at } t]\\
&= Pr[\alpha_{i+1}=1\cap(n_{i,1}=0\cup n_{i+1,1}=N_{i+1,1}) \text{ at } t+1|\\
&\qquad \alpha_{i+1}=0\cup(\alpha_{i+1}=1\cap n_{i,1}>0\cup n_{i+1,1}<N_{i+1,1})\cap n_{i+1,2}=N_{i+1,2} \text{ at } t]\\
&\quad \times Pr[n_{i+1,2}=N_{i+1,2} \text{ at } t+1|\\
&\qquad \alpha_{i+1}=0\cup(\alpha_{i+1}=1\cap n_{i,1}>0\cup n_{i+1,1}<N_{i+1,1})\cap n_{i+1,2}=N_{i+1,2} \text{ at } t]\\
&\approx Pr[\alpha_{i+1}=1\cap(n_{i,1}=0\cup n_{i+1,1}=N_{i+1,1}) \text{ at } t+1|\\
&\qquad \alpha_{i+1}=0\cup(\alpha_{i+1}=1\cap n_{i,1}>0\cup n_{i+1,1}<N_{i+1,1}) \text{ at } t]\\
&\quad \times Pr[n_{i+1,2}=N_{i+1,2} \text{ at } t+1|n_{i+1,2}=N_{i+1,2} \text{ at } t]\\
&= r_1^d(i,2)(1-R^d(i+1,2))
\end{aligned}$$

and

$$\begin{aligned}
z_{3,1}^d(i,1) &= Pr[\Delta_1^d(i,2) \text{ at } t+1|\Delta_3^d(i,2) \text{ at } t] \qquad (6.4.7)\\
&= Pr[\alpha_{i+1}=0\cup(n_{i,1}=1\cup n_{i,1}>0\cap n_{i+1,1}<N_{i+1,1})\cap n_{i+1,2}<N_{i+1,2} \text{ at } t+1|\\
&\qquad \alpha_{i+1}=0\cup(\alpha_{i+1}=1\cap n_{i,1}>0\cup n_{i+1,1}<N_{i+1,1})\cap n_{i+1,2}=N_{i+1,2} \text{ at } t]\\
&= Pr[\alpha_{i+1}=0\cup(n_{i,1}=1\cup n_{i,1}>0\cap n_{i+1,1}<N_{i+1,1}) \text{ at } t|\\
&\qquad \alpha_{i+1}=0\cup(\alpha_{i+1}=1\cap n_{i,1}>0\cup n_{i+1,1}<N_{i+1,1})\cap n_{i+1,2}=N_{i+1,2} \text{ at } t]\\
&\quad \times Pr[n_{i+1,2}<N_{i+1,2} \text{ at } t+1|\\
&\qquad \alpha_{i+1}=0\cup(\alpha_{i+1}=1\cap n_{i,1}>0\cup n_{i+1,1}<N_{i+1,1})\cap n_{i+1,2}=N_{i+1,2} \text{ at } t]\\
&= Pr[\alpha_{i+1}=0\cup(n_{i,1}=1\cup n_{i,1}>0\cap n_{i+1,1}<N_{i+1,1}) \text{ at } t+1|\\
&\qquad \alpha_{i+1}=0\cup(\alpha_{i+1}=1\cap n_{i,1}>0\cup n_{i+1,1}<N_{i+1,1}) \text{ at } t]\\
&\quad \times Pr[n_{i+1,1}<N_{i+1,1}) \text{ at } t+1|n_{i+1,1}<N_{i+1,1})\cap n_{i+1,2}=N_{i+1,2} \text{ at } t]\\
&= (1-r_1^d(i,2))R^d(i+1,2)
\end{aligned}$$

With similar approaches, the failure mode changes for $M^u(i,1)$ are:

$$
\begin{aligned}
z^u_{1,2}(i,2) &= 0 && (6.4.8)\\
z^u_{1,3}(i,2) &= 0 && (6.4.9)\\
z^u_{2,1}(i,2) &= q^d(i,2)R^u(i,2) && (6.4.10)\\
z^u_{2,3}(i,2) &= (1-R^u(i,2))q^d(i,2) && (6.4.11)\\
z^u_{3,1}(i,2) &= R^u(i,2)(1-r^u(i,2)) && (6.4.12)\\
z^u_{3,2}(i,2) &= (1-R^u(i,2))r^u(i,2) && (6.4.13)
\end{aligned}
$$

6.5 Interruption of Flow

Since there are three down states, three interruption of flow probabilities are needed to be derived. First, the probability $p^d_1(i,2)$ is the transition from $\Upsilon^d(i,2)$ to $\Delta^d_1(i,2)$, which represents the failure of $M^d(i,2)$ when $B_{i,2}$ is not empty. This transition probability is

$$
\begin{aligned}
p^d_1(i,2) &= Pr[\Delta^d_1(i,2) \text{ at } t+1 | \Upsilon^d(i,2) \cap n_{i,2} > 0 \text{ at } t] && (6.5.1)\\
&= Pr[\alpha_{i+1} = 0 \cup (\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cup n_{i+1,1} < N_{i+1,1})\\
&\qquad \cap n_{i+1,2} < N_{i+1,2} \text{ at } t+1 |\\
&\qquad \alpha_{i+1} = 1 \cap (n_{i,2} = 0 \cup n_{i+1,1} = N_{i+1,1}) \cap n_{i+1,2} < N_{i+1,2}\\
&\qquad \cap n_{i,2} > 0 \text{ at } t]
\end{aligned}
$$

This is equivalent to

$$
\begin{aligned}
p^d_1(i,2) &= Pr[(\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1}) \cup && (6.5.2)\\
&\qquad (\alpha_{i+1} = 0 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1}) \cup\\
&\qquad (\alpha_{i+1} = 0 \cap n_{i,1} = 0) \cup (\alpha_{i+1} = 0 \cap n_{i+1,1} = N_{i+1,1}) \text{ at t+1} |\\
&\qquad ((\alpha_{i+1} = 1 \cap n_{i,1} = 0) \cup (\alpha_{i+1} = 1 \cap n_{i+1,1} = N_{i+1,1}))\\
&\qquad \cap(\alpha_{i+1} = 1 \cap n_{i,2} > 0 \cup n_{i+1,2} < N_{i+1,2}) \text{ at } t]
\end{aligned}
$$

Let us define the following events:

$$\begin{aligned}
V &= (\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1}) \\
D_0 &= (\alpha_{i+1} = 0 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1}) \\
B &= (\alpha_{i+1} = 1 \cap n_{i+1,1} = N_{i+1,1}) \\
S &= (\alpha_{i+1} = 1 \cap n_{i,1} = 0) \\
D_b &= (\alpha_{i+1} = 0 \cap n_{i+1,1} = N_{i+1,1}) \\
D_s &= (\alpha_{i+1} = 0 \cap n_{i,1} = 0) \\
W_2 &= (\alpha_{i+1} = 1 \cap n_{i,2} > 0 \cap n_{i+1,2} < N_{i+1,2})
\end{aligned} \tag{6.5.3}$$

The event B is the set of states in which M_{i+1} is blocked for a Type 1 part, while S is the set of states in which M_{i+1} is starved for a Type 1 part. These two sets are disjoint events because of the approximation that the probability of a machine being blocked and starved is zero as we stated in (3.2.3). Then

$$p_1^d(i,2) = Pr[V \cup D_0 \cup D_s \cup D_b | (S \cap W_2) \cup (B \cap W_2)] \tag{6.5.4}$$

By expanding this, we have

$$\begin{aligned}
&p_1^d(i,2) = \qquad (6.5.5)\\
&\quad \frac{Pr[S \cap W_2]}{Pr[B \cap W_2] + Pr[S \cap W_2]} \times \\
&\qquad \Big(Pr[V|S \cap W_2] + Pr[D_0|S \cap W_2] + Pr[D_s|S \cap W_2] + Pr[D_b|S \cap W_2] \Big) \\
&\quad + \frac{Pr[B \cap W_2]}{Pr[B \cap W_2] + Pr[S \cap W_2]} \times \\
&\qquad \Big(Pr[V|B \cap W_2] + Pr[D_0|S \cap W_2] + Pr[D_s|B \cap W_2] + Pr[D_b|B \cap W_2] \Big)
\end{aligned}$$

The fractions in the equations can be approximated as follows

$$\begin{aligned}
\frac{Pr[S \cap W_2]}{Pr[B \cap W_2] + Pr[S \cap W_2]} &\approx \frac{Pr[S]}{Pr[B] + Pr[S]} \\
\frac{Pr[B \cap W_2]}{Pr[B \cap W_2] + Pr[S \cap W_2]} &\approx \frac{Pr[B]}{Pr[B] + Pr[S]}
\end{aligned} \tag{6.5.6}$$

It remains for us to calculate the individual probabilities. We already have defined in (5.2.4) that M being up and starved is the same event as $M^u(i+1,1)$ being down at mode $\Delta_2^u(i+1,1)$. Also it is stated that M being up and blocked is equivalent to $M^d(i+1,1)$ being down at mode $\Delta_2^d(i+1,1)$. Therefore,

$$\begin{aligned}
Pr[S] &= X_2^u(i+1,1) \\
Pr[B] &= X_2^d(i,1)
\end{aligned} \tag{6.5.7}$$

Now, we need to calculate conditional probabilities in (6.5.5). First, note that the following conditional probabilities are zero:

$$\begin{aligned}
Pr[D_s|B \cap W_2] &= 0 \\
Pr[D_b|S \cap W_2] &= 0
\end{aligned} \tag{6.5.8}$$

This is due to our approximation in (3.2.3).

Next, $Pr[V|S \cap W_2]$ is the probability that M_{i+1} is working on a Type 1 part at time $t+1$, given that it was up and starved for Type 1, but not starved nor blocked for Type 2 at time t. This probability is the same as the probability that $M^u(i+1,1)$ is up and not blocked in time $t+1$ give that it was down at mode $\Delta_2^u(i,1)$. Therefore, this conditional probability is the transition probability from $\Delta_2^u(i+1,1)$ to $\Upsilon^u(i+1,1)$, which is given by.

$$\begin{aligned} Pr[V|S \cap W_2] &= Pr[\Upsilon^u(i+1,1) \cap n_{i+1,1} < N_{i,1} \text{ at } t+1| \\ &\qquad \Delta_2^u(i+1,1) \text{ at } t] \\ &= R^u(i,1)(1-q^d(i,1)) \end{aligned} \tag{6.5.9}$$

In a similar manner, $P[V|B \cap W_2]$, the conditional probability that M_{i+1} is working on Type 1 part at $t+1$, given that M_{i+1} was blocked, but was not blocked nor starved for Type 2 at t, can be written with building blcok parameters such that

$$\begin{aligned} Pr[V|B \cap W_2] &= Pr[\Upsilon^d(i,1) \cap n_{i,1} > 0 \text{ at } t+1| \\ &\qquad \Delta_2^d(i,1) \text{ at } t] \\ &= (1-q^u(i+1,1))R^d(i+1,1) \end{aligned} \tag{6.5.10}$$

This is because when $M^d(i+1,1)$ is repaired and $M^u(i+1,1)$ does not go into the idleness failure — fail while it is blocked — at the end of time step t, M_{i+1} will be no longer blocked and process a Type 1 part at time $t+1$.//

Next $P[D_s|S \cap W_2]$ is the probability that M_{i+1} goes down while it is working on Type 2. This is equivalent that $M^u(i+1,1)$ is initially at the down mode $\Delta_2^u(i+1,1)$ and $n_{i+1,1} < N_{i+1,1}$, but the down mode change takes place from $\Delta_2^u(i+1,1)$ to $\Delta_3^u(i+1,1)$ at the end of time step. Therefore,

$$\begin{aligned} Pr[D_s|S \cap W_2] &= Pr[\Delta_3^u(i+1,1) \cap n_{i+1,1} < N_{i+1,1} \text{ at } t+1| \\ &\qquad \Delta_2^u(i+1,1) \text{ at } t] \\ &= (1-R^u(i,1))q^d(i,1) \end{aligned} \tag{6.5.11}$$

For the similar reason, $P[D_b|B \cap W_2]$ is

$$
\begin{aligned}
Pr[D_b|B \cap W_2] &= Pr[\Delta_3^d(i,1) \cap n_{i,1} > 0 \text{ at } t+1|\Delta_2^d(i,1) \text{ at } t] \\
&= q^u(i+1,1)(1 - R^d(i+1,1))
\end{aligned} \tag{6.5.12}
$$

Next $P[D_0|S \cap W_2]$ is the probability that M_{i+1} gets not starved for Type 1 and at the same time it goes down, while it is working on Type 2. This is equivalent that $M^u(i+1,1)$ is initially at the down mode $\Delta_2^u(i+1,1)$ and $n_{i+1,1} < N_{i+1,1}$, but the down mode change takes place from $\Delta_2^u(i+1,1)$ to $\Delta_1^u(i+1,1)$ at the end of time step. Therefore,

$$
\begin{aligned}
Pr[D_0|S \cap W_2] &= Pr[\Delta_1^u(i+1,1) \cap n_{i+1,1} < N_{i+1,1} \text{ at } t+1| \\
&\qquad\qquad\qquad\qquad \Delta_2^u(i+1,1) \text{ at } t] \\
&= R^u(i,1)q^d(i,1)
\end{aligned} \tag{6.5.13}
$$

For the similar reason, $P[D_0|B \cap W_2]$ is

$$
\begin{aligned}
Pr[D_0|B \cap W_2] &= Pr[\Delta_1^d(i,1) \cap n_{i,1} > 0 \text{ at } t+1|\Delta_2^d(i,1) \text{ at } t] \\
&= q^u(i+1,1)R^d(i+1,1)
\end{aligned} \tag{6.5.14}
$$

Putting everything together, we have

$$
\begin{aligned}
p_1^d(i,2) \;=\;& \frac{1}{P_s(i,1)+P_b(i+1,1)} \times \qquad (6.5.15)\\
& \Bigg(P_s(i,1)\Big(R^u(i,1)q^d(i,1)+R^u(i,1)(1-q^d(i,1))+R^u(i,1)q^d(i,1)\Big)\\
& +P_b(i+1,1)\Big(q^u(i+1,1)(1-R^d(i+1,1))+(1-q^u(i+1,1))R^d(i+1,1)\\
& +q^u(i+1,1)R^d(i+1,1)\Big)\Bigg)
\end{aligned}
$$

Next, we derive the interruption of flow equation for $p_2^d(i,2)$, the transition probability from the up state to the state in which $M^d(i+1,2)$ is down and $B_{i+1,2}$ is full. This transition probability is,

$$
p_2^d(i,2) = Pr[\Delta_2^d(i,2) \text{ at } t+1 \,|\Upsilon^d(i,2)\cap n_{i,1}>0 \text{ at } t]
$$

We first start with the derivation of this equation by applying the fact that the probability of going out of a state is equal to the probability of going into that state.

$$
\begin{aligned}
X_2^d(i,2) \quad & \Big((1-q^u(i+1,2))R^d(i+1,2)+q^u(i+1,2)R^d(i+1,2)\\
& +q^u(i+1,2)(1-R^d(i+1,2))\Big)\\
& = W^d(i,2)p_2^d(i,2)+X_3^d(i+1,2)r_1^d(i,2)(1-R^d(i+1,2))
\end{aligned}
$$

If we simplify this equation, then

$$
\begin{aligned}
& X_2^d(i,2)\Big(R^d(i+1,2)+q^u(i+1,2)(1-R^d(i+1,2))\Big)\\
& = W^d(i,2)p_2^d(i,2)+X_3^d(i,2)r_1^d(i,2)(1-R^d(i+1,2))
\end{aligned}
$$

That is,

$$p_2^d(i,2) = \frac{1}{W^d(i,2)} \times \tag{6.5.16}$$
$$\left(X_2^d(i,2)\Big(R^d(i+1,2) + q^u(i+1,2)(1-R^d(i+1,2))\Big) - X_3^d(i,2) r_1^d(i,2)(1-R^d(i+1,2))\right)$$

As a next step, we derive the interruption of flow equation for $p_3^d(i,2)$, the transition probability from the up state to the state in which $M^d(i+1,2)$ is down and $B_{i+1,1}$ is full and at the same time M_{i+1} gets failed or switches to Type 1 production. Note that when M_{i+1} is up and not blocked for Type 2, it is impossible for M_{i+1} to get blocked for Type 2 when it either gets failed or switched to Type 1. Therefore,

$$\begin{aligned} p_3^d(i,2) &= Pr[\Delta_3^d(i,2) \text{ at } t+1 \,|\Upsilon^d(i,2) \cap n_{i,2} > 0 \text{ at } t] \\ &= 0 \end{aligned} \tag{6.5.17}$$

Similarly, the interruption of flow equations for $M^u(i,2)$ are:

$$\begin{aligned} p_1^d(i,2) &= \frac{1}{P_s(i,1) + P_b(i+1,1)} \times \\ & \left(P_s(i,1)\Big(R^u(i,1)q^d(i,1) + R^u(i,1)(1-q^d(i,1)) + R^u(i,1)q^d(i,1)\Big) \right. \\ & \quad + P_b(i+1,1)\Big(q^u(i+1,1)(1-R^d(i+1,1)) + (1-q^u(i+1,1))R^d(i+1,1) \\ & \qquad \left. + q^u(i+1,1)R^d(i+1,1)\Big)\right) \\ p_2^d(i,2) &= \frac{1}{W^d(i,2)} \times \\ & \left(X_2^d(i,2)\Big(R^d(i+1,2) + q^u(i+1,2)(1-R^d(i+1,2))\Big) \right. \\ & \qquad \left. - X_3^d(i,2) r_1^d(i,2)(1-R^d(i+1,2))\right) \\ p_3^d(i,2) &= Pr[\Delta_3^d(i,2) \text{ at } t+1 \,|\Upsilon^d(i,2) \cap n_{i,2} > 0 \text{ at } t] \\ &= 0 \end{aligned} \tag{6.5.18}$$

6.6 Idleness failure

Now we need to derive expressions for the idleness failure of the building block. The idleness failure is the transition from an up state to a down state while it is idle. The probability $q_1^d(i,2)$ represents the probability that M^d(i,2) gets failed to $\Delta_1^d(i,2)$ at $t+1$ given that it was up and starved at t. That is,

$$\begin{aligned} q_1^d(i,2) \quad &= \quad Pr[\Delta_1^d(i,1) \text{ at } t+1 \,|\Upsilon^d(i,1) \cap n_{i,1} = 0 \text{ at } t] \\ &= Pr[\alpha_{i+1} = 0 \cup \{\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cup n_{i+1,1} < N_{i+1,1}\}| \\ &\quad \alpha_{i+1} = 1 \cap (n_{i,1}(t) = 0 \cup n_{i+1,1} = N_{i+1,1}) \cap n_{i,2} = 0] \end{aligned} \tag{6.6.1}$$

That is, the idleness failure occurs when M_i gets failed while it is working on Type 3, or M_i switches a process to Type 1. Then (6.6.1) can be rewritten as

$$\begin{aligned} q_1^d(i,2) \quad &= \quad Pr[\alpha_{i+1} = 0 \text{ at } t+1| \\ &\quad \alpha_{i+1} = 1 \cap (n_{i,1}(t) = 0 \cup n_{i+1,1} = N_{i+1,1}) \cap n_{i,2} = 0 \text{ at } t] \\ &+Pr[\{\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cup n_{i+1,1} < N_{i+1,1}\} \text{ at } t+1| \\ &\quad \alpha_{i+1} = 1 \cap (n_{i,1}(t) = 0 \cup n_{i+1,1} = N_{i+1,1}) \cap n_{i,2} = 0 \text{ at } t] \end{aligned} \tag{6.6.2}$$

The first term represents the case that M_{i+1} gets failed while it is processing a Type 3 part. This quantity is related with parameters for Type 3 since M_{i+1} is allowed to fail while it is working Type 3. Note that the probability that M_{i+1} is starved or blocked for Type 1 as well as it is starved for Type 2 is expressed with the parameters for a Type 3 building block such that $W^d(i,3) + P_s(i,2) + X_2^d(i,2)$. That is, the observer in $L(i,3)$ may find that $M^d(i,3)$ is either working or starved or down in Mode 2. However, between these events, M_{i+1} can get failed only when it is working on Type 3. Therefore, the first term in (6.6.2) is a fraction of the failure probability, p_{i+1}, and we approximate it such that

$$\begin{aligned} & Pr[\alpha_{i+1} = 0 \text{ at } t+1| \quad (6.6.3) \\ & \qquad \alpha_{i+1} = 1 \cap (n_{i,1}(t) = 0 \cup n_{i+1,1} = N_{i+1,1}) \cap n_{i,2} = 0 \text{ at } t] \\ = & \; p_{i+1} \frac{W_d(i,2)}{W_d(i,2) + X_2(i,2) + P_s(i,2)} \end{aligned}$$

The second term in (6.6.2) is the case that M_{i+1} switches a process from Type 2 to Type 3. This quantity is similar with the failure probability, $p_1^d(i,2)$. Note that $M^d(i,1)$ get failure to Mode 1 when M_{i+1} is down or M_{i+1} switches a production from Type 2 to Type 1. Therefore, we can derived the quantity by eliminating the terms related to the local failure in the expression of $p_1^d(i,2)$. Then the second term of (6.6.2) is

$$\begin{aligned} & Pr[\{\alpha_{i+1} = 1 \cap n_{i,1} > 0 \cup n_{i+1,1} < N_{i+1,1}\} \text{ at } t+1| \quad (6.6.4) \\ & \qquad \alpha_{i+1} = 1 \cap (n_{i,1}(t) = 0 \cup n_{i+1,1} = N_{i+1,1}) \cap n_{i,2} = 0 \text{ at } t] \\ = & \; \frac{1}{P_s(i,1) + P_b(i+1,1)} \Big(P_s(i,1) R^u(i,1) + P_b(i+1,1) R^d(i+1,1) \Big) \end{aligned}$$

Therefore,

$$\begin{aligned} q_1^d(i,2) = & \; p_{i+1} \frac{W^d(i,2)}{W^d(i,2) + X_2^d(i,2) + P_s(i,2)} \quad (6.6.5) \\ & + \frac{1}{P_s(i,1) + P_b(i+1,1)} \Big(P_s(i,1) R^u(i,1) + P_b(i+1,1) R^d(i+1,1) \Big) \end{aligned}$$

Similarly,

$$\begin{aligned} q_1^u(i,2) = & \; p_i \frac{W^u(i,2)}{W^u(i,2) + X_2^u(i,2) + P_b(i,2)} \quad (6.6.6) \\ & + \frac{1}{P_s(i-1,1) + P_b(i,1)} \Big(P_s(i-1,1) R^u(i-1,1) + P_b(i,1) R^d(i,1) \Big) \end{aligned}$$

The second failure mode $X_2^d(i,2)$ and the third failure mode $X_3^d(i,2)$ represent the case that M_{i+1} is blocked for Type 2. Note that $M^d(i,2)$ cannot get starved and blocked at the same time with our assumption in (3.2.3), the idleness failures to these failure modes do not occur. That is,

$$\begin{aligned} q_2^d(i,2) &= 0 \\ q_3^d(i,3) &= 0 \end{aligned} \tag{6.6.7}$$

6.7 Boundary Conditions

Let us consider the last building block for Type 2, $L(K,2)$. The only reason that the observer in the buffer $B_{K,2}$ do not find a part moving out of the buffer is that the machine, $M_{K+1,2}$, is down. Since $M_{K+1,2}$ is the last machine and only machine downstream from the buffer, no remote failure is presented in the machine $M^d(K,2)$. Therefore the downstream machine has only one down state and no failure mode changes are presented. Also $M_{K+1,2}$ processes only only Type 1 parts, therefore no idleness failure occurs in $M^d(K,2)$. The parameters for $M^d(K,2)$ are

$$\begin{aligned} r^d(K,2) &= r_{K+1,2} \\ p^d(K,2) &= p_{K+1,2} \\ q^d(K,2) &= 0 \end{aligned} \tag{6.7.1}$$

Similarly, the parameters for the upstream are

$$\begin{aligned} r^d(K,0) &= r_{0,2} \\ p^d(0,2) &= p_{0,2} \\ q^d(0,2) &= 0 \end{aligned} \tag{6.7.2}$$

6.8 Summary of the Decomposition Equations for Type 2

For $i = 0 \ldots K$

6.8.1 Resumption of flow

$$r_1^d(i,2) = 1/\Big(W^d(i,1) + D_s(i,1) + D_b(i+1,1) + X_1^d(i,1)\Big) \times \tag{6.8.1}$$
$$\Big(W^d(i,1)(p_2^u(i+1,1) + p_2^d(i,1)) + D_s(i,1)(1 - R^u(i,1))r_{i+1}$$
$$+ D_b(i+1,1)r_{i+1}(1 - R^d(i+1,1))\Big)$$

$$r_2^d(i,2) = (1 - q^u(i+1,2))R^d(i+1,2) \tag{6.8.2}$$

$$r_3^d(i,2) = r_1^d(i,2)R^d(i+1,2) \tag{6.8.3}$$

$$\begin{aligned} r_1^u(i+1,2) &= 1/\Big(W^u(i+1,1) + D_s(i,1) + D_b(i+1,1) + X_1^u(i+1,1)\Big) \times \\ &\quad \Big(W^u(i+1,1)(p_2^u(i+1,1) + p_2^d(i,1)) \\ &\quad + D_s(i,1)(1 - R^u(i,1))r_{i+1} \\ &\quad + D_b(i+1,1)r_{i+1}(1 - R^d(i+1,1))\Big) \end{aligned} \tag{6.8.4}$$

$$r_2^u(i+1,2) = (1 - q^u(i+1,2))R^d(i+1,2) \tag{6.8.5}$$

$$r_3^u(i+1,2) = r_1^d(i,2)R^d(i,2) \tag{6.8.6}$$

6.8.2 Failure mode change

$$z^d_{1,2}(i,2) = 0 \tag{6.8.7}$$

$$z^d_{1,2}(i,2) = 0 \tag{6.8.8}$$

$$z^d_{1,2}(i,2) = q^u(i+1,2)R^d(i+1,2) \tag{6.8.9}$$

$$z^d_{2,3}(i,2) = q^u(i+1,2)(1-R^d(i+1,2)) \tag{6.8.10}$$

$$z^d_{3,1}(i,2) = (1-r^d_1(i,2))R^d(i+1,2) \tag{6.8.11}$$

$$z^d_{3,2}(i,2) = r^d_1(i,2)(1-R^d(i+1,2)) \tag{6.8.12}$$

$$z^u_{1,2}(i+1,2) = 0 \tag{6.8.13}$$

$$z^u_{1,3}(i+1,2) = 0 \tag{6.8.14}$$

$$z^u_{2,1}(i+1,2) = q^d(i,2)R^u(i,2) \tag{6.8.15}$$

$$z^u_{2,3}(i+1,2) = (1-R^u(i,2))q^d(i,2) \tag{6.8.16}$$

$$z^u_{3,1}(i+1,2) = R^u(i+1,2)(1-r^u(i+1,2)) \tag{6.8.17}$$

$$z^u_{3,2}(i+1,2) = (1-R^u(i+1,2))r^u(i+1,2) \tag{6.8.18}$$

6.8.3 Interruption of flow

$$
\begin{aligned}
p_1^d(i,2) &= \frac{1}{P_s(i,1)+P_b(i+1,1)} \times \\
&\quad \Bigg(P_s(i,1)\Big(R^u(i,1)q^d(i,1)+R^u(i,1)(1-q^d(i,1))+R^u(i,1)q^d(i,1)\Big) \\
&\quad\quad +P_b(i+1,1)\Big(q^u(i+1,1)(1-R^d(i+1,1)) \\
&\quad\quad +(1-q^u(i+1,1))R^d(i+1,1)+q^u(i+1,1)R^d(i+1,1)\Big)\Bigg)
\end{aligned} \tag{6.8.19}
$$

$$
\begin{aligned}
p_2^d(i,2) &= \frac{1}{W^d(i,2)} \times \\
&\quad \Big(X_2^d(i,2)\Big(R^d(i+1,2)+q^u(i+1,2)(1-R^d(i+1,2))\Big) \\
&\quad\quad -X_3^d(i,2)r_1^d(i,2)(1-R^d(i+1,2))\Big)
\end{aligned} \tag{6.8.20}
$$

$$
p_3^d(i,2) = 0 \tag{6.8.21}
$$

$$
\begin{aligned}
p_1^u(i+1,2) &= \frac{1}{P_s(i,1)+P_b(i+1,1)} \times \\
&\quad \Bigg(P_s(i,1)\Big(R^u(i+1,1)q^d(i,1)+R^u(i+1,1)(1-q^d(i,1)) \\
&\quad\quad +R^u(i+1,1)q^d(i,1)\Big) \\
&\quad\quad +P_b(i+1,1)\Big(q^u(i+1,1)(1-R^d(i,1)) \\
&\quad\quad +(1-q^u(i+1,1))R^d(i,1)+q^u(i+1,1)R^d(i,1)\Big)\Bigg)
\end{aligned} \tag{6.8.22}
$$

$$
\begin{aligned}
p_2^u(i+1,2) &= \frac{1}{W^d(i+1,2)} \times \\
&\quad \Big(X_2^d(i,2)\Big(R^d(i,2)+q^u(i+1,2)(1-R^d(i,2))\Big) \\
&\quad\quad -X_3^d(i,2)r_1^d(i,2)(1-R^d(i,2))\Big)
\end{aligned} \tag{6.8.23}
$$

$$
p_3^u(i+1,2) = 0 \tag{6.8.24}
$$

6.8.4 Idleness failure

$$
\begin{aligned}
q_1^d(i,2) &= p_{i+1}\frac{W^d(i,2)}{W^d(i,2)+X_2^d(i,2)+P_s(i,2)} && (6.8.25)\\
&\quad +\frac{\Big(P_s(i,1)R^u(i,1)+P_b(i+1,1)R^d(i+1,1)\Big)}{P_s(i,1)+P_b(i+1,1)}\\
q_2^d(i,2) &= 0 && (6.8.26)\\
q_3^d(i,2) &= 0 && (6.8.27)
\end{aligned}
$$

$$
\begin{aligned}
q_1^u(i+1,2) &= p_i\frac{W^u(i+1,2)}{W^u(i+1,2)+X_2^u(i+1,2)+P_b(i+1,2)} && (6.8.28)\\
&\quad +\frac{\Big(P_s(i,1)R^u(i,1)+P_b(i+1,1)R^d(i+1,1)\Big)}{P_s(i,1)+P_b(i+1,1)}\\
q_2^u(i+1,2) &= 0 && (6.8.29)\\
q_3^u(i+1,2) &= 0 && (6.8.30)
\end{aligned}
$$

Chapter 7

Decomposition Analysis for Type 3

7.1 State definitions

The downstream machine $M^d(i,3)$, $i = 1, ..., K-1$, represents all the Type 3 flow behavior from downstream $B_{i,3}$. The upstream machine $M^d(i,3)$ is up, when M_{i+1} is up and is either blocked or starved for Type 1, and is also starved or blocked for Type 2, but is not blocked for Type 3. That is,

$$\begin{aligned} \Upsilon^d(i,3) \quad = \quad & \{\alpha_{i+1} = 1 \cap (n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}) \\ & \cap(n_{i,2} = 0 \cup n_{i+1,2} = N_{i+1,2}) \cap n_{i+1,2} < N_{i+1,2}\} \end{aligned} \tag{7.1.1}$$

From the state definitions defined in Chapter 6, this up state can be express as

$$\Upsilon^d(i,3) = \{(\Delta_2^d(i,2) \cup \Delta_2^u(i+1,2)) \cap n_{i+1,3} < N_{i+1,3}\} \tag{7.1.2}$$

A down state of $M^d(i,3)$ is the complement of the up state,

$$\sim \{(\Delta_2^d(i,2) \cup \Delta_2^u(i+1,2)) \cap n_{i+1,3} < N_{i+1,3}\} \tag{7.1.3}$$

In order to derive the decomposition equations, we categorized the down state into three different states. Before we define each down state, the following terms are defined first for our convenience.

- $A = \{\Delta_2^d(i,2) \cup \Delta_2^u(i+1,2)\}$
- $B = \{n_{i+1,3} < N_{i+1,3}\}$

The event A is the state that M_{i+1} is up and either stated or blocked for Type 1 or Type 2. The down state in (7.1.3) can be written as $\sim \{A \cap B\}$. This event is equivalent to

$$\begin{aligned}\sim (A \cap B) &= \sim A \cup \sim B \\ &= (\sim A \cap B) \cup (A \cap \sim B) \cap (\sim A \cap \sim B)\end{aligned}$$

Therefore, three separate down states are defined as following:

$$\begin{aligned}\Delta_1^d(i,3) &= (\sim A \cap B) \\ &= \{(\sim \Delta_2^d(i,2) \cap \sim \Delta_2^u(i+1,2)) \cap n_{i+1,3} < N_{i+1,3}\} \\ \Delta_2^d(i,3) &= (A \cap B) \\ &= \{(\Delta_2^d(i,2) \cup \Delta_2^u(i+1,2)) \cap n_{i+1,3} = N_{i+1,3}\} \\ \Delta_3^d(i,3) &= (\sim A \cap \sim B) \\ &= \{(\sim \Delta_2^d(i,2) \cap \sim \Delta_2^u(i+1,2)) \cap n_{i+1,3} = N_{i+1,3}\}\end{aligned} \tag{7.1.4}$$

7.2 Equalities

For convenience, we define the following building block probabilities:

$$\begin{aligned}
W^u(i,3) &= Pr[\Upsilon^u(i,3) \cap n_{i,3} < N_{i,2}] \\
W^d(i,3) &= Pr[\Upsilon^d(i,3) \cap n_{i,3} > 0] \\
X^u_m(i,3) &= Pr[\Delta^u_m(i,3) \cap n_{i,3} < N_{i,2}], \qquad m = 1,2,3 \\
X^d_n(i,3) &= Pr[\Delta^d_n(i,3) \cap n_{i,3} > 0], \qquad n = 1,2,3 \\
P_b(i,3) &= Pr[\Upsilon^u(i,3) \cap n_{i,3} = N_{i,2}] \\
P_s(i,3) &= Pr[\Upsilon^d(i,3) \cap n_{i,3} = 0] \\
D_b(i,3) &= Pr[\Delta_1(i,3) \cap n_{i,3} = N_{i,3}] \\
D_s(i,3) &= Pr[\Delta_1(i,3) \cap n_{i,3} = 0] \\
& \quad i = 1 \dots K-1
\end{aligned} \tag{7.2.1}$$

In Equation (6.1.1), the following equality can be stated.

$$\Upsilon^d(i,3) = \{(\Delta^d_2(i,3) \cup \Delta^u_2(i+1,3)) \cap n_{i+1,3} < N_{i+1,3}\}$$

Then

$$\begin{aligned}
W^d(i,3) &= Pr[\Upsilon^d(i,3) \cap n_{i,3} < N_{i,3}] \\
&= Pr[(\Delta^d_2(i,3) \cup \Delta^u_2(i+1,3)) \cap n_{i+1,3} < N_{i+1,3} \cap n_{i,3} > 0] \\
&= W^u(i+1,2)
\end{aligned}$$

Next, $X^d_1(i,3)$ is the probability that the downstream machine of $L(i,3)$ is down in Mode 1 and not starved. From the definition (6.1.3) and (7.2.1), it is

$$
\begin{aligned}
X_1^d(i,3) &= Pr[\Delta_1^d(i,3) \cap n_{i,3} > 0] \\
&= Pr[\Delta^u(i+1,3) \cap n_{i+1,3} < N_{i+1,3}] \\
&= X_1^u(i+1,3)
\end{aligned}
$$

Therefore,

$$
X_1^d(i,3) = X_1^u(i+1,3) \tag{7.2.2}
$$

Next, $X_2^d(i,3)$ is the probability that the downstream machine of $L(i,3)$ is down in Mode 2 and not starved. From the definition (6.1.3), it is

$$
\begin{aligned}
X_2^d(i,3) &= Pr[\{\alpha_{i+1} = 1 \cap \{n_{i,3} = 0 \cup n_{i+1,3} = N_{i+1,3}\} \\
&\qquad \cap n_{i+1,3} = N_{i+1,3} \cap n_{i,3} > 0] \\
&= Pr[\Upsilon^u(i+1,3) \cap n_{i+1,3} = N_{i+1,3}] \\
&= P_b(i+1,3)
\end{aligned}
$$

A similar equality can be derived for $X_2^u(i+1,1)$. Therefore,

$$
X_2^d(i,3) = P_b(i+1,3) \tag{7.2.3}
$$

$$
X_2^u(i+1,3) = P_s(i,3) \tag{7.2.4}
$$

Last,

$$
\begin{aligned}
X_3^d(i,3) &= Pr[\alpha_{i+1} = 0 \cup \{\alpha_{i+1} = 1 \cap n_{i,3} > 0 \cup n_{i+1,3} < N_{i+1,3}\} \\
&\qquad \cap n_{i,3} > 0 \cap n_{i+1,3} = N_{i+1,3}] \\
&= Pr[\Delta_3^u(i+1,3) \cap n_{i+1,3} = N_{i+1,3}] \\
&= D_b(i+1,3)
\end{aligned}
$$

Again, $X_3^u(i+1,3)$ can be derived in the similar way. Therefore,

$$X_3^d(i,3) = D_b(i+1,3) \tag{7.2.5}$$

$$X_3^u(i+1,3) = D_s(i,3) \tag{7.2.6}$$

7.3 Resumption of flow

Since there are three down states for $M^d(i,2)$, three separate repair probabilities need to be derived. First, we define the following quantities for the derivations of the resumption of flow equations.

$$R^d(i,3) = \frac{\sum_{n=1}^{3} r_n^d(i,3) X_n^d(i,3)}{\sum_{n=1}^{3} X_n^d(i,3)}$$

likewise,

$$R^u(i,3) = \frac{\sum_{m=1}^{3} r_m^u(i,3) X_m^u(i,3)}{\sum_{m=1}^{3} X_m^d(i,3)}$$

By the definition the repair probability from the first down state of $M^d(i,3)$ is

$$\begin{aligned} r_1^d(i,3) &= Pr[\Upsilon^d(i,3) \text{ at } t+1 | \Delta_1^d(i,3) \text{ at } t] \\ &= Pr[(\Delta_2^d(i,2) \cup \Delta_2^u(i+1,2)) \cap n_{i+1,3} < N_{i+1,3} \text{ at } t+1 | \\ &\qquad (\sim \Delta_2^d(i,2) \cup \sim \Delta_2^u(i+1,2)) \cap n_{i+1,3} < N_{i+1,3} \text{ at } t] \end{aligned} \tag{7.3.1}$$

Since $M^d(i,3)$ is down at t, even if it is repaired, the buffer state $n_{i+1,3} < N_{i+1,3}$ will remain as it is at $t+1$. Therefore,

$$\begin{aligned} r_1^d(i,3) &= Pr[(\Delta_2^d(i,2) \cup \Delta_2^u(i+1,2)) \text{ at } t+1 | \\ &\qquad (\sim \Delta_2^d(i,2) \cap \sim \Delta_2^u(i+1,2)) \text{ at } t] \end{aligned} \tag{7.3.2}$$

Notice that

$$\begin{aligned}
&\{(\sim \Delta_2^d(i,2) \cap \sim \Delta_2^u(i+1,2))\} \\
&\quad = \{(\Upsilon^d(i,2) \cap n_{i,2} > 0) \cup (\Delta_1^d(i,2) \cap n_{i,2} > 0) \cup \\
&\qquad\qquad (\Delta_3^d(i,2) \cap n_{i,2} > 0) \cup (\Delta_3^d(i+1,2) \cap n_{i+1,2} < N_{i+1,2})\}
\end{aligned}$$

Since the assumption stated in (3.2.3),

$$\{n_{i,2} = 0 \cap n_{i+1,2} = N_{i+1,2}\} = \emptyset$$

it can be rewritten as

$$\begin{aligned}
&\{(\sim \Delta_2^d(i,2) \cap \sim \Delta_2^u(i+1,2))\} \\
&\quad = (\Upsilon^d(i,2) \cap n_{i,2} > 0) \cup (\Delta_1^d(i,2) \cap n_{i,2} > 0) \cup (\Delta_3^d(i,2)) \cup (\Delta_3^d(i+1,2))
\end{aligned}$$

Therefore,

$$\begin{aligned}
r_1^d(i,3) \quad = \quad & 1/\Big(W^d(i,2) + D_s(i,2) + D_b(i+1,2) + X_1^d(i,2)\Big) \times \\
& \Big(W^d(i,2)(p_2^u(i+1,2) + p_2^d(i,2)) \\
& \quad + D_s(i,2)(1 - R^u(i,2))R^d(i+1,2) \\
& \quad + D_b(i+1,2)R^u(i,2)(1 - R^d(i+1,2))\Big)
\end{aligned} \tag{7.3.3}$$

Next, the resumption of flow from the second down state is derived such that

$$
\begin{aligned}
r_2^d(i,3) &= Pr[\Upsilon^d(i,2) \text{ at } t+1 | \Delta_2^d(i,2) \text{ at } t] \\
&= Pr[(\Delta_2^d(i,2) \cup \Delta^u(i+1,2)) \cap n_{i+1,3} < N_{i+1,3} \text{ at } t+1 | \\
&\qquad (\Delta_2^d(i,2) \cup \Delta_2^u(i+1,2)) \cap n_{i+1,3} = N_{i+1,3} \text{ at } t] \\
&= Pr[(\Delta_2^d(i,2) \cup \Delta^u(i+1,2)) \text{ at } t+1 | \\
&\qquad (\Delta_2^d(i,2) \cup \Delta_2^u(i+1,2)) \cap n_{i+1,3} = N_{i+1,3} \text{ at } t] \\
&\quad \times Pr[n_{i+1,3} < N_{i+1,3} \text{ at } t+1 | \\
&\qquad (\Delta_2^d(i,2) \cup \Delta_2^u(i+1,2)) \cap n_{i+1,3} = N_{i+1,3} \text{ at } t]
\end{aligned} \tag{7.3.4}
$$

Note that

$$
\begin{aligned}
&Pr[\Delta_2^d(i,2) \cup \Delta_2^u(i+1,2) \text{ at } t+1 | \\
&\qquad (\Delta_2^d(i,2) \cup \Delta_2^u(i+1,2)) \cap n_{i+1,3} = N_{i+1,3} \text{ at } t] \\
&= Pr[\Delta_2^d(i,2) \cup \Delta_2^u(i+1,2) \text{ at } t+1 | \Delta_2^d(i,2) \cup \Delta_2^u(i+1,2) \text{ at } t] \\
&= (1 - q^u(i+1,3))
\end{aligned} \tag{7.3.5}
$$

Also,

$$
\begin{aligned}
&Pr[n_{i+1,3} < N_{i+1,3} \text{ at } t+1 | \\
&\qquad (\Delta_2^d(i,2) \cup \Delta_2^u(i+1,2)) \cap n_{i+1,3} = N_{i+1,3} \text{ at } t] \\
&= R^d(i+1,3)
\end{aligned} \tag{7.3.6}
$$

Therefore,

$$
r_2^d(i,3) = (1 - q^u(i+1,3))R^d(i+1,3) \tag{7.3.7}
$$

Next the derivation of $r_3^d(i,3)$ is as follows,

$$\begin{aligned}
r_3^d(i,3) &= Pr[\Upsilon^d(i,3) \text{ at } t+1 | \Delta_3^d(i,3) \text{ at } t] \quad (7.3.8)\\
&= Pr[\Upsilon^d(i,3) \text{ at } t+1 |\\
&\qquad (\sim \Delta_2^d(i,2) \cap \sim \Delta_2^u(i+1,2)) \cap n_{i+1,3} = N_{i+1,3} \text{ at } t]\\
&= Pr[(\Delta_2^d(i,2) \cup \Delta_2^u(i+1,2)) \text{ at } t+1 |\\
&\qquad (\sim \Delta_2^d(i,2) \cap \sim \Delta_2^u(i+1,2)) \cap n_{i+1,3} = N_{i+1,3} \text{ at } t]\\
&\quad \times Pr[n_{i+1,3} < N_{i+1,3} \text{ at } t+1 |\\
&\qquad (\sim \Delta_2^d(i,2) \cap \sim \Delta_2^u(i+1,2)) \cap n_{i+1,3} = N_{i+1,3} \text{ at } t]\\
&= r_1^d(i,3) R^d(i+1,3)
\end{aligned}$$

7.4 Failure Mode Changes

Again, the down state,$\Delta_1^d(i,3)$, represents the state that either M_{i+1} is down or M_{i+1} is working on Type 1 or Type 2. From the observer in $B_{i,3}$, any state change further downstream of M_{i+1}, except the resumption of flow, will not change the state of $M^d(i,3)$. Therefore, there in no transition from $\Delta_1^d(i,3)$ to any other down state. Therefore,

$$z_{1,2}^d(i,3) = 0 \quad (7.4.1)$$

$$z_{1,3}^d(i,3) = 0 \quad (7.4.2)$$

Next, once $M^d(i,3)$ is in the down state $\Delta_2^d(i,3)$ — M_{i+1} is up and is able to work for Type 3 due to starvation or blockage of Type 1 or Type 2 but idle due to blockage for Type 3, it can make a transition into two other down states. First, let us consider the transition to $\Delta_3^d(i,3)$

$$
\begin{aligned}
z^d_{2,3}(i,3) &= Pr[\Delta^d_3(i,3) \text{ at } t+1|\Delta^d_2(i,3) \text{ at } t] \qquad (7.4.3)\\
&= Pr[(\sim \Delta^d_2(i,2) \cap \sim \Delta^u_2(i+1,2)) \cap n_{i+1,3} = N_{i+1,3} \text{ at } t+1|\\
&\qquad (\Delta^d_2(i,2) \cup \Delta^u_2(i+1,2)) \cap n_{i+1,3} = N_{i+1,3} \text{ at } t]\\
&= Pr[(\sim \Delta^d_2(i,2) \cap \sim \Delta^u_2(i+1,2)) \text{ at } t|\\
&\qquad (\Delta^d_2(i,2) \cup \Delta^u_2(i+1,2)) \cap n_{i+1,3} = N_{i+1,3} \text{ at } t]\\
&\quad \times Pr[n_{i+1,3} = N_{i+1,3} \text{ at } t+1|\\
&\qquad (\Delta^d_2(i,2) \cup \Delta^u_2(i+1,2)) \cap n_{i+1,3} = N_{i+1,3} \text{ at } t]\\
&= q^u(i+1,3)(1 - R^d(i+1,3))
\end{aligned}
$$

With the similar approach, for $z^d_{2,1}(i,3)$,

$$
\begin{aligned}
z^d_{2,1}(i,3) &= Pr[\Delta^d_1(i,3) \text{ at } t+1|\Delta^d_2(i,3) \text{ at } t] \qquad (7.4.4)\\
&= Pr[(\sim \Delta^d_2(i,2) \cap \sim \Delta^u_2(i+1,2)) \cap n_{i+1,3} = N_{i+1,3} \text{ at } t+1|\\
&\qquad (\Delta^d_2(i,2) \cup \Delta^u_2(i+1,2)) \cap n_{i+1,3} = N_{i+1,3} \text{ at } t]\\
&= Pr[(\sim \Delta^d_2(i,2) \cap \sim \Delta^u_2(i+1,2)) \text{ at } t+1|\\
&\qquad (\Delta^d_2(i,2) \cup \Delta^u_2(i+1,2)) \cap n_{i+1,3} = N_{i+1,3} \text{ at } t]\\
&\quad \times Pr[n_{i+1,3} < N_{i+1,3} \text{ at } t+1|\\
&\qquad (\Delta^d_2(i,2) \cup \Delta^u_2(i+1,2)) \cap n_{i+1,3} = N_{i+1,3} \text{ at } t]\\
&= q^u(i+1,3)R^d(i+1,3)
\end{aligned}
$$

For $z^d_{3,1}(i,3)$

$$\begin{aligned}
z^d_{3,1}(i,3) &= Pr[\Delta^d_1(i,3) \text{ at } t+1 | \Delta^d_3(i,3) \text{ at } t] \quad (7.4.5)\\
&= Pr[(\sim \Delta^d_2(i,2) \cap \sim \Delta^u_2(i+1,2)) \cap n_{i+1,3} < N_{i+1,3} \text{ at } t+1 |\\
&\qquad (\sim \Delta^d_2(i,2) \cap \sim \Delta^u_2(i+1,2)) \cap n_{i+1,3} = N_{i+1,3} \text{ at } t]\\
&= Pr[(\sim \Delta^d_2(i,2) \cap \sim \Delta^u_2(i+1,2)) \text{ at } t+1 |\\
&\qquad (\sim \Delta^d_2(i,2) \cap \sim \Delta^u_2(i+1,2)) \cap n_{i+1,3} = N_{i+1,3} \text{ at } t]\\
&\quad \times Pr[n_{i+1,3} < N_{i+1,3} \text{ at } t+1 |\\
&\qquad (\sim \Delta^d_2(i,2) \cap \sim \Delta^u_2(i+1,2)) \cap n_{i+1,3} = N_{i+1,3} \text{ at } t]\\
&= (1 - r^d_1(i,3)) R^d(i+1,3)
\end{aligned}$$

Also for $z^d_{3,2}(i,3)$

$$\begin{aligned}
z^d_{3,2}(i,3) &= Pr[\Delta^d_1(i,2) | \Delta^d_2(i,3)] \quad (7.4.6)\\
&= Pr[(\Delta^d_2(i,2) \cup \Delta^u_2(i+1,2)) \cap n_{i+1,3} = N_{i+1,3} |\\
&\qquad (\sim \Delta^d_2(i,2) \cap \sim \Delta^u_2(i+1,2)) \cap n_{i+1,3} = N_{i+1,3}]\\
&= Pr[(\Delta^d_2(i,2) \cup \Delta^u_2(i+1,2)) |\\
&\qquad (\sim \Delta^d_2(i,2) \cap \sim \Delta^u_2(i+1,2)) \cap n_{i+1,3} = N_{i+1,3}]\\
&\quad \times Pr[n_{i+1,3} = N_{i+1,3} |\\
&\qquad (\sim \Delta^d_2(i,2) \cap \sim \Delta^u_2(i+1,2)) \cap n_{i+1,3} = N_{i+1,3}]\\
&= r^d_1(i,3)(1 - R^d(i+1,3))
\end{aligned}$$

7.5 Interruption of Flow

Since there are three down states, three interruption of flow probabilities need to be derived. First, the probability $p^d_1(i,3)$ is the transition from $\Upsilon^d(i,2)$ to $\Delta^d_1(i,3)$, which represents the failure of $M^d(i,3)$ when $B_{i,3}$ is not empty. This transition probability is

$$\begin{aligned} p_1^d(i,3) &= Pr[\Delta_1^d(i,3) \text{ at } t+1 | \Upsilon^d(i,3) \cap n_{i,3} > 0 \text{ at } t] \\ &= Pr[(\sim \Delta_2^d(i,2) \cap \sim \Delta_2^u(i+1,2)) \cap n_{i+1,3} < N_{i+1,3} \text{ at } t+1 | \\ &\quad (\Delta_2^d(i,2) \cup \Delta_2^u(i+1,2)) \cap n_{i+1,3} < N_{i+1,3} \text{ at } t] \\ &= Pr[(\sim \Delta_2^d(i,2) \cap \sim \Delta_2^u(i+1,2)) \cap n_{i+1,3} < N_{i+1,3} \text{ at } t+1 | \\ &\quad (\Delta_2^d(i,2) \cap n_{i+1,3} < N_{i+1,3}) \cup \\ &\quad (\Delta_2^u(i+1,2)) \cap n_{i+1,3} < N_{i+1,3} \text{ at } t] \end{aligned} \tag{7.5.1}$$

Note that if M_i is working on a Type 3 part at time t, it cannot get failed and blocked for a Type 3 part at the same time at $t+1$. Therefore,

$$\begin{aligned} p_1^d(i,3) &= Pr[(\sim \Delta_2^d(i,2) \cap \sim \Delta_2^u(i+1,2)) \text{ at } t+1 | \\ &\quad (\Delta_2^d(i,2) \cup (\Delta_2^u(i+1,2) \text{ at } t] \\ &= Pr[(\Upsilon^d(i,2) \cap n_{i,2} > 0) \cup (\Delta_1^d(i,2) \cap n_{i,2} > 0) \\ &\quad \cup \Delta_3^d(i,2) \cup \Delta_3^u(i+1,2) \text{ at } t+1 | \\ &\quad (\Delta_2^d(i,2)) \cup (\Delta_2^u(i+1,2) \text{ at } t] \end{aligned} \tag{7.5.2}$$

By applying the probability quantities defined in Section 7.2,

$$\begin{aligned} p_1^d(i,3) &= \frac{1}{P_s(i+1,3) + P_b(i,3)} \times \\ &\quad \Big(P_b(i,3) Pr[(\Upsilon^d(i,2) \cap n_{i,2} > 0) \cup (\Delta_1^d(i,2) \cap n_{i,2} > 0) \\ &\quad \cup \Delta_3^d(i,2) \cup \Delta_3^u(i+1,2) \text{ at } t+1 | \Delta_2^d(i,2) \text{ at } t] \\ &\quad + P_s(i+1,3) Pr[(\Upsilon^d(i,2) \cap n_{i,2} > 0) \cup (\Delta_1^d(i,2) \cap n_{i,2} > 0) \\ &\quad \cup \Delta_3^d(i,2) \cup \Delta_3^u(i+1,2) \text{ at } t+1 | \Delta_2^u(i+1,2) \text{ at } t] \Big) \end{aligned} \tag{7.5.3}$$

The following identities can be defined.

$$
\begin{aligned}
Pr[\Upsilon^d(i,2) \cap n_{i,2} > 0 \text{ at } t+1 | \Delta_2^d(i,2) \text{ at } t] &= q^u(i+1,2)R^d(i+1,2) \\
Pr[\Delta_1^d(i,2) \cap n_{i,2} > 0 \text{ at } t+1 | \Delta_2^d(i,2) \text{ at } t] &= (1-q^u(i+1,2))R^d(i+1,2) \\
Pr[\Delta_3^d(i,2) \text{ at } t+1 | \Delta_2^d(i,2) \text{ at } t] &= q^u(i+1,2)(1-R^d(i+1,2)) \\
Pr[\Delta_3^u(i+1,2) \text{ at } t+1 | \Delta_2^d(i,2) \text{ at } t] &= 0 \\
Pr[\Upsilon^d(i,2) \cap n_{i,2} > 0 \text{ at } t+1 | \Delta_2^u(i+1,2) \text{ at } t] &= q^d(i,2)R^u(i,2) \\
Pr[\Delta_1^d(i,2) \cap n_{i,2} > 0 \text{ at } t+1 | \Delta_2^u(i+1,2) \text{ at } t] &= (1-q^d(i,2))R^u(i,2) \\
Pr[\Delta_3^d(i,2) \text{ at } t+1 | \Delta_2^u(i+1,2) \text{ at } t] &= q^d(i,2)(1-R^u(i,2)) \\
Pr[\Delta_3^u(i+1,2) \text{ at } t+1 | \Delta_2^u(i+1,2) \text{ at } t] &= 0
\end{aligned}
$$

Then

$$
\begin{aligned}
p_1^d(i,3) &= \frac{1}{P_s(i+1,2)+P_b(i,2)} \times \qquad (7.5.4) \\
&\Bigg(P_s(i+1,2)\Big(R^u(i,2)q^d(i,2) + R^u(i,2)(1-q^d(i,2)) + (1-R^u(i,2))q^d(i,2) \Big) \\
&\quad + P_b(i,2)\Big(q^u(i+1,2)(1-R^d(i+1,2)) \\
&\quad + (1-q^u(i+1,2))R^d(i+1,2) + q^u(i+1,2)R^d(i+1,2) \Big) \Bigg)
\end{aligned}
$$

Next, we derive the interruption of flow equation for $p_2^d(i,3)$, the transition probability from the up state to the state in which $M^d(i+1,3)$ is down and $B_{i+1,3}$ is full. This transition probability is,

$$p_2^d(i,3) = Pr[\Delta_2^d(i,3) \text{ at } t+1 \; | \Upsilon^d(i,3) \cap n_{i,3} > 0 \text{ at } t]$$

We first start with the derivation of this equation by applying the fact that the probability of going out of a state is equal to the probability of going into that state.

$$
\begin{aligned}
X_2^d(i,3) \quad & \Big((1-q^u(i+1,3))R^d(i+1,3)+q^u(i+1,3)R^d(i+1,3) \\
& \qquad +q^u(i+1,3)(1-R^d(i+1,3))\Big) \\
& = W^d(i,3)p_2^d(i,3)+X_3^d(i+1,3)r_1^d(i,3)(1-R^d(i+1,3))
\end{aligned}
$$

If we simplify this equation, then

$$
\begin{aligned}
& X_2^d(i,3)\Big(R^d(i+1,3)+q^u(i+1,3)(1-R^d(i+1,3))\Big) \\
& \quad = W^d(i,3)p_2^d(i,3)+X_3^d(i,3)r_1^d(i,3)(1-R^d(i+1,3))
\end{aligned}
$$

That is,

$$
\begin{aligned}
p_2^d(i,3) = & \frac{1}{W^d(i,3)}\times \\
& \Big(X_2^d(i,3)\Big(R^d(i+1,3)+q^u(i+1,3)(1-R^d(i+1,3))\Big) \\
& \qquad -X_3^d(i,3)r_1^d(i,3)(1-R^d(i+1,3))\Big)
\end{aligned} \tag{7.5.5}
$$

As a next step, we derive the interruption of flow equation for $p_3^d(i,3)$, the transition probability from the up state to the state in which $M^d(i+1,3)$ is down and $B_{i+1,3}$ is full and at the same time M_{i+1} gets failed or switches to Type 1 production. Note that when M_{i+1} is up and not blocked for Type 2, it is impossible for M_{i+1} to get blocked for Type 2 when it either gets failed or switched to Type 1. Therefore,

$$
\begin{aligned}
p_3^d(i,3) &= Pr[\Delta_3^d(i,3) \text{ at } t+1 \mid \Upsilon^d(i,3)\cap n_{i,3}>0 \text{ at } t] \\
&= 0
\end{aligned} \tag{7.5.6}
$$

7.6 Idleness Failure

Again, the probability $q_1^d(i,3)$ represents the probability that M^d(i,1) is down at $t+1$ given that it was up and starved at t. That is,

$$q^d(i,1) = Pr[\Delta_1^d(i,1) \text{ at } t+1 \,|\Upsilon^d(i,1) \cap n_{i,1} = 0 \text{ at } t] \tag{7.6.1}$$

Note that the idleness failure of Type 3 is different from those of Type 1 and Type 2, since Type 3 is the lowest priority type. For example, the idleness failure in Type 1 occurs when M_i gets failed while it is either blocked or starved for Type 1. Likewise, for Type 2, the idleness failure happens when M_i is either get failed or it switches back to Type 1 production, while it is blocked or starved for Type 2. Suppose M_i is blocked and starved for Type 1 and Type 2, and it works for Type 3. While it is working for a Type 3 part, it is also blocked or starved for Type 3. In this case, M_i is blocked and starved for all the part types and is idle. Although, with our assumption, the machine cannot not get failed at this condition, the idleness failure can still occur for Type 3. That is, a Type 3 observer will find an idleness failure if M_i starts making any higher priority part type. We argue that the idleness failure for $M^d(i,3)$ is as follows:

$$\begin{aligned} q_1^d(i,3) \quad &= \quad \frac{P_b(i+1,2)}{P_s(i+1,2)+P_b(i,2)} R^d(i+1,2) \\ &\qquad + \frac{P_s(i,2)}{P_s(i+1,2)+P_b(i,2)} R^u(i,2) \end{aligned} \tag{7.6.2}$$

That is, $q_1^d(i,3)$ is the weighted sum of the repair probabilities of the higher priority part. If M_i is no longer blocked or starved for Type 1 or Type 2, while M_i is idle, it will start making the higher priority part type. Similarly for $q_1^u(i,3)$ is

$$\begin{aligned} q_1^u(i,3) \quad &= \quad \frac{P_b(i,2)}{P_s(i-1,2)+P_b(i,2)} R^d(i,2) \\ &\qquad + \frac{P_s(i-1,2)}{P_s(i-1,2)+P_b(i,2)} R^u(i-1,2) \end{aligned} \tag{7.6.3}$$

7.7 Summary of the Decomposition Equations for Type 3

For $i = 1 \dots K$

7.7.1 Resumption of flow

$$
\begin{aligned}
r_1^d(i,3) &= 1/\Big(W^d(i,2) + D_s(i,2) + D_b(i+1,2) + X_1^d(i,2)\Big) \times \\
&\Big(W^d(i,2)(p_2^u(i+1,2) + p_2^d(i,2)) \\
&\qquad + D_s(i,2)(1 - R^u(i,2))R^d(i+1,2) \\
&\qquad + D_b(i+1,2)R^u(i,2)(1 - R^d(i+1,2))\Big) \qquad (7.7.1) \\
r_2^d(i,3) &= (1 - q^u(i+1,3))R^d(i+1,3) \qquad (7.7.2) \\
r_3^d(i,3) &= r_1^d(i,3)R^d(i+1,3) \qquad (7.7.3)
\end{aligned}
$$

$$
\begin{aligned}
r_1^u(i+1,3) &= 1/\Big(W^u(i+1,2) + D_s(i,2) + D_b(i+1,2) + X_1^d(i,2)\Big) \times \\
&\Big(W^u(i+1,2)(p_2^u(i+1,2) + p_2^d(i,2)) \\
&\qquad + D_s(i,2)(1 - R^d(i,2))r_{i+1} \\
&\qquad + D_b(i+1,2)r_{i+1}(1 - R^d(i+1,2))\Big) \qquad (7.7.4) \\
r_2^u(i+1,3) &= (1 - q^d(i,3))R^d(i,3) \qquad (7.7.5) \\
r_3^u(i+1,3) &= r_1^d(i,3)R^u(i,3) \qquad (7.7.6)
\end{aligned}
$$

7.7.2 Failure mode changes

$$
\begin{aligned}
z^d_{1,2}(i,3) &= 0 && (7.7.7)\\
z^d_{1,3}(i,3) &= 0 && (7.7.8)\\
z^d_{2,3}(i,3) &= q^u(i+1,3)(1-R^d(i+1,3)) && (7.7.9)\\
z^d_{2,1}(i,3) &= q^u(i+1,3)R^d(i+1,3) && (7.7.10)\\
z^d_{3,1}(i,3) &= (1-r^d_1(i,3))R^d(i+1,3) && (7.7.11)\\
z^d_{3,2}(i,3) &= r^d_1(i,3)(1-R^d(i+1,3)) && (7.7.12)
\end{aligned}
$$

$$
\begin{aligned}
z^u_{1,2}(i+1,3) &= 0 && (7.7.13)\\
z^u_{1,3}(i+1,3) &= 0 && (7.7.14)\\
z^u_{2,3}(i+1,3) &= q^d(i,3)(1-R^u(i,3)) && (7.7.15)\\
z^u_{2,1}(i+1,3) &= q^d(i+1,3)R^u(i,3) && (7.7.16)\\
z^u_{3,1}(i+1,3) &= (1-r^u_1(i+1,3))R^u(i,3) && (7.7.17)\\
z^u_{3,2}(i+1,3) &= r^u_1(i+1,3)(1-R^u(i,3)) && (7.7.18)
\end{aligned}
$$

7.7.3 Interruption of flow

$$
\begin{aligned}
p_1^d(i,3) &= \frac{1}{P_s(i+1,2)+P_b(i,2)} \times \qquad (7.7.19)\\
&\Bigg(P_s(i+1,2)\Bigg(R^u(i,2)q^d(i,2)+R^u(i,2)(1-q^d(i,2))\\
&\qquad +(1-R^u(i,2))q^d(i,2)\Bigg)\\
&\quad +P_b(i,2)\Bigg(q^u(i+1,2)(1-R^d(i+1,2))\\
&\qquad +(1-q^u(i+1,2))R^d(i+1,2)+q^u(i+1,2)R^d(i+1,2)\Bigg)\Bigg)\\
p_2^d(i,3) &= \frac{1}{W^d(i,2)} \times \qquad (7.7.20)\\
&\Bigg(X_2^d(i,3)\Bigg(R^d(i+1,3)+q^u(i+1,3)(1-R^d(i+1,3))\Bigg)\\
&\qquad -X_3^d(i,3)r_1^d(i,3)(1-R^d(i+1,3))\Bigg)\\
p_3^d(i,3) &= 0 \qquad (7.7.21)\\
p_1^u(i+1,3) &= \frac{1}{P_s(i,2)+P_b(i+1,2)} \times \qquad (7.7.22)\\
&\Bigg(P_s(i,2)\Bigg(R^d(i+1,2)q^u(i+1,2)+R^d(i+1,2)(1-q^u(i+1,2))\\
&\qquad +(1-R^d(i+1,2))q^u(i+1,2)\Bigg)\\
&\quad +P_b(i+1,2)\Bigg(q^d(i,2)(1-R^u(i,2))\\
&\qquad +(1-q^d(i,2))R^u(i,2)+q^d(i,2)R^u(i,2)\Bigg)\Bigg)\\
p_2^u(i+1,3) &= \frac{1}{W^u(i+1,2)} \times \qquad (7.7.23)\\
&\Bigg(X_2^u(i+1,3)\Bigg(R^u(i,3)+q^d(i,3)(1-R^u(i,3))\Bigg)\\
&\qquad -X_3^u(i+1,3)r_1^u(i+1,3)(1-R^u(i,3))\Bigg)\\
p_3^u(i+1,3) &= 0 \qquad (7.7.24)
\end{aligned}
$$

7.7.4 Idleness failure

$$q_1^d(i,3) = \frac{P_b(i,2)R^d(i,2) + P_s(i-1,2)R^u(i-1,2)}{P_s(i-1,2) + P_b(i,2)} \tag{7.7.25}$$

$$q_2^d(i,3) = 0 \tag{7.7.26}$$

$$q_3^d(i,3) = 0 \tag{7.7.27}$$

$$q_1^u(i+1,3) = \frac{P_b(i+1,2)R^d(i+1,2) + P_s(i,2)R^u(i,2)}{P_s(i,2) + P_b(i,2)} \tag{7.7.28}$$

$$q_2^u(i+1,3) = 0 \tag{7.7.29}$$

$$q_3^u(i+1,3) = 0 \tag{7.7.30}$$

Chapter 8

Generalization of Decomposition

Although the decomposition equations introduced in Chapter 5, 6, and 7 are constructed for the line processing three part types, the state classifications in the building blocks are designed in such a way that the equations can be generalized and applied to a production line processing more than three-part types. This section presents an idea of the general decomposition based on the equations derived in the previous chapters. The total number of part types that the general flexible line processes is denoted by Z. In this section, we let j be the index for a specific part type. The decomposition is categorized into three different cases: the highest priority ($j = 1$), the intermediate priority ($j = 2, \ldots, Z$), and the lowest priority part type ($j = Z$).

8.1 Idea of Generalization

The decomposition equations in Section 5.8, 6.8, and 7.7 are constructed so that they can be generalized for a production line processing more then three part types. Note that the machines in the building blocks for all the part types share the same number of up and down states as well as the identical structure of the Markov transitions, regardless of the number of machines in the line and the number of part types. Moreover, comparing the equations in Section 5.8, 6.8, and 7.7, considerable similarity is found in those equations within part types. Observe that those quantities are the function of the adjacent building blocks within the part types and also the function of the immediately higher or lower part type building blocks. For example, the $p_1^d(i,2)$ in the building block, $L(i,2)$ is function of the parameters in $L(i+1,2), L(i,1), L(i,3), L(i+1,1)$, and $L(i+1,3)$. If these equations are generalized for Z-part-type line, the behavior of the lower or higher priority part type is observed by the immediately lower or higher part type building blocks. That is, for the line $L(i,j_0)$, the flow behavior of the lower priority part types $(j = j_0+1 \dots Z)$ is captured by $L(i, j_0+1)$, while the flow behavior of the higher priority part types $(j = 1 \dots j_0 - 1)$ is captured by $L(i, j_0 - 1)$.

8.2 General Z Part Type Decomposition Equations

8.2.1 Highest priority part type

Comparing the equations in Section 5.8, 6.8, and 7.7, it can be observed that the failure and repair probabilities for the first down state are different from those in other part types. In the first part type building block, the first down state represents only the failure of a real machine in the line. Therefore, as stated in (5.8.1), (5.8.4), (5.8.20), and (5.8.23), the machine parameters are identical to the real machine parameters. Also note that for the highest priority part, the existence of a lower priority part type is made apparent to Type 1 only through the existence of idleness failures. Note that idleness failure equations in the first part type (8.2.29) and (5.8.26) are the function of building blocks in Type 2. The decomposition equations for the highest priority part are the following:

Resumption of flow

$$r^d(i,1) = r_{i+1} \tag{8.2.1}$$
$$r_2^d(i,1) = (1-q^u(i+1,1))R^d(i+1,1)) \tag{8.2.2}$$
$$r_3^d(i,1) = r_{i+1}R^d(i+1,1) \tag{8.2.3}$$
$$r_1^u(i+1,1) = r_{i+1} \tag{8.2.4}$$
$$r_2^u(i+1,1) = \Big(1-q^d(i,1)\Big)R^u(i,1) \tag{8.2.5}$$
$$r_3^u(i+1,1) = R^u(i,1)r_{i+1} \tag{8.2.6}$$

Failure mode change

$$z_{1,2}^d(i,1) = 0 \tag{8.2.7}$$
$$z_{1,3}^d(i,1) = 0 \tag{8.2.8}$$
$$z_{2,1}^d(i,1) = q_1^u(i+1,1)R^d(i+1,1) \tag{8.2.9}$$
$$z_{2,3}^d(i,1) = q_1^u(i+1,1)((1-R^d(i+1,1)) \tag{8.2.10}$$
$$z_{3,1}^d(i,1) = \Big(1-r_{i+1}\Big)R^d(i+1,1) \tag{8.2.11}$$
$$z_{3,2}^d(i,1) = r_{i+1}\Big(1-R^d(i+1,1)\Big) \tag{8.2.12}$$
$$z_{1,2}^u(i+1,1) = 0 \tag{8.2.13}$$
$$z_{1,3}^u(i+1,1) = 0 \tag{8.2.14}$$
$$z_{2,1}^u(i+1,1) = q_1^d(i,1)R^u(i,1) \tag{8.2.15}$$
$$z_{2,3}^u(i+1,1) = \Big(1-R^u(i,1)\Big)q_1^d(i,1) \tag{8.2.16}$$
$$z_{2,1}^u(i+1,1) = q_1^d(i,1)R^u(i,1) \tag{8.2.17}$$
$$z_{3,1}^u(i+1,1) = R^u(i,1)\Big(1-r_{i+1}\Big) \tag{8.2.18}$$
$$z_{3,2}^u(i+1,1) = \Big(1-R^u(i,1)\Big)r_{i+1} \tag{8.2.19}$$

Interruption of flow

$$
\begin{aligned}
p_1^d(i,1) &= p_{i+1} && (8.2.20)\\
p_2^d(i,1) &= \frac{1}{W^d(i,1)} \times && (8.2.21)\\
& \Bigg[X_2^d(i,1)\Bigg(R^d(i+1,1) + q^u(i+1,1)\Big(1 - R^d(i+1,1)\Big)\Bigg) \\
& \qquad -X_3^d(i,1) r_{i+1}(1 - R^d(i+1,1))\Bigg] \\
p_3^d(i,1) &= 0 && (8.2.22)\\
p_1^u(i+1,1) &= p_{i+1} && (8.2.23)\\
p_2^u(i+1,1) &= \frac{1}{W^u(i+1,1)} \times && (8.2.24)\\
& \Bigg[X_2^d(i+1,1)\Bigg(R^u(i,1) + \Big(1 - R^u(i,1)\Big) q^d(i,1)\Bigg) \\
& \qquad -X_2^d(i+1,1)\Big(1 - R^u(i,1)\Big) r_{i+1}\Bigg] \\
p_3^u(i+1,1) &= 0 && (8.2.25)
\end{aligned}
$$

Idleness failure

$$
\begin{aligned}
q_1^d(i,1) &= p_{i+1} \frac{W_d(i,2)}{W_d(i,2) + X_2(i,2) + P_s(i,2)} && (8.2.26)\\
q_2^d(i,1) &= 0 && (8.2.27)\\
q_3^d(i,1) &= 0 && (8.2.28)\\
q^u(i+1,1) &= p_{i+1} \frac{W^u(i+1,2)}{W^u(i+1,2) + P_b(i+1,2) + X_2(i+1,2)} && (8.2.29)\\
q_2^u(i+1,1) &= 0 && (8.2.30)\\
q_3^u(i+1,1) &= 0 && (8.2.31)
\end{aligned}
$$

8.2.2 Intermediate priority part type

For the intermediate priority part type ($j = 2 \ldots Z-1$), we have to account for the possibility that an observer in a intermediate part type building block will see flow into her buffer cease because a machine switches to a higher priority part. The first down states in the building block not only represents the failure of the real machine in the line, but it also represent the interruption of the flow caused by switching a production to a higher priority part type. Therefore, the failure and repair probabilities for the first down states, $\Delta_1^u(i,j)$ and $\Delta_1^d(i,j)$, are more complicated than those in the first part type building blocks. Like the first part type line, the the existence of a lower priority part type is felt through the existence of idleness failure. Note that idleness failure equations in the second part type, (8.2.59) and (8.2.59), are the function of Type 1 as well as Type 3.

The decomposition equations for intermediate priority parts are following ($i = 0 \ldots K, j = 2 \ldots Z-1$).

Resumption of flow

$$\begin{aligned} r_1^d(i,j) = 1/\Big(&W^d(i,j-1) + D_s(i,j-1) + D_b(i+1,j-1) + X_1^d(i,j-1)\Big) \times \\ &\Big(W^d(i,1)(p_2^u(i+1,j-1) + p_2^d(i,j-1)) + D_s(i,j-1)(1-R^u(i,j-1))r_{i+1} \\ &+ D_b(i+1,j-1)r_{i+1}(1-R^d(i+1,j-1))\Big) \end{aligned} \tag{8.2.32}$$

$$r_2^d(i,j) = (1-q^u(i+1,j))R^d(i+1,j) \tag{8.2.33}$$

$$r_3^d(i,j) = r_1^d(i,j)R^d(i+1,j) \tag{8.2.34}$$

$$
\begin{aligned}
r_1^u(i+1,j) &= 1/\Big(W^u(i+1,j-1)+D_s(i,j-1)+D_b(i+1,j-1)+X_1^u(i+1,j-1)\Big)\times \\
&\Big(W^u(i+1,j-1)(p_2^u(i+1,j-1)+p_2^d(i,j-1)) \\
&\quad +D_s(i,j-1)(1-R^u(i,j-1))r_{i+1} \\
&\quad +D_b(i+1,j-1)r_{i+1}(1-R^d(i+1,j-1))\Big) \qquad (8.2.35)\\
r_2^u(i+1,j) &= (1-q^u(i+1,j))R^d(i+1,j) \qquad (8.2.36)\\
r_3^u(i+1,j) &= r_1^d(i,j)R^d(i,j) \qquad (8.2.37)
\end{aligned}
$$

8.2.3 Failure mode change

$$
\begin{aligned}
z_{1,2}^d(i,j) &= 0 \qquad (8.2.38)\\
z_{1,2}^d(i,j) &= 0 \qquad (8.2.39)\\
z_{1,2}^d(i,j) &= q^u(i+1,j)R^d(i+1,j) \qquad (8.2.40)\\
z_{2,3}^d(i,j) &= q^u(i+1,j)(1-R^d(i+1,j)) \qquad (8.2.41)\\
z_{3,1}^d(i,j) &= (1-r_1^d(i,2))R^d(i+1,j) \qquad (8.2.42)\\
z_{3,2}^d(i,j) &= r_1^d(i,j)(1-R^d(i+1,j)) \qquad (8.2.43)\\
z_{1,2}^u(i+1,j) &= 0 \qquad (8.2.44)\\
z_{1,3}^u(i+1,j) &= 0 \qquad (8.2.45)\\
z_{2,1}^u(i+1,j) &= q^d(i,j)R^u(i,j) \qquad (8.2.46)\\
z_{2,3}^u(i+1,j) &= (1-R^u(i,j))q^d(i,j) \qquad (8.2.47)\\
z_{3,1}^u(i+1,j) &= R^u(i+1,j)(1-r^u(i+1,j)) \qquad (8.2.48)\\
z_{3,2}^u(i+1,j) &= (1-R^u(i+1,j))r^u(i+1,j) \qquad (8.2.49)
\end{aligned}
$$

8.2.4 Interruption of flow

$$
\begin{aligned}
p_1^d(i,j) &= \frac{1}{P_s(i,j-1)+P_b(i+1,j-1)} \times && (8.2.50)\\
&\Bigg(P_s(i,j-1)\Bigg(R^u(i,j-1)q^d(i,j-1)+R^u(i,j-1)(1-q^d(i,j-1))\\
&\qquad +R^u(i,j-1)q^d(i,j-1)\Bigg)\\
&\quad +P_b(i+1,j-1)\Bigg(q^u(i+1,j-1)(1-R^d(i+1,j-1))\\
&\qquad +(1-q^u(i+1,j-1))R^d(i+1,j-1)+q^u(i+1,j-1)R^d(i+1,j-1)\Bigg)\Bigg)\\
p_2^d(i,j) &= \frac{1}{W^d(i,j)} \times && (8.2.51)\\
&\Bigg(X_2^d(i,j)\Bigg(R^d(i+1,j)+q^u(i+1,j)(1-R^d(i+1,j))\Bigg)\\
&\qquad\qquad -X_3^d(i,j)r_1^d(i,j)(1-R^d(i+1,j))\Bigg)\\
p_3^d(i,j) &= 0 && (8.2.52)
\end{aligned}
$$

$$
\begin{aligned}
p_1^u(i+1,j) &= \frac{1}{P_s(i,j-1)+P_b(i+1,j-1)} \times \\
&\quad \Bigg(P_s(i,j-1)\Big(R^u(i+1,j-1)q^d(i,j-1) + R^u(i+1,j-1)(1-q^d(i,j-1)) \\
&\qquad + R^u(i+1,j-1)q^d(i,j-1)\Big) \\
&\quad + P_b(i+1,j-1)\Big(q^u(i+1,j-1)(1-R^d(i,j-1)) \\
&\quad + (1-q^u(i+1,j-1))R^d(i,j-1) + q^u(i+1,j-1)R^d(i,j-1)\Big)\Bigg) && (8.2.53)\\
p_2^u(i+1,j) &= \frac{1}{W^d(i+1,j)} \times \\
&\quad \Bigg(X_2^d(i,j)\Big(R^d(i,j) + q^u(i+1,j)(1-R^d(i,j))\Big) \\
&\qquad - X_3^d(i,j)r_1^d(i,j)(1-R^d(i,j))\Bigg) && (8.2.54)\\
p_3^u(i+1,2) &= 0 && (8.2.55)
\end{aligned}
$$

8.2.5 Idleness failure

$$
\begin{aligned}
q_1^d(i,j) &= p_{i+1}\frac{W^d(i,j)}{W^d(i,j)+X_2^d(i,j)+P_s(i,j)} && (8.2.56)\\
&\quad + \frac{\Big(P_s(i,j-1)R^u(i,j-1)+P_b(i+1,j-1)R^d(i+1,j-1)\Big)}{P_s(i,j-1)+P_b(i+1,j-1)} \\
q_2^d(i,j) &= 0 && (8.2.57)\\
q_3^d(i,j) &= 0 && (8.2.58)
\end{aligned}
$$

$$
\begin{aligned}
q_1^u(i+1,j) &= p_i\frac{W^u(i+1,j)}{W^u(i+1,j)+X_2^u(i+1,j)+P_b(i+1,j)} && (8.2.59)\\
&\quad + \frac{\Big(P_s(i,j-1)R^u(i,j-1)+P_b(i+1,j-1)R^d(i+1,j-1)\Big)}{P_s(i,j-1)+P_b(i+1,j-1)} \\
q_2^u(i+1,j) &= 0 && (8.2.60)\\
q_3^u(i+1,j) &= 0 && (8.2.61)
\end{aligned}
$$

8.2.6 Lowest priority part type

The unique feature of the lowest priority part type ($j = Z$) is its idleness failure. Suppose that M_i is idle due to blockage of starvation for all the part types. At this moment, the observer in Type Z believes that a machine she watches is down like the observers in the rest of the part type building blocks. Note that with our assumption, M_i is not allowed to fail while it is idle. Therefore, the Type Z observer never sees the failure of her machine due to a failure of M_i while it is idle. However, the idleness failure still can be observed in a different way. Consider the following case; at this moment when M_i is idle, M_i becomes neither starved nor blocked for some higher priority part types ($j = 1 \ldots Z - 1$). Then the Type Z observer believes that the machine that has been idle fails. This failure of the machine in the building block occurs because this machine switches a production to a higher priority part type. Note that this is the only case that the idleness failure can occur. Therefore, the idleness failure for Type Z is different from the rest of the part types.

Resumption of flow

$$\begin{aligned}
r_1^d(i, Z) &= 1/\Big(W^d(i, Z-1) + D_s(i, Z-1) + D_b(i+1, Z-1) + X_1^d(i, Z-1)\Big) \times \\
&\quad \Big(W^d(i, Z-1)(p_2^u(i+1, Z-1) + p_2^d(i, Z-1)) \\
&\qquad + D_s(i, Z-1)(1 - R^u(i, Z-1))R^d(i+1, Z-1) \\
&\qquad + D_b(i+1, Z-1)R^u(i, Z-1)(1 - R^d(i+1, Z-1))\Big) \qquad (8.2.62)
\end{aligned}$$

$$r_2^d(i, Z) = (1 - q^u(i+1, Z))R^d(i+1, Z) \qquad (8.2.63)$$

$$r_3^d(i, Z) = r_1^d(i, Z)R^d(i+1, Z) \qquad (8.2.64)$$

$$
\begin{aligned}
r_1^u(i+1,Z) &= 1/\Big(W^u(i+1,Z-1)+D_s(i,Z-1)+D_b(i+1,Z-1)+X_1^d(i,Z-1)\Big)\times \\
&\quad \Big(W^u(i+1,Z-1)(p_2^u(i+1,Z-1)+p_2^d(i,Z-1)) \\
&\qquad +D_s(i,Z-1)(1-R^d(i,Z-1))r_{i+1} \\
&\qquad +D_b(i+1,Z-1)r_{i+1}(1-R^d(i+1,Z-1))\Big) \quad (8.2.65)
\end{aligned}
$$

$$r_2^u(i+1,Z) = (1-q^d(i,Z))R^d(i,Z) \quad (8.2.66)$$

$$r_3^u(i+1,Z) = r_1^d(i,Z)R^u(i,Z) \quad (8.2.67)$$

Failure mode changes

$$z_{1,2}^d(i,Z) = 0 \quad (8.2.68)$$

$$z_{1,3}^d(i,Z) = 0 \quad (8.2.69)$$

$$z_{2,3}^d(i,Z) = q^u(i+1,Z)(1-R^d(i+1,Z)) \quad (8.2.70)$$

$$z_{2,1}^d(i,Z) = q^u(i+1,Z)R^d(i+1,Z) \quad (8.2.71)$$

$$z_{3,1}^d(i,Z) = (1-r_1^d(i,Z))R^d(i+1,Z) \quad (8.2.72)$$

$$z_{3,2}^d(i,Z) = r_1^d(i,Z)(1-R^d(i+1,Z)) \quad (8.2.73)$$

$$z_{1,2}^u(i+1,Z) = 0 \quad (8.2.74)$$

$$z_{1,3}^u(i+1,Z) = 0 \quad (8.2.75)$$

$$z_{2,3}^u(i+1,Z) = q^d(i,Z)(1-R^u(i,Z)) \quad (8.2.76)$$

$$z_{2,1}^u(i+1,Z) = q^d(i+1,Z)R^u(i,Z) \quad (8.2.77)$$

$$z_{3,1}^u(i+1,Z) = (1-r_1^u(i+1,Z))R^u(i,Z) \quad (8.2.78)$$

$$z_{3,2}^u(i+1,Z) = r_1^u(i+1,Z)(1-R^u(i,Z)) \quad (8.2.79)$$

Interruption of flow

$$\begin{aligned}
p_1^d(i,Z) &= \frac{1}{P_s(i+1,Z-1)+P_b(i,Z-1)} \times \\
&\Bigg(P_s(i+1,Z-1)\Bigg(R^u(i,Z-1)q^d(i,Z-1)+R^u(i,Z-1)(1-q^d(i,Z-1)) \\
&\qquad +(1-R^u(i,Z-1))q^d(i,Z-1)\Bigg) \\
&\quad +P_b(i,Z-1)\Bigg(q^u(i+1,Z-1)(1-R^d(i+1,Z-1)) \\
&\qquad +(1-q^u(i+1,Z-1))R^d(i+1,Z-1) \\
&\qquad +q^u(i+1,Z-1)R^d(i+1,Z-1)\Bigg)\Bigg)
\end{aligned} \tag{8.2.80}$$

$$\begin{aligned}
p_2^d(i,Z) &= \frac{1}{W^d(i,Z-1)} \times \\
&\Bigg(X_2^d(i,Z)\Bigg(R^d(i+1,Z)+q^u(i+1,Z)(1-R^d(i+1,Z))\Bigg) \\
&\qquad -X_3^d(i,Z)r_1^d(i,Z)(1-R^d(i+1,Z))\Bigg)
\end{aligned} \tag{8.2.81}$$

$$p_3^d(i,Z) = 0 \tag{8.2.82}$$

$$
\begin{aligned}
p_1^u(i+1,Z) &= \frac{1}{P_s(i,Z-1)+P_b(i+1,Z-1)} \times \\
&\Bigg(P_s(i,Z-1)\Bigg(R^d(i+1,Z-1)q^u(i+1,Z-1) \\
&\quad +R^d(i+1,Z-1)(1-q^u(i+1,Z-1)) \\
&\quad +(1-R^d(i+1,Z-1))q^u(i+1,Z-1)\Bigg) \\
&\quad +P_b(i+1,Z-1)\Bigg(q^d(i,Z-1)(1-R^u(i,Z-1)) \\
&\quad +(1-q^d(i,Z-1))R^u(i,Z-1)+q^d(i,Z-1)R^u(i,Z-1)\Bigg)\Bigg) \qquad (8.2.83)
\end{aligned}
$$

$$
\begin{aligned}
p_2^u(i+1,Z) &= \frac{1}{W^u(i+1,Z-1)} \times \\
&\Bigg(X_2^u(i+1,Z)\Bigg(R^u(i,Z)+q^d(i,Z)(1-R^u(i,Z))\Bigg) \\
&\quad -X_3^u(i+1,Z)r_1^u(i+1,Z)(1-R^u(i,Z))\Bigg) \qquad (8.2.84)
\end{aligned}
$$

$$
p_3^u(i+1,Z) = 0 \qquad (8.2.85)
$$

Idleness failure

$$
q_1^d(i,Z) = \frac{P_b(i,Z-1)R^d(i,Z-1)+P_s(i-1,Z-1)R^u(i-1,Z-1)}{P_s(i-1,Z-1)+P_b(i,Z-1)} \qquad (8.2.86)
$$

$$
q_2^d(i,Z) = 0 \qquad (8.2.87)
$$

$$
q_3^d(i,Z) = 0 \qquad (8.2.88)
$$

$$
q_1^u(i+1,Z) = \frac{P_b(i+1,Z-1)R^d(i+1,Z-1)+P_s(i,Z-1)R^u(i,Z-1)}{P_s(i,Z-1)+P_b(i,Z-1)} \qquad (8.2.89)
$$

$$
q_2^u(i+1,Z) = 0 \qquad (8.2.90)
$$

$$
q_3^u(i+1,Z) = 0 \qquad (8.2.91)
$$

Chapter 9

Algorithm and Numerical Behavior

Once the decomposition equations are constructed, an algorithm that will solve the equations needs to be developed. This chapter presents an algorithm to solve the decomposition equations. Then it provides numerical data for determining the accuracy of the algorithm. Although no mathematical proof of convergence is presented in this thesis, Numerical experiments show that the developed algorithm converges well for lines with reasonable line parameters. Also, it always converges to the same values, regardless of the starting point.

9.1 Algorithm

This section presents an algorithm for solving the decomposition equations derived in Chapter 8. The algorithm presented in the thesis is based on the DDX algorithm which won first developed for a single-part-type production line introduced in [6]. The new algorithm is distinguished from the previous one since it consists of two separate iterative loops. The inner loop sweeps down the line within a part type, using parameter values from previous iterations within a part type to calculate parameters at the current iteration, and then sweeps up the line in the reverse direction. The outer loop iteration sweeps up the line between part types, using parameter values from previous iteration between part types. The inner loop termination criterion is considered met when Conservation of Flow holds among all two-machine lines to within a certain tolerance. This method exploits the recursive nature of the interruption and resumption of flow equations.

Step 0: Initialization

Initialize upstream parameters

for i = 2 to NumMachines

for j = 1 to NumParts

$p_1^u(i,j) = p_2^u(i,j) = p_i$

$r_1^u(i,j) = r_2^u(i,j) = r_3^u(i,j) = r_i$

$q^u(i,j) = p_i$

$z_{2,1}^u(i,j) = z_{2,3}^u(i,j) = z_{3,2}^u(i,j) = z_{3,1}^u(i,j) = p_i$

end;

end;

Initialize downstream parameters

for i = 1 to NumMachines-1

for j = 1 to NumParts

$p_1^d(i,j) = P_2^d(i,j) = p_i$

$r_1^d(i,j) = r_2^d(i,j) = r_3^d(i,j) = r_i$

$q^d(i,j) = p_i$

$z_{2,1}^d(i,j) = z_{2,3}^d(i,j) = z_{3,2}^d(i,j) = z_{3,1}^d(i,j) = p_i$

end;

end;

Initialize boundary conditions

for j = 1 to NumParts

$p^u(1,j) = p_{0,j}$

$r^u(1,j) = r_{0,j}$

$q^u(1,j) = 0$

$p^d(NumMachines + 1, j) = p_{NumMachines+1,j}$

$r^d(NumMachines + 1, j) = r_{NumMachines+1,j}$

$q^d(NumMachines + 1, j) = 0;$

end;

While (**Step C.1** criterion is not met) **do Step 1** through **Step J**

Outer loop iteration

While (**Step C.2** criterion is not met) **do Step 1.1** and **Step 1.2**

Inner loop iteration

Step 1.1: Upstream Sweep for Type 1

for i = 1 to NumMachines

Evaluate Two Machine Line $L(i-1,1)$

Calculate $p_1^u(i,1)$ and $p_1^u(i,1)$

Calculate $r_1^u(i,1), r_2^u(i,1)$ and $r_3^u(i,1)$

Calculate $z_{2,1}^u(i,1), z_{2,3}^u(i,1), z_{3,1}^u(i,1)$ and $z_{3,2}^u(i,1)$

end

Step 1.2: Downstream Sweep for Type 1

for i = NumMachines-1 to 1

Evaluate Two Machine Line $L(i+1,1)$

Calculate $p_1^d(i,1)$ and $p_1^u(i,1)$

Calculate $r_1^d(i,1), r_2^u(i,j)$ and $r_3^u(i,1)$

Calculate $z_{2,1}^d(i,1), z_{2,3}^u(i,1), z_{3,1}^u(i,1)$ and $z_{3,2}^u(i,1)$

end

end

for j=2 to NumParts-1

While (**Step** 5 criterion is not met) **do Step j.1** and **Step j.2**

Step j.1: Upstream Sweep for Type j

for i = 1 to NumMachines

Evaluate Two Machine Line $L(i-1,j)$

Calculate $p_1^u(i,j)$ and $p_1^u(i,j)$

Calculate $r_1^u(i,j), r_2^u(i,j)$ and $r_3^u(i,j)$

Calculate $z^u_{2,1}(i,j), z^u_{2,3}(i,j), z^u_{3,1}(i,j)$ and $z^u_{3,2}(i,j)$

Calculate $q(i,j-1)$

end

Step j.2: Downstream Sweep for Type j

for i = NumMachines-1 to 1

Evaluate Two Machine Line $L(i+1,j)$

Calculate $p^d_1(i,j)$ and $p^u_1(i,j)$

Calculate $r^d_1(i,j), r^u_2(i,j)$ and $r^u_3(i,j)$

Calculate $z^d_{2,1}(i,j), z^u_{2,3}(i,j), z^u_{3,1}(i,j)$ and $z^u_{3,2}(i,j)$

Calculate $q(i,j-1)$

end;

Step C.1: Evaluate Inner Loop Stopping Criterion

Terminate the inner loop when the maximum value of

$$\|E(i,j) - E(0,j)\|$$

for $i = 1, \ldots$ NumMachines is less than some pre-specified ϵ *for each* part type j.

Step C.2: Evaluate Outer Loop Stopping Criterion

Terminate the algorithm when the maximum value of

$$\|E^{(m-1)}(i,j) - E^{(m)}(i,j)\|$$

where $i = 1, \ldots, NumMachines$ and $j = 1, \ldots, NumParts$ and (m) is the outer loop iteration repetition number,
is less than some pre-specified δ.

9.2 Algorithm Behavior

It is observed that the algorithm seems to converge most reliably when the mean time to fail (MTTF) and mean time to repair (MTTR) of all the machines in the line, including supply, processing, and demand machines are of the same order of magnitude as each other, and the MTTR one order of magnitude less than the MTTF. It is found that the algorithm is does not converge reliably when the MTTF and MTTR are radically different from one machine to the next, and if the algorithm does converge in these cases, it is usually not accurate. In physical terms, this corresponds to cases where adjacent machines fail and are repaired at radically different rates. It is found that, if the algorithm converges, it takes about five to ten iterations for the inner loop while it take less then three iterations for the our loop.

9.3 Randomly Generated Cases

It is hard to provide a mathematical proof of convergence of the algorithm, we follow the test procedure of Burman [3] to verify its reliability and accuracy. In the test, multiple randomly generate cases, where the parameters of the random systems are within certain pre-set tolerances, are evaluated with the algorithm. The results from the algorithm are compared with those from discrete-event simulations. This section presents the accuracy of the algorithm with respect to the simulation results for production rate and average buffer levels.

The randomly generated cases have machines that have similar, though not identical, characteristics. The machines are allowed to have different isolated efficiencies, but we do not generate lines where one machine is an extreme bottleneck. Note that the isolated production rate or isolated machine efficiency of an unreliable machine is:

$$e_i = \frac{r_i}{r_i + p_i}$$

In general, the isolated efficiencies of the supply and processing machines vary from

0.8 and 1. For the demand machines, repair probabilities are generated such that they are of the same order of magnitude as those generated for the supply and processing machines, but the failure probabilities of the demand machines are higher; they are of the same order of magnitude as the repair probabilities. This ensures that the demand rate for each part type is below the capacity of the line. This is because a system in which the demand machine for the highest priority has an isolated efficiency similar to that of the other machines in the line tends to be uninteresting, as the line spends all of its time producing type one products, and none producing lower priority parts. For example, a two-part-type processing line with eight processing machines behaves like a single-part transfer line with ten machines (where the additional machines are the supply machine upstream, and the demand machine downstream). Moreover, it is ensured that the combined rate for the demand machines for all the types is less than the capacity of the line. The reason for this is that if the line does have the capacity to meet demand for all the part types, then the estimation process is trivial; production rate will equal demand, and all intermediate buffers will be nearly full. Therefore, we ensure that

$$e_{d,j} < min(e_i) < \sum_j e_{d,j}$$

where $e_{d,j}$ is the isolated demand rate for Type j.

9.4 Computation of Approximation Error

Each simulation consisted of 10 independent simulations runs of 1,500,000 time periods each, where the first 500,000 time periods were discarded to ensure data was only collected on a system in steady state. The length of the simulation runs was chosen so that the transient period did not affect the results. The performance measures simulated were the production rate for each part type, and the average inventory level for each type in all buffers. Approximate 95% confidence intervals (see [2, 9]) were calculated for all performance measures.

For production rates, the percent error of the approximated production rate from the simulated production rate is evaluated in the following manner.

$$\% \text{ Error} = 100 \times \frac{E_{decomp} - E_{sim}}{E_{sim}}. \tag{9.4.1}$$

This metric is standard in the literature, and provides easy recognition of whether the approximation is under- or over-estimating the simulated production rate.

For average buffer levels, the percent error of the approximated average buffer level from that of the simulated buffer level is calculated using a metric that took the difference between the approximated average buffer level ($\overline{N}_{decomp}$) and the simulated average buffer level ($\overline{N}_{sim}$), divided the difference by the buffer size (N), and multiplied the quotient by 100. In other words,

$$\% \text{ Error} = 100 \times \frac{\overline{N}_{decomp} - \overline{N}_{sim}}{N}. \tag{9.4.2}$$

This measurement is standard in the literature cited as well.

9.5 Numerical Results for Two-Part-Type Lines

Production lines with five processing machines processing two part types were test with the algorithm. For the test, 300 random lines are generated. The first 100 random lines are lines where demand for Type 1 and Type 2 are roughly the same. The second 100 cases are of the line where the demand for part Type 1 exceeds that of Type 2 by up to 30%. The remaining 100 cases are of the line where the demand for part Type 2 exceeds that of part Type 1 by up to 30%. Buffers size vary from 5 to 20. For the machines modeling the supply process, and the processing machines, the repair probabilities were generated from a tight triangular distribution with mean 0.1 and a range of values from 0.08 to 0.12. The failure probabilities (the p parameters) were generated from a tight triangular distribution with mean 0.01

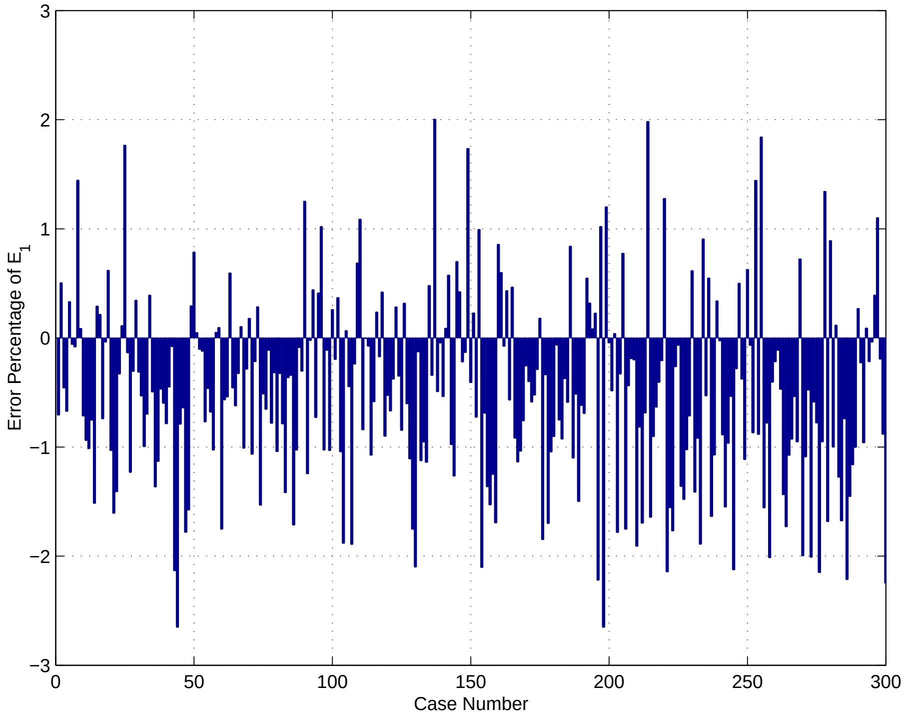

Figure 9.5.1: The errors in the decomposition approximation for Type 1 production rates

and a range of values from 0.05 to 0.015. Using these values gives an average isolated efficiency of approximately 0.91, taking values ranging from 0.8 to 1.0. The repair and failure probabilities for the demand machines were generated from similar triangular distributions, and are summarized in Table 9.6. There, $\overline{e}$ refers to the average isolated production rate of the supply machines and processing machines, and $\overline{e}_{d1}$ and $\overline{e}_{d2}$ are the average demand rates for part types one and two, respectively.

	Case 1-100	Case 101-200	Case 201-300
$\overline{e}$	0.91	0.91	0.91
$\overline{e}_{d1}$	0.55	0.55	0.4
$\overline{e}_{d2}$	0.55	0.4	0.55

Table 9.5.1: Machine parameters for the randomly generated two-part type processing lines.

The percent errors calculated for all 300 cases are shown in Figure 9.5.1 and 9.5.2. The average absolute errors for Type 1 is -0.52% while Type 2 is 2.2%. The

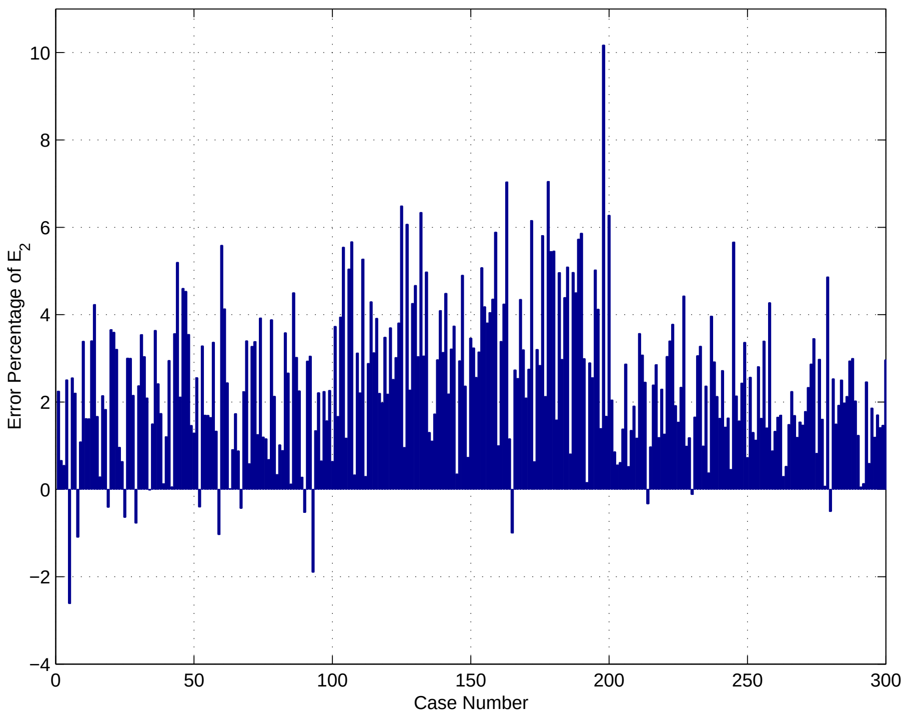

Figure 9.5.2: The errors in the decomposition approximation for Type 2 production rates

average errors for buffer levels are 7.3% and 8.2% for Type 1 and Type 2, respectively. As shown in the figures, the algorithm tends to slightly underestimate the Type 1 production rate, while overestimating for Type 2 parts.

9.6 Numerical Results for Three-Part Type Lines

Production lines with five processing machines processing three-part types is tested with the algorithm. For the test, 400 random lines are generated. The first 100 random lines are lines where demand for part Type 1, Type 2, and Type 3 are roughly the same. The second 100 cases are of the line where the demand for Type 2 and Type 3 are roughly the same while the demand for part Type 1 exceeds those of Type 2 and Type 3 by up to 30%. For the next 100 cases, the demand for Type 1 and Type 3 are roughly the same while the demand for part Type 2 exceeds those of Type 1 and Type 3 by up to 30%. The remaining 100 cases are of the line where the demand for Type 1 and Type 2 are roughly the same the demand for part Type

	Percent Error
Production rate for Type 1	-0.52
Production rate for Type 2	2.2
Average Buffer Level for Type 1	7.3
Average Buffer Level for Type 2	8.2

Table 9.5.2: Absolute percent errors for production rates and average buffer levels for the two-part type processing machines.

3 exceeds that of part Type 1 by up to 30%. Buffers size vary from 5 to 20 for all the cases. For the machines modeling the supply process, and the processing machines, the repair probabilities were generated from a tight triangular distribution ranging of values from 0.08 to 0.12. The failure probabilities (the p parameters) were generated from a tight triangular distribution with mean 0.01 and a range of values from 0.05 to 0.015. Using these values gives an average isolated efficiency of approximately 0.91, taking values ranging from 0.8 to 1.0. The repair and failure probabilities for the demand machines were generated from similar triangular distributions, and are summarized in Table 9.6.

	Case 1-100	Case 101-200	Case 201-300	Case 301-400
$\overline{e}$	0.91	0.91	0.91	0.91
$\overline{e}_{d1}$	0.4	0.4	0.3	0.3
$\overline{e}_{d2}$	0.4	0.3	0.4	0.3
$\overline{e}_{d3}$	0.4	0.3	0.3	0.4

Table 9.6.1: Machine parameters for the randomly generated three-part type processing lines.

The percent errors calculated for all 200 cases are shown in Figure 9.6.1, 9.6.2 and 9.6.3. The average absolute errors for Type 1 is -3.7%, Type 2 is 1.7%, and Type 3 is -8.7% The average errors for buffer levels are 8.5%, 7.4%, and 11.2%, for Type 1, Type 2, and Type 3 respectively. As shown in the figures, the algorithm tends to slightly underestimate the Type 1 and Type 3 production rate, while overestimating for Type 2 parts. It is interesting to note that the Type 2 results are the most accurate. More

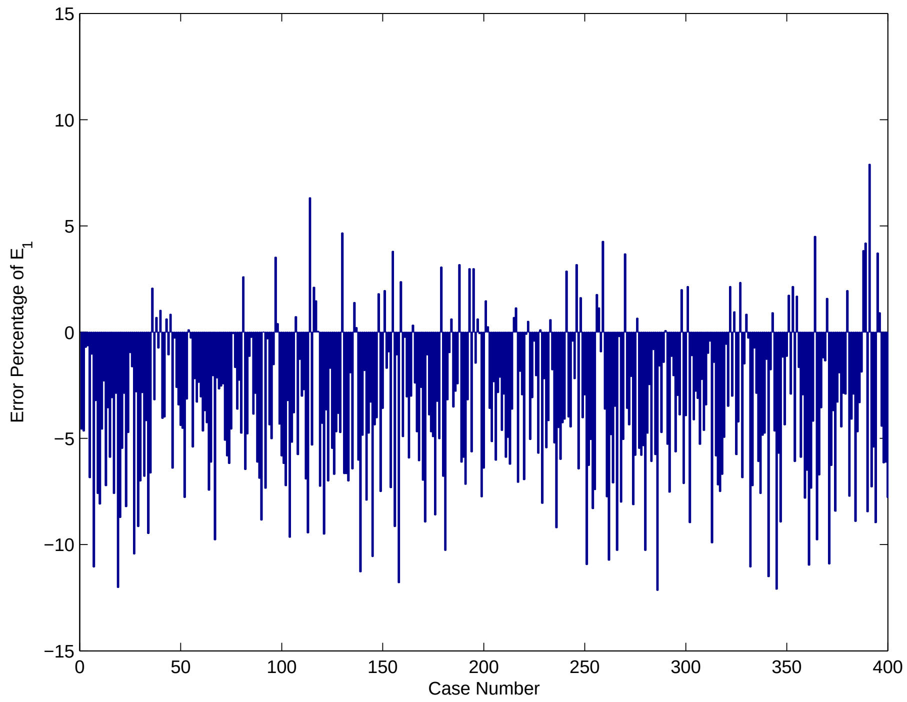

Figure 9.6.1: The errors in the decomposition approximation for Type 1 production rates

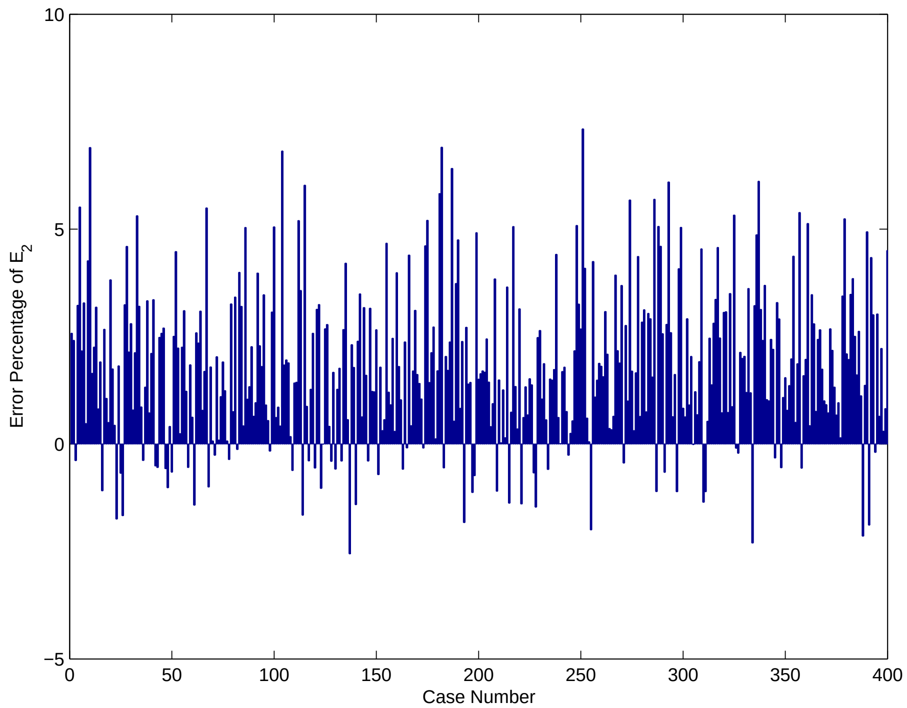

Figure 9.6.2: The errors in the decomposition approximation for Type 2 production rates

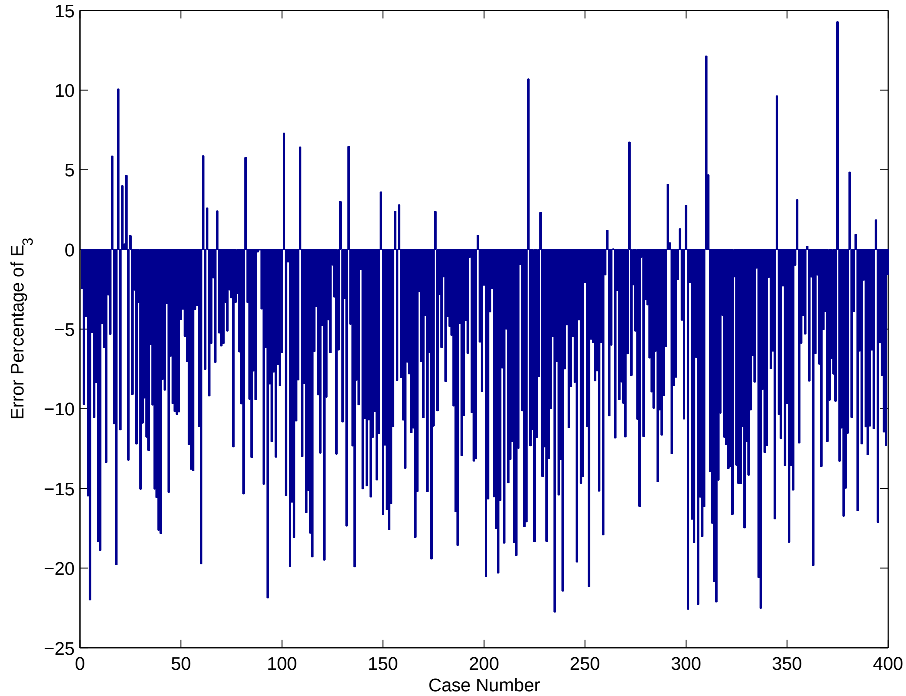

Figure 9.6.3: The errors in the decomposition approximation for Type 2 production rates

	Percent Error
Production rate for Type 1	-3.7
Production rate for Type 2	1.7
Production rate for Type 3	-8.7
Average Buffer Level for Type 1	8.5
Average Buffer Level for Type 2	7.4
Average Buffer Level for Type 3	11.2

Table 9.6.2: Absolute percent errors for production rates and average buffer levels for the three-part type processing machines.

research needs to be conducted to investigate this behavior.

Chapter 10

System Behavior

In order to understand the system behavior of the multiple-part-type line, system sensitivity analysis were performed with a line processing three different part types. This line consists of one processing machine, three demand machines, and three supply machines. All buffers are finite and machines are unreliable. Although the line has only one processing machine, it is long enough to capture the interesting dynamic behavior of the system presented in a multiple-part-type line. The processing machine in the line is denoted by M_1, and the supply and demand machines are denoted by $B_{0,j}$ and $B_{1,j}$. The line processes three different part types ($Z = 3$) and Type 1 ($j = 1$) is the highest priority part type, while Type 3 ($j = 3$) is the lowest priority part type. With our priority rule, the processing machine M_1 works on a Type 1 part when it finds a part in the Type 1 upstream buffer $B_{0,1}$ and finds space in the Type 1 downstream buffer, $B_{1,1}$. If not, it switches to the next priority part type, Type 2. Again, if the processing machine is blocked or starved for both Type 1 and Type 2, it switches to the lowest priority part type, Type 3. The throughput of Type j is denoted by E_j.

10.1 Changing Demand

We first investigate the system behavior of the line with a changing demand rate. We assume that the supply rates are high enough that the processing machine is rarely starved for input material. By choosing the high supply rates, we can concentrate on the effect of varying demand rates. Moreover, if we consider the processing machine as a flexible production line processing multiple products, it is more realistic to assume that demand rates have higher uncertainty compared to supply rates.

Let us consider the following hypothetical example. The machine in the line M_1 represents a production line processing three different sedans — a luxury sedan, a full size sedan, and an economy size sedan. The line is fully automated and flexible so that it can switch production from one model to another at a tiny fraction of the total production cost. The line is operating according to a strict priority rule. Since the luxury model is sold with the highest profit margin, the highest priority is given to the luxury model. If there is no demand for the luxury model, the line produces the full size model,which generates the second highest profit margin. The production of the economy sedan is performed when there is no demand for both the luxury and full size models. In this hypothetical case, the luxury, full size, and economy models are considered Type 1, Type 2, and Type 3, respectively.

Let us consider the case where the demand for Type 1 varies as shown in Figure 10.1.1. The corresponding system parameters are shown in Table 10.1.

Machines		Buffers	
$e_{0,1}$	0.91	$B_{0,1}$	15
$e_{0,2}$	0.91	$B_{0,2}$	15
$e_{0,3}$	0.91	$B_{0,3}$	15
$e_{1,1}$	Varying	$B_{1,1}$	15
$e_{1,2}$	0.5	$B_{1,2}$	15
$e_{1,3}$	0.5	$B_{1,3}$	15
e_1	0.833		

Table 10.1.1: Machine and buffer parameters for Figure 10.1.1

The supply machine ($M_{0,j}$) parameters are chosen in such a way that each supply machine has an isolated production rate of 0.91. The isolated production rate of the processing machine M_1 is 0.833. The isolated production rates of the demand

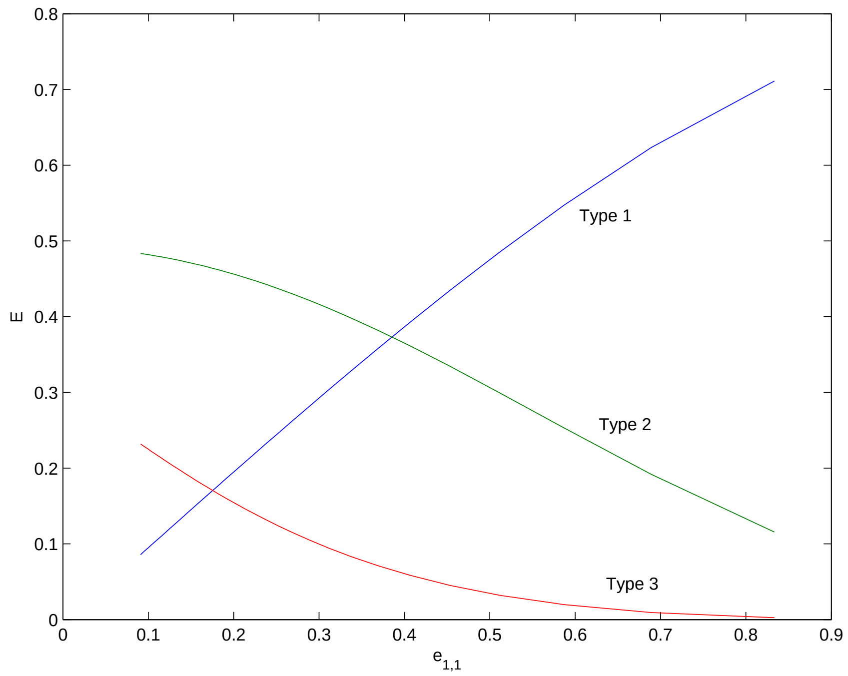

Figure 10.1.1: Throughput vs Demand for Type 1

machines for Type 2 and Type 3 are 0.5 and 0.5, respectively. The demand rate for Type 1 varies from 0.08 to 0.85. The sizes of the buffers are identical.

Changing rate for Type 1

In Figure 10.1.1, lines represent the throughput of a part type in response to the changing demand rate for Type 1. As shown in the figure, the production rate for Type 1 increases linearly at an almost a 45 degree angle. This is because Type 1 is the highest priority part and the processing machine always tries to meet the demand for Type 1 whenever possible. Therefore, as the demand for Type 1 increases, the processing machine spends more time to meet the demand for Type 1. However, as the Type 1 demand rate increases, the throughput rates for Type 1 and Type 2 decrease. Since the processing machines needs to spend more time on processing Type 1, it spends less time on processing Type 2 and Type 3. Note that the throughput of Type 2 is always higher than that of Type 3 due to our priority rule.

Machines		Buffers	
$e_{0,1}$	0.91	$B_{0,1}$	15
$e_{0,2}$	0.91	$B_{0,2}$	15
$e_{0,3}$	0.91	$B_{0,3}$	15
$e_{1,1}$	0.14	$B_{1,1}$	15
$e_{1,2}$	varying	$B_{1,2}$	15
$e_{1,3}$	0.33	$B_{1,3}$	15
e_1	0.83		

Table 10.1.2: Machine and buffer parameters for Figure 10.1.2

Changing rate for Type 2

Figure 10.1.2 illustrates the system behavior as the Type 2 demand rate varies. The system parameters are shown in Table 10.1. The demand rates for Type 1 and Type 3 are fixed at 0.14 and 0.33, respectively. In the figure, the throughput rate for Type 2 increases linearly as the demand rate for Type 2 increases. However, the throughput rate for Type 1 remains constant. Since Type 1 has priority over Type 2, the increasing demand rate for Type 2 does not affect the production of Type 1 parts. However, if the machine spends more time on processing Type 2, it will spend less time on processing Type 3. This is why the throughput rate for Type 3 drops as the demand rate for Type 2 increases.

Machines		Buffers	
$e_{0,1}$	0.91	$B_{0,1}$	15
$e_{0,2}$	0.91	$B_{0,2}$	15
$e_{0,3}$	0.91	$B_{0,3}$	15
$e_{1,1}$	0.25	$B_{1,1}$	15
$e_{1,2}$	0.25	$B_{1,2}$	15
$e_{1,3}$	Varying	$B_{1,3}$	15
e_1	0.833		

Table 10.1.3: Machine and buffer parameters for Figure 10.1.3

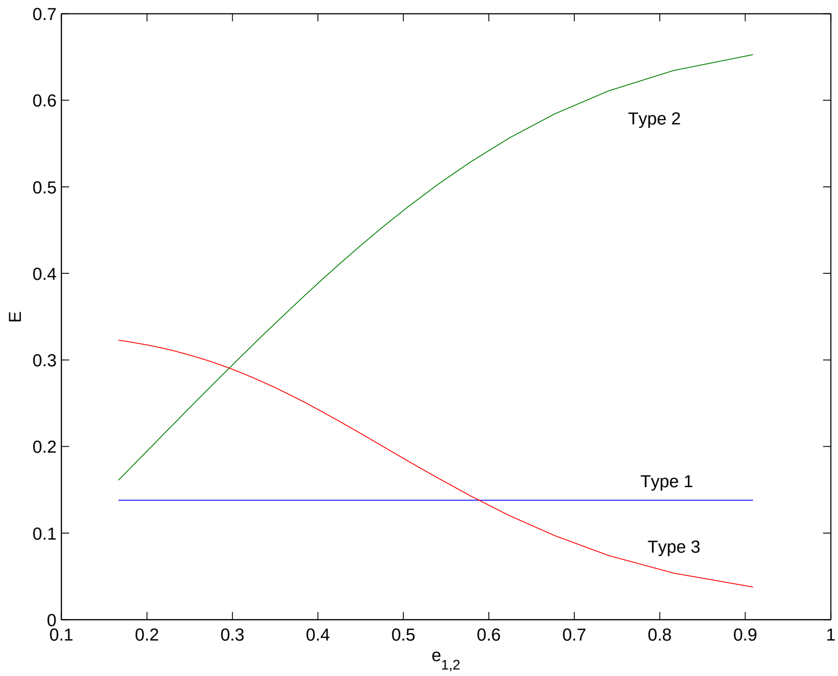

Figure 10.1.2: Throughput vs Demand for Type 2

Changing rate for Type 3

The next case is the system behavior due to the variation of the Type 3 demand rate shown in Figure 10.1.3. In this case, the demand rates for Type 1 and Type 2 are fixed, while the demand rate for Type 3 varies. In the figure, the throughput rate for Type 3 increases in response to the increasing demand rate for Type 3. However, the throughput rates for Type 1 and Type 2 remain constant. This is because Type 1 and Type 2 are higher priority part types, so they are not affected by the demand rate of the lower priority part. Note that the demand rates for Type 1 and Type 2 are 0.25. However, their throughput rates, E_1 and E_2, are less than this demand rates. This is because the size of the buffer is finite and not large enough to absorb the random behavior of the line. Also note that, due to the priority rule, the Type 1 throughput is slightly higher than the Type 2 throughput rate.

Another interesting behavior is observed in Figure 10.1.3. The throughput rate of Type 3 increases almost linearly for a while, but the rate of the increase drops slowly and gets saturated as the demand rate further increases. Note that the isolated

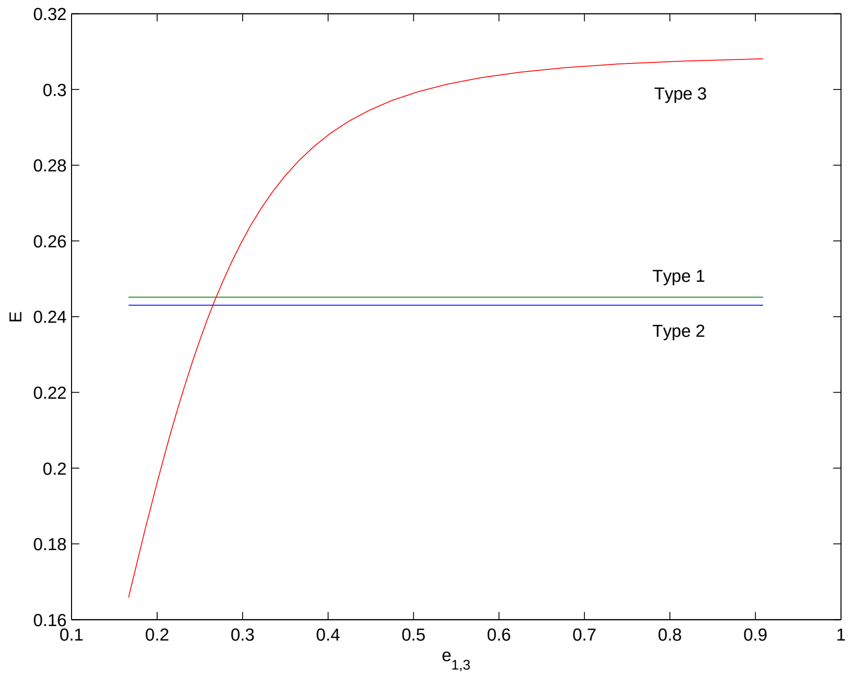

Figure 10.1.3: Throughput vs Demand for Type 3

production rate of the processing machine, e_1, is 0.833. Let us call this isolated production rate *line capacity.* When the demand rate for Type 3 is low, the total demand rate is less than the line capacity. That is,

$$e_{0,1} + e_{0,2} + e_{0,3} < e_1$$

In this case, the processing machine is under-utilized, and there is room to accommodate the increasing demand rate for the lowest priority part type. However, as the demand rate further increases, the processing machine gets to the line capacity and the throughput rate becomes saturated to the limit.

Machines		Buffers	
$e_{0,1}$	0.75	$B_{0,1}$	20
$e_{0,2}$	0.75	$B_{0,2}$	20
$e_{0,3}$	0.75	$B_{0,3}$	20
$e_{1,1}$	0.28	$B_{1,1}$	20
$e_{1,2}$	0.28	$B_{1,2}$	20
$e_{1,3}$	0.28	$B_{1,3}$	20
e_1	Varying		

Table 10.2.1: Machine and buffer parameters for Figure 10.2.1

10.2 Changing Processing Machine Capacity and Buffer Size

Changing machine capacity

This section investigates the system behavior due to the varying line capacity. The demand rates are fixed as shown in Table 10.2. The rate of the processing machine e_1 varies from 0.4 to 98. Figure 10.2.1 illustrates the system behavior resulting from changing e_1. Note that since a machine cannot produce more than its capacity, the combined throughput rate must be less then the line capacity, That is,

$$E_1 + E_2 + E_3 < e_1$$

As the line capacity increases, the throughput rate for each part type also increases. Note that once the line capacity, e_1, becomes larger than the Type 1 demand rate,

$$e_1 > e_{1,1}$$

the increase in the line capacity does not influence the Type 1 throughput rate much, because the processing machine already has a sufficient rate to meet the highest priority part type. Likewise, once the line capacity is larger than the combined Type

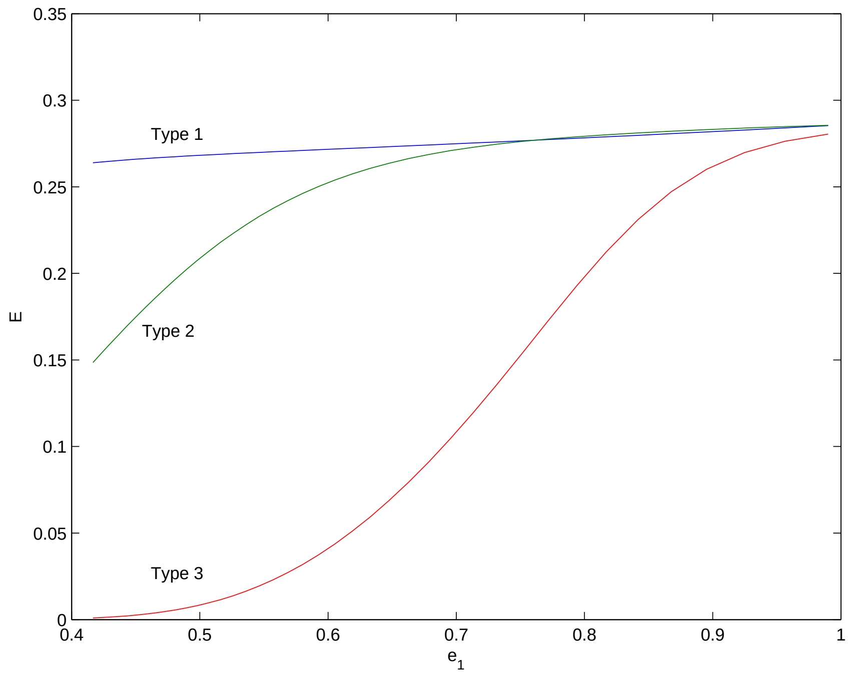

Figure 10.2.1: Throughput vs Capacity of M_1

1 and Type 2 demand rates,

$$e_1 > e_{1,1} + e_{1,2}$$

the increase in the line capacity does not impact the both throughput rate. The processing machine has sufficient capacity to meet the demand for both part types. In the case shown in Figure 10.2.1, e_1 varies from 0.4 to 0.98. The smallest value of e_1 in the figure, which is 0.4, is less than $e_{1,1} + e_{1,2}$. At this capacity, the Type 1 demand has almost met already and therefore, as the further increase in the capacity does not influence E_1, but the increase does influence E_2. Once both E_1 and E_1 come close to their corresponding demand rates, the Type 3 throughput rate begins to increase faster. Once all the demand rates are met, the throughput rates do not increase and they get saturated at their corresponding demand rates.

Changing buffer size

Figure 10.2.2 shows the throughput rate for each part type in response to the variation of the size of buffer $B_{1,2}$. The size of the buffer changes from 5 to 15. As shown in

Machines		Buffers	
$e_{0,1}$	0.99	$B_{0,1}$	5
$e_{0,2}$	0.99	$B_{0,2}$	5
$e_{0,3}$	0.99	$B_{0,3}$	5
$e_{1,1}$	0.25	$B_{1,1}$	10
$e_{1,2}$	0.33	$B_{1,2}$	Varying
$e_{1,3}$	0.33	$B_{1,3}$	10
e_1	0.91		

Table 10.2.2: Machine and buffer parameters for Figure 10.2.2

Table 10.2, the total demand rate is close to the rate of the processing machine. That is,

$$e_{1,1} + e_{1,2} + e_{1,3} \approx e_1$$

In general, the size of a buffer is the most influential when the upstream and downstream machines have similar isolated processing rates. In the figure, as $N_{1,2}$ increases, E_2 increases while E_3 decreases. When the size of the buffer $B_{1,2}$ gets bigger, M_1 becomes less blocked for Type 2 and therefore spends more time processing Type 2, Consequently, it spends less time on Type 3 and E_3 decreases. However, Type 1 is not affected since it is a higher priority part type. Note that the Type 2 throughput rate is saturated at 0.33 because the processing machine does not process parts more than the demand rate for Type 2.

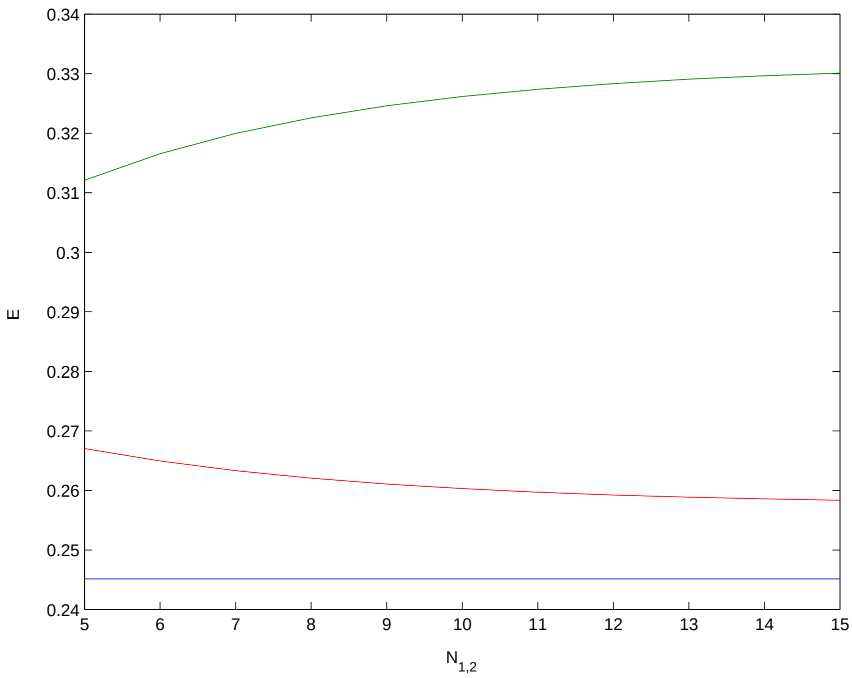

Figure 10.2.2: Throughput vs Size of

Chapter 11

Extension: Re-entrant Line

In this chapter we present a decomposition for a re-entrant production line. In developing equations for the building blocks for the re-entrant production line, we modify the decomposition model that has been created for the multiple-part-type line. Semiconductor device and liquid crystal display (LCD) fabrication processes are characterized as re-entrant processes, in which a similar sequence of processing step is always repeated several times.

11.1 Re-entrant Production Line

Figure 11.1.1 represents a re-entrant production line with a single re-entrant loop. In this example, a supply machine, M_0 and a demand machine, $M_{K+1,1}$ are placed at the beginning and end of the line. Once parts enter the line through the supply machine, they are first processed by machines from M_1 to M_K. During these processing steps, parts are stored in buffers $B_{i,2},\ i = 0 \dots K\}$. Then they are processed by the *re-entrant machine*, $M_{K+1,2}$. The function of this machine is to send parts back to

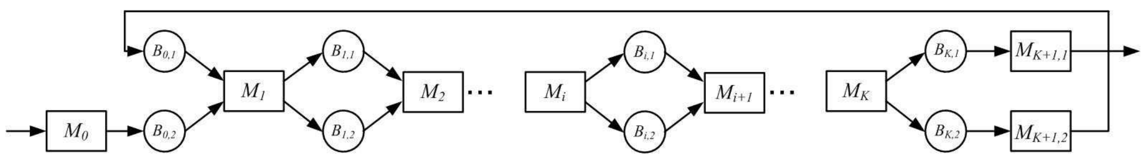

Figure 11.1.1: Re-entrant production line model

machine M_1 so that the parts can go through the same processing steps again. This re-entrant machine can be either an actual machine or an imaginary machine that logically creates the re-entrant loop.

11.2 Decomposition Equations

Note that in Figure 11.1.1 the configuration of the re-entrant line is similar to that of the production line processing multiple part types. The only difference is that the Type j flow is sent back as Type $j-1$ through $M_{K+1,j}$ and $M_{0,j-1}$. Therefore, all the decomposition equations developed for the multiple part type line can be applied to the re-entrant line with modifications. Suppose that in the two-part type production line, the parameters for the demand machine for Type 2, $M_{k+1,2}$, are given such that the machine imitates the flow behavior of Type 1 part in the line. Also, at the same time, the parameters for the supply machine for Type 1, $M_{0,1}$, are assigned such that the machine imitates the flow behavior of Type 2 parts in the line. Then we can require that the flow rate for Type 1 is equal to that of Type 2.

$$E_1 = E_2 \tag{11.2.1}$$

If we apply the above approach to the decomposition method for the re-entrant line, all the decomposition equations are the same as those constructed in two-part type line except the decomposition equations for $L(k,2)$ and $L(0,1)$.

Interruption of flow

For the interruption of flow for $M^u(0,1)$, we use the balance equation:

$$\sum_{i=1}^{3} Ps_i(K,2) r_i^u(K,2) = W^d(K-1,2) p_*^d \tag{11.2.2}$$

,where p_*^d is the probability that $M_{K+1,2}$ becomes starved due to any machine failure upstream of $B_{K,2}$. Then

$$\begin{aligned} p_1^u(0,j) &= p_*^d + p_{K+1,j+1} \\ &= \frac{1}{W^d(1,j+1)} \sum_{i=1}^{3} Ps_i(K,j+1) r_i^u(K,j+1) + p_{K+1,2} \end{aligned} \tag{11.2.3}$$

Similarly, for $M^d(2,2)$

$$\sum_{i=1}^{3} Pb_i(0,j) r_i^d(0,j) = W^u(1,j) p_*^u \tag{11.2.4}$$

$$\begin{aligned} p_1^d(K,j+1) &= p_*^u + p_{K+1,j+1} \\ &= \frac{1}{W^u(1,j)} \sum_{i=1}^{3} Pb_i(0,j) r_i^d(0,j) + p_{K+1,j+1} \end{aligned} \tag{11.2.5}$$

Resumption of flow

Flow rate idle time is used for the derivations of the resumption of flow equations.

$$E = e_{K+1,j+1}\Big(1 - \widetilde{Ps}(K,j+1) - \widetilde{Pb}(0,j)\Big) \tag{11.2.6}$$

where $e_{K+1,j+1} = \frac{r_{K+1,j+1}}{r_{K+1,j+1}+p_{K+1,j+1}}$, $\widetilde{Ps} = \sum_{i=1}^{3} Ps_i$ and $\widetilde{Pb} = \sum_{i=1}^{3} Pb_i$. Also we know that

$$E^u(0,j) = e^u(0,j)\Big(1 - \widetilde{Pb}(0,j)\Big)$$

$$E^d(K,j+1) = e^d(K,j+1)\Big(1 - \widetilde{Ps}(K,j+1)\Big)$$

These can be written

$$\widetilde{Ps}(K,j+1) = 1 - \frac{E^d(K,j+1)}{e^d(K,j+1)}$$

$$\widetilde{Pb}(0,j) = 1 - \frac{E^u(0,j)}{e^u(0,j)}$$

Then (11.2.6) becomes

$$E = e_{K+1,j+1}\left(\frac{E^d(K,j+1)}{e^d(K,j+1)} + \frac{E^u(0,j)}{e^u(0,j)} - 1\right)$$

or since $E = E^d(K,j+1) = E^u(1,j)$,

$$1 = e_{K+1,j+1}\left(\frac{1}{e^d(K,j+1)} + \frac{1}{e^u(0,j)} - \frac{1}{E}\right) \tag{11.2.7}$$

We know that

$$\frac{1}{e^u(0,j)} = \frac{p^u(0,j) + r^u(0,j)}{r^u(0,j)}$$

$$\frac{1}{e^d(K,j+1)} = \frac{p^d(K,2) + r^d(K,j+1)}{r^d(K,j+1)}$$

Then (11.2.7) becomes

$$1 = e_{K+1,j+1}\left(\frac{p^u(0,j) + r^u(0,j)}{r^u(0,j)} + \frac{p^d(K,j+1) + r^d(K,j+1)}{r^d(K,j+1)} - \frac{1}{E}\right)$$

$$1 = e_{K+1,j+1}\left(\frac{p^u(0,j)}{r^u(0,j)} + \frac{p^d(K,j+1)}{r^d(K,j+1)} - \frac{1}{E} + 2\right)$$

That is,

$$\frac{1}{e_{K+1,j+1}} + \frac{1}{E} - 2 = \frac{p^u(0,j)}{r^u(0,j)} + \frac{p^d(K,j+1)}{r^d(K,j+1)} \tag{11.2.8}$$

Two equation are introduced

$$I^u(0,1) = \frac{p^u(0,j)}{r^u(0,j)} \qquad \text{and} \qquad I^d(K,j+1) = \frac{p^d(K,j+1)}{r^d(K,j+1)} \tag{11.2.9}$$

Then we can rewrite (11.2.8) such that,

$$\begin{aligned} I^u(0,j) &= \frac{1}{E^d(K,j+1)} + \frac{1}{e_{K+1,j+1}} - I^d(K,j+1) - 2 \qquad (11.2.10) \\ I^d(K,j+1) &= \frac{1}{E^u(0,j)} + \frac{1}{e_{K+1,j+1}} - I^u(0,j) - 2 \end{aligned}$$

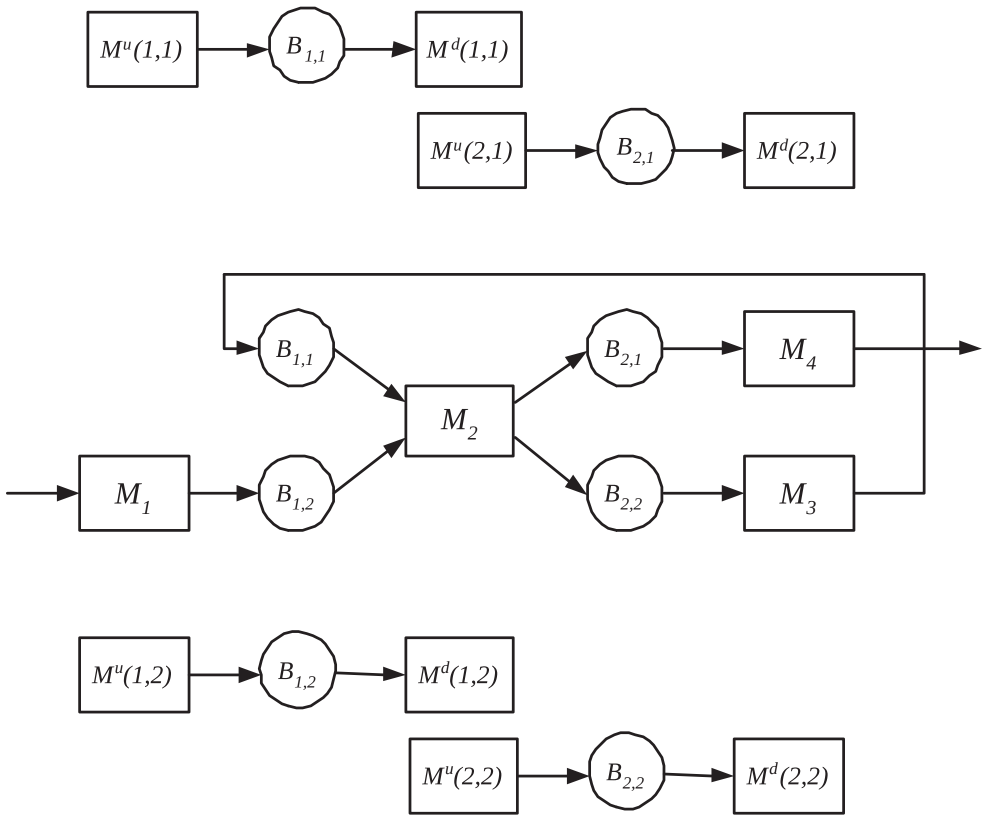

Figure 11.3.1: Simple re-entrance production line model

11.3 Numerical Results

In order to verify the analytic equations derived in the previous section, we compare the numerical results of a small system with four machines and four buffers with simulations. The small system is shown in Figure 11.3.1. Two separate cases are presented. For both cases, machine parameters of M_4 vary, while the rest of machine parameters remain constant. We examine the response of the production rate of the system to the varying parameters and compare the results with simulations.

11.3.1 Case1: Varying p_4 and r_4 (e_4 constant)

The system parameters are shown in Table 11.3.1. For this case, we increase the failure rate of M_4 from 0.3 to 0.52. At the same time we vary the repair rate of M_4 to keep the isolated production rate of M_4 constant at 0.48. The rest of the parameters are unchanged. The results of this case are shown in Figure 11.3.2. In the figure, the

Machine	Parameter	Value	Isolated Prod Rate
M_1	r_1	0.48	0.48
	p_1	0.52	
M_2	r_2	0.1	0.9091/2
	p_2	0.11	= 0.4545
M_3	r_3	0.48	0.48
	p_3	0.52	
M_4	r_3	varying	0.48
	p_3	0.3∼0.52	

Table 11.3.1: Failure and repair parameters. ($N_1 = N_2 = N_3 = N_4 = 20$)

straight line represents the the production rate of the analytical result and the stars and circles represent the upper and lower bounds of 95% confidence intervals evaluated from simulation runs. As shown in the figure, the production rate of the system is little bit below 0.45. This result matches our expectations, because although the parameters of M_4 are changed, the isolated production rate of the machine remains the same. Also the bottleneck machine of the system is M_2 and therefore the parameter change of the non-bottleneck machine M_4 has little influence the production rate of the system.

From the figure, we can see that the analytical results are within the upper and lower bounds of the 95% confidence intervals. We calculated the percent error of the production rate from the simulated production rate in the following manner.

$$\%Error = 100 \times \frac{E_{analytical} - E_{sim}}{E_{sim}}$$

The result is shown in Figure 11.3.3. As shown in the figure, most of errors are within 1.5% and the maximum error is about 2.5%.

11.3.2 Case 2: Varying p_4 with constant r_4

The system parameters of the second case are shown in Table 11.3.2. In this case, we vary p_4 from 0.1 to 0.8. However, unlike the first case, we fix the value r_4, therefore,

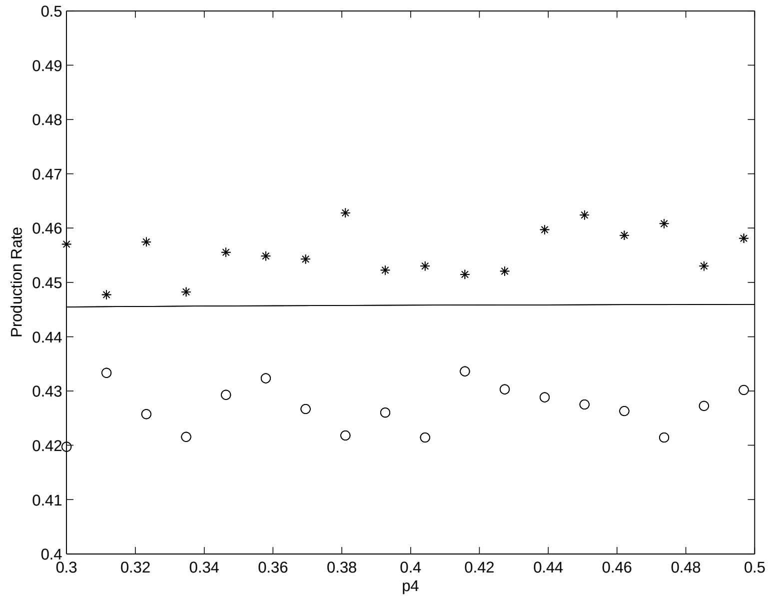

Figure 11.3.2: Production rate vs. p_4 (e_4 fixed)

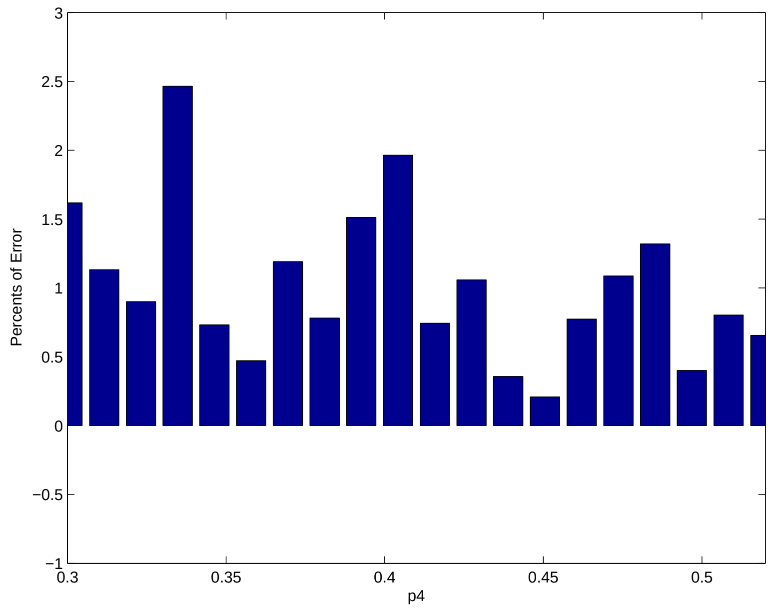

Figure 11.3.3: Percent of Error vs. p_4

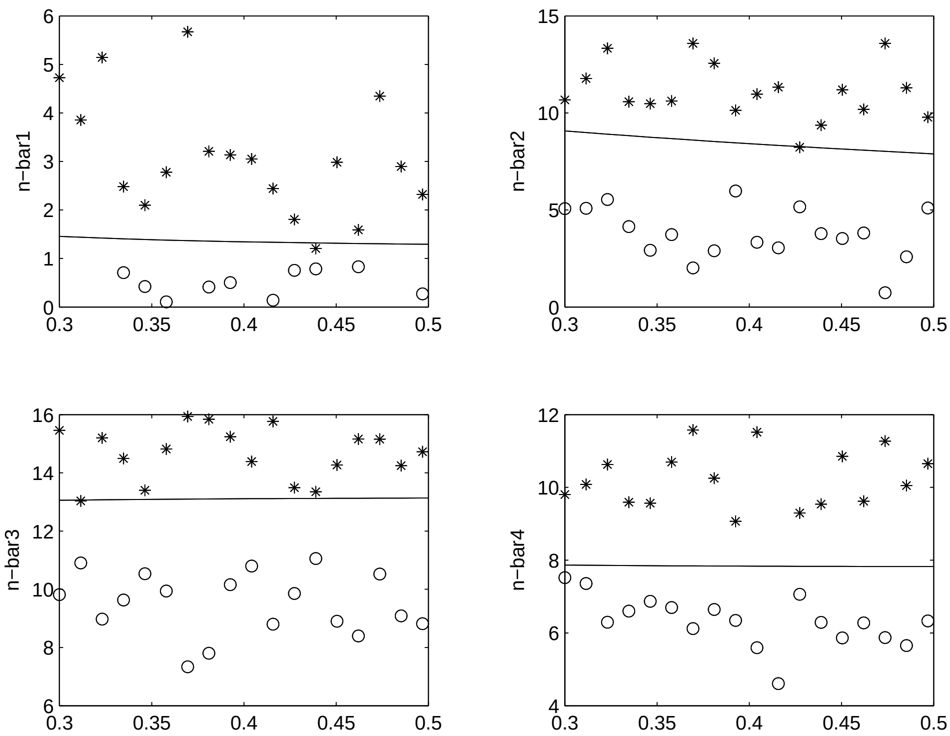

Figure 11.3.4: Average buffer level vs. p_4

the isolated production rate of M_4 decreases as p_4 increases. The result is of the case is shown in Figure 11.3.5. The production rate of the system is unchanged until p_4 is bigger than 0.58. This is because when p_4 is than 0.58, the bottleneck machine is M_2 and any parameter changes of the non-bottleneck machine do not influence the system production rate. However, if p_4 is bigger than 0.58 the bottleneck machine becomes M_4 and the production rate decreases as the bottleneck machine deceases its capacity. Again, the analytical results are also within the range of the confidence intervals evaluated from the simulation runs.

Figure 11.3.6 shows the percent of error of the production rate of case 2. As shown in the figure, all the errors remain within 3%. Notice that the analytical result tends to over-estimate the production rate when M_2 is bottleneck, while it under-estimate it when M_4 is bottleneck. This behavior should be investigated.

Machine	Parameter	Value	Isolated Prod Rate
M_1	r_1	0.48	0.48
	p_1	0.52	
M_2	r_2	0.1	0.9091/2
	p_2	0.11	= 0.4545
M_3	r_3	0.48	0.48
	p_3	0.52	
M_4	r_3	0.48	varying
	p_3	0.1∼0.8	

Table 11.3.2: Failure and repair parameters. $(N_1 = N_2 = N_3 = N_4 = 20)$

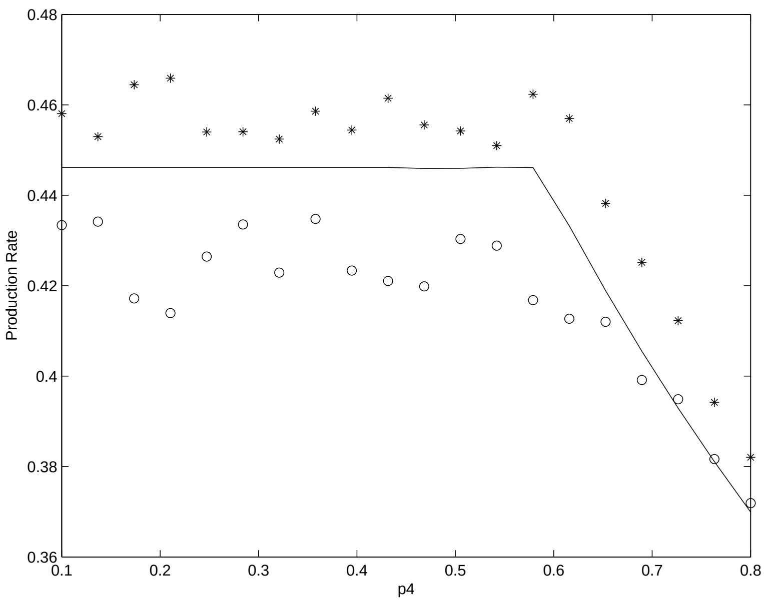

Figure 11.3.5: Production rate vs. p_4 (r_4 fixed)

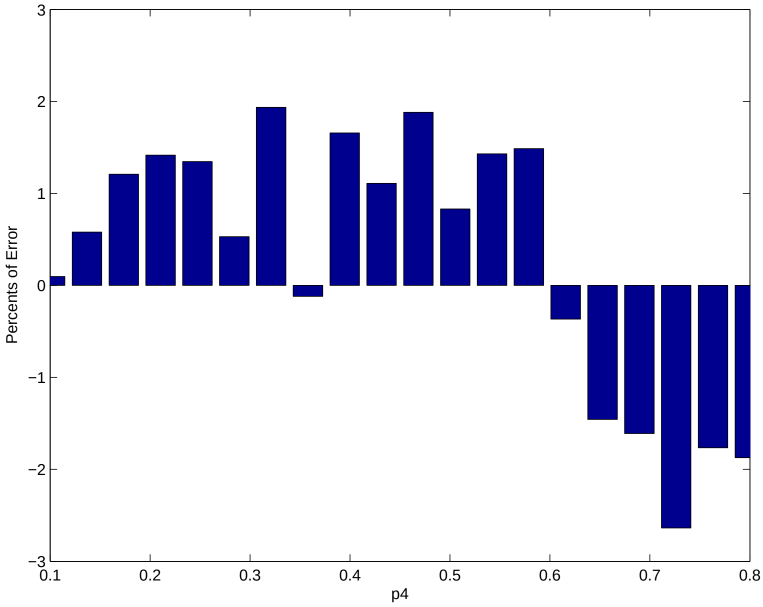

Figure 11.3.6: Percent of Error vs. p_4

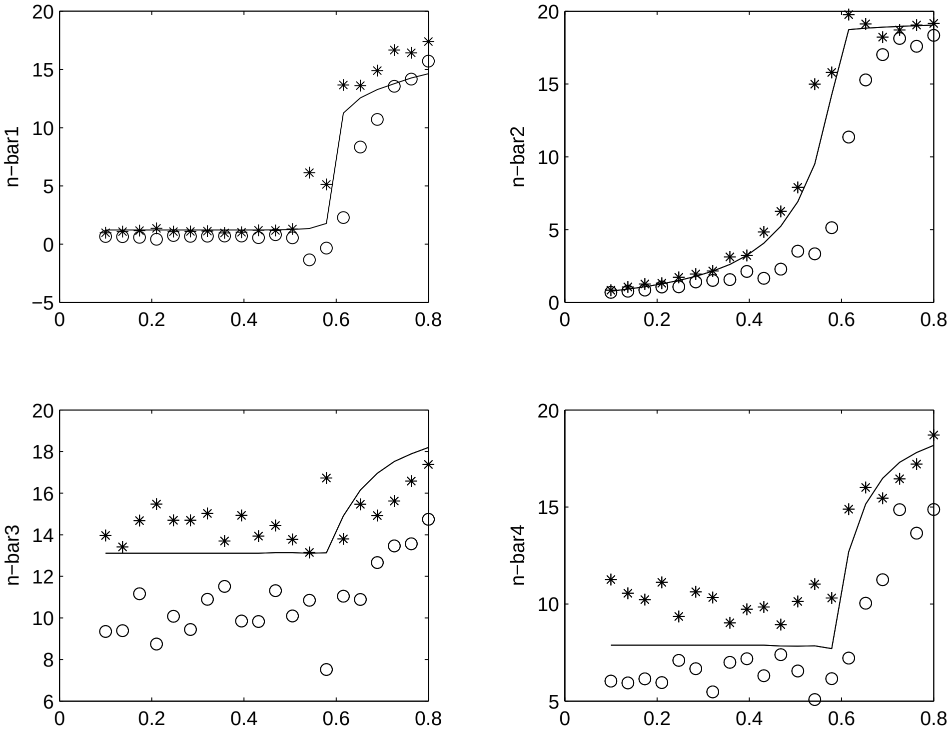

Figure 11.3.7: Average buffer level vs. p_4

Chapter 12

Conclusion

In this thesis, we present a model and analysis of a flexible production line with unreliable machines and finite buffers, processing different part types. In the line, the machines operate according to a static priority rule, working on the highest priority part whenever possible, and on lower priority parts only when unable to process those with higher priorities due to either blockage or starvation. We construct decomposition equations to analyze the line and then develop a solution algorithm to solve the decomposition equations. The algorithm converges reliably with reasonable line parameters. Simulations confirmed that the results generated with the decomposition were accurate. System analysis with various parameter shows interesting behaviors of the line.

In the thesis, we also introduce decomposition equations for a re-entrant flow line. We modify the decomposition equations for the multiple-part type production line for the re-entrant system. For verification, the results from the analytical model are compared with results from simulations runs. From the verification, we found that the analytical results were well matched with our intuitions and with results from the simulation.

Based on the decomposition equations, an optimal buffer allocation algorithm for the multiple-part-type line can be developed. Also, the size of buffer can be used to control the flow of each product type, and therefore, this work can be modified to develop a control policy for multiple products. We leave these topics for future research.

Appendix A

Two-Machine One-Buffer Line Transition Equations

A.1 Internal States

The transition equations for the internal states are defined by

$$
\begin{aligned}
P(n, \Delta_j^u, \Delta_l^d) \quad = \quad & P(n, \Delta_j^u, \Delta_l^d)(1 - r_j^u - \widetilde{Z}_j^u)(1 - r_l^d - \widetilde{Z}_l^d) \\
& + \sum_{g=1}^{L} P(n, \Delta_j^u, \Delta_g^d)(1 - r_j^u - \widetilde{Z}_j^u) z_{g,l}^d \\
& + \sum_{f=1}^{J} P(n, \Delta_f^u, \Delta_l^d) z_{f,j}^u (1 - r_l^d - \widetilde{Z}_l^d) \\
& + P(n, \Delta_j^u, \Upsilon^d)(1 - r_j^u - \widetilde{Z}_j^u) p_l^d \\
& + \sum_{f=1}^{J} p(n, \Delta_f^u, \Upsilon^d) z_{f,j}^u p_l^d \\
& + P(n, \Upsilon^u, \Delta_l^d) p_j^u (1 - r_l^d - \widetilde{Z}_l^d) \\
& + \sum_{g=1}^{L} p(n, \Upsilon^u, \Delta_g^d) p_j^u z_{g,l}^d \\
& + \sum_{f=1}^{J} \sum_{g=1}^{L} P(n, \Delta_f^u, \Delta_g^d) z_{f,j}^u z_{g,l}^d \\
& + P(n, \Upsilon^u, \Upsilon^d) p_j^u p_l^d
\end{aligned}
\qquad (A.1.1)
$$

$$
\begin{aligned}
P(n, \Delta_j^u, \Upsilon^d) \quad &= \sum_{l=1}^{L} P(n+1, \Delta_j^u, \Delta_l^d)(1 - r_j^u - \widetilde{Z}_j^u) r_l^d \\
&+ \sum_{f=1}^{J} \sum_{l=1}^{L} P(n+1, \Delta_f^u, \Delta_l^d) z_{f,j}^u r_g^d \\
&+ P(n+1, \Delta_j^u, \Upsilon^d)(1 - r_j^u - \widetilde{Z}_j^u)(1 - P^d) \\
&+ \sum_{f=1}^{L} P(n+1, \Delta_f^u, \Upsilon^d) z_{f,j}^u (1 - P^d) \\
&+ \sum_{l=1}^{L} P(n+1, \Upsilon^u \Delta_l^d) p_j^u r_l^d \\
&+ P(n+1, \Upsilon^u \Upsilon^d) p_j^u (1 - P^d)
\end{aligned} \tag{A.1.2}
$$

$$
\begin{aligned}
P(n, \Upsilon^u, \Delta_l^d) \quad &= \sum_{j=1}^{J} P(n-1, \Delta_j^u, \Delta_l^d) r_j^u (1 - r_l^d - \widetilde{Z}_l^d) \\
&= \sum_{j=1}^{J} \sum_{g=1}^{L} P(n-1, \Delta_j^u, \Delta_g^d) r_j^u z_{g,l}^d \\
&+ \sum_{j=1}^{J} P(n-1, \Delta_j^u, \Upsilon^d) r_j^u p_l^d \\
&+ P(n-1, \Upsilon^u, \Delta_l^d)(1 - P^u)(1 - r_l^d - \widetilde{Z}_l^d) \\
&+ \sum_{g=1}^{L} P(n-1, \Upsilon^u \Delta_g^d)(1 - P^u) z_{g,l}^d \\
&+ P(n-1, \Upsilon^u \Upsilon^d)(1 - P^u) p_l^d
\end{aligned} \tag{A.1.3}
$$

$$
\begin{aligned}
P(n, \Upsilon^u, \Upsilon^d) \quad &= \sum_{j=1}^{J} \sum_{l=1}^{L} P(n, \Delta_j^u, \Delta_l^d) r_j^u r_l^d \\
&+ \sum_{j=1}^{J} P(n, \Delta_j^u, \Upsilon^d) r_j^u (1 - P^d) \\
&+ \sum_{l=1}^{L} P(n, \Upsilon^u \Delta_l^d)(1 - P^u) r_l^d \\
&+ P(n, \Upsilon^u \Upsilon^d)(1 - P^u)(1 - P^d)
\end{aligned} \tag{A.1.4}
$$

A.2 Upper Boundary States

The transition equations for the Upper boundary states are the following:

$$
\begin{aligned}
P(0, \Delta_j^u, \Delta_l^d) \quad &= P(0, \Delta_j^u, \Delta_l^d)(1 - r_j^u - \widetilde{Z}_j^u)(1 - r_l^d - \widetilde{Z}_l^d) \\
&+ \sum_{f=1}^{J} P(0, \Delta_f^u, \Delta_l^d) z_{f,j}^u (1 - r_l^d - \widetilde{Z}_l^d) \\
&+ \sum_{g=1}^{L} P(0, \Delta_j^u, \Delta_g^d)(1 - r_j^u - \widetilde{Z}_j^u) z_{g,l}^d \\
&+ \sum_{f=1}^{J} \sum_{g=1}^{L} P(0, \Delta_f^u, \Delta_g^d) z_{f,j}^u z_{g,l}^d \\
&+ P(0, \Delta_j^u, \Upsilon^d)(1 - r_j^u - \widetilde{Z}_j^u) q_l^d \\
&+ \sum_{f=1}^{J} P(0, \Delta_f^u, \Upsilon^d) z_{f,j}^u q_l^d
\end{aligned}
\tag{A.2.1}
$$

$$
\begin{aligned}
P(0,\Delta_j^u,\Upsilon^d) \quad &= \sum_{l=1}^{L} P(0,\Delta_j^u,\Delta_l^d)(1-r_j^u-\widetilde{Z}_j^u)r_l^d \\
&+ \sum_{l=1}^{L} P(1,\Delta_j^u,\Delta_l^d)(1-r_j^u-\widetilde{Z}_j^u)r_l^d \\
&+ \sum_{f=1}^{L}\sum_{l=1}^{L} P(0,\Delta_f^u,\Delta_l^d)z_{f,j}^u r_l^d \\
&+ \sum_{f=1}^{L}\sum_{l=1}^{L} P(1,\Delta_f^u,\Delta_l^d)z_{f,j}^u r_l^d \\
&+ P(0,\Delta_j^u,\Upsilon^d)(1-r_j^u-\widetilde{Z}_j^u)(1-Q^d) \\
&+ \sum_{f=1}^{L} P(0,\Delta_f^u,\Upsilon^d)z_{f,j}^u(1-Q^d) \\
&+ P(1,\Delta_j^u,\Upsilon^d)(1-r_j^u-\widetilde{Z}_j^u)(1-P^d) \\
&+ \sum_{f=1}^{L} P(1,\Delta_f^u,\Upsilon^d)z_{f,j}^u(1-P^d) \\
&+ \sum_{l=1}^{L} P(1,\Upsilon^u,\Delta_l^d)p_j^u r_l^d \\
&+ P(1,\Upsilon^u\Upsilon^d)p_j^u(1-P_l^d)
\end{aligned} \tag{A.2.2}
$$

$$
\begin{aligned}
P(1,\Delta_j^u,\Delta_l^d) \quad &= P(1,\Delta_j^u,\Delta_l^d)(1-r_j^u-\widetilde{Z}_j^u)(1-r_l^d-\widetilde{Z}_l^d) \\
&+\sum_{g=1}^{L} P(1,\Delta_j^u,\Delta_g^d)(1-r_j^u-\widetilde{Z}_j^u)z_{g,l}^d \\
&+\sum_{f=1}^{J} P(1,\Delta_f^u,\Delta_l^d)z_{f,j}^u(1-r_l^d-\widetilde{Z}_l^d) \\
&+\sum_{f=1}^{J}\sum_{g=1}^{L} P(1,\Delta_f^u,\Delta_g^d)z_{f,j}^u z_{g,l}^d \\
&+P(1,\Delta_j^u,\Upsilon^d)(1-r_j^u-\widetilde{Z}_j^d)p_l^d \\
&+\sum_{f=1}^{J} P(1,\Delta_f^u,\Upsilon^d)z_{f,j}^u p_l^d \\
&+P(1,\Upsilon^u,\Delta_l^d)p_j^u(1-r_l^d-\widetilde{Z}_l^d) \\
&+\sum_{g=1}^{L} P(1,\Upsilon^u\Delta_g^d)p_j^y z_{g,l}^d \\
&+P(1,\Upsilon^u,\Upsilon^d)p_j^u p_l^d
\end{aligned} \tag{A.2.3}
$$

$$
\begin{aligned}
P(1,\Delta_j^u,\Upsilon^d) \quad &= \sum_{l=1}^{L} P(2,\Delta_j^u,\Delta_l^d)(1-r_j^u-\widetilde{Z}_j^u)r_l^d \\
&+\sum_{f=1}^{J} P(2,\Delta_f^u,\Delta_l^d)z_{f,j}^u r_l^d \\
&+P(2,\Delta_j^u,\Upsilon^d)(1-r_j^u-\widetilde{Z}_j^u)(1-P^d) \\
&+\sum_{f=1}^{J} P(2,\Delta_f^u,\Upsilon^d)z_{f,j}^u(1-P^d) \\
&+\sum_{l=1}^{L} P(2,\Upsilon^u\Delta_l^d)p_j^u r_l^d \\
&+P(2,\Upsilon^u,\Upsilon^d)p_j^u(1-P^d)
\end{aligned} \tag{A.2.4}
$$

$$
\begin{aligned}
P(1,\Upsilon^u,\Delta_l^d) \quad &= \sum_{j=1}^{J} P(0,\Delta_j^u,\Delta_l^d) r_j^u (1 - r_l^d - \widetilde{Z}_l^d) \\
&+ \sum_{j=1}^{J}\sum_{g=1}^{L} P(0,\Delta_j^u,\Delta_g^d) r_j^u z_{g,l}^d \\
&+ \sum_{j=1}^{J} P(0,\Delta_j^u,\Upsilon^d) r_j^u q_l^d
\end{aligned} \tag{A.2.5}
$$

$$
\begin{aligned}
P(1,\Upsilon^u,\Upsilon^d) \quad &= \sum_{j=1}^{J}\sum_{l=1}^{L} P(0,\Delta_j^u,\Delta_l^d) r_j^u r_l^d \\
&+ \sum_{j=1}^{J}\sum_{l=1}^{L} P(1,\Delta_j^u,\Delta_l^d) r_j^u r_l^d \\
&+ \sum_{j=1}^{J} P(0,\Delta_j^u,\Upsilon^d) r_j^u (1 - Q^d) \\
&+ \sum_{j=1}^{J} P(1,\Delta_j^u,\Upsilon^d) r_j^u (1 - P^d) \\
&+ \sum_{l=1}^{L} P(1,\Upsilon^u,\Delta_l^d)(1 - P^u) r_l^d \\
&+ P(1,\Upsilon^u,\Upsilon^d)(1 - P^u)(1 - P^d)
\end{aligned} \tag{A.2.6}
$$

$$
\begin{aligned}
P(2,\Upsilon^u,\Delta_l^d) \quad &= \sum_{j=1}^{J} P(1,\Delta_j^u,\Delta_l^d) r_j^u (1 - r_l^d - \widetilde{Z}_l^d) \\
&+ \sum_{j=1}^{J}\sum_{g=1}^{L} P(1,\Delta_j^u,\Delta_g^d) r_j^u z_{g,l}^d \\
&+ \sum_{j=1}^{J} P(1,\Delta_j^u,\Upsilon^d) r_j^u p_l^d \\
&+ P(1,\Upsilon^u,\Delta_l^d)(1 - P^u)(1 - r_l^d - \widetilde{Z}_l^d) \\
&+ \sum_{g=1}^{L} P(1,\Upsilon^u,\Delta_g^d)(1 - P^u) z_{g,l}^d \\
&+ P(1,\Upsilon^u,\Upsilon^d)(1 - P^u) p_l^d
\end{aligned} \tag{A.2.7}
$$

A.3 Lower Boundary States

The transition equations for the lower boundary states are the followings:

$$
\begin{aligned}
P(N-2, \Delta_j^u, \Upsilon_l^d) &= \sum_{l=1}^{L} P(N-1, \Delta_j^u, \Delta_l^d)(1 - r_j^u - \widetilde{Z}_j^d) r_l^d \\
&+ \sum_{f=1}^{J} \sum_{l=1}^{L} P(N-1, \Delta_f^u, \Delta_l^d) z_{f,j}^u r_l^d \\
&+ P(N-1, \Delta_j^u, \Upsilon^d)(1 - r_j^u - \widetilde{Z}_j^u)(1 - P^d) \\
&+ \sum_{f=1}^{J} P(N-1, \Delta_f^u, \Upsilon^d) z_{f,j}^u (1 - P^d) \\
&+ \sum_{l=1}^{L} P(N-1, \Upsilon^u \Delta_l^d) p_j^u r_l^d \\
&+ P(N-1, \Upsilon^u, \Upsilon^d) p_j^u (1 - P^d)
\end{aligned}
\tag{A.3.1}
$$

$$
\begin{aligned}
P(N-1, \Delta_j^u, \Delta_l^d) &= P(N-1, \Delta_j^u, \Delta_l^d)(1 - r_j^u - \widetilde{Z}_j^u)(1 - r_l^d - \widetilde{Z}_l^d) \\
&+ \sum_{f=1}^{J} P(N-1, \Delta_f^u, \Delta_l^d) z_{f,j}^u (1 - r_l^d - \widetilde{Z}_l^d) \\
&+ \sum_{g=1}^{L} P(N-1, \Delta_j^u, \Delta_g^d)(1 - r_j^u - \widetilde{Z}_j^u) z_{g,l}^d \\
&+ \sum_{f=1}^{J} \sum_{g=1}^{L} P(N-1, \Delta_f^u, \Delta_g^d) z_{f,j}^u z_{g,l}^d \\
&+ P(N-1, \Delta_j^u, \Upsilon^d)(1 - r_j^u - \widetilde{Z}_j^u) p_l^d \\
&+ \sum_{f=1}^{J} P(N-1, \Delta_f^u, \Upsilon^d) z_{f,j}^u p_l^d \\
&+ P(N-1, \Upsilon^u, \Delta_l^d) p_j^u (1 - r_l^d - \widetilde{Z}_l^d) \\
&+ \sum_{g=1}^{L} P(N-1, \Upsilon^u, \Delta_g^d) p_j^u z_{g,l}^d \\
&+ P(N-1, \Upsilon^u, \Upsilon^d) p_j^u p_l^d
\end{aligned}
\tag{A.3.2}
$$

$$
\begin{aligned}
P(N-1,\Delta_j^u,\Upsilon^d) \quad &= \sum_{l=l}^{L} P(N,\Delta_j^u,\Delta_l^d)(1-r_j^u-\widetilde{Z}_j^u)r_l^d \\
&= \sum_{f=l}^{J} P(N,\Delta_f^u,\Delta_l^d)z_{f,j}^u r_l^d \\
&+ \sum_{l=l}^{L} P(N,\Upsilon^u,\Delta_l^d)q_j^u r_l^d
\end{aligned} \tag{A.3.3}
$$

$$
\begin{aligned}
P(N-1,\Upsilon^u,\Delta_l^d) \quad &= \sum_{j=1}^{J} P(N-2,\Delta_j^u,\Delta_l^d)r_j^u(1-r_l^d-\widetilde{Z}_l^d) \\
&+ \sum_{j=1}^{J}\sum_{g=1}^{L} P(N-2,\Delta_j^u,\Delta_g^d)r_j^u z_{g,l}^d \\
&+ \sum_{j=1}^{J} P(N-2,\Delta_j^u,\Upsilon^d)r_j^u p_l^d \\
&+ P(N-2,\Upsilon^u,\Delta_l^d)(1-P^u)(1-r_l^d-\widetilde{Z}_l^d) \\
&+ \sum_{g=1}^{L} P(N-2,\Upsilon^u,\Delta_g^d)(1-P^u)z_{g,l}^d \\
&+ P(N-2,\Upsilon^u,\Upsilon^d)(1-P^u)p_l^d
\end{aligned} \tag{A.3.4}
$$

$$
\begin{aligned}
P(N-1,\Upsilon^u,\Upsilon^d) \quad &= \sum_{j=1}^{J}\sum_{l=1}^{L} P(N-1,\Delta_j^u,\Delta_l^d)r_j^u r_l^d \\
&+ \sum_{j=1}^{J}\sum_{l=1}^{L} P(N,\Delta_j^u,\Delta_l^d)r_j^u r_l^d \\
&+ \sum_{j=1}^{J} P(N-1,\Delta_j^u,\Upsilon^d)r_j^u(1-P^d) \\
&+ \sum_{l=1}^{L} P(N-1,\Upsilon^u,\Delta_l^d)(1-P^u)r_l^d \\
&+ \sum_{l=1}^{L} P(N,\Upsilon^u,\Delta_l^d)(1-Q^u)r_l^d \\
&+ P(N-1,\Upsilon^u,\Upsilon^d)(1-P^u)(1-P^d)
\end{aligned} \tag{A.3.5}
$$

$$
\begin{aligned}
P(N,\Delta_j^u,\Delta_l^d) \quad & +P(N,\Delta_j^u,\Delta_l^d)(1-r_j^u-\widetilde{Z}_j^u)(1-r_l^d-\widetilde{Z}_l^d) \\
& +\sum_{f=1}^{J} P(N,\Delta_f^u,\Delta_l^d) z_{f,j}^u (1-r_l^d-\widetilde{Z}_l^d) \\
& +\sum_{g=1}^{L} P(N,\Delta_j^u,\Delta_g^d)(1-r_j^u-\widetilde{Z}_j^u) z_{g,l}^d \\
& +\sum_{f=1}^{J}\sum_{g=1}^{L} P(N,\Delta_f^u,\Delta_g^d) z_{f,j}^u z_{g,l}^d \\
& +P(N,\Upsilon^u,\Delta_l^d) q_j^u (1-r_l^d-\widetilde{Z}_l^d) \\
& +\sum_{g=1}^{L} P(N,\Upsilon^u,\Delta_g^d) q_j^u z_{g,l}^d
\end{aligned} \tag{A.3.6}
$$

$$
\begin{aligned}
P(N,\Upsilon^u,\Delta_l^d) \quad & =\sum_{j=1}^{J} P(N-1,\Delta_j^u,\Delta_l^d) r_j^u (1-r_l^d-\widetilde{Z}_l^d) \\
& +\sum_{j=1}^{J} P(N,\Delta_j^u,\Delta_l^d) r_j^u (1-r_l^d-\widetilde{Z}_l^d) \\
& +\sum_{j=1}^{J}\sum_{g=1}^{L} P(N-1,\Delta_j^u,\Delta_g^d) r_j^u z_{g,l}^d \\
& +\sum_{j=1}^{J}\sum_{g=1}^{L} P(N,\Delta_j^u,\Delta_g^d) r_j^u z_{g,l}^d \\
& +\sum_{j=1}^{J} P(N-1,\Delta_j^u,\Upsilon^d) r_j^u p_l^d \\
& +P(N-1,\Upsilon^u,\Delta_l^d)(1-P^u)(1-r_l^d-\widetilde{Z}_l^d) \\
& +\sum_{g=1}^{L} P(N-1,\Upsilon^u,\Delta_g^d)(1-P^u) z_{g,l}^d \\
& +P(N,\Upsilon^u,\Delta_l^d)(1-Q^u)(1-r_l^d-\widetilde{Z}_l^d) \\
& +\sum_{g=1}^{L} P(N,\Upsilon^u,\Delta_g^d)(1-Q^u) z_{g,l}^d \\
& +P(N-1,\Upsilon^u,\Upsilon^d)(1-P^u) p_l^d
\end{aligned} \tag{A.3.7}
$$

Bibliography

[1] Mostafa H. Ammar and Stanley B. Gershwin. Equivalence relations in queuing models of fork/join networks. *Performance Evaluation Journal*, 10:233–245, 1989. Special Issue on Queueing Networks with Finite Capacities.

[2] Donald W. Berry and Bernard W. Lindgren. *Statistics, Theory and Methods.* Brooks/Cole Publishing Company, Pacific Grove, CA, 1990.

[3] M. H. Burman. *New Results in Flow Line Analysis.* PhD thesis, Massachusetts Institute of Technology, Operations Research Center, June 1995.

[4] J. A. Buzacott and L. E. Hanifin. Models of automatic transfer lines with inventory banks — a review and comparison. *AIIE Transactions*, 10(2):197–207, 1978.

[5] John A. Buzacott and J. George Shanthikumar. *Stochastic Models of Manufacturing Systems.* Prentice Hall, Englewood Cliffs, NJ, 1993.

[6] Yves Dallery, R. David, and X.-L. Xie. An efficient algorithm for analysis of transfer lines with unreliable machines and finite buffers. *IIE Transactions*, 20(3):280–283, 1988.

[7] S. B. Gershwin. An efficient decomposition method for the approximate evaluation of tandem queues with finite storage space and blocking. *Operations Research*, 35(2):291–305, 1987.

[8] S. B. Gershwin. *Manufacturing Systems Engineering.* Prentice-Hall, Englewood Cliffs, NJ, 1994. For corrections, see `http://web.mit.edu/manuf-sys/www/gershwin.errata.html`.

[9] Averill M. Law and W. David Kelton. *Simulation Modeling and Analysis.* McGraw-Hill, New York, NY, second edition, 1991.

[10] N. Maggio, A. Matta, S. B. Gershwin, and T. Tolio. An approximate analytical method for evaluating the performance of closed loop flow systems with unreliable machines and finite buffers — Part I: Small loops. Unpublished manuscript, 2003.

[11] Micheline Maynard. Ford moving toward to flexible manufacturing. *New York Times*, April 14, 2006.

[12] J. E. Nemec. *Diffusion and Decomposition Approximations of Stochastic Models of Multiclass Processing Networks.* Mit operations research center ph. d. thesis, Massachusetts Institute of Technology, February 1999.

[13] T. Tolio and Stanley B. Gershwin. Deterministic two-machine lines with multiple failure modes. Technical report, Politicnico di Milano — Dipartimento di Meccanica, 1996.