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Collusion-Resilient Revenue In Combinatorial Auctions

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Abstract

In auctions of a single good, the second-price mechanism achieves, in dominant strategies, a revenue benchmark that is naturally high and resilient to any possible collusion.

We show how to achieve, to the maximum extent possible, the same properties in combinatorial auctions.

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1 The Problem of Guaranteeing Revenue in Combinatorial Auctions

1.1 Combinatorial Auctions and Their Basic Revenue Problem

A (non-Bayesian, $n \times m$) combinatorial-auction *context* is described as follows. There is a set of *players* $N = \{1, \ldots, n\}$ and a set of m goods G. A *valuation* is a function from G's subsets to $\mathbb{R}^+$, and each player i has a secret valuation TV_i, which we refer to as i's *true valuation.* An *outcome* consists of (1) a *profile* (i.e., a vector indexed by the players) $P = P_1, \ldots, P_n$, where $P_i \in \mathbb{R}^+$ is the *price* to be paid by player i, and (2) an *allocation* $A = A_0, A_1, \ldots, A_n$, where A_i is the subset of goods allocated to player i, and A_0 the set of unallocated goods. For each outcome (A, P), the *utility* of each player i is defined to be $TV_i(A_i) - P_i$; that is, i's true value of the goods allocated to him minus the price he pays. Note that such a context is fully described by just N, G, and the true-valuation profile TV, since these determine outcomes and utilities.

For such a context, a combinatorial-auction *mechanism* is a (possibly probabilistic) function M mapping a profile of valuations V to an outcome (A, P) such that $P_i = 0$ whenever V_i is the *null valuation* —the *opt-out condition.* An $n \times m$ context $\mathcal{C} = (N, G, TV)$ and an $n \times m$ mechanism $\mathcal{M}$ define a $(n \times m)$ *combinatorial auction* $(\mathcal{C}, \mathcal{M})$: namely, the game $\mathcal{G}$ envisaged to be played as follows. First, each player i (independently of the others) chooses a valuation BID_i on inputs TV_i, N, and G. Then, an outcome (A, P) is obtained by evaluating $\mathcal{M}$ on BID. We refer to the so chosen valuations as *bids*, to emphasize that they need not coincide with the players' true valuations. In such a game, a *strategy* is a (possibly probabilistic) way for a player to choose his bid.

Since the players' true valuation are not publicly known, generating high revenue requires a careful design of the mechanism. As for any game, in a rational play of an auction the players end up selecting an *equilibrium*, that is, a profile of strategies $\sigma = (\sigma_1, \ldots, \sigma_n)$ such that no player i has an incentive to deviate from his strategy σ_i if the other players stick to theirs. Accordingly, the basic revenue problem can be informally stated as follows: *find a mechanism M such that, for every context C, the resulting combinatorial auction (C, M) yields high revenue for every equilibrium σ the players may reasonably select.* Since computational efficiency is not a traditional concern of game theory, we shall not be concerned about M's efficiency either.

1.2 The Worst Natural Setting and Natural Solution-Pairs

The basic revenue problem for combinatorial auctions is already difficult. Yet, we wish to investigate and provide solutions to a more adversarial version of it: namely, the problem of generating revenue in the *worst natural setting.*

We characterize such a setting by the following three assumptions.

- *Total Equilibrium Uncertainty.* An auction having an equilibrium (or several equilibria) σ yielding high revenue may not "guarantee" high revenue at all. Indeed, as for any other game, an auction typically has plenty of equilibria, and rational players may ultimately select one other than σ. In the worst (rational) setting, we assume that the equilibrium ultimately selected will be the one generating least revenue among those equilibria τ having a "modicum of rationality," that is, those composed by weakly non-dominated strategies.[1]
- *Total Bayesian Ignorance.* In a Bayesian setting, the players' true valuations are assumed to be drawn from a distribution D. Clearly, knowledge of such a distribution can be very helpful for designing high-revenue mechanisms, particularly when D is of a suitable form.[2] In the worst setting, however, no Bayesian information is available.

[1] Informally, a strategy τ_i is weakly dominated for player i if i has another strategy ψ_i that is always at least as good as τ_i and sometimes strictly better.

[2] For single-good auctions, Myerson [18] has put forward mechanisms that generate optimal revenue essentially whenever the players have independent valuations of the good, a quite general condition. For such a case, he is able to optimally set "minimum winning bids" without undue risk of pricing out players. By contrast, in combinatorial auctions no general distributions for the players' true valuations are known for which optimal revenue mechanisms exist.

- *Devilish Collusion.* Nash equilibria are defined in terms of "individual-player deviations," and offer no guarantees when *multiple* players deviate from their prescribed strategies. This is already true when such players act *independently,* but even more so when they *collude*, that is, when they coordinate their deviation strategies. Unfortunately, this is not a theoretical possibility: collusion is endemic problem in real auctions.[3] The severity of this problem decreases when we can assume some restrictions on the ability of collusive players to coordinate themselves. In the worst setting, however, there are no such restrictions. In particular, collusive players can make side payments to one another, and enter any secret and binding agreement of their choosing. More ominously, the worst natural setting envisages a malicious *external* entity (the Devil) which, with the sole purpose of lowering the auction's revenue and with full knowledge of the players' true valuations, is free to introduce any number of additional players whose bids he freely chooses.[4]

We consider any revenue achievable in such an adversarial setting as *guaranteed revenue,* because it is a fortiori achievable in any more realistic one. Accordingly, the problem addressed this paper can be rephrased as follows: *How much revenue can be guaranteed in combinatorial auctions?*

Let us now argue that a natural solution to this problem consists of a revenue benchmark and a dominant-strategy truthful (DST) auction mechanism satisfying some simple properties. Here, by a "revenue benchmark" we mean a function from sequences of valuations for the same set of goods to real numbers; and by a "DST auction mechanism" we mean an auction mechanism for which, for any player, bidding his true valuation is at least as good as any other strategy, no matter what bids the other players might choose.

Definition 1. *Let B be a revenue benchmark, M a DST auction mechanism, and $g(n,m)$ a real-valued function. Then, we say that (B,M) is a* natural solution with revenue guarantee *$g(n,m)$ if the following two properties hold:*

1. *B is* player monotone: *For any valuation profile S and any sub-profile T of S, $B(S) \geq B(T)$; and*
2. *$\mathcal{M}$ $g(n,m)$-achieves B: For any valuation profile V in a $n \times m$ auction, the revenue of $M(V_1,\ldots,V_n)$ is $\geq g(n,m) \cdot B(V_1,\ldots,V_n)$.*

If Condition 2 holds only for all $n,m \geq k$, we say that $(B,\mathcal{M})$ has revenue guarantee $g(n,m)$ for $n,m \geq k$.

Notice that, if (B,M) is a natural solution pair, then mechanism M is ideally suited to generate revenue in the worst natural setting. In fact, M's dominant-strategy truthfulness essentially successfully handles equilibrium uncertainty: when the auction mechanism is DST, no rational player player has reason to prefer any other bid to his true valuation.[5] Further, M does not rely on any Bayesian information: the only inputs of M (and of benchmark B) solely consist of *specific* valuations, even if these were indeed drawn from a suitable distribution D. Finally, Player monotonicity supports the fundamental economic principle that increased competition among buyers is good for the seller. And a mechanism achieving a player-monotone benchmark guarantees that the seller is always better off when additional players join the auction, even if they are coordinated by the Devil himself. This is formally stated by the following, trivially proven lemma, where the function D models the Devil and, being universally quantified, all possible ways to lower revenue.

Lemma 1. *For any set of goods G of cardinality m, any sequence $V_1,\ldots,V_s$ of G's valuations, and any function D whose range consists of tuples of G's valuations, letting $(V_{s+1},\ldots,V_n) = D(V_1,\ldots,V_s)$, the revenue of $M(V_1,\ldots,V_n)$ is $\geq g(n,m) \cdot B(V_1,\ldots,V_s)$.*

In other words,

[3]

[4] I.e., the Devil has a totally different type of utility, and can force the collusive players to bid against their own interests in the auction, by offering "external" compensation/punishment.

[5] From the perspective of an individual player i, DST mechanisms also remove any "strategy uncertainty" as well. In a general equilibrium σ, σ_i is i's best course of action only *assuming* that he believes that every other player j will chose to stick to his prescribed strategy σ_j. Such beliefs, however, may be hard to justify and constitute an additional condition that weakens the strength of the equilibrium. Quite differently, when σ is a dominant-strategy equilibrium, no reliance on such beliefs is necessary to support σ: σ_i is i's best response to any possible strategies of the other players.

A player-monotone revenue benchmark guarantees that if a set of players, I, bids independently, then the benchmark relative to the entire set of players N (possibly including collusive or irrational *players) will be at least as high as the benchmark relative to the set I.*

Of course, the interest of a natural solution pair ultimately depends on the quality of its revenue guarantee. Of course too, natural solution pairs are *sufficient conditions* to guarantee revenue in the worst natural setting. As such, they might very well be unnecessarily demanding, but are certainly appealingly simple.

1.3 Natural Solution Pairs: Single-Good vs. Combinatorial Auctions

When there is a single good for sale, the true valuation of a player i is a non-negative real number TV_i, representing the "value to i" of the item for sale; an outcome specifies just the winning player w and a price P_i for each player i; and the utility of each player i in an outcome $(w, P_1, \ldots, P_n)$ consists of $TV_i - P_i$ if $i = w$, and $-P_i$ otherwise. For such auctions, denote by $2P$ the famous *second-price* mechanism (i.e., the mechanism returning the player whose bid is highest as the winner, the second-highest bid as the winner's price, and 0 as the price of every other player); and by $2V$ the *Second-Valuation* benchmark (i.e., the function returning the second highest valuation).[6] Now, the benchmark $2V$ is clearly player-monotone, and the mechanism $2P$ is well known to be DST. Therefore, not only is $(2V, 2P)$ a natural solution pair with guarantee 1, but the revenue benchmark $2V$ achieved by $2P$ is reasonably high.

The situation ceases to be so rosy in combinatorial auctions. To be sure, the 2P mechanism is a special case of the famous VCG mechanism which indeed applies to combinatorial auctions. Let us quickly recall the definition of this more general mechanism, starting with some familiar concepts.

In a combinatorial auction, the *social welfare* relative to a profile of valuations V and an allocation A —denoted by $Value(V, A)$— is the sum of the values actually accrued by the players. That is, $Value(V, A) = V_1(A_1) + \cdots + V_n(A_n)$. The *best allocation* relative to a valuation profile V, denoted by $BestAlloc(V)$, is the allocation whose social welfare is maximum. The actual social welfare of this allocation is called the *maximum social welfare,* and is denoted by $MSW(V)$. With this preamble, given a valuation profile V for the goods for sale, the VCG mechanism returns an allocation A and a price profile P as follows: $A = BestAlloc(V)$ and $P_i = MSW(V_{-i}) - Value(V_{-i}, BestAlloc(V))$, that is the maximum social welfare relative to all valuations in V except i's one, minus the social welfare of all players except i in the best allocation relative to V. (Any "tie" is broken arbitrarily.)

From its definition, it is clear that the primary objective of the VCG mechanism is *efficiency* in the *economic* sense.[7] However, although revenue and efficiency are not unrelated, and although it too is dominant-strategy truthful, the VCG mechanism cannot be the second component of any natural solution pair with non-trivial revenue guarantees. This is so because the revenue generated by the VCG mechanism suffers from three (interrelated) deficiencies, illustrated below via three separate examples of auction contexts (or rather, via the same context viewed from three different angles). In each example there are two goods for sale, a and b, and two or three players, 1, 2 and, if necessary, 3. The true valuations of each context are represented as a matrix, with as many rows as players and as many columns as subsets of the goods, whose entry (i, j) entry represents the value to the ith player of the jth subset. Whenever this value is 0, for visual clarity entry (i, j) is left empty.

- *Revenue Unattainability.* Consider the following profile of true valuations.

	{a}	{b}	{a,b}
1	1000		
2			1000
3		1000	

Example A

[6] In either case, "ties are broken arbitrarily."

[7] Indeed, best allocations appear to be computationally hard, but the amount of computation is orthogonal to the traditional goals of game theory, and is not a concern of this paper either.

On input the true valuations of Example A —which is the case at the envisaged DST equilibrium— the VCG mechanism allocates a to player 1 for a price of 0, and b to player 2 for a price of 0. Thus the revenue generated is 0. Perhaps not much can be deduced on the basis of a single "bad" example. But, Example A can be trivially turned into an infinite sequence of examples by having all its non-zero entries go to infinity, while remaining equal to each other. Now notice that the VCG mechanism continues to return revenue 0 in each example in this sequence. This would not be a problem if no or minimal revenue were obtainable for the so constructed examples, but this is far from being the case: there is plenty of revenue to be obtained, and in dominant-strategies too.[8]

- *Revenue Fragility.* Consider the following profile of true valuations.

	{a}	{b}	{a,b}
1	1000		
2			1000

Example B

On input the true valuations of Example B —which is the case at the envisaged DST equilibrium— the VCG mechanism returns revenue of 1000 (no matter how it break ties —i.e., no matter which of the two best allocations it selects). Consider now adding a third player with the following valuation: 1000 for $\{b\}$ and 0 for all other subsets. Then, one obtains Example A, and thus the revenue generated by the VCG mechanism suddenly drops from 1000 to 0. Accordingly, in a combinatorial auction whose mechanism is the VCG, if the seller knows that he is going to fetch good revenue from a group of players (e.g., because he knows their valuations well enough), he should be wary of —and indeed prevent— the participation of additional players. Indeed, the arrival of a single "unknown" player may cause his revenue to decrease dramatically or vanish altogether. In other words, the revenue of the VCG mechanism is *not* player-monotone.[9]

- *Collusion Vulnerability.* Consider the following profile of true valuations.

	{a}	{b}	{a,b}
1	500		
2			1000
3		500	

Example C

On input the true valuations of Example C —which is the case at the envisaged DST equilibrium— the VCG mechanism returns revenue of 1000. Such revenue is actually optimal here, but is also very vulnerable to collusion, despite the fact that the VCG is DST. Indeed, the DST property solely guarantees that an *independent* player has nothing better to do than bidding his true valuation. But it guarantees nothing of the sort about two or more players! Letting the context be one whose true valuations are as in Example C, assume that players 1 and 3 are collusive and know that player 2 values only the subset $\{a, b\}$, and for 1000. Then, player 1 and 2 may coordinate their bids as follows: 1 bids 1000 for a, and 2 bids 1000 for b. With these two bids, each of the collusive players obtains his desired set while paying 0. In fact, together with player 2's truthful bid, one gets exactly the valuations of Example A.

Note that player 1 and 2 need not know exactly how much player 2 values the subset $\{a, b\}$. It suffices that they know an upper-bound for it. For instance, if they know that he values $\{a, b\}$ for no more than one hundred thousand, then if each of players 1 and 3 bids 1 million for his own valued set, both get what they want for nothing! In sum,

[8] For instance, consider the following mechanism M: for each player i, compute the personal welfare to i of all goods (i.e., the maximum social welfare of just i's valuation, that is $MSW(V_i)$); allocate all goods to the player, w, with the highest personal welfare; let w pay the second highest personal welfare, and let every other player pay 0. It is easy to show that M is dominant-strategy truthful, and that the revenue it generates along the above sequence of example tends to infinity.

[9] Indeed the revenue of the VCG mechanism is itself a benchmark, that is, a function from multi-sets of valuations to reals.

For combinatorial auctions, not only no natural solutions with non-trivial revenue guarantees are known, but not even general results about revenue are known, even in the Bayesian setting.

2 Our Results

We make various contributions to the guaranteed-revenue problem in combinatorial auctions. The easiest to appreciate is our "positive" one. Namely, we explicitly put forward a natural solution pair $(\mathbb{B}, \mathbb{M})$ with a reasonable revenue guarantee. This solution is attractive due to the conceptual simplicity of its probabilistic mechanism $\mathbb{M}$, and the fact that $\mathbb{B}$ is a very natural generalization of the second-valuation benchmark of single-good auctions. Personally, however, we appreciate more our "negative" results. In particular, we prove that no DST mechanism can significantly outperform $\mathbb{M}$ on our benchmark, and that an exponential performance gap exists between $\mathbb{M}$ and any deterministic DST mechanism. Therefore, while our positive result establishes reasonably attractive properties of a specific DST mechanism, our negatives ones establish intrinsic limitations of all possible DST auction mechanisms, and advance our understanding of what revenue can be guaranteed in combinatorial auctions. (The usefulness of such understanding is perhaps evidenced by the fact that we have first proved our negative results, and that these have then guided us to discover our positive one!)

2.1 Our Benchmark

What revenue benchmarks should we choose for natural solution pairs in combinatorial auctions? An obvious temptation is to consider MSW. In fact, MSW is both player-monotone and the highest benchmark rationally achievable, since rational players will never collectively pay more than the goods are collectively worth to them. To be sure, MSW cannot be achieved with guarantee 1 in a robust way,[10] but natural solution pairs are allowed to work with lower guarantees, and it is thus legitimate to ask whether solution pairs (MSW, M) with less than 1 but positive revenue guarantees exist. We prove that the answer is no. That is,

Theorem 1. *For any DST mechanism M and any positive function $g(n,m)$, there exists a valuation profile V for an $n \times m$ auction context such that the revenue of $M(V_1, \ldots, V_n) < g(n,m) \cdot MSW(V_1, \ldots, V_n)$.*

In other words, MSW is "too perfect:" aiming to approximate it in combinatorial auctions in a sufficiently adversarial setting is tantamount to aiming for eternal life in a national health-care system. But in looking for alternatives, we must resist another natural temptation: namely, choosing a benchmark so "next-to-perfect" as to be achieved or reasonably approximated only in the rarest of auctions. It is in fact our goal to discover *revenue guarantees that apply to all combinatorial auctions.* And it is with this goal in mind that we put forward the following benchmark $MSW_{-\star}$.

Definition 2. *Let $V = (V_1, \ldots, V_n)$ be a sequence of valuations of a finite set of goods G, and let i and S be such that $V_i(S) \geq V_j(T)$ for all $j \in [1,n]$ and all $T \subset G$. Then, we define $MSW_{-\star}(V) = MSW(V_{-\star})$, where $V_\star = V_i$ and $V_{-\star} = (V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_n)$.*

We refer to such a player i as the *star player,* and denote him by the symbol $\star$, so as to capitalize on standard game theoretic notation in our symbolic manipulations. (If two or more such players i exist, the star player is the "smallest" one among them.)

Translating valuations into players, $MSW_{-\star}$ first removes the star player, and then compute the maximum social welfare for the rest of the players. It should then be obvious that $MSW_{-\star}$ is player monotone, and that it is a generalization of the second-valuation benchmark: that is, $MSW_{-\star} = 2V$ in single-good auctions. Indeed, in single-good auctions, removing the highest bid is tantamount to removing the star player, and the social welfare of the rest is the highest of the remaining bids, and thus the second highest of the original bids.

[10] Indeed, if (MSW, M) were a solution pair with guarantee 1, then M could not, on input the valuations $V = (V_1, \ldots, V_n)$, but return the allocation $BestAlloc(V)$. Consequently, at the end, M should coincide with the VCG mechanism that, as per Example B, is very fragile.

A "Surprising" Example. Consider the following context, where each player i only values the ith good.

	{a}	{b}	{c}	{d}	{e}	{f}	{g}
1	1001						
2		1000					
3			1000				
4				1000			
5					1000		
6						1000	
7							1000

Example D
(Subsets of two or more goods are not valued, and their corresponding columns not shown.)

Traditional general mechanisms would have hard time generating revenue given the valuations of Example D, because no competition for any single subset of the goods exists. In particular, the VCG mechanism would return revenue of 0. By contrast, for the valuations of Example D the value of $MSW_{-\star}$ is 6000 and thus we have the right to expect reasonably high revenue from a mechanism that, like our $\mathbb{M}$, is always guaranteed to approximate $MSW_{-\star}$ reasonably. Indeed, as we shall become clear from its definition, the expected revenue of $\mathbb{M}$ in such example is greater than 1250. This performance is perhaps surprising in view of the fact that Example D is quite special and difficult, while our $\mathbb{M}$ is designed to perform in any combinatorial auction.

2.2 Our Mechanism

Our mechanism $\mathbb{M}$ is simple and probabilistic. In constructing it, some of our choices are dictated by our desire for $\mathbb{M}$ to be DST; others by our desire for $\mathbb{M}$ to generate revenue approximating $MSW_{-\star}$.

At the highest level, the idea is that of *trading efficiency for revenue.* We obtain $\mathbb{M}$ by starting with an underlying, deterministic, DST, and high-efficiency mechanism $\mathcal{M}$, and then modifying it so as convert some of its efficiency to revenue. The first approach to implement such a plan is that of running $\mathcal{M}$ on the profile of bids provided by the players so as to obtain an allocation A' and a profile of prices P', and then raise the prices of the players who win goods. By doing so, efficiency may suffer, because when the raised price exceed a winner' bid, the player may refuse to pay, leaving the goods he wins unallocated. At the same time, if the prices are raised judiciously, then this loss will be compensated by other players paying more so that the total revenue generated will increase. But: How should prices be raised? And by how much?

As we shall argue later on, the revenue of *any* deterministic mechanism can only poorly approximate our benchmark. Thus, we shall raise prices *probabilistically.* Further, in light of our benchmark, one easy way to do this is to add to the prices produced by $\mathcal{M}$ in response to a given sequence of valuations BID a fraction α of $MSW_{-\star}(BID)$, where the scaling factor α is probabilistically chosen between 0 and 1. This approach, however, needs to be refined. To begin with, to ensure that $\mathbb{M}$ is DST, we do not want that "a player's price depend on his bid," and $MSW_{-\star}(BID)$ may indeed depend on a player i's bid. This problem is traversed by increasing i's bid by $\alpha MSW_{-i}(BID)$ instead. Our analysis will support that this small change does not alter our ability to achieve our benchmark. At the same time, such a modification of $\mathcal{M}$ is guaranteed to remain DST.

Two choices now remain to be made: that of the mechanism $\mathcal{M}$ and that of scaling factor α. For $\mathcal{M}$, as we plan to turn efficiency into revenue, it is natural to choose the VCG mechanism, since it has optimal efficiency. (However, any $\mathcal{M}$ whose efficiency is a "sufficiently high" fraction of MSW would work too, leaving room for computationally more tractable auction mechanisms and other desiderata.) As for choosing α, we are actually guided by one of our upper-bound results —discussed in the next subsection, and indeed discovered before $\mathbb{M}$. Informally, this result states that, in a $n \times m$ auction, no DST mechanism can guarantee revenue greater than a logarithmic (in $\mu = min\{n, m\}$) fraction of $MSW_{-\star}(BID)$. With this limitation in mind, as a first approximation, we start by choosing α uniformly among $\{0, 1, \frac{1}{2}, \frac{1}{4}, \ldots, 2^{-\lfloor \log \mu \rfloor}\}$. (Discrete exponential distributions were used in simpler settings by [13] and [3].[11]) In expectation this distribution of α will essentially $\frac{1}{\log \mu}$-approximate the optimal revenue: let $\bar{\alpha}$ be the scale factor which would induce a price for player i of exactly his bid; the distribution of α just described has the property that with logarithmic probability the α chosen either multiplicatively 2-approximates $\bar{\alpha}$, or additively $\frac{1}{2\mu}$-approximates $\bar{\alpha}$. Thus a logarithmic fraction of the time, the distribution "guesses" almost exactly the right price to charge each player.

In the mechanism $\mathbb{M}$ below, we refine this distribution so as to optimize the worst-case bound on its revenue. Specifically, instead of the discrete exponential distribution above, we use a "continuous exponential" distribution, and we adjust the relative probability of selecting $\alpha = 0$. (Our specific selection of constants is solely justified by our desire to optimize our solution.)

Definition 3. *We denote by* $\mathbb{M}$ *the auction mechanism that on input* BID*, a profile of* n *bids for a set of* m *goods, computes an outcome* (A, P) *as follows:*

1. *Pick a* scaling factor $\alpha \in [0, 1]$ *as follows:*
   - *Let* $\mu = \min\{n, m\}$ *and* $c_{n,m}$ *be the constant* > 2 *that solves the equation* $e^{x-2} = x\mu$.
   - *Flip a coin whose probability of Heads is* $\frac{1}{c_{n,m}-1}$. *If Heads, choose* $\alpha = 0$. *If Tails, draw* r *uniformly from* $[-(c_{n,m} - 2), 0]$ *and choose* $\alpha = e^r$.
2. *Compute the* provisional allocation A' *and the profile of* provisional prices $P' = VCG_p(BID)$ *— respectively the allocation and the prices of the VCG mechanism for the bid profile* BID*— and then the set of* provisional winners W' *consisting of all players that obtain a non-empty subset of goods in* A'.
3. *For each* $i \in W'$ *compute* i*'s* offer price $P'_i + \alpha MSW(BID_{-i})$. *If* i*'s bid* $BID_i(A'_i)$ *exceeds* i*'s offer price, set* $A_i = A'_i$ *and* $P_i = P'_i + \alpha MSW(BID_{-i})$*; otherwise set* $A_i = \emptyset$ *and* $P_i = 0$.

REMARKS

- We note that $c_{n,m}$ is uniquely defined since for $\mu \geq 1$ the function $f_{n,m}(x) = e^{x-2} - x\mu$ is negative at $x = 2$, goes to infinity as x increases, and has positive second derivative everywhere.
- Notice that although each price P_i is personalized, it is obtained via the same choice of scaling factor α. Were we in a Bayesian setting, where different players have different distributions for their valuations, then we would optimally choose a separate scaling factor α_i for each player i.

For all combinatorial auctions, we prove the following about our mechanism and our benchmark.

Theorem 2. $(MSW_{-\star}, \mathbb{M})$ *is a natural solution pair with revenue guarantee* $1/c_{n,m}$.

CONCRETE REVENUE CONSEQUENCES. As implied by $c_{n,m}$'s very definition, $1/c_{n,m}$ is a very slowly decreasing function of n and m; namely: $1/c_{n,m} = \Theta(1/\log min\{n, m\})$. That is, disregarding constants, $1/c_{n,m}$ equals $1/\log min\{n, m\}$. Therefore, when either n or m is reasonably small, $\mathbb{M}$'s revenue is expected to be a reasonable portion of the "potential value" expressed by our $MSW_{-\star}$ benchmark. For instance,

- In the case of Example D, $n = 7$ and $1/c_{n,m} > 17\%$. Accordingly, because here the value of our benchmark is 6000, Theorem 2 implies that $\mathbb{M}$'s expected revenue in Example D *must* be greater than 1000. This performance is indeed guarantee for any $n \times m$ auction with a $MSW_{-\star}$ benchmark of 6000 and $min\{n, m\} = 7$. But Example D specifies a very special case of such auctions, and $\mathbb{M}$'s expected revenue with Example D's valuations is even better. Indeed, it can be separately shown to be greater than 1250.

[11] For instance, the goal in [3] was to obtain high-revenue in the *unlimited supply* model. Recall that in this model there are arbitrarily many copies of each good for sale, and thus competition among players for the same good is not an issue. Consequently, in this simpler setting their mechanism is content with producing a single "global" price, paying which any player can obtain as many copies of every good he wants. They call their pricing structure the *amusement-park model*, in analogy to what happens in amusement parks where payment of the fixed entrance fee enables one to enjoy as many rides as he wants.

- If there are fewer than 299 goods for sale (perhaps a plausible scenario), then it can be shown (see Corollary 1) that, regardless of whether the number of players is in the tens, the thousands, or the millions, that $\mathbb{M}$ always returns as revenue 10% of $MSW_{-\star}$ "relative to all independent players," no matter how combinatorially complex the valuations may be, and no matter how collusively, irrationally, or maliciously the other players may bid.

2.3 Optimality of Our Mechanism

We prove two results about $\mathbb{M}$'s revenue performance, relative to our benchmark and any other possible DST mechanism. Namely, (1) it is precisely optimal asymptotically and (2) it is optimal within a constant factor in every auction. Crudely put, we prove there is no way, in dominant strategies, to extract significantly more revenue than $\mathbb{M}$ for *all* combinatorial auctions. Let us now to express these properties more precisely.

Definition 4. *Let $opt(n, m)$ denote the greatest $x \in \mathbb{R}^+$ for which there exists a DST mechanism $\mathcal{M}^{n,m}$ whose (expected) revenue is at least $x \cdot MSW_{-\star}(BID)$ for any $n \times m$ bid profile BID.*

Theorem 3.

$$\lim_{n,m\to\infty} \frac{c_{n,m}}{opt(n,m)} = 1.$$

Theorem 4. *For all n and m,*

$$c_{n,m} \geq \frac{1}{5.5} \cdot opt(n,m).$$

2.4 The Necessity of Probabilism

Our mechanism demonstrates once again the power of randomization. Indeed, we prove the following.

Theorem 5. *For any deterministic auction mechanism $\mathcal{M}$, $(MSW_{-\star}, \mathcal{M})$ has revenue guarantee $\frac{1}{\min\{n,m\}-1}$ for $n, m \geq 2$.*

Thus an exponential gap exists between the revenue guaranteed by our $\mathbb{M}$ and that guaranteed by any deterministic DST mechanism.

3 Conclusions

The simplicity of $\mathbb{M}$ was for us a pleasant surprise. We believe that such simplicity is not only due to $\mathbb{M}$'s crucial probabilism, but also to our choice of benchmark. That is, we interpret $\mathbb{M}$'s simplicity as a possible sign of the naturalness of the $MSW_{-\star}$ benchmark.

Finally, we believe and hope that the robustness achieved by our natural solution pair $(MSW_{-\star}, \mathbb{M})$ can be exported to other mechanism-design settings. In particular, the traditional game-theoretic assumption of the perfect rationality of all players strikes us as being often too risky in practice.[12] We thus look forward to some more thorough investigation of the degree of protection achievable against somewhat irrational players.

[12]For instance, consider the well known game of choosing an integer between 0 and 100 and rewarding the player whose integer is closest to half of the average of chosen integers. Would you really choose 0 when playing it among strangers on the street?

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Appendix

A Preliminaries

A.1 Additional Notation

We consistently denote the set of players by $N = \{1, \ldots, n\}$ and the set of goods for sale by $G = \{g_1, \ldots, g_m\}$.

Sub-profiles. Recall that a profile is a vector V indexed by N. If $C \subset N$, the sub-vector of V indexed by C is denoted by V_C, and is referred to as a sub-profile. If $i \in N$, we more simply write V_i rather than $V_{\{i\}}$; V_{-i} rather than $V_{N-\{i\}}$; and, if $C \subset N$, V_{-C} rather then V_{N-C}. If S and T are disjoint subsets of G, then by $V_S \sqcup V_T$ be denote the sub-profile mapping each player $i \in S \cup T$ to $(V_S)_i$ if $i \in S$, and to $(V_T)_i$ otherwise.

We extend to sub-profiles the functions $Value$, $BestAlloc$ and MSW as follows. For each valuation sub-profile V_C and allocation A: $Value(V_C, A) = \sum_{i \in C} V_i(A_i)$; $BestAlloc(V_C) = argmax_{A \in \mathbb{A}(G)} Value(V_C, A)$; and $MSW(V_C) = Value(V_C, BestAlloc(V_C))$; where $\mathbb{A}(G)$ denotes the set of all possible allocations of G.

By convention, $argmax$'s ties are broken lexicographically. Also by convention, let $BestAlloc(V_C)_i \neq X$ whenever $V_i(X) = 0$, even if $i \in C$ and $X \subset G$.

Mechanisms and Their Revenues. Recall that an auction mechanism is a probabilistic function mapping a bid profile BID to pair (A, P), where A is an allocation and P a price profile, satisfying the *opt-out condition*: $P_i = 0$ whenever BID_i is the null valuation. We thus view each mechanism $\mathcal{M}$ as two separate functions: an *allocation function* $\mathcal{M}_a$ and a *price function* $\mathcal{M}_p$. That is, for all bid profiles BID: $\mathcal{M}(BID) = (\mathcal{M}_a(BID), \mathcal{M}_p(BID))$. The *expected revenue* of mechanism $\mathcal{M}$ on bid profile BID is $E[\sum_{i \in N} \mathcal{M}_p(BID)_i]$.

DST Equilibria. Let $\mathcal{C} = (N, G, TV)$ be an auction context, and $\mathcal{G} = (\mathcal{C}, \mathcal{M})$ an auction. Then, we say that a profile of bids BID^* is a *dominant-strategy truthful* (DST) equilibrium of $\mathcal{G}$ if (1) $BID^* = TV$ and for all bid profiles BID' and (2) players $i \in N$: and $E[u_i(TV_i, \mathcal{M}(TV_i \sqcup BID'_{-i}))] \geq E[u_i(TV_i, \mathcal{M}(BID'))]$. We say that an $n \times m$ mechanism $\mathcal{M}$ is dominant-strategy truthful if for all $n \times m$ auction contexts $\mathcal{C}$, the auction $\mathcal{G} = (\mathcal{C}, \mathcal{M})$ has a DST equilibrium.

Winners. We define the set of *winners* in an allocation A as follows: $Wins(A) = \{i \in N : A_i \neq \emptyset\}$.

A.2 The Opt-Out Condition and the DST-1 and DST-2 Properties

Note that the opt-out condition, recalled just above, is a *model* constraint —*not* a mechanism constraint. Indeed, it captures the fact that participation in an auction is optional. To model this reality we have two choices. The first is adding to any auction an initial round in which players choose whether or not to participate. (Those who so choose will then bid in the following round, where they can be "charged" for their participation whether or not they win any goods.) The second, adopted here, is assuming that "everyone participates," but letting those who do not submit any bids —or submit the null bid— be charged nothing. This latter choice is clearly preferable in auctions —as eBay ones— where every Internet user is a potential bidder, and a round of interaction involving all users is inconceivable.

Note that the opt-out condition is crucial for the first of the following two, clearly holding, properties.

DST-1: For all (probabilistic or not) DST mechanisms $\mathcal{M}$, players i, and bid profile BID, we have: $0 \leq E[\mathcal{M}_p(BID)_i] \leq E[BID_i(\mathcal{M}_a(BID)_i)]$.

DST-2: For all deterministic DST mechanisms $\mathcal{M}$, players i, and bid profiles BID and BID' such that $BID_{-i} = BID'_{-i}$, we have: $\mathcal{M}_a(BID)_i = \mathcal{M}_a(BID')_a$ implies $\mathcal{M}_p(BID)_i = \mathcal{M}_p(BID')_i$.

B Proof of Theorem 1

We actually prove a slightly stronger result; namely,

Theorem 1′: *For any $n \times m$, probabilistic, DST mechanism $\mathcal{M}$, any player i, any bid sub-profile BID_{-i}, and any positive constant c, there exists a bid BID_i such that, letting $BID = BID_{-i} \sqcup BID_i$, we have*

$$E\left[\sum_{j\in N} \mathcal{M}_p(BID)_j\right] < \frac{MSW(BID)}{c}.$$

Proof. Fix arbitrarily a bid profile BID_{-i} and a subset S of the goods. Then, for any $\epsilon > 0$, define

$\mathbb{P} = sup_x\, Pr[\mathcal{M}_a(BID_{-i} \sqcup (S,x))_i = S]$.

That is, $\mathbb{P}$ is the "maximum probability with which player i can win set S, with a single-minded bid for S, when all other players bid BID_{-i}."

x_ϵ to be a positive real number such that $Pr[\mathcal{M}_a(BID_{-i} \sqcup (S,x_\epsilon))_i = S] \geq \mathbb{P} - \epsilon$.

That is, the probability that player i wins S single-mindedly bidding (S, x_ϵ) is greater than $\mathbb{P}(S) - \epsilon$.

$x'_\epsilon = \frac{x_\epsilon + MSW(BID_{-i})}{\epsilon}$.

The definition of DST states that, for any profiles TV and BID', the following inequality holds:

$$E[u_i(TV_i, \mathcal{M}(TV_i \sqcup BID'_{-i}))] \geq E[u_i(TV_i, \mathcal{M}(BID'))]. \tag{1}$$

Letting $TV_i = (S, x'_\epsilon)$, $BID'_{-i} = BID_{-i}$, and $BID'_i = (S, x_\epsilon)$, we can reexpress Inequality 1 as

$$[E[u_i((S,x'_\epsilon), \mathcal{M}((S,x'_\epsilon) \sqcup BID_{-i}))] \geq E[u_i((S,x'_\epsilon), \mathcal{M}((S,x_\epsilon) \sqcup BID_{-i}))]. \tag{2}$$

Again assuming that all players but i bid BID_{-i}, let $\mathbb{P}'_\epsilon$ be the probability that player i wins S when he bids (S, x'_ϵ), and let p'_ϵ be the corresponding expected price he pays. Similarly, let $\mathbb{P}_\epsilon$ be the probability that i wins S when he bids (S, x_ϵ), and p_ϵ the expected price he pays. Then we may reexpress Inequality 2 as

$$x'_\epsilon \mathbb{P}'_\epsilon - p'_\epsilon \geq x'_\epsilon \mathbb{P}_\epsilon - p_\epsilon. \tag{3}$$

Because $x'_\epsilon = \frac{x_\epsilon + MSW(BID_{-i})}{\epsilon}$; because $\mathbb{P}'_\epsilon \leq \mathbb{P}$ (since $\mathbb{P}$ is i's "highest probability of winning S"); because $\mathbb{P}_\epsilon > \mathbb{P} - \epsilon$ (by the definitions of x_ϵ and $\mathbb{P}_\epsilon$); and because $p_\epsilon \leq x_\epsilon$ (by property DST-1); Inequality 3 becomes

$$\frac{x_\epsilon + MSW(BID_{-i})}{\epsilon}\mathbb{P} - p'_\epsilon > \frac{x_\epsilon + MSW(BID_{-i})}{\epsilon}(\mathbb{P} - \epsilon) - x_\epsilon. \tag{4}$$

Simplifying Inequality 4 yields

$$p'_\epsilon < 2x_\epsilon + MSW(BID_{-i}). \tag{5}$$

Now notice that the following Inequality also holds:

$$E\left[\sum_j \mathcal{M}_p(BID)_j\right] < 2(x'_\epsilon + MSW(BID_{-i})). \tag{6}$$

In fact, $E\left[\sum_j \mathcal{M}_p(BID)_j\right] \leq MSW(BID)$ by Property DST-1; $MSW(BID) \leq MSW(BID_{-i}) + MSW(BID_i)$ by the player monotonicity of MSW; $MSW(BID_{-i}) \leq p'_\epsilon$ by Property DST-1; and $p'_\epsilon < 2x_\epsilon + MSW(BID_{-i})$ by Inequality 6.

At the same time, however, we have $\frac{x_\epsilon + MSW(BID_{-i})}{\epsilon} = x'_\epsilon = MSW(BID_i) \leq MSW(BID)$; that is, $x_\epsilon + MSW(BID_{-i}) \leq \epsilon MSW(BID)$. Thus, combining this inequality with Inequality 6 yields

$$E[\sum_j \mathcal{M}_p(BID)_j] < 2\epsilon MSW(BID). \tag{7}$$

Since ϵ can be chosen arbitrarily small, Equation 7 proves our thesis. *Q.E.D.*

C Proof of Theorem 2

We break Theorem 2 into two separate theorems. Namely,

Theorem 2a: $(MSW_{-\star}, \mathbb{M})$ *is a natural solution pair.*
and
Theorem 2b: $(MSW_{-\star}, \mathbb{M})$ *has revenue guarantee* $1/c_{n,m}$.

C.1 Proof of Theorem 2a

Theorem 2a is quite straightforward. The key observation is that $\mathbb{M}$ is DST because it has been obtained from the VCG mechanism by modifications that trivially preserve the dominant-strategy truthfulness. The proof below is mostly given for completeness and self-containment sake.

Definition 5. *A function* $f : N \times \mathbb{V}(G)^N \to \mathbb{R}^+$ *is called* stable *if, for each* $i \in N$*, there exists a function* g_i *such that, for each* $BID \in \mathbb{V}(G)^N$*,* $f(i, BID) = g_i(BID_{-i})$ *—i.e.,* f *is "independent" of* BID_i*.*

Definition 6. *Given a deterministic mechanism* $\mathcal{M}$ *and a stable function* f *define* $\mathcal{M}^{+f}$ *to be the mechanism defined as follows: on input* $BID \in \mathbb{V}(G)^N$*,*

1. *Compute the* provisional allocation $A' = \mathcal{M}_a(BID)$*, the profile of* provisional prices $P' = \mathcal{M}_p(BID)$*, and the set of* provisional winners $W' = Wins(A')$*.*
2. *For each* $i \in W'$*, if* $BID_i(A'_i) \geq P'_i + f(i, BID)$ *then let* $P_i = P'_i + f(i, BID)$ *and* $A_i = A'_i$*; otherwise let* $P_i = 0$ *and* $A_i = \emptyset$*.*

Lemma 2. *If* $\mathcal{M}$ *is DST, so is* $\mathcal{M}^{+f}$ *for any stable* f*.*

Proof. Given an auction context $\mathcal{C} = (N, G, TV)$, a bid profile $BID \in \mathbb{V}(G)^N$, and a player $i \in N$ we have

$$\begin{aligned}
u_i(TV_i, \mathcal{M}^{+f}(TV_i \sqcup BID_{-i})) &\overset{(1)}{=} TV_i(\mathcal{M}_a^{+f}(TV_i \sqcup BID_{-i})_i) - \mathcal{M}_p^{+f}(TV_i \sqcup BID_{-i})_i \\
&\overset{(2)}{=} \max\{0\ ,\ TV_i(\mathcal{M}_a(TV_i \sqcup BID_{-i})_i) - \mathcal{M}_p(TV_i \sqcup BID_{-i})_i - f(i, BID_{-i})\} \\
&\overset{(3)}{=} max\{0\ ,\ u_i(TV_i, \mathcal{M}(TV_i \sqcup BID_{-i})) - f(i, BID_{-i})\} \\
&\overset{(4)}{\geq} max\{0\ ,\ u_i(TV_i, \mathcal{M}(BID)) - f(i, BID_{-i})\} \\
&\overset{(5)}{\geq} u_i(TV_i, \mathcal{M}^{+f}(BID)),
\end{aligned}$$

where: e4quality (1) holds by the definition of the utility function u; equality (2) holds by the definition of $\mathcal{M}^{+f}$, and can be easily checked in each of the two cases $\mathcal{M}_a^{+f}(TV_i \sqcup BID_{-i})_i = \mathcal{M}_a(TV_i \sqcup BID_{-i})_i$ and $\mathcal{M}_a^{+f}(TV_i \sqcup BID_{-i})_i = \emptyset$; equality (3) holds by the definition of the utility function u; inequality (4) holds because $\mathcal{M}$ is dominant-strategy truthful; and inequality (5) holds because $\mathcal{M}^{+f}(BID)$ either yields i utility 0 or utility $u_i(TV_i, \mathcal{M}(BID)) - f(i, BID_{-i})$, and thus at most the maximum of these two quantities. *Q.E.D.*

We now define a special class of probabilistic mechanisms.

Definition 7. *Let* $\mathcal{D}$ *a distribution over a set of deterministic* $n \times m$ *auction mechanisms. Then we denote by* $\mathbb{S}_\mathcal{D}$ *the the probabilistic* $n \times m$ *auction mechanism that, on input a profile of bids* BID*, first selects a mechanism* M *according to* $\mathcal{D}$ *and then returns the outcome* $M(BID)$*.*

We refer to such a mechanism $\mathbb{S}_\mathcal{D}$ as above as a $(n \times m)$ *sampler.*

Lemma 3. *If* $\mathbb{S}_\mathcal{D}$ *is a sampler and all mechanisms in* $\mathcal{D}$*'s support are DST, then* $\mathbb{S}_\mathcal{D}$ *is DST.*

Proof. Let our $\mathbb{S}_{\mathcal{D}}$ actually be an $n \times m$ sampler, and let $\mathcal{C} = (N, G, TV)$ be an $n \times m$ auction context, $BID \in \mathbb{V}(G)^n$, and $i \in N$. Then we have

$$\begin{aligned} E[u_i(TV_i, \mathbb{S}_{\mathcal{D}}(TV_i \sqcup BID_{-i}))] &\overset{(1)}{=} \\ \underset{M \leftarrow \mathcal{D}}{E}[u_i(TV_i, M(TV_i \sqcup BID_{-i}))] &\overset{(2)}{\geq} \\ \underset{M \leftarrow \mathcal{D}}{E}[u_i(TV_i, M(BID))] &\overset{(3)}{=} \\ E[u_i(TV_i, \mathbb{S}_{\mathcal{D}}(BID))] & \end{aligned}$$

where equality (1) holds because the definition of $\mathbb{S}_{\mathcal{D}}$; inequality (2) because M is DST; and equality (3) because of the definition of $\mathbb{S}_{\mathcal{D}}$. This chain of inequality proves that $\mathbb{S}_{\mathcal{D}}$ is DST.

Q.E.D.

Theorem 2a: $(MSW_{-\star}, \mathbb{M})$ *is a natural solution pair.*

Proof. The benchmark $MSW_{-\star}$ is player-monotone, while mechanism $\mathbb{M}$ is a sampler satisfying the hypothesis of Lemma 4.

Q.E.D.

C.2 Proof of Theorem 2b

Theorem 2b: $(MSW_{-\star}, \mathbb{M})$ *has revenue guarantee* $1/c_{n,m}$.

Proof. In virtue of Definition 2, we need to prove that, whenever BID is a valuation profile for a $n \times m$ auction, we have

$$E[\sum_{i \in N} \mathbb{M}_p(BID)_i] \geq \frac{MSW_{-\star}(BID)}{c_{n,m}}. \tag{8}$$

For each player i, let S_i be the (possibly empty) set player i provisionally wins, and let P'_i be the provisional price $VCG_p(BID)_i$. We divide our proof into two cases: in the first the star player bids huge and the revenue bound is derived from just the revenue $\mathbb{M}$ extracts from him. In the second case, no huge bidder exists, and we must sum up the revenue that $\mathbb{M}$ extracts from each set-winning player.

Case 1: $BID_\star(S_\star) > P'_\star + MSW_{-\star}(BID)$**.**

Note that the right-hand side of the inequality of this case is always ≥ 0, thus $BID_\star(S_\star) > 0$ always. This implies that $S_\star \neq \emptyset$; namely that $\star$ is a provisional winner. As such, mechanism $\mathbb{M}$ "makes $\star$ the offer" $P'_\star + \alpha \cdot MSW_{-\star}(BID)$ where $\alpha \leq 1$, the offer price will always be at most player $\star$'s bid for $S_\star$, and hence player $\star$ will always pay his offer price. Thus the expected revenue from player $\star$ is just the expected offer price, namely

$$
\begin{aligned}
E[\mathbb{M}_p(BID)_\star] &= \frac{1}{c_{n,m}-1}P'_\star + (1-\frac{1}{c_{n,m}-1})\int_{-(c_{n,m}-2)}^{0}\frac{1}{c_{n,m}-2}\left(P'_\star + e^r MSW_{-\star}(BID)\right)dr \\
&= \left(\frac{1}{c_{n,m}-1} + (1-\frac{1}{c_{n,m}-1})\right)P'_\star + (1-\frac{1}{c_{n,m}-1})\frac{1}{c_{n,m}-2}MSW_{-\star}(BID)\int_{-(c_{n,m}-2)}^{0} e^r dr \\
&= P'_\star + \frac{1}{c_{n,m}-1}MSW_{-\star}(BID)\int_{-(c_{n,m}-2)}^{0} e^r dr \\
&= P'_\star + MSW_{-\star}(BID)\frac{1-e^{-(c_{n,m}-2)}}{c_{n,m}-1} \\
&\geq MSW_{-\star}(BID)\frac{1-\mu e^{-(c_{n,m}-2)}}{c_{n,m}-1} = MSW_{-\star}(BID)\frac{1-\frac{1}{c_{n,m}}}{c_{n,m}-1} = \frac{MSW_{-\star}(BID)}{c_{n,m}},
\end{aligned}
$$

where the inequality follows because $P'_1 \geq 0$ and $\mu \geq 1$, and the second to last equality is by the definition of $c_{n,m}$, namely that $\mu c_{n,m} e^{-(c_{n,m}-2)} = 1$. Thus we have the desired result in this case.

Case 2: $BID_1(S_1) < P'_1 + MSW(BID_{-1})$**.**

Consider a provisional winner i. We note that when $\alpha = 0$ the offer price for player i is just P'_i, which is less than or equal to $BID_i(S_i)$ since the VCG mechanism never charges players more than their bid; thus when $\alpha = 0$ player i will pay P'_i. We claim that the expected price paid by player i is at least

$$
\frac{P'_1}{c_{n,m}-1} + (1-\frac{1}{c_{n,m}-1})\int_{-(c_{n,m}-2)}^{\log_e \frac{BID_i(S_i)-P'_i}{MSW(BID_{-i})}}\frac{1}{c_{n,m}-2}\left(P'_i + e^r MSW(BID_{-i})\right)dr.
$$

This expression is just the natural expression for the expected offer price when r is limited to be at most $\log_e \frac{BID_i(S_i)-P'_i}{MSW(BID_{-i})}$, namely the value of r for which the offer price of $P'_i + e^r MSW(BID_{-i})$ equals player i's bid $BID_i(S_i)$. For the above integral to equal this expectation we require $-(c_{n,m}-2) \leq \log_e \frac{BID_i(S_i)-P'_i}{MSW(BID_{-i})} \leq 0$. We prove the second inequality, which we restate as $BID_i(S_i) \leq MSW(BID_{-i}) + P'_i$. We note that for $i = 1$ this inequality is implied by the condition of this case. For $i \neq 1$ we have that $MSW(BID_{-i})$ is at least player 1's highest bid since $1 \in N - \{i\}$, which is at least $BID_i(S_i)$ since by assumption player 1 is the highest bidder.

We note that the first inequality is not necessarily true, and in this case since the limits of integration are in the "wrong order" the integral will be negative. Since the expected revenue when $\alpha > 0$ is non-negative, the integral is thus a *lower bound* for the expected revenue, as claimed. Thus we have

$$
\begin{aligned}
E[\mathbb{M}_p(BID)_i] &\geq \frac{P'_i}{c_{n,m}-1} + (1-\frac{1}{c_{n,m}-1})\int_{-(c_{n,m}-2)}^{\log_e \frac{BID_i(S_i)-P'_i}{MSW(BID_{-i})}}\frac{1}{c_{n,m}-2}\left(P'_i + e^r MSW(BID_{-i})\right)dr \\
&\geq \frac{P'_i}{c_{n,m}-1} + (1-\frac{1}{c_{n,m}-1})\int_{-(c_{n,m}-2)}^{\log_e \frac{BID_i(S_i)-P'_i}{MSW(BID_{-i})}}\frac{1}{c_{n,m}-2}\left(e^r MSW(BID_{-i})\right)dr \\
&= \frac{P'_i}{c_{n,m}-1} + MSW(BID_{-i})\frac{1}{c_{n,m}-1}\left(e^{\log_e \frac{BID_i(S_i)-P'_i}{MSW(BID_{-i})}} - e^{-(c_{n,m}-2)}\right) \\
&= \frac{1}{c_{n,m}-1}\left(BID_i(S_i) - e^{-(c_{n,m}-2)}MSW(BID_{-i})\right)
\end{aligned}
$$

Summing up this inequality over all provisional winners i, we get

$$
E[\sum_{i\in W'}\mathbb{M}_p(BID)_i] \geq \frac{1}{c_{n,m}-1}\left(\sum_{i\in W'} BID_i(S_i) - e^{-(c_{n,m}-2)}\sum_{i\in W'} MSW(BID_{-i})\right).
$$

Now notice that $\sum_{i \in W'} BID_i(S_i) = MSW(BID)$. Further since $|W'| \leq \mu$ and $MSW(BID_{-i}) \leq MSW(BID)$ we have $\sum_{i \in W'} MSW(BID_{-i}) \leq \mu \cdot MSW(BID)$. Thus we have

$$E[\sum_{i \in W'} \mathbb{M}_p(BID)_i] \geq MSW(BID)\frac{1-\mu e^{-(c_{n,m}-2)}}{c_{n,m}-1} = MSW(BID)\frac{1-\frac{1}{c_{n,m}}}{c_{n,m}-1} = \frac{MSW(BID)}{c_{n,m}} \geq \frac{MSW_{-\star}(BID)}{c_{n,m}},$$

where we invoke the definition of $c_{n,m}$ to derive the first equality. Thus we have the desired conclusion. *Q.E.D.*

Remarks.

- Notice that our mechanism $\mathbb{M}$ requires that its underlying DST mechanism be reasonably efficient. Indeed, in the analysis of Case 2, we rely on the fact that the VCG algorithm is 100% efficient: namely, we rely $\sum_{i \in W'} BID_i(S_i) = MSW(BID)$. If another DST mechanism is used, one should make sure that, for its provisional allocation A, $\sum BID_i(A_i)$ is a sufficient fraction of $MSW(BID)$.
- Notice that, when lower-bounding the revenue generated by $\mathbb{M}$, the profile of prices returned by the underlying DST mechanism are essentially ignored. However, were we to "simplify" the definition of $\mathbb{M}$ by replacing the provisional prices with zeros, the resulting mechanism would not be DST.

Corollary 1. *Given a constant $c > 2$, for every auction context where* $\min\{|N|, |G|\} \leq \frac{e^{c-2}}{c}$,

$$E[\sum_{i \in N} \mathbb{M}_p(BID)_i] \geq \frac{MSW_{-\star}(BID)}{c}.$$

Proof. This follows immediately from Theorem 2 after solving for μ in the definition of $c_{n,m}$. *Q.E.D.*

We note that for the value $c = 10$ we have $\frac{e^{c-2}}{c} > 298$ and thus this corollary says that for any auction with either at most 298 players and unlimited goods, or at most 298 goods and unlimited players, we can guarantee expected revenue of 10% of our benchmark.

Corollary 2. *Given a constant $c > 2$, for every auction context (N, G, TV), where* $\min\{|N|, |G|\} \leq \frac{e^{c-2}}{c}$,

$$E[\sum_{i \in N} \mathbb{M}_p(BID)_i] \geq \frac{MSW_{-\star}(TV)}{c}.$$

Proof. From Theorem **??**, mechanism $\mathbb{M}$ has a dominant-strategy equilibrium where $BID = TV$, so the corollary follows immediately from the previous corollary. *Q.E.D.*

C.3 Proof of Theorem 3

Let us start by recalling the notion of a "single-minded auction".

Definition 8. *A valuation x of a finite set of goods G is* single-minded *if there exists a single set of goods S and $v \in \mathbb{R}^+$ such that $x(T) = v$ whenever $S \subset T$ and 0 otherwise.*

We compactly represent such a single-minded valuation x by the pair (S, v).

C.3.1 Harmonic Distributions and Harmonic Pricing

Let us introduce a simple distribution that we have found crucial for proving several impossibility results in general auctions (this is just the first one).

Definition 9. (Bounded-Harmonic Distributions) *For any subset of goods S and positive integer k, we denote by h_S^k the distribution assigning, for each integer $i \in [1, k]$, probability $\frac{1}{k}$ to the single-minded valuation $(S, \frac{1}{i})$.*

We now prove a property of DST mechanisms that may be of independent interest. It is well known that the harmonic series, $\sum_{j=1}^{\infty} 1/i$, diverges, although "slowly." By contrast, the series of the prices paid, in any DST mechanism, by a given player who bids harmonically for a given set not only always converges, but is actually upper-bounded by 1.

Lemma 4. (Harmonic-Pricing) *For all probabilistic DST mechanisms $\mathcal{M}$, all players i, all valuation subprofiles BID_{-i}, and all subsets of goods S,*

$$\sum_{j=1}^{\infty} E[\mathcal{M}_p(BID_{-i} \sqcup (S, \frac{1}{j}))_i] \leq 1.$$

Proof. We show that the sum of over any finite set $j \in \{1, \dots k\}$ is at most 1, for any integer k. Define α_j as the expected price paid by player i, for each $j \leq k$ and let β_j be the probability that player i receives some set containing set S when $\mathcal{M}$ is evaluated on $BID_{-i} \sqcup (S, \frac{1}{j})$; let $\alpha_{k+1} = \beta_{k+1} = 0$. We show that $\sum_{j \leq k} \alpha_j \leq 1$.

We note that since $\mathcal{M}$ is DST, we may apply the definition of DST with $TV_i = (S, \frac{1}{j})$ and $BID' = BID_{-i} \sqcup (S, \frac{1}{j+1})$ to yield

$$E[u_i((S, \frac{1}{j}), \mathcal{M}((S, \frac{1}{j}) \sqcup BID_{-i}))] \geq E[u_i((S, \frac{1}{j}), \mathcal{M}((S, \frac{1}{j+1}) \sqcup BID_{-i}))].$$

Evaluating both sides by the definition of u_i yields

$$\frac{1}{j}\beta_j - \alpha_j \geq \frac{1}{j}\beta_{j+1} - \alpha_{j+1}. \tag{9}$$

Suppose for the sake of contradiction that $\sum_{j=1}^{k} \alpha_j > 1$. Thus $1 < \sum_{j=1}^{k} \alpha_j = \sum_{j=1}^{k} j(\alpha_j - \alpha_{j+1})$ and further since for each j, $0 \leq \beta_j \leq 1$, we have $\beta_1 = \sum_{j=1}^{k} (\beta_j - \beta_{j+1}) \leq 1$. Comparing these two sums term by term we note that there must exist a j such that the corresponding term from the first sum exceeds the term from the second sum, namely $j(\alpha_j - \alpha_{j+1}) > (\beta_j - \beta_{j+1})$. Dividing by j and rearranging terms yields $\frac{1}{j}\beta_j - \alpha_j < \frac{1}{j}\beta_{j+1} - \alpha_{j+1}$, which contradicts Equation 9. Thus $\sum \alpha_j \leq 1$, as desired. *Q.E.D.*

C.3.2 Our Specific Revenue Upper-Bound

Lemma 5. *For any $n \times m$, probabilistic, and DST mechanism $\mathcal{M}$ there exists a bid profile BID such that, letting $\mu = \min\{n, m\}$, we have*

$$E[\sum_{i \in N} \mathcal{M}_p(BID)_i] \leq \frac{MSW_{-\star}(BID)}{\log_e \mu - 2\sqrt{\log_e \mu} - 2}.$$

Proof. We actually show that we can choose BID to consist of just single-minded bids. Our proof is non-constructive. We endow the space of all possible bid profiles with a particular distribution, and show that, under it, the probability of an "unprofitable" profile is > 0.

Denote by G the given set of m goods, consider a subset of the players, $N' = \{1, \dots, \mu\}$ and let G' be a subset of G of cardinality μ, indexed by integers in $\{1, ..., \mu\}$. Define $S_i = \{i\}$, and let $\mathcal{BID}$ be the distribution on the set of all profiles of single-minded valuations of G where for each $i \in N'$, $\mathcal{BID}_i = h^{\mu}_{S_i}$. We show

1. With probability $< \frac{1}{\log_e \mu}$, $MSW_{-\star}(BID) \leq \log_e \mu - \frac{\pi^2}{6}\sqrt{\log_e \mu} - 1$;
2. With probability $\leq \frac{-1+\log_e \mu}{\log_e \mu}$, $E[\sum_{i \in N} \mathcal{M}_p(BID)_i] \geq \frac{\log_e \mu}{-1+\log_e \mu}$.

Proof of Property 1.

We note that the bid value of each player in N' has mean $\frac{1}{\mu}\sum_{i\le\mu}\frac{1}{i}$. Since the valued sets of the players are disjoint, the expected "total value" of the μ players in N' is $E[MSW(BID)] = \sum_{i\le\mu}\frac{1}{i} > \log_e \mu$. Recall that the variance of a random variable is always at most the expected value of its square. Thus we may bound the variance of the bid value of each player by $\frac{1}{\mu}\sum_{i\le\mu}\frac{1}{i^2} < \frac{1}{\mu}\sum_i \frac{1}{i^2} = \frac{\pi^2}{6\mu}$. Thus, the variance of $MSW(BID)$ is less than $\frac{\pi^2}{6}$, and thus $MSW(BID)$ has standard deviation less than $\frac{\pi}{\sqrt{6}}$. By Chebyshev's inequality, the probability that $MSW(BID)$ is more than $\sqrt{\log_e \mu}$ standard deviations below $\log_e \mu$ is less than $\frac{1}{\log_e \mu}$.

We note that since each bid is at most 1, $MSW_{-\star}(BID) \ge MSW(BID) - 1$, and thus with probability $< \frac{1}{\log_e \mu}$, we have $MSW_{-\star}(BID) \le \log_e \mu - \frac{\pi^2}{6}\sqrt{\log_e \mu} - 1$. ■

Proof of Property 2. From Lemma 4, for any profile BID, $\sum_{j=1}^{\mu} E[\mathcal{M}_p(BID_{-i} \sqcup (S_i, \frac{1}{j}))_i] \le 1$. Replacing the sum with an expectation we have that $\mu \cdot E_{j\leftarrow\{1,\ldots,\mu\}}[E[\mathcal{M}_p(BID_{-i} \sqcup (S_i, \frac{1}{j}))_i]] \le 1$. We take the expectation of the left hand side over $BID \leftarrow \mathcal{BID}$, and divide by μ to see that $E_{BID\leftarrow\mathcal{BID}}[E_{j\leftarrow\{1,\ldots,\mu\}}[E[\mathcal{M}_p(BID_{-i} \sqcup (S_i, \frac{1}{j}))_i]]] \le \frac{1}{\mu}$. We note that the distribution of $(S_i, \frac{1}{j})$ in the left hand side equals the distribution of BID_i in $\mathcal{BID}_i$, so we may simplify the left hand side to $E_{BID\leftarrow\mathcal{BID}}[E[\mathcal{M}_p(BID)_i]]$. Thus the expected price paid by player i is at most $\frac{1}{\mu}$. By symmetry (since $\mathcal{BID}$ is symmetric with respect to permutations π applied to the player set and the good set) the expected price paid by each player is at most $\frac{1}{\mu}$, and thus the expected total price paid by players in N' is at most 1. Since only the players in N' bid, no revenue can be collected from players in $N - N'$. Thus the total expected revenue is at most 1. By Markov's inequality, the revenue will exceed $\frac{\log_e \mu}{-1+\log_e \mu}$ with probability at most $\frac{-1+\log_e \mu}{\log_e \mu}$.■

Combining Properties 1 and 2 via the union bound, we conclude that there exists a profile BID such that $MSW_{-\star}(BID) \ge \log_e \mu - \frac{\pi^2}{6}\sqrt{\log_e \mu} - 1$ and $E[\sum_{i\in N}\mathcal{M}_p(BID)_i] \le \frac{\log_e \mu}{-1+\log_e \mu}$. Thus we have

$$\begin{aligned}\frac{MSW_{-\star}(BID)}{E[\sum_{i\in N}\mathcal{M}_p(BID)_i]} &\ge \frac{1}{\log_e \mu}\left(\log_e^2 \mu - \frac{\pi^2}{6}\log_e^{1.5}\mu - 2\log_e \mu + \frac{\pi^2}{6}\log_e^{.5}\mu + 1\right)\\ &\ge \frac{1}{\log_e \mu}\left(\log_e^2 \mu - \frac{\pi^2}{6}\log_e^{1.5}\mu - 2\log_e \mu\right) \ge \log_e \mu - 2\sqrt{\log_e \mu} - 2.\end{aligned}$$

Q.E.D.

D Proof of Theorem 4

The proof of this theorem will be provided in the final paper.

E Proof of Theorem 5

We actually prove a slightly stronger version of Theorem 5: namely

Theorem 5′: *For any deterministic $n \times m$ auction algorithm $\mathcal{M}$ and any $\nu < \min\{n, m\} - 1$, there exists a single-minded bid profile BID such that*

$$\sum_i \mathcal{M}_p(BID)_i \le \frac{MSW_{-\star}(BID)}{\nu}.$$

Proof. Setting $\mu = \min\{n.m\}$, we construct the desired bid profile within three steps.

Step 1. Define the single-minded bid profile BID^0 as follows: BID^0_i equals $(\{i\}, 1)$ if player $i \leq \mu$, and the null valuation otherwise. It is thus clear that $MSW_{-\star}(BID^0) = \mu - 1$ and, due to our choice of μ and ν,

$$1 < \frac{MSW_{-\star}(BID^0)}{\nu}. \tag{10}$$

On the price side, we distinguish two cases: namely, (1) $\mathcal{M}_p(BID^0)_i > 0$ for no i and (2) $\mathcal{M}_p(BID^0)_i > 0$ for some i. In the first case,

$$\sum_i \mathcal{M}_p(BID^0)_i = 0 \tag{11}$$

so that Inequalities 10 and 11 imply that BID^0 satisfies our thesis. Otherwise, we proceed to Step 2.

Step 2. Let j be a player such that $\mathcal{M}_p(BID^0)_j > 0$, and define for each integer $\alpha \geq 2$ the bid profile BID^α as follows: BID^α_i equals $(\{j\}, \mu^\alpha)$ if $i = j$, and $(\{i\}, 1)$ otherwise. It is thus evident that, for all $\alpha \geq 2$, $MSW_{-\star}(BID^\alpha) = \mu - 1$ and, due to our choice of μ and ν,

$$1 < \frac{MSW_{-\star}(BID^\alpha)}{\nu}. \tag{12}$$

Let us now analyze the price side. Notice three facts: by construction, $BID^0_{-j} = BID^\alpha_{-j}$; by Property DST-1, j is allocated $\{j\}$ in BID^0; and, for all $\alpha \geq 2$, j's bid value for $\{j\}$ is higher in BID^α than in BID^0. Thus, because $\mathcal{M}$ is deterministic, Property DST-2 implies that, for all $\alpha \geq 2$, j continues to win the set $\{j\}$ in BID^α and to pay the same price he pays in BID^0, which at most 1 —because of Property DST-1 and because $BID^0_j(\{j\}) = 1$.

We now distinguish two cases: (a) there is some integer $\bar{\alpha} \geq 2$ such that $\mathcal{M}_p(BID^\alpha)_i = 0$ for all $i \neq j$, and (b) for each integer $\alpha \geq 2$ there is a player k_α, $k_\alpha \neq j$, such that $\mathcal{M}_p(BID^\alpha)_{k_\alpha} > 0$. In the first case,

$$\sum_i \mathcal{M}_p(BID^{\bar{\alpha}}) \leq 1 \tag{13}$$

and thus Inequalities 12 and 13 imply that $BID^{\bar{\alpha}}$ satisfies our thesis. Otherwise, we proceed to Step 3.

Step 3. By the opt-out condition, $k_\alpha \in \{1, \mu\} \setminus \{j\}$ for all $\alpha \geq 2$. Thus, the pigeonhole principle implies the existence of $\beta, \gamma \in \{2, \ldots, \mu + 1\}$ such that $k_\beta = k_\gamma$ and $\beta < \gamma$. Define now

$$k = k_\beta (= k_\gamma) \quad \text{and} \quad BID' = BID^\gamma_{-k} \sqcup (\{k\}, \mu^\gamma).$$

Since the star player in BID' is either j of k, it is clear that $MSW_{-\star}(BID') = \mu^\gamma + \mu - 2 \geq \mu^\gamma$. Further, because γ and β are integers, $\gamma > \beta$, and $\beta \geq 2$, we have $\mu^\gamma > (\mu^\beta + \mu)(\mu - 1)$ and, due to our choice of μ and ν,

$$\mu^\beta + \mu < \frac{MSW_{-\star}(BID')}{\nu}. \tag{14}$$

Let us now analyze the price situation. We consider the following two mutually exclusive cases.

Case 1: $\mathcal{M}_p(BID')_j \leq \mu^\beta$.

The definition of BID' and Property DST-1 clearly imply that $\sum_{i \in -\{j,k\}} \mathcal{M}_p(BID')_i \leq \mu - 1$. As for player k $(= k_\gamma)$, note the following four facts: $\mathcal{M}_a(BID^\gamma)_k = \{k\}$ (because $\mathcal{M}_p(BID^\gamma)_k > 0$); $\mathcal{M}_p(BID')_k \leq 1$ (because $BID'_k = (\{k\}, 1)$); $BID'_{-k} = BID^\gamma_{-k}$; and k's bid value for $\{k\}$ is higher in BID' than in BID^γ. Thus, $\mathcal{M}_p(BID')_k = \mathcal{M}_p(BID^\gamma) \leq 1$ so that, in the case under consideration,

$$\sum_i \mathcal{M}_p(BID') \leq \mu^\beta + \mu_1. \tag{15}$$

Thus Inequalities 14 and 15 imply that the bid profile BID' satisfies our thesis.

Case 2: $\mathcal{M}_p(BID')_j > \mu^\beta$**.**

Define

$$BID'' = BID^\beta_{-k} \sqcup (\{k\}, \mu^\gamma).$$

As for BID', it is clear that $MSW_{-\star}(BID'') = \mu^\gamma + \mu - 2 \geq \mu^\gamma$ and thus that

$$\mu^\beta + \mu < \frac{MSW_{-\star}(BID'')}{\nu}. \tag{16}$$

Turning our attention to prices, as for BID', it is clear that $\sum_{i \in -\{j,k\}} \mathcal{M}_p(BID'')_i \leq \mu - 1$.

Let us now analyze the price of player j. Notice that BID' and BID'' differ only in the bid of player j, and that j bids higher for $\{j\}$ in BID' than in BID''. Thus, if j won $\{j\}$ in BID'', then he would win it too in BID' would still pay $\mathcal{M}_p(BID'')_j$. But since j's bid value in BID'' is μ^β, we would have $\mathcal{M}_p(BID')_j = \mathcal{M}_p(BID'')_j \leq \mu^\beta$, that is a contradiction in the case under consideration. The contradiction shows that $\mathcal{M}_p(BID^{''})_j = 0$.

Finally, let us analyze the price of player k. Notice that BID'' and BID^β differ only on the bid of k; that k wins his set $\{k\}$ under BID^β paying at most 1; and that k bids higher for $\{k\}$ in BID'' than in BID^β. Thus k continue to win $\{k\}$ in BID'' and $\mathcal{M}_p(BID'')_k \leq 1$, so that

$$\sum_i \mathcal{M}_p(BID'')_i \leq \mu^\beta + \mu - 1. \tag{17}$$

Thus Inequalities 16 and 17 imply that the bid profile BID'' satisfies our thesis. *Q.E.D.*