

Reduced Basis Method for Boltzmann Equation

by

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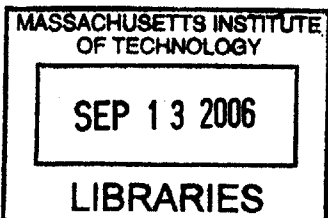
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Submitted to the CDO Programme
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Abstract

The main aim of the project is to solve the BGK model of the Knudsen parameterized Boltzmann equation which is 1-d with respect to both space and velocity. In order to solve the Boltzmann equation, we first transform the original differential equation by replacing the dependent variable with another variable, weighted with function $\tau(y)$; next we obtain a Petrov Galerkin weak form of this new transformed equation. To obtain a stable and accurate solution of this weak form, we perform a transformation of the velocity variable y , such that the semi-infinite domain is mapped into a finite domain; we choose the weighting function $\tau(y)$, to balance contributions at infinity. Once we obtain an accurate and well defined finite element solution of the problem. The next step is to perform the reduced basis analysis of the equation using these accurate finite element solutions. We conclude the project by verifying that the orthonormal reduced Basis method based on the greedy algorithm converges rapidly over the chosen test space.

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Chapter 1

Introduction

1.1 Background

The Boltzmann equation for dilute gases is the more general equation from which the Navier-Stokes equations are derived. The Navier-Stokes equations are the limit in which the mean free path is much smaller than the spatial extent. However, this assumption is violated, when the spatial extent is comparable to the mean free path. These days, plenty of applications are based on smaller technologies like microchannels. In order to model such phenomena, one must solve the Boltzmann equation instead of the Navier-Stokes equations. The reference [2], explores both these equations and gives us a good insight in to the difference between both these equations.

1.2 Parameters involved and the quantities of Interest

The key parameter in the Boltzmann equation is the Knudsen number, Kn , defined as the ratio of the mean free path to the macroscale. This implies that the Navier-Stokes is valid in the limit of small Knudsen number. The Boltzmann equation is a Partial Differential Equation for a distribution function F , as a function of x, y, z, c_x, c_y , and c_z , where the x, y , and z indicate positions and c_x, c_y, c_z are molecular velocities. Since, in general the equa-

tion is defined over six independent variables, classical methods to solve Partial Differential Equations are not typically applicable. Once, we solve the Boltzmann equation, macro quantities like velocity and flow rate can be evaluated as appropriate expectations. It could be interesting to be able to rapidly predict these quantities as a function of molecular quantities like Knudsen number.

1.3 Related work done previously

Carlo Cercignani and Adelia Daneri have numerically analysed the poiseuille flow of a rarefied gas between two parallel plates for an inverse Knudsen number ranging from 0.01 to 10.5 in the reference [1]. The Bhatnagar, Gross, and Krook model of the Boltzmann equation was used in the study and the transport integrodifferential equation was reduced to a purely integral one, which in turn was solved numerically by the discrete ordinate method. The plot of the volume flow rate vs Knudsen number obtained, was found to have an expected minimum and the results also seemed to fit well with the experimental results and previous approximate calculations. These results will be compared with the results obtained by solving the same Boltzmann equation using Finite element methods in chapter 5.

1.4 Motivation for Reduced Basis methods

The Boltzmann equation was often used in the past, to model the flow in the high Mach number cases. Here, We are only interested in the low Mach number cases, that arise in various micro-engineering situations. Thus, the solutions derived from the Boltzmann equation are essentially smooth, which is important for the application of reduced-basis approach. In general, statistical particle methods are most often used for the solving the Boltzmann equation. The Monte Carlo methods in particular, become more efficient for more numbers of coordinates. The reference [3] is one such work in which Monte Carlo methods were applied to solve the Boltzmann equation. However, the reduced-basis methods might also prove efficient, since at least for the online stage, the underlying dimensions of the problem

are reasonably small. A lot of work has been done by the Professor Anthony Patera group and others in this regard. The research outputs of the group like [4, 5, 6, 7, 8, 9, 10, 11] give us insight to the application of Reduced Basis methods for real-time, reliable computation. In general, the distribution function F , may not be smooth even at low mach number, but it appears that in some interesting cases either F is smooth or at least the singularity location is not a function of the parameters. In such cases, the reduced-basis methods might indeed work well. The references [12, 13, 14, 15] are few other research papers that deal with the methodology of Reduced Basis methods . In this project we pursue the finite element "truth" approach. A good description of the finite element method is provided in[16]. The "truth" approximation is the solution upon which we build the reduced-basis approximation; hence a variational approach for the former leads us naturally, to the variational approach of the latter. The ultimate goal of the project is real-time, reliable prediction of microflows for educational purposes, for design and optimization, for in-the-field parameter estimation etc by the reduced-basis methods.

Chapter 2

Strong form of the Boltzmann equation

2.1 The original differential equation and output

In this project we consider the strong form of the Boltzmann equation which is 1-dimensional with respect to both space and velocity. The corresponding independent variables are x in $[-1, 1]$ (the spatial coordinate across the channel) and y (which is a molecular velocity in a particular direction) in $]-\infty, \infty[$. We will denote our dependent variable related to the distribution function as $u(x, y; \theta)$. The equation governing $u(x, y; \theta)$ is

$$y \frac{\partial u}{\partial x} + \frac{1}{\theta} \left[u(x, y; \theta) - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y'^2} u(x, y'; \theta) dy' \right] = f, \quad (2.1)$$

with boundary conditions

$$u(x = +1, y) = 0 \quad \forall y < 0, \quad (2.2)$$

$$u(x = -1, y) = 0 \quad \forall y > 0. \quad (2.3)$$

where θ is a parameter (related to the Knudsen number) and $f = \frac{1}{2}$. In fact, for this particular scaling of the problem θ is equal to twice the Knudsen number. We will take our

output to be

$$S(\theta) = \frac{1}{2} \left(\frac{1}{\sqrt{\pi}} \right) \int_{-1}^{+1} \int_{-\infty}^{\infty} e^{-y^2} u(x, y; \theta) dy dx. \quad (2.4)$$

$S(\theta)$ represents the flow rate through the channel for a particular θ (for our particular scaling). We can also observe that the boundary conditions of the equation are defined for $y > 0$ and $y < 0$, but not $y = 0$; as we can see, at $x = \pm 1$ the limit of $u(x, y)$ as y tends to zero is different from the top ($y > 0$) and the bottom ($y < 0$). In general, u may be discontinuous when $y = 0$, at points other than $x = \pm 1$. Hence, there is a singularity in the domain of interest at $y = 0$. For future reference, we denote the open domain $y > 0$ and $x = [-1, 1]$ as Ω_I and the open domain $y < 0$ and $x = [-1, 1]$ as Ω_{II} .

2.2 The transformed differential equation and output

We define a weighting function $\tau(y) > 0, \forall y \in R$ and let

$$U(x, y) = \rho(y)^{\frac{1}{2}} \tau(y)^{-\frac{1}{2}} u(x, y), \quad (2.5)$$

where

$$\rho(y) = \frac{e^{-y^2}}{\sqrt{\pi}}. \quad (2.6)$$

As we have already discussed, there is a singularity in the differential equation at $y = 0$. In order to get a more accurate discrete solution, we need to transform the differential equation as shown above. In this project, we solve the transformed differential equation for a particular choice of $\tau(y)$ and compute the output in terms of this solution. There are two distinct advantages of applying this transformation. One is that, using the function $\tau(y)$, we can get a finer mesh in the vicinity of the singularity i.e. at $y = 0$. And the second advantage is that, we can avoid the singularities in the weak form by choosing a $\tau(y)$, such that it also cancels the terms depending on y that lead to a singularity in the weak form. In chapter 4, we will discuss in detail, the choice of the $\tau(y)$ which achieves both the above

objectives. From equation(2.5), we have

$$u(x, y) = \rho(y)^{-\frac{1}{2}}\tau(y)^{\frac{1}{2}}U(x, y). \quad (2.7)$$

Once we substitute for u(x,y) in the differential equation in (2.1)with the expression in equation (2.7)and simplify, we get

$$\begin{aligned} y \frac{\partial\{\rho(y)^{-\frac{1}{2}}\tau(y)^{\frac{1}{2}}U(x, y)\}}{\partial x} + \frac{1}{\theta}[\rho(y)^{-\frac{1}{2}}\tau(y)^{\frac{1}{2}}U(x, y) \\ - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y'^2} \rho(y')^{-\frac{1}{2}}\tau(y')^{\frac{1}{2}}U(x, y')dy'] = f. \end{aligned} \quad (2.8)$$

From (2.5) we have

$$\begin{aligned} y\rho(y)^{-\frac{1}{2}}\tau(y)^{\frac{1}{2}}\frac{\partial U(x, y)}{\partial x} + \frac{1}{\theta}[\rho(y)^{-\frac{1}{2}}\tau(y)^{\frac{1}{2}}U(x, y) - \\ \int_{-\infty}^{\infty} \rho(y') \cdot \rho(y')^{-\frac{1}{2}}\tau(y')^{\frac{1}{2}}U(x, y')dy'] = f. \end{aligned} \quad (2.9)$$

$$y\rho(y)^{-\frac{1}{2}}\tau(y)^{\frac{1}{2}}\frac{\partial U(x, y)}{\partial x} + \frac{1}{\theta}[\rho(y)^{-\frac{1}{2}}\tau(y)^{\frac{1}{2}}U(x, y) - \int_{-\infty}^{\infty} \rho(y')^{\frac{1}{2}}\tau(y')^{\frac{1}{2}}U(x, y')dy'] = f, \quad (2.10)$$

Multiplying both sides of (2.10) with $\rho(y)^{\frac{1}{2}}\tau(y)^{\frac{1}{2}}$, we have

$$\tau(y)y\frac{\partial U(x, y)}{\partial x} + \frac{1}{\theta}[\tau(y)U(x, y) - \rho(y)^{\frac{1}{2}}\tau(y)^{\frac{1}{2}} \int_{-\infty}^{\infty} \rho(y')^{\frac{1}{2}}\tau(y')^{\frac{1}{2}}U(x, y')dy'] = \rho(y)^{\frac{1}{2}}\tau(y)^{\frac{1}{2}}f; \quad (2.11)$$

For the sake of convenience, we will represent $\tau(y)$, $\rho(y)$, $U(x, y)$, $\frac{\partial U(x, y)}{\partial x}$ as τ , ρ , U , U_x respectively. So, the final form of the differential equation becomes

$$\tau y U_x + \frac{1}{\theta}[\tau U - \rho^{\frac{1}{2}}\tau^{\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}}\tau^{\frac{1}{2}}U dy'] = \rho^{\frac{1}{2}}\tau^{\frac{1}{2}}f. \quad (2.12)$$

Similarly, when we substitute for $u(x,y)$ in the boundary conditions in the equations in (2.4) and (2.5) with the expression in equation (2.7) and simplify, we get

$$\begin{aligned} u(x = +1, y) &= 0 \quad \forall y < 0, \\ \Rightarrow \rho(y)^{-\frac{1}{2}}\tau(y)^{\frac{1}{2}}U(x = +1, y) &= 0. \quad \forall y < 0 \end{aligned} \tag{2.13}$$

From the definition of $\rho(y)$ in (2.6) such that $\rho(y) > 0, \forall y \in R$ and the definition of $\tau(y)$ such that $\tau(y) > 0, \forall y \in R$ we can see that $u(x = +1, y) = 0$, implies that

$$U(x = +1, y) = 0 \quad \forall y < 0. \tag{2.14}$$

similarly,

$$\begin{aligned} u(x = -1, y) &= 0 \quad \forall y > 0, \\ \Rightarrow \rho(y)^{-\frac{1}{2}}\tau(y)^{\frac{1}{2}}U(x = -1, y) &= 0. \quad \forall y > 0 \end{aligned} \tag{2.15}$$

From the definition of $\rho(y)$ in (2.6) such that $\rho(y) > 0, \forall y \in R$ and the definition of $\tau(y)$ such that $\tau(y) > 0, \forall y \in R$ we can see that $u(x = -1, y) = 0$ implies that

$$U(x = -1, y) = 0 \quad \forall y > 0. \tag{2.16}$$

Now, summarizing the transformed differential equation and the boundary conditions in equations (2.17),(2.18)and (2.19) respectively, we have

$$\tau y U_x + \frac{1}{\theta} [\tau U - \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy'] = \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} f. \tag{2.17}$$

$$U(x = +1, y) = 0 \quad \forall y < 0, \tag{2.18}$$

$$U(x = -1, y) = 0 \quad \forall y > 0. \quad (2.19)$$

As far as output is concerned, on substituting the expression for $u(x, y)$ from (2.5), in the expression for output in (2.4), we get

$$\begin{aligned} S(\theta) &= \frac{1}{2} \frac{1}{\sqrt{\pi}} \int_{-1}^{+1} \int_{-\infty}^{\infty} e^{-y^2} \rho(y)^{-\frac{1}{2}} \tau(y)^{\frac{1}{2}} U(x, y; \theta) dy dx, \\ \Rightarrow S(\theta) &= \frac{1}{2} \int_{-1}^{+1} \int_{-\infty}^{\infty} \rho(y) \cdot \rho(y)^{-\frac{1}{2}} \tau(y)^{\frac{1}{2}} U(x, y; \theta) dy dx, \\ \Rightarrow S(\theta) &= \frac{1}{2} \int_{-1}^{+1} \int_{-\infty}^{\infty} \rho(y)^{\frac{1}{2}} \tau(y)^{\frac{1}{2}} U(x, y; \theta) dy dx; \end{aligned} \quad (2.20)$$

which according to our simplified notation becomes

$$S(\theta) = \frac{1}{2} \int_{-1}^{+1} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U(x, y; \theta) dy dx. \quad (2.21)$$

Chapter 3

Weak form of the Boltzmann equation

3.1 Definition of the Hilbert space of test function

Let X represent the continuous Hilbert space which satisfies the weak form of the Boltzmann equation. For a well defined solution, $U(\theta)$ to exist, $U(\theta)$ must satisfy the following conditions. $U(\theta) \in X \equiv \{v \mid \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau v^2 dy dx < \infty, \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau y^2 v_x^2 dy dx < \infty\}$. The choice of this Hilbert space will be discussed later in this section.

3.2 Definition of weak form and inner product

The weak form of the Boltzmann equation has been defined as shown below. It has been motivated by the Stream-line Upwind Petrov Galerkin scheme: The references [17, 18, 19, 20, 21, 22, 23, 24] give us a good insight into the SUPG scheme.

$$a^0(U, v; \theta) = \hat{f}^0[v + \theta y v_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy], \forall U(\theta), v \in X. \quad (3.1)$$

and the output

$$S(\theta) = \frac{1}{2} \int_{-1}^{+1} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U(\theta) dy dx. \quad (3.2)$$

where

$$a^0(U, v; \theta) = \theta d(U, v) + \frac{1}{\theta} m(U, v) + \sigma(U, v) + b(U, v), \quad (3.3)$$

$$d^0(U, v) = \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau y^2 U_x v_x dy dx, \quad (3.4)$$

$$b^0(U, v) = \int_{-1}^{+1} \left[\int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} y U_x dy \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy'' \right] dx - \int_{-1}^{+1} \left[\int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} y v_x dy \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right] dx, \quad (3.5)$$

$$m^0(U, v) = m0(U, v) - m1(U, v), \quad (3.6)$$

$$m0^0(U, v) = \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau U v dy dx, \quad (3.7)$$

$$m1^0(U, v) = \int_{-1}^{+1} \left[\int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy'' \right] dx, \quad (3.8)$$

$$\sigma^0(U, v) = \int_{0_{x=1}}^{\infty} \tau y U v dy - \int_{-\infty_{x=-1}}^0 \tau y U v dy, \quad (3.9)$$

$$\hat{f}^0[v] = \int_{-1}^1 \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} f v dy dx. \quad (3.10)$$

For the evaluation of the output in terms of the weak form, by putting $v = U$ in (3.1) and also using (3.10) and the fact that $f = \frac{1}{2}$, we get

$$a^0(U, U; \theta) = \frac{1}{2} \int_{-1}^1 \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \left\{ U + \theta y U_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right\} dy dx, \quad (3.11)$$

and then we have

$$a^0(U, U; \theta) = S(\theta) + \frac{1}{2} \int_{-1}^1 \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \theta y U_x dy dx + \frac{1}{2} \int_{-1}^1 \int_{-\infty}^{\infty} \rho \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right\} dy dx, \quad (3.12)$$

Using the fact that $\int_{-\infty}^{\infty} \rho dy = 1$ and further simplifying we have

$$a^0(U, U; \theta) = 2S(\theta) + \frac{1}{2} \int_{-1}^1 \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \theta y U_x dy dx, \quad (3.13)$$

Considering $\int_{-1}^1 \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \theta y U_x dy dx$ and simplifying it using the strong form in (2.17) and the fact that $\int_{-\infty}^{\infty} \rho dy = 1$, we have

$$\begin{aligned} \int_{-1}^1 \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \theta y U_x dy dx &= \int_{-1}^1 \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \theta \{-\tau U + \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' + \frac{1}{2} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}}\} dy dx, \\ &= \theta \int_{-1}^1 \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy dx - \theta \int_{-1}^1 \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' dx \\ &\quad + \frac{\theta}{2} \int_{-1}^1 \int_{-\infty}^{\infty} \rho dy dx \Rightarrow, \\ \int_{-1}^1 \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \theta y U_x dy dx &= \theta. \end{aligned} \quad (3.14)$$

Using (3.14) and (3.13), we can express the output as

$$S(\theta) = \frac{1}{2} a^0(U, U) - \frac{\theta}{4}. \quad (3.15)$$

The proper inner product/norm for this problem can be defined as

$$(U, v)_x = d^0(U, v) + m^0(U, v) + \sigma^0(U, v), \quad (3.16)$$

such that

$$\|v\|_x = (v, v)_x^{\frac{1}{2}}. \quad (3.17)$$

3.3 Choice of the space X

The choice of the continuous Hilbert space X , has been made such that each of the terms defining the weak form, are well defined and finite. Since, both $U(\theta)$ and v are functions belonging to the same space and also v is an arbitrary function . We can always choose v to be same as $U(\theta)$. In other words $v = U(\theta)$ or $U(\theta) = v$. According to the definition of space X, we know $U(\theta) \in X \equiv \{v \mid \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau v^2 dy dx < \infty, \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau y^2 v_x^2 dy dx < \infty\}$.

From the conditions used in defining X, we show below that each of the terms used in defining the weak form are well defined.

$$d^0(U, v) = \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau y^2 U_x v_x dy dx, \quad (3.18)$$

choosing $U_x \in X$ and $v_x \in X$ to be equal to some $\tilde{v}_x \in X$, we have

$$d^0(\tilde{v}, \tilde{v}) = \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau y^2 \tilde{v}_x^2 dy dx. \quad (3.19)$$

which is obviously finite and well defined from the definition of X. Now considering

$$m0^0(U, v) = \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau U v dy dx, \quad (3.20)$$

Now considering $m1^0(U, v)$ and choosing $U(\theta) \in X$ and $v \in X$ to be equal to some $\tilde{v} \in X$ then we have

$$m0^0(\tilde{v}, \tilde{v}) = \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau \tilde{v}^2 dy dx, \quad (3.21)$$

which is obviously finite and well defined from the definition of X. Choosing $U(\theta) \in X$ and $v \in X$ to be equal to some $\tilde{v} \in X$ then we have

$$m1^0(\tilde{v}, \tilde{v}) = \int_{-1}^{+1} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \tilde{v} dy' \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \tilde{v} dy'' \right\} dx, \quad (3.22)$$

Using the Cauchy Schwarz inequality we have

$$\begin{aligned} m1^0(\tilde{v}, \tilde{v}) &= \int_{-1}^{+1} \left(\int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \tilde{v} dy' \right)^2 dx \\ &\leq \int_{-1}^{+1} \left\{ \left(\int_{-\infty}^{\infty} \tau \tilde{v}^2 dy' \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \rho dy' \right)^{\frac{1}{2}} \right\}^2 dx, \end{aligned} \quad (3.23)$$

We already know from (2.6) that $(\int_{-\infty}^{\infty} \rho dy) = 1$, so we have

$$m1^0(\tilde{v}, \tilde{v}) \leq \left(\int_{-1}^{+1} \left(\int_{-\infty}^{\infty} \tau \tilde{v}^2 dy' \right)^{\frac{1}{2}} dx \right)^2. \quad (3.24)$$

Since, $(\int_{-\infty}^{\infty} \tau v^2 dy)$ is finite, by definition of the space X , $m1^0(\tilde{v}, \tilde{v})$ is also finite. Now considering $b^0(U, v)$ and choosing $U_x \in X$, $v \in X$ to be equal to some $\tilde{v}1_x \in X$, $\tilde{v}2 \in X$ respectively and U, v_x to be equal to $\tilde{v}1, \tilde{v}2_x$ then we have

$$b^0(U, v) = b_1^0(U, v) - b_2^0(U, v), \quad (3.25)$$

$$b_1^0(\tilde{v}1, \tilde{v}2) = \int_{-1}^{+1} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} y \tilde{v}1_x dy \right\} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \tilde{v}2 dy \right\} dx, \quad (3.26)$$

$$b_2^0(\tilde{v}1, \tilde{v}2) = \int_{-1}^{+1} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} y \tilde{v}1 dy \right\} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \tilde{v}2_x dy \right\} dx, \quad (3.27)$$

Applying Cauchy Schwarz inequality on the terms $b_1^0(\tilde{v}1, \tilde{v}2)$ and $b_2^0(\tilde{v}1, \tilde{v}2)$, we have

$$\begin{aligned} b_1^0(\tilde{v}1, \tilde{v}2) &= \int_{-1}^{+1} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} y \tilde{v}1_x dy \right\} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \tilde{v}2 dy \right\} dx \\ &\leq \int_{-1}^{+1} \left\{ \left(\int_{-\infty}^{\infty} \rho dy' \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \tau y^2 \tilde{v}1_x^2 dy \right)^{\frac{1}{2}} \right\} \left\{ \left(\int_{-\infty}^{\infty} \rho dy' \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \tau \tilde{v}2^2 dy \right)^{\frac{1}{2}} \right\} dx, \end{aligned} \quad (3.28)$$

$$b_2^0(\bar{v}_1, \bar{v}_2) = \int_{-1}^{+1} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} y \bar{v}_1 dy \right\} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v \bar{v}_2 dx dy \right\} dx, \quad (3.29)$$

$$b_2^0(\bar{v}_1, \bar{v}_2) \leq \int_{-1}^{+1} \left\{ \left(\int_{-\infty}^{\infty} \rho dy' \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \tau y^2 \bar{v}_1^2 dy \right)^{\frac{1}{2}} \right\} \left\{ \left(\int_{-\infty}^{\infty} \rho dy' \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \tau v \bar{v}_2^2 dy \right)^{\frac{1}{2}} \right\} dx, \quad (3.30)$$

Since, $\int_{-\infty}^{\infty} \rho dy' = 1$ we have

$$b_1^0(\tilde{v}_1, \tilde{v}_2) \leq \int_{-1}^{+1} \left\{ \left(\int_{-\infty}^{\infty} \tau y^2 \tilde{v}_1^2 dy \right)^{\frac{1}{2}} \right\} \left\{ \left(\int_{-\infty}^{\infty} \tau v \tilde{v}_2^2 dy \right)^{\frac{1}{2}} \right\} dx. \quad (3.31)$$

$$b_2^0(\bar{v}_1, v \bar{v}_2) \leq \int_{-1}^{+1} \left\{ \left(\int_{-\infty}^{\infty} \tau y^2 \bar{v}_1^2 dy \right)^{\frac{1}{2}} \right\} \left\{ \left(\int_{-\infty}^{\infty} \tau v \bar{v}_2^2 dy \right)^{\frac{1}{2}} \right\} dx. \quad (3.32)$$

Since, $(\int_{-1}^{+1} (\int_{-\infty}^{\infty} \tau y^2 v^2 dy) dx)$ and $(\int_{-1}^{+1} (\int_{-\infty}^{\infty} \tau v^2 dy) dx)$ are finite by definition, $b_1^0(U, v)$ and $b_2^0(U, v)$ should be finite. Hence, $b^0(U, v)$ is also finite. Consider the term $\sigma^0(w, v)$ from the equation (3.9) we have

$$\sigma^0(U, v) = \int_{0_{x=1}}^{\infty} \tau y U v dy - \int_{-\infty_{x=-1}}^0 \tau y U v dy, \quad (3.33)$$

subtracting $\int_{0_{x=-1}}^{\infty} \tau y U v dy$ and adding $\int_{-\infty_{x=1}}^0 \tau y U v dy$ from the Right hand side of this equation, does not change the equation as both these terms are zero according to the boundary conditions described in the equations (2.14) and (2.16). So, we have

$$\sigma^0(U, v) = \int_{0_{x=1}}^{\infty} \tau y U v dy - \int_{0_{x=-1}}^{\infty} \tau y U v dy + \int_{-\infty_{x=1}}^0 \tau y U v dy - \int_{-\infty_{x=-1}}^0 \tau y U v dy, \quad (3.34)$$

Since τ is only a function of y , this equation, in turn can be written as

$$\begin{aligned}
\sigma^0(U, v) &= \int_0^\infty \tau y [Uv]_{-1}^1 dy + \int_{-\infty}^0 \tau y [Uv]_{-1}^1 dy, \\
\Rightarrow \sigma(U, v) &= \int_{-\infty}^\infty \tau y [Uv]_{-1}^1 dy, \\
\Rightarrow \sigma(U, v) &= \int_{-\infty}^\infty \tau y \int_{-1}^1 \frac{\partial [Uv]}{\partial x} dx dy,
\end{aligned} \tag{3.35}$$

$$\sigma^0(U, v) = \int_{-\infty}^\infty \int_{-1}^1 \tau y U v_x dx dy + \int_{-\infty}^\infty \int_{-1}^1 \tau y U_x v dx dy, \tag{3.36}$$

So we have,

$$\sigma^0(U, v) = \sigma_1^0(U, v) + \sigma_2^0(U, v), \tag{3.37}$$

Where

$$\sigma_1^0(U, v) = \int_{-1}^1 \int_{-\infty}^\infty \tau y U v_x dy dx, \tag{3.38}$$

$$\sigma_2^0(U, v) = \int_{-1}^1 \int_{-\infty}^\infty \tau y U_x v dy dx, \tag{3.39}$$

In the simplification shown above, we can notice that the boundary conditions are implicit in the term $\sigma^0(U, v)$. This means that the boundary conditions are imposed weakly, in a Neumann sense rather than Dirichlet sense. Choosing U, v to be equal to \tilde{v} we have

$$\sigma_1^0(\tilde{v}, \tilde{v}) = \int_{-1}^1 \int_{-\infty}^\infty \tau y \tilde{v} \tilde{v}_x dy dx, \tag{3.40}$$

$$\sigma_2^0(\tilde{v}, \tilde{v}) = \int_{-1}^1 \int_{-\infty}^{\infty} \tau y \tilde{v} \tilde{v}_x dy dx, \quad (3.41)$$

From Cauchy Schwarz inequality we have

$$\sigma_1^0(\tilde{v}, \tilde{v}) \leq \int_{-1}^1 \left(\int_{-\infty}^{\infty} \tau y^2 \tilde{v}_x^2 dy \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \tau \tilde{v}^2 dy \right)^{\frac{1}{2}} dx, \quad (3.42)$$

similarly,

$$\sigma_2^0(\tilde{v}, \tilde{v}) \leq \int_{-1}^1 \left(\int_{-\infty}^{\infty} \tau y^2 \tilde{v}^2 dy \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \tau \tilde{v}^2 dy \right)^{\frac{1}{2}} dx. \quad (3.43)$$

From, the definition of space X, we can see that $\sigma_1^0(U, v)$ and $\sigma_2^0(U, v)$ are finite and well defined. Considering the Right hand side of the weak form in (3.1) and (3.10) we have putting

$$\hat{v} = v + \theta y v_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy', \quad (3.44)$$

$$\hat{f}^0[\hat{v}] = \int_{-1}^1 \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} f \left\{ v + \theta y v_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy' \right\} dy dx, \quad (3.45)$$

On expanding we have

$$\begin{aligned} \hat{f}^0[\hat{v}] &= \int_{-1}^1 \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} f \{ v + \theta y v_x \} dy dx + \\ &\int_{-1}^1 f \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy' \right\} \left\{ \int_{-\infty}^{\infty} \rho dy \right\} dx, \end{aligned} \quad (3.46)$$

simplifying the second term on the right hand side of the equation further by using the fact that $\int_{-\infty}^{\infty} \rho dy = 1$, we get

$$\hat{f}^0[\hat{v}] = \int_{-1}^1 \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} f\{v + \theta y v_x\} dy dx + \int_{-1}^1 f\left\{\int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy'\right\} dx, \quad (3.47)$$

By changing the variable y' to y in the second term on the right hand side, we get

$$\hat{f}^0[\hat{v}] = \int_{-1}^1 \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} f\{v + \theta y v_x\} dy dx + \int_{-1}^1 f\left\{\int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy\right\} dx, \quad (3.48)$$

Combining the 2 terms on the right hand side we get

$$\begin{aligned} \hat{f}^0[v + \theta y v_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy'] &= \int_{-1}^1 \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} f\{2v + \theta y v_x\} dy dx \\ &= \hat{f}^0[2v + \theta y v_x], \end{aligned} \quad (3.49)$$

Now assuming $2v + \theta y v_x = \tilde{v} \in X$ we get

$$\hat{f}^0[\tilde{v}] = \int_{-1}^1 \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} f \tilde{v} dy dx, \quad (3.50)$$

Applying Cauchy Schwarz inequality on the above equation we get

$$\hat{f}^0[\tilde{v}] \leq \int_{-1}^1 f\left\{\int_{-\infty}^{\infty} \tau \tilde{v}^2 dy\right\}^{\frac{1}{2}} \left\{\int_{-\infty}^{\infty} \rho dy'\right\}^{\frac{1}{2}} dx, \quad (3.51)$$

From the definition of space X , we know that $\int_{-1}^1 \int_{-\infty}^{\infty} \tau \tilde{v}^2 dy dx$ is finite and also $\int_{-\infty}^{\infty} \rho dy' = 1$.

Hence, from

$$\hat{f}^0[\tilde{v}] \leq \int_{-1}^1 f\left\{\int_{-\infty}^{\infty} \tau \tilde{v}^2 dy\right\}^{\frac{1}{2}} dx, \quad (3.52)$$

we can conclude that $\hat{f}^0[v + \theta y v_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy']$ is finite and well defined.

3.4 Equivalence of the Strong form and weak form

From the definition of $a^0(U, v, \theta)$ in the weak form (3.1) and equations (3.4), (3.5), (3.4), (3.5), (3.6), (3.7), (3.8), (3.9), (3.10) and (3.36), we have

$$\begin{aligned} a^0(U, v; \theta) &= \theta \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau y^2 U_x v_x dy dx + \\ &\quad \frac{1}{\theta} \left\{ \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau U v dy dx - \int_{-1}^{+1} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy'' dx \right\} + \\ &\quad \int_{-\infty}^{\infty} \int_{-1}^1 \tau y U v_x dx dy + \\ &\quad \int_{-\infty}^{\infty} \int_{-1}^1 \tau y U_x v dx dy + \\ &\quad \int_{-1}^{+1} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} y U_x dy \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy'' \right\} dx - \\ &\quad \int_{-1}^{+1} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} y v_x dy \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right\} dx, \end{aligned} \quad (3.53)$$

Rearranging,rewriting and combining various terms in the above equation we get

$$\begin{aligned}
a^0(U, v; \theta) &= \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau y U_x \{v + \theta y v_x\} dy dx + \frac{1}{\theta} \left\{ \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau U v dy dx \right\} - \\
&\frac{1}{\theta} \left\{ \int_{-1}^{+1} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy'' dx \right\} + \\
&\frac{1}{\theta} \int_{-\infty}^{\infty} \int_{-1}^1 \tau \theta y U v_x dx dy + \\
&\int_{-1}^{+1} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \tau y U_x dy \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy'' \right\} dx - \\
&\frac{1}{\theta} \left\{ \int_{-1}^{+1} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \theta y v_x dy \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right\} dx \right\}, \tag{3.54}
\end{aligned}$$

Further rearranging terms and combining them

$$\begin{aligned}
a^0(U, v; \theta) &= \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau y U_x \{v + \theta y v_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy''\} dy dx + \\
&\frac{1}{\theta} \left\{ \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau U \{v + \theta y v_x\} dy dx \right\} - \\
&\frac{1}{\theta} \left\{ \int_{-1}^{+1} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right\} \{v + \theta y v_x\} dy dx \right\}, \tag{3.55}
\end{aligned}$$

adding and subtracting $\frac{1}{\theta} \left\{ \int_{-1}^{+1} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right\} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy'' \right\} dx \right\}$ from the RHS , we have

$$\begin{aligned}
a^0(U, v; \theta) &= \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau y U_x \{v + \theta y v_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy''\} dy dx + \\
&\frac{1}{\theta} \left\{ \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau U \{v + \theta y v_x\} dy dx \right\} \\
&- \frac{1}{\theta} \left\{ \int_{-1}^{+1} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right\} \{v + \theta y v_x\} dy dx \right\} + \\
&\frac{1}{\theta} \int_{-1}^{+1} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right\} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy'' \right\} dx - \\
&\frac{1}{\theta} \int_{-1}^{+1} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right\} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy'' \right\} dx, \tag{3.56}
\end{aligned}$$

Rewriting the terms further

$$\begin{aligned}
a^0(U, v; \theta) &= \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau y U_x \{v + \theta y v_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy''\} dy dx \\
&\quad + \frac{1}{\theta} \left\{ \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau U \{v + \theta y v_x\} dy dx \right\} \\
&\quad - \frac{1}{\theta} \left\{ \int_{-1}^{+1} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right\} \{v + \theta y v_x\} dy dx \right\} \\
&\quad + \frac{1}{\theta} \int_{-1}^{+1} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \tau U dy \right\} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy'' \right\} dx \\
&\quad - \frac{1}{\theta} \int_{-1}^{+1} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right\} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy'' \right\} dx, \tag{3.57}
\end{aligned}$$

Combining the 4th term and 2nd terms in the RHS we get

$$\begin{aligned}
a^0(U, v; \theta) &= \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau y U_x \{v + \theta y v_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy''\} dy dx \\
&\quad + \frac{1}{\theta} \left\{ \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau U \{v + \theta y v_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy''\} dy dx \right\} \\
&\quad - \frac{1}{\theta} \left\{ \int_{-1}^{+1} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right\} \{v + \theta y v_x\} dy dx \right\} \\
&\quad - \frac{1}{\theta} \int_{-1}^{+1} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right\} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy'' \right\} dx, \tag{3.58}
\end{aligned}$$

Rewriting the last term of the RHS using the fact that $\int_{-\infty}^{\infty} \rho dy = 1$

$$\begin{aligned}
a^0(U, v; \theta) &= \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau y U_x \{v + \theta y v_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy''\} dy dx + \\
&\quad \frac{1}{\theta} \left\{ \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau U \{v + \theta y v_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy''\} dy dx \right\} - \\
&\quad \frac{1}{\theta} \left\{ \int_{-1}^{+1} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right\} \{v + \theta y v_x\} dy dx \right\} \\
&\quad - \frac{1}{\theta} \left\{ \int_{-1}^{+1} \left\{ \int_{-\infty}^{\infty} \rho \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right\} dy \right\} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy'' \right\} dx, \tag{3.59}
\end{aligned}$$

Rewriting the last term of the RHS

$$\begin{aligned}
a^0(U, v; \theta) &= \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau y U_x \{v + \theta y v_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy''\} dy dx \\
&+ \frac{1}{\theta} \left\{ \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau U \{v + \theta y v_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy''\} dy dx \right\} \\
&- \frac{1}{\theta} \left\{ \int_{-1}^{+1} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right\} \{v + \theta y v_x\} dy dx \right\} \\
&- \frac{1}{\theta} \int_{-1}^{+1} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right\} dy \right\} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy'' \right\} dx,
\end{aligned} \tag{3.60}$$

Combining the last two terms we have

$$\begin{aligned}
a^0(U, v; \theta) &= \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau y U_x \{v + \theta y v_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy''\} dy dx + \\
&\frac{1}{\theta} \left\{ \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau U \{v + \theta y v_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy''\} dy dx \right\} \\
&- \frac{1}{\theta} \left\{ \int_{-1}^{+1} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \left\{ \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right\} \{\hat{v}\} dy dx \right\},
\end{aligned} \tag{3.61}$$

where

$$\hat{v} = v + \theta y v_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v dy', \tag{3.62}$$

Combining all the terms in to one, we have

$$a^0(U, v; \theta) = \int_{-1}^{+1} \int_{-\infty}^{\infty} \left\{ \tau y U_x + \frac{1}{\theta} \left\{ \tau U - \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right\} \right\} \{\hat{v}\} dy dx, \tag{3.63}$$

From the definition of weak form in (3.1),(3.63) and equation(3.10) we have

$$\begin{aligned}
a^0(U, v; \theta) &= \int_{-1}^{+1} \int_{-\infty}^{\infty} \left\{ \tau y U_x + \frac{1}{\theta} \left\{ \tau U - \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right\} \right\} \{ \hat{v} \} dy dx \\
&= \hat{f}[\hat{v}] \\
&= \int_{-1}^{+1} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} f \{ \hat{v} \} dy dx,
\end{aligned} \tag{3.64}$$

On simplification we have

$$\int_{-1}^{+1} \int_{-\infty}^{\infty} \left\{ \tau y U_x + \frac{1}{\theta} \left\{ \tau U - \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} U dy' \right\} - f \right\} \times \{ \hat{v} \} dy dx = 0. \tag{3.65}$$

As, v is any arbitrary function belonging to X. We can always choose a non zero v. The same should hold good for \hat{v} as well. Because, the above double integral is equal to zero for all such arbitrary functions belonging to X. That implies that, the differential equation in (2.12) is satisfied by $U(\theta)$.

3.5 Mapping Transformation of weak form

3.5.1 Decomposed definition of the weak form

The problem can be decomposed into two different sub-domains as shown below.

$$\Omega_I^0 \equiv x \in [-1, 1] \times y \in] - \infty, 0[\tag{3.66}$$

$$\Omega_{II}^0 \equiv x \in [-1, 1] \times y \in]0, \infty[\tag{3.67}$$

The original Hilbert space

$$U(\theta) \in X^0 \equiv \left\{ v \mid \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau v^2 dy dx < \infty, \int_{-1}^{+1} \int_{-\infty}^{\infty} \tau y^2 v_x^2 dy dx < \infty \right\}, \tag{3.68}$$

can be decomposed and expressed as

$$X_I^0 \equiv \{v \mid \int_{-1}^{+1} \int_{-\infty}^0 \tau v^2 dy dx < \infty, \int_{-1}^{+1} \int_{-\infty}^0 \tau y^2 v_x^2 dy dx < \infty\}, \quad (3.69)$$

$$X_{II}^0 \equiv \{v \mid \int_{-1}^{+1} \int_0^{\infty} \tau v^2 dy dx < \infty, \int_{-1}^{+1} \int_0^{\infty} \tau y^2 v_x^2 dy dx < \infty\}. \quad (3.70)$$

The solution U can be decomposed as shown below.

$$U = (U_I, U_{II}), \text{ where } U_I \in X_I^0, U_{II} \in X_{II}^0; \forall v_I \in X_I^0 \text{ and } \forall v_{II} \in X_{II}^0, \quad (3.71)$$

The weak form

$$a^0(U_I, U_{II}, v_I, v_{II}) = \check{f}^0(v_I, v_{II}), \quad (3.72)$$

has to be satisfied such that

$$\begin{aligned} a^0 : ((X_I \times X_{II}) \times (X_I \times X_{II})) &\longrightarrow \mathcal{R} \\ \check{f}^0 : ((X_I \times X_{II})) &\longrightarrow \mathcal{R}. \end{aligned} \quad (3.73)$$

where

$$\begin{aligned} a^0(w_1, w_2), (v_1, v_2) &= \theta d^0((w_1, w_2), (v_1, v_2)) + \frac{1}{\theta} m 0^0((w_1, w_2), (v_1, v_2)) \\ &\quad - \frac{1}{\theta} m 1^0((w_1, w_2), (v_1, v_2)) + \sigma^0((w_1, w_2), (v_1, v_2)) \\ &\quad + b^0((w_1, w_2), (v_1, v_2)), \end{aligned} \quad (3.74)$$

further decomposing it we have

$$\begin{aligned}
a^0((w_1, w_{II}), (v_1, v_2)) &= \theta \{d_I^0(w_1, v_1) + d_{II}^0(w_2, v_2)\} + \frac{1}{\theta} \{m0_I^0(w_1, v_1) + m0_{II}^0(w_2, v_2)\} \\
&\quad - \frac{1}{\theta} \{(m1^0)_I^I(w_1, v_1) + (m1^0)_I^{II}(w_1, v_2)\} \\
&\quad - \frac{1}{\theta} \{(m1^0)_{II}^I(w_2, v_1) + (m1^0)_{II}^{II}(w_2, v_2)\} \\
&\quad + \sigma_I^0(w_1, v_1) + \sigma_{II}^0(w_1, v_1) \\
&\quad + (b^0)_I^I(w_1, v_1) + (b^0)_I^{II}(w_1, v_2) + (b^0)_{II}^I(w_2, v_1) + (b^0)_{II}^{II}(w_2, v_2) \\
&\quad - (b^0)_I^I(v_1, w_1) - (b^0)_{II}^{II}(v_2, w_1) - (b^0)_I^I(v_1, w_2) - (b^0)_{II}^{II}(v_2, w_2)
\end{aligned} \tag{3.75}$$

$$\begin{aligned}
\check{f}^0(v_1, v_2) &= \hat{f}_I^0[v_1 + \theta y(v_1)_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v_1 dy'] \\
&\quad + \hat{f}_{II}^0[v_2 + \theta y(v_2)_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v_2 dy']
\end{aligned} \tag{3.76}$$

where

$$\begin{aligned}
d^0((w_1, w_2), (v_1, v_2)) &= d_I^0(w_1, v_1) + d_{II}^0(w_2, v_2) \\
d_I^0(w_1, v_1) &= \int_{-1}^1 \int_{-\infty}^0 \tau y^2 (w_1)_x (v_1)_x dy dx \\
d_{II}^0(w_2, v_2) &= \int_{-1}^1 \int_0^{\infty} \tau y^2 (w_2)_x (v_2)_x dy dx
\end{aligned} \tag{3.77}$$

$$\begin{aligned}
m0^0((w_1, w_2), (v_1, v_2)) &= m0_I^0(w_1, v_1) + m0_{II}^0(w_2, v_2) \\
m0_I^0(w_1, v_1) &= \int_{-1}^1 \int_{-\infty}^0 \tau w_1 v_1 dy dx \\
m0_{II}^0(w_2, v_2) &= \int_{-1}^1 \int_0^{\infty} \tau w_2 v_2 dy dx
\end{aligned} \tag{3.78}$$

$$\begin{aligned}
\sigma^0((w_1, w_2), (v_1, v_2)) &= \sigma_I^0(w_1, v_1) + \sigma_{II}^0(w_2, v_2) \\
\sigma_I^0(w_1, v_1) &= - \int_{-\infty_{x=-1}}^0 \tau y w_1 v_1 dy \\
\sigma_{II}^0(w_2, v_2) &= \int_{0_{x=1}}^{\infty} \tau y w_2 v_2 dy
\end{aligned} \tag{3.79}$$

Now lets consider the Right Hand Side from (3.1),(3.10),(3.87)

$$\begin{aligned}
\check{f}^0(v_1, v_2) &= \hat{f}_I^0[v_1 + \theta y(v_1)_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v_1 dy'] + \\
&\quad \hat{f}_{II}^0[v_2 + \theta y(v_2)_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v_2 dy']
\end{aligned} \tag{3.80}$$

where

$$\begin{aligned}
\hat{f}_I^0[v_1 + \theta y(v_1)_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v_1 dy'] &= \int_{-1}^1 \int_{-\infty}^0 \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} f\{2v_1 + \theta y(v_1)_x\} dy dx \\
\hat{f}_{II}^0[v_2 + \theta y(v_2)_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v_2 dy'] &= \int_{-1}^1 \int_0^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} f\{2v_2 + \theta y(v_2)_x\} dy dx
\end{aligned} \tag{3.81}$$

$$\begin{aligned}
m1^0((w_1, w_2), (v_1, v_2)) &= (m1^0)_I^I(w_1, v_1) + (m1^0)_I^{II}(w_1, v_2) + \\
&\quad (m1^0)_{II}^I(w_2, v_1) + (m1^0)_{II}^{II}(w_2, v_2)
\end{aligned} \tag{3.82}$$

$$\begin{aligned}
(m1^0)_I^I(w_1, v_1) &= \int_{-1}^1 m1_I^0[w_1(x, \cdot)]m1_I^0[v_1(x, \cdot)]dx \\
(m1^0)_I^H(w_1, v_2) &= \int_{-1}^1 m1_I^0[w_1(x, \cdot)]m1_{II}^0[v_2(x, \cdot)]dx \\
(m1^0)_{II}^I(w_2, v_1) &= \int_{-1}^1 m1_{II}^0[w_2(x, \cdot)]m1_I^0[v_1(x, \cdot)]dx \\
(m1^0)_{II}^H(w_2, v_2) &= \int_{-1}^1 m1_{II}^0[w_2(x, \cdot)]m1_{II}^0[v_2(x, \cdot)]dx
\end{aligned} \tag{3.83}$$

where

$$\begin{aligned}
m1_I^0[w_1(x, \cdot)] &= \int_{-\infty}^0 \rho^{\frac{1}{2}}\tau^{\frac{1}{2}}w_1(x, y')dy' \\
m1_{II}^0[w_2(x, \cdot)] &= \int_0^{\infty} \rho^{\frac{1}{2}}\tau^{\frac{1}{2}}w_2(x, y')dy' \\
m1_I^0[v_1(x, \cdot)] &= \int_{-\infty}^0 \rho^{\frac{1}{2}}\tau^{\frac{1}{2}}v_1(x, y'')dy'' \\
m1_{II}^0[v_2(x, \cdot)] &= \int_0^{\infty} \rho^{\frac{1}{2}}\tau^{\frac{1}{2}}v_2(x, y'')dy''
\end{aligned} \tag{3.84}$$

Rewriting (3.5) as shown below

$$\begin{aligned}
b^0((w_1, w_2), (v_1, v_2)) &= (b^0)_I^I(w_1, v_1) + (b^0)_{II}^{II}(w_1, v_2) + (b^0)_{II}^I(w_2, v_1) + (b^0)_{II}^{II}(w_2, v_2) \\
&\quad - (b^0)_I^I(v_1, w_1) - (b^0)_{II}^{II}(v_2, w_1) - (b^0)_{II}^I(v_1, w_2) - (b^0)_{II}^{II}(v_2, w_2) \\
(b^0)_I^I(w_1, v_1) &= \int_{-1}^1 b1_I^0[(w_1(x, \cdot))_x] b2_I^0[v_1(x, \cdot)] dx \\
(b^0)_{II}^{II}(w_1, v_2) &= \int_{-1}^1 b1_{II}^0[(w_1(x, \cdot))_x] b2_{II}^0[v_2(x, \cdot)] dx \\
(b^0)_{II}^I(w_2, v_1) &= \int_{-1}^1 b1_{II}^0[(w_2(x, \cdot))_x] b2_I^0[v_1(x, \cdot)] dx \\
(b^0)_{II}^{II}(w_2, v_2) &= \int_{-1}^1 b1_{II}^0[(w_2(x, \cdot))_x] b2_{II}^0[v_2(x, \cdot)] dx \\
(b^0)_I^I(v_1, w_1) &= \int_{-1}^1 b2_I^0[w_1(x, \cdot)] b1_I^0[(v_1(x, \cdot))_x] dx \\
(b^0)_{II}^{II}(v_2, w_1) &= \int_{-1}^1 b2_{II}^0[w_1(x, \cdot)] b1_{II}^0[(v_2(x, \cdot))_x] dx \\
(b^0)_{II}^I(v_1, w_2) &= \int_{-1}^1 b2_{II}^0[w_2(x, \cdot)] b1_I^0[(v_1(x, \cdot))_x] dx \\
(b^0)_{II}^{II}(v_2, w_2) &= \int_{-1}^1 b2_{II}^0[w_2(x, \cdot)] b1_{II}^0[(v_2(x, \cdot))_x] dx \tag{3.85}
\end{aligned}$$

where

$$\begin{aligned}
b1_I^0[(w_1(x, \cdot))_x] &= \int_{-\infty}^0 \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} y (w_1(x, y'))_x dy' \\
b1_{II}^0[(w_2(x, \cdot))_x] &= \int_0^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} y (w_2(x, y'))_x dy' \\
b2_I^0[v_1(x, \cdot)] &= \int_{-\infty}^0 \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v_1(x, y'') dy'' \\
b2_{II}^0[v_2(x, \cdot)] &= \int_0^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v_2(x, y'') dy'' \\
b1_I^0[(v_1(x, \cdot))_x] &= \int_{-\infty}^0 \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} y (v_1(x, y'))_x dy' \\
b1_{II}^0[(v_2(x, \cdot))_x] &= \int_0^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} y (v_2(x, y'))_x dy' \\
b2_I^0[w_1(x, \cdot)] &= \int_{-\infty}^0 \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} w_1(x, y'') dy'' \\
b2_{II}^0[w_2(x, \cdot)] &= \int_0^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} w_2(x, y'') dy''
\end{aligned} \tag{3.86}$$

In this section, we choose the parameter function $\tau(y)$ as

$$\tau(y) = \frac{4}{(|y| + 1)^4} \tag{3.87}$$

and also apply a mapping transformation on the various terms in the weak form in order to get a well defined and accurate solution . We will apply two different transformations for mapping two different sub domains in (3.66) and (3.67). We apply transformations T_I and T_{II} , such that the original sub -domains Ω_I^0 , Ω_{II}^0 are transformed to the mapped sub-domains Ω_I and Ω_{II} , which are defined in (3.88) and (3.89).

$$\Omega_I \equiv x \in [-1, 1] \times \eta_I \in] - 2, 0[\tag{3.88}$$

$$\Omega_{II} \equiv x \in [-1, 1] \times \eta_{II} \in]0, 2[\tag{3.89}$$

$$T_I : \Omega_I^0 \rightarrow \Omega_I \quad (3.90)$$

$$T_{II} : \Omega_{II}^0 \rightarrow \Omega_{II} \quad (3.91)$$

where

$$T_I : (x, y) = \left(x, \frac{(2 + \eta_I)}{\eta_I}\right) \quad (3.92)$$

$$T_{II} : (x, y) = \left(x, \frac{(\eta_{II})}{2 - \eta_{II}}\right) \quad (3.93)$$

Below, we consider each term of the weak form and apply these transformations. Lets start with the computation of $\tau(y)$ and the jacobians $\frac{\partial y}{\partial \eta_I}$ and $\frac{\partial y}{\partial \eta_{II}}$ in terms of η_I and η_{II} . From (3.87),(3.92) and (3.93), Since in $\Omega_I^0, y < 0$ we have

$$\begin{aligned} \tau(\eta_I) &= \frac{4}{\left(\frac{2+\eta_I}{\eta_I} + 1\right)^4} \\ \Rightarrow \tau(\eta_I) &= \frac{4}{\left(\frac{2+2\eta_I}{\eta_I}\right)^4} \\ \Rightarrow \tau(\eta_I) &= \frac{4\eta_I^4}{16(1 + \eta_I)^4} \end{aligned} \quad (3.94)$$

$$\tau(\eta_I) = \frac{\eta_I^4}{4(1 + \eta_I)^4} \quad (3.95)$$

Since in $\Omega_{II}^0, y > 0$ we have

$$\begin{aligned} \tau(\eta_{II}) &= \frac{4}{\left(\frac{\eta_{II}}{2-\eta_{II}} + 1\right)^4} \\ \Rightarrow \tau(\eta_{II}) &= \frac{4}{\left(\frac{\eta_{II}+2-\eta_{II}}{2-\eta_{II}}\right)^4} \\ \Rightarrow \tau(\eta_{II}) &= \frac{4(2 - \eta_{II})^4}{16} \end{aligned} \quad (3.96)$$

$$\tau(\eta_{II}) = \frac{(2 - \eta_{II})^4}{4} \quad (3.97)$$

Since, only one variable is being transformed in both the sub-domains, it is easy to see that the jacobian is a simple derivative as shown below.

$$\begin{aligned} y(\eta_I) &= \frac{2 + \eta_I}{\eta_I} \\ \Rightarrow y(\eta_I) &= 1 + \frac{2}{\eta_I} \end{aligned} \quad (3.98)$$

$$\frac{\partial y}{\partial \eta_I} = -\frac{2}{\eta_I^2} \quad (3.99)$$

$$\begin{aligned} y(\eta_{II}) &= \frac{\eta_{II}}{2 - \eta_{II}} \\ \Rightarrow y(\eta_{II}) &= -1 + \frac{2}{2 - \eta_{II}} \end{aligned} \quad (3.100)$$

$$\frac{\partial y}{\partial \eta_{II}} = \frac{2}{(1 - \eta_{II})^2} \quad (3.101)$$

Using (2.6),(3.92) and (3.93) we have

$$\rho(\eta_I) = \frac{e^{-\frac{(2+\eta_I)^2}{\eta_I^2}}}{\sqrt{\pi}} \quad (3.102)$$

$$\rho(\eta_{II}) = \frac{e^{-\frac{\eta_{II}^2}{(2-\eta_{II})^2}}}{\sqrt{\pi}} \quad (3.103)$$

Considering the various terms of the weak form as shown below. We have

Firstly, we will transform $d_I^0(w_1, v_1), d_{II}^0(w_2, v_2)$ to $d_I(w_1, v_1), d_{II}(w_2, v_2)$ using

(3.95), (3.97),(3.99),(3.101).

$$\begin{aligned}
d_I(w_1, v_1) &= \int_{-1}^1 \int_0^{-2} \tau(\eta_I) \left\{ \frac{2 + \eta_I}{\eta_I} \right\}^2 (w_1)_x (v_1)_x \frac{\partial y}{\partial \eta_I} d\eta_I dx \\
\Rightarrow d_I(w_1, v_1) &= \int_{-1}^1 \int_0^{-2} \left\{ \frac{\eta_I^4}{4(1 + \eta_I)^4} \right\} \frac{\{2 + \eta_I\}^2}{\{\eta_I\}^2} (w_1)_x (v_1)_x \left\{ -\frac{2}{\eta_I^2} \right\} d\eta_I dx
\end{aligned} \tag{3.104}$$

On simplifying further, we have

$$d_I(w_1, v_1) = \int_{-1}^1 \int_{-2}^0 \frac{(2 + \eta_I)^2}{2(1 + \eta_I)^4} (w_1)_x (v_1)_x d\eta_I dx \tag{3.105}$$

$$\begin{aligned}
d_{II}(w_2, v_2) &= \int_{-1}^1 \int_0^2 \tau(\eta_{II}) \left\{ \frac{\eta_{II}}{2 - \eta_{II}} \right\}^2 (w_2)_x (v_2)_x \frac{\partial y}{\partial \eta_{II}} d\eta_{II} dx \\
\Rightarrow d_{II}(w_2, v_2) &= \int_{-1}^1 \int_0^2 \left\{ \frac{(2 - \eta_{II})^4}{4} \right\} \frac{\{\eta_{II}\}^2}{\{2 - \eta_{II}\}^2} (w_2)_x (v_2)_x \left\{ \frac{2}{(2 - \eta_{II})^2} \right\} d\eta_{II} dx
\end{aligned} \tag{3.106}$$

On simplifying further, we have

$$d_{II}(w_2, v_2) = \int_{-1}^1 \int_0^2 \frac{\eta_{II}^2}{2} (w_2)_x (v_2)_x d\eta_{II} dx \tag{3.107}$$

Firstly, we will transform $m0_I^0(w_1, v_1), m0_{II}^0(w_2, v_2)$ to $m0_I(w_1, v_1), m0_{II}(w_2, v_2)$ using (3.95),(3.97),(3.99),(3.101).

$$\begin{aligned}
m0_I(w_1, v_1) &= \int_{-1}^1 \int_0^{-2} \tau(\eta_I) w_1 v_1 \frac{\partial y}{\partial \eta_I} d\eta_I dx \\
\Rightarrow m0_I(w_1, v_1) &= \int_{-1}^1 \int_0^{-2} \left\{ \frac{\eta_I^4}{4(1 + \eta_I)^4} \right\} w_1 v_1 \left\{ -\frac{2}{\eta_I^2} \right\} d\eta_I dx
\end{aligned} \tag{3.108}$$

On simplifying further, we have

$$m0_I(w_1, v_1) = \int_{-1}^1 \int_{-2}^0 \frac{\eta_I^2}{2(1 + \eta_I)^4} w_1 v_1 d\eta_I dx \quad (3.109)$$

$$\begin{aligned} m0_{II}(w_2, v_2) &= \int_{-1}^1 \int_0^2 \tau(\eta_{II}) w_2 v_2 \frac{\partial y}{\partial \eta_{II}} d\eta_{II} dx \\ \Rightarrow m0_{II}(w_2, v_2) &= \int_{-1}^1 \int_0^2 \left\{ \frac{(2 - \eta_{II})^4}{4} \right\} w_2 v_2 \left\{ \frac{2}{(2 - \eta_{II})^2} \right\} d\eta_{II} dx \end{aligned} \quad (3.110)$$

On simplifying further, we have

$$m0_{II}(w_2, v_2) = \int_{-1}^1 \int_0^2 \frac{(2 - \eta_{II})^2}{2} w_2 v_2 d\eta_{II} dx \quad (3.111)$$

Firstly, we will transform $\sigma_I^0(w_1, v_1), \sigma_{II}^0(w_2, v_2)$ to $\sigma_I(w_1, v_1), \sigma_{II}(w_2, v_2)$ using (3.95),(3.97),(3.99),(3.101).

$$\begin{aligned} \sigma_I(w_1, v_1) &= - \int_{0_{x=-1}}^{-2} \tau(\eta_I) \left\{ \frac{2 + \eta_I}{\eta_I} \right\} w_1 v_1 \frac{\partial y}{\partial \eta_I} d\eta_I \\ \Rightarrow \sigma_I(w_1, v_1) &= - \int_{0_{x=-1}}^{-2} \left\{ \frac{\eta_I^4}{4(1 + \eta_I)^4} \right\} \left\{ \frac{2 + \eta_I}{\eta_I} \right\} w_1 v_1 \left\{ -\frac{2}{\eta_I^2} \right\} d\eta_I \end{aligned} \quad (3.112)$$

On simplifying further, we have

$$\sigma_I(w_1, v_1) = - \int_{-2_{x=-1}}^0 \frac{\eta_I(2 + \eta_I)}{2(1 + \eta_I)^4} w_1 v_1 d\eta_I \quad (3.113)$$

$$\begin{aligned}
\sigma_{II}(w_2, v_2) &= \int_{0_{x=1}}^2 \tau(\eta_{II}) \left\{ \frac{\eta_{II}}{2 - \eta_{II}} \right\} w_2 v_2 \frac{\partial y}{\partial \eta_{II}} d\eta_{II} \\
\Rightarrow \sigma_{II}(w_2, v_2) &= \int_{0_{x=1}}^2 \left\{ \frac{(2 - \eta_{II})^4}{4} \right\} \left\{ \frac{\eta_{II}}{2 - \eta_{II}} \right\} w_2 v_2 \left\{ \frac{2}{(2 - \eta_{II})^2} \right\} d\eta_{II}
\end{aligned} \tag{3.114}$$

On simplifying further, we have

$$\sigma_{II}(w_2, v_2) = \int_{0_{x=1}}^2 \frac{(2 - \eta_{II})\eta_{II}}{2} w_2 v_2 d\eta_{II} \tag{3.115}$$

$$\hat{v}_1 = v_1 + \theta y (v_1)_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v_1 dy', \tag{3.116}$$

we have

$$\begin{aligned}
\hat{f}_I[\hat{v}_1] &= \int_{-1}^1 \int_0^{-2} \rho(\eta_I)^{\frac{1}{2}} \tau(\eta_I)^{\frac{1}{2}} f \left\{ 2v_1 + \theta \left\{ \frac{2 + \eta_I}{\eta_I} \right\} (v_1)_x \right\} \frac{\partial y}{\partial \eta_I} d\eta_I dx \\
\Rightarrow \hat{f}_I[\hat{v}_1] &= \int_{-1}^1 \int_0^{-2} \frac{e^{-\frac{(2+\eta_I)^2}{2\eta_I^2}}}{\pi^{\frac{1}{4}}} \left\{ \frac{\eta_I^2}{2(1 + \eta_I)^2} \right\} f \left\{ 2v_1 + \theta \left\{ \frac{2 + \eta_I}{\eta_I} \right\} (v_1)_x \right\} \left\{ -\frac{2}{\eta_I^2} \right\} d\eta_I dx
\end{aligned} \tag{3.117}$$

on further simplification we have

$$\hat{f}_I[\hat{v}_1] = \int_{-1}^1 \int_{-2}^0 \frac{e^{-\frac{(2+\eta_I)^2}{2\eta_I^2}}}{\pi^{\frac{1}{4}}} \left\{ \frac{1}{(1 + \eta_I)^2} \right\} f \left\{ 2v_1 + \theta \left\{ \frac{2 + \eta_I}{\eta_I} \right\} (v_1)_x \right\} d\eta_I dx \tag{3.118}$$

$$\hat{v}_2 = v_2 + \theta y (v_2)_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v_2 dy', \tag{3.119}$$

Using the relationships (3.92),(3.93),(3.95),(3.97),(3.99)and (3.101)in

$$\begin{aligned}\hat{f}_{II}[\hat{v}_2] &= \int_{-1}^1 \int_0^2 \rho(\eta_{II})^{\frac{1}{2}} \tau(\eta_{II})^{\frac{1}{2}} f\{2v_2 + \theta\{\frac{\eta_{II}}{2 - \eta_{II}}\}(v_2)_x\} \frac{\partial y}{\partial \eta_{II}} d\eta_{II} dx \\ \Rightarrow \hat{f}_{II}[\hat{v}_2] &= \int_{-1}^1 \int_0^2 \frac{e^{-\frac{\eta_{II}^2}{2(2-\eta_{II})^2}}}{\pi^{\frac{1}{4}}} \left\{ \frac{(2 - \eta_{II})^2}{2} \right\} f\{2v_2 + \theta\{\frac{\eta_{II}}{2 - \eta_{II}}\}(v_2)_x\} \left\{ \frac{2}{(2 - \eta_{II})^2} \right\} d\eta_{II} dx\end{aligned}\quad (3.120)$$

on further simplification we have

$$\hat{f}_{II}[\hat{v}_2] = \int_{-1}^1 \int_0^2 \frac{e^{-\frac{\eta_{II}^2}{2(2-\eta_{II})^2}}}{\pi^{\frac{1}{4}}} f\{2v_2 + \theta\{\frac{\eta_{II}}{2 - \eta_{II}}\}(v_2)_x\} d\eta_{II} dx \quad (3.121)$$

On Using the relationships (3.92),(3.93),(3.95),(3.97),(3.99)and (3.101)in (3.84) we get

$$\begin{aligned}m1_I[w_1] &= \int_0^{-2} \left\{ \frac{e^{-\frac{(2+\eta_I'')^2}{2\eta_I''^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{\eta_I''^2}{2(1 + \eta_I'')^2} \right\} w_1(x, \eta_I'') \left\{ \frac{-2}{\eta_I''^2} \right\} d\eta_I'' \\ m1_{II}[w_2] &= \int_0^2 \left\{ \frac{e^{-\frac{(\eta_{II}'')^2}{2(2-\eta_{II}'')^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(2 - \eta_{II}'')^2}{2} \right\} w_2(x, \eta_{II}'') \left\{ \frac{2}{(2 - \eta_{II}'')^2} \right\} d\eta_{II}'' \\ m1_I[v_1] &= \int_0^{-2} \left\{ \frac{e^{-\frac{(2+\eta_I'')^2}{2\eta_I''^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{\eta_I''^2}{2(1 + \eta_I'')^2} \right\} v_1(x, \eta_I'') \left\{ \frac{-2}{\eta_I''^2} \right\} d\eta_I'' \\ m1_{II}[v_2] &= \int_0^2 \left\{ \frac{e^{-\frac{\eta_{II}'')^2}{2(2-\eta_{II}'')^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(2 - \eta_{II}'')^2}{2} \right\} v_2(x, \eta_{II}'') \left\{ \frac{2}{(2 - \eta_{II}'')^2} \right\} d\eta_{II}''\end{aligned}\quad (3.122)$$

on further simplification we have

$$m1_I[w_1] = \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta_I'')^2}{2\eta_I''^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{1}{(1 + \eta_I'')^2} \right\} w_1(x, \eta_I'') d\eta_I'' \quad (3.123)$$

$$m1_{II}[w_2] = \int_0^2 \left\{ \frac{e^{-\frac{\eta''_I{}^2}{2(2-\eta''_I)^2}}}{\pi^{\frac{1}{4}}} \right\} w_2(x, \eta''_I) d\eta''_I \quad (3.124)$$

$$m1_I[v_1] = \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta''_I)^2}{2\eta''_I{}^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{1}{(1+\eta''_I)^2} \right\} v_1(x, \eta''_I) d\eta''_I \quad (3.125)$$

$$m1_{II}[v_2] = \int_0^2 \left\{ \frac{e^{-\frac{\eta''_I{}^2}{2(2-\eta''_I)^2}}}{\pi^{\frac{1}{4}}} \right\} v_2(x, \eta''_I) d\eta''_I \quad (3.126)$$

$$\begin{aligned} m1_I^I(w_1, v_1) &= \int_{-1}^1 m1_I[w_1(x, \cdot)] m1_I[v_1(x, \cdot)] dx \\ m1_I^{II}(w_1, v_2) &= \int_{-1}^1 m1_I[w_1(x, \cdot)] m1_{II}[v_2(x, \cdot)] dx \\ m1_{II}^I(w_2, v_1) &= \int_{-1}^1 m1_{II}[w_2(x, \cdot)] m1_I[v_1(x, \cdot)] dx \\ m1_{II}^{II}(w_2, v_2) &= \int_{-1}^1 m1_{II}[w_2(x, \cdot)] m1_{II}[v_2(x, \cdot)] dx \end{aligned} \quad (3.127)$$

$$m1((w_1, w_2), (v_1, v_2)) = m1_I^I(w_1, v_1) + m1_I^{II}(w_1, v_2) + m1_{II}^I(w_2, v_1) + m1_{II}^{II}(w_2, v_2) \quad (3.128)$$

On Using the relationships (3.92),(3.93),(3.95),(3.97),(3.99)and (3.101)in (3.84) we get

$$\begin{aligned}
b1_I[(w_1)_x] &= \int_0^{-2} \left\{ \frac{e^{-\frac{(2+\eta'_I)^2}{2\eta_I'^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{\eta_I'^2}{2(1+\eta_I')^2} \right\} \left\{ \frac{(2+\eta'_I)}{\eta'_I} \right\} (w_1(x, \eta'_I))_x \left\{ \frac{-2}{\eta_I'^2} \right\} d\eta'_I \\
b1_{II}[(w_2)_x] &= \int_0^2 \left\{ \frac{e^{-\frac{\eta_{II}'^2}{2(2-\eta_{II}')^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(2-\eta_{II}')^2}{2} \right\} \left\{ \frac{\eta_{II}'}{(2-\eta_{II}')} \right\} (w_2(x, \eta_{II}'))_x \left\{ \frac{2}{(2-\eta_{II}')^2} \right\} d\eta_{II}' \\
b2_I[v_1] &= \int_0^{-2} \left\{ \frac{e^{-\frac{(2+\eta''_I)^2}{2\eta_I''^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{\eta_I''^2}{2(1+\eta_I'')^2} \right\} v_1(x, \eta_I'') \left\{ \frac{-2}{\eta_I''^2} \right\} d\eta_I'' \\
b2_{II}[v_2] &= \int_0^2 \left\{ \frac{e^{-\frac{\eta_{II}''^2}{2(2-\eta_{II}'')^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(2-\eta_{II}'')^2}{2} \right\} v_2(x, \eta_{II}'') \left\{ \frac{2}{(2-\eta_{II}'')^2} \right\} d\eta_{II}'' \\
b1_I[(v_1)_x] &= \int_0^{-2} \left\{ \frac{e^{-\frac{(2+\eta'_I)^2}{2\eta_I'^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{\eta_I'^2}{2(1+\eta_I')^2} \right\} \left\{ \frac{(2+\eta'_I)}{\eta'_I} \right\} (v_1(x, \eta'_I))_x \left\{ \frac{-2}{\eta_I'^2} \right\} d\eta'_I \\
b1_{II}[(v_2)_x] &= \int_0^2 \left\{ \frac{e^{-\frac{\eta_{II}'^2}{2(2-\eta_{II}')^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(2-\eta_{II}')^2}{2} \right\} \left\{ \frac{\eta_{II}'}{(2-\eta_{II}')} \right\} (v_2(x, \eta_{II}'))_x \left\{ \frac{2}{(2-\eta_{II}')^2} \right\} d\eta_{II}' \\
b2_I[w_1] &= \int_0^{-2} \left\{ \frac{e^{-\frac{(2+\eta''_I)^2}{2\eta_I''^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{\eta_I''^2}{2(1+\eta_I'')^2} \right\} w_1(x, \eta_I'') \left\{ \frac{-2}{\eta_I''^2} \right\} d\eta_I'' \\
b2_{II}[w_2] &= \int_0^2 \left\{ \frac{e^{-\frac{(\eta_{II}'')^2}{2(2-\eta_{II}'')^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(2-\eta_{II}'')^2}{2} \right\} w_2(x, \eta_{II}'') \left\{ \frac{2}{(2-\eta_{II}'')^2} \right\} d\eta_{II}''
\end{aligned} \tag{3.129}$$

$$b1_I[(w_1)_x] = \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta'_I)^2}{2\eta_I'^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(2+\eta'_I)}{\eta'_I(1+\eta_I')^2} \right\} (w_1(x, \eta'_I))_x d\eta'_I \tag{3.130}$$

$$b1_{II}[(w_2)_x] = \int_0^2 \left\{ \frac{e^{-\frac{(\eta_{II}')^2}{2(2-\eta_{II}')^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{\eta_{II}'}{(2-\eta_{II}')} \right\} (w_2(x, \eta_{II}'))_x d\eta_{II}' \tag{3.131}$$

$$b2_I[v_1] = \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta''_I)^2}{2\eta_I''^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{1}{(1+\eta_I'')^2} \right\} v_1(x, \eta_I'') d\eta_I'' \tag{3.132}$$

$$b2_{II}[v_2] = \int_0^2 \left\{ \frac{e^{-\frac{\eta_{II}''^2}{2(2-\eta_{II}'')^2}}}{\pi^{\frac{1}{4}}} \right\} v_2(x, \eta_{II}'') d\eta_{II}'' \tag{3.133}$$

$$b1_I[(v_1)_x] = \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta'_I)^2}{2\eta_I'^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(2+\eta'_I)}{\eta'_I(1+\eta_I'^2)} \right\} (v_1(x, \eta'_I))_x d\eta'_I \quad (3.134)$$

$$b1_{II}[(v_2)_x] = \int_0^2 \left\{ \frac{e^{-\frac{(2+\eta''_{II})^2}{2\eta_{II}''^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(\eta''_{II})}{(2-\eta''_{II})} \right\} (v_2(x, \eta''_{II}))_x d\eta''_{II} \quad (3.135)$$

$$b2_I[w_1] = \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta''_I)^2}{2\eta_I''^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{1}{(1+\eta_I''^2)} \right\} w_1(x, \eta''_I) d\eta''_I \quad (3.136)$$

$$b2_{II}[w_2] = \int_0^2 \left\{ \frac{e^{-\frac{\eta_{II}''^2}{2(2-\eta_{II}'')^2}}}{\pi^{\frac{1}{4}}} \right\} w_2(x, \eta''_{II}) d\eta''_{II} \quad (3.137)$$

$$\begin{aligned} b((w_1, w_2), (v_1, v_2)) &= b_I^I(w_1, v_1) + b_I^{II}(w_1, v_2) + b_{II}^I(w_2, v_1) + b_{II}^{II}(w_2, v_2) \\ &\quad - b_I^I(v_1, w_1) - b_I^{II}(v_2, w_1) - b_{II}^I(v_1, w_2) - b_{II}^{II}(v_2, w_2) \\ b_I^I(w_1, v_1) &= \int_{-1}^1 b1_I[(w_1(x, \cdot))_x] b2_I[v_1(x, \cdot)] dx \\ b_I^{II}(w_1, v_2) &= \int_{-1}^1 b1_I[(w_1(x, \cdot))_x] b2_{II}[v_2(x, \cdot)] dx \\ b_{II}^I(w_2, v_1) &= \int_{-1}^1 b1_{II}[(w_2(x, \cdot))_x] b2_I[v_1(x, \cdot)] dx \\ b_{II}^{II}(w_2, v_2) &= \int_{-1}^1 b1_{II}[(w_2(x, \cdot))_x] b2_{II}[v_2(x, \cdot)] dx \\ b_I^I(v_1, w_1) &= \int_{-1}^1 b2_I[w_1(x, \cdot)] b1_I[(v_1(x, \cdot))_x] dx \\ b_I^{II}(v_2, w_1) &= \int_{-1}^1 b2_I[w_1(x, \cdot)] b1_{II}[(v_2(x, \cdot))_x] dx \\ b_{II}^I(v_1, w_2) &= \int_{-1}^1 b2_{II}[w_2(x, \cdot)] b1_I[(v_1(x, \cdot))_x] dx \\ b_{II}^{II}(v_2, w_2) &= \int_{-1}^1 b2_{II}[w_2(x, \cdot)] b1_{II}[(v_2(x, \cdot))_x] dx \end{aligned} \quad (3.138)$$

3.5.2 Definition of the transformed weak form

From the above discussion, the Hilbert spaces of the transformed weak form can be written as

$$X_I \equiv \left\{ v \mid \int_{-1}^{+1} \int_{-2}^0 \frac{\eta_I^2}{2(1+\eta_I)^4} v^2 d\eta_I dx < \infty, \int_{-1}^{+1} \int_{-2}^0 \frac{\{\eta_I + 2\}^2}{2(1+\eta_I)^4} v_x^2 d\eta_I dx < \infty \right\} \quad (3.139)$$

and

$$X_{II} \equiv \left\{ v \mid \int_{-1}^{+1} \int_0^2 \frac{\{2-\eta_2\}^2}{2} v^2 d\eta_2 dx < \infty, \int_{-1}^{+1} \int_{-\infty}^0 \frac{\eta_2^2}{2} v_x^2 d\eta_2 dx < \infty \right\} \quad (3.140)$$

The solution vector in the transformed domain should satisfy

$$U = (U_I, U_{II}), \text{ where } U_I \in X_I, U_{II} \in X_{II}; \forall v_I \in X_I \text{ and } \forall v_{II} \in X_{II} \quad (3.141)$$

and the weak form

$$a(U_I, U_{II}, v_I, v_{II}) = \check{f}(v_I, v_{II}) \quad (3.142)$$

where

$$\begin{aligned} a : ((X_I \times X_{II}) \times (X_I \times X_{II})) &\longrightarrow \mathcal{R} \\ \check{f} : ((X_I \times X_{II})) &\longrightarrow \mathcal{R} \end{aligned} \quad (3.143)$$

The LHS of the transformed weak form in the decomposed form can be written as

$$\begin{aligned}
a((w_1, w_2), (v_1, v_2)) &= \theta d((w_1, w_2), (v_1, v_2)) + \frac{1}{\theta} m_0((w_1, w_2), (v_1, v_2)) - \\
&\quad \frac{1}{\theta} m_1((w_1, w_2), (v_1, v_2)) + \sigma((w_1, w_2), (v_1, v_2)) + b((w_1, w_2), (v_1, v_2))
\end{aligned} \tag{3.144}$$

Then the complete transformed weak form can be written as

$$\begin{aligned}
a((w_1, w_2), (v_1, v_2)) &= \theta \{d_I(w_1, v_1) + d_{II}(w_2, v_2)\} + \frac{1}{\theta} \{m_{0I}(w_1, v_1) + m_{0II}(w_2, v_2)\} \\
&\quad - \frac{1}{\theta} \{m_{1I}^I(w_1, v_1) + m_{1I}^{II}(w_1, v_2) + m_{1II}^I(w_2, v_1) + m_{1II}^{II}(w_2, v_2)\} \\
&\quad + \sigma_I(w_1, v_1) + \sigma_{II}(w_1, v_1) + b_I^I(w_1, v_1) + b_I^{II}(w_1, v_2) \\
&\quad + b_{II}^I(w_2, v_1) + b_{II}^{II}(w_2, v_2) \\
&\quad - b_I^I(v_1, w_1) - b_I^{II}(v_2, w_1) - b_{II}^I(v_1, w_2) - b_{II}^{II}(v_2, w_2) \\
\check{f}(v_1, v_2) &= \hat{f}_I[v_1 + \theta y(v_1)_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v_1 dy'] \\
&\quad + \hat{f}_{II}[v_2 + \theta y(v_2)_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} v_2 dy'].
\end{aligned} \tag{3.145}$$

Chapter 4

Finite Element Discretization

4.1 Definition of discrete Spaces

We define the discrete space as follows

$$X_h = \{v \in X, v|_{T_h} = a + bx, \forall T_h \in \mathcal{T}_h\} \quad (4.1)$$

Where \mathcal{T}_h represents a set of all quadrilaterals that can be used to discretize the domain. Where as T_h represents the set of rectangular elements used to discretize the problem's transformed sub-domains as shown in the Figure 4-1.

4.2 Definition of Basis function used

We choose the two node rectangular elements to represent our problem. as shown in the Figure 4-1. The basis function corresponding to any node varies linearly with position x and is a step function with respect to η_I/η_{II} with in the element containing that node in the sub-domains Ω_I / Ω_{II} respectively. We can define the 2 basis functions corresponding to the 2 different arbitrary nodes in the sub-domains Ω_I and Ω_{II} as shown in (4.6). $\psi_j^I(\eta_I), \psi_j^{II}(\eta_{II})$ represent the step functions in the sub-domains Ω_I and Ω_{II} respectively. Where as $\phi_i(x)$ represents the standard one dimensional linear shape function. Mathematically, $\phi_i(x_j)$ can

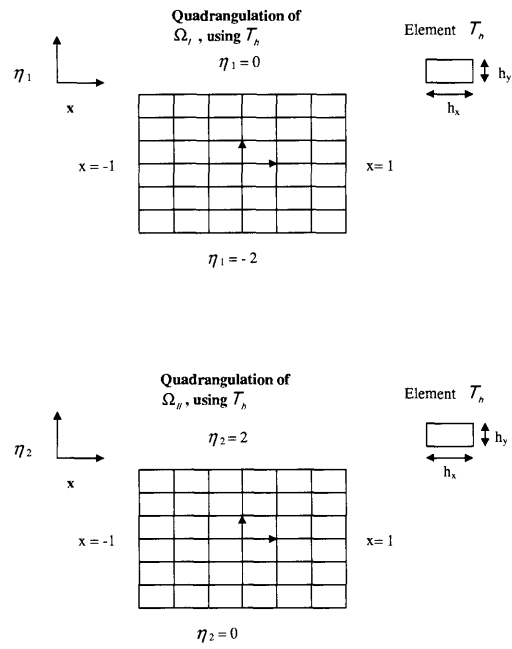


Figure 4-1: Discretization of the space using finite elements

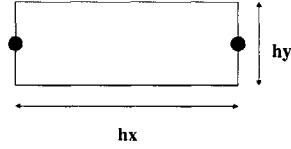


Figure 4-2: Node locations in a typical element

be represented as

$$\phi_i(x_j) = \delta_{ij}, \quad s.t. \quad \phi_i(x) \in X_h \quad (4.2)$$

where δ_{ij} represents the Kronecker delta function, which in turn is defined as

$$\begin{aligned} \delta_{ij} &= 1, & \text{if } i &= j \\ \delta_{ij} &= 0, & \text{if } i &\neq j \end{aligned} \quad (4.3)$$

where as the definition of steps functions $\psi_j^I(\eta_I)$ and $\psi_j^{II}(\eta_{II})$ is as follows

$$\begin{aligned} \psi_j^I(\eta_I) &= 1 & \text{with in the element in } \Omega_I \\ \psi_j^I(\eta_I) &= 0 & \text{outside the element} \end{aligned} \quad (4.4)$$

$$\begin{aligned}
\psi_j^{II}(\eta_{II}) &= 1 \quad \text{with in the element in } \Omega_{II} \\
\psi_j^{II}(\eta_{II}) &= 0 \quad \text{outside the element}
\end{aligned} \tag{4.5}$$

Then the basis functions v_I and v_{II} in the sub-domains Ω_I and Ω_{II} can be defined as

$$\begin{aligned}
v_I &= \phi_i(x)\psi_j^I(\eta_I) \\
v_{II} &= \phi_i(x)\psi_j^{II}(\eta_{II})
\end{aligned} \tag{4.6}$$

4.3 Discussion of Node Numbering

The main aim of this chapter is describe the node numbering scheme and also introduce some vectors which will be useful in evaluating the Stiffness matrix and the Right Hand side. In each of the sub-domains, we need 2 indices, i and j to represent any node. The x-index, i of a node with x-coordinate x_i is defined in both sub-domains Ω_I and Ω_{II} as

$$i = \frac{x_i + 1}{h_x} + 1 \tag{4.7}$$

where as the velocity indices, j and j' are defined differently in both the sub-domains Ω_I and Ω_{II} respectively as

$$j = \frac{(\eta_I)_j^t + 2}{h_y} \tag{4.8}$$

and

$$j' = \frac{(\eta_{II})_{j'}^t}{h_y} \tag{4.9}$$

where $(\eta_I)_j^t$ represents the velocity coordinate of the line bounding the element j containing the node, on the top, in the sub-domain Ω_I and $(\eta_{II})_{j'}^t$ represents the velocity coordinate of the line bounding the element j' , containing the node on the top in the sub-domain Ω_{II} . Let us say there are N_y velocity strips in each sub-domain and N_x nodes in each velocity strip.

This implies that there are $N_x N_y$ nodes in each of these sub-domains. If we start the global node number count from the sub-domain Ω_I and then proceed to Ω_{II} . Then the following functions will map the local node indices of each node to the corresponding Global node number. For the nodes in Ω_I , the function $k\mathcal{N}_I^*(i, j)$ gives the global node number as

$$\mathcal{N}_I^*(i, j) = i + (j - 1) * N_x \quad 1 \leq j \leq N_y, \quad 1 \leq i \leq N_x \quad (4.10)$$

For the nodes in Ω_{II} , the function $\mathcal{N}_{II}^*(i, j')$ gives the global node number as

$$\mathcal{N}_{II}^*(i, j') = N_x N_y + i + (j' - 1) N_x \quad 1 \leq j' \leq N_y, \quad 1 \leq i \leq N_x \quad (4.11)$$

Now let us define the vectors containing the corresponding velocity (transformed) values of the lines bounding the velocity strips in either domain. The vector containing the corresponding velocity values of the lines bounding the strips from below in the domain Ω_I is

$$t1^I = \begin{bmatrix} -2 \\ -2 + h_y \\ -2 + 2h_y \\ \cdot \\ \cdot \\ -2h_y \\ -h_y \end{bmatrix}_{N_y \times 1}$$

(4.12)

and the vector containing the corresponding velocity values of the lines bounding the strips from above, in the domain Ω_I is

$$t2^I = \begin{bmatrix} -2 + h_y \\ -2 + 2h_y \\ \cdot \\ \cdot \\ -h_y \\ 0 \end{bmatrix}_{N_y \times 1}$$

(4.13)

where as the vector containing the corresponding velocity values of the lines bounding the strips from below, in the domain Ω_{II}

$$t1^{II} = \begin{bmatrix} 0 \\ 2h_y \\ 3h_y \\ \cdot \\ \cdot \\ 2 - 2h_y \\ 2 - h_y \end{bmatrix}_{N_y \times 1}$$

(4.14)

and the vector containing the corresponding velocity values of the lines bounding the strips from above, in the domain Ω_{II}

$$t2^{II} = \begin{bmatrix} h_y \\ 2h_y \\ \cdot \\ \cdot \\ 2 - h_y \\ 2 \end{bmatrix}_{N_y \times 1} \quad . \quad (4.15)$$

These vectors will prove useful while evaluating the quantities depending on the velocity, in order to set up the global stiffness matrix and the Right hand side vector.

4.4 Generation of local Matrices and vectors

To generate the entries of the global stiffness matrix we perform the following substitutions.

$$\begin{aligned} w_1 &= \phi_k \psi_l^I \\ v_1 &= \phi_i \psi_j^I \\ w_2 &= \phi_k \psi_l^{II} \\ v_2 &= \phi_i \psi_j^{II} \end{aligned} \quad (4.16)$$

using (4.16)in (4.17) we have

$$\begin{aligned}
d_I(\phi_k \psi_l^I, \phi_i \psi_j^I) &= \int_{-1}^1 \int_{-2}^0 \frac{(2 + \eta_I)^2}{2(1 + \eta_I)^4} (\phi_k \psi_l^I)_x (\phi_i \psi_j^I)_x d\eta_I dx \\
d_I(\phi_k \psi_l^I, \phi_i \psi_j^I) &= \left\{ \int_{-1}^1 (\phi_i)_x (\phi_k)_x dx \right\} \left\{ \int_{-2}^0 \frac{(2 + \eta_I)^2}{2(1 + \eta_I)^4} (\psi_l^I)(\psi_j^I) d\eta_I \right\} \Rightarrow \\
\bar{D}(\mathcal{N}_I^*(i, j), \mathcal{N}_I^*(k, l)) &= \tilde{D}(i, k) T_d^I(j) \delta_{jl}
\end{aligned} \tag{4.17}$$

using (4.16)in we have

$$\begin{aligned}
d_{II}(\phi_k \psi_l^{II}, \phi_i \psi_j^{II}) &= \int_{-1}^1 \int_{-2}^0 \frac{\eta_{II}^2}{2} (\phi_k \psi_l^{II})_x (\phi_i \psi_j^{II})_x d\eta_{II} dx \\
d_{II}(\phi_k \psi_l^{II}, \phi_i \psi_j^{II}) &= \left\{ \int_{-1}^1 (\phi_i)_x (\phi_k)_x dx \right\} \left\{ \int_{-2}^0 \frac{\eta_{II}^2}{2} (\psi_l^{II})(\psi_j^{II}) d\eta_{II} \right\} \Rightarrow \\
\bar{D}(\mathcal{N}_{II}^*(i, j), \mathcal{N}_{II}^*(k, l)) &= \tilde{D}(i, k) T_d^{II}(j) \delta_{jl}
\end{aligned} \tag{4.18}$$

On simplifying further, we have

$$\begin{aligned}
m_{0I}(\phi_k \psi_l^I, \phi_i \psi_j^I) &= \int_{-1}^1 \int_{-2}^0 \frac{\eta_I^2}{2(1 + \eta_I)^4} \phi_k \psi_l^I \phi_i \psi_j^I d\eta_I dx \Rightarrow \\
m_{0I}(\phi_k \psi_l^I, \phi_i \psi_j^I) &= \left\{ \int_{-1}^1 \phi_k \phi_i dx \right\} \left\{ \int_{-2}^0 \frac{\eta_I^2}{2(1 + \eta_I)^4} \psi_l^I \psi_j^I d\eta_I \right\} \\
\bar{M}0(\mathcal{N}_I^*(i, j), \mathcal{N}_I^*(k, l)) &= \tilde{M}(i, k) T_{m0}^I(j) \delta_{jl}
\end{aligned} \tag{4.19}$$

On simplifying further, we have

$$\begin{aligned}
m_{0II}(w_2, v_2) &= \int_{-1}^1 \int_0^2 \frac{(2 - \eta_{II})^2}{2} \phi_k \psi_l^{II} \phi_i \psi_j^{II} d\eta_{II} dx \\
m_{0II}(\phi_k \psi_l^{II}, \phi_i \psi_j^{II}) &= \left\{ \int_{-1}^1 \phi_k \phi_i dx \right\} \left\{ \int_{-2}^0 \frac{(2 - \eta_{II})^2}{2} \psi_l^{II} \psi_j^{II} d\eta_{II} \right\} \\
\bar{M}0(\mathcal{N}_{II}^*(i, j), \mathcal{N}_{II}^*(k, l)) &= \tilde{M}(i, k) T_{m0}^{II}(j) \delta_{jl}
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
\sigma_I(\phi_k \psi_l^I, \phi_i \psi_j^I) &= - \int_{-2x=-1}^0 \frac{\eta_I(2+\eta_I)}{2(1+\eta_I)^4} \phi_k \psi_l^I \phi_i \psi_j^I d\eta_I \Rightarrow \\
\bar{\sigma}(\mathcal{N}_I^*(i, j), \mathcal{N}_I^*(k, l)) &= \{\phi_k \phi_i|_{x=-1}\} \left\{ - \int_{-2}^0 \frac{\eta_I(2+\eta_I)}{2(1+\eta_I)^4} \psi_l^I \psi_j^I d\eta_I \right\} \\
\bar{\sigma}(\mathcal{N}_I^*(i, j), \mathcal{N}_I^*(k, l)) &= \tilde{\sigma}_I(i, k) T_\sigma^I(j) \delta_{jl}
\end{aligned} \tag{4.21}$$

On simplifying further, we have

$$\begin{aligned}
\sigma_{II}(w_2, v_2) &= \int_{0x=1}^2 \frac{(2-\eta_{II})\eta_{II}}{2} w_2 v_2 d\eta_{II} \\
\sigma_{II}(\phi_k \psi_l^{II}, \phi_i \psi_j^{II}) &= \int_{0x=1}^2 \frac{(2-\eta_{II})\eta_{II}}{2} \phi_k \psi_l^{II} \phi_i \psi_j^{II} d\eta_{II} \Rightarrow \\
\bar{\sigma}(\mathcal{N}_{II}^*(i, j), \mathcal{N}_{II}^*(k, l)) &= \{\phi_k \phi_i|_{x=1}\} \left\{ \int_0^2 \frac{(2-\eta_{II})\eta_{II}}{2} \psi_l^{II} \psi_j^{II} d\eta_{II} \right\} \\
\bar{\sigma}(\mathcal{N}_{II}^*(i, j), \mathcal{N}_{II}^*(k, l)) &= \tilde{\sigma}_{II}(i, k) T_\sigma^{II}(j) \delta_{jl}
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
\check{f}(\phi_i \psi_j^I, \phi_i \psi_j^{II}) &= \hat{f}_I[\phi_i \psi_j^I + \theta y(\phi_i \psi_j^I)_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \phi_i \psi_j^I dy'] + \\
&\quad \hat{f}_{II}[\phi_i \psi_j^{II} + \theta y(\phi_i \psi_j^{II})_x + \rho^{\frac{1}{2}} \tau^{-\frac{1}{2}} \int_{-\infty}^{\infty} \rho^{\frac{1}{2}} \tau^{\frac{1}{2}} \phi_i \psi_j^{II} dy'] \\
\check{f}(\phi_i \psi_j^I, \phi_i \psi_j^{II}) &= \int_{-1}^1 \int_{-2}^0 \frac{e^{-\frac{(2+\eta_I)^2}{2\eta_I^2}}}{\pi^{\frac{1}{4}}} \left\{ \frac{1}{(1+\eta_I)^2} \right\} f \left\{ 2\phi_i \psi_j^I + \theta \left\{ \frac{2+\eta_I}{\eta_I} \right\} (\phi_i \psi_j^I)_x \right\} d\eta_I dx + \\
&\quad \int_{-1}^1 \int_0^2 \frac{e^{-\frac{\eta_{II}^2}{2(2-\eta_{II})^2}}}{\pi^{\frac{1}{4}}} f \left\{ 2\phi_i \psi_j^{II} + \theta \left\{ \frac{\eta_{II}}{2-\eta_{II}} \right\} (\phi_i \psi_j^{II})_x \right\} d\eta_{II} dx
\end{aligned} \tag{4.23}$$

$$\begin{aligned}
\check{f}(\phi_i \psi_j^I, \phi_i \psi_j^{II}) &= \left\{ \int_{-1}^1 \phi_i dx \right\} \int_{-2}^0 \frac{e^{-\frac{(2+\eta_I)^2}{2\eta_I^2}}}{\pi^{\frac{1}{4}}} \left\{ \frac{2f}{(1+\eta_I)^2} \right\} \psi_j^I d\eta_I + \\
&\quad \left\{ \int_{-1}^1 (\phi_i)_x dx \right\} \int_{-2}^0 \frac{e^{-\frac{(2+\eta_I)^2}{2\eta_I^2}}}{\pi^{\frac{1}{4}}} \left\{ \frac{\theta f}{(1+\eta_I)^2} \right\} \left\{ \frac{2+\eta_I}{\eta_I} \right\} \psi_j^I d\eta_I + \\
&\quad \left\{ \int_{-1}^1 \phi_i dx \right\} \int_0^2 \frac{e^{-\frac{\eta_{II}^2}{2(2-\eta_{II})^2}}}{\pi^{\frac{1}{4}}} 2f \psi_j^{II} d\eta_{II} + \\
&\quad \left\{ \int_{-1}^1 (\phi_i)_x dx \right\} \int_0^2 \frac{e^{-\frac{\eta_{II}^2}{2(2-\eta_{II})^2}}}{\pi^{\frac{1}{4}}} \left\{ \frac{\eta_{II}}{2-\eta_{II}} \right\} \theta f \psi_j^{II} d\eta_{II}
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
\Rightarrow \check{F}(\mathcal{N}_I^*(i, j), 1) &= \hat{F}_I^1(\mathcal{N}_I^*(i, j), 1) + \theta \hat{F}_I^2(\mathcal{N}_I^*(i, j), 1) \\
\check{F}(\mathcal{N}_I^*(i, j), 1) &= 2f \tilde{R}1(i) C^I(j) + \theta f \tilde{R}2(i) D_b^I(j) \\
\check{F}(\mathcal{N}_{II}^*(i, j), 1) &= \hat{F}_{II}^1(\mathcal{N}_{II}^*(i, j), 1) + \theta \hat{F}_{II}^2(\mathcal{N}_{II}^*(i, j), 1) \\
\Rightarrow \check{F}(\mathcal{N}_{II}^*(i, j), 1) &= 2f \tilde{R}1(i) C^{II}(j) + \theta f \tilde{R}2(i) D_b^{II}(j)
\end{aligned} \tag{4.25}$$

on further simplification we have

$$\begin{aligned}
m1_I[\phi_k \psi_i^I] &= \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta'_I)^2}{2\eta'^2_I}}}{\pi^{\frac{1}{4}}(1+\eta'_I)^2} \right\} \phi_k \psi_i^I d\eta'_I \Rightarrow \\
m1_I[\phi_k \psi_i^I] &= \{\phi_k\} \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta'_I)^2}{2\eta'^2_I}}}{\pi^{\frac{1}{4}}(1+\eta'_I)^2} \right\} \psi_i^I d\eta'_I
\end{aligned} \tag{4.26}$$

$$\begin{aligned}
m1_{II}[\phi_k \psi_i^{II}] &= \int_0^2 \left\{ \frac{e^{-\frac{(\eta'_{II})^2}{2(2-\eta'_{II})^2}}}{\pi^{\frac{1}{4}}} \right\} \phi_k \psi_i^{II} d\eta'_{II} \Rightarrow \\
m1_{II}[\phi_k \psi_i^{II}] &= \{\phi_k\} \int_0^2 \left\{ \frac{e^{-\frac{(\eta'_{II})^2}{2(2-\eta'_{II})^2}}}{\pi^{\frac{1}{4}}} \right\} \psi_i^{II} d\eta'_{II}
\end{aligned} \tag{4.27}$$

$$\begin{aligned}
m1_I[\phi_i \psi_j^I] &= \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta_I'')^2}{2\eta_I''^2}}}{\pi^{\frac{1}{4}}(1+\eta_I'')^2} \right\} \phi_i \psi_j^I d\eta_I'' \Rightarrow \\
m1_I[\phi_i \psi_j^I] &= \{\phi_i\} \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta_I'')^2}{2\eta_I''^2}}}{\pi^{\frac{1}{4}}(1+\eta_I'')^2} \right\} \psi_j^I d\eta_I''
\end{aligned} \tag{4.28}$$

$$\begin{aligned}
m1_{II}[\phi_i \psi_j^{II}] &= \int_0^2 \left\{ \frac{e^{-\frac{(\eta_{II}')^2}{2(2-\eta_{II}')^2}}}{\pi^{\frac{1}{4}}} \right\} \phi_i \psi_j^{II} d\eta_{II}' \Rightarrow \\
m1_{II}[\phi_i \psi_j^{II}] &= \{\phi_i\} \int_0^2 \left\{ \frac{e^{-\frac{(\eta_{II}')^2}{2(2-\eta_{II}')^2}}}{\pi^{\frac{1}{4}}} \right\} \psi_j^{II} d\eta_{II}'
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
m1_I^I(\phi_k \psi_l^I, \phi_i \psi_j^I) &= \int_{-1}^1 m1_I[\phi_k \psi_l^I] m1_I''[\phi_i \psi_j^I] dx \Rightarrow \\
m1_I^I(\phi_k \psi_l^I, \phi_i \psi_j^I) &= \left\{ \int_{-1}^1 \phi_k \phi_i dx \right\} \left\{ \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta_I')^2}{2\eta_I'^2}}}{\pi^{\frac{1}{4}}(1+\eta_I')^2} \right\} \psi_l^I d\eta_I' \right\} \left\{ \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta_I'')^2}{2\eta_I''^2}}}{\pi^{\frac{1}{4}}(1+\eta_I'')^2} \right\} \psi_j^I d\eta_I'' \right\}
\end{aligned} \tag{4.30}$$

$$\bar{M}1(\mathcal{N}_I^*(i, j), \mathcal{N}_I^*(k, l)) = \tilde{M}(i, k) C^I(j) C^I(l) \tag{4.31}$$

$$\begin{aligned}
m1_I^{II}(\phi_k \psi_l^I, \phi_i \psi_j^{II}) &= \int_{-1}^1 m1_I'[\phi_k \psi_l^I] m1_{II}''[\phi_i \psi_j^{II}] dx \Rightarrow \\
m1_I^{II}(\phi_k \psi_l^I, \phi_i \psi_j^{II}) &= \left\{ \int_{-1}^1 \phi_k \phi_i dx \right\} \left\{ \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta_I')^2}{2\eta_I'^2}}}{\pi^{\frac{1}{4}}(1+\eta_I')^2} \right\} \psi_l^I d\eta_I' \right\} \left\{ \int_0^2 \left\{ \frac{e^{-\frac{(\eta_{II}')^2}{2(2-\eta_{II}')^2}}}{\pi^{\frac{1}{4}}} \right\} \psi_j^{II} d\eta_{II}' \right\}
\end{aligned} \tag{4.32}$$

$$\bar{M}1(\mathcal{N}_{II}^*(i, j), \mathcal{N}_I^*(k, l)) = \tilde{M}(i, k) C^{II}(j) C^I(l) \tag{4.33}$$

$$\begin{aligned}
m1_{II}^I(\phi_k \psi_l^{II}, \phi_i \psi_j^I) &= \int_{-1}^1 m1'_{II}[\phi_k \psi_l^{II}] m1''_I[\phi_i \psi_j^I] dx \Rightarrow \\
m1_{II}^I(\phi_k \psi_l^{II}, \phi_i \psi_j^I) &= \left\{ \int_{-1}^1 \phi_k \phi_i dx \right\} \left\{ \int_0^2 \left\{ \frac{e^{-\frac{(\eta'_{II})^2}{2(2-\eta'_{II})^2}}}{\pi^{\frac{1}{4}}} \right\} \psi_l^{II} d\eta'_{II} \right\} \left\{ \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta''_I)^2}{2\eta''_I{}^2}}}{\pi^{\frac{1}{4}}(1+\eta''_I)^2} \right\} \phi_i \psi_j^I d\eta''_I \right\}
\end{aligned} \tag{4.34}$$

$$\bar{M}1(\mathcal{N}_I^*(i, j), \mathcal{N}_{II}^*(k, l)) = \tilde{M}(i, k) C^I(j) C^{II}(l) \tag{4.35}$$

$$\begin{aligned}
m1_{II}^{II}(\phi_k \psi_l^{II}, \phi_i \psi_j^{II}) &= \int_{-1}^1 m1'_{II}[\phi_k \psi_l^{II}] m1''_{II}[\phi_i \psi_j^{II}] dx \\
m1_{II}^{II}(\phi_k \psi_l^{II}, \phi_i \psi_j^{II}) &= \left\{ \int_{-1}^1 \phi_k \phi_i dx \right\} \left\{ \int_0^2 \left\{ \frac{e^{-\frac{(\eta'_{II})^2}{2(2-\eta'_{II})^2}}}{\pi^{\frac{1}{4}}} \right\} \psi_l^{II} d\eta'_{II} \right\} \left\{ \int_0^2 \left\{ \frac{e^{-\frac{(\eta''_{II})^2}{2(2-\eta''_{II})^2}}}{\pi^{\frac{1}{4}}} \right\} \psi_j^{II} d\eta''_{II} \right\}
\end{aligned} \tag{4.36}$$

$$\bar{M}1(\mathcal{N}_{II}^*(i, j), \mathcal{N}_{II}^*(k, l)) = \tilde{M}(i, k) C^{II}(j) C^{II}(l) \tag{4.37}$$

$$\begin{aligned}
b1_I[(\phi_k \psi_l^I)_x] &= \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta'_I)^2}{2\eta'_I{}^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(2+\eta'_I)}{\eta'_I(1+\eta'_I)^2} \right\} (\phi_k \psi_l^I)_x d\eta'_I \Rightarrow \\
b1_I[(\phi_k \psi_l^I)_x] &= (\phi_k)_x \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta'_I)^2}{2\eta'_I{}^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(2+\eta'_I)}{\eta'_I(1+\eta'_I)^2} \right\} \psi_l^I d\eta'_I
\end{aligned} \tag{4.38}$$

$$\begin{aligned}
b1_{II}[(\phi_k \psi_l^{II})_x] &= \int_0^2 \left\{ \frac{e^{-\frac{(\eta'_{II})^2}{2(2-\eta'_{II})^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{\eta'_{II}}{(2-\eta'_{II})} \right\} (\phi_k \psi_l^{II})_x d\eta'_{II} \Rightarrow \\
b1_{II}[(\phi_k \psi_l^{II})_x] &= (\phi_k)_x \int_0^2 \left\{ \frac{e^{-\frac{(\eta'_{II})^2}{2(2-\eta'_{II})^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{\eta'_{II}}{(2-\eta'_{II})} \right\} \psi_l^{II} d\eta'_{II}
\end{aligned} \tag{4.39}$$

$$\begin{aligned}
b2_I[\phi_i \psi_j^I] &= \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta_I'')^2}{2\eta_I''^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{1}{(1+\eta_I'')^2} \right\} \phi_i \psi_j^I d\eta_I'' \Rightarrow \\
b2_I[\phi_i \psi_j^I] &= \phi_i \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta_I'')^2}{2\eta_I''^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{1}{(1+\eta_I'')^2} \right\} \psi_j^I d\eta_I'' \quad (4.40)
\end{aligned}$$

$$\begin{aligned}
b2_{II}[\phi_i \psi_j^{II}] &= \int_0^2 \left\{ \frac{e^{-\frac{\eta_{II}'^2}{2(2-\eta_{II}')^2}}}{\pi^{\frac{1}{4}}} \right\} \phi_i \psi_j^{II} d\eta_{II}' \Rightarrow \\
b2_{II}[\phi_i \psi_j^{II}] &= \phi_i \int_0^2 \left\{ \frac{e^{-\frac{\eta_{II}'^2}{2(2-\eta_{II}')^2}}}{\pi^{\frac{1}{4}}} \right\} \psi_j^{II} d\eta_{II}' \quad (4.41)
\end{aligned}$$

$$\begin{aligned}
b1_I[(\phi_i \psi_j^I)_x] &= \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta_I')^2}{2\eta_I'^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(2+\eta_I')}{\eta_I'(1+\eta_I')^2} \right\} (\phi_i \psi_j^I)_x d\eta_I' \Rightarrow \\
b1_I[(\phi_i \psi_j^I)_x] &= (\phi_i)_x \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta_I')^2}{2\eta_I'^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(2+\eta_I')}{\eta_I'(1+\eta_I')^2} \right\} \psi_j^I d\eta_I' \quad (4.42)
\end{aligned}$$

$$\begin{aligned}
b1_{II}[(\phi_i \psi_j^{II})_x] &= \int_0^2 \left\{ \frac{e^{-\frac{(2+\eta_{II}')^2}{2\eta_{II}'^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(\eta_{II}')}{(2-\eta_{II}')^2} \right\} (\phi_i \psi_j^{II})_x d\eta_{II}' \Rightarrow \\
b1_{II}[(\phi_i \psi_j^{II})_x] &= (\phi_i)_x \int_0^2 \left\{ \frac{e^{-\frac{(2+\eta_{II}')^2}{2\eta_{II}'^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(\eta_{II}')}{(2-\eta_{II}')^2} \right\} \psi_j^{II} d\eta_{II}' \quad (4.43)
\end{aligned}$$

$$\begin{aligned}
b2_I[\phi_k \psi_i^I] &= \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta_I'')^2}{2\eta_I''^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{1}{(1+\eta_I'')^2} \right\} \phi_k \psi_i^I d\eta_I'' \Rightarrow \\
b2_I[\phi_k \psi_i^I] &= \phi_k \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta_I'')^2}{2\eta_I''^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{1}{(1+\eta_I'')^2} \right\} \psi_i^I d\eta_I'' \quad (4.44)
\end{aligned}$$

$$\begin{aligned}
b_{2II}[\phi_k \psi_l^{II}] &= \int_0^2 \left\{ \frac{e^{-\frac{\eta_{II}''^2}{2(2-\eta_{II}'')^2}}}{\pi^{\frac{1}{4}}} \right\} \phi_k \psi_l^{II} d\eta_{II}'' \Rightarrow \\
b_{2II}[\phi_k \psi_l^{II}] &= \phi_k \int_0^2 \left\{ \frac{e^{-\frac{\eta_{II}''^2}{2(2-\eta_{II}'')^2}}}{\pi^{\frac{1}{4}}} \right\} \psi_l^{II} d\eta_{II}'' \quad (4.45)
\end{aligned}$$

$$\begin{aligned}
b((w_1, w_2), (v_1, v_2)) &= b_I^I(w_1, v_1) + b_I^{II}(w_1, v_2) + b_{II}^I(w_2, v_1) + b_{II}^{II}(w_2, v_2) \\
&\quad - b_I^I(v_1, w_1) - b_I^{II}(v_2, w_1) - b_{II}^I(v_1, w_2) - b_{II}^{II}(v_2, w_2) \quad (4.46)
\end{aligned}$$

$$\begin{aligned}
b_I^I(\phi_k \psi_l^I, \phi_i \psi_j^I) &= \left\{ \int_{-1}^1 (\phi_k)_x \phi_i dx \right\} \left\{ \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta_I')^2}{2\eta_I'^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(2+\eta_I')}{\eta_I'(1+\eta_I')^2} \right\} \psi_l^I d\eta_I' \right\} \\
&\quad \times \left\{ \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta_I'')^2}{2\eta_I''^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{1}{(1+\eta_I'')^2} \right\} \psi_j^I d\eta_I'' \right\} \Rightarrow \\
\bar{B}(\mathcal{N}_I^*(i, j), \mathcal{N}_I^*(k, l)) &= \tilde{P}(i, k) D_b^I(l) C^I(j) \quad (4.47)
\end{aligned}$$

$$\begin{aligned}
b_I^{II}(\phi_k \psi_l^I, \phi_i \psi_j^{II}) &= \left\{ \int_{-1}^1 (\phi_k)_x \phi_i dx \right\} \left\{ \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta_I')^2}{2\eta_I'^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(2+\eta_I')}{\eta_I'(1+\eta_I')^2} \right\} \psi_l^I d\eta_I' \right\} \\
&\quad \times \left\{ \int_0^2 \left\{ \frac{e^{-\frac{\eta_{II}''^2}{2(2-\eta_{II}'')^2}}}{\pi^{\frac{1}{4}}} \right\} \psi_j^{II} d\eta_{II}'' \right\} \Rightarrow \\
\bar{B}(\mathcal{N}_{II}^*(i, j), \mathcal{N}_I^*(k, l)) &= \tilde{P}(i, k) D_b^I(l) C^{II}(j) \quad (4.48)
\end{aligned}$$

$$\begin{aligned}
b_{II}^I(\phi_k \psi_l^{II}, \phi_i \psi_j^I) &= \left\{ \int_{-1}^1 (\phi_k)_x \phi_i dx \right\} \left\{ \int_0^2 \left\{ \frac{e^{-\frac{(\eta_{II}')^2}{2(2-\eta_{II}')^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{\eta_{II}'}{(2-\eta_{II}')^2} \right\} \psi_l^{II} d\eta_{II}' \right\} \\
&\quad \times \left\{ \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta_I'')^2}{2\eta_I''^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{1}{(1+\eta_I'')^2} \right\} \psi_j^I d\eta_I'' \right\} \Rightarrow \\
\bar{B}(\mathcal{N}_I^*(i, j), \mathcal{N}_{II}^*(k, l)) &= \tilde{P}(i, k) D_b^{II}(l) C^I(j) \quad (4.49)
\end{aligned}$$

$$\begin{aligned}
b_{II}^{II}(\phi_k \psi_l^{II}, \phi_i \psi_j^{II}) &= \left\{ \int_{-1}^1 (\phi_k)_x \phi_i dx \right\} \left\{ \int_0^2 \left\{ \frac{e^{-\frac{(\eta'_{II})^2}{2(2-\eta'_{II})^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{\eta'_{II}}{(2-\eta'_{II})} \right\} \psi_l^{II} d\eta'_{II} \right\} \\
&\quad \times \left\{ \int_0^2 \left\{ \frac{e^{-\frac{\eta''_I^2}{2(2-\eta''_I)^2}}}{\pi^{\frac{1}{4}}} \right\} \psi_j^{II} d\eta''_I \right\} \Rightarrow \\
\bar{B}(\mathcal{N}_{II}^*(i, j), \mathcal{N}_{II}^*(k, l)) &= \tilde{P}(i, k) D_b^{II}(l) C^{II}(j)
\end{aligned} \tag{4.50}$$

$$\begin{aligned}
b_I^I(\phi_i \psi_j^I, \phi_k \psi_l^I) &= \left\{ \int_{-1}^1 (\phi_i)_x \phi_k dx \right\} \left\{ \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta'_I)^2}{2\eta'_I{}^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(2+\eta'_I)}{\eta'_I(1+\eta'_I)^2} \right\} \psi_j^I d\eta'_I \right\} \\
&\quad \times \left\{ \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta''_I)^2}{2\eta''_I{}^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{1}{(1+\eta''_I)^2} \right\} \psi_l^I d\eta''_I \right\} \Rightarrow \\
\bar{B}(\mathcal{N}_I^*(k, l), \mathcal{N}_I^*(i, j)) &= \tilde{P}(k, i) D_b^I(j) C^I(l)
\end{aligned} \tag{4.51}$$

$$\begin{aligned}
b_{II}^{II}(\phi_i \psi_j^{II}, \phi_k \psi_l^I) &= \left\{ \int_{-1}^1 (\phi_i)_x \phi_k dx \right\} \left\{ \int_0^2 \left\{ \frac{e^{-\frac{(2+\eta'_{II})^2}{2\eta'_{II}{}^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(\eta'_{II})}{(2-\eta'_{II})} \right\} \psi_j^{II} d\eta'_{II} \right\} \\
&\quad \times \left\{ \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta''_I)^2}{2\eta''_I{}^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{1}{(1+\eta''_I)^2} \right\} \psi_l^I d\eta''_I \right\} \Rightarrow \\
\bar{B}(\mathcal{N}_I^*(k, l), \mathcal{N}_{II}^*(i, j)) &= \tilde{P}(k, i) D_b^{II}(j) C^I(l)
\end{aligned} \tag{4.52}$$

$$\begin{aligned}
b_{II}^I(\phi_i \psi_j^I, \phi_k \psi_l^{II}) &= \left\{ \int_{-1}^1 (\phi_i)_x \phi_k dx \right\} \left\{ \int_0^2 \left\{ \frac{e^{-\frac{\eta''_I^2}{2(2-\eta''_I)^2}}}{\pi^{\frac{1}{4}}} \right\} \psi_l^{II} d\eta''_I \right\} \\
&\quad \times \left\{ \int_{-2}^0 \left\{ \frac{e^{-\frac{(2+\eta'_I)^2}{2\eta'_I{}^2}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(2+\eta'_I)}{\eta'_I(1+\eta'_I)^2} \right\} \psi_j^I d\eta'_I \right\} \Rightarrow \\
\bar{B}(\mathcal{N}_{II}^*(k, l), \mathcal{N}_I^*(i, j)) &= \tilde{P}(k, i) C^{II}(l) D_b^I(j)
\end{aligned} \tag{4.53}$$

$$\begin{aligned}
b_{II}^{II}(\phi_i \psi_j^{II}, \phi_k \psi_l^{II}) &= \left\{ \int_{-1}^1 (\phi_i)_x \phi_k dx \right\} \left\{ \int_0^2 \left\{ \frac{e^{-\frac{\eta_{II}^{\prime 2}}{2(2-\eta_{II}^{\prime})^2}}}{\pi^{\frac{1}{4}}} \right\} \psi_l^{II} d\eta_{II}^{\prime} \right\} \\
&\quad \times \left\{ \int_0^2 \left\{ \frac{e^{-\frac{(2+\eta_{II}^{\prime})^2}{2\eta_{II}^{\prime 2}}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(\eta_{II}^{\prime})}{(2-\eta_{II}^{\prime})} \right\} \psi_j^{II} d\eta_{II}^{\prime} \right\} \Rightarrow \\
\bar{B}(\mathcal{N}_{II}^*(k, l), \mathcal{N}_{II}^*(i, j)) &= \tilde{P}(k, i) C^{II}(l) D_b^{II}(j)
\end{aligned} \tag{4.54}$$

$$\tilde{M}(\phi_k, \phi_i) = \int_{-1}^1 \phi_k \phi_i dx \quad 1 \leq k \leq N_x, \quad 1 \leq i \leq N_x \tag{4.55}$$

$$\tilde{M} = \begin{bmatrix} \frac{h_x}{3} & \frac{h_x}{6} & & & & \\ \frac{h_x}{3} & \frac{2h_x}{3} & \frac{h_x}{3} & & & \mathbf{0} \\ & \frac{h_x}{6} & \frac{2h_x}{3} & \frac{h_x}{6} & & \\ & & & \cdot & & \\ & & & & \cdot & \\ \mathbf{0} & & & & & \frac{h_x}{6} & \frac{2h_x}{3} & \frac{h_x}{6} \\ & & & & & \frac{h_x}{6} & \frac{h_x}{3} & \\ & & & & & & & \end{bmatrix}_{N_x \times N_x} \tag{4.56}$$

$$\tilde{D}(\phi_k, \phi_i) = \int_{-1}^1 (\phi_k)_x (\phi_i)_x dx \quad 1 \leq k \leq N_x, \quad 1 \leq i \leq N_x \tag{4.57}$$

$$\tilde{D} = \begin{bmatrix} \frac{1}{h_x} & -\frac{1}{h_x} & & & & \\ -\frac{1}{h_x} & \frac{2}{h_x} & -\frac{1}{h_x} & & & \mathbf{0} \\ & -\frac{1}{h_x} & \frac{2}{h_x} & -\frac{1}{h_x} & & \\ & & & \cdot & & \\ & & & & \cdot & \\ \mathbf{0} & & & & & -\frac{1}{h_x} & \frac{2}{h_x} & -\frac{1}{h_x} \\ & & & & & -\frac{1}{h_x} & \frac{1}{h_x} & \\ & & & & & & & \end{bmatrix}_{N_x \times N_x}$$

(4.58)

$$\tilde{\sigma}_I(\phi_k, \phi_i) = \{\phi_k \phi_i|_{x=-1}\} \quad 1 \leq k \leq N_x, \quad 1 \leq i \leq N_x \quad (4.59)$$

$$\tilde{\sigma}_I = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & & 0 & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix}_{N_x \times N_x}$$

(4.60)

$$\tilde{\sigma}_{II}(\phi_k, \phi_i) = \{\phi_k \phi_i|_{x=1}\} \quad 1 \leq k \leq N_x, \quad 1 \leq i \leq N_x \quad (4.61)$$

$$\tilde{\sigma}_{II} = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & & 0 & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 \end{bmatrix}_{N_x \times N_x}$$

(4.62)

$$\tilde{R}1(\phi_i) = \left\{ \int_{-1}^1 \phi_i dx \right\} \quad 1 \leq i \leq N_x \quad (4.63)$$

$$\tilde{R}1 = \begin{bmatrix} \frac{h_x}{2} \\ h_x \\ \cdot \\ \cdot \\ \cdot \\ h_x \\ \frac{h_x}{2} \end{bmatrix}_{N_x \times 1} \quad (4.64)$$

$$\tilde{R}2(\phi_i) = \left\{ \int_{-1}^1 (\phi_i)_x dx \right\} \quad 1 \leq i \leq N_x \quad (4.65)$$

$$\tilde{R}2 = \begin{bmatrix} -1 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 1 \end{bmatrix}_{N_x \times 1} \quad (4.66)$$

$$\tilde{P}(\phi_k, \phi_i) = \int_{-1}^1 (\phi_k)_x (\phi_i) dx \quad 1 \leq k \leq N_x, \quad 1 \leq i \leq N_x \quad (4.67)$$

$$\begin{aligned}
T_{m0}^I(j) &= \int_{t1_j^I}^{t2_j^I} \frac{(2 + \eta_I)^2}{2(1 + \eta_I)^4} d\eta_I \Rightarrow \\
T_{m0}^I(j) &= IT_{m0}^I(t2_j^I) - IT_{m0}^I(t1_j^I), \\
IT_{m0}^I(t) &= \frac{1}{2(t+1)^2} - \frac{1}{6(t+1)^3} - \frac{1}{2(t+1)}
\end{aligned} \tag{4.71}$$

$$\begin{aligned}
T_{m0}^{II}(j) &= \int_{t1_j^{II}}^{t2_j^{II}} \frac{(1 - \eta_{II})^2}{2} d\eta_{II} \Rightarrow \\
T_{m0}^{II}(j) &= IT_{m0}^{II}(t2_j^{II}) - IT_{m0}^{II}(t1_j^{II}) \\
IT_{m0}^{II}(t) &= -\frac{(2-t)^3}{6}
\end{aligned} \tag{4.72}$$

$$\begin{aligned}
T_{\sigma}^I(j) &= \int_{t1_j^I}^{t2_j^I} -\frac{\eta_I(2 + \eta_I)}{2(1 + \eta_I)^4} d\eta_I \Rightarrow \\
T_{\sigma}^I(j) &= IT_{\sigma}^I(t2_j^I) - IT_{\sigma}^I(t1_j^I), \\
IT_{\sigma}^I(t) &= -\frac{1}{6(t+1)^3} + \frac{1}{2(t+1)}
\end{aligned} \tag{4.73}$$

$$\begin{aligned}
T_{\sigma}^{II}(j) &= \int_{t1_j^{II}}^{t2_j^{II}} \frac{(2 - \eta_{II})\eta_{II}}{2} d\eta_{II} \Rightarrow \\
T_{\sigma}^{II}(j) &= IT_{\sigma}^{II}(t2_j^{II}) - IT_{\sigma}^{II}(t1_j^{II}), \\
IT_{\sigma}^{II}(t) &= -\frac{t^3}{6} + \frac{t^2}{2}
\end{aligned} \tag{4.74}$$

$$\begin{aligned}
C^I(j) &= \int_{t1_j^I}^{t2_j^I} \left\{ \frac{e^{-\frac{(2+\eta'_I)^2}{2\eta_I'^2}}}{\pi^{\frac{1}{4}}(1+\eta'_I)^2} \right\} d\eta'_I \Rightarrow \\
\eta'_I &= \frac{2t - t1_j^I - t2_j^I}{t2_j^I - t1_j^I} \\
C^I(j) &= \int_{-1}^1 \frac{e^{-0.5\left(\frac{2t+t1_j^I-3t2_j^I}{2t-t2_j^I-t1_j^I}\right)^2}}{\pi^{\frac{1}{4}}} \left\{ \frac{(t2_j^I - t1_j^I)^2}{4(t - t1_j^I)^2} \right\} \frac{2}{(t2_j^I - t1_j^I)} dt \\
C^I(j) &= \int_{-1}^1 \frac{(t2_j^I - t1_j^I) e^{-0.5\left(\frac{2t-t1_j^I-t2_j^I}{3t2_j^I-t1_j^I-2t}\right)^2}}{2\pi^{\frac{1}{4}}(t - t1_j^I)^2} dt \\
C^I(j) &= \int_{-1}^1 IC^I(t, t1_j^I, t2_j^I) dt \\
IC^I(t, t1_j^I, t2_j^I) &= \frac{(t2_j^I - t1_j^I) e^{-0.5\left(\frac{2t-t1_j^I-t2_j^I}{3t2_j^I-t1_j^I-2t}\right)^2}}{2\pi^{\frac{1}{4}}(t - t1_j^I)^2} \\
(C^*)^I(j) &= IC^I\left(-\frac{1}{\sqrt{3}}, t1_j^I, t2_j^I\right) + IC^I\left(\frac{1}{\sqrt{3}}, t1_j^I, t2_j^I\right) \tag{4.75}
\end{aligned}$$

In (4.75) we calculate $(C^*)^I(j)$ using 2 point gauss quadrature to approximate $C^I(j)$

$$\begin{aligned}
C^{II}(j) &= \int_{t1_j^{II}}^{t2_j^{II}} \left\{ \frac{e^{-\frac{(\eta''_{II})^2}{2(2-\eta''_{II})^2}}}{\pi^{\frac{1}{4}}} \right\} d\eta''_{II} \Rightarrow \\
\eta''_{II} &= \frac{2t - t1_j^{II} - t2_j^{II}}{t2_j^{II} - t1_j^{II}} \\
C^{II}(j) &= \int_{-1}^1 \frac{e^{-0.5\left(\frac{2t-t1_j^{II}-t2_j^{II}}{3t2_j^{II}-t1_j^{II}-2t}\right)^2}}{\pi^{\frac{1}{4}}} \frac{2}{(t2_j^{II} - t1_j^{II})} dt \\
C^{II}(j) &= \int_{-1}^1 \frac{2e^{-0.5\left(\frac{2t-t1_j^{II}-t2_j^{II}}{3t2_j^{II}-t1_j^{II}-2t}\right)^2}}{\pi^{\frac{1}{4}}(t2_j^{II} - t1_j^{II})} dt \\
C^{II}(j) &= \int_{-1}^1 IC^{II}(t, t1_j^{II}, t2_j^{II}) dt \\
IC^{II}(t, t1_j^{II}, t2_j^{II}) &= \frac{2e^{-0.5\left(\frac{2t-t1_j^{II}-t2_j^{II}}{3t2_j^{II}-t1_j^{II}-2t}\right)^2}}{\pi^{\frac{1}{4}}(t2_j^{II} - t1_j^{II})} \\
(C^*)^{II}(j) &= IC^{II}\left(-\frac{1}{\sqrt{3}}, t1_j^{II}, t2_j^{II}\right) + IC^{II}\left(\frac{1}{\sqrt{3}}, t1_j^{II}, t2_j^{II}\right) \quad (4.76)
\end{aligned}$$

In (4.75) we calculate $(C^*)^{II}(j)$ using 2 point gauss quadrature to approximate $C^{II}(j)$

$$\begin{aligned}
D_b^I(j) &= \int_{t1_j^I}^{t2_j^I} \left\{ \frac{e^{-\frac{(2+\eta'_I)^2}{2\eta'^2_I}}}{\pi^{\frac{1}{4}}(1+\eta'_I)^2} \right\} \left\{ \frac{(2+\eta'_I)}{2\eta'_I} \right\} d\eta'_I \Rightarrow \\
\eta'_I &= \frac{2t - t1_j^I - t2_j^I}{t2_j^I - t1_j^I} \\
D_b^I(j) &= \int_{-1}^1 \frac{e^{-0.5\left(\frac{2t+t1_j^I-3t2_j^I}{2t-t2_j^I-t1_j^I}\right)^2}}{\pi^{\frac{1}{4}}} \left\{ \frac{(t2_j^I - t1_j^I)^2}{4(t - t1_j^I)^2} \right\} \frac{2}{(t2_j^I - t1_j^I)} \left\{ \frac{2t + t1_j^I - 3t2_j^I}{2t - t2_j^I - t1_j^I} \right\} dt \\
D_b^I(j) &= \int_{-1}^1 \frac{(t2_j^I - t1_j^I) e^{-0.5\left(\frac{2t-t1_j^I-t2_j^I}{3t2_j^I-t1_j^I-2t}\right)^2}}{2\pi^{\frac{1}{4}}(t - t1_j^I)^2} \left\{ \frac{2t + t1_j^I - 3t2_j^I}{2t - t2_j^I - t1_j^I} \right\} dt \\
D_b^I(j) &= \int_{-1}^1 ID^I(t, t1_j^I, t2_j^I) dt \\
ID^I(t, t1_j^I, t2_j^I) &= \frac{(t2_j^I - t1_j^I) e^{-0.5\left(\frac{2t-t1_j^I-t2_j^I}{3t2_j^I-t1_j^I-2t}\right)^2}}{2\pi^{\frac{1}{4}}(t - t1_j^I)^2} \left\{ \frac{2t + t1_j^I - 3t2_j^I}{2t - t2_j^I - t1_j^I} \right\} \\
(D_b^*)^I(j) &= ID^I\left(-\frac{1}{\sqrt{3}}, t1_j^I, t2_j^I\right) + ID^I\left(\frac{1}{\sqrt{3}}, t1_j^I, t2_j^I\right) \tag{4.77}
\end{aligned}$$

In (4.77) we calculate $(D_b^*)^I(j)$ using 2 point gauss quadrature to approximate $D_b^I(j)$

$$\begin{aligned}
D_b^{II}(j) &= \int_{t1_j^{II}}^{t2_j^{II}} \left\{ \frac{e^{-\frac{(\eta_{II}'')^2}{2(2-\eta_{II}'')}}}{\pi^{\frac{1}{4}}} \right\} \left\{ \frac{(\eta_{II}'')}{(2-\eta_{II}'')} \right\} d\eta_{II}'' \Rightarrow \\
\eta_{II}'' &= \frac{2t - t1_j^{II} - t2_j^{II}}{t2_j^{II} - t1_j^{II}} \\
D_b^{II}(j) &= \int_{-1}^1 \frac{e^{-0.5\left(\frac{2t-t1_j^{II}-t2_j^{II}}{3t2_j^{II}-t1_j^{II}-2t}\right)^2}}{\pi^{\frac{1}{4}}} \left\{ \frac{2}{(t2_j^{II} - t1_j^{II})} \right\} \left\{ \frac{2t - t1_j^{II} - t2_j^{II}}{3t2_j^{II} - t1_j^{II} - 2t} \right\} dt \\
D_b^{II}(j) &= \int_{-1}^1 \frac{2e^{-0.5\left(\frac{2t-t1_j^{II}-t2_j^{II}}{3t2_j^{II}-t1_j^{II}-2t}\right)^2}}{\pi^{\frac{1}{4}}(t2_j^{II} - t1_j^{II})} \left\{ \frac{2t - t1_j^{II} - t2_j^{II}}{3t2_j^{II} - t1_j^{II} - 2t} \right\} dt \\
D_b^{II}(j) &= \int_{-1}^1 ID^{II}(t, t1_j^{II}, t2_j^{II}) dt \\
ID^{II}(t, t1_j^{II}, t2_j^{II}) &= \frac{2e^{-0.5\left(\frac{2t-t1_j^{II}-t2_j^{II}}{3t2_j^{II}-t1_j^{II}-2t}\right)^2}}{\pi^{\frac{1}{4}}(t2_j^{II} - t1_j^{II})} \left\{ \frac{2t - t1_j^{II} - t2_j^{II}}{3t2_j^{II} - t1_j^{II} - 2t} \right\} \\
(D_b^*)^{II}(j) &= ID^{II}\left(-\frac{1}{\sqrt{3}}, t1_j^{II}, t2_j^{II}\right) + ID^{II}\left(\frac{1}{\sqrt{3}}, t1_j^{II}, t2_j^{II}\right) \tag{4.78}
\end{aligned}$$

In (4.78) we calculate $(D_b^*)^{II}(j)$ using 2 point gauss quadrature to approximate $D_b^{II}(j)$

4.5 Stamping procedure used to general global quantities

The stamping procedure is shown below

```
Nx=2/hx + 1; %total number of nodes in each strip.
```

```
Ny=2/hy; %number of strips in each sub-domain.
```

```
% The generation of D_bar is computed by superimposing the
contributions of D_I(w1,v1) and D_II(w2,v2).
```

```
function[I1]= N_I(i,j)
```



```

    for k=1:Nx
        for l=1:Ny
            I1=N_I(i,j);%generation of global node numbers
            J1=N_I(k,l);
            I2=N_II(i,j);
            J2=N_II(k,l);
            if(j==1)
                M0_bar(I1,J1)=M0_bar(I1,J1)+ M_til(i,k)*TI_m0(j);
                M0_bar(I2,J2)=M0_bar(I2,J2)+ M_til(i,k)*TII_m0(j);
            end
        end
    end
end
end
end
end
% The generation of Sg_bar is computed by superimposing the
% contributions of Sigma(w1,v1)
%and Sigma(w2,v2)
Sg_bar = sparse(2*Nx*Ny,2*Nx*Ny);
for i=1:Nx
    for j=1:Ny
        for k=1:Nx
            for l=1:Ny
                I1=N_I(i,j);%generation of global node numbers
                J1=N_I(k,l);
                I2=N_II(i,j);
                J2=N_II(k,l);
                if(j==1)
                    Sg_bar(I1,J1)=Sg_bar(I1,J1)+ SgI_til(i,k)*TI_Sg(j);
                end
            end
        end
    end
end

```



```

%b_I_II(w1,v2),b_II_I(w2,v1),b_II_II(w2,v2),
b_I_I(v1,w1), b_I_II(v2,w1),b_II_I(v1,w2)
%and b_II_II(v2,w2)
B_bar= sparse(2*Nx*Ny,2*Nx*Ny);
for i=1:Nx
    for j=1:Ny
        for k=1:Nx
            for l=1:Ny
                I1=N_I(i,j);%generation of global node numbers
                J1=N_I(k,l);
                I2=N_II(i,j);
                J2=N_II(k,l);
                B_bar(I1,J1)=B_bar(I1,J1)+ P_til(i,k)*C_I(j)*Db_I(l);
                B_bar(I1,J2)=B_bar(I1,J2)+ P_til(i,k)*C_I(j)*Db_II(l);
                B_bar(I2,J1)=B_bar(I2,J1)+ P_til(i,k)*C_II(j)*Db_I(l);
                B_bar(I2,J2)=B_bar(I2,J2)+ P_til(i,k)*C_II(j)*Db_II(l);
                B_bar(J1,I1)=B_bar(J1,I1)+ P_til(k,i)*C_I(l)*Db_I(j);
                B_bar(J2,I1)=B_bar(J2,I1)+ P_til(k,i)*C_I(l)*Db_II(j);
                B_bar(J1,I2)=B_bar(J1,I2)+ P_til(k,i)*C_II(l)*Db_I(j);
                B_bar(J2,I2)=B_bar(J2,I2)+ P_til(k,i)*C_II(l)*Db_II(j);
            end
        end
    end
end
end
% The generation of RHS
F_check=zeros(2*NX*Ny,1);
for i=1:Nx
    for j=1:Ny

```

```

I1=N_I(i,j);
I2=N_II(i,j);
F_check(I1,1)=F_hat(I1,1)+2f*R1_til(i)*C_I(j)+
              theta*f*R2_til(i)*Db_I(j);
F_check(I2,1)=F_hat(I2,1)+2f*R1_til(i)*C_II(j)+
              theta*f*R2_til(i)*Db_II(j);
end
end

% Computation the stiffness matrix
A = sparse(2*Nx*Ny,2*Nx*Ny);
A = theta*D_bar+(1/theta)*(M0_bar-M1_bar)+Sg_bar+B_bar ;
% Solve for U=(U_I;U_{II})
U = A\F_check;

```

4.6 Computation of output

The output can be computed in terms of the solution U as

$$\begin{aligned}
S^e(\theta) &= \frac{1}{2}a((U_I, U_{II})^e, (U_I, U_{II})^e; \theta) - \frac{\theta}{4} \\
S^e(\theta) &= \frac{1}{2}U^T A U - \frac{\theta}{4}.
\end{aligned} \tag{4.79}$$

Chapter 5

Finite Element Results

5.1 Result showing the weak imposition of the boundary conditions

The results in Figure 5-1 and Figure 5-2 are obtained by setting $hx = 0.0714$ and $hy = 0.05$ in the Finite element code, when $\theta = 0.1905$. The dependent variable U is plotted on Y-axis at $x = \pm 1$, where as the transformed velocity is plotted on X-axis. From the graph we can clearly tell that the Dirichlet boundary conditions are imposed weakly, in the Neumann sense. From Figure 5-1, we can see that $U(x = -1, y) \neq 0$ for all velocities, $y > 0$, due to the imposition of this boundary condition. For the first few points when $y > 0$, though $U(x = -1, y) \neq 0$, it quickly decays to zero as the value of y starts to increase in magnitude. This is a definite indicator of the weak imposition of the boundary conditions. Similar argument holds for the other boundary condition $U(x = +1, y) = 0$ for all $y < 0$.

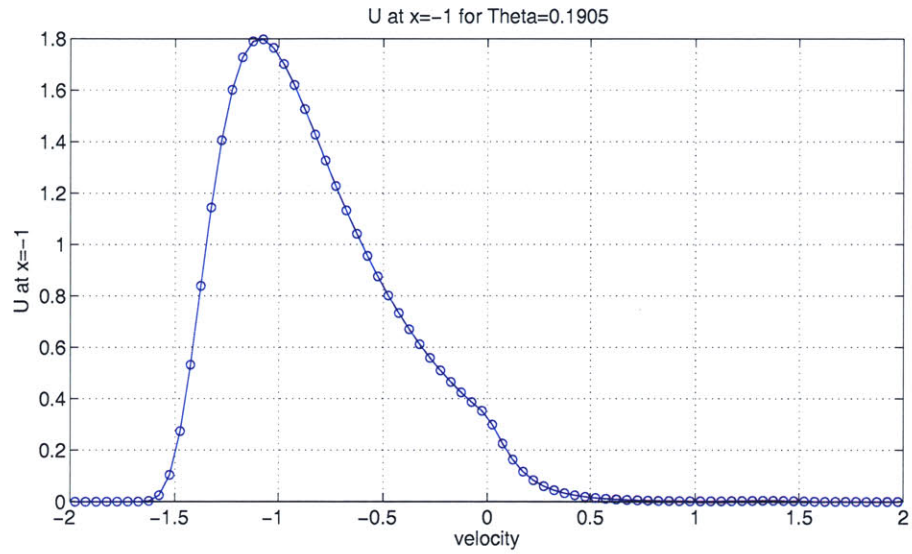


Figure 5-1: The evidence of weak imposition of dirichlet conditions at $x=-1$

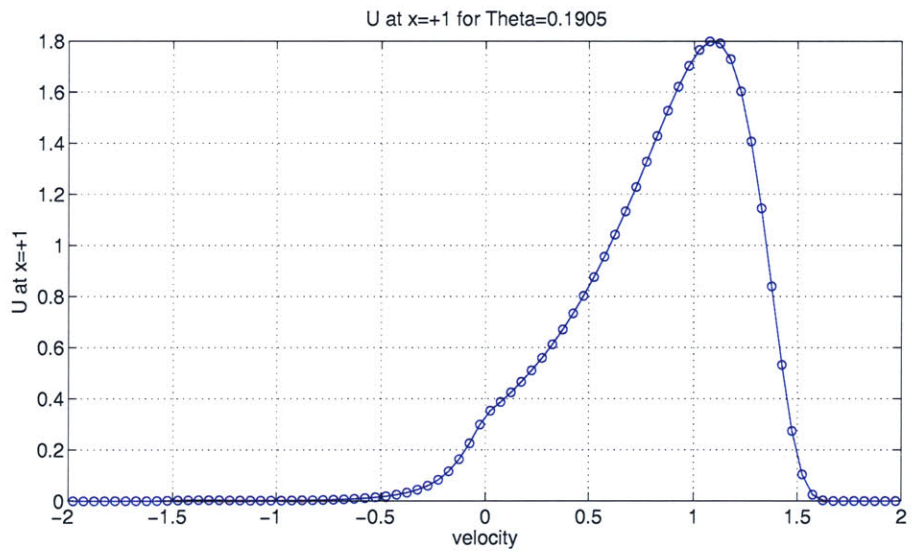


Figure 5-2: The evidence of weak imposition of dirichlet conditions at $x=+1$

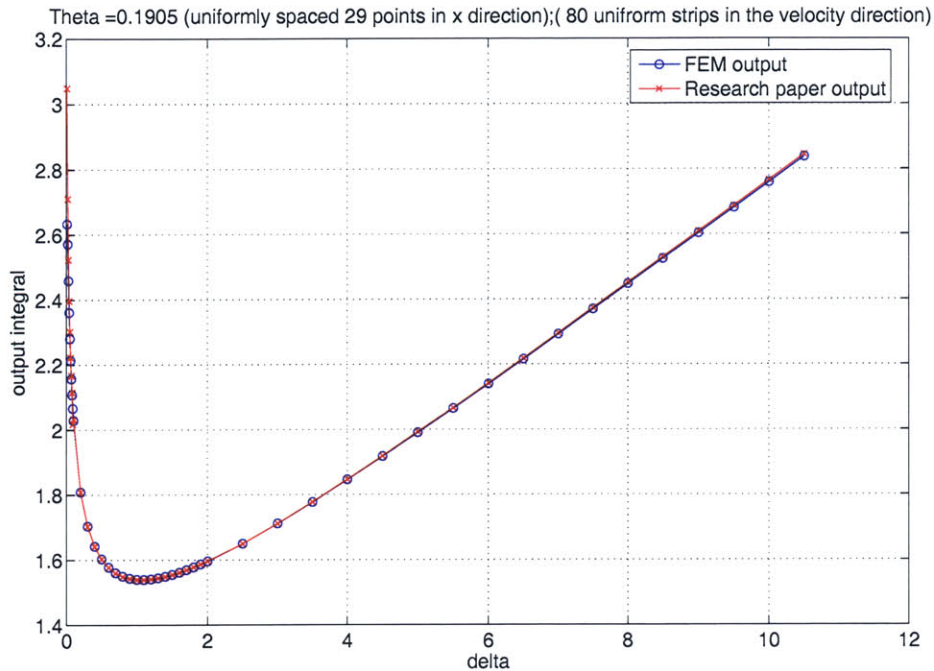


Figure 5-3: Comparison of the FEM output results with the output results published in the paper [1]

5.2 Comparison of the output results with the ones in the literature

The results in Figure 5-3 are obtained by setting $hx = 0.0714$ and $hy = 0.05$ in the Finite element code for the theta values ranging from 0.1905 to 200, as given in the research paper published by Daneri and Cercigiani. In this plot, we compare our FEM output results with the output values published in the research paper. As we can see, the results seem to be in good agreement.

5.3 Convergence results of the output

The results in Figure 5-4 and Figure 5-5 can be plotted in order to verify the convergence rates by comparing with the expected convergence rates. In the Figure 5-4, we plot the

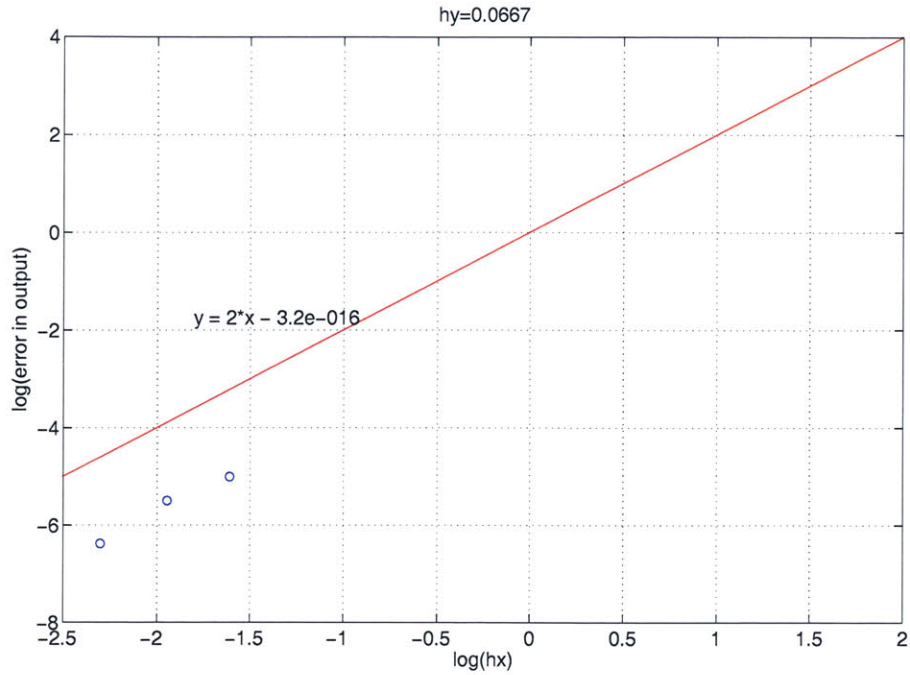


Figure 5-4: Convergence in the output with respect to h_x : The 'o' are data points from the results, where as the line is a reference line of slope 2.

logarithm of absolute error in the output with respect to a very fine mesh ($h_x = 0.05, h_y = 0.0667$) against $\log(h_x)$ by fixing $h_y = 0.0667$ and varying h_x as $h_x = 0.1, 0.1333, 0.2$. In the Figure 5-5, we plot the logarithm of absolute error in the output with respect to a very fine mesh, ($h_x = 0.04, h_y = 0.0667$) against $\log(h_y)$ by fixing $h_x = 0.04$ and varying h_y as $h_y = 0.08, 0.1, 0.1333$. From both Figure 5-4 and Figure 5-5, we can tell that the numerical convergence rate of output agrees with the theoretical expectation with respect to h_x and is $O(h_x)^2$. Where as, we get super-convergence with respect h_y , i.e. $O(h_x)^p$, where $p > 2$ and not $O(h_y)^2$. It is not very clear, why the we get super-convergence with respect to h_y .

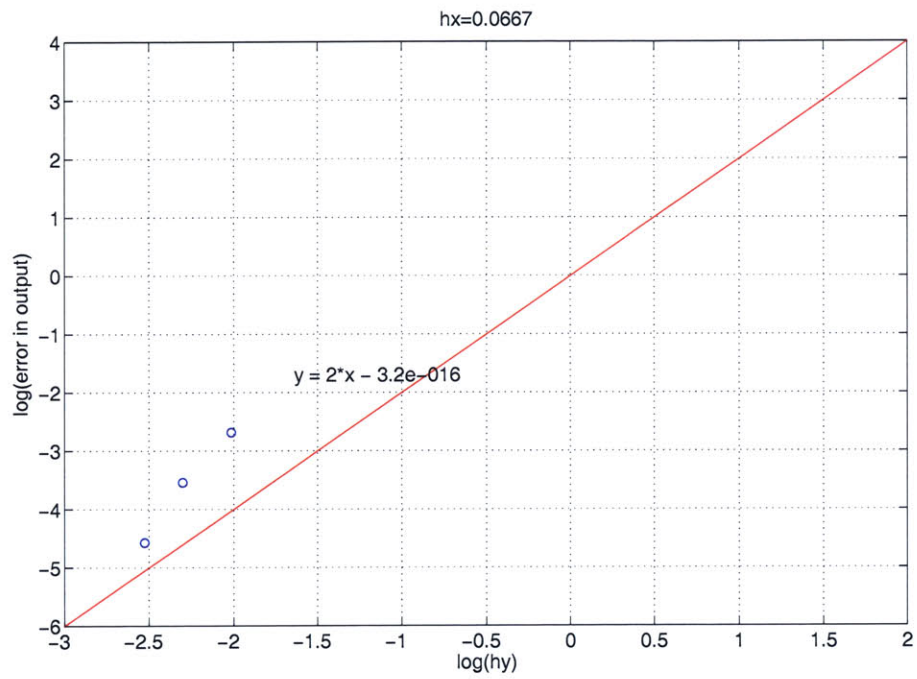


Figure 5-5: Convergence in the output with respect to h_y : The 'o' are data points from the results, whereas the line is a reference line of slope 2.

Chapter 6

Reduced Basis Methods for Boltzmann equation

6.1 Reduced Basis estimation of output

6.1.1 Parameter Grids

It shall be convenient to introduce parameter grids that shall serve us subsequently in several contexts. We first define as $\mathcal{I} \equiv [\theta^{\min}, \theta^{\max}]$, where $\theta^{\min} = 0.1905$ and $\theta^{\max} = 200$. We can now express our parameter domain as $\mathcal{D}_p \equiv \mathcal{I}$.

We then consider standard (“linear”) grids. Towards that end, we introduce the set of M' equi-spaced points between z^{\min} and z^{\max} ,

$$\begin{aligned} G_{M'}^{\text{lin}}[z^{\min}, z^{\max}] &= \{z_{M'}^1, z_{M'}^2, \dots, z_{M'}^{M'}\} \\ z_{M'}^j &= z^{\min} + \frac{j-1}{M'-1} (z^{\max} - z^{\min}), \quad 1 \leq j \leq M'. \end{aligned} \tag{6.1}$$

We then define the grid over \mathcal{D}_p , $\Xi_{M'}^{\text{lin}} \subset \mathcal{D}_p \subset \mathbb{R}^1$, as

$$\Xi_{M'}^{\text{lin}} = G_{M'}^{\text{lin}}[\theta^{\min}, \theta^{\max}] \tag{6.2}$$

Note that there are M' points in $\Xi_{M'}^{\text{lin}}$. We now consider logarithmic grids. We first introduce $G_{M'}^{\text{log}}[z^{\text{min}}, z^{\text{max}}]$,

$$\begin{aligned} G_{M'}^{\text{log}}[z^{\text{min}}, z^{\text{max}}] &= \{z_{M'}^1, z_{M'}^2, \dots, z_{M'}^{M'}\} \\ \ln z_{M'}^j &= \ln z^{\text{min}} + \frac{j-1}{M'-1} \ln\left(\frac{z^{\text{max}}}{z^{\text{min}}}\right), \quad 1 \leq j \leq M'. \end{aligned} \tag{6.3}$$

This grid is equi-spaced in “log,” which is often advantageous within the reduced basis approximation context; more generally, the “log” spacing represents equal relative increments, and thus represents better coverage for parameters that vary over a large range. We can then define grids over \mathcal{D}_p , $\Xi_{M'}^{\text{log}} \subset \mathcal{D}_p \subset \mathbb{R}^1$, as

$$\Xi_{M'}^{\text{log}} = G_{M'}^{\text{log}}[\theta^{\text{min}}, \theta^{\text{max}}] \tag{6.4}$$

note $\Xi_{M'}^{\text{log}}$ contains M' points. We also define a particular test grid (biased neither towards “log” nor “lin”).

$$\Xi^{\text{test}} = \Xi_{M'=200}^{\text{lin}} \cup \Xi_{M'=200}^{\text{log}} ; \tag{6.5}$$

Note that Ξ^{test} contains $2M' - 2$ points.

6.1.2 Projection

We are interested only in solutions that reside on the *low-dimensional* and *smooth* parametrically induced manifold, $\mathcal{M} \equiv \{(u_I, u_{II})(\theta) \mid \theta \in \mathcal{D}_p\}$. Hence we can hope to achieve significant dimension reduction, and ultimately significant computational economies, if we focus our approximation space “around” \mathcal{M} . (Our finite element space, X , even if adaptively generated, is unnecessarily general — X can well approximate many functions *not* on our manifold of interest, \mathcal{M} — and hence unnecessarily large and ultimately unnecessarily expensive for purposes of prediction of $(u_I, u_{II})(\theta)$ and $s(\theta)$.) In particular, we can hope that any element of the manifold \mathcal{M} , $(u_I, u_{II})(\theta)$, can be well represented by some very small number N of pre-computed elements of \mathcal{M} , $(u_I, u_{II})(\theta^1), \dots, (u_I, u_{II})(\theta^n)$; the $(u_I, u_{II})(\theta^n)$,

$1 \leq n \leq N$, are often referred to as “snapshots.” We first introduce, for given positive integer N , a parameter sample $S_N \equiv \{\theta^1, \theta^2, \dots, \theta^N\}$. We then define the reduced basis space W_N (of dimension N) as the span of the snapshots,

$$W_N \equiv \text{span}\{(u_I, u_{II})(\theta^n), 1 \leq n \leq N\} . \quad (6.6)$$

The reduced basis approximation $(u_I, u_{II})_N(\theta) \in W_N \subset X$ is then given by simple Galerkin projection,

$$a((u_I, u_{II})_N(\theta), (v_I, v_{II}); \theta) = \tilde{f}((v_I, v_{II})), \quad \forall (v_I, v_{II}) \in W_N , \quad (6.7)$$

We may then evaluate our approximation to the output as

$$s_N(\theta) = \frac{1}{2}a((u_I, u_{II})_N(\theta), (u_I, u_{II})_N(\theta)); \theta) - \frac{\theta}{4} \quad (6.8)$$

6.1.3 Orthonormal Basis

For the reduced basis space W_N , the conditioning of the reduced basis stiffness matrix — with basis set $\{(u_I, u_{II})(\theta^n)\}_{n=1, \dots, N}$ — increases (and must increase) quite rapidly as N increases and the reduced basis error decreases. This can, in fact, limit the attainable accuracy. It is thus of interest to choose a different set of basis functions for our space W_N . In fact, as we shall see, a simple (appropriate) orthogonalization suffices: we shall pursue standard Gram-Schmidt orthonormalization with respect to the (\cdot, \cdot) inner product of (3.16).

Our new basis set for W_N shall be denoted $\{\zeta_i\}_{i=1, \dots, N}$. We first set

$$\zeta_1 = (u_I, u_{II})(\theta^1) / \|(u_I, u_{II})(\theta^1)\| . \quad (6.9)$$

Then, for $i = 2, \dots, N$,

$$\begin{aligned} z_i &= (u_I, u_{II})(\theta^i) - \sum_{j=1}^{i-1} (\zeta_j, (u_I, u_{II})(\theta^i)) \zeta_j, \\ \zeta_i &= z_i / \|z_i\|. \end{aligned} \quad (6.10)$$

(In practice, (6.10) can be replaced with full orthogonalization.) It is clear that, through this construction, we ensure $(\zeta_i, \zeta_j) = \delta_{ij}$, $1 \leq i, j \leq N$. Note that we must still assume that $\dim(W_N) = N$, as otherwise the orthogonalization will break down.

We now express $(u_I, u_{II})_N(\theta)$ as

$$(u_I, u_{II})_N(\theta) = \sum_{n=1}^N \beta_{Nn}(\theta) \zeta(\theta^n); \quad (6.11)$$

the $\beta_{Nn}(\theta)$, $1 \leq n \leq N$, are the coefficients of our orthonormal basis functions ζ_n , $1 \leq n \leq N$. It then follows that the $\beta_{Nn}(\theta)$, $1 \leq n \leq N$, satisfy the algebraic equation

$$\sum_{j=1}^N a(\zeta_j, \zeta_i; \theta) \beta_{Nj}(\theta) = \tilde{f}(\zeta_i), \quad 1 \leq i \leq N; \quad (6.12)$$

the reduced basis output can then be evaluated as

$$s_N(\theta) = \frac{1}{2} a((u_I, u_{II})(\theta), (u_I, u_{II})(\theta); \theta) - \frac{\theta}{4} \quad (6.13)$$

6.1.4 Offline-Online Computational procedure

We now express $(u_I, u_{II})_N(\theta)$ as

$$(u_I, u_{II})_N(\theta) = \sum_{j=1}^N \beta_{Nn}(\theta) \zeta(\theta^n); \quad (6.14)$$

the $\beta_{Nn}(\theta)$, $1 \leq n \leq N$, are the coefficients of our orthonormalized pre-computed solutions/snapshots $\zeta(\theta^n)$, $1 \leq n \leq N$, in terms of which we define W_N and which serve as our

basis functions. It then follows that the $\beta_{Nn}(\theta)$, $1 \leq n \leq N$, satisfy the algebraic equation

$$\sum_{j=1}^N a(\zeta(\theta^j), \zeta(\theta^i); \theta) \beta_{Nj}(\theta) = \tilde{f}(\zeta(\theta^i)), \quad 1 \leq i \leq N ; \quad (6.15)$$

Here a involves the \mathcal{N} -dimensional pre-computed (finite element “truth”) solutions, and hence the operation count for the formation of the stiffness matrix (and right-hand side) in (6.15) will ostensibly scale with \mathcal{N} . However, in the offline-online context, we can “localize” this \mathcal{N} -dependence to just the offline stage. From (3.3), we can rewrite the term $a(\zeta(\theta^j), \zeta(\theta^i); \theta)$ of (6.15) as shown in (6.16)

$$\begin{aligned} a(\zeta(\theta^j), \zeta(\theta^i); \theta) &= \theta d(\zeta(\theta^j), \zeta(\theta^i); \theta) + \\ &\quad \frac{1}{\theta} m_0(\zeta(\theta^j), \zeta(\theta^i); \theta) - \\ &\quad \frac{1}{\theta} m_1(\zeta(\theta^j), \zeta(\theta^i); \theta) + \\ &\quad \sigma(\zeta(\theta^j), \zeta(\theta^i); \theta) + \\ &\quad b(\zeta(\theta^j), \zeta(\theta^i); \theta) \end{aligned} \quad (6.16)$$

Note that, in general, the reduced basis stiffness matrix $a(\zeta(\theta^j), \zeta(\theta^i); \theta)$, $1 \leq i, j \leq N$, will not be sparse. In (6.17) We define a matrix Z which has the orthonormalized solutions $\zeta(\theta^i)$, $1 \leq i \leq N$ as its columns.

$$Z = [\zeta(\theta^1), \dots, \zeta(\theta^N)]_{\mathcal{N} \times N} \quad (6.17)$$

From (6.15), we can see that the reduced stiffness matrix in terms of the stiffness matrix and Z as shown in (6.18)

$$A_N(\theta) = Z^T A(\theta) Z \quad (6.18)$$

Using (6.16) and (6.18), we can express the reduced stiffness matrix in terms of the stiffness matrix contribution of each term in the weak form.

$$\begin{aligned}
A_N(\theta) &= \theta D_N + \frac{1}{\theta} M0_N - \frac{1}{\theta} M1_N + \hat{\sigma}_N + B_N \\
\text{where } D_N &= Z^T D Z \\
M0_N &= Z^T M0 Z \\
M1_N &= Z^T M1 Z \\
\hat{\sigma}_N &= Z^T \hat{\sigma} Z \\
B_N &= Z^T B Z
\end{aligned} \tag{6.19}$$

$$\tilde{F}_N(\theta) = Z^T F_I^1 + \theta Z^T F_I^2 + Z^T F_{II}^1 + \theta Z^T F_{II}^2 \tag{6.20}$$

$$A_N(\theta) U_N(\theta) = \tilde{F}_N(\theta) \tag{6.21}$$

The reduced basis output can then be evaluated as

$$s_N(\theta) = \frac{1}{2} a((u_I, u_{II})(\theta), (u_I, u_{II})(\theta); \theta) - \frac{\theta}{4} \tag{6.22}$$

$$s_N(\theta) = \frac{1}{2} U_N(\theta)^T A_N(\theta) U_N(\theta) - \frac{\theta}{4} \tag{6.23}$$

6.1.5 Offline-Online Computational strategy

Here we present the offline-online computational scheme

- Offline

1. Choose N
2. Choose sample S_N

3. Construct orthonormal basis Z
4. Construct $D_N, M0_N, M1_N, \hat{\sigma}_N, B_N, Z^T F_I^1, Z^T F_I^2, Z^T F_{II}^1$ and $Z^T F_{II}^2$.

- Online

1. Form $A_N(\theta), \tilde{F}_N(\theta)$
2. Solve $A_N(\theta)U_N(\theta) = \tilde{F}_N(\theta)$
3. Evaluate the output $S_N(\theta)$

6.1.6 Online operation count for output evaluation

From the computational strategy, we can see that, the formation of the reduced basis stiffness matrix from the reduced matrices requires $5N^2$ multiplications and $4N^2$ additions. Where as, the formation of the reduced Right hand side requires $2N$ multiplications and $3N$ additions . Then, the solution of the reduced basis coefficients , requires $\frac{2}{3}N^3$ operations, assuming that we use gauss elimination . The evaluation of the output requires $N + 1$ multiplications, $N - 1$ additions, 1subtraction, 1 division. So, the total online operation count is $\frac{2}{3}N^3 + 9N^2 + 7N + 2$

6.2 Greedy Algorithm

6.2.1 Motivation

For tensor product parameter samples/reduced basis spaces, our *field* dimension reduction from \mathcal{N} to N will often be eroded as the *parameter* dimension P increases. There in fact is no way in which we can completely eliminate the dependence of N on P (for some fixed error tolerance) for our class of problems. However, fortunately, there is an “adaptive” fashion by which to rationally create non-tensor-product parameter samples/reduced basis spaces; and, even more fortunately, it appears (empirically) that, even for larger P , these unstructured spaces provide very rapid convergence in N — more rapid convergence than tensor-product samples which, in effect, ignore the underlying parametric manifold \mathcal{M} .

The ability of reduced basis methods to converge exponentially on these unstructured/sparse samples/spaces is yet another reason that reduced basis approximation is often preferable to

“connecting the dots” and directly approximating $s: \mathcal{D}_p \rightarrow \mathbb{R}$; the latter requires scattered data approximations that, in particular in higher dimensions P , are difficult to construct and often perform/converge quite poorly. In some sense, we would like to choose our “snapshots” — the $(u_I, u_{II})(\theta^n)$, $1 \leq n \leq N$ — to be maximally different (a concept that can be readily though not uniquely articulated mathematically). There are two issues that must be addressed in order to transform this idea into a practical algorithm. The first issue is that it may be very difficult to find N maximally different functions on \mathcal{M} : the problem is combinatorial in nature, and hence very (very) expensive. To address this issue we settle for a “greedy” algorithm that sequentially finds a “*next* best — maximally different — snapshot”; although clearly suboptimal, greedy algorithms often perform quite well — and in some cases can be shown to roughly preserve the good convergence properties of the optimal construction. The second issue is that it may be quite expensive to evaluate the “difference” in order to choose the next snapshot: greedy algorithms require a training sample (also known as a “dictionary”) of candidates from which to choose the next snapshot; for larger P , this dictionary can or at least should be very large, leading to prohibitively large offline computational expense. In future work, the later should be addressed with a-posteriori error bound.

6.2.2 Algorithm

With this preamble, we can now present the algorithm. We will require a large training sample Ξ^{train} (for example, $\Xi^{\text{train}} = \Xi_{M'=40}^{\log}$, or perhaps a Monte Carlo sample) and a desired/prescribed minimum error tolerance $\varepsilon_{\text{tol},\text{min}}$. Then, given a sample S_N , associated reduced basis space W_N (and orthonormal basis set $\{\zeta_i\}_{i=1,\dots,N}$), we choose θ^{N+1} — which is then appended to S_N to form S_{N+1} and hence W_{N+1} — as that parameter value in Ξ^{train} that maximizes our error $e_N^s(\theta)$, defined as

$$e_N^s(\theta) = s(\theta) - s_N(\theta), \tag{6.24}$$

from which We conclude the process at $N = N_{\max}$ for which the maximum error $e_N^s(\theta)$ over Ξ^{train} is equal to the desired minimum tolerance, $\varepsilon_{\text{tol},\min}$.

Mathematically, we first choose (in some perhaps arbitrary fashion) θ^1 and hence S_1 and W_1 and ζ_1 ; we also compute $\hat{\varepsilon}_1 = \max_{\Xi^{\text{train}}} \Delta_1^s(\theta)$. Then

- G1. While $\hat{\varepsilon}_N > \varepsilon_{\text{tol},\min}$:

- G2. $\theta^{N+1} = \arg \max_{\Xi^{\text{train}}} e_N^s(\theta)$;

- G3. $\hat{\varepsilon}_{N+1} = e_N^s(\theta^{N+1})$;

- G4. $N_{\max} = N + 1$;

- G5. $N \leftarrow N + 1$.

There are many steps implicit in this simple loop. In particular, after G2., we must update $S_{N+1} = S_N + \theta_{N+1}$, calculate $(u_I, u_{II})(\theta^{N+1})$ to “form” W_{N+1} , construct the new contribution to our orthonormal basis set, ζ_{N+1} , and — in anticipation of G2. in the next pass through the loop — calculate all the necessary “ S_{N+1} ” online quantities for both our reduced basis prediction and associated error. We note here a practical point: as we proceed from N to $N + 1$, we should only compute the necessary *incremental* quantities — the incremental “arrowhead” contributions to the various online inner-product arrays required for the reduced basis prediction and error with respect to Finite element truth.

Related to this last point, we observe that the algorithm in fact generates not a single space ($S_{N_{\max}}$), but rather — at no additional offline cost — a whole sequence of nested samples ($S_1 \subset S_2 \subset \dots \subset S_{N_{\max}}$), nested spaces ($W_1 \subset W_2 \subset \dots \subset W_{N_{\max}}$), and nested basis sets. In the online stage, we may choose $N \in \{1, \dots, N_{\max}\}$ “on the fly” to match the desired accuracy requirements (e.g., in the parameter estimation context), and simply extract the necessary reduced basis online “inner products” (for the stiffness matrix, output evaluation, and the reduced basis error as subarrays of the stored (N_{\max}) quantities. We note that $\hat{\varepsilon}_N$ in fact provides us with a guideline for how we might choose N as a function of the desired error ε_{tol} ($\geq \varepsilon_{\text{tol},\min}$).

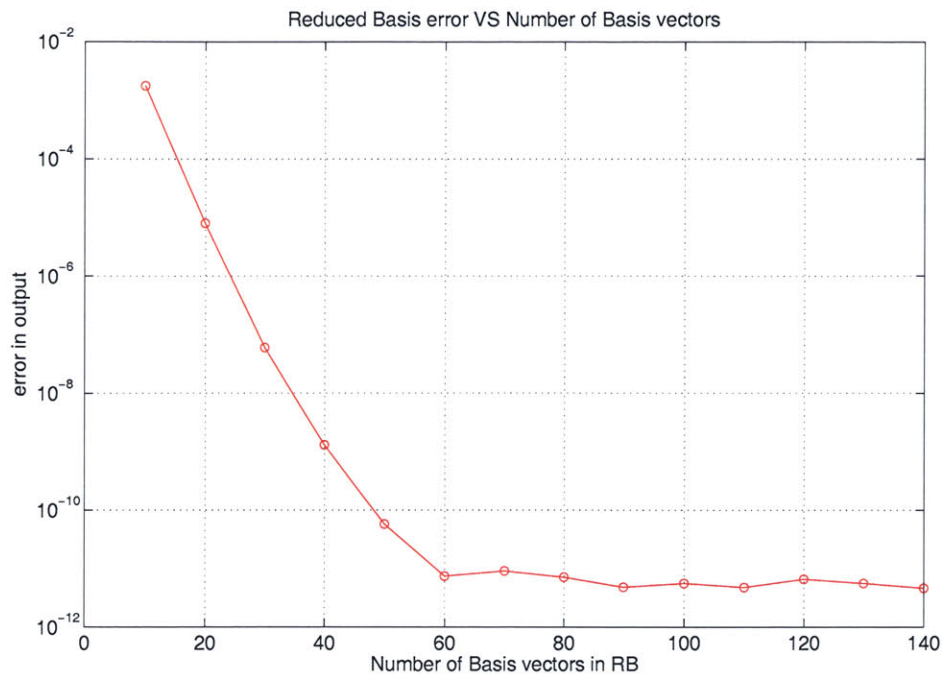


Figure 6-1: Convergence of Reduced Basis results with Classical Orthogonalized basis vectors

6.3 Reduced Basis Results

6.3.1 Convergence of Orthogonalized Reduced Basis

In the Figure 6-1, we plot the Convergence of the absolute maximum online error versus the number of orthogonal basis vectors chosen from the orthonormal sample space S_N . The parameter Sample spaces S_N are chosen such that $S_N = \Xi_{M'}^{\log}$ for $M' = 10, 20, \dots, 140$. Where as the online test space is defined as $\Xi^{\text{test}} = \Xi_{M'=200}^{\text{lin}} \cup \Xi_{M'=200}^{\log}$

6.3.2 Convergence of Greedy Algorithm-based Reduced Basis

In the Figure 6-2, we plot the Convergence of the absolute maximum online error versus the number of orthonormal basis vectors chosen by the greedy algorithm from the training sample space $S_N = \Xi_{M'=100}^{\text{lin}} \cup \Xi_{M'=100}^{\log}$ for tolerance values $\epsilon_{\text{tol}} = 10^{-4}, 10^{-6}, 10^{-8}, 10^{-10}$. Where as, the online test space is defined as $\Xi^{\text{test}} = \Xi_{M'=200}^{\text{lin}} \cup \Xi_{M'=200}^{\log}$

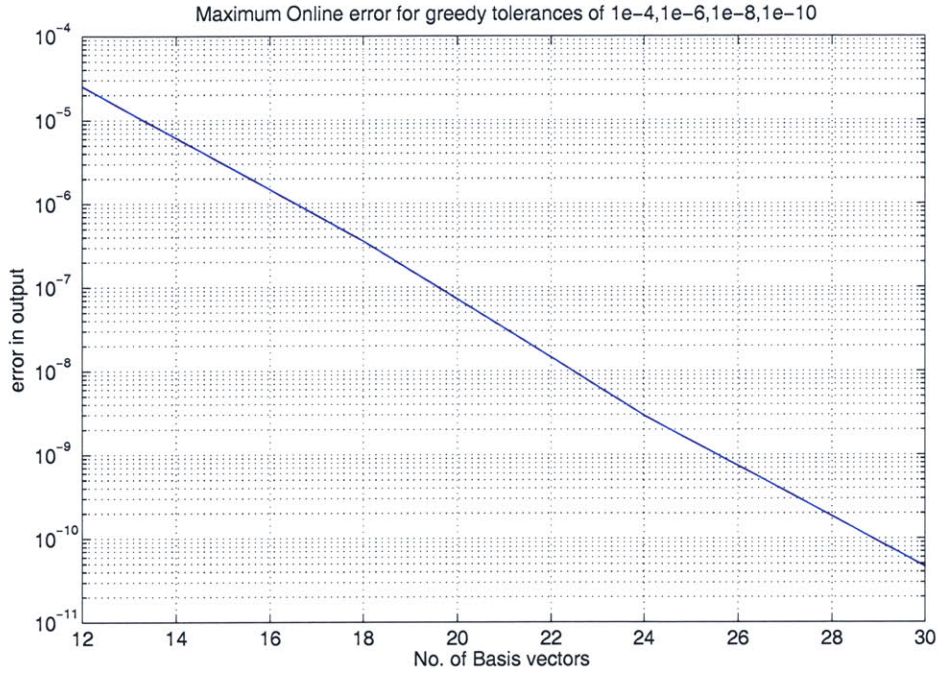


Figure 6-2: Convergence of Reduced Basis results using the Greedy Basis vectors

6.3.3 Comparison of online timing with the Finite element solution timing

In the following table, we compare the average time taken to compute the Finite element output to the average time taken for online output computation. Solving the mesh where $h_x = 0.0714$ and $h_y = 0.0500$ using both Finite element method and Greedy Algorithm based Reduced Basis and comparing the timings

Avg. FEM time	greedy tol	Avg.online time
1.0119e+3 s	1e-4	0.0785e-3 s
	1e-6	0.1178e-3 s
	1e-8	0.3926e-3 s
	1e-10	0.4711e-3 s

,we can see that a lot of computational time is saved.

6.3.4 Scope for future work

There is a lot of scope for further research on this problem. Firstly, we can save a lot of offline computational effort by developing rigorous *a posteriori* bounds. This way, we can avoid computing the true error. We can also extend the problem to higher dimensions. Currently, with the 1-d model of the Boltzmann equation, we can model very few practical problems. To make Boltzmann equation more applicable to practical problems, we need to solve it in higher dimensions.

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