SPACE CHARGE DYNAMICS OF LIQUIDS

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SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF SCIENCE
at the
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

August, 1970

Signature of Author ______________________________
Department of Electrical Engineering, August 24, 1970

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Submitted to the Department of Electrical Engineering on September 24, 1970, in partial fulfillment of the requirements for the Degree of Doctor of Science.

ABSTRACT

The propagation and instability characteristics of small-signal electro-fluid-mechanical space charge and polarization waves in non-homogeneous fluids is developed. Stratified equilibrium configurations with fluid properties and space charge density constant within planar, cylindrical, and spherical surfaces, are emphasized. Equilibrium electric fields due both to charges in the fluid and to externally imposed potentials are normal to these surfaces. The liquid is modeled as incompressible, inviscid, and perfectly insulating, with domains of sufficiently high frequency or growth rate to validate the last assumption defined in terms of electrical conductivity or mobility.

Two types of stratification for the mass density, dielectric constant, and space charge density are distinguished and related: discrete layers and continuous distributions. A general set of relations for perturbation field and flow variables on the perturbed surfaces of fluid layers having constant properties and space charge are derived in each of the configurations. Detailed description of wave dispersion and instability for interactions in the following situations exemplifies how these relations are used in representing a broad class of discretely stratified equilibria: a) Perfectly-conducting interface stressed by normal field and bounded from above by fluid supporting uniform space charge; b) Two planar layers of differing properties and space charge; c) Uniformly charged liquid jet; d) Uniformly charged liquid drop. It is shown that the general relations can be used to represent systems of coupled layers which approximate continuous distribution by a series of step functions. Specific examples of weak-gradient and exponential distributions are presented showing that the solution found directly from the distributed theory is approached by the system of coupled layers, if the limit is taken in which the number of layers becomes large while the layer thickness approaches zero. Experiments are described which attempt to delineate the coupling of space charge to electrohydrodynamic surface waves on a perfectly-conducting interface in the configuration of (a), above.

THESIS SUPERVISOR: James R. Melcher
TITLE: Professor of Electrical Engineering

ACKNOWLEDGMENT

My association with Professor J. R. Melcher over the past four years has greatly enhanced the value of my graduate education. He introduced me to the area of electromechanics, and has inspired and stimulated my advancement. In particular, the problems treated in this thesis were initiated and aided by him. The help and guidance of my readers, Professors A. Bers and L. N. Howard, were invaluable in the early development of this work. Many thanks are due to all members of the Continuum Electromechanics group for their friendship and support. The financial support of the Department of Electrical Engineering is acknowledged.

The excellent typing of Mia Brown has added to the appearance and organization of this work.

I am grateful to my parents for all their encouragement throughout the years.

I wish to thank my wife, Linda, for her love and understanding.

The work was carried out under NASA Grant No. NGL-22-009-014.

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CHAPTER I
Background

1.1 Examples of Space Charge Distributions

In various disciplines, the dynamics of fluids in the presence of bulk space charge has assumed considerable importance. Bohr and Wheeler successfully used a charged liquid drop as a model for the atomic nucleus.[1,2] They considered the dynamics of a spherical drop which was incompressible, inviscid, and perfectly insulating, with charge uniformly distributed throughout its bulk. Since some of the properties of nuclear forces are analogous to the properties of forces which hold a liquid drop together, these concepts, together with classical concepts such as electrostatic forces and surface tension, were used to set up a semi-empirical formula for the mass or binding energy of a nucleus in its ground state.[3]

Chemical interactions illustrate how the competing mechanisms between Coulomb and diffusion forces can produce a space-charge distribution. Often, charge density gradients are very large near abrupt discontinuities like an interface or electrode surface, where these gradients extend over small distances on the order of Debye lengths ($\sim 1000Å$). Because the net charge density is large over these very small length scales (even though the net charge is zero), the electric fields are very large, and thus the electric forces can be large enough to cause macroscopic fluid motions. With a potential difference on the order of two volts across these space charge junctions (so-called 'electrical double layers'), steady-state convection of the fluid is possible.[4]

These double-layer interactions are important in electrolytic solutions because of the large ion densities present. However, even in very insulating fluids like hexane or transformer oils, dissolved ions are present due to impurities or from ionic emission from electrodes when very high field strengths are present. If a sufficiently large amount of net charge is present, the electrical forces on the fluid can induce various motions.[5-7] The bulk cellular convection described by Avsec and Luntz is probably related to such space charge effects.[6]

Forces in both electrolytic and poorly conducting fluids can be of some use, and certainly are fundamental in a scientific sense. If the electrical forces are large enough to overcome stabilizing forces due to hydrostatic pressure, diffusion and viscosity, it may be possible to make the fluid statically unstable, putting it in a constant state of random motion, thus enhancing the heat and mass transfer. In an electrolytic solution, this could be important in influencing reaction rates or in the mass transfer through a semi-permeable membrane. In oils which are used for cooling purposes, like transformer oil, a constant agitation of the fluid will greatly increase the rate at which heat can be carried away. Similar processes can increase the efficiences of other heat transfer systems.[8-12]

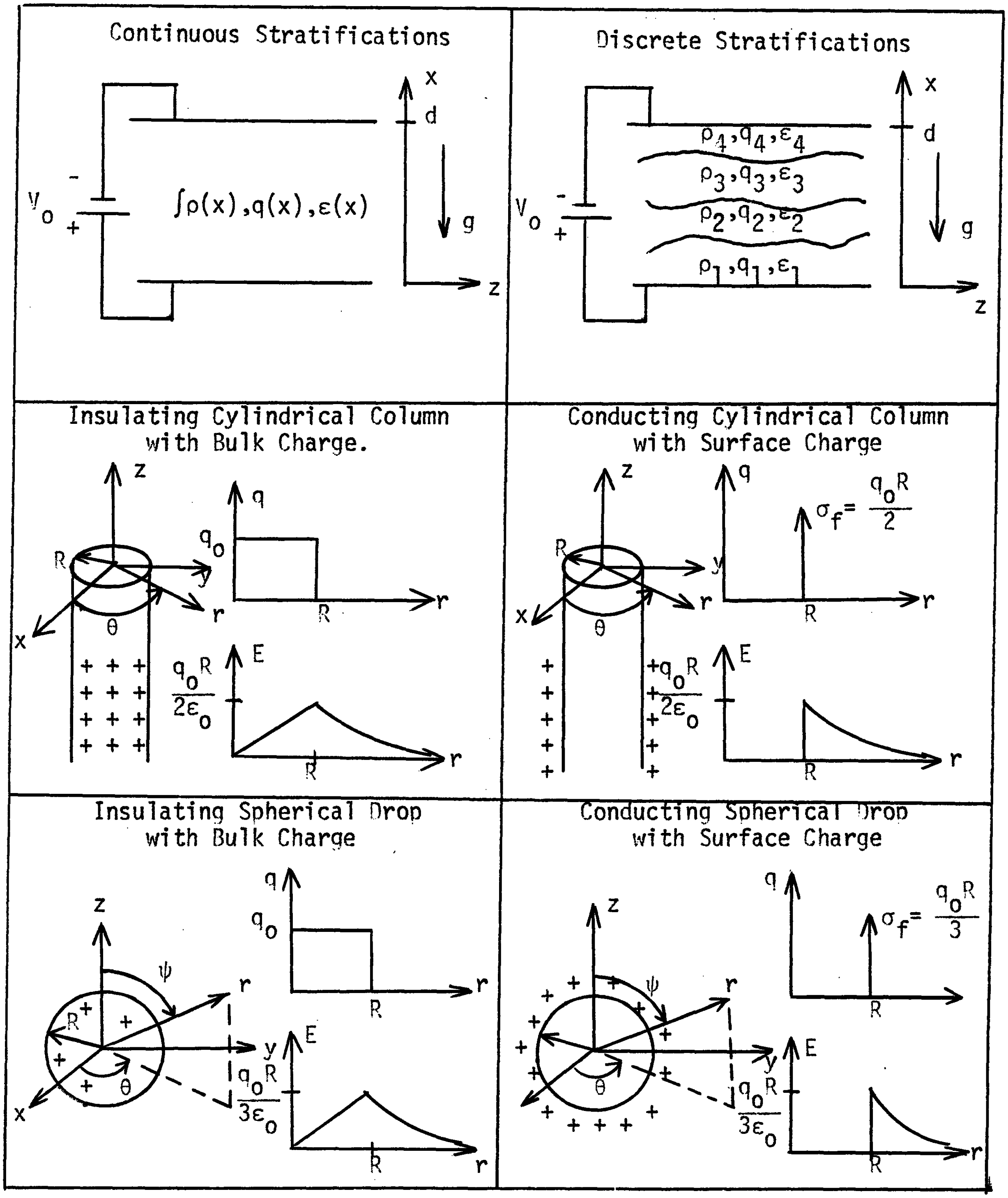
On a larger scale, the origin of thunderstorms and other meteorological phenomena has been attributed to the effects of space charge throughout the atmosphere.[13] The fluid dynamics may be analagous to

Lord Kelvin's water dripper, wherein the motion of liquid drops is responsible for the generation of high voltages with no external electrical excitations. (See Ref. 14, Part II, pp. 388-392) These and similar processes can be useful for aiding in the transfer or conversion of energy. The presence of space charge in a fluid is the common denominator, yet the scales of these phenomena differ greatly. In electrolytic systems, and hence in double layers, potential differences on the order of volts are necessary for electrical forces to be important. In typical situations involving insulating fluids, potential differences on the order of kilovolts are necessary, while atmospheric phenomena require hundreds of kilovolts. In each physical situation, different parameters are necessary for the complete description of the problem. Compressibility, temperature, diffusion, viscosity, and electrical conduction are just a few parameters which may or may not be important. The inclusion of these parameters greatly complicates the analysis and understanding of the dynamics of these fluids. Although steady state behavior may be solvable, questions of stability and temporal response are very difficult to answer.

1.2 Scope of Thesis

Space charge equilibrium distributions often have dependence on only one spatial dimension, being uniform in the other directions. This may be in rectangular, cylindrical, or spherical geometry.

We wish to discuss here the propagation and instabilities of small signal space charge waves through such waveguide—like structures composed of charged fluids and electrodes as illustrated in Fig. 1. These



Examples of charged liquid structures which support small signal space charge waves.

Figure 1

include superposed fluid layers with properties of mass density, charge density, permittivity, and convection velocity differing between layers and the analagous problem of continuous distributions of these properties. Similar analysis has been performed in the past for internal gravity wave systems.[15-17]

The analysis proceeds with the usual theory of small oscillations about an equilibrium state. In order to describe a wide class of interactions, we make the simplifying assumptions from the outset that for dynamical times of interest the fluid is inviscid, incompressible and perfectly insulating. It will be shown in section 1.4 that for many fluids this approximation is valid over a wide frequency range. The subtle point here is that we can rely on the conduction mechanism to create the equilibrium space charge, but for small signal interactions, we can assume the fluid to be perfectly insulating. Within this frequency range, we can ignore conduction and assume that the charges are literally tied to the fluid.

Melcher has previously considered single interface problems of this type including electric and magnetic coupling.[18] However, the electrical forces only acted at the interface and had no contribution in the fluid bulk. Our analysis will include these surface forces and also emphasize the bulk force due to space charge for any number of layers.

The general problem of a fluid whose properties are continuously stratified throughout the bulk is troublesome, because the resulting differential equations have space-varying coefficients, for which there are no general solutions. However, we will model these continuous stratifications by many thin layers, each with constant properties. We can then greatly simplify the description of the dynamics determined by the dispersion relation. The more layers we use as an approximation, the more accurate the resulting dispersion relation will be. In fact, for a finite height system for which we approximate a continuous stratification by many thin layers, if we let the number of layers tend toward infinity as the thickness of each layer tends to zero, the dispersion relation will be shown to become exactly correct. This will be illustrated in a number of special cases for which we can exactly solve the differential equation with the continuous stratifications. A technique will be shown of how to solve the infinite number of layers problem exactly for these special cases.

The layered approach is of great value in analyzing space-varying systems, since we change from solving a difficult space-varying coefficient differential equation to many simpler linear constant-coefficient difference equations. These equations can easily be solved on a computer, although a few special cases can be solved in closed form, which will be illustrated here.

In considering a large number of layers, we must develop a systematic approach so as to avoid confusion. This can be done by considering a prototype layer of inviscid, incompressible and perfectly insulating fluid. We will describe the layer with sufficient generality so that our analysis provides a prototype relation between variables at the upper and lower surfaces of the layer, which can be used to describe systems composed of many such layers. Our objective of providing a prototype relation between variables evaluated adjacent to the interfaces amounts to relating the respective surface potentials and normal perturbation electric fields just inside the interfaces, and the perturbation pressures and surface deflections just adjacent to the interfaces.

The value of these "terminal relations" is that one avoids solving the bulk equations again for every region. Rather, since they directly relate pertinent interfacial variables which are related through boundary conditions, we can easily determine the dispersion relation for the electromechanical interfacial waves by simply "interconnecting the terminals."

These methods will also be applied to cylindrical columns and spherical shells of constant mass and charge density. Analogous terminal relations will be derived for these geometries.

The examples presented will compare the dynamics between a perfectly insulating fluid with bulk charge and a perfectly conducting fluid with surface charge. This will be done for a spherical drop and a cylindrical column. This comparison is important, for in any experiment where we try to change a charge distribution, for example from zero to a finite value, the fluid will act like a perfect insulator for the first few instants. As time increases further, the charges will relax to the surface of the fluid, shielding out any fields and thus acting like a perfect conductor. Depending on the time scale of the experiment, the fluid dynamics will be much different.

We cannot self-consistently allow any surface charge on an interface in an inviscid fluid model unless the interface is perfectly conducting.[19,20] If we allow surface charge on a finitely conducting interface, any motions of the interface will result in electrical shear forces which can only be balanced by viscosity. If the fluid is a perfect conductor, the electric field will come in normal to the interface, resulting in no electrical shear force.

1.3 Equilibrium Space Charge Distributions

The conduction mechanism establishes the equilibrium charge distribution. Although the details of this process are not necessary for the linearized analysis, since the frequencies of interest will be much higher than the response time of the charges, it is important to know the equilibrium distributions as these terms do appear in the linearized analysis. A direct way to measure the distributions is desirable. In highly polar fluids like nitrobenzene or chlorobenzene, electro-optical observations using the Kerr effect may be used to map the electric field and thus the charge density.[21,22]

If these fluids are stressed by an electric field, light polarized parallel to this electric field will travel at a different phase velocity than light polarized normal (the polarization in the plane of the electric field). If the resultant light signal passes between crossed polarizers, an interference pattern will result, with the maxima and minima related to the imposed electric field.

However, in most liquids, the Kerr effect is small making this procedure impractical. Thus, in order to determine the equilibrium distributions, a model must be proposed and these distributions calculated. The two most useful models are ohmic and unipolar conduction.

1.3.1 Ohmic Conduction

The simplest conduction law often used is Ohm's law, where the current and electric field are linearly related

$$\overline{J}'_{cond} = \sigma\,\overline{E}' \tag{1}$$

where the prime (') refers to quantities measured in the frame of reference of the medium.

The model from which this law is derived, assumes that there are at least two species of charges present of opposite signs, where one species moves relative to the other with a drift velocity determined by the collision frequency. Thus, even though the net charge is zero, we can have a net conduction current. We use this constitutive law in Maxwell's equations

$$\nabla \cdot (\varepsilon\overline{E}) = q \tag{2}$$

$$\nabla \cdot \overline{J} \frac{+\partial q}{\partial t} = 0 \tag{3}$$

$$\overline{J} = \overline{J}_{cond} + q\overline{v} = \sigma\overline{E} + q\overline{v} \tag{4}$$

to obtain the relation

$$\frac{\partial q}{\partial t} + \overline{v} \cdot \nabla q + q\nabla \cdot \overline{v} + \overline{E} \cdot \nabla\sigma + \frac{\sigma}{\varepsilon} q = 0 \tag{5}$$

where we assume the permittivity ε to be constant.

If there is no convection and the conductivity is a constant

$$\overline{v} = 0; \sigma = \text{constant} \tag{6}$$

the solution to (5) is simply

$$q = q\ (x, y, z, t = 0)\ e^{-(\frac{\sigma}{\varepsilon})t} \tag{7}$$

This relation shows that in the absence of convection, and with a constant conductivity and permittivity, we can have no equilibrium space charge. Any initial charge density will decay with characteristic time $\tau = \frac{\varepsilon}{\sigma}$, so that in the steady state there will be zero charge in the bulk, with the total charge now distributed around the boundary as a surface charge.

Similarly, if the convection velocity is a constant, and the medium has a constant dielectric constant and conductivity

$$\overline{v} = U\overline{i}_x; \varepsilon = \text{constant}; \sigma = \text{constant} \tag{8}$$

we find the one dimensional steady state solution ($\frac{\partial}{\partial t} = 0$) with the boundary condition of a charge source at x = 0 to be

$$q = q\ (x=0)\ e^{-\frac{x}{Re\,l}} \tag{9}$$

where

$$Re = \frac{\varepsilon U}{\sigma l}$$

which represents a ratio of the electrical relaxation time to the fluid transit time.

In the absence of convection, the only way to have a steady state charge distribution is if the conductivity is a function of position. The one dimensional steady state solution to (5) is

$$q = \varepsilon J_o \frac{\partial}{\partial x}\left(\frac{1}{\sigma}\right) \tag{10}$$

where J_o is the current/area through the fluid. The extreme case is at the interface of two materials with different conductivities, where we have surface charge.

Wong and Melcher built a pump using this mechanism to create space charge.[23] They applied a temperature gradient across a rectangular section of a channel filled with corn oil. Since the conductivity of corn oil is a strong function of temperature, the resulting conductivity gradient generated space charge as given by (10). The bulk electrical force thus induced, caused the liquid to flow around the channel.

This model is usually appropriate for metals and other highly conducting materials. In most homogeneous insulating liquids like hexane, freon, kerosene, transformer oils, silicone oils, and in highly polar liquids like nitrobenzene, chlorobenzene, acetophenone, or xylene, it is well known that when these fluids are stressed by an electric field, a space charge develops, usually in the vicinity of the electrodes. This effect determined by the non-linearity of the voltage-current relation and by optical techniques using the Kerr effect, cannot be accounted for with an ohmic constitutive law with constant conductivity as shown by (7). A mobility model is necessary, where we allow the charge density at any point to be non-zero, (although the whole system must be neutral).

1.3.2 Unipolar Conduction-Mobility Model

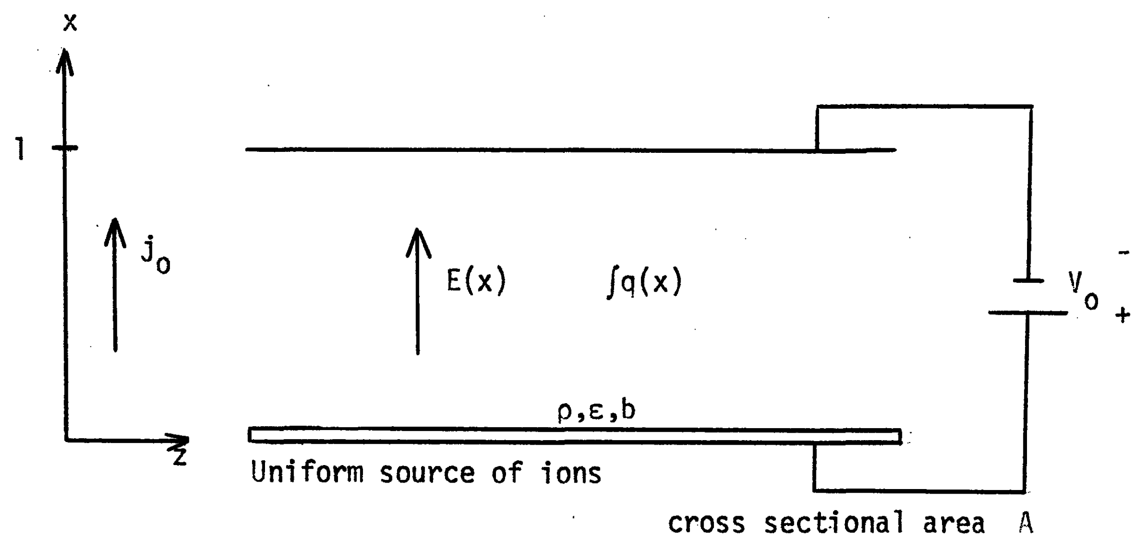
The mobility model is a generalized version of ohmic conduction, where the charge density of each carrier is a function of the electric field present, through Gauss's Law. For ohmic conduction, the charge density of each carrier is not a function of the electric field, although the conductivity could be a function of position.

For many species, the conduction law is given by

$$\overline{J}_{cond} = \sum_i (q_i b_i)\,\overline{E} \qquad (1)$$

where q_i is the magnitude of the charge density of each carrier, and b_i is its mobility (q_i and b_i are always positive). We wish to consider the presence of only one carrier, which we assume to be positive.

An Example. In particular, suppose we specify a pair of equipotential surfaces, having a potential difference V_0 over a spacing l as shown in Fig. 1. The lower electrode at $x = 0$, is a source of ions which have a mobility b in the fluid medium.



Geometry considered for unipolar conduction. The lower electrode at x = 0 is a source of ions with mobility b in the fluid medium.

Figure 1

The general equations of the system are:

$$\bar{J} = q\,b\,\bar{E} + q\,\bar{v} \tag{2}$$

$$\nabla \times \bar{E} = 0\ ;\ \bar{E} = -\nabla\phi \tag{3}$$

$$\varepsilon\nabla \cdot \bar{E} = q \tag{4}$$

$$\nabla \cdot \bar{J} + \frac{\partial q}{\partial t} = 0 \tag{5}$$

$$\frac{D\rho}{Dt} + \rho\,\nabla \cdot \bar{v} = 0 \tag{6}$$

$$\rho\,\frac{D\bar{v}}{Dt} + \nabla p = q\bar{E} - \rho g\,\bar{i}_x + \mu\,\nabla^2\,\bar{v} \tag{7}$$

$$\varepsilon,\ b\ \text{constant} \tag{8}$$

We wish to consider the equilibrium distributions, when the fluid is stationary. Thus, in equilibrium we have

$$\bar{v} = 0 \qquad \bar{J} = j_o \bar{i}_x = \text{constant}$$

$$q_o(x) = \frac{j_o}{bE_o} = \frac{\varepsilon dE_o}{dx} \tag{9}$$

Solving, we obtain

$$E_o(x) = \sqrt{E^2(o) + \frac{2j_o x}{\varepsilon b}} \tag{10}$$

$$q_o(x) = \frac{j_o}{b\sqrt{E^2(o) + \frac{2j_o x}{\varepsilon b}}} \tag{11}$$

$$V_o = \frac{1}{3}\frac{\varepsilon b}{j_o}\left[\left(E^2(o) + \frac{2j_o l}{\varepsilon b}\right)^{3/2} - E^3(o)\right] \tag{12}$$

where E(o) is the electric field at the emission electrode.

In most experiments, the potential difference V_o is imposed, and the current density j_o is whatever it has to be. Thus, we would like to eliminate j_o from these equations by using (12). However, this cannot be easily done due to the cumbersome form of (12). In addition, we must also specify E(o) before the solution is complete.

To simplify the presentation, we define the parameter

$$\alpha = \frac{2j_o l}{\varepsilon b\ E^2(o)} \tag{13}$$

Then

$$E(o) = \frac{3\ \alpha\ V_o/l}{2[(1.+\alpha)^{3/2} - 1.]} \tag{14}$$

$$E(x) = E(o)\ [1. + \frac{\alpha x}{l}\ ]^{1/2} \tag{15}$$

$$q(x) = \frac{\varepsilon E(o)\ \frac{\alpha}{l}}{2\ [1. + \frac{\alpha x}{l}\ ]^{1/2}} \tag{16}$$

$$(\frac{V_o}{l})^2 = \frac{8}{9}\ \frac{j_o l}{\varepsilon b}\ \frac{[(1. + \alpha)^{3/2} - 1.]^2}{\alpha^3} \tag{17}$$

Note that the current is always proportional to the square of the voltage. See Fig. 2.

In Figs. 3 and 4 we have plotted the electric field and charge density as a function of position for various values of E(o) (or equivalently α).

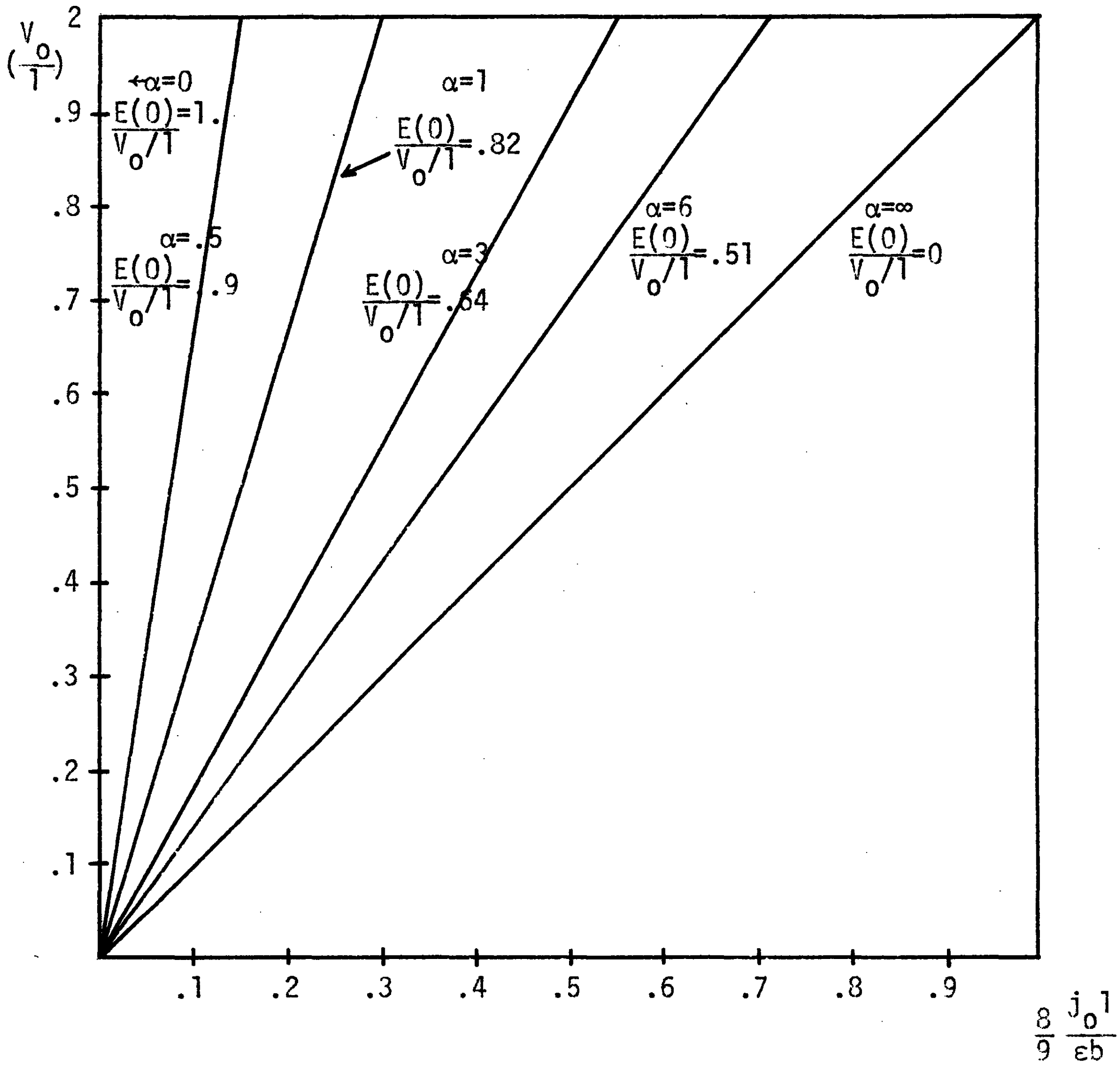
The plots show how sensitive the field and charge distributions are to the injection process due to the strong dependence on the electric field at the injection electrode.

The net electrical force on the fluid is

$$f_x = \frac{\varepsilon}{2}\ \{[E(l)]^2 - [E(o)]^2\}\ A \tag{18}$$

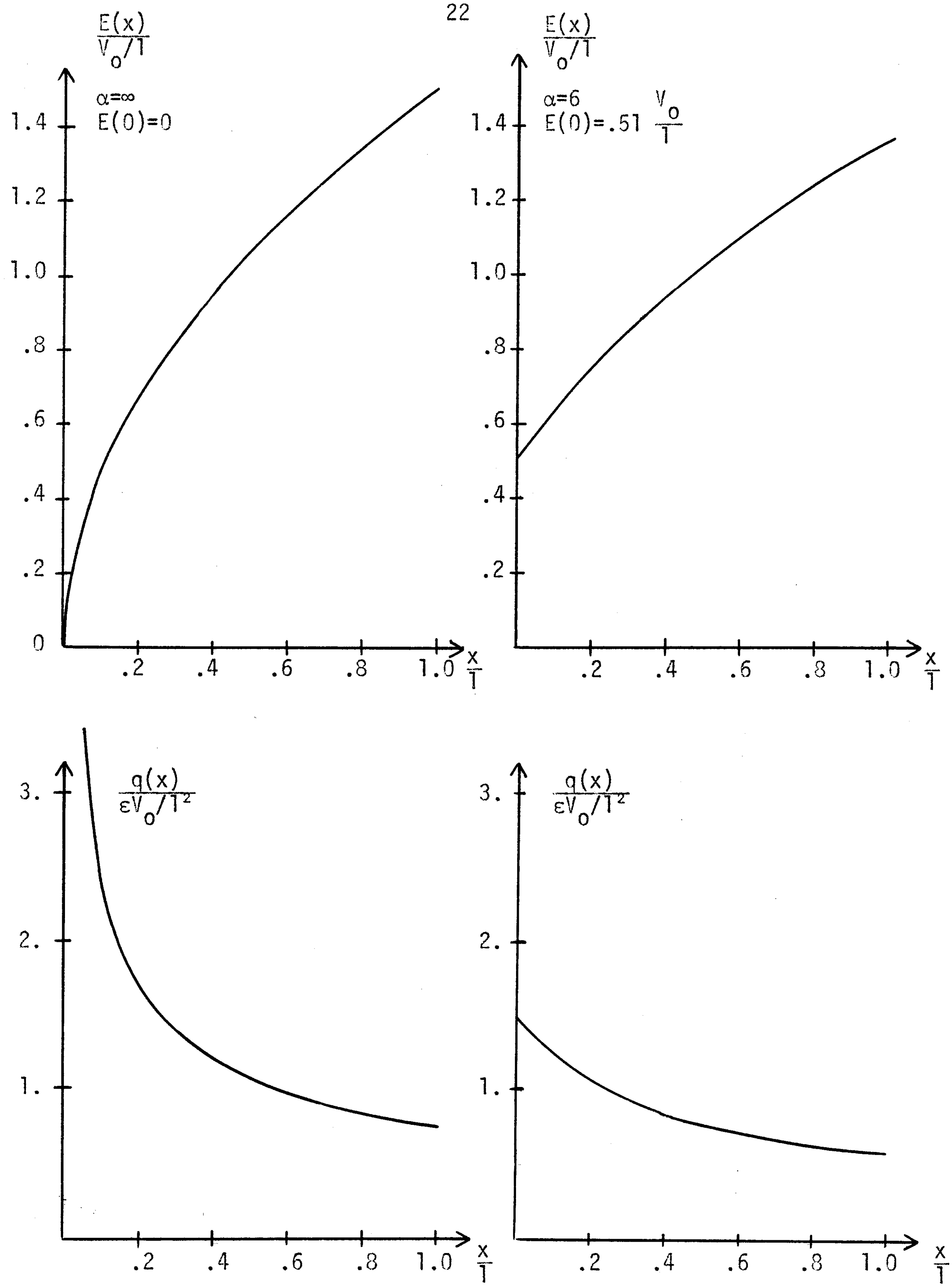
To maximize this force, we would prefer the space charge limited case $E(o) = 0$ where we have all the charge in the fluid bulk, and no surface charge on the emission electrode.

We might suspect that such a charge distribution is unstable at high enough electric fields, since the bulk charge is being attracted



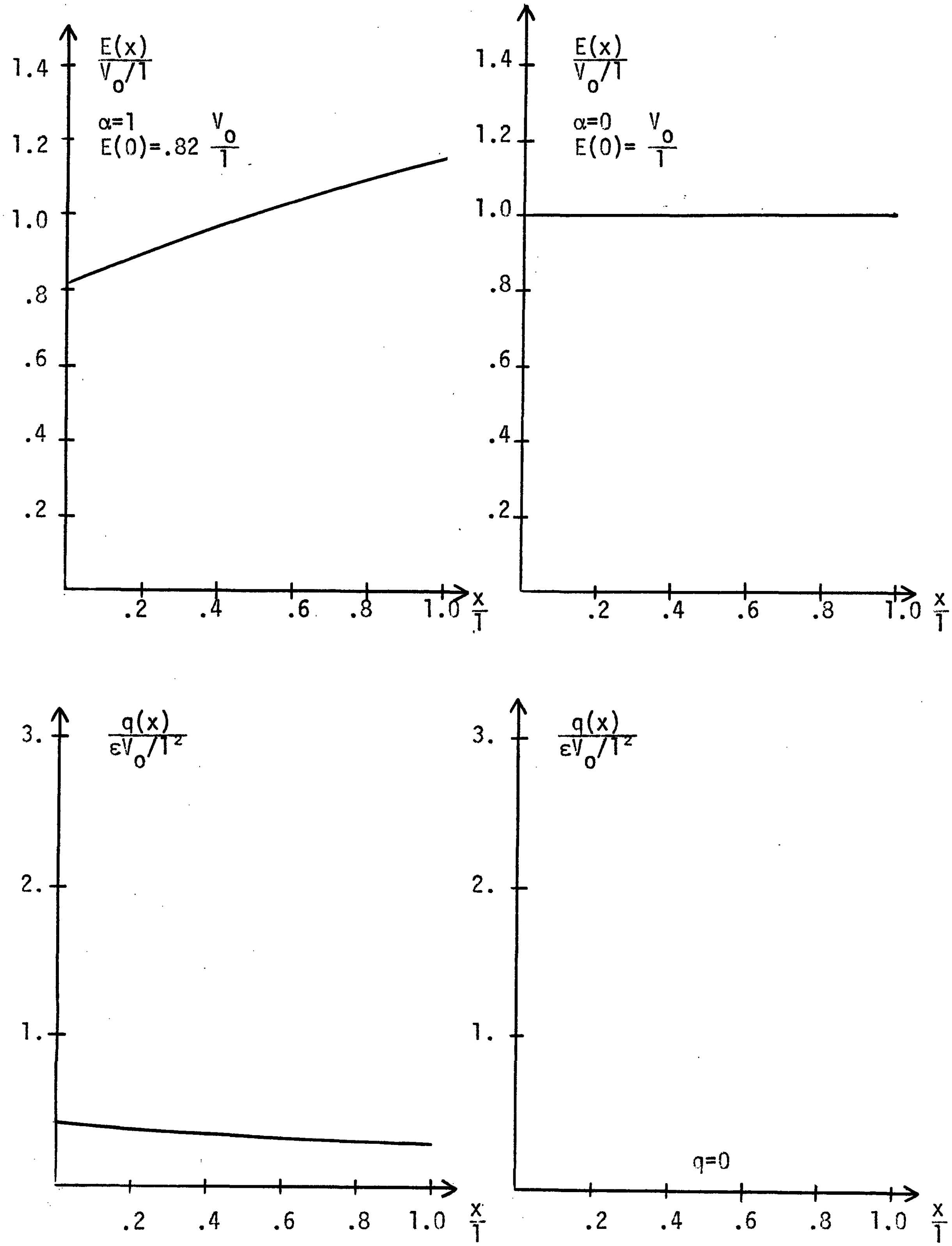
Square law voltage-current relation for unipolar conduction.

Figure 2



Unipolar conduction. Electric field and charge density as a function of position for various emission conditions.

Figure 3



Unipolar Conduction. Electric field and charge density as a function of position for various emission conditions.

Figure 4

by the surface charge on the other electrode. If this attractive force overcomes the fluid pressure and viscous forces, this distribution will be unstable. This instability has been analyzed by other workers and will be discussed in section 5.

Ion Drag Pumps

Stuetzer has built ion drag pumps similar in geometry to Fig. 1 where charge is injected into the fluid through points at $x = 0$ and and collected at a permeable grid at $x = l$, such that the fluid could flow unimpeded. The electrical force given by (18), pumps the fluid. He used such fluids as freon 113, silicone oil, castor oil, and kerosene. His analysis assumed space charge limited conduction, which agreed well with his experiments. However, there is good reason to suspect that within the pump, the fluid was unstable, upsetting the equilibrium charge distribution, as it has been shown that for unipolar conduction, voltages in excess of a few hundred volts cause most insulating liquids to be unstable. This will be discussed in section 5.

1.4 Significant Parameters

In order to carry our analysis to sufficient depth, we will mainly concern ourselves here with a model of a liquid which is inviscid, incompressible, and perfectly insulating. Our results will then only be useful for real liquids during an appropriate time scale. However, we will show that this time scale covers a wide range for many fluids.

All fluids are slightly compressible, which is the mechanism by which acoustic waves can propagate. The phase velocity of acoustic waves in liquids is on the order of 1500 meters/sec. To ignore the effects of this slight compressibility, the acoustic wavelengths must be much longer than the size of the system. For systems with a characteristic length on the order of 10 cm., the frequency range must be much less than 10 KHz.

We can neglect the effects of fluid viscosity if the fluid inertia is large enough so that

$$\left| \rho \frac{\partial \overline{v}}{\partial t} \right| >> \left| \mu \nabla^2 \overline{v} \right| \qquad (1)$$

which will be true if the Reynolds number

$$R = \frac{\rho L^2 \omega}{\mu} \qquad (2)$$

is much greater than unity. This will occur in a frequency range where

$$\omega >> \frac{\mu}{\rho L^2} \qquad (3)$$

where L is some characteristic length. This condition is easily met for frequencies greater than .1 Hz if we use relatively inviscid fluids like freon or hexane as indicated in Table I.

One of the most stringent conditions is that on the conduction mechanism. For ohmic fluids, the frequencies should be high enough that the charges cannot undergo appreciable conduction. This will be in a frequency range where

$$\omega >> \frac{\sigma}{\varepsilon} \qquad (4)$$

Examining Table I, we see that the electrical relaxation time ($\tau = \frac{\varepsilon}{\sigma}$) for the fluids of interest can vary from milliseconds to tens of seconds,

Material	Dielectric Constant ϵ/ϵ_o.	Viscosity ($Nts\text{-}sec/m^2$) μ	Mobility ($m^2/volt\text{-}sec$) b	Conductivity (mhos/m) σ	Density (kg/m^3) ρ	$\frac{\mu}{\rho L^2}$ ($L = .01m$)	$\frac{\sigma}{\epsilon}$	qb/ϵ ($q = 10^{-3} coul/m^3$)
Air[24]	1	1.7×10^{-5}	+ ions-1.4×10^{-4} - ions-2.0×10^{-4}		1.3	.13		+ 1.6×10^4 - 2.25×10^4
Water (tap)*	80	10^{-3}		4×10^{-6}	1000	10^{-2}	5.6×10^3	
Hexane[7]	1.9	3×10^{-4}	1.4×10^{-7}		670	4.5×10^{-3}		8.3
Freon 113[24]	2.4	$.62 \times 10^{-3}$	6×10^{-7}	2×10^{-11}	1565	4×10^{-3}	.94	28
Corn Oil*	3.1	5.46×10^{-2}		$\sim 5 \times 10^{-11}$	940	.585	1.8	
Silicone Oil[24] (Dow 550)	2.8	.13	35×10^{-7}		1040	1.25		140
Silicone Oil* (Dow 200)	2.63	$\sim 10^{-2}$		$\sim 3 \times 10^{-13}$	940	$\sim 10^{-1}$	~.015	
Transformer Oil* GE - 10 - C	2.4	$\sim 10^{-2}$		$\sim 5 \times 10^{-13}$	~870	$\sim 10^{-1}$	~.025	
Castor Oil[24]	4.5	.95	2.7×10^{-7}		960	10		6.8
Kerosene[24]	2.04	1.9×10^{-3}	3×10^{-7}		750	2.5×10^{-2}		17
SF_6[24]	1		1.4×10^{-4}		6.2			$1.6 \times 10^{+4}$
Liquid N_2 (80°C)[24]	1.5	1.7×10^{-4}	10^{-6}		880	1.9×10^{-3}		75
CCl_4[24]	2.15	10^{-3}	7×10^{-7}		1600	6.2×10^{-3}		37

* Properties measured by members of the Continuum Electromechanics Group at M.I.T., Department of Electrical Engineering.

TABLE I

so that we can easily find fluids to meet this condition. Note that, in order to have an equilibrium space charge in ohmic fluids, it is necessary to have a conductivity gradient, otherwise any equilibrium space charge will decay with time constant $\tau = \varepsilon/\sigma$. For fluids governed by mobility law, the analogous condition is

$$\omega >> \frac{qb}{\varepsilon} \qquad (5)$$

The maximum charge density that can occur over a centimeter length in a fluid which is space-charge-limited with a potential difference of 20KV is

$$q_{max} \simeq 10^{-3} \text{ coulombs/m}^3 \qquad (6)$$

Examining Table I, we find many fluids which will obey our approximations in the frequency range

$$1 \leq f \leq 1000 \quad \text{Hz} \qquad (\omega = 2\pi f) \qquad (7)$$

1.5 Past Work On Space Charge in Fluids

Turnbull and Melcher were the first to carry through, in depth, a general electrohydrodynamic problem.[11] They considered an incompressible fluid in a gravity field g, directed in the -x direction with an imposed colinear electric field in an initially stationary fluid which was perfectly insulating. The equilibrium density ρ, viscosity μ, space charge density q, and permittivity ε were functions of the vertical coordinate, x. A variational principle was used to describe the stability of the system, with the analysis similar to approaches used in hydrodynamic problems like the Rayleigh-Taylor or Bénard problems.[30] The principle of exchange of stabilities was shown, thus making it possible to reduce the prediction of incipience of instability to an eigenvalue problem in the electric field. They derived a sufficient condition for stability to be

$$gD\rho - EDq - EDED\varepsilon < 0 \tag{1}$$

where D = d/dx. Barston has shown that this condition is both necessary and sufficient.[31]

These results provide a limiting case, regardless of the electrical conduction model. If an ohmic conduction model is used, then these results are valid in the zero conductivity limit, and if a mobility model is used, then these results are true if the mobility is zero. However, as we have previously discussed, since all real fluids have some finite conduction mechanism, these results will be only approximately true for high enough frequency so that any free charge cannot respond to the excitation. Turnbull and Melcher focused their attention on conditions for instability, which occur at zero frequency, and so take an infinite time to grow. On this long time scale, the conduction rate processes have a chance to exert their influence, so this limit at zero frequency is the place where many of their approximations are least appropriate for a real fluid. Previously, Turnbull has shown that, for a fluid with finite ohmic conductivity, instability is not incipient at zero frequency, although for small conductivity the condition for incipience approaches $\omega = 0$.[12]

In his experiments, Turnbull applied a vertical temperature gradient in a poorly conducting liquid like corn oil or castor oil. The gradient in conductivity resulting from the temperature gradient caused free charge to accumulate in the fluid when an electric field was applied. In the cases which were considered, the gradient in dielectric constant was negligible with the only significant electric force being due to space charge. If the fluid is inviscid and perfectly insulating, the necessary and sufficient condition for instability is given by (1) with ε constant. If the fluid is inviscid with instantaneous relaxation, the condition for stability is

$$\frac{\varepsilon E^2 D\left(\frac{\sigma}{\sigma_0}\right)}{(-gD\rho)} < \left[(ka)^2 + (n\pi)^2\right] \qquad (2)$$

where a is the distance between the parallel electrodes, σ is the electrical conductivity with σ_0 being the average conductivity. This result shows that for any given field strength, the fluid is unstable for short enough wavelengths. However, viscosity, which was neglected, has its greatest effect at short wavelengths, which indicates that the inviscid limit is unrealistic for highly conducting fluids. Including viscosity, he obtains a condition for stability to be

$$H \equiv \frac{\varepsilon E^2 a\, D\left(\frac{\sigma}{\sigma_0}\right)}{\mu\,(-gD\rho/\rho_0 a)^{1/2}} < \frac{n\pi 3\sqrt{3}}{2} \qquad n = 1, 2, \ldots \qquad (3)$$

which for n = 1 becomes

$$H < 8.15 \qquad (4)$$

The instability is caused by the electrical forces acting through the conductivity gradient overcoming the viscous damping of the internal gravity waves.

He measured the onset of instability using the Schmidt-Milverton principle. The heat transfer Q, across a tank with a known temperature gradient $\Delta T/d$ is measured and the Nusselt number is calculated

$$Nu = \frac{Q}{(k\Delta T/d)}$$

where k is the thermal conductivity. Q is the actual heat flow and $k\Delta T/d$ is the heat flow in the absence of convection. If the Nusselt number exceeds 1, convection is present. The convection is due to the onset of fluid instability. His experimental data showed that the heat transfer suddenly increased at a critical value of electric field which agreed well with his theory.

Watson and Schneider have approximately solved for the stability of a poorly conducting liquid with a single charge carrier of finite mobility.[32] Their stability conditions agreed well with experiments.[33] They assumed that the principle of exchange of stabilities was valid, although it was not proved. Their experimental agreement seems to indicate that, for fluids with slight mobility, incipience of instability occurs near $\omega = 0$. With this assumption, they derived a critical parameter which they termed the 'electric Rayleigh number',

$$R = \frac{3}{8}\frac{\varepsilon V_0}{\mu b} = 99 \qquad (5)$$

where b is the mobility, V_0 the applied potential, μ the viscosity and ε the permittivity.

This critical number was derived using the mechanical boundary conditions that one surface was fixed, requiring both the normal and shear fluid velocities to be zero, while the other surface to which ions were injected, was assumed to be flat with zero tangential stress. The electrical boundary conditions were that both surfaces be equipotential surfaces with space charge limited conduction.

Atten and Moreau studied a similar problem, also assuming instability to be incipient at zero frequency.[34] Their boundary conditions were that the perturbation velocities and potential be zero at the electrodes and that the perturbation charge at the injector electrode be zero. They found the critical value for instability to be

$$\frac{\varepsilon V_o}{\mu b} \simeq 161 \qquad (6)$$

Using the above condition for hexane (see Table I) we find the critical voltage to be

$$V_{crit} \simeq 400 \text{ volts}$$

This critical value is typical for most insulating liquids. These results indicate that with moderate voltages, a unipolar conduction mechanism becomes unstable, and so any equilibrium distributions calculated, as in section 1.3.2, will be upset, with the fluid trying to reach a new equilibrium which cannot be stationary.

These results indicate that when measuring the mobilities of insulating liquids, one must only use voltages less than the critical voltage. If this voltage is exceeded, the fluid is no longer stationary and so convection currents in addition to conduction currents are being measured. If these facts are not accounted for, erroneous values of the mobility will result.

1.6 Relation to Past Work in Fluid Mechanics

The analyses presented in this thesis, are electrohydrodynamic versions of the classic Rayleigh-Taylor and Kelvin-Helmholtz problems.[30] In the past, simple two layer versions of these problems have laid the groundwork for more extensive work concerning many fluid layers and continuum distributions of mass density and velocity.

The properties of fluids with density gradients and velocity distributions have been important in meteorology, geology, and oceanography. We have included the added complication that the fluid is charged and so each fluid element is coupled through the electric field in addition to gravity and inertia.

Our method follows the chronology of the fluid mechanical problem. We use simple layer problems as clues for the general properties of complex systems. Eventually, we generalize to continuous distributions and offer the convergence of many thin layers as an alternate method for computing system dynamics.

Greenhill in 1887 set up the problem of (N+1) stationary superposed liquids of differing densities which he solved for a special case in the long-wavelength limit.[35] Long, in 1953, solved a similar problem ignoring the density gradients in the inertia term.[15] He mentions that the discrete layers yield similar results as an analogous continuous distribution, but he does not go into any depth.

A controversy has existed for many years which only lately has been resolved concerning the correctness of modeling a continuous velocity distribution by many thin layers, each with constant velocity. Taylor and Goldstein, in 1931, concluded that this procedure was wrong, and that the two methods would give different answers even as the number of layers go to infinity with the thickness of each going to zero.[36,37] However, in 1960, Case resolved the dilemma. He considered the stability of inviscid Couette flow (linear velocity distribution).[38,39] The stability of the system was investigated in accordance with the usual theory of small oscillations about a state of steady motion. He found that there were no solutions. He then considered an initial value problem and watched the solution evolve in time. The solution was stable with an asymptotic response that went like 1/t. He attributed this result to a continuum of modes with the frequency extremes determined by the velocity limits and the wavenumber.

The results of his analysis indicate that the general solution to problems of this type are the ordinary discrete modes and a continuum of modes. The continuum of modes represent solutions which grow (or decay) differently than exponential, in this case 1/t. In Fourier space, the 1/t time dependence is represented by a continuous frequency spectrum. It has since been proved that if performed correctly, a large number of thin layers does approximate the continuous problem.[40]

The crucial question in the problems considered here is to what order of singularity are we allowed to use in approximating continuous distributions. One might guess that we may use many interfaces with surface charge as an approximation to a continuous charge distribution, yet as described earlier in section 1.2, such a formulation is inconsistent with an inviscid fluid model. For the continuous formulation, it will be shown that the spatial derivative of the charge density is the pertinent parameter. For a step-wise approximation, which we will use, this term is a set of impulses, while a model using surface charge on an interface results in a set of doublets. This is too high a singularity to work with, so contradictions result. Because our formulation avoids these higher order singularities, we only have discrete modes.

CHAPTER II

GENERAL FORMULATION OF THE DYNAMICS OF UNIFORMLY CHARGED LAYERS

2.1 Introduction

We will examine the propagation characteristics and stability of stratified equilibrium configurations by the usual method of considering small perturbations about the equilibrium state. In general, the electromechanical equations of motion, although linear, have space varying coefficients.

The types of stratification which are of interest here may be superposed layers, where each layer has constant properties of mass density, dielectric constant, viscosity, charge density and convection velocity or continuous distributions of these properties. In the case of layers the space varying coefficients in the differential equation are constant within each layer, although they will be different for each layer with possible singularities at the interface.

The solutions within each layer will be simple because of the constant coefficients. Our approach will be to solve the equations within each layer and obtain the total solution by appropriately splicing the solutions at each interface using boundary conditions.

This method is also useful in approximating the description for continuous distributions of material properties. The continuum is replaced by many discrete layers of constant properties. The more layers we use as an approximation, the more accurate the solution will be. If we let the number of layers go to infinity, with the thickness of each layer tending to zero, the resulting answers will be exactly correct.

Thus our method has traded a difficult linear space varying coefficient differential equation for many simpler linear constant coefficient difference equations. This alternate method can be of special value with computer analysis.

In considering a large number of layers, we will develop a systematic approach to aid the analysis. We will describe a prototype layer with sufficient generality so that our analysis provides a prototype relation between interfacial variables for which adjacent layers can be "married" together through the boundary conditions.

2.2 Equations of Motion

The general equations for an incompressible and perfectly insulating fluid are:

Conservation of momentum —

$$\rho \frac{D\overline{v}}{Dt} + \nabla p = q\overline{E} - \frac{1}{2}\overline{E} \cdot \overline{E}\,\nabla\,\varepsilon + \rho\overline{g} + \mu\nabla^2\overline{v} \tag{1}$$

Conservation of mass —

$$\nabla \cdot \overline{v} = 0 \tag{2}$$

$$\frac{D\rho}{Dt} = 0 \tag{3}$$

Maxwell's equations —

$$\nabla \times \overline{E} = 0 \;;\quad \overline{E} = -\nabla\phi \tag{4}$$

Gauss's Law:

$$\nabla \cdot (\varepsilon\overline{E}) = q \tag{5}$$

$$\frac{D\varepsilon}{Dt} = 0 \tag{6}$$

Conservation of charge: $\frac{Dq}{Dt} = 0$ (7)

No matter what the geometry, in the prototype layer we assume the fluid to be homogeneous such that the properties of mass density, charge density, dielectric constant, viscosity and convection velocity are constant.

To simplify the mathematics, we will assume from the outset that the fluid is inviscid ($\mu = 0$). After presenting the inviscid analysis, it should be clear to the reader how to extend this work to include viscosity.

Thus our approach is to assume all quantities of the form:

$$\overline{v} = \overline{U}_\Delta + \overline{v}' \tag{8}$$

$$p = p_o(\overline{r}) + p' \tag{9}$$

$$\rho = \rho_\Delta + \rho' \tag{10}$$

$$\phi = \Phi(\overline{r}) + \phi' \; ; \quad \overline{E} = -\nabla\phi \tag{11}$$

$$q = q_\Delta + q' \tag{12}$$

$$\varepsilon = \varepsilon_\Delta + \varepsilon' \tag{13}$$

$$\varepsilon_\Delta, \rho_\Delta, q_\Delta, \overline{U}_\Delta \quad \text{constant} \tag{14}$$

and

$$\overline{r} = x\overline{i}_x + y\overline{i}_y + z\overline{i}_z \tag{15}$$

The primed quantities are small perturbations from the equilibrium.

Equilibrium

The equilibrium variables must obey the time independent form of (1) - (7), subject to the constraint of a homogeneous medium. Thus the equilibrium variables in each region must obey the relations:

$$q = q_\Delta \tag{16}$$

$$\rho = \rho_\Delta \tag{17}$$

$$\varepsilon = \varepsilon_\Delta \tag{18}$$

$$\overline{v} = \overline{U}_\Delta \tag{19}$$

$$\nabla p_o\ (\overline{r}) = q_\Delta\ \overline{E}\ (\overline{r}) + \rho_\Delta \overline{g} \tag{20}$$

$$\nabla \cdot \overline{E}\ (\overline{r}) = \frac{q_\Delta}{\varepsilon_\Delta} \tag{21}$$

$$\nabla \times \overline{E}\ (\overline{r}) = 0\ ;\quad \overline{E}\ (\overline{r}) = -\nabla\Phi\ (\overline{r}) \tag{22}$$

Moreover, they must satisfy the appropriate boundary conditions between regions. For example, the pressure distribution must be such that the equilibrium conditions are satisfied.

In particular, we will examine equilibrium distributions which obey (16) - (22) for prototype layers in planar, cylindrical and spherical geometry.

<u>Perturbation Equations</u>

Because we have assumed the fluid layer to be homogeneous, (3), (6), and (7) may be written as:

$$\left(\frac{\partial}{\partial t} + \overline{U}_\Delta \cdot \nabla\right) \begin{bmatrix} q' \\ \rho' \\ \varepsilon' \end{bmatrix} = 0 \tag{23}$$

for which we conclude that if any point in question can only be joined through particle lines to points of constant properties, we have solutions:

$$\rho' = 0 \tag{24}$$

$$q' = 0 \tag{25}$$

$$\varepsilon' = 0 \tag{26}$$

For these conditions, the charge density, mass density, and dielectric constant remain constant, in spite of the fluid motion. This is to be expected, because any transport of material into a given region leads to a transport of material which has the same properties as that previously occupying the given region. This statement only applies to those portions of the fluid not swept out by motions of an interface. If a point of interest is adjacent to an interface, an excursion of the interface could result in an abrupt change of properties. However, if surface deflections are considered at a given instant, all properties everywhere between interfaces are uniform.

The other perturbation equations of interest are:

$$\nabla \cdot \overline{v}' = 0 \tag{27}$$

$$\rho_\Delta \left[\frac{\partial \overline{v}'}{\partial t} + \overline{U}_\Delta \cdot \nabla \overline{v}'\right] + \nabla \left(p' + q_\Delta \, \phi'\right) = 0 \tag{28}$$

$$\nabla^2 \phi' = 0 \tag{29}$$

If we take the divergence of (28), use (27) and define the quantity

$$\pi' = p' + q_\Delta \phi' \tag{30}$$

we obtain the concise equation

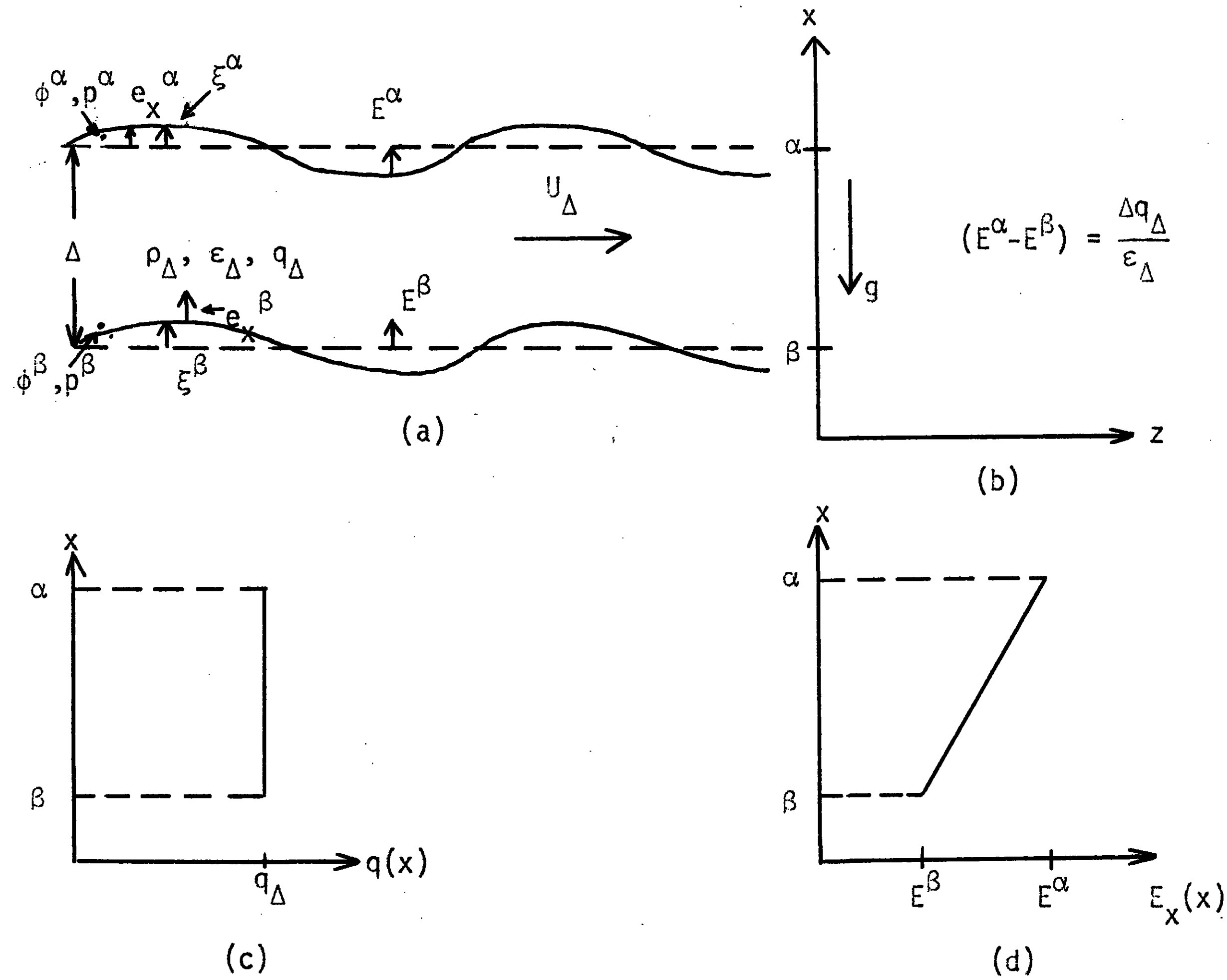
$$\nabla^2 \pi' = 0 \tag{31}$$

Thus, regardless of the geometry, this problem reduces to solutions of Laplace's equations, both for the perturbation potential and for the quantity π'.

In particular, we will solve (29) and (31) in rectangular, cylindrical and spherical geometry, but this analysis can be extended to any co-ordinate system.

2.3 Rectangular Geometry

The physical situation to be characterized is shown in Fig. 1. A planar layer of inviscid, incompressible and perfectly insulating fluid assumes an equilibrium thickness, Δ. We will describe the layer with sufficient generality so that our analysis provides a prototype relation between variables at the upper and lower surfaces of the layer, which can be used to describe systems composed of many such layers. There is a distribution of charge throughout the bulk of the layer given by q_Δ; this charge is uniform, as depicted by Fig 1c. Because of this charge, there is an equilibrium electric field intensity at the upper surface which is generally different from that at the lower surface, and these are designated by E^α and E^β, respectively. E^α and E^β as defined here are evaluated inside the bounding surfaces. In general, they are not continuous through the interface. Our objective of providing a prototype relation between variables evaluated adjacent to the interfaces amounts to relating the respective surface potentials and normal perturbation electric field intensities just inside the interfaces, and the perturbation pressures and surface deflections just adjacent to the interfaces.



Definition of variables for a prototype layer of incompressible, inviscid, perfectly insulating fluid supporting a uniformly distributed charge density:

a) cross-sectional view

b) definition of coordinates and relation between equilibrium electric field intensities at the surface

c) distribution of volume charge density

d) distribution of electric field intensity in equilibrium

Figure 1

For this geometry, the equilibrium quantities are:

$$q = q_\Delta \tag{1}$$

$$\rho = \rho_\Delta \tag{2}$$

$$\varepsilon = \varepsilon_\Delta \tag{3}$$

$$\overline{v} = U_\Delta \overline{i}_z \tag{4}$$

$$p = -\rho_\Delta g x - q_\Delta \phi_o(x) + \text{constant} \tag{5}$$

$$E_x = \frac{q_\Delta x}{\varepsilon_\Delta} + \text{constant} \tag{6}$$

We let all perturbation variables be of the form

$$\pi' = \text{Re}\ \hat{\pi}\ (x)\ e^{j(\omega t - k_y y - k_z z)} \tag{7}$$

Solutions of this form to (2.2.29) and (2.2.31) are

$$\begin{bmatrix} \hat{\pi}\ (x) \\ \hat{\phi}\ (x) \end{bmatrix} = A \sinh kx + B \cosh kx \tag{8}$$

where

$$k = \sqrt{k_y^{\,2} + k_z^{\,2}} \tag{9}$$

Now the velocities at α and β are related to the displacements as

$$\hat{v}_x\ (\alpha) = j\ (\omega - k_z U_\Delta)\ \hat{\xi}^\alpha = j\omega' \hat{\xi}^\alpha \tag{10}$$

$$\hat{v}_x(\beta) = j\ (\omega - k_z U_\Delta)\ \hat{\xi}^\beta = j\omega'\ \hat{\xi}^\beta \tag{11}$$

where

$$\omega' = \omega - k_z U_\Delta \tag{12}$$

From (2.2.28) we have that

$$\rho_\Delta j\omega' \hat{v}_x + \frac{\partial \hat{\pi}}{\partial x} = 0 \tag{13}$$

Using (8) - (13) we then obtain the relations

$$\hat{v}_x = j\omega' \left[\frac{\sinh k\,(x - \beta)}{\sinh k\Delta} \hat{\xi}^\alpha - \frac{\sinh k\,(x - \alpha)}{\sinh k\Delta} \hat{\xi}^\beta \right] \tag{14}$$

and

$$\hat{\pi} = \frac{\rho_\Delta \omega'^2}{k \sinh k\Delta} \left[\cosh k\,(x - \beta)\, \hat{\xi}^\alpha - \cosh k\,(x - \alpha) \hat{\xi}^\beta\right] \tag{15}$$

where

$$\Delta = \alpha - \beta \tag{16}$$

Our analyses implicitly assumed the interfacial displacements to be small, for in (14) we used (10) and (11) evaluated at the equilibrium positions rather than at the interface itself. Fortunately, because the velocity itself is a perturbation, the difference between evaluating it at the interface or at the equilibrium position is second order in the perturbation amplitudes. This illustrates the general approach used in linearized surface deformation problems. The boundary condition at the moving interface is replaced by one at the equilibrium position of the boundary, thus greatly simplifying the analysis.

Then using the solution of (8) with (10), (11), and (2.2.28), we find that $\hat{\pi}$ (α) and $\hat{\pi}$ (β) are related to the interfacial displacements as

$$\begin{bmatrix} \hat{\pi}(\alpha) \\ \\ \hat{\pi}(\beta) \end{bmatrix} = \begin{bmatrix} \omega'^2 \rho_\Delta \dfrac{\coth k\Delta}{k} & \dfrac{-\omega'^2 \rho_\Delta}{k \sinh k\Delta} \\ \dfrac{\omega'^2 \rho_\Delta}{k \sinh k\Delta} & \dfrac{-\rho_\Delta \omega'^2 \coth k\Delta}{k} \end{bmatrix} \begin{bmatrix} \hat{\xi}^\alpha \\ \\ \hat{\xi}^\beta \end{bmatrix} \quad (17)$$

However, as the interfaces deform, in addition to perturbing all variables, the equilibrium quantities acting on the interfaces also change. Thus to compute the total first order change in all variables evaluated at the interface, we must include linear changes of equilibrium quantities. For example, the total linear changes in the pressures are:

$$\hat{p}^\alpha = \hat{p}\ (\alpha) + \frac{\partial p_o}{\partial x}\bigg|_{(x\,=\,\alpha)} \hat{\xi}^\alpha \quad (18)$$

$$\hat{p}^\beta = \hat{p}\ (\beta) + \frac{\partial p_o}{\partial x}\bigg|_{(x\,=\,\beta)} \hat{\xi}^\beta \quad (19)$$

Similarly

$$\hat{\phi}^\alpha = \hat{\phi}\ (\alpha) + \frac{\partial \phi_o}{\partial x}\bigg|_{(x\,=\,\alpha)} \hat{\xi}^\alpha = \hat{\phi}\ (\alpha) - E^\alpha \hat{\xi}^\alpha \quad (20)$$

$$\hat{\phi}^\beta = \hat{\phi}\ (\beta) + \frac{\partial \phi_o}{\partial x}\bigg|_{(x\,=\,\beta)} \hat{\xi}^\beta = \hat{\phi}\ (\beta) - E^\beta \hat{\xi}^\beta \quad (21)$$

Using these definitions, we have

$$\hat{p}^\alpha \equiv \hat{\pi}\ (\alpha) - q_\Delta \hat{\phi}\ (\alpha) + [E^\alpha q_\Delta - \rho_\Delta g]\hat{\xi}^\alpha \quad (22)$$

$$\hat{p}^\beta \equiv \hat{\pi}\ (\beta) - q_\Delta \hat{\phi}\ (\beta) + [E^\beta q_\Delta - \rho_\Delta g]\hat{\xi}^\beta \quad (23)$$

and so the mechanical transfer relations are from (17), (22), and (23)

$$\begin{bmatrix} \hat{p}^{\alpha} \\ \\ \hat{p}^{\beta} \end{bmatrix} = \begin{bmatrix} \dfrac{\rho_{\Delta}\omega'^2 \coth k\Delta}{k} - \rho_{\Delta} g & \dfrac{-\rho_{\Delta}\omega'^2}{k \sinh k\Delta} \\ \\ \dfrac{\rho_{\Delta}\omega'^2}{k \sinh k\Delta} & \dfrac{-\rho_{\Delta}\omega'^2 \coth k\Delta}{k} - \rho_{\Delta} g \end{bmatrix} \begin{bmatrix} \hat{\xi}^{\alpha} \\ \\ \hat{\xi}^{\beta} \end{bmatrix} - q_{\Delta} \begin{bmatrix} \hat{\phi}^{\alpha} \\ \\ \hat{\phi}^{\beta} \end{bmatrix} \tag{24}$$

There remains the task of finding the perturbation electric fields. The charges in the layer of fluid have an effect on the fields represented by the normal component of the electric field intensity just inside the respective interfaces. By relating perturbation potentials to perturbation normal electric fields, we are able to give a solution which is of general applicability.

We have already introduced the convention of the total linear changes in the potential at the interfaces as given by (20) and (21). In writing force equilibrium for the interfaces, it is convenient to similarly define

$$\hat{Q}^{\alpha} = -\,\varepsilon_{\Delta}\hat{e}_x^{\alpha} = -\,[q_{\Delta}\hat{\xi}^{\alpha} + \varepsilon_{\Delta}\hat{e}_x(\alpha)] \tag{25}$$

$$\hat{Q}^{\beta} = +\,\varepsilon_{\Delta}\hat{e}_x^{\beta} = q_{\Delta}\hat{\xi}^{\beta} + \varepsilon_{\Delta}\hat{e}_x(\beta) \tag{26}$$

where we use the fact that

$$\frac{\partial E_x}{\partial x} = \frac{q_{\Delta}}{\varepsilon_{\Delta}}$$

Using (8), perturbation fields are related at the equilibrium positions of the interfaces by

$$\begin{bmatrix} \varepsilon_\Delta \hat{e}_x(\alpha) \\ \\ \varepsilon_\Delta \hat{e}_x(\beta) \end{bmatrix} = \begin{bmatrix} -\varepsilon_\Delta k \coth k\Delta & \dfrac{\varepsilon_\Delta k}{\sinh k\Delta} \\ \\ \dfrac{-\varepsilon_\Delta k}{\sinh k\Delta} & \varepsilon_\Delta k \coth k\Delta \end{bmatrix} \begin{bmatrix} \hat{\phi}(\alpha) \\ \\ \hat{\phi}(\beta) \end{bmatrix} \tag{27}$$

Then, using the definitions provided by (25) and (26), (27) becomes

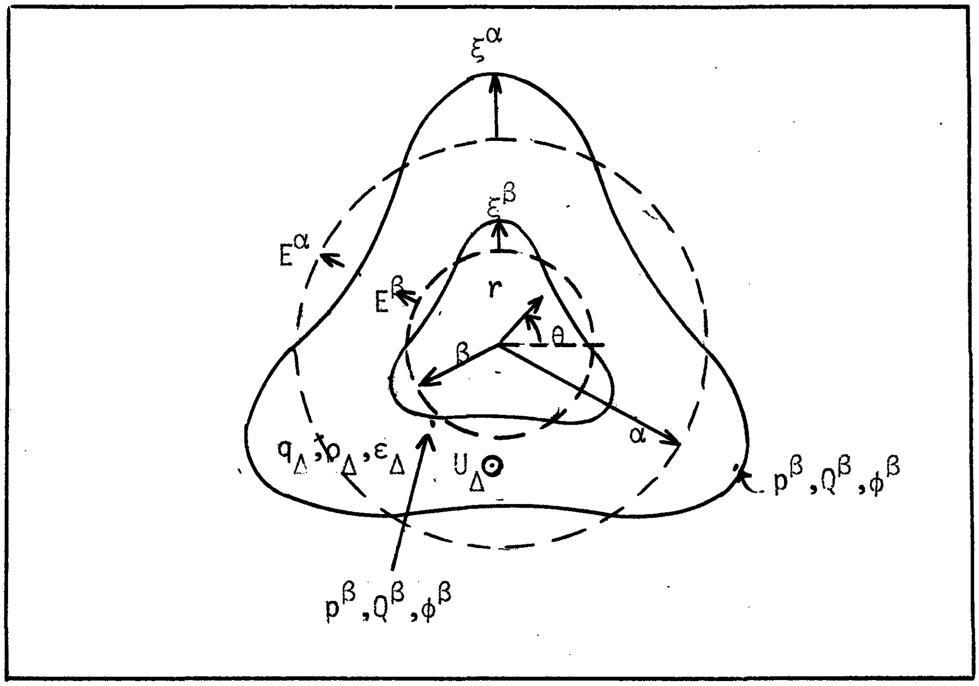
$$\begin{bmatrix} \hat{Q}^\alpha \\ \\ \hat{Q}^\beta \end{bmatrix} = \begin{bmatrix} \varepsilon_\Delta k \coth k\Delta & \dfrac{-\varepsilon_\Delta k}{\sinh k\Delta} \\ \\ \dfrac{-\varepsilon_\Delta k}{\sinh k\Delta} & \varepsilon_\Delta k \coth k\Delta \end{bmatrix} \begin{bmatrix} \hat{\phi}^\alpha + E^\alpha \hat{\xi}^\alpha \\ \\ \hat{\phi}^\beta + E^\beta \hat{\xi}^\beta \end{bmatrix} + q_\Delta \begin{bmatrix} -\hat{\xi}^\alpha \\ \\ +\hat{\xi}^\beta \end{bmatrix} \tag{28}$$

The electromechanical relations for the prototype layer are summarized by (24) and (28). In a sense, problems of this type are surface coupled, although the coupling is actually through volume forces. We are successful in formally uncoupling the electrical and mechanical equations in the volume only because the electric field intensity can be so simply described in terms of a potential.

In Chapter III, the value of these "terminal relations" will be seen when applied to specific examples.

2.4 Cylindrical Geometry

We will follow the same reasoning here, as we had in rectangular geometry. In particular we consider a cylindrical shell of perfectly-insulating, inviscid, incompressible fluid with mass density ρ_Δ and constant charge density q_Δ, which is moving with velocity U_Δ in the z direction. We neglect the effects due to gravity.



Definition of terminal variables for a prototype cylindrical shell of incompressible, inviscid, perfectly-insulating fluid with uniform charge density, q_Δ.

Figure 1

In equilibrium, conservation of momentum has

$$p_o + q_\Delta \Phi_o = \text{constant} \tag{1}$$

with

$$\Phi_o = -\frac{q_\Delta r^2}{4\varepsilon_\Delta} - V_o \ln \frac{r}{R_o} \; ; \; V_o, R_o \text{ some constants} \tag{2}$$

The second term in (2) represents the Laplacian part of the equilibrium potential which is due to charges located in other regions of space (perhaps ∞). Similarly, we have that

$$E_o = \frac{q_\Delta r}{2\varepsilon_\Delta} + \frac{V_o}{r} \tag{3}$$

In general, these Laplace solutions are non-zero and are essential terms in our analysis, which will be seen in some later examples.

In rectangular geometry, the Laplacian solution was the constant in 2.3.6, and was unimportant in the perturbation equations. Here the Laplacian solution is a function of radial position, and so will appear in our terminal relations.

Terminal Relations with No z-Dependence

If wavelengths of interest in the z-direction are much longer than the length of the column, then the modes of the system are essentially independent of z. Thus, we assume all variables of the form

$$\pi' = \text{Re}\ \hat{\pi}(r)\ e^{j(\omega t - m\theta)} \tag{4}$$

$$\overline{v}' = \text{Re}\ \hat{v}(r)\ e^{j(\omega t - m\theta)} \tag{5}$$

where m must be an integer, since the system closes on itself, and

$$\hat{v}_r(\alpha) = j\omega\hat{\xi}^\alpha \tag{6}$$

$$\hat{v}_r(\beta) = j\omega\hat{\xi}^\beta \tag{7}$$

From 2.231, $\pi'(\pi' = p' + q_\Delta\phi')$ must obey Laplace's equation, for which solutions in cylindrical coordinates are

$$\hat{\pi}(r) = Ar^m + Br^{-m} \tag{8}$$

Then proceeding as we had in rectangular geometry, we can easily derive the transfer relation between $\hat{\pi}^\alpha, \hat{\pi}^\beta$ and the displacements $\hat{\xi}^\alpha$, $\hat{\xi}^\beta$. Since our boundary conditions are phrased in terms of the pressures, we write the terminal relations as

$$\begin{bmatrix} \hat{p}^{\alpha} \\ \\ \hat{p}^{\beta} \end{bmatrix} = \frac{\rho_{\Delta}\omega^2}{m[(\frac{\beta}{\alpha})^m - (\frac{\alpha}{\beta})^m]} \begin{bmatrix} -\alpha[(\frac{\alpha}{\beta})^m + (\frac{\beta}{\alpha})^m] & +2\beta \\ \\ -2\alpha & \beta[(\frac{\beta}{\alpha})^m + (\frac{\alpha}{\beta})^m] \end{bmatrix} \begin{bmatrix} \hat{\xi}^{\alpha} \\ \\ \hat{\xi}^{\beta} \end{bmatrix} - q_{\Delta} \begin{bmatrix} \hat{\phi}^{\alpha} \\ \\ \hat{\phi}^{\beta} \end{bmatrix} \tag{9}$$

where $m \neq 0$ since the fluid is incompressible.

For the special case where

$$\beta = 0$$

(9) simplifies to

$$\hat{p}^{\alpha} = \frac{\rho_{\Delta}\omega^2\alpha}{|m|} \hat{\xi}^{\alpha} - q_{\Delta} \hat{\phi}^{\alpha} \tag{10}$$

and when

$$\alpha = \infty$$

$$\hat{p}^{\beta} = \frac{-\rho_{\Delta}\omega^2\beta}{|m|} \hat{\xi}^{\beta} - q_{\Delta}\hat{\phi}^{\beta} \tag{11}$$

As before, all quantities are to be evaluated at their respective interfaces. Linear changes due to change in equilibrium position are included. The only possible quantities which may be confusing as to where they are evaluated are the potentials and normal electric fields, which are redefined below for clarity.

$$\hat{\phi}^{\alpha} = \frac{\partial \Phi_o}{\partial r}\Big|_{r=\alpha} \hat{\xi}^{\alpha} + \hat{\phi}\ (\alpha) = \hat{\phi}\ (\alpha) - E^{\alpha}\hat{\xi}^{\alpha} \qquad (12)$$

$$\hat{\phi}^{\beta} = \frac{\partial \Phi_o}{\partial r}\Big|_{r=\beta} \hat{\xi}^{\beta} + \hat{\phi}\ (\beta) = \hat{\phi}\ (\beta) - E^{\beta}\hat{\xi}^{\beta} \qquad (13)$$

$$\hat{Q}^{\alpha} = -\varepsilon_{\Delta}\hat{e}_r^{\ \alpha} = -\varepsilon_{\Delta}\hat{e}_r\ (\alpha) - \varepsilon_{\Delta}\frac{\partial E_o}{\partial r}\Big|_{r=\alpha} \hat{\xi}^{\alpha} \qquad (14)$$

$$\hat{Q}^{\beta} = \varepsilon_{\Delta}\hat{e}_r^{\ \beta} = \varepsilon_{\Delta}\hat{e}_r(\beta) + \varepsilon_{\Delta}\frac{\partial E_o}{\partial r}\Big|_{r=\beta} \hat{\xi}^{\beta} \qquad (15)$$

All variables with superscripts α,β (like $\hat{\phi}^{\alpha}$, $\hat{\phi}^{\beta}$, $\hat{Q}^{\alpha}$, $\hat{Q}^{\beta}$, ...) mean that the quantity is evaluated at the respective interface. Quantities, such as $\hat{\phi}(\alpha)$, $\hat{\phi}(\beta)$, $\hat{e}_r(\alpha)$, $\hat{e}_r(\beta)$, ... are solutions of the perturbation equations and so may be evaluated at their equilibrium position. The difference between the latter quantities and superscripted quantities is due to linear variations due to change in equilibrium position. $\hat{Q}^{\alpha}$ and $\hat{Q}^{\beta}$ in some sense take the form of a surface charge to terminate the normal electric field at the inside surface of each interface, but it must be remembered that, physically, they are just another way of describing the radial field at each interface.

With these terms defined, and from 2.2.29 it is easily shown that the electrical terminal relation is

$$\begin{bmatrix} \hat{Q}^{\alpha} \\ \\ \hat{Q}^{\beta} \end{bmatrix} = \frac{-m\varepsilon_{\Delta}}{[(\frac{\beta}{\alpha})^m - (\frac{\alpha}{\beta})^m]} \begin{bmatrix} (\beta^{-m}\alpha^{m-1} + \beta^{m}\alpha^{-m-1}) & -\frac{2}{\alpha} \\ \\ -\frac{2}{\beta} & (\beta^{-m-1}\alpha^{m} + \beta^{m-1}\alpha^{-m}) \end{bmatrix} \begin{bmatrix} \hat{\phi}^{\alpha} + E^{\alpha}\hat{\xi}^{\alpha} \\ \\ \hat{\phi}^{\beta} + E^{\beta}\hat{\xi}^{\beta} \end{bmatrix}$$

$$+ \begin{bmatrix} -\varepsilon_{\Delta} \left.\frac{\partial E_o}{\partial r}\right|_{r=\alpha} \hat{\xi}^{\alpha} \\ \\ \varepsilon_{\Delta} \left.\frac{\partial E_o}{\partial r}\right|_{r=\beta} \hat{\xi}^{\beta} \end{bmatrix} \qquad (16)$$

For the special case

$$\beta = 0$$

(16) simplifies to

$$\hat{Q}^{\alpha} = \frac{m\varepsilon_{\Delta}}{\alpha} (\hat{\phi}^{\alpha} + E^{\alpha}\hat{\xi}^{\alpha}) - \varepsilon_{\Delta} \left.\frac{\partial E_o}{\partial r}\right|_{r=\alpha} \hat{\xi}^{\alpha} \qquad (17)$$

and when

$$\alpha = \infty$$

we obtain

$$\hat{Q}^{\beta} = \frac{m\varepsilon_{\Delta}}{\beta} (\hat{\phi}^{\beta} + E^{\beta}\hat{\xi}^{\beta}) + \varepsilon_{\Delta} \left.\frac{\partial E_o}{\partial r}\right|_{r=\beta} \hat{\xi}^{\beta} \qquad (18)$$

In (9) and (16) we see no effect due to convection. This is because we assumed no z variation and thus the wavenumber is zero such that

$$\omega' = \omega - kU_{\Delta} = \omega$$

We note that our terminal relations are even in m, so when only coupling these relations together we can assume m to be positive. If the geometry or excitation has a preferential θ variation, we must keep in mind that m is generally both positive and negative.

Terminal Relations with z and θ Variation

For an arbitrary length cylindrical column, the modes take the form

$$\hat{\xi}^{\alpha} = \text{Re}\ \hat{\xi}^{\alpha} e^{j(\omega t - m\theta - kz)} \tag{19}$$

$$\hat{\xi}^{\beta} = \text{Re}\ \hat{\xi}^{\beta} e^{j(\omega t - m\theta - kz)} \qquad (m \text{ integer}) \tag{20}$$

where

$$\hat{v}_r\ (\alpha) = j(\omega - kU_{\Delta})\ \hat{\xi}^{\alpha} = j\omega' \hat{\xi}^{\alpha} \tag{21}$$

$$\hat{v}_r(\beta) = j(\omega - kU_{\Delta})\ \ \hat{\xi}^{\alpha} = j\omega' \hat{\xi}^{\beta} \tag{22}$$

$$\nabla^2 \pi' = 0\ ; \quad \pi' = p' + q_{\Delta}\phi' \tag{23}$$

$$\nabla^2 \phi' = 0 \tag{24}$$

The solutions to Laplace's equation here are

$$\hat{\phi}\ (r) = A\ J_m\ (jkr) + B\ H_m\ (jkr) \tag{25}$$

where J_m (jkr) is the m^{th} order Bessel's function and H_m (jkr) is the m^{th} order Hankel function of first kind, both with pure imaginary arguments.

Then using the momentum equation and (21) - (25) we obtain the terminal relations

$$\begin{bmatrix} \hat{p}^{\alpha} \\ \hat{p}^{\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \hat{\xi}^{\alpha} \\ \hat{\xi}^{\beta} \end{bmatrix} - q_{\Delta} \begin{bmatrix} \hat{\phi}^{\alpha} \\ \hat{\phi}^{\beta} \end{bmatrix} \tag{26}$$

where

$$A_{11} = \frac{j\omega^2 \rho_{\Delta}}{k\mathrm{Det}_A} [J_m(jk\alpha)H_m'(jk\beta) - H_m(jk\alpha)J_m'(jk\beta)]$$

$$A_{12} = \frac{j\omega^2 \rho_{\Delta}}{k\mathrm{Det}_A} [H_m(jk\alpha)J_m'(jk\alpha) - J_m(jk\alpha)H_m'(jk\alpha)]$$

$$A_{21} = \frac{j\omega^2 \rho_{\Delta}}{\mathrm{Det}_A} [J_m(jk\beta)H_m'(jk\beta) - H_m(jk\beta)J_m'(jk\beta)]$$

$$A_{22} = \frac{j\omega^2 \rho_{\Delta}}{k\mathrm{Det}_A} [H_m(jk\beta)J_m'(jk\alpha) - J_m(jk\beta)H_m'(jk\alpha)]$$

$$\mathrm{Det}_A = [J_m'(jk\beta)\, H_m'(jk\alpha) - H_m'(jk\beta)J_m'(jk\alpha)]$$

and $$J_m'(jkx) = \frac{dJ_m(jkx)}{d(jkx)} \quad ; \quad H_m'(jkx) = \frac{dH_m(jkx)}{d(jkx)}$$

$$\begin{bmatrix} \hat{Q}^{\alpha} \\ \hat{Q}^{\alpha} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \hat{\phi}^{\alpha} + E^{\alpha}\hat{\xi}^{\alpha} \\ \hat{\phi}^{\beta} + E^{\beta}\hat{\xi}^{\beta} \end{bmatrix} + \begin{bmatrix} -\varepsilon_{\Delta} \left.\frac{\partial E_o}{\partial r}\right|_{r=\alpha} \hat{\xi}^{\alpha} \\ \varepsilon_{\Delta} \left.\frac{\partial E_o}{\partial r}\right|_{r=\beta} \hat{\xi}^{\beta} \end{bmatrix} \tag{27}$$

where

$$B_{11} = \frac{jk\varepsilon_\Delta}{Det_B} [J_m(jk\beta)H_m'(jk\alpha) - H_m(jk\beta)J_m'(jk\alpha)]$$

$$B_{12} = \frac{jk\varepsilon_\Delta}{Det_B} [H_m(jk\alpha)J_m'(jk\alpha) - J_m(jk\alpha)H_m'(jk\alpha)]$$

$$B_{21} = \frac{jk\varepsilon_\Delta}{Det_B} [H_m(jk\beta)J_m'(jk\beta) - J_m(jk\beta)H_m'(jk\beta)]$$

$$B_{22} = \frac{jk\varepsilon_\Delta}{Det_B} [J_m(jk\alpha)H_m'(jk\beta) - H_m(jk\alpha)J_m'(jk\beta)]$$

$$Det_B = J_m(jk\beta)H_m(jk\alpha) - J_m(jk\alpha)H_m(jk\beta)$$

For the special case when

$$\beta = 0$$

(26) and (27) reduce to

$$\hat{p}^\alpha = \omega'^2 \rho_\Delta \alpha \frac{J_m(jk\alpha)}{jk\alpha J_m'(jk\alpha)} \hat{\xi}^\alpha \tag{28}$$

$$\hat{Q}^\alpha = \frac{\varepsilon_\Delta}{\alpha} \frac{jk\alpha J_m'(jk\alpha)}{J_m(jk\alpha)} (\hat{\phi}^\alpha + E^\alpha \hat{\xi}^\alpha) - \varepsilon_\Delta \frac{\partial E_o}{\partial r}\Big|_{r=\alpha} \hat{\xi}_\alpha \tag{29}$$

and when

$$\alpha = \infty$$

$$\hat{p}^\beta = \omega'^2 \rho_\Delta \beta \frac{H_m(jk\beta)\ \hat{\xi}^\beta}{jk\beta H_m'(jk\beta)} \tag{30}$$

$$\hat{\Omega}^{\beta} = \frac{-\varepsilon_{\Delta}}{\beta} \frac{jk\beta H_m'(jk\beta)}{H_m(jk\beta)} (\hat{\phi}^{\beta} + E^{\beta}\hat{\xi}^{\beta}) + \varepsilon_{\Delta} \frac{\partial E_o}{\partial r}\bigg|_{r=\beta} \hat{\xi}^{\beta} \tag{31}$$

We list below some properties of the Bessel functions and plot in Fig. 2 some combinations which often appear.[41]

$$J_{-m}(jx) = (-1)^m J_m(jx) \quad (m \text{ integer}) \tag{32}$$

$$H_{-m}(jx) = e^{jm\pi} H_m(jx) \quad [\Gamma(m) = (m-1)!] \tag{33}$$

$$J_m(-jx) = e^{jm\pi} J_m(jx) \tag{34}$$

$$H_m(-jx) = -e^{-jm\pi}[2J_m(jx) - H_m(jx)] \tag{35}$$

$$J_m'(jx) = \frac{m}{jx} J_m(jx) - J_{(m+1)}(jx) \tag{36}$$

$$H_m'(jx) = \frac{m}{jx} H_m(jx) - H_{(m+1)}(jx) \tag{37}$$

lim x → 0 (x real)

$$J_m(jx) \to \frac{(\frac{1}{2} jx)^m}{\Gamma(m+1)} \tag{38}$$

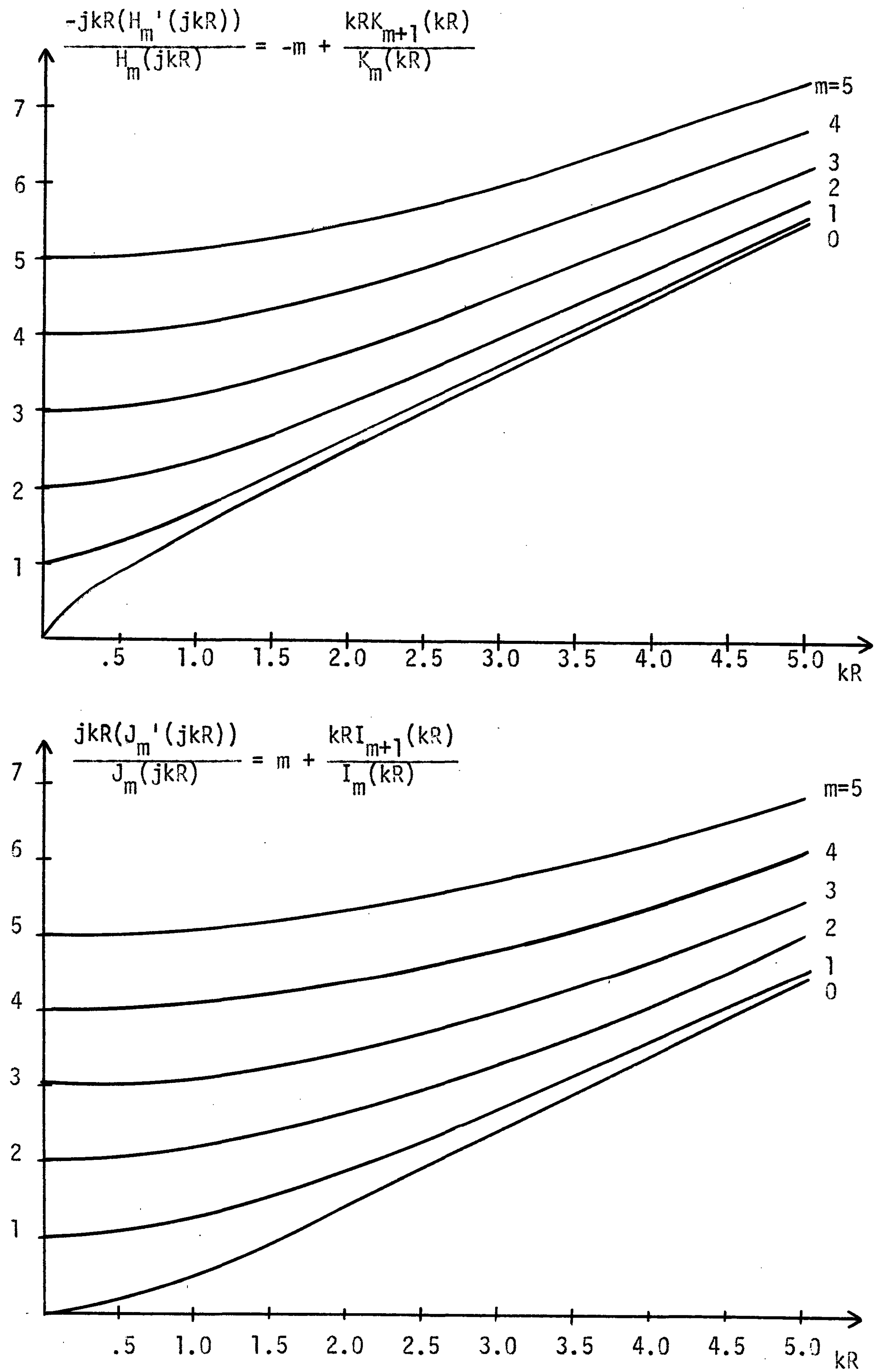
$$H_m(jx) \to -\frac{j}{\pi} \Gamma(m)(\frac{1}{2} jx)^{-m} \tag{39}$$

lim x → ∞ (x real)

$$J_m(jx) = \frac{(j)^m e^x}{\sqrt{2\pi x}} \tag{40}$$

$$H_m(jx) = \frac{2}{\pi} j^{-(m+1)} \sqrt{\frac{\pi}{2x}} e^{-x} \tag{41}$$

It is also sometimes useful to define modified Bessel functions



Plots of Bessel function combinations which often appear.

Figure 2

which have real arguments as below.

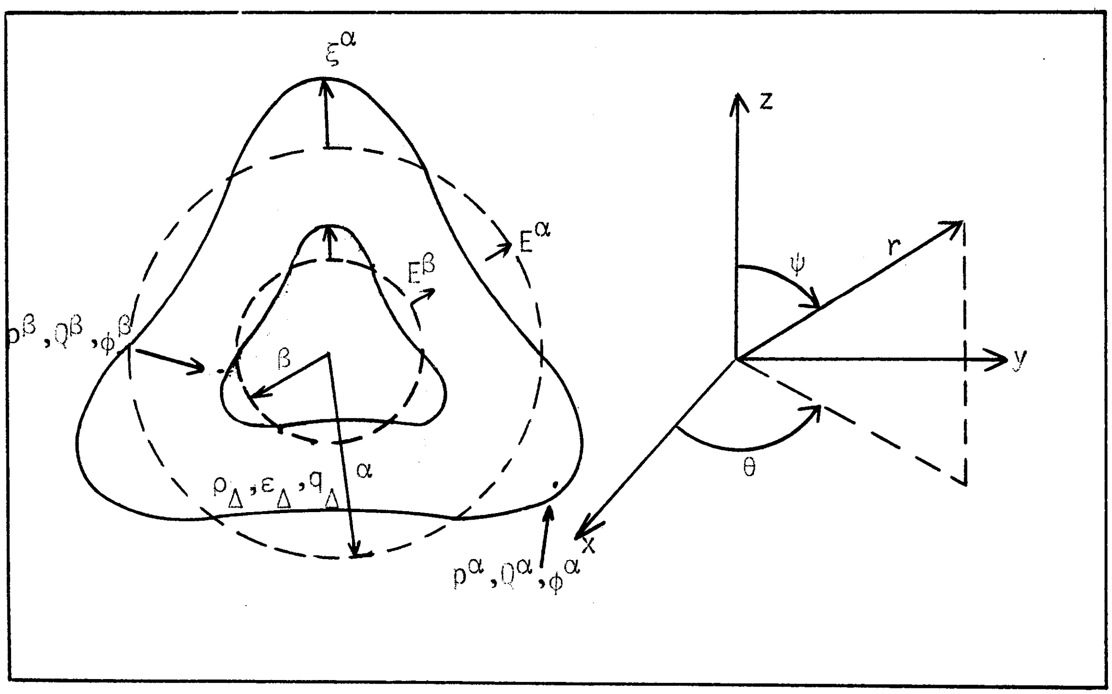
$$I_m(x) = j^{-m} J_m(jx) \tag{42}$$

$$K_m(x) = \frac{\pi}{2} j^{m+1} H_m(jx) \tag{43}$$

Using the previous relations, we see that both terminal relations are even in m and k. In the absence of geometry or excitations which have preferential directions, we can assume m and k to be positive.

2.5 Spherical Geometry

We now consider a spherical shell of perfectly-insulating, inviscid, incompressible fluid with constant mass density ρ_Δ and charge density q_Δ, as shown below. Our analysis proceeds in the same fashion as for the previous geometries, but we neglect the effect of gravity.



Definition of terminal variables for a prototype spherical shell of incompressible, inviscid, perfectly-insulating fluid with uniform charge density q_Δ.

Figure 1

In equilibrium, the net force must be zero,

$$p_o + q_\Delta \Phi_o = \text{constant} \tag{1}$$

with

$$\Phi_o = -\frac{q_\Delta r^2}{6\varepsilon_\Delta} + \frac{V_o R_o}{r} + \text{constant} \tag{2}$$

V_o, R_o some constants

The last two terms in (2) represent the Laplacian part of the equilibrium potential. Then

$$E_o = -\nabla\Phi_o = \frac{q_\Delta r}{3\varepsilon_\Delta} + \frac{V_o R_o}{r^2} \tag{3}$$

We proceed as usual with a perturbation analysis. The method is to solve the bulk equations, which amounts to solving Laplace's equation operating on $\pi'(\pi' = p' + q_\Delta\phi')$ and ϕ'. We obtain the bulk solution in terms of two arbitrary constants, which we then relate to the interfacial displacements ξ^α and ξ^β. In spherical geometry, we assume that the perturbations take the form

$$p' = \text{Re}\ \hat{p}(r)\ Y_{mn}(\psi,\theta)\ e^{j\omega t} \tag{4}$$

$$\xi = \text{Re}\ \hat{\xi} Y_{mn}(\psi,\theta)\ e^{j\omega t} \tag{5}$$

where Y_{mn} are the spherical harmonics, which are related to Legendre's functions as

$$Y_{mn} = e^{-jm\theta}\ P_n^m\ (\cos\psi) \tag{6}$$

with m and n positive integers and $m \leq n$.

Again, we wish to express the mechanical terminal relation in terms of the pressure, displacement and potential at each interface, for which we obtain

$$\begin{bmatrix} \hat{p}^{\alpha} \\ \\ \hat{p}^{\beta} \end{bmatrix} = C \begin{bmatrix} \dfrac{-\alpha^{n}\beta^{-(n+2)}}{n} - \dfrac{\alpha^{-(n+1)}\beta^{n-1}}{n+1} & \dfrac{2n+1}{\alpha^{2}n(n+1)} \\ \\ \dfrac{-(2n+1)}{\beta^{2}n(n+1)} & \dfrac{\beta^{n}\alpha^{-(n+2)}}{n} + \dfrac{\beta^{-(n+1)}\alpha^{n-1}}{n+1} \end{bmatrix} \begin{bmatrix} \hat{\xi}^{\alpha} \\ \\ \hat{\xi}^{\beta} \end{bmatrix} - q_{\Delta} \begin{bmatrix} \hat{\phi}^{\alpha} \\ \\ \hat{\phi}^{\beta} \end{bmatrix} \tag{7}$$

where

$$C = \frac{\omega^{2}\rho_{\Delta}}{(\beta^{n-1}\alpha^{-(n+2)} - \beta^{-(n+2)}\alpha^{n-1})}$$

For the special case when

$$\beta = 0$$

(7) reduces to

$$\hat{p}^{\alpha} = \frac{\omega^{2}\rho_{\Delta}\alpha}{n}\hat{\xi}^{\alpha} - q_{\Delta}\hat{\phi}^{\alpha} \tag{8}$$

and when

$$\alpha = \infty$$

we obtain

$$\hat{p}^{\beta} = \frac{-\omega^{2}\rho_{\Delta}\beta}{n+1}\hat{\xi}^{\beta} - q_{\Delta}\hat{\phi}^{\beta} \tag{9}$$

Again, all variables are evaluated at the interfacial position, since we have accounted for linear changes in the equilibrium. We define

$$\hat{Q}^{\alpha} = -\varepsilon \hat{e}_r^{\alpha} \tag{10}$$

$$\hat{Q}^{\beta} = +\varepsilon \hat{e}_r^{\beta} \tag{11}$$

whereupon we obtain the electrical terminal relation as

$$\begin{bmatrix} \hat{Q}^{\alpha} \\ \\ \hat{Q}^{\beta} \end{bmatrix} = A \begin{bmatrix} [-n\beta^{-(n+1)}\alpha^{n-1} - (n+1)\alpha^{-(n+2)}\beta^{n}] & \dfrac{2n+1}{\alpha^2} \\ \\ \dfrac{2n+1}{\beta^2} & [-n\beta^{n-1}\alpha^{-(n+1)} - (n+1)\alpha^{n}\beta^{-(n+2)}] \end{bmatrix} \begin{bmatrix} \hat{\phi}^{\alpha} + E^{\alpha}\hat{\xi}^{\alpha} \\ \\ \phi^{\beta} + E^{\beta}\hat{\xi}^{\beta} \end{bmatrix}$$

$$+ \begin{bmatrix} -\varepsilon_{\Delta} \left.\dfrac{\partial E_o}{\partial r}\right|_{r=\alpha} \hat{\xi}^{\alpha} \\ \\ +\varepsilon_{\Delta} \left.\dfrac{\partial E_o}{\partial r}\right|_{r=\beta} \hat{\xi}^{\beta} \end{bmatrix} \tag{12}$$

where

$$A = \frac{\varepsilon_{\Delta}}{[+\beta^{n}\alpha^{-(n+1)} - \alpha^{n}\beta^{-(n+1)}]}$$

For the special case

$$\beta = 0$$

(12) reduces to

$$\hat{Q}^{\alpha} = \frac{\varepsilon_{\Delta} n}{\alpha} (\hat{\phi}^{\alpha} + E^{\alpha}\hat{\xi}^{\alpha}) - \varepsilon_{\Delta} \left.\frac{\partial E_o}{\partial r}\right|_{r=\alpha} \hat{\xi}^{\alpha}$$

and when

$$\alpha = \infty$$

$$\hat{Q}^{\beta} = \frac{\varepsilon_{\Delta}(n+1)}{\beta}\left(\hat{\phi}^{\beta} + E^{\beta}\hat{\xi}^{\beta}\right) + \varepsilon_{\Delta}\left.\frac{\partial E_{o}}{\partial r}\right|_{r=\beta}\hat{\xi}^{\beta}$$

The transfer relations given by (7) and (12) complete our description in spherical geometry. The value of these relations will be illustrated with some examples in Chapter III.

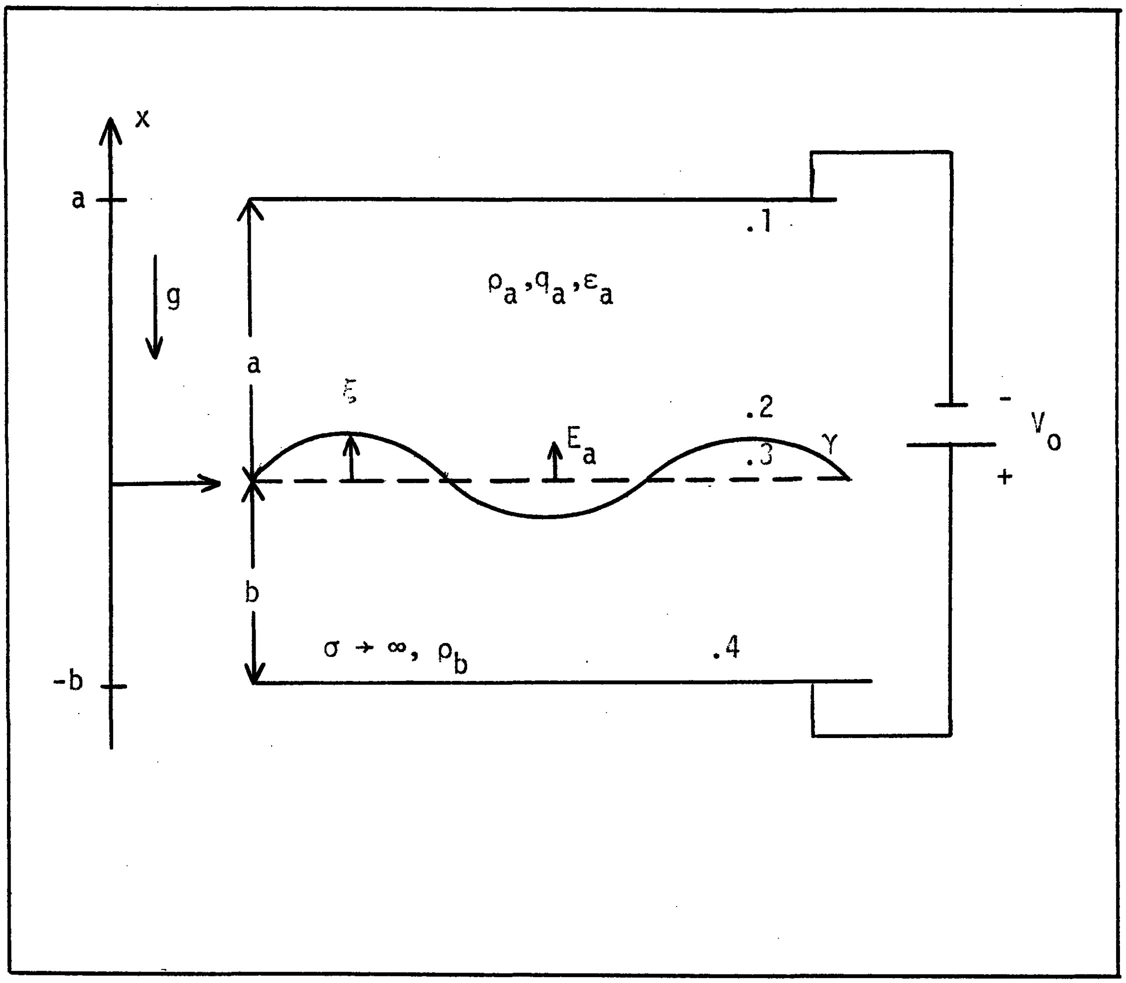
CHAPTER III
PROPAGATION CHARACTERISTICS AND STABILITY OF SIMPLE STRATIFIED EQUILIBRIA

In this section we will consider problems which will illustrate the use and value of the transfer relations derived in the previous chapter. These problems will have two regions with the boundary conditions at the interface dictating the manner in which the two regions are joined together. This is particularly convenient since our transfer relations already relate interfacial variables.

3.1 Perfectly Conducting Fluid Bounded from Above by a Layer of Uniformly Charged Fluid

Extensive analysis has been performed for this configuration, excluding the effects of space charge.[18] If the upper fluid is air, while the lower fluid is conducting (like water), measured frequency shifts as a function of applied voltage agree well with a theory that does not include space charge in the air. (See ref. 18, p.75) For a two liquid system, where the upper fluid is very insulating (like a hexane-water system), experimental results often do not agree with this theory.[42] The effects of space charge in the insulating liquid, either due to the conduction mechanism, impurities or ionic emission from the electrodes, may account for this discrepancy. Although such a charge distribution probably has spatial variations, our first attempt at analysis assumes the liquid to be uniformly charged.

In particular, we consider a perfectly-conducting fluid with density ρ_b and depth b which is bounded from below by a rigid wall and



Definition of variables for a perfectly conducting interface stressed by a normal field and bounded from above by an insulating fluid supporting uniform space charge.

Figure 1

above by a stationary perfectly-insulating fluid of height a, mass density ρ_a, charge density q_a, and dielectric constant ε_a. The interface has surface tension γ. (A table for surface tension tractions may be found in the Appendix.) We wish to determine the dispersion equation for the electromechanical waves. The boundary conditions for this problem are

$$\hat{\xi}_1 = 0 \qquad \hat{\phi}_1 = 0 \qquad \hat{\phi}_2 = \hat{\phi}_3 = 0$$
$$\hat{\xi}_4 = 0 \qquad \hat{\phi}_4 = 0 \qquad \hat{\xi}_2 = \hat{\xi}_3 = \hat{\xi}$$

$$\hat{p}_3 - \hat{p}_2 + E_a\varepsilon_a\hat{e}_x(\hat{\xi}) - \gamma k^2\hat{\xi} = 0 \tag{1}$$

From our terminal relations of 2.3.24 we obtain

$$\hat{p}_2 = \left(\frac{-\rho_a\omega^2}{k} \coth ka - \rho_a g\right)\hat{\xi} \tag{2}$$

$$\hat{p}_3 = \left(\frac{\rho_b\omega^2}{k} \coth kb - \rho_b g\right)\hat{\xi} \tag{3}$$

The electrical terminal relation of 2.3.28 yields

$$\hat{\Omega}_2 = \varepsilon_a\hat{e}_x(\hat{\xi}) = \varepsilon_a k \coth ka \,(E_a\hat{\xi}) + q_a\hat{\xi} \tag{4}$$

Combining (1) - (4) we obtain

$$\frac{\omega^2}{k}(\rho_a \coth ka + \rho_b \coth kb) = g(\rho_b - \rho_a) - \varepsilon_a k E_a^2 \coth ka$$
$$- q_a E_a + \gamma k^2 \tag{5}$$

where

$$E_a = \frac{V_o}{a} - \frac{q_a a}{2\varepsilon_a} \tag{6}$$

In the absence of electric fields, our problem reduces to the classic Rayleigh-Taylor problem. If the heavier fluid is on the bottom,

$$\rho_b > \rho_a \tag{7}$$

the gravity term is stabilizing. Surface tension is always stabilizing in planar geometry, so we have gravity capillary waves at the interface.

If the heavier fluid is on top ($\rho_b < \rho_a$), then the system will be unstable for long enough wavelengths. This first occurs at the Taylor wavelength

$$\lambda^* = 2\pi\sqrt{\frac{\gamma}{g(\rho_b - \rho_a)}} \tag{8}$$

This is known as the Rayleigh-Taylor instability.

We note that for the product $q_a E_a$ negative, we have a stabilizing force. The question arises as to whether this term can ever exceed the destabilizing effect of the other electrical force term due to the surface charge on the interface. We use (6) to rewrite (5) as

$$\frac{\omega^2}{k}(\rho_a \coth ka + \rho_b \coth kb) = g(\rho_b - \rho_a) - \frac{\varepsilon_a V_o^2}{a^3} ka \coth ka + q_a \left(\frac{V_o}{a} - \frac{q_a a}{2\varepsilon_a}\right)(ka \coth ka - 1) + \gamma k^2 \tag{9}$$

Now, (ka coth ka - 1) is always positive, so for the space charge to stabilize the system, we must have

$$\frac{\varepsilon V_o^3}{a^3} \frac{ka \coth ka}{(ka \coth ka-1)} + \frac{q_a^2 a}{2\varepsilon_a} - \frac{q_a V_o}{a} < 0 \tag{10}$$

We find that there are no real values of q_a or V_o to satisfy this inequality for any k, so we conclude that the net electrical force is always destabilizing.

If the heavier fluid is on the bottom such that gravity stabilized the system, as we increase the electric field there is a critical value when the interface first becomes unstable. If we define the right-hand side of (5) as

$$\eta = g(\rho_b - \rho_a) + \gamma k^2 - \varepsilon_a k E_a^2 \coth ka - q_a E_a \tag{11}$$

the threshold for instability will occur when

$$\eta = 0 \tag{12}$$

and

$$\frac{d\eta}{dk} = 0 \tag{13}$$

which results in the relations

$$W = \frac{(ka)^{*2} + G}{(ka)^* \coth (ka)^*} \tag{14}$$

and

$$\sinh 2(ka)^* [(ka)^{*2} - G] + 2(ka)^* [(ka)^{*2} + G] = 0 \tag{15}$$

where

$$G = [g(\rho_b - \rho_a) - q_a E_a] \frac{a^2}{\gamma} \tag{16}$$

and

$$W = \frac{\varepsilon_a E_a^{\,2} a}{\gamma} \tag{17}$$

We use the results and similar notation defined by Melcher in his solution of the simpler problem when $q_a = 0$. (See ref. 18, p.61-65) Fig. 2 plots the wavenumber and parameter W for impending instability as a function of the parameter G. However, we can obtain some closed form solutions for the long and short wavelength limits.

Long Wavelength Limit

When the distance to the upper electrode is closer than wavelengths of interest

$$(ka)^* << 1 \tag{18}$$

for which

$$\coth (ka)^* \simeq \frac{1}{(ka)^*} \tag{19}$$

$$\sinh (ka)^* \simeq (ka)^* \tag{20}$$

Then the solution to (15) is

$$(ka)^* \simeq 0 \tag{21}$$

which from (14), (16) and (17) yields

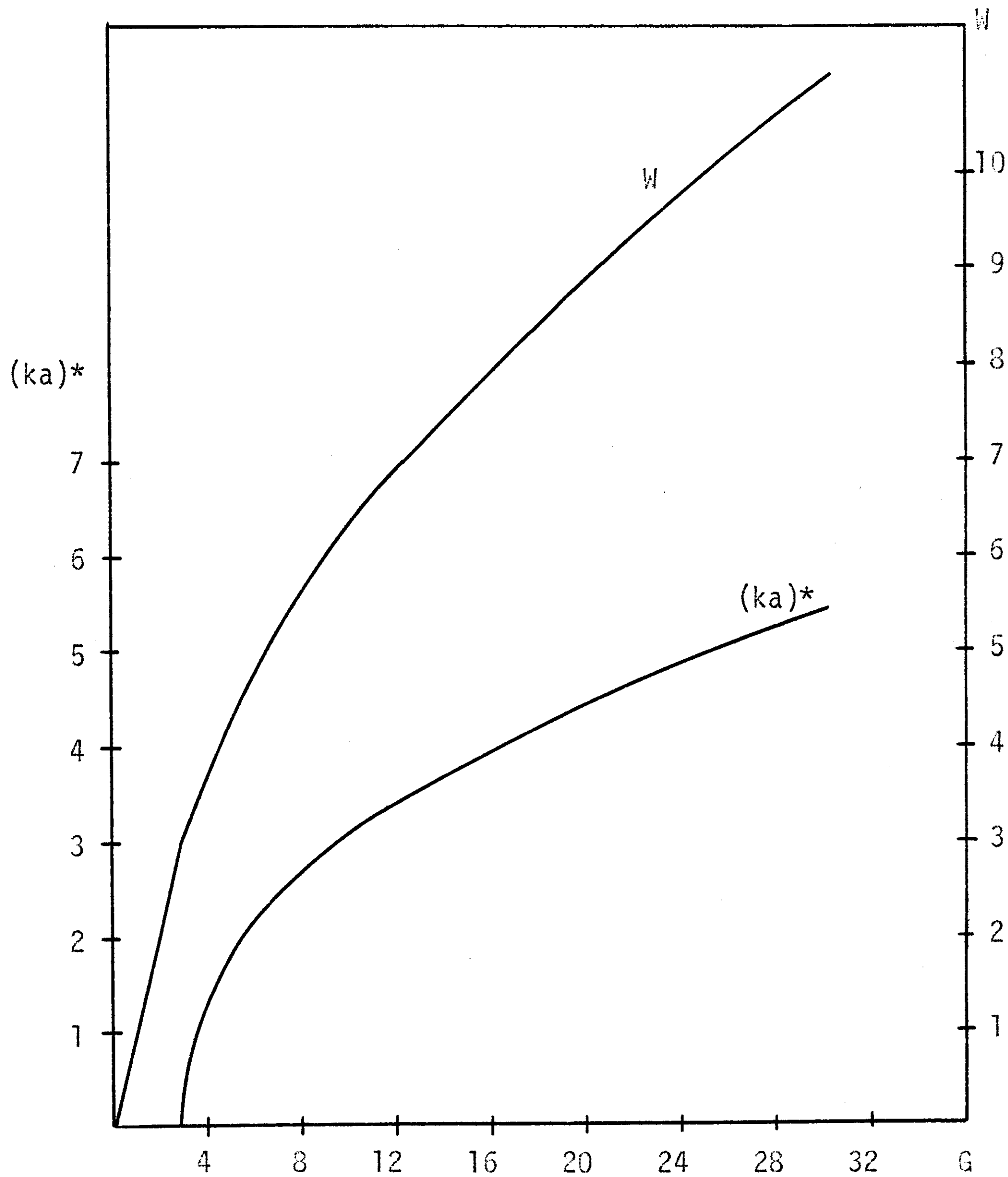
$$W = G \tag{22}$$

or

$$\frac{\varepsilon_a E_a^{\,2}}{a} + q_a E_a - g\,(\rho_b - \rho_a) = 0 \tag{23}$$

Upon solving and using (6) we obtain

$$V_o = \pm\, a \sqrt{\left(\frac{q_a a}{2\varepsilon_a}\right)^2 + \frac{g(\rho_b - \rho_a)\, a}{\varepsilon_a}} \tag{24}$$



Relation between the parameter G, W and (ka)* at incipience of instability.

Figure 2

Short Wavelength Limit

If the upper electrode is farther away than wavelengths of interest

$$(ka)^* \gg 1 \tag{25}$$

for which

$$\coth\,(ka)^* \simeq 1 \tag{26}$$

and

$$\sinh\,(ka)^* \gg 1 \tag{27}$$

Then the solutions to (14) and (15) are

$$(ka)^* = \sqrt{G} = \sqrt{[g(\rho_b - \rho_a) - q_a E_a]\,\frac{a^2}{\gamma}} \tag{28}$$

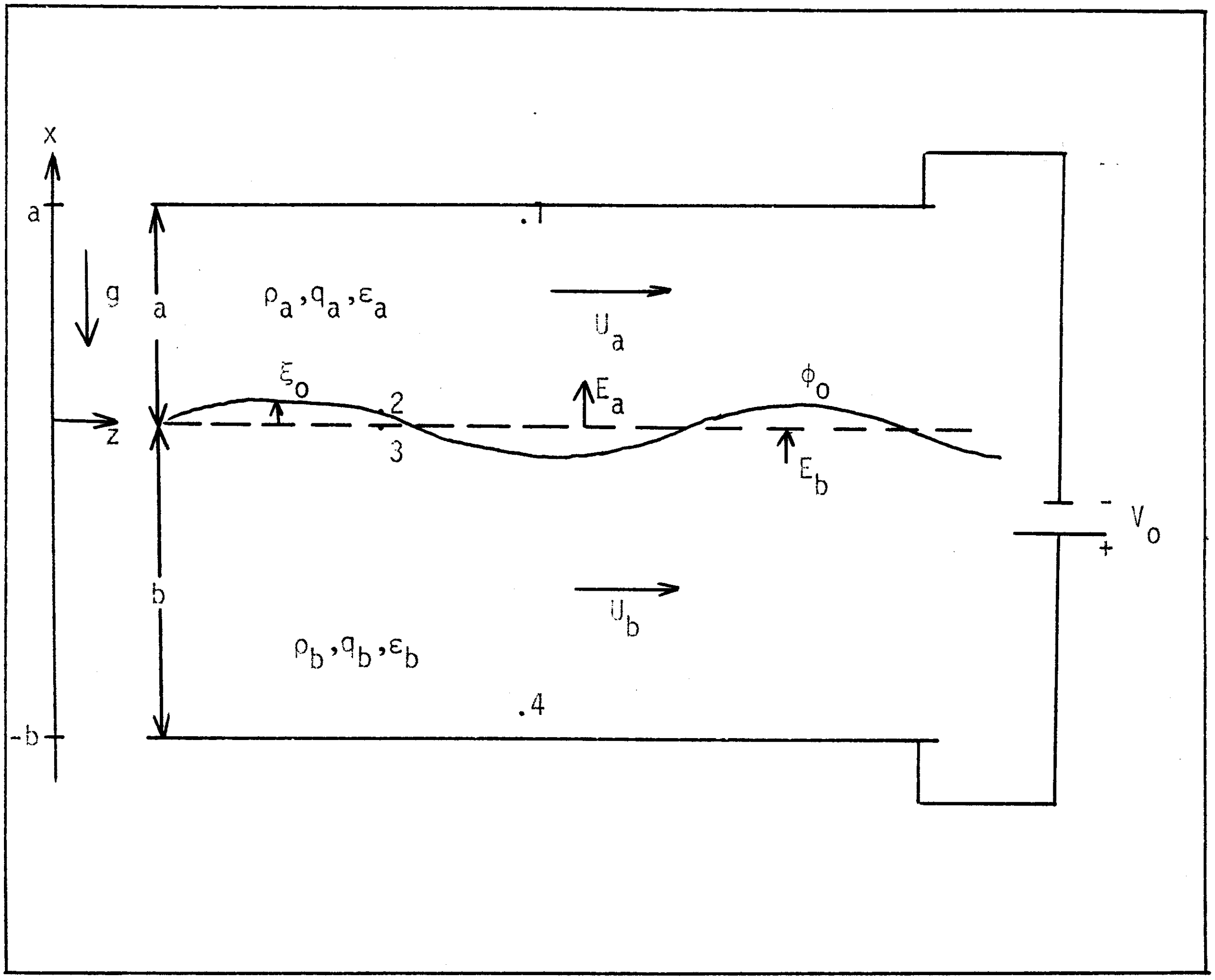
and

$$W = 2\sqrt{G} = 2\,(ka)^* \tag{29}$$

3.2 Two Superposed Charged Layers In Relative Motion

This problem is of interest for it will demonstrate the existence of space charge and polarization waves, in addition to the classic capillary waves and Rayleigh-Taylor and Kelvin-Helmholtz instabilities.[30] Aspects of this problem also model some conceivable charge distributions that appear when a liquid is stressed by an electric field. This general problem provides the first approximation to continuously stratified distributions.

We consider two planar layers of uniformly charged fluids bounded from above and below by rigid perfectly conducting plates as shown in Figure 1. These layers each have their own uniform equilibrium mass and space charge density, convection velocity, and dielectric constant



Definition of variables for two uniformly charged layers in relative motion.

Figure 1

with a potential difference of V_0 across the system. The interface has surface tension γ. (A table for surface tension tractions in rectangular, cylindrical and spherical geometry may be found in the appendix.) We wish to determine the dispersion relation for the electromechanical waves propagating on the interface.

The electromechanical waves depend on the equilibrium electric field intensities to each side of the equilibrium interface. For the system shown in Fig. 1, these quantities are found from an integration of 2.2.4 and 2.2.5 to be

$$E_a = \frac{\varepsilon_a}{\varepsilon_a b + \varepsilon_b a}\left[V_0 + \frac{q_b b^2}{2\varepsilon_b} - \frac{q_a a^2}{2\varepsilon_a}\right] \tag{1}$$

$$E_b = \frac{\varepsilon_a}{\varepsilon_a b + \varepsilon_b a}\left[V_0 + \frac{q_b b^2}{2\varepsilon_b} - \frac{q_a a^2}{2\varepsilon_a}\right] \tag{2}$$

Note that we do not allow any surface charge on the interface. Because of the rigid plates and constant voltage we have

$$\begin{aligned} \hat{\xi}_1 &= 0 & \hat{\phi}_1 &= 0 \\ \hat{\xi}_4 &= 0 & \hat{\phi}_4 &= 0 \end{aligned} \tag{3}$$

At the interface, conservation of mass requires that the fluid displacement be continuous and the irrotationality of $\overline{E}$ requires that the potential be continuous

$$\begin{aligned} \hat{\xi}_2 &= \hat{\xi}_3 = \hat{\xi}_0 \\ \hat{\phi}_2 &= \hat{\phi}_3 = \hat{\phi}_0 \end{aligned} \tag{4}$$

Because there is no surface charge on the interface, in writing force equilibrium we require that the pressure be discontinuous only because of polarization and surface tension effects which implies

$$\hat{p}_2 - \hat{p}_3 = \varepsilon_a E_a \hat{e}_{x_2} - \varepsilon_b E_b \hat{e}_{x_3} - \gamma k^2 \hat{\xi}_0 = E_a \hat{Q}_2 + E_b \hat{Q}_3 - \gamma k^2 \hat{\xi}_0 \tag{5}$$

and because there is no surface charge, we also must have

$$\hat{Q}_2 + \hat{Q}_3 = 0 \tag{6}$$

We use these boundary conditions in connection with the transfer relations of section 2.3 which are now identified first with one of the fluid layers and then with the other. From 2.3.28 in conjunction with (3) and (4) we have

$$\hat{Q}_2 = \varepsilon_a k \coth ka \, (\hat{\phi}_0 + E_a \hat{\xi}_0) + q_a \hat{\xi}_0 \tag{7}$$

$$\hat{Q}_3 = \varepsilon_b k \coth kb \, (\hat{\phi}_0 + E_b \hat{\xi}_0) - q_b \hat{\xi}_0 \tag{8}$$

If we add these relations and use (6) we find

$$\hat{\phi}_0 = \frac{-\hat{\xi}_0 \, [(q_a - q_b) + k(\varepsilon_a E_a \coth ka + \varepsilon_b E_b \coth kb)]}{k(\varepsilon_a \coth ka + \varepsilon_b \coth kb)} \tag{9}$$

To express force equilibrium on the interface, we use 2.3.24 for the upper and then lower regions, to obtain

$$\hat{p}_2 = - [\frac{\rho_a}{k} (\omega - kU_a)^2 \coth ka + \rho_a g] \, \hat{\xi}_0 - q_a \hat{\phi}_0 \tag{10}$$

$$\hat{p}_3 = [\frac{\rho_b}{k} (\omega - kU_b)^2 \coth kb - \rho_b g] \, \hat{\xi}_0 - q_b \hat{\phi}_0 \tag{11}$$

Then using (7) through (11) in (5) we obtain the desired dispersion relation as

$$\frac{(\omega - kU_a)^2}{k} \rho_a \coth ka + \frac{(\omega - kU_b)^2}{k} \rho_b \coth kb = g(\rho_b - \rho_a) + \gamma k^2$$

$$+ (E_a q_a - E_b q_b) + \frac{(q_a - q_b)^2}{k(\varepsilon_a \coth ka + \varepsilon_b \coth kb)}$$

$$-2(\varepsilon_b - \varepsilon_a) \frac{(q_a E_a \coth kb + q_b E_b \coth ka)}{\varepsilon_a \coth ka + \varepsilon_b \coth kb}$$

$$\frac{-k\,(\varepsilon_a - \varepsilon_b)^2 E_a E_b}{(\varepsilon_a \tanh kb + \varepsilon_b \tanh ka)} \qquad (12)$$

Within (12) there are a variety of interesting cases.

Kelvin-Helmholtz Instability

In the absence of electric fields, our problem reduces to the classic hydrodynamic problem of the instability due to the relative motion of two adjacent fluids. For this case, when the boundaries are much farther away than wavelengths of interest, such that

$$ka \gg 1$$

$$kb \gg 1$$

(12) reduces to

$$(\rho_a + \rho_b)\omega^2 - 2\omega k(\rho_a U_a + \rho_b U_b) - g(\rho_b - \rho_a)\,k - \gamma k^3$$

$$+ k^2\,(\rho_a U_a{}^2 + \rho_b U_b{}^2) = 0 \qquad (13)$$

which has solutions

$$\omega = \frac{k(\rho_a U_a + \rho_b U_b)}{\rho_a + \rho_b} \pm \sqrt{\frac{\gamma k^3 + g(\rho_b - \rho_a)k - \frac{k^2 \rho_a \rho_b}{(\rho_a + \rho_b)} (U_a - U_b)^2}{(\rho_a + \rho_b)}} \tag{14}$$

We see that no matter how small the difference of velocities, its effect is always destabilizing. For stability, we must have

$$\frac{\rho_a \rho_b (U_a - U_b)^2}{\rho_a + \rho_b} < \frac{g}{k}(\rho_b - \rho_a) + \gamma k \tag{15}$$

For $\rho_b > \rho_a$, the right hand side of (15) has a minimum at the Taylor wavelength, (3.1.8), and so surface tension will suppress instability if

$$(U_a - U_b)^2 < \frac{2}{\rho_a \rho_b} [\gamma g(\rho_b - \rho_a)]^{1/2} \tag{16}$$

The effects of convection only enter through the inertia term in (12), having no influence on the electrical terms. Thus, in the following analysis we will assume both layers to be initially stationary, such that $U_a = U_b = 0$.

Electrical Coupling $(U_a = U_b = 0)$

Melcher has previously considered this problem in the absence of free charge. Then from (12), we see that the polarization effects are always destabilizing. The new terms introduced because of the space charge change the character of the electrical coupling. We will examine some special cases to illustrate these space charge waves.

$\varepsilon_a = \varepsilon_b$

If the dielectric constant is the same for each region, then $E_a = E_b$ and (12) (in the absence of convection) becomes

$$\frac{\omega^2}{k}(\rho_a \coth ka + \rho_b \coth kb) = g(\rho_b - \rho_a) + \gamma k^2$$

$$+ E_a (q_a - q_b) + \frac{(q_a - q_b)^2}{k\varepsilon_a (\coth ka + \coth kb)} \qquad (17)$$

For simplicity we assume $a = b$

If we then define

$$\eta = G + (ka)^2 + \frac{Q}{ka \coth ka} \qquad (18)$$

where

$$G = \left[\frac{g(\rho_b - \rho_a)}{\gamma} + E_a (q_a - q_b)\right] \frac{a^2}{\gamma} \qquad (19)$$

and

$$Q = \frac{(q_a - q_b)^2 a^3}{2\varepsilon\gamma} \qquad (20)$$

instability is incipient when

$$\eta = 0 \qquad (21)$$

and

$$\frac{d\eta}{d(ka)} = 0 \qquad (22)$$

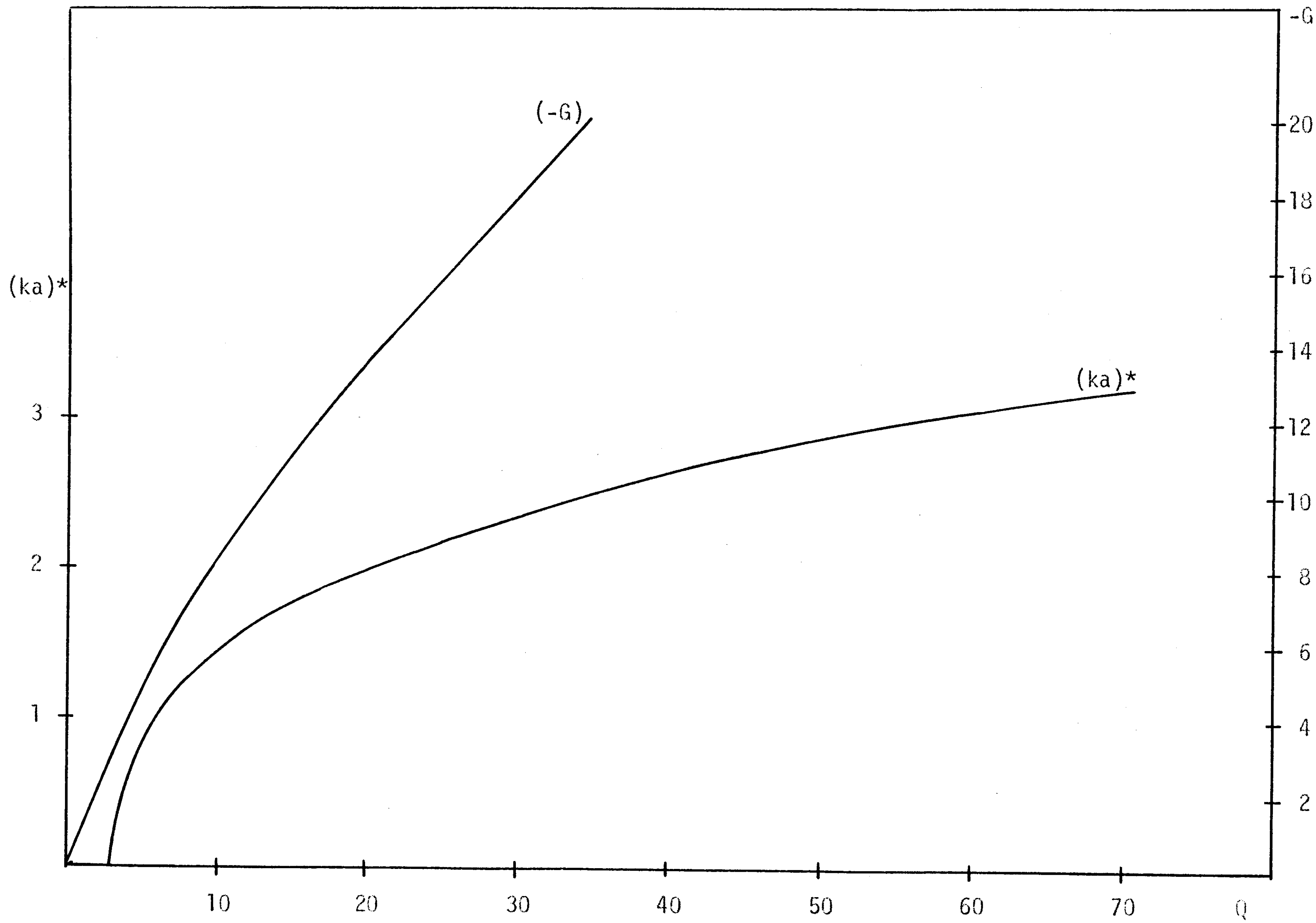
This last condition results in the relation

$$2(ka)^{*3} \cosh^2 (ka)^* + Q\,[(ka)^* - \frac{1}{2} \sinh 2(ka)^*] = 0 \qquad (23)$$

Fig. 2 shows the solution to (21) - (23). For instability G must be negative, either because of adverse gravity of because

$$E_a (q_a - q_b) < 0$$

The parameter Q tends to stabilize the system, so as Q increases, the



Relation between the parameters (-G), Q and (ka)* at incipience of instability.

Figure 2

magnitude of G must increase for instability to set in, for which (ka)* also increases.

For very long or short wavelengths, we can be more explicit.

Short Wavelength Limit

If

$$(ka)^* \gg 1 \tag{24}$$

we approximately obtain from (23) that

$$(ka)^{*3} = \frac{Q}{2} \tag{25}$$

and from (18)

$$G = -3 \left(\frac{Q}{2}\right)^{2/3} \tag{26}$$

Long Wavelength Limit

If

$$(ka)^* \ll 1 \tag{27}$$

(23) yields

$$(ka)^* = 0 \tag{28}$$

and then from (18)

$$G = -Q \tag{29}$$

"Space Charge Limited" Interface Condition

If the total bulk charge in each region is just the image of the corresponding surface charge on the electrodes, Gauss's Law requires that the electric field at the interface be zero. This condition will be true if

$$\sigma_f^{\,a} = -\varepsilon_a E(x=a) = \frac{-\varepsilon_a \varepsilon_b}{\varepsilon_a b + \varepsilon_b a}\left[V_0 + \frac{q_b b^2}{2\varepsilon_b} - \frac{q_a a^2}{2\varepsilon_a}\right] - q_a a = -q_a a \tag{30}$$

$$\sigma_f{}^b = \varepsilon_b E(x=-b) = \frac{\varepsilon_a \varepsilon_b}{\varepsilon_a b + \varepsilon_b a} \left[V_o + \frac{q_b b^2}{2\varepsilon_b} - \frac{q_a a^2}{2\varepsilon_a} \right] - q_b b = - q_b b \qquad (31)$$

which implies that

$$\frac{q_a a^2}{2\varepsilon_a} - \frac{q_b b^2}{2\varepsilon_b} = V_o \qquad (32)$$

which from (1) and (2) yield

$$E_a = E_b = 0 \qquad (33)$$

as expected.

Even though the electric field is zero at the interface, we still have a stabilizing contribution in (12) from the self-fields due to the space charge

$$\frac{\omega^2}{k} (\rho_a \coth ka + \rho_b \coth kb) = g(\rho_b - \rho_a) + \gamma k^2 + \frac{(q_a - q_b)^2}{k(\varepsilon_a \coth ka + \varepsilon_b \coth kb)} \qquad (34)$$

In the long wavelength limit (ka << 1, kb << 1), we see that the waves are non-dispersive with phase velocity

$$\frac{\omega}{k} = \frac{\pm}{\left(\frac{\rho_a}{a} + \frac{\rho_b}{b}\right)^{1/2}} \left[g(\rho_b - \rho_a) + \frac{(q_a - q_b)^2}{\left(\frac{\varepsilon_a}{a} + \frac{\varepsilon_b}{b}\right)} \right]^{1/2} \qquad (35)$$

$q_a = q_b$

If the charge density in the two regions are equal, (12) becomes

$$\frac{\omega^2}{k}(\rho_a + \rho_b)\coth ka = g(\rho_b - \rho_a) + \gamma k^2 - \frac{q_a E_a(\varepsilon_b - \varepsilon_a)}{\varepsilon_b} - \frac{k(\varepsilon_a - \varepsilon_b)^2 \frac{\varepsilon_a}{\varepsilon_b} E_a{}^2}{(\varepsilon_a + \varepsilon_b)\tanh ka} \tag{36}$$

where for simplicity, we assume

$$a = b \tag{37}$$

We note that if

$$\frac{q_a E_a(\varepsilon_b - \varepsilon_a)}{\varepsilon_b} < 0 \tag{38}$$

the space charge tends to stabilize the system, although for short wavelengths (ka >> 1) the destabilizing effects of the polarization forces dominate.

If we define

$$\eta = (ka)^2 + G - W\, ka \coth ka \tag{39}$$

where

$$G = \left[g(\rho_b - \rho_a) - \frac{q_a E_a(\varepsilon_b - \varepsilon_a)}{\varepsilon_b}\right]\frac{a^2}{\gamma} \tag{40}$$

and

$$W = \frac{(\varepsilon_a - \varepsilon_b)^2 \frac{\varepsilon_a}{\varepsilon_b} E_a{}^2}{(\varepsilon_a + \varepsilon_b)} \frac{a}{\gamma} \tag{41}$$

incipience of instability occurs when

$$\eta = 0 \tag{42}$$

and

$$\frac{d\eta}{d(ka)} = 0 \tag{43}$$

However, we note that the form of (39) is identical to (3.1.11) - (3.1.17). The results of that section including Fig. 3.1.2 can be used for this case if we use G and W as given by (40) and (41).

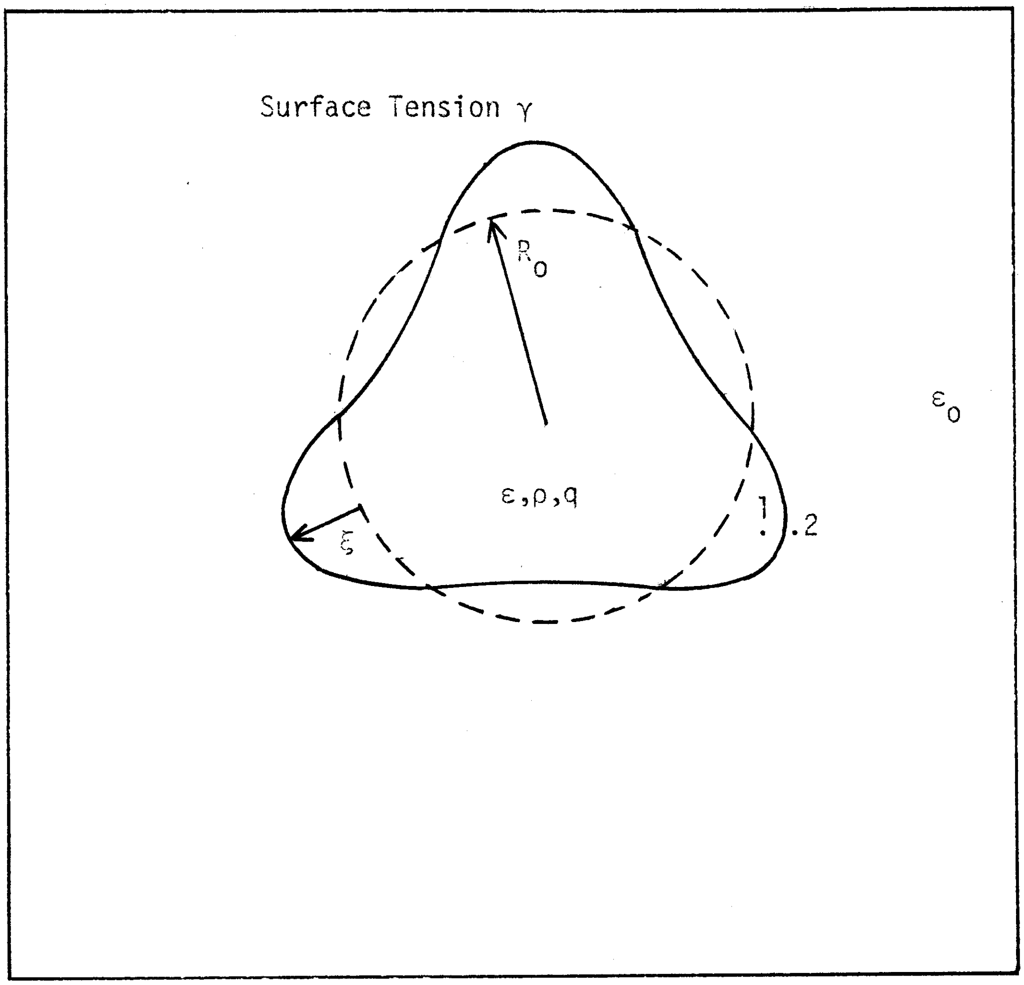
3.3 Stability of an Insulating Charged Fluid Cylinder in Free Space - No z Dependence

Investigators in the late 1800's found that the instability of a capillary jet could be partially arrested by inducing charges on the surface of the jet, such as by placing an electrified comb near the stream.[43] However, as will be shown in section 3.4, the surface charge on a perfectly conducting jet does not completely stabilize the system, but rather makes different wavelengths become more unstable. If the jet were insulating with a bulk charge, the behavior of the jet may be different. The problems treated in sections (3.3) - (3.6) address themselves to this problem, where we consider the equilibrium jet to be stationary. If the jet is convecting with velocity U in the z direction we simply let

$$\omega \rightarrow \omega - kU$$

We first consider a charged cylinder where wavelengths in the z direction are very long compared to dimensions of interest. This charged column is placed in free space and is shown in cross section in Fig. 1.

We include the effect of polarization forces by giving the cylinder a dielectric constant ε, which may be different from the free space permittivity. The fluid is described by a mass density ρ, charge density q and a surface tension γ. (A table for surface tension tractions in



Definition of terminal variables for charged cylinder in free space.

Figure 1

rectangular, cylindrical and spherical geometry may be found in the Appendix.) This is a two-region problem, since the free space region has electric fields due to the charged cylinder, even though there is no charge in free space. We can use our terminal relations of section 2.4, first for the charged column of fluid where $\alpha \to R_0$ and $\beta \to 0$, and then for the infinite free-space region where $\alpha \to \infty$ and $\beta \to R_0$.

We first calculate the equilibrium electric field to be

$$E_0 = \begin{cases} \dfrac{qr}{2\varepsilon} & r < R_0 \\[2ex] \dfrac{qR_0^{\,2}}{2\varepsilon_0 r} & r > R_0 \end{cases} \qquad (1)$$

Note that the electric field is discontinuous at $r = R_0$ if $\varepsilon \neq \varepsilon_0$, but of course the displacement vector $\overline{D} = \varepsilon\overline{E}$ is continuous, since we cannot allow any surface charge at the interface, because it is inconsistent with an inviscid model for the fluid.

The boundary conditions for this problem are:

$$\hat{\xi}_1 = \hat{\xi}_2 = \hat{\xi} \qquad (2)$$

$$\hat{\phi}_1 = \hat{\phi}_2 = \hat{\phi} \qquad (3)$$

$$\hat{Q}_1 + \hat{Q}_2 = 0 \qquad (4)$$

$$\hat{p}_1 - \hat{p}_2 + \varepsilon_0 E_2 \hat{e}_{r_2} - \varepsilon E_1 \hat{e}_{r_1} + \frac{\gamma}{R_0^2}(1-m^2)\hat{\xi} = 0 \qquad (5)$$

where

$$\varepsilon_o \hat{e}_{r_2} = \hat{Q}_2$$
$$-\varepsilon \hat{e}_{r_1} = \hat{Q}_1 \tag{6}$$

and

$$E_1 = \frac{q}{2\varepsilon} R_o \text{ and } E_2 = \frac{qR_o}{2\varepsilon_o} \tag{7}$$

From our terminal relations of section 2.4, we find

$$\hat{p}_1 = \frac{\rho\omega^2 R_o}{m} \hat{\xi} - q \hat{\phi} \tag{8}$$

$$\hat{p}_2 = 0 \tag{9}$$

$$\hat{Q}_1 = \frac{m\varepsilon}{R_o} (\hat{\phi} + E_1 \hat{\xi}) - \frac{q}{2} \hat{\xi} \tag{10}$$

$$\hat{Q}_2 = \frac{m\varepsilon_o}{R_o} (\hat{\phi} + E_2 \hat{\xi}) - \frac{q}{2} \hat{\xi} \tag{11}$$

Since there is no surface charge on the interface, we use (4) to obtain

$$\hat{\phi} = \frac{\frac{R_o}{m}[q - \frac{m}{R_o}(\varepsilon E_1 + \varepsilon_o E_2)]}{(\varepsilon + \varepsilon_o)} \hat{\xi} \tag{12}$$

Using relations (8) - (11) in (5) and using the facts that

$$\varepsilon E_1 = \varepsilon_o E_2 = \frac{qR_o}{2} \tag{13}$$

and

$$E_2 - E_1 = \frac{qR_o}{2\varepsilon_o \varepsilon} (\varepsilon - \varepsilon_o) \tag{14}$$

we find our dispersion relation to be

$$\frac{\rho\omega^2 R_0}{m} = (m-1) \left[\frac{\gamma(m+1)}{R_0^{\ 2}} - \frac{q^2 R_0}{(\varepsilon+\varepsilon_0)} \left(\frac{1}{m} + \frac{(\varepsilon-\varepsilon_0)^2}{4\varepsilon_0\varepsilon} \right) \right] \quad (15)$$

The m = 0 solution is not allowed, since this implies a pure expansion of the cylinder which is not allowed by mass conservation if there is no z-dependence. The m = 1 solution is stable, since this implies just a pure translation of the fluid cylinder; m > 1 may be either stable or unstable, depending on the relative strengths of surface tension and electrical forces. The electrical forces are destabilizing both for free charge forces and polarization forces. We can check our results with the general results derived by Turnbull and Melcher[11], that

$$-EDq - EDED\varepsilon > 0 \quad (16)$$

is destabilizing, where here $D = d/dr$. We can easily see that for q positive, $E > 0$ and $Dq < 0$ at the interface, so $EDq < 0$. If $\varepsilon > \varepsilon_0$, then $E_2 > E_1$, and vice versa if $\varepsilon < \varepsilon_0$, $E_2 < E_1$. In either case, $EDED\varepsilon < 0$, so this check does indeed agree with our results. Similar arguments hold for q negative. Surface tension always tends to stabilize the system.

As we increase the value of q, the system verges on instability when

$$q^2 = \frac{(\varepsilon + \varepsilon_0)\ \gamma\ (m + 1)}{R_0^{\ 3}\left[\frac{1}{m} + \frac{(\varepsilon-\varepsilon_0)^2}{4\varepsilon_0\varepsilon}\right]} \quad (17)$$

The first m value which becomes unstable is m = 2, so the critical value of charge is

$$q_{crit}^2 = \frac{(\varepsilon+\varepsilon_0)\ 6\gamma}{R_0^{\ 3}\ [1 + \frac{(\varepsilon-\varepsilon_0)^2}{2\varepsilon_0\varepsilon}\ ]} = \frac{(\varepsilon+\varepsilon_0)\ 12\varepsilon_0\varepsilon\gamma}{R_0^{\ 3}\ (\varepsilon^2 + \varepsilon_0^{\ 2})} \tag{18}$$

For the special case when $\varepsilon = \varepsilon_0$ we find

$$q_{crit}^2 = \frac{12\varepsilon\gamma}{R_0^{\ 3}} \tag{19}$$

3.4 Stability of a Perfectly Conducting Fluid Cylinder with Surface Charge - No z Dependence.

The question arises as to how the stab ility condition for the previous problem differs from the problem of a surface charge σ_f spread uniformly on the surface of a perfectly-conducting fluid cylinder. The terminal relations of section 2.4 are still correct; however, we must set q = 0, since there is no volume charge. We modify our boundary conditions to be

$$\hat{\xi}_1 = \hat{\xi}_2 = \hat{\xi} \tag{1}$$

$$\hat{\phi}_1 = \hat{\phi}_2 = 0 \tag{2}$$

$$\hat{p}_1 - \hat{p}_2 + E_2\hat{D}_2 + \frac{\gamma}{R_0^{\ 2}}(1-m^2)\ \hat{\xi} = 0 \tag{3}$$

Note that this problem is still consistent with an inviscid description of the fluid because it is perfectly-conducting. The electric fields must terminate normal to the interface, and so cannot create shear forces. The equilibrium electric field is

$$E_o = \begin{cases} 0 & r < R_o \\ \dfrac{\sigma_f}{\varepsilon_o}\dfrac{R_o}{r} & r \geq R_o \end{cases} \quad (4)$$

Thus, from our terminal relations, we obtain

$$\hat{p}_1 = \frac{\rho\omega^2 R_o}{m}\hat{\xi} \quad (5)$$

$$\hat{p}_2 = 0 \quad (6)$$

$$\hat{Q}_2 = \frac{m\varepsilon_o}{R_o} E_2\hat{\xi} - \frac{\sigma_f}{R_o}\hat{\xi} \quad (7)$$

Substituting these relations into (3), we obtain

$$\frac{\rho\omega^2 R_o}{m} = (m-1)\left[\frac{\gamma}{R_o^2}(m+1) - \frac{\sigma_f^2}{\varepsilon_o R_o}\right] \quad (8)$$

There are no polarization effects since there are no electric fields within the fluid.

If we assume that the net charge for each problem was the same, such that

$$q\pi R_o^2 = \sigma_f 2\pi R_o \quad (9)$$

we can write (8) as

$$\frac{\rho\omega^2 R_o}{m} = (m-1)\left[\frac{\gamma}{R_o^2}(m+1) - \frac{q^2 R_o}{4\varepsilon_o}\right] \quad (10)$$

The threshold for stability occurs with m = 2 with critical charge value

$$q^2_{crit} = \frac{12\varepsilon_o\gamma}{R_o^{\,3}} \qquad (11)$$

The critical value for the surface charge problem is identical to that for the bulk charge problem, with $\varepsilon = \varepsilon_o$, (3.3.19), although the mode dependences differ. The bulk charge cylinder will become unstable for lesser values of q if $\varepsilon \neq \varepsilon_o$, because the polarization forces are destabilizing.

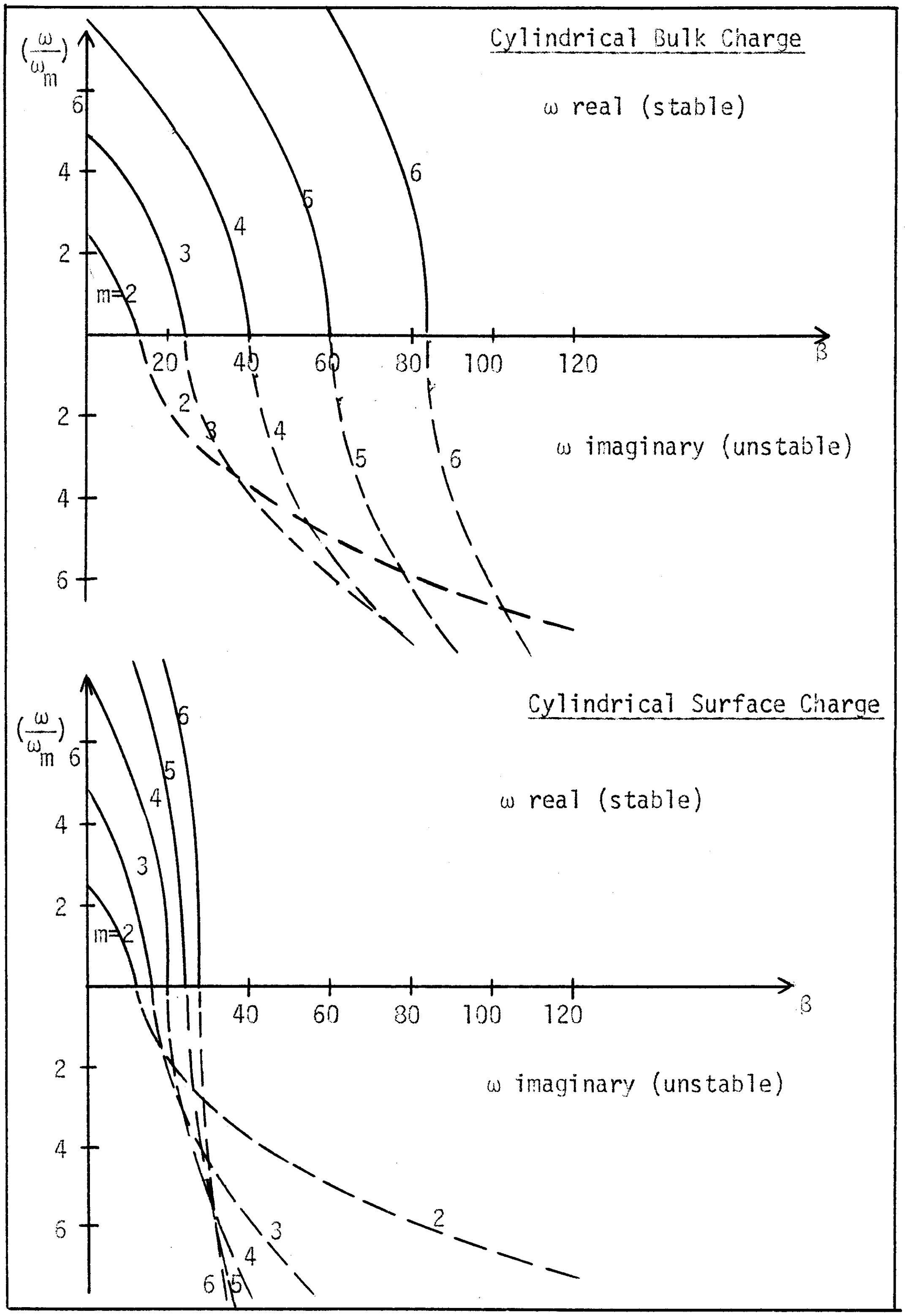
Comparison ($\varepsilon=\varepsilon_o$)

In Fig. 1 we plot the resonant frequency and growth rate for the two cases considered in sections (3.3) and (3.4) as a function of the non-dimensional parameter $\beta = \frac{q^2 R_o^3}{\varepsilon\gamma}$, which represents a ratio of electric forces to surface tension forces. The frequency is normalized to

$$\omega_m = \left[\frac{\gamma}{\rho R_o^{\,3}}\right]^{1/2}$$

The dotted lower part of each plot represents imaginary ω, and thus the growth and decay rates. The upper part is for real ω, and thus represents stable oscillations of the cylinder.

We see that the dispersion relation for the m = 2 mode is identical for the conducting and insulating cylinders, but that for higher modes, the conducting cylinder with surface charge becomes unstable sooner.



Regions of stability and instability for various modes for the cylindrical bulk and surface charge distributions as a function of the parameter $\beta = \frac{q^2 R_o^3}{\epsilon\gamma}$. $(\epsilon=\epsilon_o)$

Figure 1

3.5 Stability of an Insulating Stationary Charged Fluid Cylinder in Free Space - z and θ - Dependence Included

We consider the same problem as in section 3.3 including z variations. The boundary conditions are identical to those previously used in section 3.4, with a slight modification to the surface traction because of the z-dependence of the surface tension (see Appendix). We repeat the boundary conditions as

$$\hat{\xi}_1 = \hat{\xi}_2 = \hat{\xi} \tag{1}$$

$$\hat{\phi}_1 = \hat{\phi}_2 = \hat{\phi} \tag{2}$$

$$\hat{Q}_1 + \hat{Q}_2 = 0 \tag{3}$$

$$\hat{p}_1 - \hat{p}_2 + (E_1 - E_2)\,\hat{Q}_1 - \frac{\gamma}{R_o^2}(m^2-1+(kR_o)^2)=0 \tag{4}$$

$$E_1 = \frac{qR_o}{2\varepsilon} \quad ; \quad E_2 = \frac{qR_o}{2\varepsilon_o} \tag{5}$$

Using the terminal relations of section 2.4 we have

$$\hat{p}_1 = \frac{\omega^2 \rho R_o J_m(jkR_o)\,\hat{\xi}}{jkR_o J_m'(jkR_o)} - q\,\hat{\phi} \tag{6}$$

$$\hat{p}_2 = 0 \tag{7}$$

$$\hat{Q}_1 = \frac{jk\varepsilon J_m'(jkR_o)}{J_m(jkR_o)}\,(\hat{\phi} + E_1\hat{\xi}) - \frac{q}{2}\,\hat{\xi} \tag{8}$$

$$\hat{O}_2 = \frac{-jk\varepsilon_o H_m'(jkR_o)}{H_m(jkR_o)} \quad (\hat{\phi} + E_2\hat{\xi}) - \frac{q}{2}\,\hat{\xi} \tag{9}$$

From (3), (5), (8), and (9) we obtain

$$\hat{\phi} = \frac{qR_o\left[1 - \frac{jkR_o}{2}\left(\frac{J_m'(jkR_o)}{J_m(jkR_o)} - \frac{H_m'(jkR_o)}{H_m(jkR_o)}\right)\right]}{jkR_o\left[\varepsilon\,\frac{J_m'(jkR_o)}{J_m(jkR_o)} - \frac{\varepsilon_o H_m'(jkR_o)}{H_m(jkR_o)}\right]}\;\hat{\xi} \tag{10}$$

Using these relations in (4), we obtain the dispersion relation as

$$\frac{\omega^2\rho R_o J_m(jkR_o)}{jkR_o J_m'(jkR_o)} = \frac{+q^2 R_o}{4}\;\frac{(\varepsilon-\varepsilon_o)}{\varepsilon\varepsilon_o}\left[\frac{jkR_o J_m'(jkR_o)}{J_m(jkR_o)} - 1\right]$$
$$+\, q^2 R_o\left[1 + \frac{(\varepsilon-\varepsilon_o)}{2\varepsilon_o}\;\frac{jkR_o J_m'(jkR_o)}{J_m(jkR_o)}\right]\left[\frac{1 - \frac{jkR_o}{2}\left[\frac{J_m'(jkR_o)}{J_m(jkR_o)} - \frac{H_m'(jkR_o)}{H_m(jkR_o)}\right]}{jkR_o\left[\frac{\varepsilon J_m'(jkR_o)}{J_m(jkR_o)} - \frac{\varepsilon_o H_m'(jkR_o)}{H_m(jkR_o)}\right]}\right]$$
$$+\, \frac{\gamma}{R_o^2}\;(m^2 - 1 + (kR_o)^2) \tag{11}$$

Note that all terms in (11) are real. Fig. 2.4.2 plots the conbination of Bessel functions which recur in (11).

We can check this answer to see if it agrees with 3.3.15 in the limit as $k \to 0$. Thus

$$\lim_{k\to 0} \frac{J_m(jkR_o)}{jkR_oJ'_m(jkR_o)} = \frac{(\frac{1}{2}\,jkR_o)^m\,\Gamma(m+1)}{\Gamma(m+1)(jkR_o)(\frac{1}{2}\,jkR_o)^{m-1}\,\frac{m}{2}} = \frac{1}{m}$$

$$\lim_{k\to 0} \frac{H_m(jkR_o)}{jkR_oH'_m(jkR_o)} = \frac{-\frac{j}{\pi}\,\Gamma(m)(\frac{1}{2}\,jkR_o)^{-m}}{(jkR_o)(+\frac{j}{\pi}\,\Gamma(m))\frac{m}{2}\,(\frac{1}{2}\,jkR_o)^{-m-1}} = -\frac{1}{m} \qquad (12)$$

Substituting (12) into (11), we find that we recover 3.3.15.

The short wavelength limit is another interesting case. Then from section 2.4, equations (40) - (41), we find that when kR_o is very large, with $x = kR_o$

$$\lim_{k\to\infty} \frac{J_m(jkR_o)}{jkR_oJ'_m(jkR_o)} = \frac{(j)^m\,e^x}{\sqrt{2\pi x}(j)^m e^x\left[\frac{1}{\sqrt{2\pi x}} - \frac{\pi}{(2\pi x)^{3/2}}\right]}$$

$$= \frac{2}{2x-1} = \frac{2}{2kR_o-1} \simeq \frac{1}{kR_o} \qquad (13)$$

$$\lim_{k\to\infty} \frac{H_m(jkR_o)}{jkR_oH'_m(jkR_o)} = \frac{\frac{2}{\pi}\,j^{-(m+1)}\sqrt{\frac{\pi}{2x}}\;e^{-x}}{\left[x\,\frac{2}{\pi}\,j^{-(m+1)}e^{-x}\left(-\frac{\pi}{2x} - \frac{1}{2}(\frac{\pi}{2})^{1/2}\;(\frac{1}{x})^{3/2}\right)\right]}$$

$$= -\frac{2}{(2x-1)} = \frac{-2}{2kR_o-1} \simeq \frac{-1}{kR_o}$$

In this short wavelength limit ($kR_o \gg 1$), (11) is approximately

$$\frac{\omega^2\rho}{k} = \frac{-q^2R_o(\varepsilon-\varepsilon_o)^2\,(kR_o)}{4(\varepsilon+\varepsilon_o)\,\varepsilon\varepsilon_o} \qquad (14)$$

$$\frac{-q^2R_o}{(\varepsilon+\varepsilon_o)} + \frac{\gamma}{R_o^2}\,(m^2 - 1 + (kR_o)^2)$$

In this limit, the interface looks planar to these very short waves. We can check (14) with the dispersion relation derived in section 3.2 for two planar layers, where the upper layer will have no space charge. When $kR_o \gg 1$, (14) does agree with (3.2.12).

3.6 Stability of a Perfectly-Conducting Cylinder with Surface Charge in Free Space; z- and θ- Dependence Included

Again we can use the terminal relations of section 2.4 with boundary conditions

$$\hat{\xi}_1 = \hat{\xi}_2 = \hat{\xi} \qquad (1)$$

$$\hat{\phi}_1 = \hat{\phi}_2 = 0 \qquad (2)$$

$$\hat{p}_1 - \hat{p}_2 + E_o(R_o)\hat{Q}_2 + \frac{\gamma}{R_o^2}\,[1-m^2-(kR_o)^2] = 0 \qquad (3)$$

where

$$\hat{p}_1 = \omega^2\rho R_o\,\frac{J_m(jkR_o)\hat{\xi}}{jkR_oJ_m'(jkR_o)} \qquad (4)$$

$$\hat{p}_2 = 0 \qquad (5)$$

$$\hat{Q}_2 = \frac{-jk\varepsilon_o H_m'(jkR_o)}{H_m(jkR_o)} \; E_o(R_o)\hat{\xi} + \varepsilon_o \left.\frac{\partial E_o}{\partial r}\right|_{R_o} \hat{\xi} \qquad (6)$$

with E_o being the equilibrium electric field due to the surface charge σ_f

$$E_o = \begin{cases} 0 & r < R_o \\ \dfrac{\sigma_f R_o}{\varepsilon_o r} & r \geq R_o \end{cases} \qquad (7)$$

Substituting these relations into the boundary conditions yields the dispersion relation as

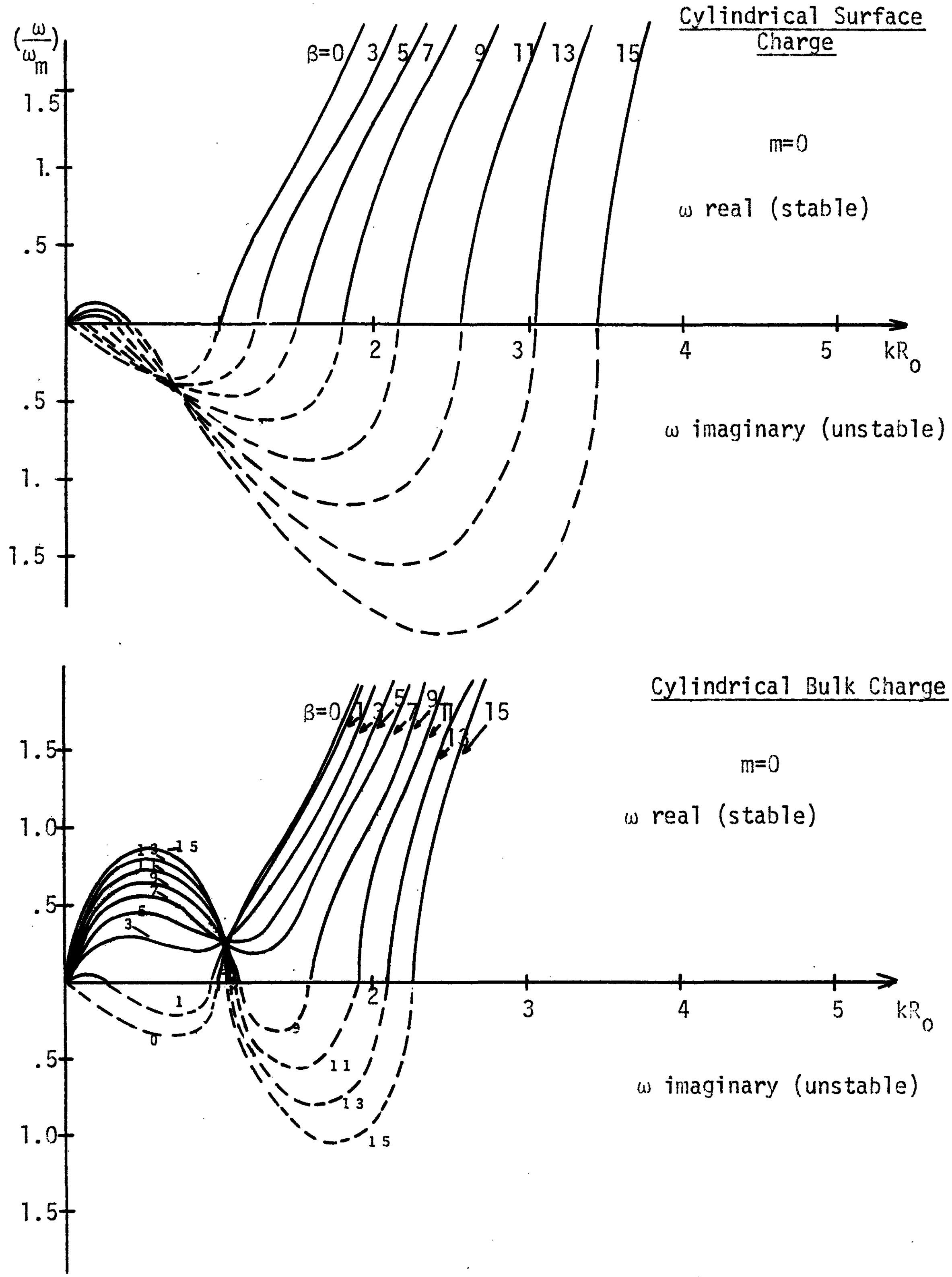
$$\frac{\omega^2 \rho R_o J_m(jkR_o)}{jkR_o J_m'(jkR_o)} = \frac{\sigma_f^2}{\varepsilon_o R_o}\left[jkR_o \frac{H_m'(jkR_o)}{H_m(jkR_o)} + 1 \right] + \frac{\gamma}{R_o^2}\left[m^2 - 1 + (kR_o)^2\right] \qquad (8)$$

This relation agrees with the one previously derived by Melcher[18].

<u>Comparison</u> ($\varepsilon = \varepsilon_o$)

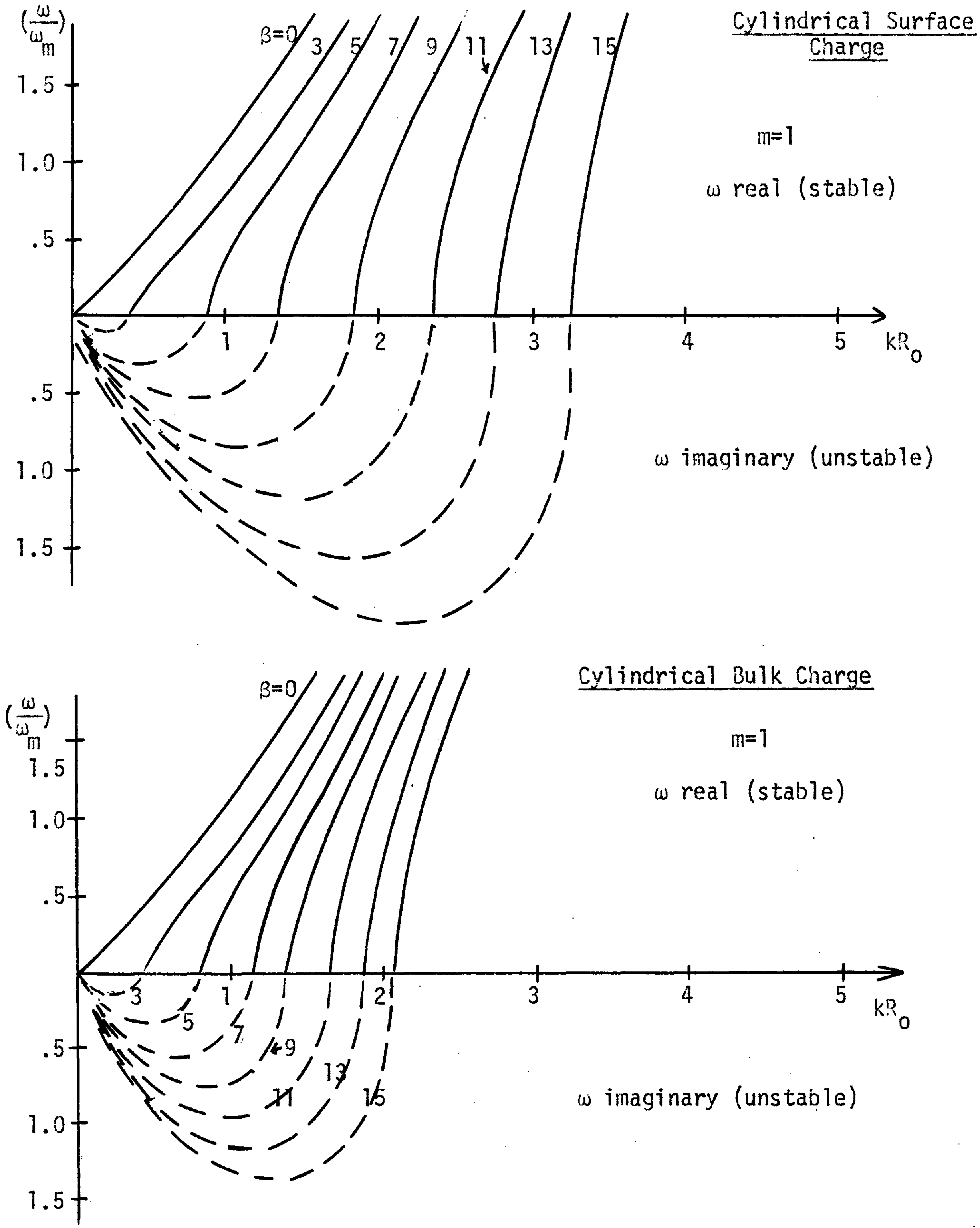
Figs. 1 - 5 plot the dispersion relation for various values of the parameter $\beta = \frac{q^2 R^2}{\varepsilon \gamma}$, for the two cases considered here.

It should be noted that the upper portion of each plot represents real ω, while the dotted lower half is for imaginary ω. This allows us to put on one plot, the dynamics of stable and unstable waves. The normalization is the same used in Fig. 3.4.1,



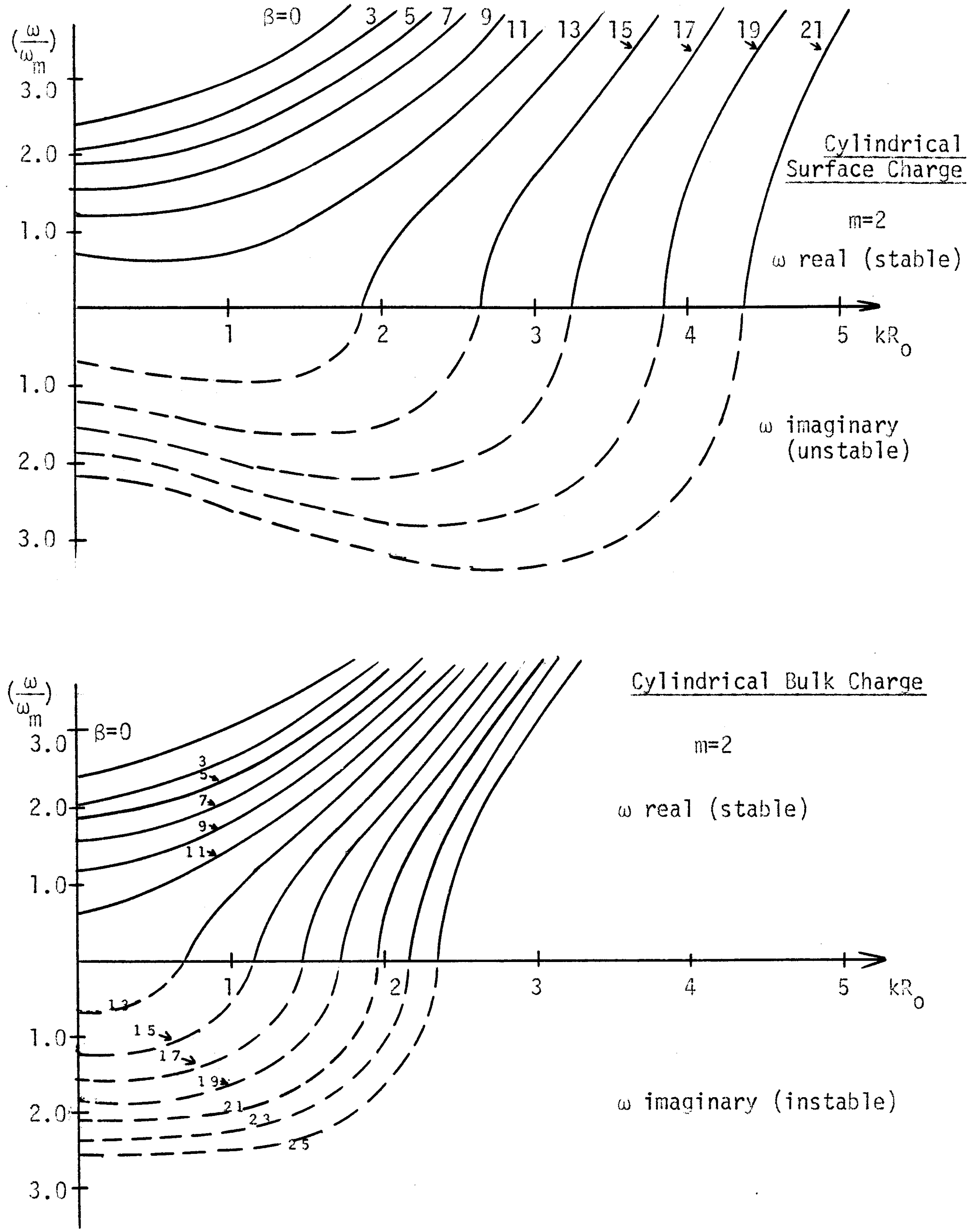
Dispersion relation for cylindrical surface and bulk charge distributions in the m=0 mode. $(\varepsilon=\varepsilon_0)$

Figure 1



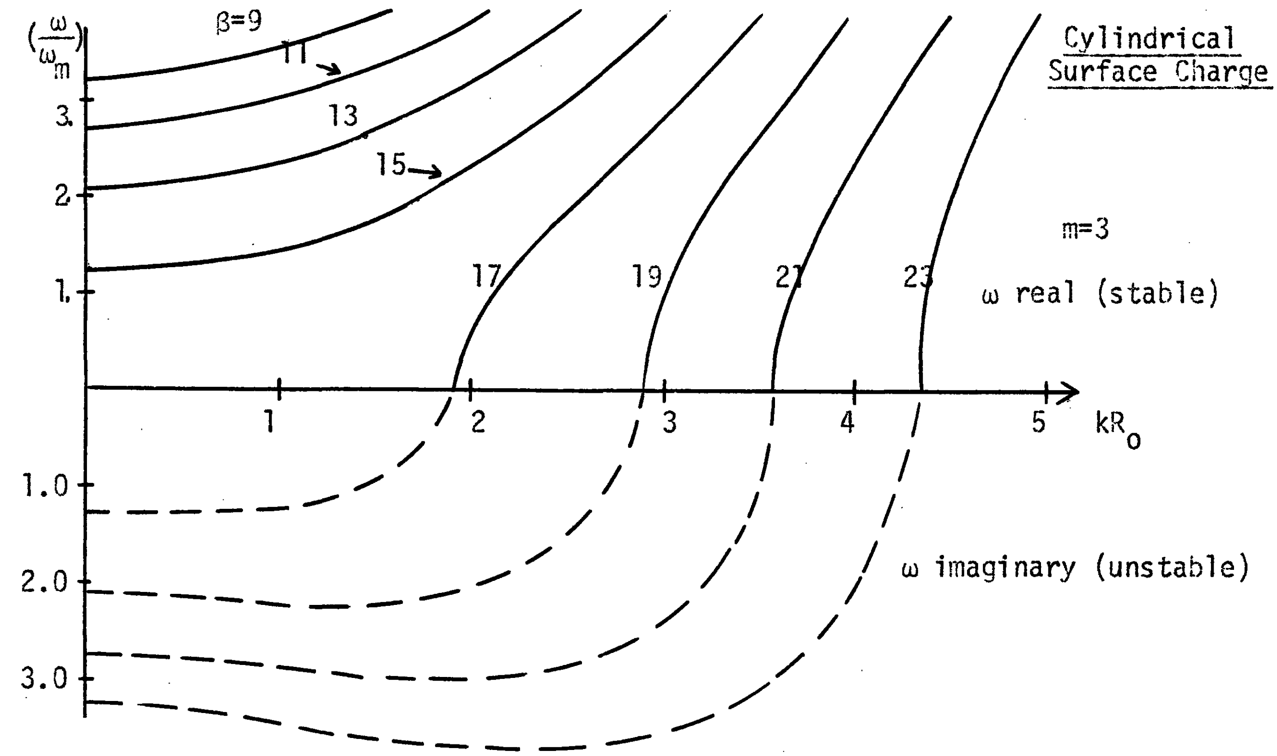
Dispersion relation for cylindrical surface and bulk charge distributions in the m=1 mode. ($\varepsilon=\varepsilon_0$)

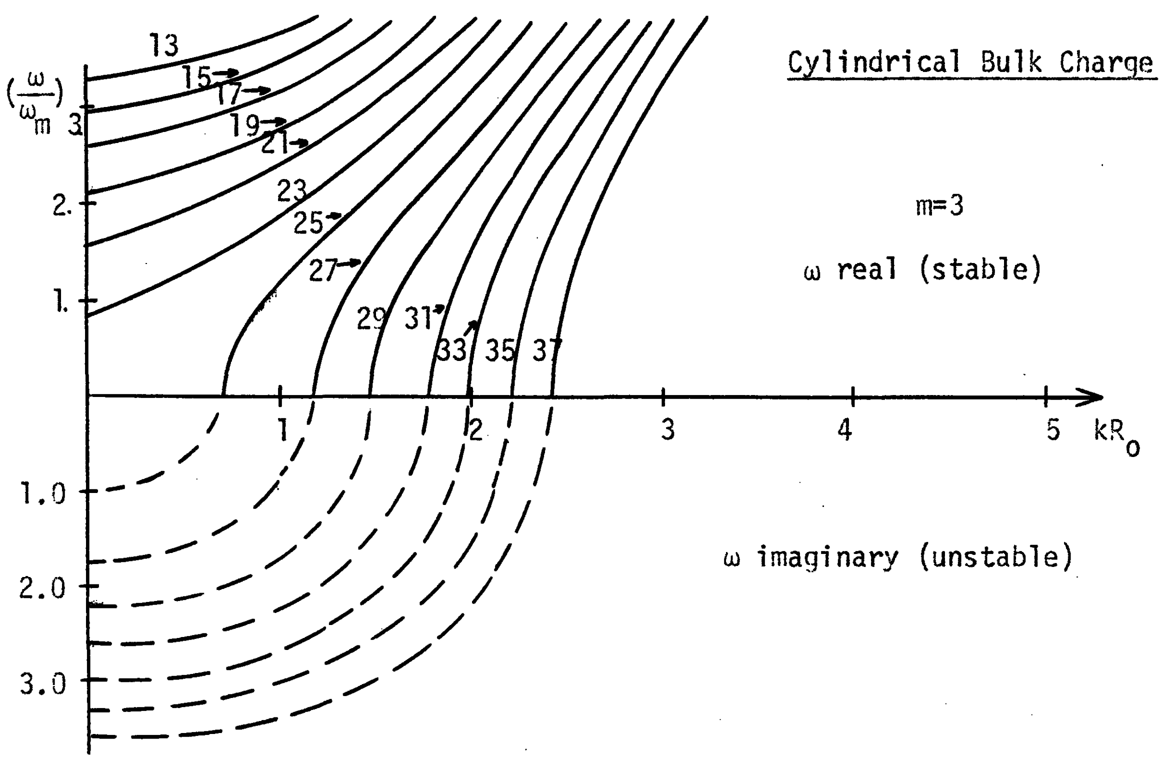
Figure 2



Dispersion relation for cylindrical surface and bulk charge distributions in the m=2 mode. ($\varepsilon=\varepsilon_0$)

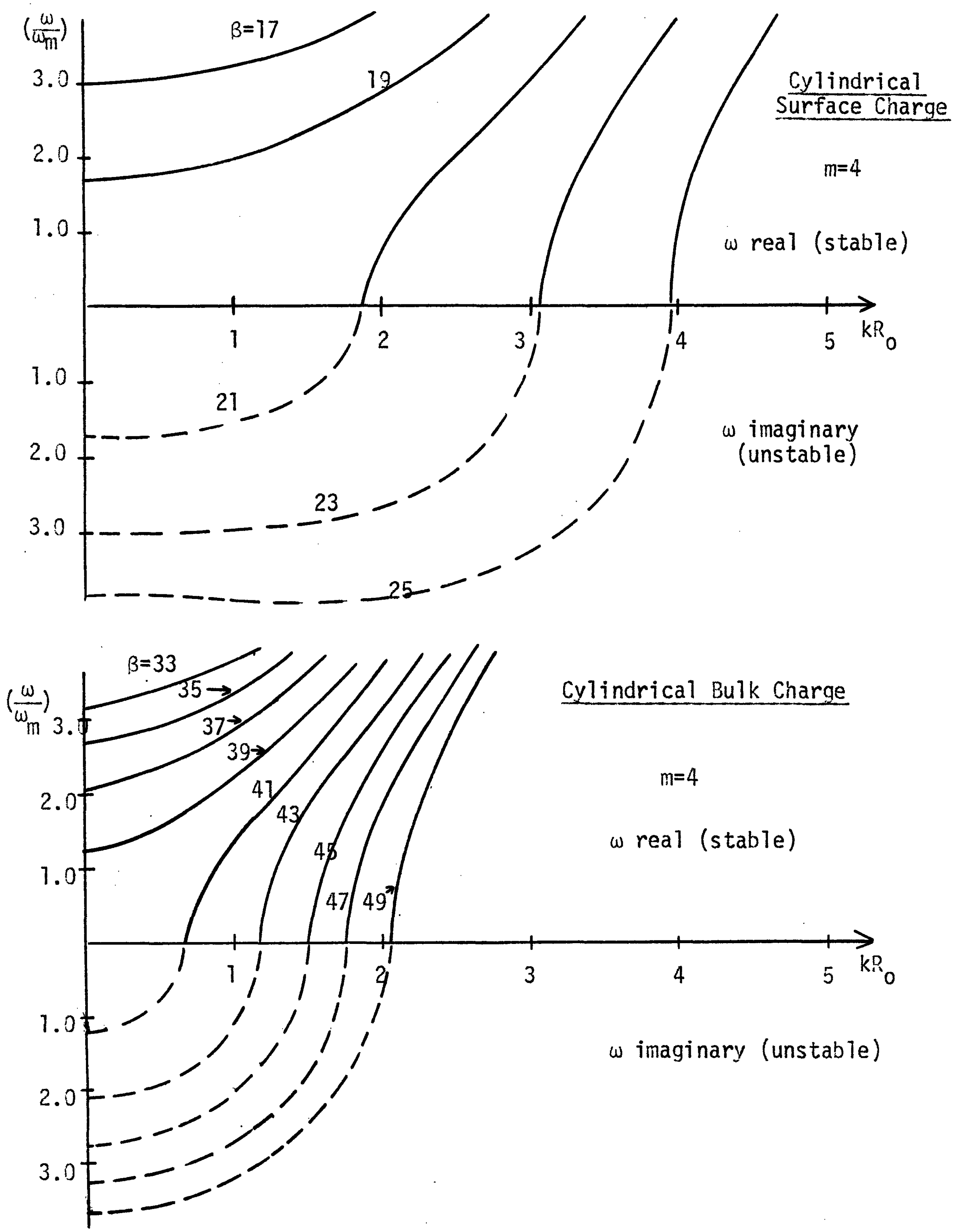
Figure 3





Dispersion relation for cylindrical surface and bulk charge distributions in the m=3 mode. ($\varepsilon=\varepsilon_o$)

Figure 4



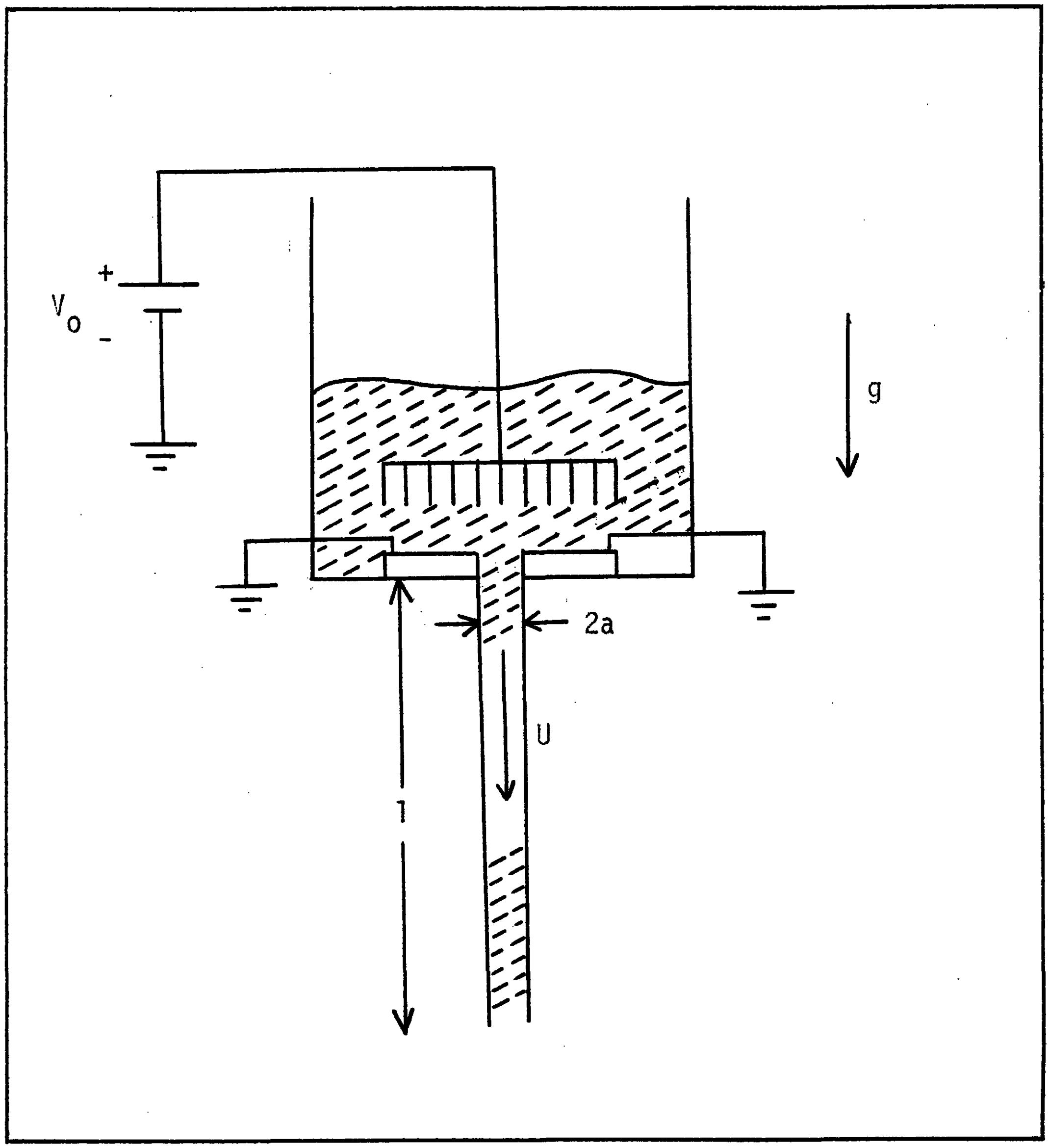
Dispersion relation for cylindrical surface and bulk charge distributions in the m=4 mode. ($\varepsilon=\varepsilon_o$)

Figure 5

$$\omega_m = \left[\frac{\gamma}{\rho R_o^3}\right]^{1/2} \tag{9}$$

In the m = 0 mode, there is always a range of k where the conducting cylinder with surface charge is unstable, no matter the amount of charge. However, for the insulating cylinder with bulk charge, we find some values of charge density which make the m = 0 mode stable. These correspond on the plot to $\beta = 3, 5$, and 7. For lesser or larger values of β, there again is a range of k where ω is imaginary. However, for those values of β where the m = 0 mode is stable, we see from Fig. 2 that the m = 1 mode is unstable. Thus, there is no way to completely stabilize this system. As we increase the charge density, correspondingly β, higher modes become unstable.

Because the propagation and instability characteristics between these two cases differ, we can devise an experiment to determine whether a cylinder supports bulk or surface charge. The easiest case is to consider a convecting stream which is supersonic. Then our analysis is still appropriate replacing ω by ω - kU. Then the imaginary ω plots in Figs. 1 - 5, will correspond to spatial growth rates of the convective instabilities. We consider a jet of radius a , and characteristic length l. Fig. 6 illustrates an appropriate geometry. We introduce charges into an insulating fluid through the use of razor blades or sharp points by a corona emission process. We then let the liquid fall through an orifice. For the perfectly insulating approximation to be valid, the transit time of the ion, $\tau_{ion} = \frac{a}{bE}$, must be



Geometry for possible experiment to distinguish the instability characteristics of cylindrical bulk or surface charge. Depending on the ion time constants, this jet could act either like a conductor or an insulator.

Figure 6

much longer than the transit time of the jet, $\tau_{jet} = \frac{1}{U}$

$$Re = \frac{a}{l} \frac{U}{bE} >> 1$$

For a very insulating jet we typically have

$$a = 2 \times 10^{-3} \text{ m}$$

$$l = 10^{-1} \text{m}$$

$$U = 1 \text{ m/sec}$$

$$b = 10^{-7} \text{ m}^2\text{/volt-sec.}$$

$$E = 10^5 \text{ Volts/m}$$

for which

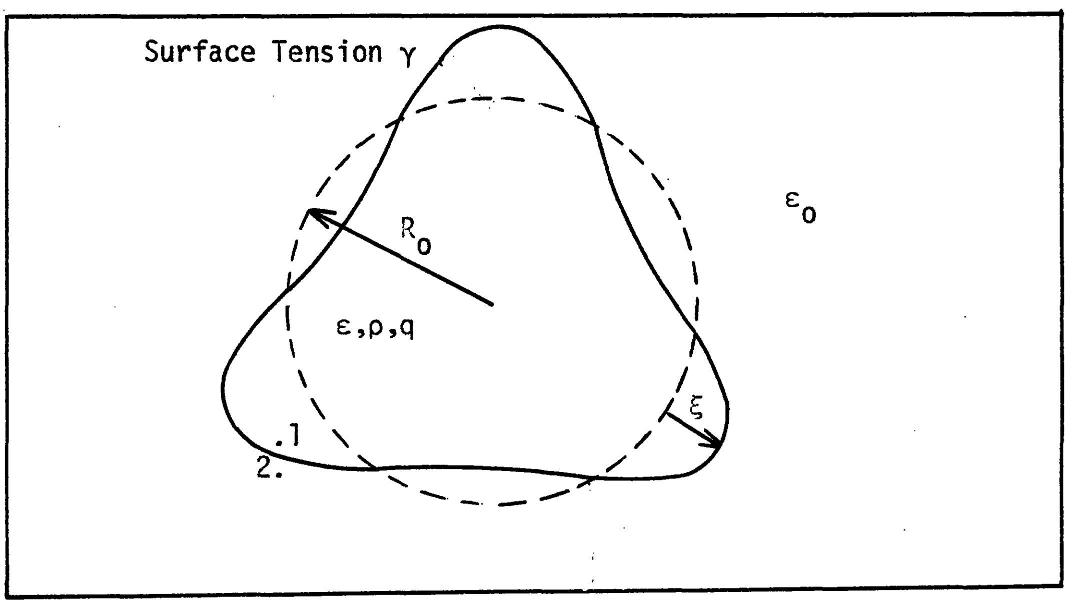
$$Re = 2$$

With these parameters, the experiment is borderline, but if we can find fluids with a lower mobility, or if we increase the convection velocity and alter the geometry we can arrange the appropriate time constants. If Re is too low, the ions will have a chance to reach the surface and accumulate surface charge. The jet will then act like a perfect conductor.

3.7 Stability of a Charged Spherical Drop In Free Space

This case is of historical interest, since the liquid drop model of the atomic nucleus first used by Bohr and Wheeler[1-3] which models the nucleus as an incompressible, inviscid, perfectly-insulating liquid drop with uniform charge density q, is often used in nuclear physics to explain the nuclear fission process. Some of the properties of nuclear forces are analagous to the forces which hold a liquid drop together. This concept, together with other classical ideas such as electrostatic repulsion and surface tension, are used to set up a semi-empirical formula for the mass or binding energy of a nucleus in its ground state. The formula is developed by considering the different factors which contribute to the binding energy and weighting these factors where possible with constants derived from theory or from experimental data.[3]

In particular, we consider the geometry of Fig. 1



Definition of terminal variables for a charged spherical drop in free space.

Figure 1

For such a drop, the equilibrium electric field is

$$E_0 = \begin{cases} \dfrac{qr}{3\varepsilon} & r < R_0 \\[2ex] \dfrac{qR_0^{\;3}}{3\varepsilon_0 r^2} & r > R_0 \end{cases} \tag{1}$$

The boundary conditions for this problem are

$$\hat{\xi}_1 = \hat{\xi}_2 = \hat{\xi} \tag{2}$$

$$\hat{\phi}_1 = \hat{\phi}_2 = \hat{\phi} \tag{3}$$

$$\hat{Q}_1 + \hat{Q}_2 = 0 \tag{4}$$

$$\hat{p}_1 - \hat{p}_2 - \varepsilon E_1 \hat{e}_{r_1} + \varepsilon_0 E_2 \hat{e}_{r_2} - \frac{\gamma}{R_0^{\;2}} (n-1)(n+2)\hat{\xi} = 0 \tag{5}$$

where

$$E_1 = \frac{qR_0}{3\varepsilon}\ ; \qquad E_2 = \frac{qR_0}{3\,\varepsilon_0} \tag{6}$$

We can determine the interfacial quantities by using the terminal relations of section 2.5, first in the inner region with $\beta = 0$ and $\alpha = R_0$, and then for the outer region with $\beta = R_0$ and $\alpha = \infty$. We then obtain

$$\hat{p}_1 = \frac{\omega^2 \rho R_0}{n}\ \hat{\xi} - q\hat{\phi}\ ; \qquad n \neq 0 \tag{7}$$

$$\hat{p}_2 = 0 \tag{8}$$

$$-\varepsilon \hat{e}_{r_1} = \hat{Q}_1 = \frac{\varepsilon n}{R_o} (\hat{\phi} + E_1 \hat{\xi}) - \frac{q}{3} \hat{\xi} \tag{9}$$

$$\varepsilon_o \hat{e}_{r_2} = \hat{Q}_2 = \frac{\varepsilon_o (n+1)}{R_o} (\hat{\phi} + E_2 \hat{\xi}) - \frac{2}{3} q \hat{\xi} \tag{10}$$

From (4), (9), and (10) we obtain

$$\hat{\phi} = \frac{2q R_o (1 - n)}{3[n(\varepsilon + \varepsilon_o) + \varepsilon_o]} \hat{\xi} \tag{11}$$

Then using (7) - (11) in (5), we obtain the dispersion relation as

$$\frac{\rho \omega^2 R_o}{n} = \frac{q^2 R_o (\varepsilon - \varepsilon_o)(n-1)}{9 \varepsilon_o \varepsilon} \left[\frac{n(\varepsilon_o - \varepsilon) + \varepsilon_o}{n(\varepsilon + \varepsilon_o) + \varepsilon_o} \right] - \frac{2q^2 R_o (n-1)}{3[n(\varepsilon + \varepsilon_o) + \varepsilon_o]} + \frac{\gamma}{R_o^2} (n-1)(n+2) \tag{12}$$

The n = 0 solution is not allowed, due to the incompressibility of the drop. The n = 1 solution is stable, because to first order, this solution represents a pure translation of the drop. The first mode which can become unstable is for n = 2. The critical value of charge density is then

$$q^2_{crit} = \frac{36 \gamma \varepsilon_o \varepsilon (2\varepsilon + 3\varepsilon_o)}{R_o^3 [2\varepsilon^2 + 3\varepsilon_o^2 + \varepsilon_o \varepsilon]} \tag{13}$$

If

$$\varepsilon = \varepsilon_o$$

this condition becomes

$$q_{crit} = \left[\frac{30\gamma\varepsilon}{R_o^3}\right]^{1/2} \tag{14}$$

This problem can be compared to the stability of a perfectly-conducting drop with surface charge, the so-called "Rayleigh's limit."[44]

3.8 Stability of a Perfectly-Conducting Drop with Surface Charge -- Rayleigh's Limit[44]

The equilibrium electric field is now

$$E_o = \begin{cases} 0 & r < R_o \\ \dfrac{\sigma_f R_o^2}{\varepsilon_o r^2} & r \geq R_o \end{cases} \tag{1}$$

Our boundary conditions are

$$\hat{\xi}_1 = \hat{\xi}_2 = \hat{\xi} \tag{2}$$

$$\hat{\phi}_1 = \hat{\phi}_2 = 0 \tag{3}$$

$$\hat{p}_1 - \hat{p}_2 + E_o(R_o)Q_2 - \frac{\gamma}{R_o^2}(n-1)(n+2)\,\hat{\xi} = 0 \tag{4}$$

From the terminal relations of section 2.5, we have

$$\hat{p}_1 = \frac{\omega^2 \rho R_o}{n}\,\hat{\xi} \qquad n \neq 0 \tag{5}$$

$$\hat{p}_2 = 0 \tag{6}$$

$$\hat{Q}_2 = \frac{(n+1)}{R_o} \sigma_f \hat{\xi} - \frac{2\sigma_f}{R_o} \hat{\xi} \tag{7}$$

Thus, our dispersion relation is

$$\frac{\omega^2 \rho R_o}{n} = - \frac{(n-1)}{R_o} \frac{\sigma_f^2}{\varepsilon_o} + \frac{\gamma}{R_o^2} (n-1)(n+2)$$

$$= (n-1) \left[\frac{\gamma(n+2)}{R_o^2} - \frac{\sigma_f^2}{\varepsilon_o R_o} \right] \tag{8}$$

The threshold for instability occurs when

$$\sigma_{f\ crit} = 2 \sqrt{\frac{\gamma \varepsilon_o}{R_o}} \tag{9}$$

<u>Comparison</u> ($\varepsilon = \varepsilon_o$)

To compare this result to the previous bulk charge problem of section 3.7, we assume $\varepsilon = \varepsilon_o$ so that there is no polarization force, and that the total charge for each problem is the same, for which

$$\frac{4}{3} \pi R_o^3 q = 4\pi R_o^2 \sigma_f \tag{10}$$

We then can rewrite (9) as

$$q_{crit} = 6 \left[\frac{\gamma \varepsilon}{R_o^3} \right]^{1/2} \tag{11}$$

Comparing (11) and (3.7.14), we see that the bulk charge system with $\varepsilon = \varepsilon_0$ becomes unstable slightly earlier as the total amount of charge is increased.

Physically, the insulating bulk charged drop and the conducting drop with surface charge have much in common. As was discussed in the analagous cylindrical problems of sections (3) - (6), in any real experiment where a drop is charged, on a time scale short compared to the fluid relaxation time $(\frac{\varepsilon}{\sigma})$, this drop can approximately be considered perfectly insulating with a bulk charge. For longer periods of time, the bulk charge will relax to the surface, shielding out the electric field as if the drop were perfectly conducting.

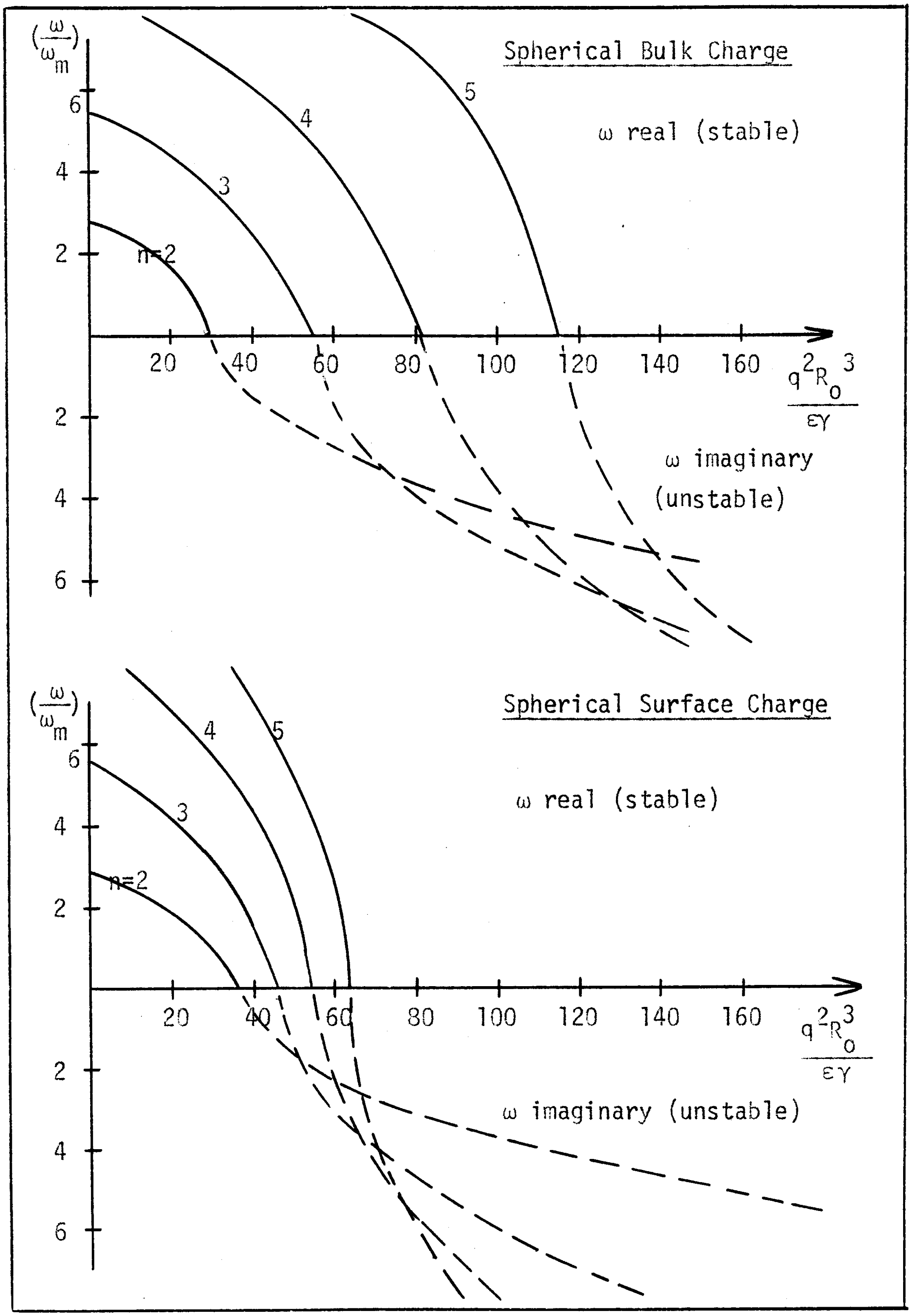
For the case when $\varepsilon = \varepsilon_0$, Fig. 1 plots the resonant frequencies (and growth rates) for both these cases as a function of the non-dimensional parameter

$$\frac{q^2 R_o^{\ 3}}{\varepsilon \gamma}$$

where we assume the total charge for each case is the same. The frequency is normalized to

$$\omega_m = \left[\frac{\gamma}{\rho R_o^{\ 3}}\right]^{1/2}$$

The two situations have different behavior for a given mode. Although the insulating drop becomes unstable first, in the n = 2 mode, we see that increasing the amount of charge further results in the higher modes of the conducting drop becoming unstable sooner.



Stability regions for a spherical drop with bulk or surface charge. ($\varepsilon=\varepsilon_o$)

Figure 1

CHAPTER IV
GENERAL FORMULATION OF CONTINUOUS STRATIFICATIONS

4.1 Introduction

The discrete stratifications discussed in the previous chapters are a subset of the general class of problems of continuous stratifications. The interfacial waves which propagate on the discrete structures are replaced by bulk waves. In the past, much attention has been given to gravity waves because of their presence in the oceans and the atmosphere.[15-17] The waves present at the water-air interface of the ocean exemplify the class of interfacial waves, while the smooth density gradients present in the atmosphere and in the oceans represent internal gravity waves. Bulk properties of liquid systems are usually temperature dependent, so any temperature variations will induce property gradients. Since temperature variations are usually smooth, property distributions are also smooth. Interfacial waves are a special case where property gradients are particularly sharp at the interface.

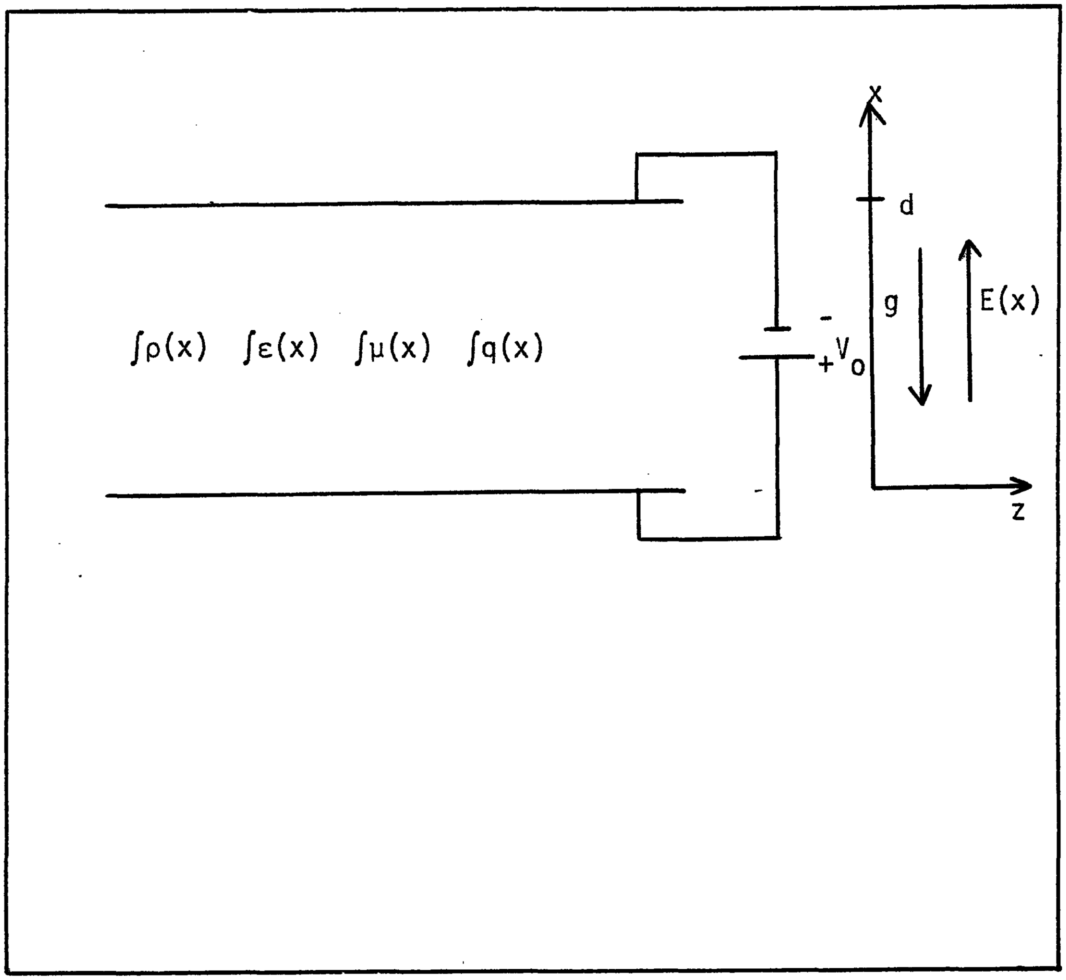
In charged fluid systems, each fluid element is coupled through the electric field in addition to gravity, so that the new class of "space charge" waves arise. In the previous chapters we have discussed interfacial versions of these new waves, but in systems where there are no surfaces, as when electrodes are submerged in a liquid, bulk space charge waves will also propagate.

For such systems which are stratified in mass density, charge density, and dielectric constant, we will derive the small signal equations of motion in planar geometry. We will then consider the special cases of weak gradient and exponential distributions, since the coefficients of the differential equation reduce to constants, for which it is easy to obtain solutions. In Chapter V, we will consider many thin layers as approximations to these distributions, and show that as the number of approximating layers approach infinity, with the thickness of each layer tending to zero, the continuous and discrete results are identical.

4.2 Equations of Motion

Turnbull and Melcher[11] considered the general case of an initially static, incompressible fluid which is perfectly insulating and stratified with continuous distributions of mass density, viscosity, permittivity, space charge and electric field in planar geometry, as illustrated in Fig. 1. For completeness, the general analysis will be repeated here. Their analysis emphasized conditions for incipience of instability which occur at zero frequency. As described in sections (1.4) and (1.5), the approximation of a perfectly insulating fluid is not appropriate for real liquids at this frequency, and so we wish to emphasize the dynamics of such systems, in addition to the stability. In particular, we will use the case studies of weak gradient and exponential stratifications, to emphasize the wave propagation characteristics of such systems. In Chapter V, we will approximate these distributions by many thin layers to illustrate the convergence.

The necessary equations are:



An initially static fluid, stratified in the x direction, is stressed by an electric field intensity E(x) imposed by means of perfectly conducting electrodes (at x=0,d) constrained to a constant potential difference, V_o. There is an equilibrium distribution of mass density ρ(x), viscosity μ(x), permittivity ε(x), and space charge q(x) in the vertical direction.

Figure 1

Conservation of Momentum

$$\rho \frac{D\overline{v}}{Dt} + \nabla p = q\overline{E} - \frac{1}{2}\overline{E}\cdot\overline{E}\,\nabla\varepsilon + \nabla\left(\frac{1}{2}\rho\frac{\partial\varepsilon}{\partial\rho}\,\overline{E}\cdot\overline{E}\right)$$

$$+ \mu\nabla^2\overline{v} + 2(\nabla\mu\cdot\nabla)\,\overline{v} + \rho\overline{g}$$

$$+ \nabla\mu \times (\nabla\times\overline{v}) \tag{1}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \overline{v}\cdot\nabla \tag{2}$$

Conservation of Mass

$$\nabla \cdot \overline{v} = 0 \tag{3}$$

$$\frac{D\rho}{Dt} = 0 \tag{4}$$

Maxwell's Equations

$$\nabla \times \overline{E} = 0 \;;\quad \overline{E} = -\nabla\phi \tag{5}$$

Gauss's Law

$$D\,(\varepsilon\overline{E}) = q \tag{6}$$

$$\frac{D\varepsilon}{Dt} = 0 \tag{7}$$

Conservation of Charge

$$\frac{Dq}{Dt} = 0 \tag{8}$$

where

$$D = \frac{d}{dx} \tag{9}$$

We have included electrostrictive forces in (1), however, since it is a gradient of a quantity, we can include it with the pressure, and because the fluid is presumed incompressible, electrostriction will have no effect on the dynamics.

All quantities have an equilibrium and perturbation part as listed below:

$$\rho = \rho_o + \rho' \tag{10}$$

$$\overline{v} = \overline{v}' \tag{11}$$

$$p = p_o + p' \tag{12}$$

$$q = q_o + q' \tag{13}$$

$$\overline{E} = \overline{E}_o + e' \tag{14}$$

$$\phi = \phi_o + \phi' \tag{15}$$

$$\varepsilon = \varepsilon_o + \varepsilon' \tag{16}$$

The primed quantities are presumed much smaller than the equilibrium values.

Once we specify the equilibrium, we consider small perturbations to be of the form

$$\phi = \text{Re}\ \hat{\phi}\ (x)\ e^{j(\omega t\ -\ k_y y\ -\ k_z z)} \tag{17}$$

with

$$k = \sqrt{k_y^{\ 2} + k_z^{\ 2}} \tag{18}$$

The linearized form of our equations become:

$$\rho_o \frac{\partial \bar{v}}{\partial t} = -\rho' g \bar{i}_x - \nabla p' + \mu \nabla^2 \bar{v} + 2(\nabla\mu \cdot \nabla)\, \bar{v}$$

$$+ \nabla\mu x\ (\nabla x \bar{v}) + q' E_o \bar{i}_x - q_o \nabla\phi - \frac{1}{2}\,(E_o \cdot E_o) \nabla\varepsilon'$$

$$+ E_o \nabla\varepsilon_o \frac{\partial\phi}{\partial x} \qquad (19)$$

$$\nabla \cdot \bar{v} = 0 \qquad (20)$$

$$\frac{\partial\rho'}{\partial t} = -v_x D\rho_o \qquad (21)$$

$$\frac{\partial\varepsilon'}{\partial t} = -v_x D\varepsilon_o \qquad (22)$$

$$\frac{\partial q'}{\partial t} = -v_x Dq_o \qquad (23)$$

$$-\varepsilon_o \nabla^2 \phi' + D(E_o \varepsilon') - D\varepsilon_o D\phi' = q' \qquad (24)$$

It is convenient to work with the variables v_x and ϕ, so we eliminate all variables in terms of these two. The resulting equations are thus

$$j\omega\,[D(\rho_o D\hat{v}_x) - \rho_o k^2\hat{v}_x] = D\,[\mu(D^2-k^2)D\hat{v}_x)] - \mu k^2(D^2-k^2)\hat{v}_x$$

$$+ D\,[D\mu(D^2+k^2)\hat{v}_x] - 2k^2 D\mu D\hat{v}_x - k^2\,\frac{gD\rho_o}{j\omega}\,\hat{v}_x$$

$$- k^2 Dq_o\hat{\phi} + k^2\,\frac{E_o Dq_o}{j\omega}\,\hat{v}_x + k^2 E_o\,\frac{DE_o D\varepsilon_o \hat{v}_x}{j\omega}$$

$$- k^2 E_o D\varepsilon_o D\hat{\phi} \tag{25}$$

and

$$D(\varepsilon_o D\hat{\phi}) - k^2\varepsilon_o\hat{\phi} + \frac{DE_o D\varepsilon_o}{j\omega}\,\hat{v}_x + \frac{E_o D(D\varepsilon_o \hat{v}_x)}{j\omega}$$

$$\frac{-Dq_o}{j\omega}\,\hat{v}_x = 0 \tag{26}$$

Typical boundary conditions for these equations are:

$$\begin{array}{lll} \hat{v}_x(0) = 0 & D\hat{v}_x(0) = 0 & \hat{\phi}(0) = 0 \\ \hat{v}_x(+d) = 0 & D\hat{v}_x(+d) = 0 & \hat{\phi}\,(+d) = 0 \end{array} \tag{27}$$

Equations (25) and (26), with boundary conditions such as (27), summarize the eigenvalue problem for ω. It has been shown by Turnbull and Melcher,[11] and Barston[31] that the necessary and sufficient condition for stability is

$$gD\rho_o - E_o Dq_o - E_o DE_o D\varepsilon_o < 0 \tag{28}$$

with instability incipient at zero frequency.

If only conditions at incipience of instability are needed, we simply solve (25) - (27) when

$$\omega = 0.$$

Thus, we have traded an eigenvalue problem for ω for an eigen value problem in the system parameters, like the electric field. However, if wave propagation characteristics or growth rates of an instability are needed, we must solve (25) - (27). In general this is difficult since the coefficients of the differential equations are functions of position. However, for weak gradient and exponential distributions (25) and (26) reduce to constant coefficient differential equations, which can be easily solved.

For these case studies, we assume from the outset that the fluid is inviscid, and that the permittivity is a constant. Then (25) and (26) reduce to

$$j\omega\,[D(\rho_o D\hat{v}_x) - \rho_o k^2 \hat{v}_x] = \frac{-k^2}{j\omega}(gD\rho_o - E_o Dq_o)\,\hat{v}_x - k^2 Dq_o \hat{\phi} \tag{29}$$

and

$$D^2\hat{\phi} - k^2\hat{\phi} - \frac{Dq_o}{j\omega\varepsilon_o}\hat{v}_x = 0 \tag{30}$$

4.3 Weak Gradient Stratifications

If the gradient in fluid properties is very weak, we can use the Boussinesq approximation that every coefficient in (29) and (30) is constant. In particular, we assume the equilibrium mass density to increase very slightly with fluid depth such that

$$\rho_o = \rho_T(1 + \alpha x) \qquad 0 \leq x \leq d \tag{1}$$

with

$$\alpha d << 1 \tag{2}$$

Then we can ignore density gradients in the inertial terms, but we still keep density gradients in the gravity term of equation (4.2.29).

We also assume that the equilibrium bulk charge density is a linear function in space such that

$$Dq_o = \text{constant} \tag{3}$$

but that the magnitude of the charge density is very small, such that we can ignore its contributions to the electric field. The imposed fields due to the external battery are much larger than the self-fields due to the space charge. (Equivalently we can say that the total surface charge on the electrodes is much larger than the total volume charge present in the liquid). With these approximations

$$E_o = \frac{V_o}{d} \tag{4}$$

Then from (4.2.30), we have

$$\hat{v}_x = \frac{j\omega\varepsilon_o}{Dq_o} (D^2 - k^2)\,\hat{\phi} \tag{5}$$

Note that in (5) we retain the effects of the fluid motion on the potential. This will result in self-field effects being included. A less rigorous analysis would forget (5) and ignore the last term in (4.2.29). Note that there would still be electrical coupling in (4.2.29) due to the imposed fields.

With these approximations in mind, we use (5) in (4.2.29) to obtain

$$(D^2 - k^2)^2\hat{\phi} + \frac{k^2}{\omega^2}\left(-\alpha g + \frac{E_o D q_o}{\rho_T}\right)(D^2 - k^2)\hat{\phi} - \frac{k^2(Dq_o)^2}{\omega^2 \rho_T \varepsilon}\hat{\phi} = 0 \quad (6)$$

The last term in (6) represents the reaction back on the potential due to a fluid displacement. A less rigorous analysis would neglect this term.

We assume solutions to (6) of the form

$$\hat{\phi} = Ae^{sx} \quad (7)$$

where s must satisfy the characteristic equation

$$(s^2 - k^2)^2 + \beta(s^2 - k^2) - \gamma = 0 \quad (8)$$

with

$$\beta = \frac{k^2}{\omega^2}\left(-\alpha g + \frac{E_o D q_o}{\rho_T}\right) \quad (9)$$

and

$$\gamma = \frac{k^2(Dq_o)^2}{\omega^2 \rho_T \varepsilon} \quad (10)$$

Solutions to (8) are

$$s^2 - k^2 = -\frac{\beta}{2} \pm \sqrt{(\frac{\beta}{2})^2 + \gamma} \tag{11}$$

Thus, the solutions to (6) are of the form

$$\hat{\phi} = A_1 e^{s_1 x} + A_2 e^{s_2 x} + A_3 e^{s_3 x} + A_4 e^{s_4 x} \tag{12}$$

with

$$s_1 = + [k^2 - \frac{\beta}{2} + \sqrt{(\frac{\beta}{2})^2 + \gamma}\ ]^{1/2} \tag{13}$$

$$s_2 = - s_1 \tag{14}$$

$$s_3 = + [k^2 - \frac{\beta}{2} - \sqrt{(\frac{\beta}{2})^2 + \gamma}\ ]^{1/2} \tag{15}$$

$$s_4 = - s_3 \tag{16}$$

From (5) we also have

$$\hat{v}_x = \frac{j\omega\varepsilon}{Dq_0} [A_1 (s_1{}^2 - k^2)\ e^{s_1 x} + A_2 (s_2{}^2 - k^2)\ e^{s_2 x} + A_3 (s_3{}^2 - k^2)\ e^{s_3 x} + A_4 (s_4{}^2 - k^2)\ e^{s_4 x}] \tag{17}$$

Because of the rigid, perfectly-conducting boundaries at x = 0 and x = +d, the boundary conditions are

$$\hat{\phi}(0) = 0 \qquad \hat{v}_x(0) = 0$$
$$\hat{\phi}(+d) = 0 \qquad \hat{v}_x(+d) = 0 \tag{18}$$

or

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ e^{-s_1 d} & e^{s_1 d} & e^{-s_3 d} & e^{s_3 d} \\ (s_1^{\,2}-k^2) & (s_1^{\,2}-k^2) & (s_3^{\,2}-k^2) & (s_3^{\,2}-k^2) \\ (s_1^{\,2}-k^2)e^{-s_1 d} & (s_1^{\,2}-k^2)e^{s_1 d} & (s_3^{\,2}-k^2)e^{-s_3 d} & (s_3^{\,2}-k^2)e^{s_3 d} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = 0 \qquad (19)$$

For solution, the determinant of the coefficients must be zero, for which

$$(s_1^{\,2} - s_3^{\,2})^2 \, (e^{s_3 d} - e^{-s_3 d}) \, (e^{s_1 d} - e^{-s_1 d}) = 0 \qquad (20)$$

So, for non-trivial solutions to (20), we must have either

$$s_1^{\,2} = -\left(\frac{n\pi}{d}\right)^2 \qquad (21)$$

or

$$s_2^{\,2} = -\left(\frac{n\pi}{d}\right)^2 \qquad (22)$$

Note that either (21) or (22) is allowed. Once, one solution is picked, the other is determined from (13) - (16). Using (13) - (16), we can treat both possibilities simultaneously

$$k^2 - \frac{\beta}{2} \pm \sqrt{\left(\frac{\beta}{2}\right)^2 + \gamma} = -\left(\frac{n\pi}{d}\right)^2 \qquad (23)$$

or

$$\pm \sqrt{\left(\frac{\beta}{2}\right)^2 + \gamma} = -\left(\frac{n\pi}{d}\right)^2 - k^2 + \left(\frac{\beta}{2}\right) \qquad (24)$$

We square both sides of (24) and simplify, to obtain

$$\gamma = [(\frac{n\pi}{d})^2 + k^2]\,[(\frac{n\pi}{d})^2 + k^2 - \beta] \tag{25}$$

Then from (9) and (10) we obtain the dispersion relation as

$$\omega^2 = \frac{k^2[\frac{(Dq_o)^2}{\rho_T\varepsilon} + (-\alpha g + \frac{E_o Dq_o}{\rho_T})\,(\{\frac{n\pi}{d}\}^2 + k^2)]}{[(\frac{n\pi}{d})^2 + k^2]^2} \tag{26}$$

$$n = 1, 2, 3, \ldots$$

The self-field term on the right hand side of (26) results from including the effects of the fluid motion on the potential as previously discussed. Its effect is always stabilizing. However, for very short waves ($k \to \infty$) the other term dominates. We see that we will have instability if

$$(-\alpha g + \frac{E_o Dq_o}{\rho_T}) < 0 \tag{27}$$

This agrees with the general criterion of (4.2.28).

If the inequality of (27) is not met, the system supports internal gravity and space charge waves. Plots of the wave dispersion given by (26) will be presented in section 4.5.

4.4 Exponential Stratifications

In the previous section we ignored the contributions to the electric field due to the space charge. Thus, our results were only applicable when the quantity of space charge was much less than the total surface charge due to the battery. In addition, we assumed the mass density change to be much smaller than the average mass density.

We consider here an exact problem with no approximations. We assume that

$$\rho = \rho_o e^{+\alpha x} \tag{1}$$

and

$$q = q_o e^{+\alpha/2 \, x} . \tag{2}$$

These equilibrium distributions are quite particular, since the exponential factors in (1) and (2) are similar. This problem is useful for it provides an exact case study, for which approximate analyses can be checked.

The previous analysis of section 4.3 can be checked when

$$\alpha d << 1$$

It will lend credence to keeping the self-field terms in the previous section. In Chapter V, we will check the convergence of many thin layers as an approximation to these exponential distributions. This analysis will also be exact, to which approximate analyses can be compared.

With these justifications, we proceed by using (2) with Gauss's law to obtain the electric field in the absence of any other field sources.

$$E = + \frac{2q_o}{\varepsilon\alpha} e^{+\alpha/2 \, x} \tag{3}$$

We then substitute these equilibrium distributions into equations (4.2.29) and (4.2.30). After eliminating $\hat{v}_x$ in favor of $\hat{\phi}$, we find that all the exponential factors cancel out, leaving us with a constant coefficient differential equation. This fact enables us to solve exactly for the dynamics which justify our choice of distributions as given by (1) - (3). Thus, we have obtained

$$[D^2 - (\frac{\alpha}{2})^2 - k^2]\,(D^2 - k^2)\,\hat{\phi} + k^2\,\frac{(D^2 - k^2)\hat{\phi}}{\omega^2 \varepsilon \rho_o}\,[-g\rho_o\alpha\varepsilon + q_o{}^2] - \frac{k^2(\frac{\alpha}{2}\,q_o)^2}{\omega^2 \varepsilon \rho_o}\,\hat{\phi} = 0 \tag{4}$$

We let

$$\beta = \frac{k^2[-\alpha g\rho_o\varepsilon + q_o{}^2]}{\omega^2 \varepsilon \rho_o} \tag{5}$$

and

$$\gamma = \frac{k^2\,(\frac{\alpha}{2}\,q_o)^2}{\omega^2 \varepsilon \rho_o} \tag{6}$$

We then can assume solutions to (4) of the form

$$\hat{\phi} = Ae^{sx} \tag{7}$$

for which we obtain the characteristic equation as

$$\left(s^2 - \left(\frac{\alpha}{2}\right)^2 - k^2\right)\left(s^2 - k^2\right) + \beta\left(s^2 - k^2\right) - \gamma = 0 \tag{8}$$

which we fortunately can rewrite as

$$\left(s^2 - k^2\right)^2 + \left(s^2 - k^2\right)\left(\beta - \left(\frac{\alpha}{2}\right)^2\right) - \gamma = 0 \tag{9}$$

and has solutions

$$s^2 - k^2 = -\left[\frac{\beta - \left(\frac{\alpha}{2}\right)^2}{2}\right] \pm \sqrt{\left[\frac{\beta - \left(\frac{\alpha}{2}\right)^2}{2}\right]^2 + \gamma} \tag{10}$$

We denote the four roots as

$$s_1 = \left[k^2 - \left[\frac{\beta - \left(\frac{\alpha}{2}\right)^2}{2}\right] + \sqrt{\left[\frac{\beta - \left(\frac{\alpha}{2}\right)^2}{2}\right]^2 + \gamma}\,\right]^{1/2} \tag{11}$$

$$s_2 = -s_1 \tag{12}$$

$$s_3 = \left[k^2 - \left[\frac{\beta - \left(\frac{\alpha}{2}\right)^2}{2}\right] - \sqrt{\left[\frac{\beta - \left(\frac{\alpha}{2}\right)^2}{2}\right]^2 + \gamma}\,\right]^{1/2} \tag{13}$$

$$s_4 = -s_3 \tag{14}$$

Thus, the most general solution to (4) is

$$\hat{\phi} = A_1 e^{s_1 x} + A_2 e^{s_2 x} + A_3 e^{s_3 x} + A_4 e^{s_4 x} \tag{15}$$

and, from (4.2.30)

$$\hat{v}_x = \frac{+2j\omega\varepsilon}{\alpha q_0}\,[A_1(s_1^{\,2} - k^2)e^{(s_1-\alpha/2)x} + A_2(s_2^{\,2}-k^2)e^{(s_2-\alpha/2)x}$$

$$+ A_3(s_3^{\,2} - k^2)e^{(s_3-\alpha/2)x} + A_4(s_4^{\,2} - k^2)e^{(s_4-\alpha/2)x} \qquad (16)$$

Because of the rigid, perfectly-conducting boundaries, our boundary conditions are

$$\begin{matrix} \hat{\phi}(0) = 0 & \hat{v}_x(0) = 0 \\ \hat{\phi}(+d) = 0 & \hat{v}_x(+d) = 0 \end{matrix} \qquad (17)$$

which results in the relations

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ (s_1^{\,2}-k^2) & (s_1^{\,2}-k^2) & (s_3^{\,2}-k^2) & (s_3^{\,2}-k^2) \\ e^{-s_1 d} & e^{+s_1 d} & e^{-s_3 d} & e^{+s_3 d} \\ (s_1^{\,2}-k^2)e^{-s_1 d} & (s_1^{\,2}-k^2)e^{+s_1 d} & (s_3^{\,2}-k^2)e^{-s_3 d} & (s_3^{\,2}-k^2)e^{+s_3 d} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = 0 \qquad (18)$$

For solution, the determinant of the coefficients must be zero, which results in the relation

$$(s_3^{\,2} - s_1^{\,2})^2\,(e^{s_3 d} - e^{-s_3 d})\,(e^{s_1 d} - e^{-s_1 d}) = 0 \qquad (19)$$

Thus, for non-trivial solution, either

$$(s_3)^2 = -\left(\frac{n\pi}{d}\right)^2 \tag{20}$$

or

$$(s_1)^2 = -\left(\frac{n\pi}{d}\right)^2 \tag{21}$$

We can handle both situations simultaneously using (10) - (14)

$$k^2 - \left[\frac{\beta - \left(\frac{\alpha}{2}\right)^2}{2}\right] \pm \sqrt{\frac{\beta - \left(\frac{\alpha}{2}\right)^2}{2} + \gamma} = -\left(\frac{n\pi}{d}\right)^2 \tag{22}$$

Rearranging (22) squaring and reducing, we finally arrive at

$$\gamma - \left(\beta - \left(\frac{\alpha}{2}\right)^2\right)\left(k^2 + \left(\frac{n\pi}{d}\right)^2\right) = \left[\left(\frac{n\pi}{d}\right) + k^2\right]^2 \tag{23}$$

Then, using (5) and (6), we write (23) as

$$\omega^2 = \frac{k^2}{\rho_o}\ \frac{\left[\frac{\left(\frac{\alpha}{2} q_o\right)^2}{\varepsilon} + \left(-\alpha g \rho_o + \frac{q_o^2}{\varepsilon}\right)\left[\left(\frac{n\pi}{d}\right)^2 + k^2\right]\right]}{\left[\left(\frac{n\pi}{d}\right)^2 + k^2\right]\left[\left(\frac{n\pi}{d}\right)^2 + k^2 + \left(\frac{\alpha}{2}\right)^2\right]} \tag{24}$$

$$n = 1, 2, \ldots$$

We note that the electrical terms are always stabilizing, so gravity can be the only destabilizing mechanism. When

$$\alpha > 0$$

the heavier fluid is on top and so the system is potentially unstable, unless the amount of charge present is enough to stabilize the system, which will occur if

$$\frac{q_o^2}{\varepsilon} > \alpha g \rho_o \tag{25}$$

If the inequality of (25) is met, the system will support purely propagating waves.

4.5 Presentation of Results

Because the dispersion relations of (4.4.24) and (4.3.26) are so similar in form, it is possibly to present the information of both these equations simultaneously.

We consider only stable systems when (4.2.28) is obeyed. For the weak gradient case (section 4.3) we define the quantities

$$\omega_m^2 = -\alpha g + \frac{E_o D q_o}{\rho_T} \tag{1}$$

and

$$\omega_p^2 = \frac{d^2 (D q_o)^2}{\rho_T \varepsilon} \tag{2}$$

Similarly, for the exponential distributions, we define

$$\omega_m^2 = -\alpha g + \frac{q_o^2}{\rho_o \varepsilon} \tag{3}$$

and

$$\omega_p^2 = \left(\frac{\alpha d q_o}{2}\right)^2 \bigg/ \rho_o \varepsilon \tag{4}$$

We see that (3) and (4) agree with (1) and (2) when

$$\alpha d << 1 \tag{5}$$

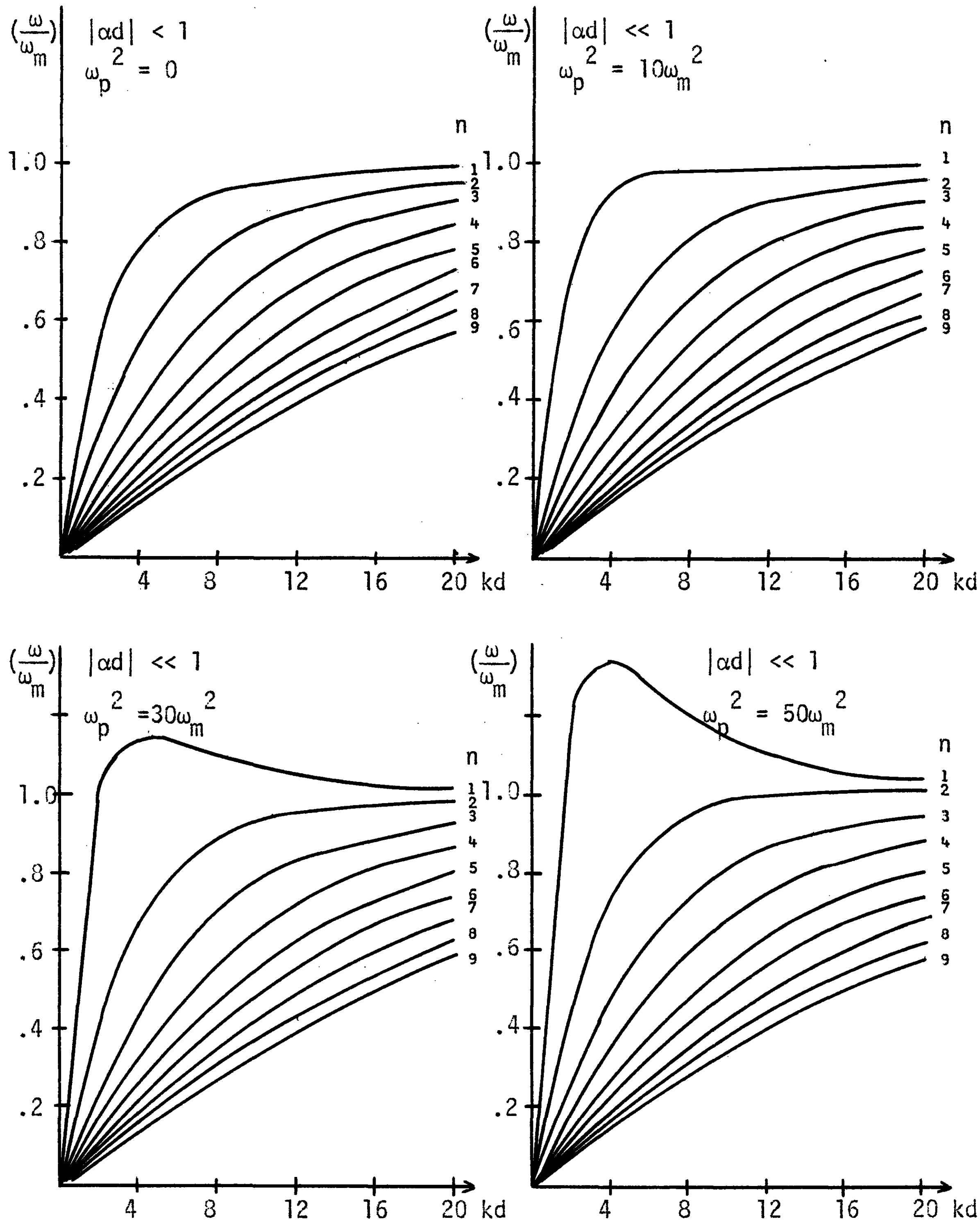
Fig. 1 plots the dispersion relation, when $\alpha d << 1$, for various values of ω_p^2, where the frequency is normalized to $\omega_m (\omega_m^2 > 0)$. If one is considering the weak gradient case, ω_m^2 and ω_p^2 are given by (1) and (2), while if the exponential case is of interest they are given by (3) and (4). Fig. 2 is appropriate for the exponential stratification when

$$\alpha d = 5.$$

When ω_p^2 exceeds ω_m^2, we find that for increasing wavenumber, the group velocity, given by the slope of the curves ($v_g = \frac{d\omega}{dk}$), decreases, eventually becomes zero, and then for higher k becomes negative. In our analysis, we neglected all dissipative mechanisms such as viscosity and conduction. Simple perturbation theory can be used to calculate corrections to the relations given by (4.3.26) and (4.4.24), for slight loss. If the system is driven at real frequency ω, the spatial damping is given by the imaginary part of k.[45]

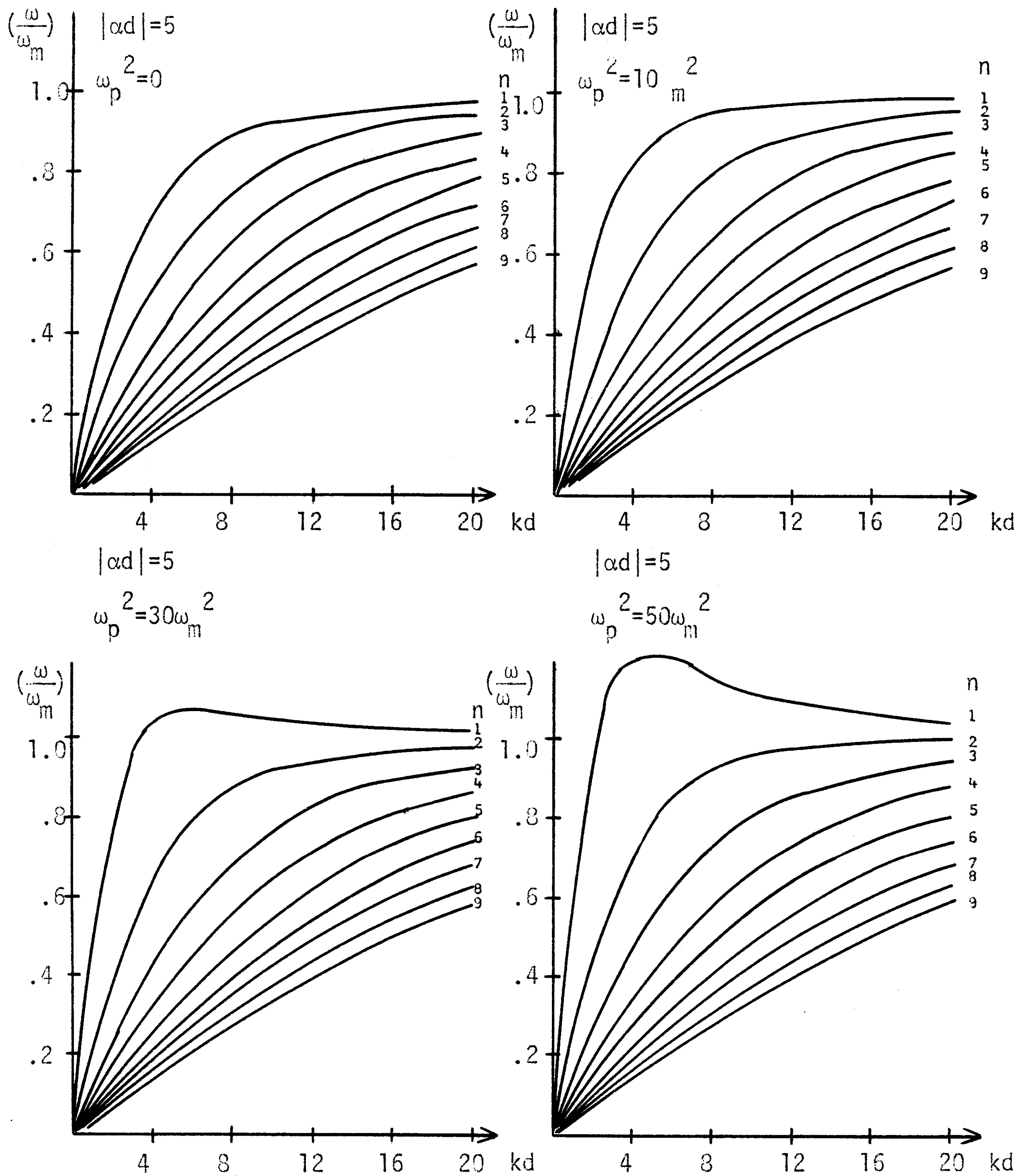
$$k_i = \frac{1}{2} \frac{\langle P_d \rangle}{v_g \langle W \rangle} \tag{6}$$

where $\langle P_d \rangle$ is the time average power dissipated and $\langle W \rangle$ is the time average power stored in the system. However, when the group velocity becomes zero, k_i becomes very large violating our approximations. This indicates that when



Dispersion relation for weak gradient, or exponential stratification with $|\alpha d| \ll 1$, for various values of $\omega_p^{\;2}$.

Figure 1



Dispersion relation for exponential stratification with $|\alpha d|=5$, for various values of ω_p^2.

Figure 2

$$v_g = 0$$

a disturbance will not propagate away, thus allowing other mechanisms a chance to exert their influence. Thus, we must keep in mind that when

$$v_g \simeq 0$$

some assumptions may be violated in a real system.

In Chapter V, the two cases treated here will be compared to analagous surface wave problems. For these continuous distributions, we have an infinite number of modes. Superposed layered systems will have only as many modes as there are interfaces. However, as the number of interfaces gets large it will be shown that the two methods approximately agree.

CHAPTER V
ANALYSIS OF MANY SUPERPOSED CHARGED LAYERS

5.1 Introduction

The analysis of Chapters II and III considered the dynamics of superposed layers of incompressible, inviscid, perfectly-insulating fluids, where each layer had constant properties of mass density, charge density, dielectric constant and convection velocity, which are different for each layer. The electromechanical interactions couple through each interface, and so can be easily analyzed by systematically applying the boundary conditions at each interface. The terminal relations derived in Chapter II aid in the analysis, since they directly relate the interfacial perturbation quantities of pressure, displacement, electric potential, and normal electric fields, which are the pertinent quantities that appear in the interfacial boundary conditions.

We will illustrate the usefulness of the "terminal relation" approach by developing the general case of (N+1) superposed layers in planar geometry. The resulting solution will be of use for any discrete strata problem of the type considered here, or if the number of layers becomes large, with the thickness of each layer tending to zero, the solution approaches the limiting case of a continuous stratification. We will illustrate this convergence by modeling the weak gradient and exponential stratifications treated in sections (4.3) and (4.4) by many thin layers.

For completeness we include the possibilities of the layers having relative velocities. There has been considerable controversy in the past as to whether continuous flow stratifications can be analyzed as the limiting case of a multi-layered system.[40] If performed correctly, such a representation is correct, but there are difficulties due to the presence of a continuum of modes as discussed in section (1.6). We avoid these difficulties in our examples by assuming the layers to be stationary in equilibrium.

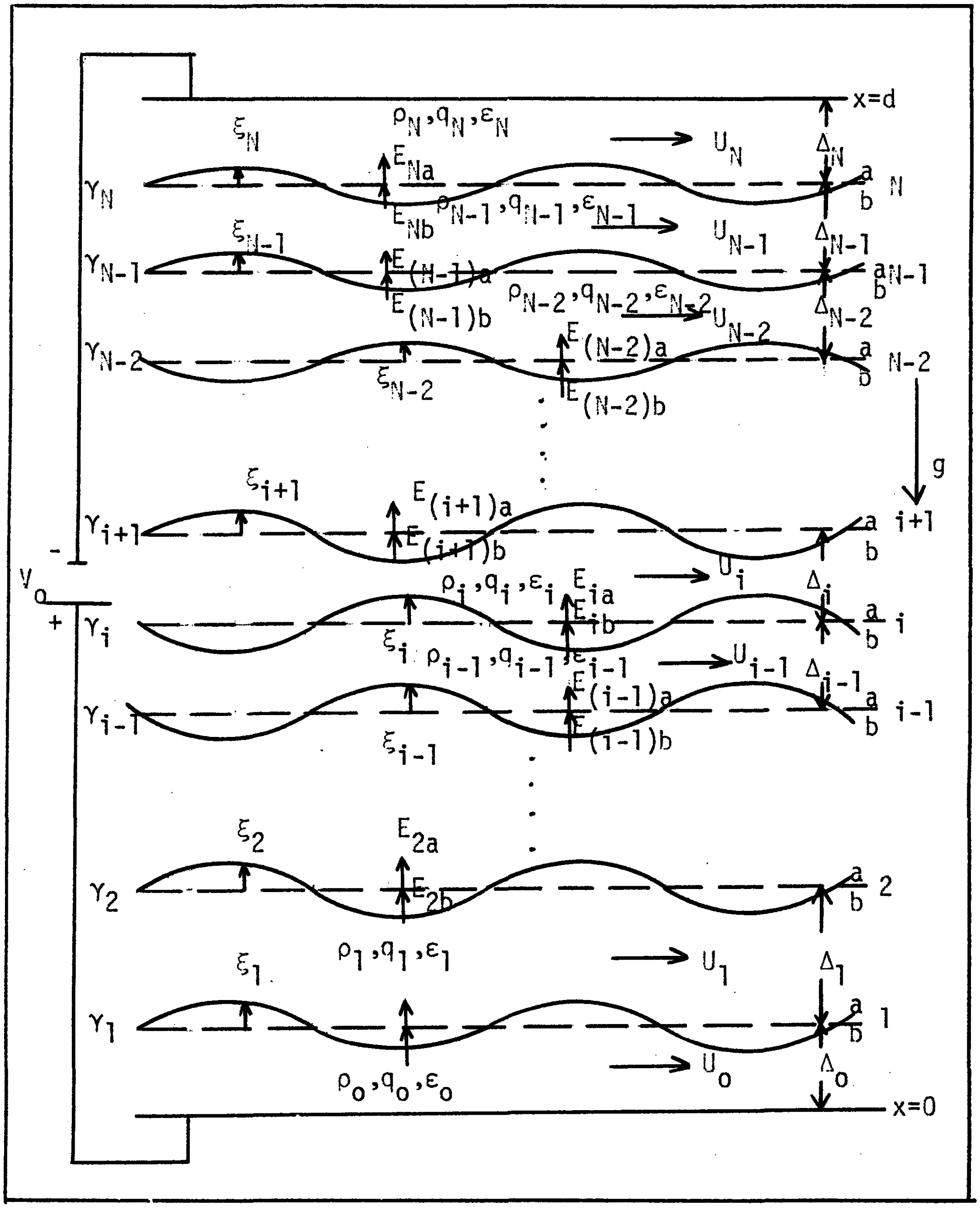
5.2 Equations of Motion for (N+1) Layers (N interfaces)

We consider (N+1) layers of incompressible, inviscid, perfectly-insulating fluid bounded from above and below by rigid, perfectly conducting plates. Each layer has its corresponding thickness Δ, mass density ρ, charge density q, dielectric constant ε, and convection velocity U, as shown in Fig. 1, Each interface also has its corresponding surface tension γ.

For clarity, we repeat here the terminal relations derived in section 2.3 for the prototype layer shown in Fig. 2.

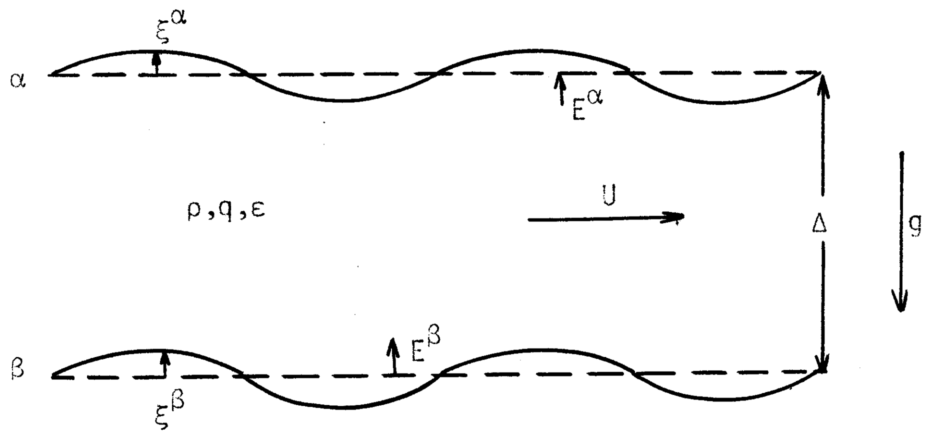
$$\begin{bmatrix} \hat{p}^{\alpha} \\ \hat{p}^{\beta} \end{bmatrix} = \begin{bmatrix} \frac{\rho\omega'^2}{k}\coth k\Delta - \rho g & \frac{-\rho\omega'^2}{k\sinh k\Delta} \\ \frac{\rho\omega'^2}{k\sinh k\Delta} & -\frac{\rho\omega'^2}{k}\coth k\Delta - \rho g \end{bmatrix} \begin{bmatrix} \hat{\xi}^{\alpha} \\ \hat{\xi}^{\beta} \end{bmatrix} - q \begin{bmatrix} \hat{\phi}^{\alpha} \\ \hat{\phi}^{\beta} \end{bmatrix} \quad (1)$$

$$\begin{bmatrix} \hat{D}^{\alpha} \\ \hat{D}^{\beta} \end{bmatrix} = \begin{bmatrix} \varepsilon k\coth k\Delta & \frac{-\varepsilon k}{\sinh k\Delta} \\ \frac{-\varepsilon k}{\sinh k\Delta} & \varepsilon k\coth k\Delta \end{bmatrix} \begin{bmatrix} \hat{\phi}^{\alpha} + E^{\alpha}\hat{\xi}^{\alpha} \\ \hat{\phi}^{\beta} + E^{\beta}\hat{\xi}^{\beta} \end{bmatrix} + q \begin{bmatrix} -\hat{\xi}^{\alpha} \\ +\hat{\xi}^{\beta} \end{bmatrix} \quad (2)$$



Definition of variables for (N+1) superposed layers. Each layer has its corresponding mass density ρ, charge density q, dielectric constant ε, thickness Δ, and convection velocity U. Each interface also has its corresponding surface tension γ.

Figure 1



$$\hat{Q}^{\alpha} = -\varepsilon \hat{e}_x^{\alpha}$$
$$\hat{Q}^{\beta} = +\varepsilon \hat{e}_x^{\beta}$$

Prototype layer for rectangular geometry.

Figure 2

We focus attention on the i^{th} interface which separates the i^{th} and $(i-1)^{th}$ layers. The boundary conditions are

$$\hat{\phi}_{ia} = \hat{\phi}_{ib} = \hat{\phi}_i \tag{3}$$

$$\hat{\xi}_{ia} = \hat{\xi}_{ib} = \hat{\xi}_i \tag{4}$$

$$\hat{p}_{ib} - \hat{p}_{ia} + \varepsilon_i E_{ai} \hat{e}_{xai} - \varepsilon_{i-1} E_{bi} \hat{e}_{xbi} - \gamma_i k^2 \hat{\xi}_i = 0 \tag{5}$$

$$\hat{Q}_{ia} + \hat{Q}_{ib} = 0 \tag{6}$$

We define

$$\omega_i' = \omega - kU_i \tag{7}$$

Using the terminal relations of (1) and (2) we have

$$\hat{p}_{ia} = - (\rho_i \omega_i'^2 \coth k\Delta_i + \rho_i g)\ \hat{\xi}_i + \frac{\rho_i \omega_i'^2}{k \sinh k\Delta_i}\ \hat{\xi}_{i+1} - q_i \hat{\phi}_i \tag{8}$$

$$\hat{p}_{ib} = (\rho_{i-1} \omega_{i-1}'^2 \coth k\Delta_{i-1} - \rho_{i-1} g)\ \hat{\xi}_i - \frac{\rho_{i-1} \omega_{i-1}'^2}{k \sinh k\Delta_{i-1}}\ \hat{\xi}_{i-1} - q_{i-1} \hat{\phi}_i \tag{9}$$

$$\hat{Q}_{ib} = \varepsilon_{i-1}\ k \coth k\Delta_{i-1}\ (\hat{\phi}_i + E_{bi} \hat{\xi}_i)$$

$$\frac{-\varepsilon_{i-1}\ k}{\sinh k\Delta_{i-1}}\ (\hat{\phi}_{i-1} + E_{a(i-1)}\ \hat{\xi}_{i-1}) - q_{i-1}\ \hat{\xi}_i \tag{10}$$

$$\hat{Q}_{ia} = \frac{-\varepsilon_i k}{\sinh k\Delta_i}\ (\hat{\phi}_{i+1} + E_{b(i+1)} \hat{\xi}_{i+1})$$

$$+ \varepsilon_i k \coth k\Delta_i\ (\hat{\phi}_i + E_{ai}\ \hat{\xi}_i) + q_i\ \hat{\xi}_i \tag{11}$$

We note that the electric fields above and below an interface are in general different, since

$$E_{ai} \neq E_{bi} \tag{12}$$

because

$$\varepsilon_i E_{ai} = \varepsilon_{i-1} E_{bi} \tag{13}$$

Now using the boundary condition expressed by (6) we obtain

$$\hat{\phi}_i \left[\varepsilon_i k \coth \Delta_i + \varepsilon_{i-1} k \coth k\Delta_{i-1}\right]$$

$$- \hat{\phi}_{i-1} \left[\frac{+\varepsilon_{i-1} k}{\sinh k\Delta_{i-1}} \right] - \hat{\phi}_{i+1} \left[\frac{+\varepsilon_i k}{\sinh k\Delta_i} \right]$$

$$+ \hat{\xi}_i \left[\varepsilon_{i-1} E_{bi} k \coth k\Delta_{i-1} + \varepsilon_i E_{ai} k \coth k\Delta_i + q_i - q_{i-1}\right]$$

$$- \hat{\xi}_{i+1} \left[\frac{+\varepsilon_i E_{b(i+1)} k}{\sinh k\Delta_i} \right] - \hat{\xi}_{i-1} \left[\frac{+\varepsilon_{i-1} E_{a(i-1)} k}{\sinh k\Delta_{i-1}} \right] = 0 \tag{14}$$

We can rewrite the boundary condition of (5) as

$$\hat{p}_{ib} - \hat{p}_{ia} + (E_{ai} - E_{bi})\ (\hat{Q}_{ia}) - \gamma_i k^2 \hat{\xi}_i = 0 \tag{15}$$

which results in the relation

$$\hat{\xi}_i \left[\frac{\omega'^2_{i-1}}{k} \rho_{i-1} \coth k\Delta_{i-1} + \frac{\omega'^2_i}{k} \rho_i \coth k\Delta_i - g(\rho_{i-1} - \rho_i) - \gamma_i k^2 \right.$$

$$\left. + (E_{ai} - E_{bi}) \epsilon_i E_{ai} k \coth k\Delta_i + q_i (E_{ai} - E_{bi}) \right]$$

$$- \frac{\omega'^2_{i-1} \rho_{i-1}}{k \sinh k\Delta_{i-1}} \hat{\xi}_{i-1}$$

$$- \left[\frac{\omega'^2_i \rho_i}{k \sinh k\Delta_i} + \frac{(E_{ai} - E_{bi}) \epsilon_i E_{b(i+1)} k}{\sinh k\Delta_i} \right] \hat{\xi}_{i+1}$$

$$- [q_{i-1} - q_i - \epsilon_i (E_{ai} - E_{bi}) k \coth k\Delta_i] \hat{\phi}_i$$

$$- \left[\frac{\epsilon_i (E_{ai} - E_{bi}) k}{\sinh k\Delta_i} \right] \hat{\phi}_{i+1} = 0 \tag{16}$$

After writing (15) and (16) as a matrix equation, and performing some matrix manipulations we can write the result concisely as (17), where

$$a_n = \frac{\rho_n \omega'^2_n}{k} \coth k\Delta_n + \frac{\rho_{n-1} \omega'^2_{n-1}}{k} \coth k\Delta_{n-1} - g(\rho_{n-1} - \rho_n) - \gamma_i k^2$$

$$+ \epsilon_n (E_{an} - E_{bn})^2 k \coth k\Delta_n + q_n (E_{an} - E_{bn})$$

$$+ E_{bn} (q_{n-1} - q_n) \tag{18}$$

$$\begin{bmatrix} a_1 & b_1 & c_2 & 0 & 0 & 0 & \cdots & & \\ b_1 & d_1 & e_2 & f_2 & 0 & 0 & \cdots & & \\ c_2 & e_2 & a_2 & b_2 & c_3 & 0 & 0 & \cdots & \\ 0 & f_2 & b_2 & d_2 & e_3 & f_3 & 0 & 0 & \cdots \\ 0 & 0 & c_3 & e_3 & a_3 & b_3 & c_4 & 0 & \cdots \\ 0 & 0 & 0 & f_3 & b_3 & d_3 & e_4 & f_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} \hat{\xi}_1 \\ \hat{\phi}_1 \\ \hat{\xi}_2 \\ \hat{\phi}_2 \\ \vdots \end{bmatrix} = 0 \qquad (17)$$

$$b_n = (q_n - q_{n-1}) + \varepsilon_n (E_{an} - E_{bn}) k \coth k\Delta_n \tag{19}$$

$$c_n = \frac{-\omega_n'^2 \rho_n}{k \sinh k\Delta_n} \tag{20}$$

$$d_n = \varepsilon_n k \coth k\Delta_n + \varepsilon_{n-1} k \coth k\Delta_{n-1} \tag{21}$$

$$e_n = \frac{-\varepsilon_n k (E_{an} - E_{bn})}{\sinh k\Delta_n} \tag{22}$$

$$f_n = \frac{-\varepsilon_n k}{\sinh k\Delta_n} \tag{23}$$

We note from (17) that the matrix is symetric with all elements real. For N interfaces, the order of the matrix is 2N x 2N. The matrix elements appear in a systematic form with the non-zero elements situated around the major diagonal. If we let each layer have the same dielectric constant, then $E_{an} = E_{bn}$, and (17) reduces to a simpler form because $e_n = 0$.

Equations (17) - (23) give the general solution for discrete strata of any number of layers. The class of problems encompassed by these equations include: (a) gravity-capillary waves and instabilities; (b) polarization waves and instabilities; (c) space charge waves and instabilities; (d) effects due to relative convection of layers.

These equations also have the extra value of approximating continuous stratifications when letting the number of layers approach infinity, with the thickness of each layer approaching zero.

We will use (17) - (23) to treat the examples of many layers when the properties between layers change slowly or obey an exponential law. This will be analagous to the continuous weak gradient and exponential stratifications considered in sections (4.3) and (4.4). We will then illustrate the convergence of many thin layers by showing the dispersion relations of the limiting discrete and continuous problems to be identical.

5.3 Discrete Weak Gradient Distribution ($\gamma_i = 0$, $U_i = 0$, ε_i = constant)

The geometry for this stratified layer problem is illustrated in Fig. 1a. We assume that each interface has surface tension equal to zero ($\gamma_i = 0$), and that in equilibrium each layer is stationary ($U_i = 0$). The dielectric constant and thickness of each layer are identical ($\varepsilon_i = \varepsilon$ = constant, $\Delta_i = \Delta$ = constant).

In particular, we consider a step-wise approximation to linear distributions of mass and charge density as illustrated in Fig. 1b and 1c. Then

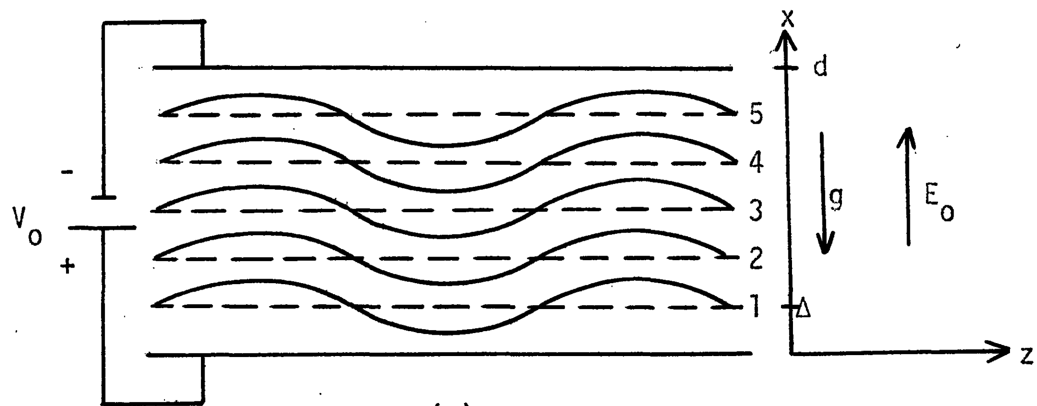
$$\rho_n = \rho_o\,(1 + \alpha n \Delta) \qquad n=0,1,\ldots N \tag{1}$$

where

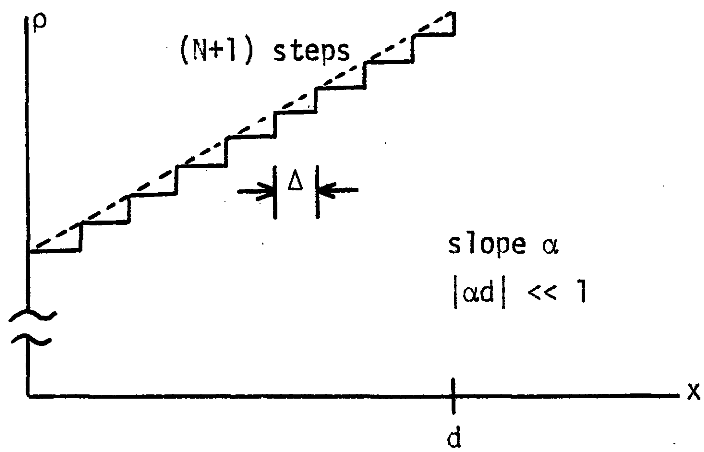
$$\alpha d << 1 \tag{2}$$

and

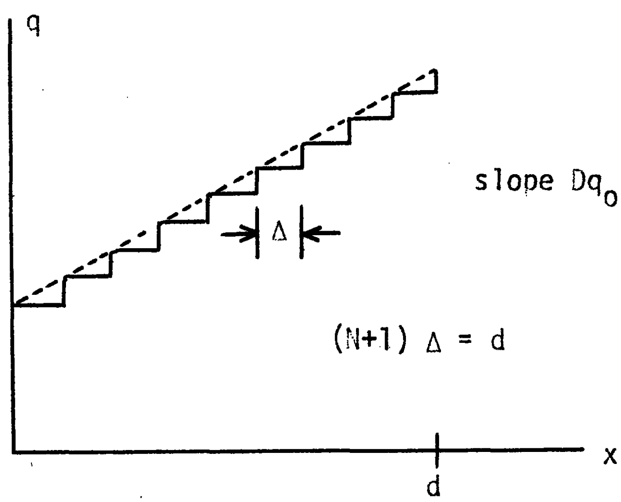
$$q_n - q_{n-1} = (Dq_o)\Delta \tag{3}$$



(a)



(b)



a) Geometry for weak gradient stratification; We approximate linear distributions in mass (b) and charge (c) densities by step functions.

Figure 1

with

$$Dq_o = \text{constant} \tag{4}$$

In analogy to the Boussinesq approximation made in section (4.3), we assume the mass density to be constant in the inertia terms, while retaining density gradients in the gravity terms. We also assume the charge density to be very small, so that the electric field is determined from the external battery. With these approximations we have

$$\rho_n + \rho_{n-1} \simeq 2\rho_o \tag{5}$$

$$\rho_{n-1} - \rho_n = -\rho_o \alpha\Delta \tag{6}$$

$$E = E_o = \frac{V_o}{d} = \text{constant} \tag{7}$$

Then within our approximations (5.2.18) - (5.2.23) become

$$a_n = \frac{\omega^2}{k} \ 2\rho_o \coth k\Delta + \alpha g \rho_o \Delta - E_o D q_o \Delta \equiv a \tag{8}$$

$$b_n = Dq_o\Delta \equiv b \tag{9}$$

$$c_n = \frac{-\omega^2 \rho_o}{k \sinh k\Delta} \equiv c \tag{10}$$

$$d_n = 2\varepsilon k \coth k\Delta \equiv d \tag{11}$$

$$e_n \equiv 0 \tag{12}$$

$$f_n = \frac{-\varepsilon k}{\sinh k\Delta} \equiv f \tag{13}$$

We see that, within the same assumptions made in section 4.3, every matrix element is a constant and does not change from row to row in (5.2.17). Thus, we can write (5.2.17) as

$$\begin{bmatrix} a & b & c & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\ b & d & 0 & f & 0 & 0 & 0 & 0 & 0 \cdots \\ c & 0 & a & b & c & 0 & 0 & 0 & 0 \cdots \\ 0 & f & b & d & 0 & f & 0 & 0 & 0 \cdots \\ 0 & 0 & c & 0 & a & b & c & 0 & 0 \cdots \\ 0 & 0 & 0 & f & b & d & 0 & f & 0 \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \hat{\xi}_1 \\ \hat{\phi}_1 \\ \hat{\xi}_2 \\ \hat{\phi}_2 \\ \hat{\xi}_3 \\ \hat{\phi}_3 \\ \vdots \end{bmatrix} = 0 \qquad (14)$$

In terms of difference equations, we have

$$c\hat{\xi}_{n-1} + a\hat{\xi}_n + c\hat{\xi}_{n+1} + b\hat{\phi}_n = 0 \qquad (15)$$

$$f\hat{\phi}_{n-1} + d\hat{\phi}_n + f\hat{\phi}_{n+1} + b\hat{\xi}_n = 0 \qquad (16)$$

We have in (15) and (16) two coupled linear, constant-coefficient difference equations. We assume solutions of the form

$$\hat{\xi}_n = \hat{\xi}\lambda^n \tag{17}$$

$$\hat{\phi}_n = \hat{\phi}\lambda^n \tag{18}$$

for which we rewrite (15) and (16) as

$$\hat{\xi}[c\lambda^2 + a\lambda + c] + b\lambda\hat{\phi} = 0 \tag{19}$$

$$\hat{\xi}b\lambda + \hat{\phi}[f\lambda^2 + d\lambda + f] = 0 \tag{20}$$

For non-trivial solutions, the determinant of the coefficients of $\hat{\phi}$ and $\hat{\xi}$ must be zero, or

$$[c\lambda^2 + a\lambda + c][f\lambda^2 + d\lambda + f] - b^2\lambda^2 = 0 \tag{21}$$

which can also be written as

$$\lambda^4 + (\frac{a}{c} + \frac{d}{f})\lambda^3 + \lambda^2\ (2 + \frac{ad}{cf} - \frac{b^2}{fc}) + (\frac{a}{c} + \frac{d}{f})\lambda + 1 = 0 \tag{22}$$

In general, it is possible to solve for the four roots of a quartic equation, but it is usually quite unwieldy.[41] In this case, (2 2) is factorable to

$$[\lambda^2 + \{\frac{A}{2} + \sqrt{(\frac{A}{2})^2 - (B - 2)}\}\lambda + 1]\ [\lambda^2 + \{\frac{A}{2} - \sqrt{(\frac{A}{2})^2 - (B-2)}\}\lambda + 1] = 0 \tag{23}$$

where

$$A = \frac{a}{c} + \frac{d}{f} \tag{24}$$

$$B = 2 + \frac{ad}{cf} - \frac{b^2}{fc} \tag{25}$$

with

$$\sqrt{\left(\frac{A}{2}\right)^2 - (B - 2)} = \sqrt{\left(\frac{a}{2c} - \frac{d}{2f}\right)^2 + \frac{b^2}{fc}} \tag{26}$$

We define

$$\beta_+ = \frac{1}{2}\left(\frac{A}{2} + \sqrt{\left(\frac{A}{2}\right)^2 - (B - 2)}\right)$$

$$= \frac{1}{2}\left(\frac{a}{2c} + \frac{d}{2f} + \sqrt{\left(\frac{a}{2c} - \frac{d}{2f}\right)^2 + \frac{b^2}{fc}}\right) \tag{27}$$

and

$$\beta_- = \frac{1}{2}\left(\frac{A}{2} - \sqrt{\left(\frac{A}{2}\right)^2 - (B - 2)}\right)$$

$$= \frac{1}{2}\left(\frac{a}{2c} + \frac{d}{2f} - \sqrt{\left(\frac{a}{2c} - \frac{d}{2f}\right)^2 + \frac{b^2}{fc}}\right) \tag{28}$$

We then see that the four roots of (23) are

$$\lambda_1 = -\beta_+ + [\beta_+^2 - 1]^{1/2} \tag{29}$$

$$\lambda_2 = -\beta_+ - [\beta_+^2 - 1]^{1/2} \tag{30}$$

$$\lambda_3 = -\beta_- + [\beta_-^2 - 1]^{1/2} \tag{31}$$

$$\lambda_4 = -\beta_- - [\beta_-^2 - 1]^{1/2} \tag{32}$$

If we further let

$$\beta_+ = \cos\theta_+ \tag{33}$$

and

$$\beta_- = \cos\theta_- \tag{34}$$

we can rewrite the roots of (29) - (32) as

$$\lambda_1 = -e^{-j\theta_+} \qquad (35)$$

$$\lambda_2 = -e^{+j\theta_+} \qquad (36)$$

$$\lambda_3 = -e^{-j\theta_-} \qquad (37)$$

$$\lambda_4 = -e^{+j\theta_-} \qquad (38)$$

Because of the rigid perfectly conducting electrodes, the boundary conditions are:

$$\hat{\xi}_o = 0 \qquad \hat{\phi}_o = 0$$
$$\hat{\xi}_{N+1} = 0 \qquad \hat{\phi}_{N+1} = 0 \qquad (39)$$

So, we assume solutions of the form

$$\xi_n = A_1\lambda_1^{\,n} + A_2\lambda_2^{\,n} + A_3\lambda_3^{\,n} + A_4\lambda_4^{\,n} \qquad (40)$$

and then, from (20)

$$\phi_n = \frac{A_1 b\lambda_1^{\,n+1}}{f\lambda_1^{\,2} + d\lambda_1 + f} + \frac{A_2 b\lambda_2^{\,n+1}}{f\lambda_2^{\,2} + d\lambda_2 + f} + \frac{A_3 b\lambda_3^{\,n+1}}{f\lambda_3^{\,2} + d\lambda_3 + f} + \frac{A_4 b\lambda_4^{\,n+1}}{f\lambda_4^{\,2} + d\lambda_4 + f}$$

(41)

Applying the boundary conditions of (39) and using the facts that

$$\frac{f\lambda_1^2 + d\lambda_1 + f}{\lambda_1} = \frac{f\lambda_2^2 + d\lambda_2 + f}{\lambda_2} = d - 2f\beta_+ \tag{42}$$

and

$$\frac{f\lambda_3^2 + d\lambda_3 + f}{\lambda_3} = \frac{f\lambda_4^2 + d\lambda_4 + f}{\lambda_4} = d - 2f\beta_- \tag{43}$$

we obtain

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1^{N+1} & \lambda_2^{N+1} & \lambda_3^{N+1} & \lambda_4^{N+1} \\ \frac{1}{[\frac{d}{f}-2\beta_+]} & \frac{1}{[\frac{d}{f}-2\beta_+]} & \frac{1}{[\frac{d}{f}-2\beta_-]} & \frac{1}{[\frac{d}{f}-2\beta_-]} \\ \frac{\lambda_1^{N+1}}{[\frac{d}{f}-2\beta_+]} & \frac{\lambda_2^{N+1}}{[\frac{d}{f}-2\beta_+]} & \frac{\lambda_3^{N+1}}{[\frac{d}{f}-2\beta_-]} & \frac{\lambda_4^{N+1}}{[\frac{d}{f}-2\beta_-]} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = 0 \tag{44}$$

For non-trivial solutions, the determinant of the coefficients must be zero, which becomes

$$\frac{2f(\beta_+ - \beta_-)^2 \, (\lambda_4^{N+1} - \lambda_3^{N+1}) \, (\lambda_2^{N+1} - \lambda_1^{N+1})}{[d-2f\beta_+] \, [d-2f\beta_-]} = 0 \tag{45}$$

We must have, from (35) - (38), that either

$$\sin (N+1)\, \theta_{+} = 0 \qquad (46)$$

or

$$\sin (N+1)\, \theta_{-} = 0 \qquad (47)$$

for which

$$\theta_{\pm} = \frac{p\pi}{N+1} \qquad p = 1, \ldots\ldots N \qquad (48)$$

(p = 0 solution is trivial)

Then, from (33) or (34) we have

$$\beta_{\pm} = \cos \frac{p\pi}{N+1} = \frac{1}{2} \left(\frac{a}{2c} + \frac{d}{2f} \pm \sqrt{\left(\frac{a}{2c} - \frac{d}{2f}\right)^2 + \frac{b^2}{fc}} \right) \qquad (49)$$

(49) can be rewritten as

$$\left(\frac{a}{2c} - \frac{d}{2f}\right)^2 + \frac{b^2}{fc} = 4 \cos^2 \frac{p\pi}{N+1} + \left(\frac{a}{2c} + \frac{d}{2f}\right)^2 - 2\left(\frac{a}{c} + \frac{d}{f}\right) \cos \frac{p\pi}{N+1} \qquad (50)$$

which can be further reduced to

$$\frac{b^2}{fc} - \frac{ad}{cf} + 2\left(\frac{a}{c} + \frac{d}{f}\right) \cos \frac{p\pi}{N+1} - 4\cos^2 \frac{p\pi}{N+1} = 0 \qquad (51)$$

Referring back to (8) - (13), we have that

$$\frac{d}{f} = -2 \cosh k\Delta \qquad (52)$$

and

$$\frac{a}{c} = -2 \cosh k\Delta + (-\alpha g \rho_o + E_o D q_o) \frac{k \sinh k\Delta}{\omega^2 \rho_o} \tag{53}$$

and

$$\frac{b^2}{fc} = \frac{(Dq_o\Delta)^2 \sinh^2 k\Delta}{\omega^2 \rho_o \varepsilon} \tag{54}$$

We can then rewrite (51) as

$$\omega^2 = \frac{\dfrac{(Dq_o\Delta)^2 \sinh^2 k\Delta}{\rho_o \varepsilon} + 2\left(-\alpha g + \dfrac{E_o D q_o}{\rho_o}\right) k \sinh k\Delta \left(\cos \frac{p\pi}{N+1} + \cosh k\Delta\right)}{4\left(\cos \frac{p\pi}{N+1} + \cosh k\Delta\right)^2}$$

$$p = 1, 2, \ldots N \tag{55}$$

Equation (55) describes the wave dispersion for (N+1) layers with small changes between layers. Gravity will be stabilizing for α negative, for then the heavier fluids will be below. A necessary and sufficient condition for stability is

$$E_o D q_o - \alpha g \rho_o > 0 \tag{56}$$

We see that the self field term in (55) is always stabilizing, but that the second term dominates when $k\Delta$ becomes very large. Plots of (55) for various values of the parameters will be presented in section 5.5.

We can compare (55) to the analagous continuous problem treated in section 4.3, if we take the limits

$$N \to \infty \tag{57}$$

$$\Delta \to 0 \tag{58}$$

$$(N+1)\Delta = d \tag{59}$$

Then

$$\lim_{k\Delta\to 0} \cosh k\Delta \simeq 1 + \frac{(k\Delta)^2}{2} + \dots . \tag{60}$$

$$\lim_{k\Delta\to 0} \sinh k\Delta \simeq k\Delta + \dots . \tag{61}$$

$$\lim_{\substack{N\to\infty \\ p\simeq N}} \left(1 + \cos \frac{p\pi}{N+1}\right) = 2 \cos^2 \frac{p\pi}{2(N+1)} \simeq \left(\frac{\pi}{2}\right)^2 \left(1 - \frac{p}{N+1}\right)^2$$

$$\simeq \left(\frac{\pi}{2}\right)^2 \left(\frac{N+1-p}{N+1}\right)^2 \tag{62}$$

Since p is a running integer, we count backwards from N

$$p = N,\ N-1,\ N-2,\ N-3,\ \dots 1 \tag{63}$$

or in general

$$p = N - r + 1 \qquad r = 1, 2, 3, \dots N \tag{64}$$

so from (62)

$$\lim_{N\to\infty} 2 \cos^2 \frac{p\pi}{2(N+1)} \simeq \left(\frac{\pi}{2}\right)^2 \left(\frac{r}{N+1}\right)^2 \qquad r = 1, 2, 3, \dots \tag{65}$$

Then in this limit, (55) becomes

$$\omega^2 = \frac{k^2\left[\frac{(Dq_o)^2}{\rho_o\varepsilon} + (-\alpha g + \frac{E_o Dq_o}{\rho_o})\ ([\frac{r\pi}{d}]^2 + k^2)\right]}{[(\frac{r\pi}{d})^2 + k^2]^2} \qquad (66)$$

$r=1, 2, 3, \ldots$

which is identical to the dispersion relation of the continuous distribution treated in section 4.3. (Eq. 4.3.26) We have shown that we can model a continuous mass and charge density distribution by a large number of thin layers, with our answers exactly agreeing in the limit as the number of layers tend to infinity, with the thickness of each layer approaching zero. We emphasize that there is no wavelength restriction, but we must keep in mind that we used the Boussinesq approximation on the mass density and ignored self field effects on the electric field, so our results are only approximate.

In the next section, we will treat the exact problem of a discrete exponential stratification. The results of this and the next section will be compared to the continuous stratification in section 5.5.

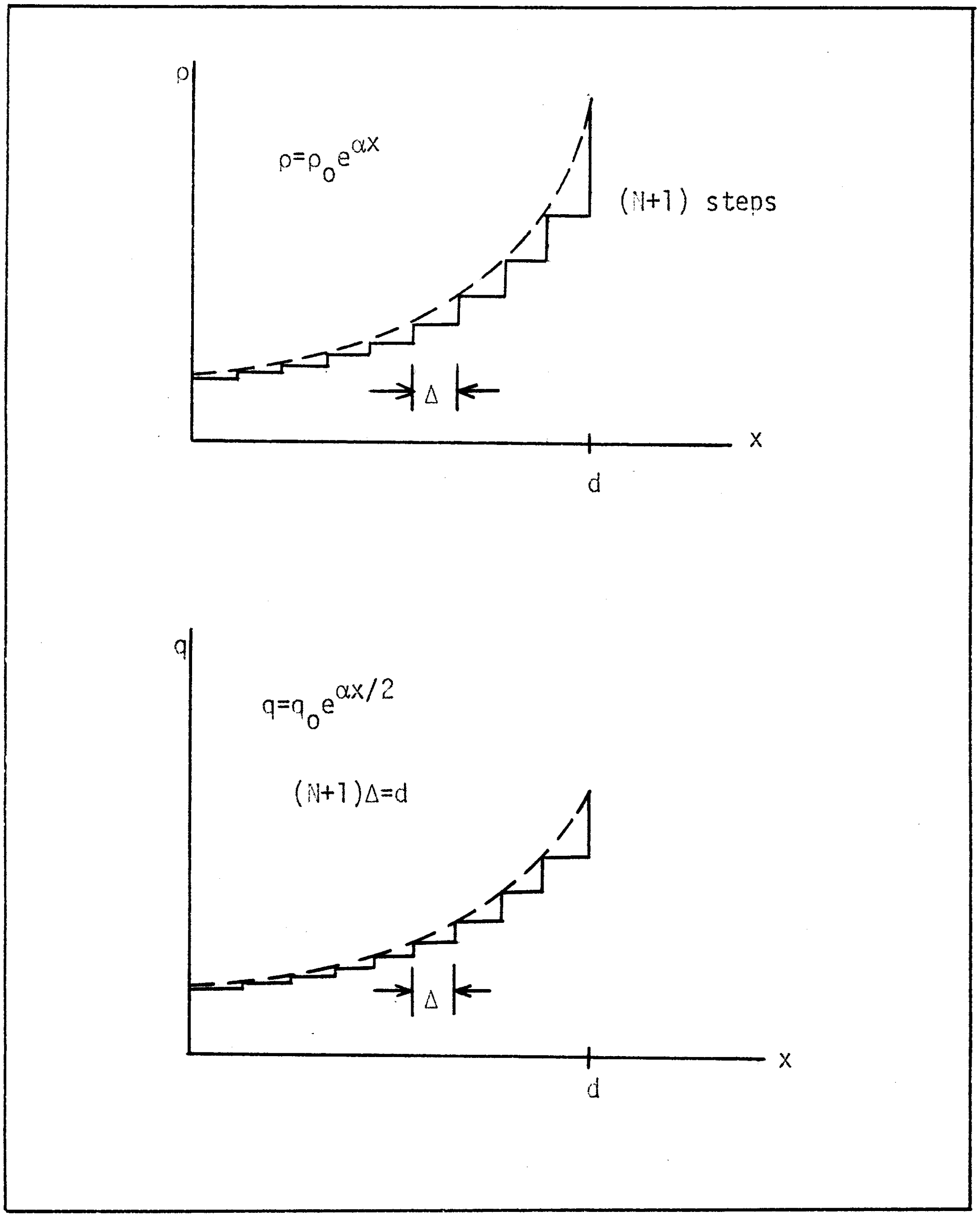
5.4 Discrete Exponential Stratifications

We will consider discrete stratifications in mass and charge density which obey an exponential law, as illustrated in Fig. 1. There will be no approximations in this section.

We assume the distribution between layers to be

$$\rho_{n-1} = \rho_n e^{-\alpha\Delta} \qquad (1)$$

$$q_{n-1} = q_n e^{-\alpha\Delta/2} \qquad (2)$$



Discrete exponential stratification. We approximate exponential distributions in mass and charge density by step functions.

Figure 1

$$E_n = \frac{2}{\varepsilon\alpha} q_n e^{\alpha\Delta/2} \tag{3}$$

Thus

$$\rho_{n-1} + \rho_n = \rho_n(1 + e^{-\alpha\Delta}) \tag{4}$$

$$\rho_{n-1} - \rho_n = \rho_n(e^{-\alpha\Delta} - 1) \tag{5}$$

$$q_{n-1} - q_n = q_n\ (e^{-\alpha\Delta/2} - 1) \tag{6}$$

Then from (5.2.18) - (5.2.23) we have

$$a_n = \frac{\omega^2}{k} \rho_n\ (1 + e^{-\alpha\Delta}) \coth k\Delta - g\rho_n\ (e^{-\alpha\Delta} - 1) + \frac{2q_n^{\ 2}}{\varepsilon\alpha} e^{-\alpha\Delta/2}\ (e^{-\alpha\Delta/2} - 1) \tag{7}$$

$$b_n = -q_n\ (e^{-\alpha\Delta/2} - 1) \tag{8}$$

$$c_n = \frac{-\omega^2\rho_n}{k \sinh k\Delta} \tag{9}$$

$$d_n = 2\varepsilon k \coth k\Delta \equiv d \tag{10}$$

$$e_n = 0 \tag{11}$$

$$f_n = \frac{-\varepsilon k}{\sinh k\Delta} \equiv f \tag{12}$$

Then the pertinent difference equations are

$$c_n\hat{\xi}_{n-1} + a_n\hat{\xi}_n + c_{n+1}\hat{\xi}_{n+1} + b_n\hat{\phi}_n = 0 \qquad (13)$$

$$b_n\hat{\xi}_n + f\hat{\phi}_{n-1} + d\hat{\phi}_n + f\hat{\phi}_{n+1} = 0 \qquad (14)$$

Now, because of the special distribution, we chose

$$q_n^{\ 2} = \frac{\rho_n q_o^{\ 2}}{\rho_o} \qquad (15)$$

Thus we can write a_n in (7) purely in terms of ρ_n. It is appropriate to define the variables

$$y_n = \rho_n\hat{\xi}_n \qquad (16)$$

and

$$x_n = q_n\hat{\phi}_n \qquad (17)$$

Then, if we multiply (14) by b_n, and use the relations of (15) - (17) we can rewrite (13) - (14) as

$$\frac{-\omega^2 e^{-\alpha\Delta}}{k \sinh k\Delta}\, y_{n-1} + \left[\frac{\omega^2}{k}\,(1+e^{-\alpha\Delta}) \coth k\Delta - g(e^{-\alpha\Delta}-1) + \frac{2q_o^{\ 2}}{\varepsilon\alpha\rho_o}\,(e^{-\alpha\Delta/2}-1)\right] y_n$$

$$\frac{-\omega^2}{k \sinh k\Delta}\; y_{n+1} - (e^{-\alpha\Delta/2} - 1)\, x_n = 0 \qquad (18)$$

and

$$\frac{q_o^2}{\varepsilon\rho_o}(e^{-\alpha\Delta/2} - 1)\, y_n + \frac{k}{\sinh k\Delta} e^{-\alpha\Delta/2} x_{n-1} - 2\,k \coth k\Delta\, x_n$$

$$\frac{+k}{\sinh k\Delta} e^{+\alpha\Delta/2} x_{n+1} = 0 \qquad (19)$$

We recognize (18) and (19) as linear constant-coefficient difference equations, for which we assume solutions of the form

$$y_n = \hat{Y}\lambda^n \qquad (20)$$

$$x_n = \hat{X}\lambda^n \qquad (21)$$

We define the parameter

$$\beta = \frac{k \sinh k\Delta}{\omega^2}\left[\frac{\omega^2}{k}(1+e^{-\alpha\Delta})\coth k\Delta - g(e^{-\alpha\Delta}-1) + \frac{2q_o^2}{\varepsilon\alpha\rho_o}(e^{-\alpha\Delta/2} - 1)\right] \qquad (22)$$

and substitute our assumed form of solution into (13) and (14) to obtain

$$[\lambda^2 - \beta\lambda + e^{-\alpha\Delta}]\,\hat{Y} + (e^{-\alpha\Delta/2} - 1)\frac{k \sinh k\Delta}{\omega^2}\lambda\hat{X} = 0 \qquad (23)$$

$$\hat{Y}\lambda\,\frac{q_o^2}{\rho_o}(e^{-\alpha\Delta/2} - 1)\frac{\sinh k\Delta}{\varepsilon k}e^{-\alpha\Delta/2} + (\lambda^2 - 2\cosh k\Delta e^{-\alpha\Delta/2}\lambda + e^{-\alpha\Delta})\hat{X} = 0$$

$$(24)$$

For non-trivial solutions to (23) and (24), the determinant of the coefficients must be zero, for which we obtain

$$[\lambda^2 - \beta\lambda + e^{-\alpha\Delta}]\,[\lambda^2 - 2\cosh k\Delta\, e^{-\alpha\Delta/2}\,\lambda + e^{-\alpha\Delta}] - \frac{\lambda^2\,(e^{-\alpha\Delta/2} - 1)^2 \sinh^2 k\Delta\, q_o^{\,2} e^{-\alpha\Delta/2}}{\omega^2 \rho_o \varepsilon} = 0 \qquad (25)$$

Expanding the terms of (25) we obtain an equation of the form

$$\lambda^4 + \lambda^3\,[-\beta-\gamma] + \lambda^2\,[2e^{-\alpha\Delta} + \beta\gamma - \delta] + \lambda\,[-\gamma-\beta]e^{-\alpha\Delta} + e^{-2\alpha\Delta} = 0 \qquad (26)$$

where

$$\gamma = 2\cosh k\Delta\, e^{-\alpha\Delta/2} \qquad (27)$$

$$\delta = \frac{(e^{-\alpha\Delta/2} - 1)^2 \sinh^2 k\Delta q_o^{\,2} e^{-\alpha\Delta/2}}{\omega^2 \rho_o \varepsilon} \qquad (28)$$

Because of the special equilibrium distribution we chose, the coefficients of the cubic and first-order term are proportional, which allows us to put (26) in the much simpler form

$$[\lambda^2 + (-\tfrac{\gamma+\beta}{2} + \sqrt{(\tfrac{\gamma+\beta}{2})^2 - \beta\gamma + \delta})\,\lambda + e^{-\alpha\Delta}] \cdot [\lambda^2 + (-\tfrac{\gamma+\beta}{2} - \sqrt{(\tfrac{\gamma+\beta}{2})^2 - \beta\gamma + \delta})\,\lambda + e^{-\alpha\Delta}] = 0 \qquad (29)$$

We note that (29) is very similar to the analogous equation (5.3.23) which was derived for a linear mass and charge density distribution. This is as expected, since the exponential distributions assumed here approach linear distributions for $\alpha d << 1$. We now proceed in the same way as in Section (4.3). We define the quantities

$$s_+ = \frac{1}{2}\left[-\frac{\gamma+\beta}{2} + \sqrt{\left(\frac{\gamma+\beta}{2}\right)^2 - \beta\gamma + \delta}\ \right] \tag{30}$$

and

$$s_- = \frac{1}{2}\left[-\frac{\gamma+\beta}{2} - \sqrt{\left(\frac{\gamma+\beta}{2}\right)^2 - \beta\gamma + \delta}\ \right] \tag{31}$$

so that the solutions to (29) can simply be expressed as

$$\lambda_1 = -S_+ + [S_+^{\,2} - e^{-\alpha\Delta}]^{1/2} \tag{32}$$

$$\lambda_2 = -S_+ - [S_+^{\,2} - e^{-\alpha\Delta}]^{1/2} \tag{33}$$

$$\lambda_3 = -S_- + [S_-^{\,2} - e^{-\alpha\Delta}]^{1/2} \tag{34}$$

$$\lambda_4 = -S_- - [S_-^{\,2} - e^{-\alpha\Delta}]^{1/2} \tag{35}$$

Let

$$S_+ = \cos\theta_+ \, e^{-\alpha\Delta/2} \tag{36}$$

and

$$S_- = \cos\theta_- \, e^{-\alpha\Delta/2} \tag{37}$$

Then (32) - (35) become

$$\lambda_1 = -e^{-j\theta} + e^{-\alpha\Delta/2} \tag{38}$$

$$\lambda_2 = -e^{+j\theta} + e^{-\alpha\Delta/2} \tag{39}$$

$$\lambda_3 = -e^{-j\theta} - e^{-\alpha\Delta/2} \tag{40}$$

$$\lambda_4 = -e^{+j\theta} - e^{-\alpha\Delta/2} \tag{41}$$

Then, the most general solutions to (18) and (19) are

$$y_n = A_1\lambda_1{}^n + A_2\lambda_2{}^n + A_3\lambda_3{}^n + A_4\lambda_4{}^n \tag{42}$$

$$x_n = \frac{-\omega^2}{k^2\Delta(e^{-\frac{\alpha}{2}\Delta}-1)} \left[\frac{A_1(\lambda_1{}^2 - \beta\lambda_1 + e^{-\alpha\Delta})}{\lambda_1}\lambda_1{}^n + A_2\frac{(\lambda_2{}^2 - \beta\lambda_2 + e^{-\alpha\Delta})}{\lambda_2}\lambda_2{}^n + A_3\frac{(\lambda_3{}^2 - \beta\lambda_3 + e^{-\alpha\Delta})}{\lambda_3}\lambda_3{}^n + A_4\frac{(\lambda_4{}^2 - \beta\lambda_4 + e^{-\alpha\Delta})}{\lambda_4}\lambda_4{}^n \right] \tag{43}$$

Equation (43) can be simplified if we use the relation from (29) that

$$\lambda^2 + e^{-\alpha\Delta} = -(2S_{\pm})\lambda \tag{44}$$

so that (43) can be rewritten as

$$x_n = \frac{\omega^2}{k^2\Delta(e^{-\frac{\alpha}{2}\Delta}-1)} [A_1(2S_+ + \beta)\lambda_1{}^n + A_2\,(2S_+ + \beta)\lambda_2{}^n$$

$$+ A_3(\,2S_- + \beta)\lambda_3{}^n + A_4\,(2S_- + \beta)\lambda_4{}^n] \tag{45}$$

The rigid perfectly conducting electrodes impose the boundary conditions

$$\begin{aligned} x_o &= 0 \qquad & y_o &= 0 \\ x_{N+1} &= 0 \qquad & y_{N+1} &= 0 \end{aligned} \tag{46}$$

which result in the relations

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1{}^{N+1} & \lambda_2{}^{N+1} & \lambda_3{}^{N+1} & \lambda_4{}^{N+1} \\ (2S_++\beta) & (2S_++\beta) & (2S_-+\beta) & (2S_-+\beta) \\ (2S_++\beta)\lambda_1{}^{N+1} & (2S_++\beta)\lambda_2{}^{N+1} & (2S_-+\beta)\lambda_3{}^{N+1} & (2S_-+\beta)\lambda_4{}^{N+1} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = 0 \tag{47}$$

For solutions to (47), the determinant of the coefficients must be zero, so that

$$4(S_+-S_-)^2\,(\lambda_4{}^{N+1} - \lambda_3{}^{N+1})\,(\lambda_2{}^{N+1} - \lambda_1{}^{N+1}) = 0 \tag{48}$$

which for non-trivial solutions yields

$$S_{\pm} = \cos \frac{p\pi}{N+1} e^{-\alpha\Delta/2} \tag{49}$$

p = 1, 2, ...N
(p = 0 is a trivial solution)

From (30) and (31) we have

$$\frac{1}{2}\left(-\frac{\gamma+\beta}{2} \pm \sqrt{(\frac{\gamma}{2})^2 + (\frac{\beta}{2})^2 + \delta}\right) = + \cos \frac{p\pi}{N+1} e^{\frac{-\alpha\Delta}{2}} \tag{50}$$

which can be reduced to

$$(2 \cos \frac{p\pi}{N+1} e^{\frac{-\alpha\Delta}{2}} + \gamma)\ (2 \cos \frac{p\pi}{N+1} e^{\frac{-\alpha\Delta}{2}} + \beta) = \delta \tag{51}$$

Substituting the definitions given by (22), (27), and (28) we finally obtain

$$\omega^2 = \frac{\frac{q_o^2}{\rho_o \varepsilon}(e^{-\alpha\Delta/2}-1)^2 \sinh^2 k\Delta + 2(g(e^{-\alpha\Delta}-1) - \frac{2q_o^2}{\varepsilon\alpha\rho_o}(e^{-\alpha\Delta/2}-1))k \sinh k\Delta (\cos\frac{p\pi}{N+1} + \cosh k\Delta)}{(2 \cos \frac{p\pi}{N+1} + 2 \cosh k\Delta)\ (2 \cos \frac{p\pi}{N+1} e^{-\alpha\Delta/2} + \cosh k\Delta\ [1+e^{-\alpha\Delta}])} \tag{52}$$

In order to compare this result with the continuous distribution problem, we take the limits given by (5.3.57) - (5.3.65) for which (52) becomes

$$\omega^2 = \frac{\frac{k^2}{\rho_o}\left[\frac{(\frac{\alpha q_o}{2})^2}{\varepsilon} + (-\alpha g\rho_o + \frac{q_o^2}{\varepsilon})\,([\frac{r\pi}{d}]^2 + k^2)\right]}{\left[(\frac{r\pi}{d})^2 + k^2\right]\left[(\frac{r\pi}{d})^2 + k^2 + (\frac{\alpha}{2})^2\right]} \qquad (53)$$

$$r = 1, 2, \ldots$$

This result is identical to the continuous exponential stratification treated in section (4.4). (Eq. 4.4.24) We see in (52) and (53) that if α is positive the system is potentially unstable only because of adverse gravity. The contributions from the electric field is always stabilizing. A necessary and sufficient condition for stability is

$$\frac{2q_o^2}{\varepsilon} > \frac{\alpha g\rho_o(e^{-\alpha\Delta} - 1)}{(e^{-\alpha\Delta/2} - 1)} \qquad (54)$$

5.5 Presentation of Results

The solution to the discrete weak gradient problem, given by (5.3.55) is very similar in form to the solution for the discrete exponential distribution given by (5.4.52).

For the weak gradient distribution, we define

$$\omega_m^2 = -\alpha g + \frac{E_o D q_o}{\rho_o} \qquad (1)$$

and

$$\omega_p^2 = \frac{d^2(Dq_o)^2}{\rho_o\varepsilon} \qquad (2)$$

Analagously, for the discrete exponential stratification, we define

$$\omega_m^2 = \frac{g}{\Delta}\left(e^{-\alpha\Delta} - 1\right) - \frac{2q_o^2}{\varepsilon\alpha\rho_o\Delta}\left(e^{-\alpha\Delta/2} - 1\right) \tag{3}$$

and

$$\omega_p^2 = \frac{q_o^2}{\rho_o\varepsilon}\left(e^{-\alpha\Delta/2} - 1\right)^2 (N+1)^2 \tag{4}$$

As $\Delta \to 0$, the definitions given by (1) - (4) approach those given for the analagous continuous distributions in section 4.5 (Eqs. 4.5.1 - 4.5.4).

Figs. 1 - 4 are valid for the weak gradient case only when $\alpha d \ll 1$. with the parameters defined by (1) and (2). The results are plotted for the exponential case when $\alpha d \ll 1$ and $\alpha d = -5$, with the parameters defined by (3) and (4). The dispersion relation (ω real for k real) is plotted for various numbers of layers and for various values of ω_p^2. The number of modes exactly equal the number of interfaces.

The first few modes for an 18 layer problem are plotted in Figs. 5 and 6 and compared to the results for the continuous case. The two results will be close when

$$k\Delta \ll 1$$

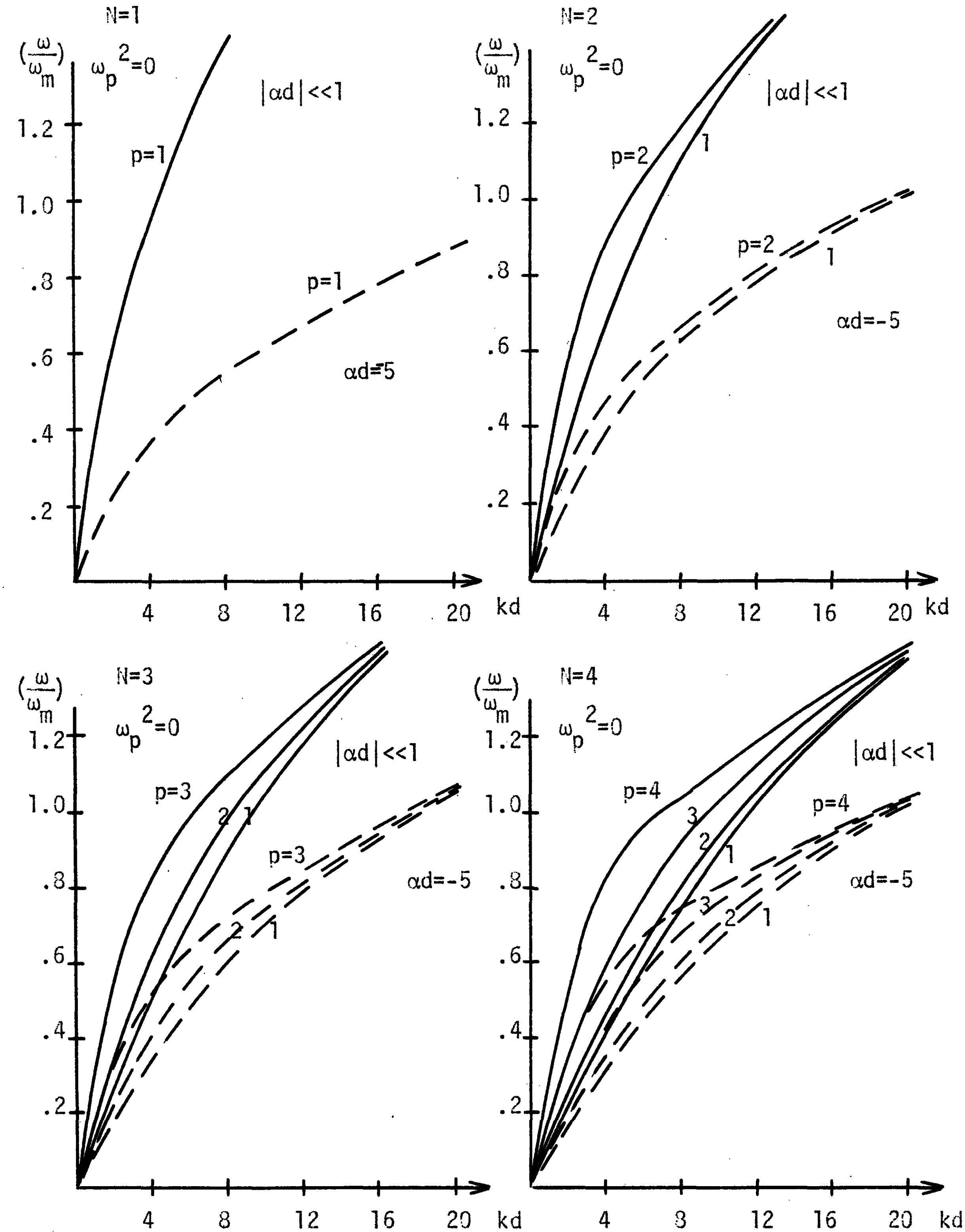
or equivalently

$$kd \ll (N+1)$$

and when

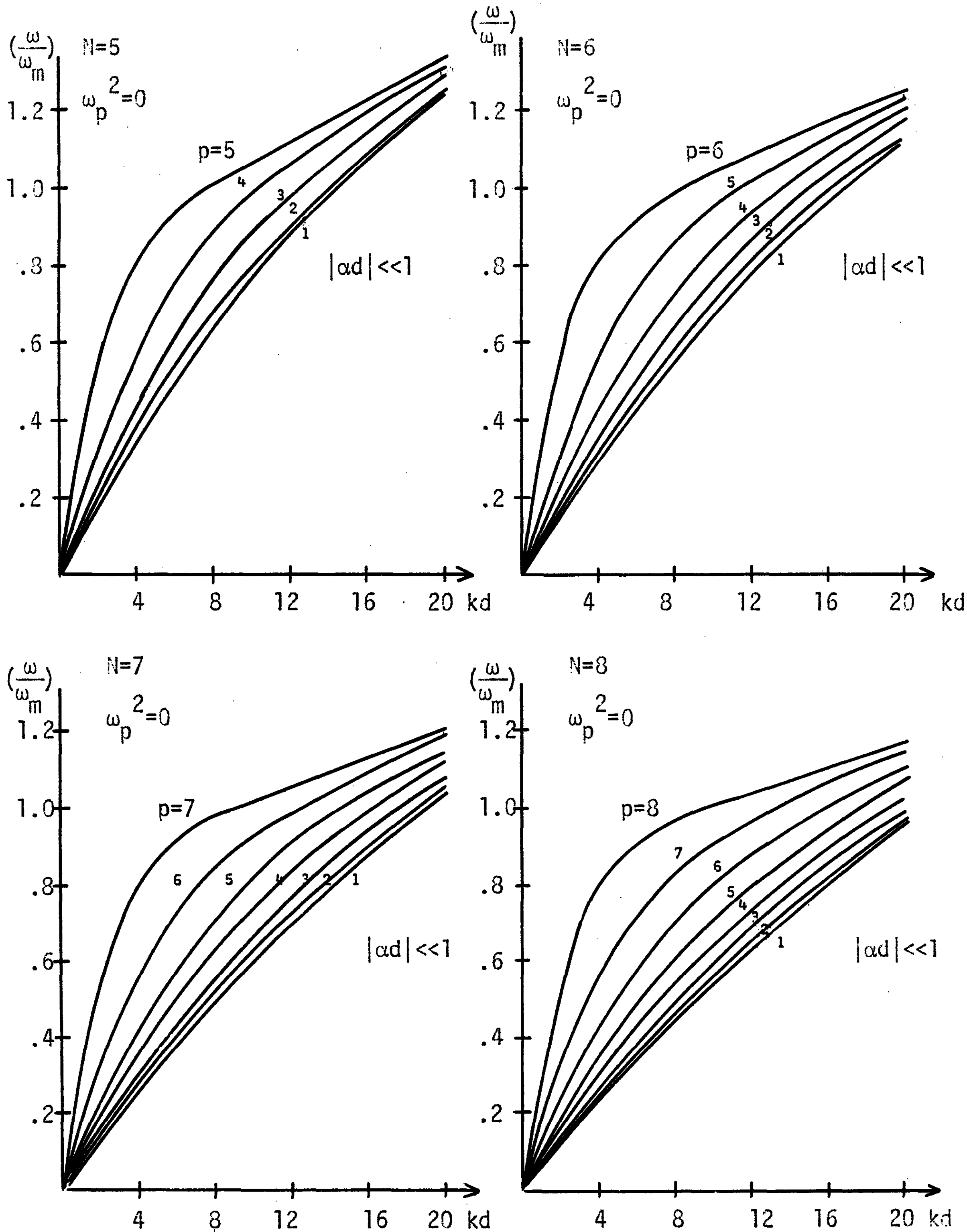
$$kd \ll \frac{2\pi}{n}$$

(n represents the number of vertical spatial variations for the continuous case.)



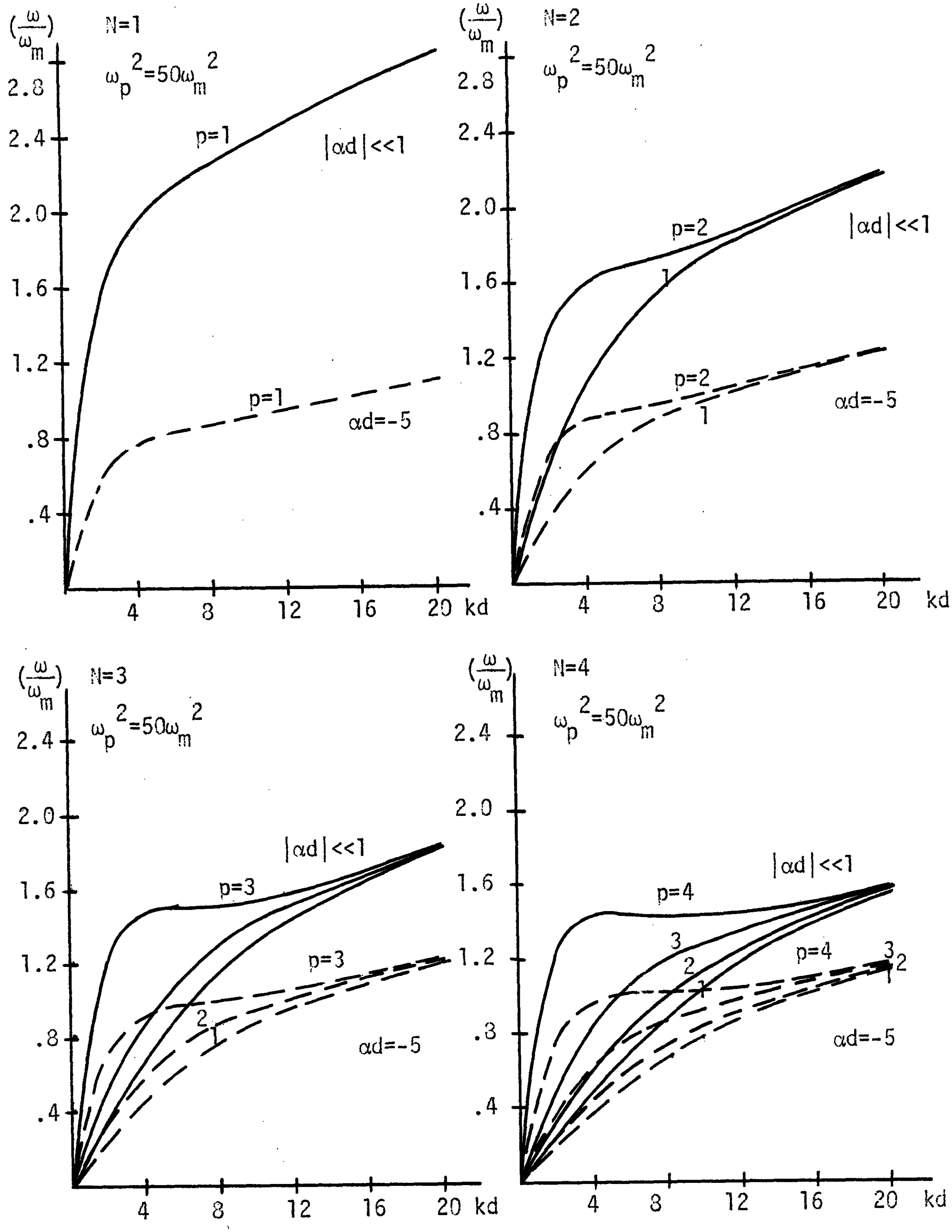
Dispersion relation for discrete weak gradient and exponential stratifications for 1-4 interfaces with $\omega_p^2=0$.

Figure 1



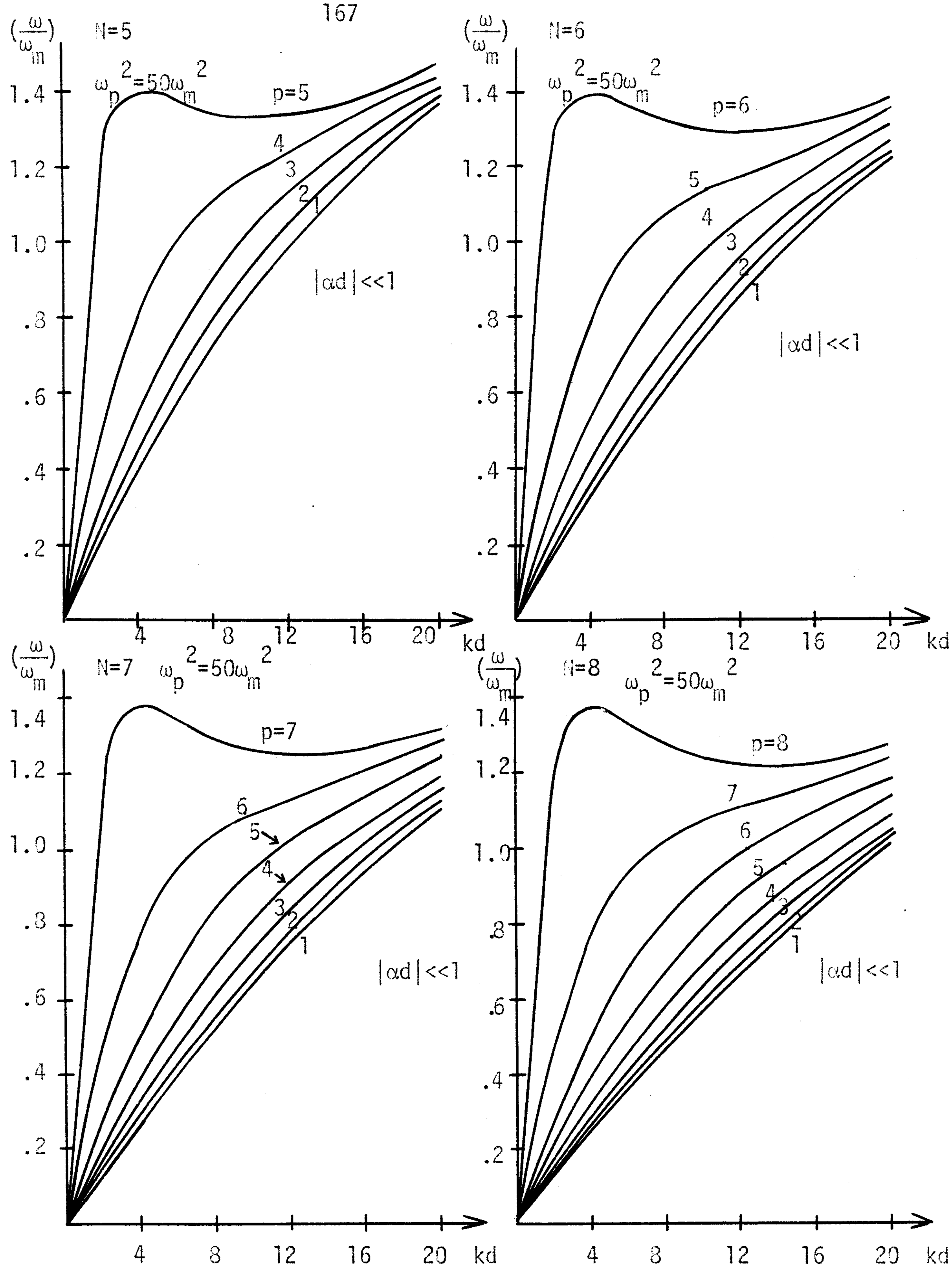
Dispersion relation for discrete weak gradient and exponential stratifications for 5-8 interfaces with $\omega_p{}^2=0$.

Figure 2



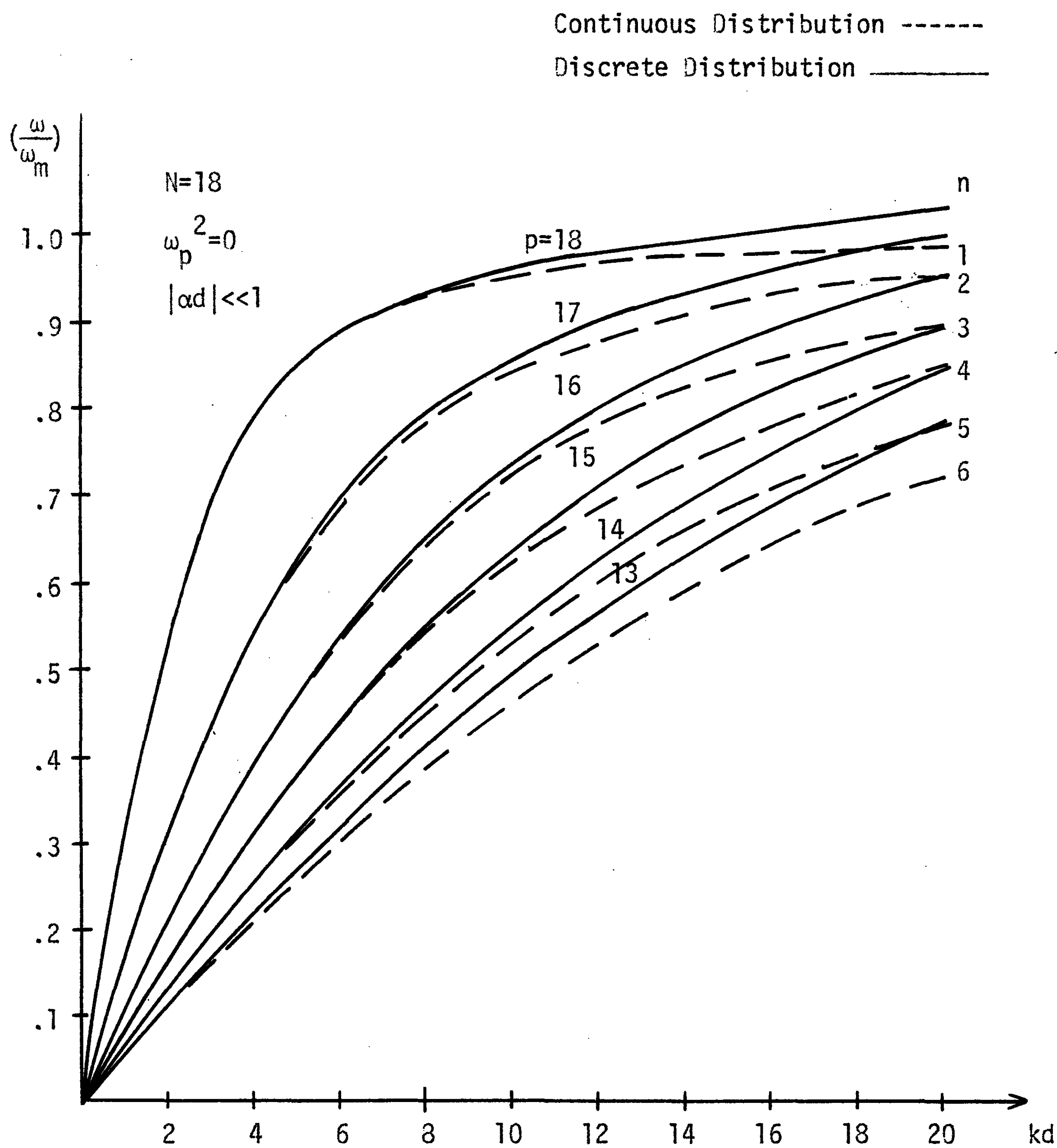
Dispersion relation for discrete weak gradient and exponential stratifications for 1-4 interfaces with $\omega_p^2=50\omega_m^2$.

Figure 3



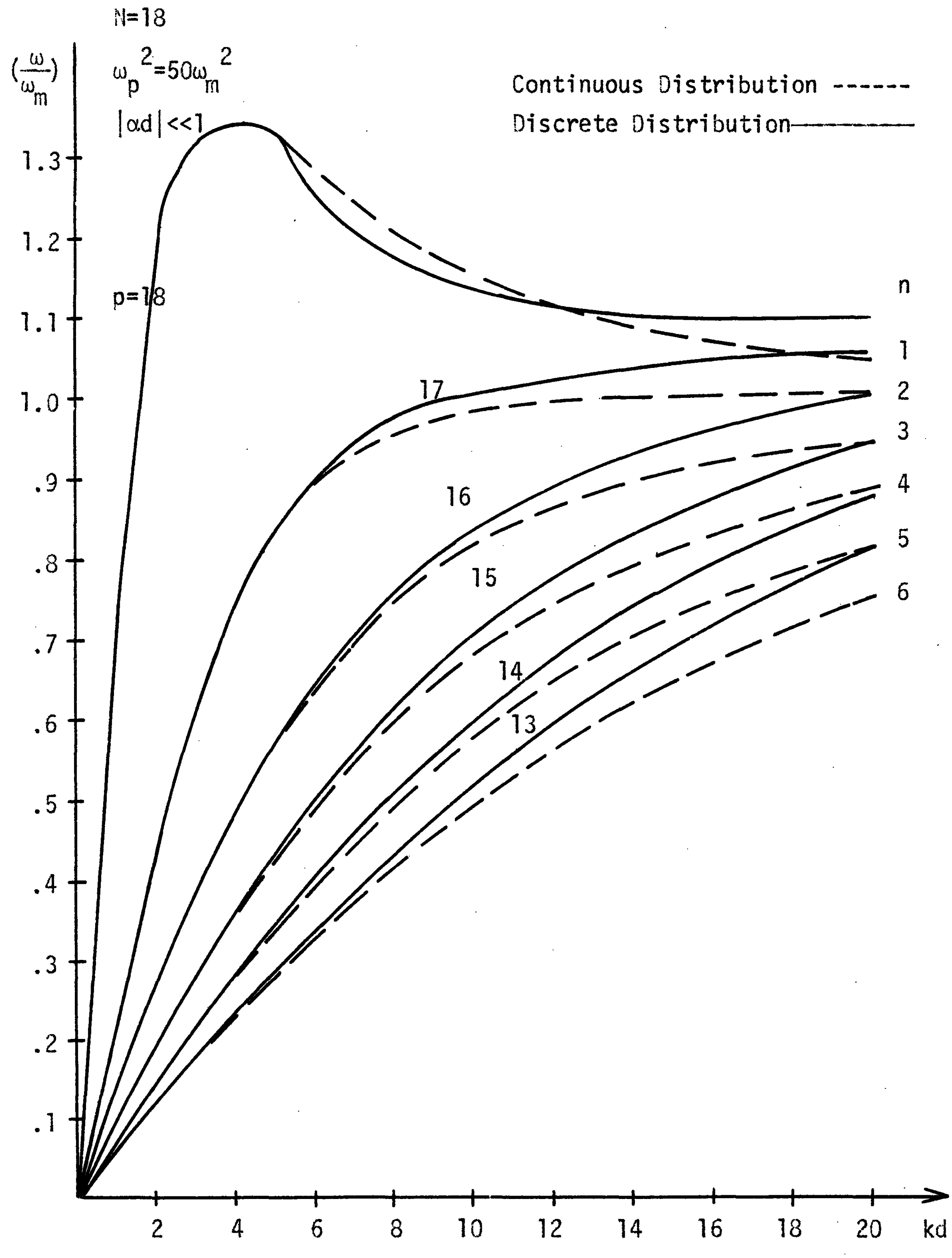
Dispersion relation for discrete weak gradient and exponential stratifications for 5-8 interfaces with $\omega_p^2=50\omega_m^2$.

Figure 4



With 18 interfaces the dispersion relation for the discrete and continuous cases of weak gradient and exponential stratification are close when wavelengths are longer than the layer thicknesses and transverse variations. $\omega_p^2=0$.

Figure 5



With 18 interfaces the dispersion relation for the discrete and continuous cases of weak gradient and exponential stratification are close when wavelengths are longer than the layer thicknesses and transverse variations. $\omega_p^2=50\omega_m^2$.

Figure 6

Under these conditions, wavelengths are much longer than the thickness of the layers and also much longer than vertical variations for the continuous case ($\frac{\lambda}{d} \gg n$).

CHAPTER VI
EXPERIMENTAL WORK

6.1 Introduction

Many anomalous results found with DC experiments using insulating liquids are often attributed to so called "space charge" effects. These anomalous effects have occurred in experimental configurations similar to that analyzed in section 3.1, where we have a perfectly conducting interface bounded from above by an insulating fluid.[42] Analysis has previously been done assuming the upper fluid to be uncharged.[18] When the upper fluid is air, the frequency decrease in the presence of a static electric field agrees well with theory.[18] This frequency shift is given by (3.1.9) with $q_a = 0$. However, if the upper fluid is an insulating liquid, the frequency shifts often do not agree with this theory. It is thought that the anomalous results are due to the presence of space charge. The analysis of section 3.1 considers the upper liquid to be uniformly charged, and so offers a new attack on this problem.

6.2 Description of Experiment

The dispersion relation given by (3.1.5) describes the motions of a single mode of frequency ω and wavenumber k.

$$\frac{\omega^2}{k}(\rho_a \coth ka + \rho_b \coth kb) = g(\rho_b - \rho_a) + \gamma k^2 - \frac{\varepsilon_a E_a^{\ 2}}{a} ka \coth ka$$

$$-q_a E_a \tag{1}$$

$$E_a = \frac{V_o}{a} - \frac{q_a a}{2\varepsilon_a} \tag{2}$$

We consider a rectangular resonator which can be shook at precisely controlled frequencies as illustrated in Fig. 1. We use tap water as the lower conducting liquid and hexane as the insulating upper liquid. Since the relaxation time of tap water is on the order of microseconds as contrasted to tenths of a second for hexane, it is appropriate to assume the water perfectly conducting compared to hexane. Properties of these liquids are listed in Table 1.

Interfacial deflections are measured by means of a pencil beam of light from a small laser. Part of the light which is incident upon the interface is reflected. As the interface moves up and down, the reflected light sweeps out a line on a screen. Even though interfacial deflections are small, the distance the light sweeps is proportional to the slope of the interface and not the actual amplitude, so the light deflections are sufficiently large for easy viewing.

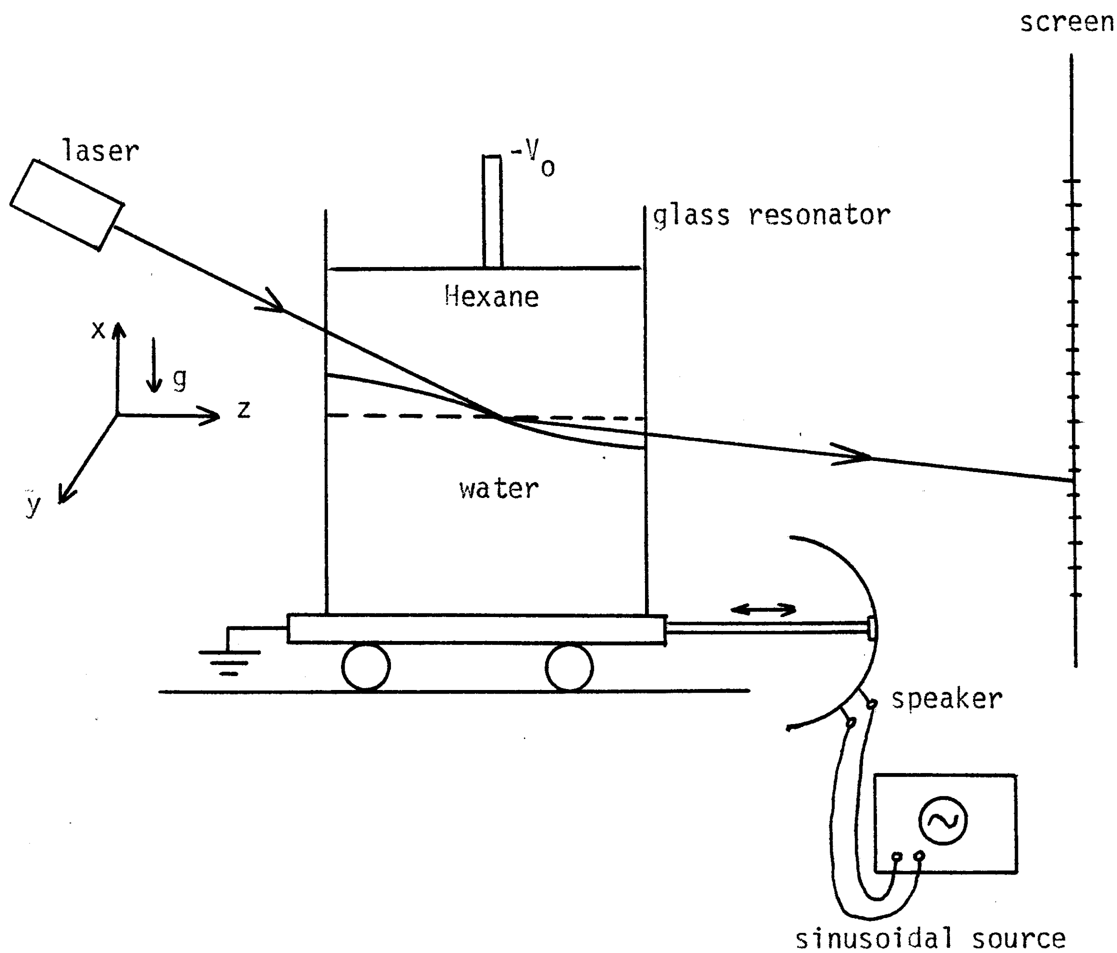
6.3 System Characteristics - No Electric Field

Since the resonator is thin in the y direction, interfacial deflections have variations only in the z direction. If we assume the fluid to slip at the walls, at resonance, an integer number of half wavelengths must fit in the box, for which

$$k_y = 0 \tag{1}$$

$$k_z = \frac{n\pi}{l} \qquad n = 1, 2, 3, \ldots \tag{2}$$

However, in the way we push the resonator, conservation of momentum only lets us excite those modes where n is odd.



Experimental configuration for electrohydrodynamic resonator.

Figure 1

	Tap Water	Hexane
Mass density	1. kg/m^3	.67 kg/m^3
Permittivity	80 ε_0	1.9 ε_0
Relaxation time	$\simeq 10^{-6}$ sec.	$\simeq$ .1 sec.
Depth	b = 7 cm.	a = 2.4 cm.

Surface tension (hexane-water interface) .038 Nts./m. [Measured by balancing weights against the downward pull on a platinum sheet placed through the interface.]

Resonator dimensions $l_z = 9.5$ cm.

$l_y = 3.5$ cm.

Physical constants and geometry for hexane-water system.

Table 1

Each value of n has its corresponding resonant frequency determined by gravity, the density difference, interfacial surface tension, and the geometry, as given by (6.2.1) with no electric field. If the system is driven at this frequency, interfacial deflections will be a maximum, limited only by viscosity. Table 2 compares the theoretical and measured resonant frequencies. In the measured frequency column, we have put those values which appear most often. Actually, from day to day, these values can increase as much as 10% for the higher modes, most likely due to variations in surface tension because of temperature changes or impurities. Since our main goal is to measure the effects of the electric field, any discrepancies here are unimportant.

A crucial parameter to know is the wavenumber k. Although calculated in (2), the assumption was made that the liquids slip at the walls. Experimentally, it is observed that there is some sticking on the walls, which could modify (1) and (2). We have checked these relations by actually measuring the wavelengths with the use of a telescopic eyepiece. In addition, the good agreement in Table 1, of the measured and theoretical resonant frequencies further confirm that (1) and (2) are valid, thus indicating that the liquid sticking at the walls has negligible effect on the wavenumber.

6.4 Frequency Shifts with a Time Varying Field

The amount of space charge present in insulating liquids is a function of the electric field, as related through Gauss's law. However, the time scale for which a charge distribution can respond is determined by the conduction mechanism. If the frequency is much higher than the

n	f (measured) $sec.^{-1}$	f (theoretical) $sec.^{-1}$
1	1.2	1.14
3	2.5	2.28
5	3.4	3.21
7	4.3	4.2
9	5.4	5.3
11	6.5	6.4
13	7.8	7.86
15	9.3	9.35
17	11.0	10.9

Comparison between theoretical and measured resonant frequencies with no electrical parameters.

$$f\ (\text{theoretical}) = \frac{1}{2\pi}\sqrt{\frac{g(\rho_b - \rho_a)k + \gamma k^3}{\rho_a \coth ka + \rho_b \coth kb}}$$

$$k = \frac{n\pi}{l}\ ; \quad l = 9.5\ \text{cm.}$$

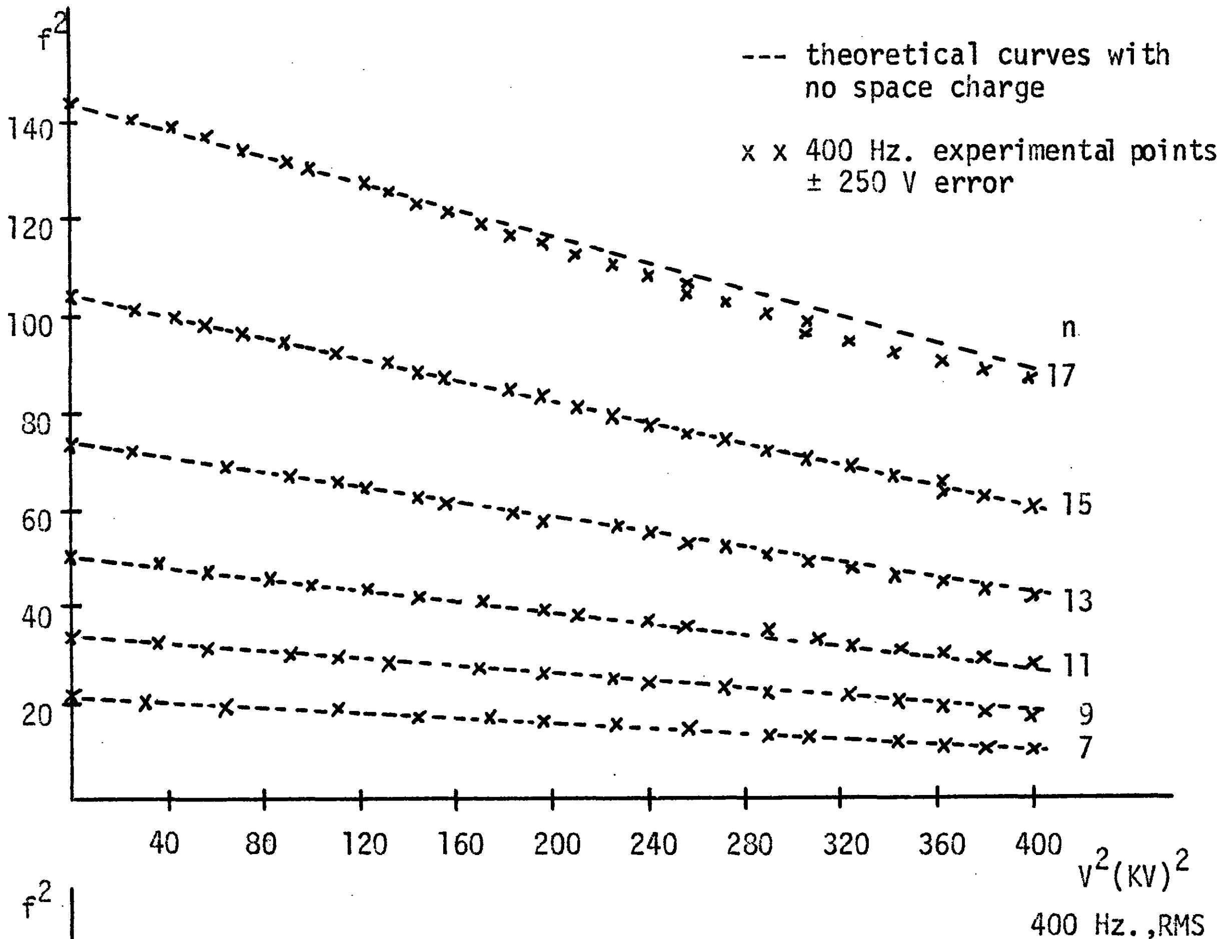
Table 2

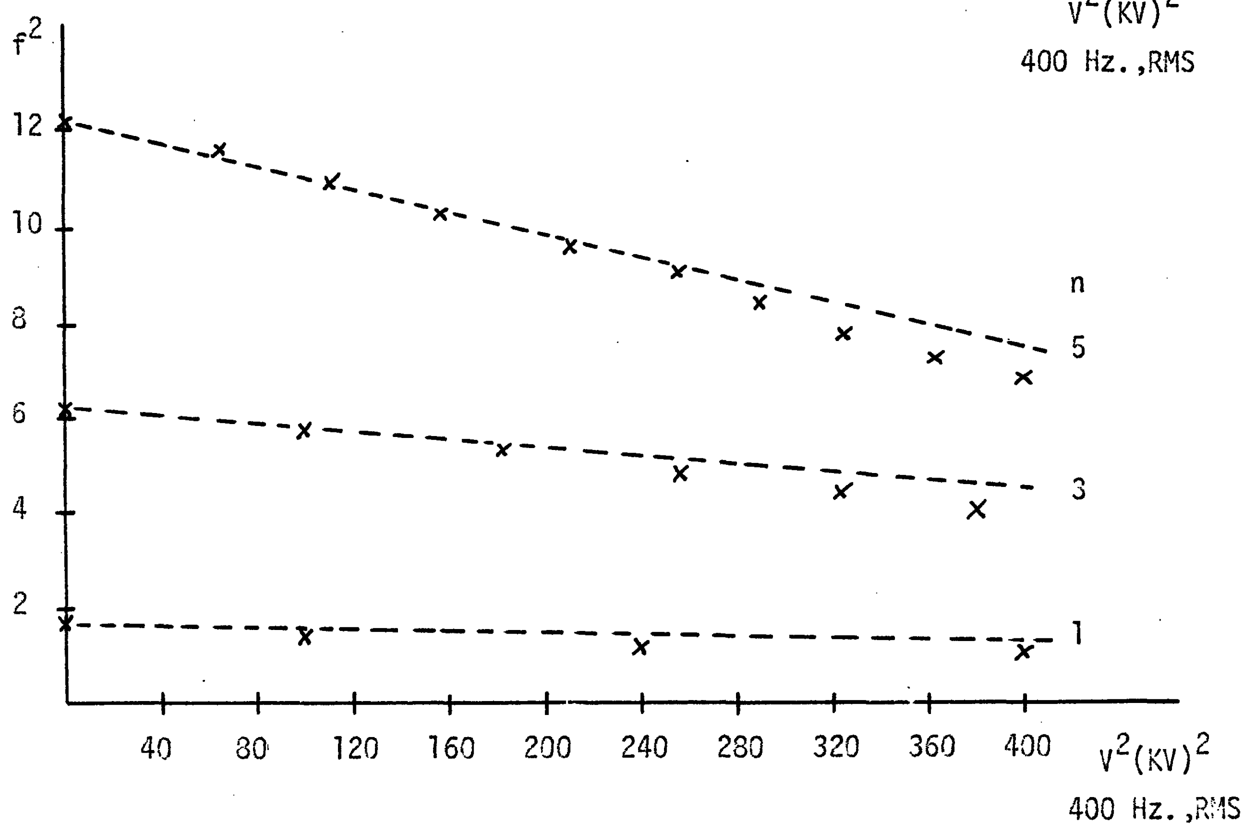
reciprocal relaxation time, the volume charge is approximately zero, as the charges cannot respond as fast as the electric field. From Table 1, in section 1.4, the relaxation time of hexane is approximately .1 sec, so with an excitation at 400 Hz, we can assume the volume charge to be negligible.

For the system of Fig. (6.2.1), the dynamics can still be represented by (6.2.1) with $q_a = 0$, if we use the root-mean square (RMS) value of the electric field. This is valid because the remaining electrical term is proportional to the square of the electric field, which has a DC and a double frequency (800 Hz.) part. The interface cannot follow the high frequency component because of the viscosity of the fluids, and thus only responds to the DC part.

Our method is to shake the resonator at a frequency below resonance. As we increase the potential, the resonance frequency will decrease until it agrees with the same frequency with which we are shaking. The resonance is determined when the light deflections on the screen are a maximum.

This procedure was performed for the first nine odd modes, as shown in Fig. 1. The data agrees very well with the theoretical straight lines. The frequency decrease is due to the destabilizing nature of the electric field pulling on the interfacial surface charge. If we go much higher in voltage, the interface will become unstable for some wavenumber, as discussed in section 3.1, equations (11) - (29). For $ka >> 1$ (and $q_a = 0$) this will occur when





With high frequency AC potential, the resonant frequency shifts agree well with a theory that does not allow any volume charge.

Figure 1

$$V_{crit} = a \left[\frac{4(\rho_b - \rho_a)g\gamma}{\varepsilon_a^2}\right]^{1/4}$$

and

$$(ka)^* = \left[\frac{(\rho_b - \rho_a)\, ga^2}{\gamma}\right]^{1/2}$$

which for the hexane water system yields

$$V_{crit} = 27.6 \text{ KV}$$

and

$$(ka)^* \simeq 7$$

6.5 Frequency Shifts with a Static Electric Field

The same procedure described in section 6.4, was repeated with a DC potential. Frequency shifts as a function of V_0^2 were measured, and were found to consistently differ over many trials with the measurements described in section 6.4. However, these results are not without question, as when the measurements were repeated in a "very clean" resonator, there were no longer any anomalous shifts. The data was then identical to that presented in section 6.4 both for DC and AC fields. The original set of data was taken in a resonator held together by a silicone sealant. Over the course of many experiments the sealant became dirty and ingrained with Flaming Red, an oil soluble dye. This was the only difference between the measurements. AC experiments were always consistent as presented in section 6.4.

It is difficult to offer an explanation for the non-reproductibility of the DC results between "clean" and "dirty" resonators, except to remark that impurities do alter the physical character of these liquids. However, intentionally adding Flaming Red dye or pieces of the silicone sealant into the "clean" resonator had no effect. Some further discussion appears in the critique, section 6.6.

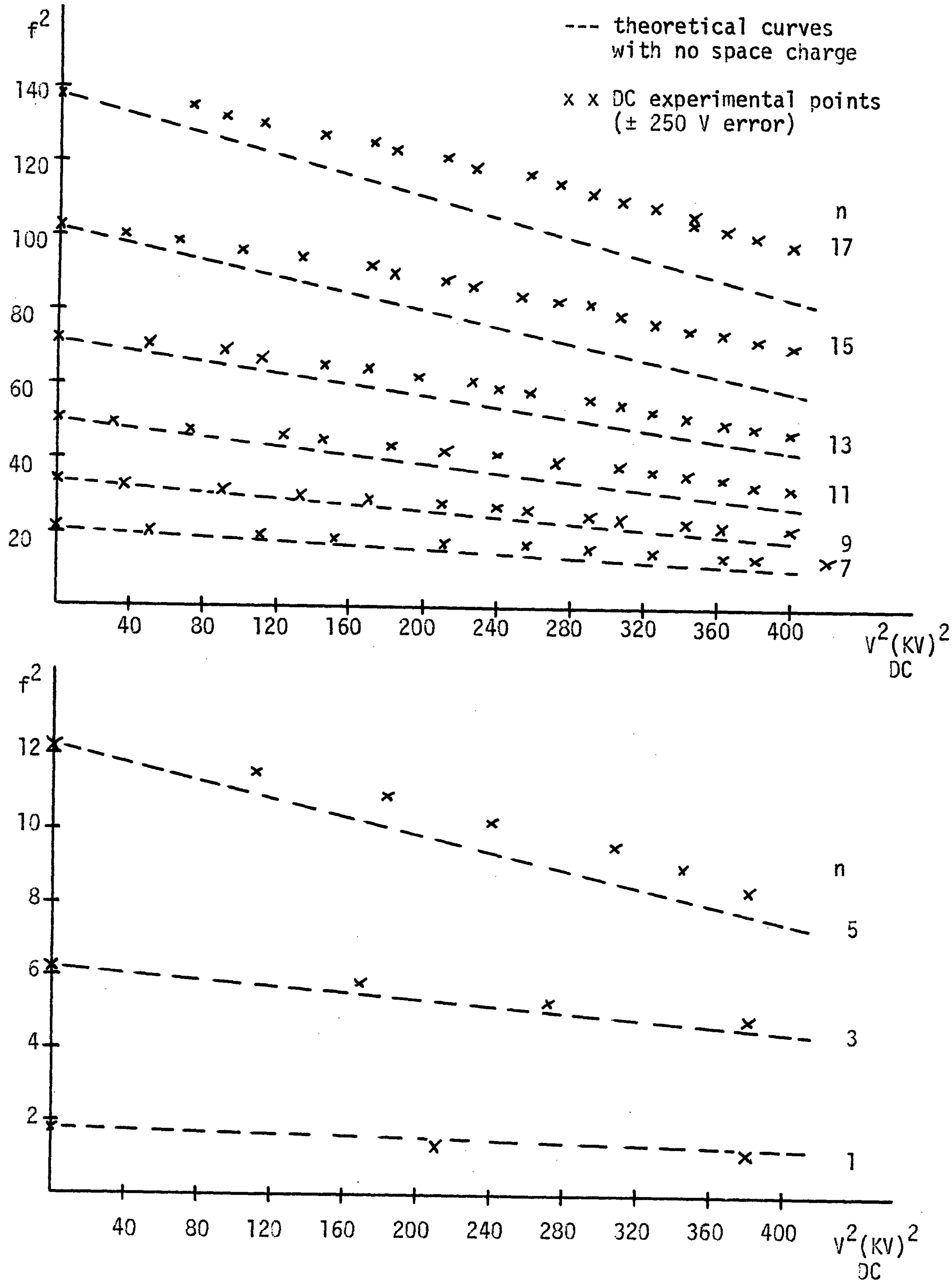
The results of the DC experiments in the "dirty" resonator are plotted in Fig. 1. While using this resonator, the results displayed in Fig. 1 were consistent and reproducible. The slopes of all the modes are consistently less than the theoretical slopes given with no space charge. The results remain the same, even when reversing polarity. We wish to discuss this anomaly in the context of space charge present in the hexane. An alternative viewpoint will be discussed in the critique, section 6.6.

For sufficiently short wavelengths (modes 3 or higher for our geometry) such that ka >> 1, the last term in 6.2.1 is small, such that

$$|q_a E_a| \ll \frac{\varepsilon_a E_a^2}{a}$$

Thus, the main effect of the space charge, at short wavelengths, is to alter the electric field, as given by (6.2.2).

Since the measured slopes have decreased, the electric field at the interface must have decreased implying that positive space charge is



The resonant frequency shifts with a DC potential do not agree with a theory that omits the effects of space charge.

Figure 1

distributed near the interface, which itself supports positive surface charge as shown in Fig. (6.2.1), (V_0 assumed positive). The space charge has partially stabilized the interface by decreasing the electric field.

The slopes of the higher modes are about 25% less than the theoretical slopes, if there were no space charge. This implies that the electric field has decreased by approximately 13.5%. This can be accounted for from (6.2.2) if

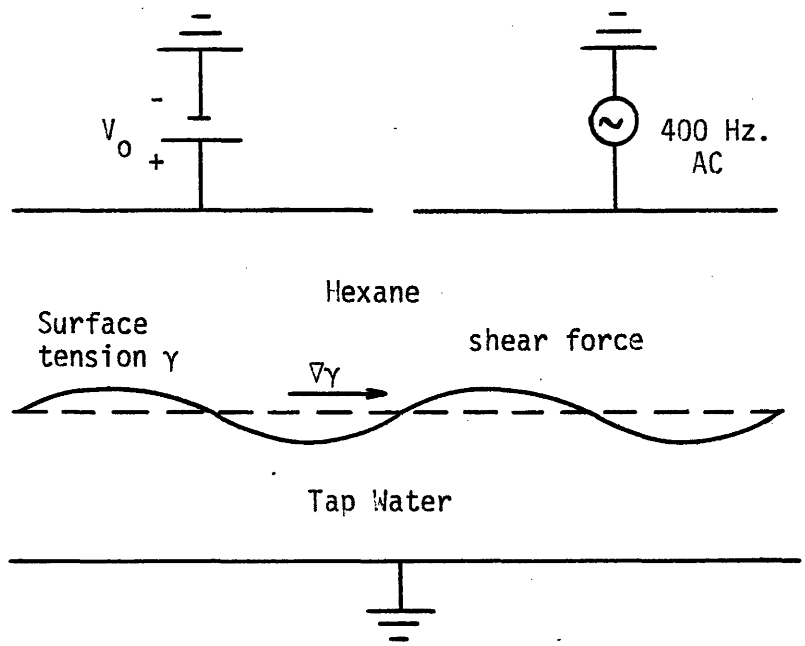
$$q_a = .27 \frac{\varepsilon_a V_o}{a^2}$$

If negative charge were distributed throughout the hexane, the electric field at the interface would be enhanced, tending to destabilize the interface. In section (1.5), we discussed the bulk instability of unipolar conduction for voltages in excess of a few hundred volts. In our experiments, we work in the kilovolt range, so by this criterion the hexane bulk has become unstable, if the upper metal electrode is the source of negative ions. The system would then try to reach a new equilibrium which cannot be stationary. It is possible that there exists a small scale turbulence in the hexane which alters the charge distribution, as the system tries to reach a new equilibrium. This may be the reason for our measurements indicating a stabilizing effect on the interface. Further work is necessary to support these hypotheses.

6.6 Critique

The DC results presented here are certainly open to question because of the lack of reproductibility in going from "dirty" to "clean" resonators. Apparently slight impurities can greatly alter the conduction properties in insulating liquids. It is of interest that the results for the "clean" resonator had no measured anomalies. Other investigators using the same liquids have found the frequency shifts to be anomalous in the other direction.[42] That is, the slopes were too large as contrasted to our measurements, which found the slopes too small. Apparently, resonator experiments of this type determine what the electric field is at the interface for a particular set of conditions. Slight impurities, ionization or ionic emission can greatly alter the electric field, so each investigator is measuring his electric field for his apparatus at that specific moment. The only way to gain complete confidence of one's results is to devise an alternate scheme for measuring the electric field at the liquid interface.

Another viewpoint would account for anomalous frequency shifts due to the surface tension being a function of the electric field.[46] To be consistent with the experimental results, the surface tension could only change with a DC field, but would remain unchanged in an AC field. To test this hypothesis, we set up the configuration shown in Fig. 1. If the surface tension were truly a function of the DC electric field, we would have a gradient in surface tension between the electrodes. There would then exist a shear force on the interface, resulting in fluid motion. However, under no conditions did we find any fluid motion,



If the surface tension γ is a function of a DC field, there will exist a surface shear force proportional to $\nabla\gamma$.

Figure 1

which indicates that this field dependent surface tension argument is not valid here.

CHAPTER VII
SUMMARY AND CONCLUSIONS

7.1 Summary of Results

We have laid the groundwork for a broad class of interactions involving liquids which have stratifications in mass density, charge density, permittivity, and convection velocity. We modeled these liquids as incompressible, inviscid and perfectly insulating, with domains of sufficiently high frequency or growth rates to validate these assumptions defined. In particular, we emphasized stratifications consisting of many homogeneous layers. Within these homogeneous layers, we are able to express the volume force due to the space charge as a gradient of a quantity and thus could group it with the hydrodynamic pressure. This fact particularly simplified the analysis, as the problem is now representable as a surface coupled interaction. This simplification will occur for any volume force which is curl free within a homogeneous layer.

Because of the surface coupled nature of these interactions, "transfer relations" are derived for prototype layers in rectangular, cylindrical, and spherical geometry which relate the interfacial variables of pressure, displacement, electric potential, and normal electric field. Since these variables appear in the boundary conditions, we can treat many layers by simply "splicing" regions together in a manner dictated by the boundary conditions, avoiding the redundancy of solving the bulk equations in every homogeneous region.

In particular, the value of this approach was exemplified in the following situations: a) Perfectly conducting interface stressed by normal electric field and bounded from above by fluid supporting uniform space charge; b) Two planar layers with differing properties and space charge; c) Uniformly charged liquid jet; and d) Uniformly charged liquid drop. The dynamics for the insulating jet and drop (c and d above) were found to be much different than that of a perfectly conducting jet and drop with surface charge. Experiments have also been described which attempt to delineate the coupling of space charge to electrohydrodynamic surface waves on a perfectly conducting interface in the configuration of a, above.

An added feature of our approach, is that it allows one to model continuous stratifications by many thin layers, each with constant properties. For the examples of weak gradient and exponential stratifications, we have shown that as the number of approximating layers approach infinity, with the thickness of each layer approaching zero, the dispersion relation for the discrete and continuous cases become identical.

7.2 State of Experimental Observations

There remains much experimental work relating to this thesis. The biggest problem is the determination of the space charge distribution. The electro-optic Kerr effect offers promise, yet fluids which have large Kerr constants are usually highly conducting, which invalidates our assumptions. In addition, electrical heating induces property gradients and convection which makes it hard to distinguish effects due solely to the space charge present. The conduction processes in liquids which determine the equilibrium charge distribution are usually so complex that they are difficult to analyze. Simple models of ohmic or unipolar conduction are often not valid in liquids. In measuring voltage-current relations, it is difficult to distinguish between conduction currents and convection currents, that might be associated with small scale turbulence in the liquids.

Electromechanical interactions as discussed in this thesis, offer the advantage of measuring dynamical responses from which charge distributions can be inferred. This is convenient because the same experimental configuration can be used to make measurements with high frequency AC or DC fields. The AC fields, if sufficiently fast will prohibit any bulk charge, while the DC field will allow the space charge time to form an equilibrium profile. By comparing responses for these two cases, we can directly infer the effects due to the space charge.

Most past work has relied on measurements for conditions at incipience of instability. As pointed out in Chapter I, instability is usually incipient near zero frequency, allowing other rate processes to exert an influence, making it difficult to distinguish between mechanisms. Experimental work described in Chapter VI, focused attention on stable waves. Measurements of resonant frequency shifts due to space charge present were attempted, but no consistent results were obtained. This fact agrees with other experimental observations relating to the voltage current relation which in insulating liquids are not reproducible from day to day or sample to sample.

Despite these initial difficulties, electromechanical interactions offer a direct way to measure the effects due to space charge. Experiments must be designed which isolate the effects due to the space charge, and in themselves do not introduce any more questions. Measurements of resonance conditions, incipience of instability, and growth rates in configurations similar to those discussed in this thesis, offer a three-way approach which hopefully can converge to aid in the understanding of the dynamics of charged liquids in the presence of electric fields.

7.3 Suggestions for Future Work

The analysis in this thesis modeled the liquids of interest as incompressible, inviscid, and perfectly insulating, and treated configurations in selected geometries. Future work can extend this description, by relaxing some of our assumptions. In the following, some directions in which one can head are discussed.

Geometry

Although our analysis considered only rectangular, cylindrical and spherical geometry, this approach can easily be extended to any other geometry. The difficulties in these new geometries amount to solving for the equilibrium variables and solving Laplace's equation for the perturbation quantities.

Viscosity

In concept, the effects of viscosity can easily be added to our analysis. However, the mathematics become difficult because the differential equations are now sixth order. For the prototype layers, our terminal relations must now include shear forces and shear displacements in addition to the normal forces and displacements. This results in the transfer relations becoming 4 x 4 matrices, rather than the simpler 2 x 2 matrices considered in this thesis. The boundary conditions to be matched at the interface now include a shear stress balance.

Compressibility

In the absence of space charge, similar analysis can be performed for compressible fluids. The new necessary ingredient is the equation of state, which relates the pressure to other variables. If the fluid can be modeled as a perfect gas, the mechanical terminal relations will be very similar to those derived in Chapter 2. It would be of interest to add the space charge effects to this analysis, to find the effect of space charge on acoustic waves. Attempts should be made to see if this addition still leaves the analysis tractable.

Effects of Conduction

Our analysis modeled the liquids of interest as perfectly insulating, with domains of sufficiently high frequency or growth rate to validate this assumption defined in terms of electrical conductivity or mobility. It would be useful to extend this work to include the effects of finite conduction. However, this extension greatly complicates the analysis as the electric force in a homogeneous layer is no longer curl free because of the presence of perturbation charge.

The charge is no longer tied to the liquid, and so we cannot lump the electric force with the pressure. Attempts to overcome these difficulties should be made.

Other Volume Forces

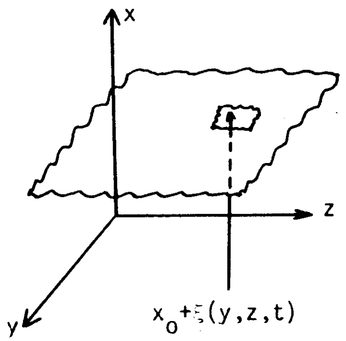
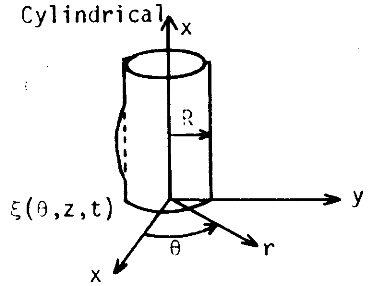
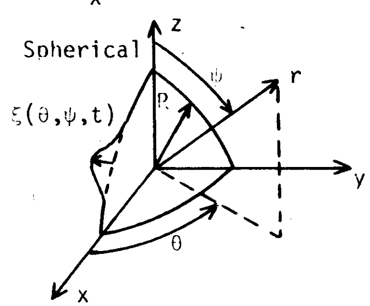
The techniques developed here may be used for other volume forces which are curl free within a homogeneous layer, or for surface forces which only act at an interface. Some systems for which this approach may prove useful are

a) Extension of the analysis here to include equilibrium electric fields which are parallel to the interface.

b) Analagous magnetic systems with the magnetic field having components both parallel and perpendicular to the interface. Homogeneous fluid layers carrying a constant current may have tractable solutions.

c) Elastic media with electric and magnetic coupling.

d) Layers of plasma or electron beam with non-uniform equilibrium ion densities and convection velocities.

Geometry	Normal Vector	Surface Tension Traction; τ_s	Dependence on Surface Coordinates	Complex Amplitude of Surface Tension Traction; $\hat{\tau}_s$
Planar 	$\bar{i}_x$ $-\frac{\partial\xi}{\partial y}\bar{i}_y$ $-\frac{\partial\xi}{\partial z}\bar{i}_z$	$\gamma\left[\frac{\partial^2\xi}{\partial y^2}+\frac{\partial^2\xi}{\partial z^2}\right]$	$\exp j(\omega t - k_y y - k_z z)$ $k = \sqrt{k_y^2 + k_z^2}$	$-\gamma k^2\hat{\xi}$
Cylindrical 	$\bar{i}_r$ $-\frac{1}{R}\frac{\partial\xi}{\partial\theta}\bar{i}_\theta$ $-\frac{\partial\xi}{\partial z}\bar{i}_z$	$\gamma\left[\frac{\xi}{R^2}+\frac{1}{R}\frac{\partial^2\xi}{\partial\theta^2}+\frac{\partial^2\xi}{\partial z^2}\right]$	$\exp j(\omega t - m\theta - kz)$	$\frac{\gamma}{R^2}[(1-m^2)-(kR)^2]\hat{\xi}$
Spherical 	$\bar{i}_r$ $-\frac{1}{R}\frac{\partial\xi}{\partial\psi}\bar{i}_\psi$ $-\frac{1}{R\sin\psi}\frac{\partial\xi}{\partial\theta}\bar{i}_\theta$	$\gamma\left[\frac{2\xi}{R^2}\right.$ $+\frac{1}{R^2\sin\psi}\frac{\partial}{\partial\psi}(\sin\psi\frac{\partial\xi}{\partial\psi})$ $\left.+\frac{1}{R^2\sin^2\psi}\frac{\partial^2\xi}{\partial\theta^2}\right]$	$Y_{mn}(\psi,\theta)e^{j\omega t}$	$-\frac{\gamma}{R^2}(n-1)(n+2)\hat{\xi}$

Summary of normal vector and surface tension traction for small perturbations from planar, circular cylindrical and spherical equilibrium.

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