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A Note on Anti-Berge Equilibrium for Bimatrix Game

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Abstract. We introduce a new concept of equilibrium based on Nash and Berge equilibriums. This equilibrium is called Anti-Berge equilibrium. We prove an existence of Anti-Berge equilibrium in the game. Based on Mills theorem [9], we reduce finding Anti-Berge equilibrium to a quadratic programming problem with linear constraints. The proposed approach has been illustrated on an example.

Keywords: Berge equilibrium, optimization, bimatrix game, Anti-Berge equlibrium.

1. Introduction

Game theory plays an important role in applied mathematics, economics and decision theory. There are many works devoted to game theory. Most of them deals with a Nash equilibrium. A global search algorithm for finding a Nash equilibrium was proposed in [13]. Also, the extraproximal and extragradient algorithms for the Nash equilibrium have been discussed in [3]. Berge equilibrium is a model of cooperation in social dilemmas, including the Prisoner's Dilemma games [15].

The Berge equilibrium concept was introduced by the French mathematician Claude Berge [5] for coalition games. The first research works of Berge equilibrium were conducted by Vaisman and Zhukovskiy [18; 19]. A method for constructing a Berge equilibrium which is Pareto-maximal with respect to all other Berge equilibriums has been examined in Zhukovskiy [10]. Also, the equilibrium was studied in [16] from a view point of differential games. Abalo and Kostreva [1; 2] proved the existence theorems for pure-strategy Berge equilibrium in strategic-form games of differential games. Nessah [11] and Larbani, Tazdait [12] provided with a new existence theorem. Applications of Berge equilibrium in social science have been discussed in [6; 17]. Also, the work [7] deals with an application of Berge equilibrium in economics. Connection of Nash and Berge equilibriums has been shown in [17]. Most recently, the Berge equilibrium was examined in Enkhbat and Batbileg [14] for Bimatrix game with its nonconvex optimization reduction. In this paper, inspired by Nash and Berge equilibriums, we introduce a new notion of equilibrium so-called Anti-Berge equilibrium. The main goal of this paper is to examine Anti-Berge equilibrium for bimatrix game.

The work is organized as follows. Section 2 is devoted to the existence of Anti-Berge equilibrium in a bimatrix game for mixed strategies. In Section 3, an optimization formulation of Anti-Berge equilibrium has been formulated.

2. Bimatrix Game

Consider the bimatrix game in mixed strategies with matrices (A, B) for players 1 and 2.

$$A = (a_{ij}),\ i = 1, \dots, m,$$
$$B = (b_{ij}),\ j = 1, \dots, n.$$

Denote by X and Y the sets

$$X = \{x \in R^m \mid \sum_{i=1}^{m} x_i = 1,\ x_i \geq 0,\ i = 1, \dots, m\},$$

$$Y = \{y \in R^n \mid \sum_{j=1}^{n} y_j = 1,\ y_j \geq 0,\ j = 1, \dots, n\}.$$

A mixed strategy for player 1 is a vector $x = (x_1, x_2, \dots, x_m)^T \in X$ representing the probability that player 1 uses a strategy i. Similarly, the mixed strategies for player 2 is $y = (y_1, y_2, \dots, y_n)^T \in Y$. Their expected payoffs are given by :

$$f_1(x, y) = x^T Ay, \qquad f_2(x, y) = x^T By.$$

First, we introduce the definitions of the equilibriums

Definition 1. *A pair strategy* $(x^1, y^1) \in X \times Y$ *is a Nash equilibrium if*

$$\begin{cases} f_1(x^1, y^1) \geq f_1(x, y^1), & \forall x \in X, \\ f_2(x^1, y^1) \geq f_2(x^1, y), & \forall y \in Y. \end{cases}$$

Definition 2. *A pair strategy* $(x^2, y^2) \in X \times Y$ *is a Berge equilibrium if*

$$\begin{cases} f_1(x^2, y^2) \geq f_1(x^2, y), & \forall y \in Y, \\ f_2(x^2, y^2) \geq f_2(x, y^2), & \forall x \in X. \end{cases}$$

Definition 3. *A pair strategy* $(x^3, y^3) \in X \times Y$ *is an Anti-Berge equilibrium(with respect to player 2) if*

$$\begin{cases} f_1(x^3, y^3) \geq f_1(x^3, y), & \forall y \in Y, \\ f_2(x^3, y^3) \leq f_2(x, y^3), & \forall x \in X. \end{cases}$$

It is clear that

$$f_1(x^3, y^3) = \max_{y \in Y} f_1(x^3, y),$$

$$f_2(x^3, y^3) = \min_{x \in X} f_2(x, y^3).$$

Definition 4. *A pair strategy* $(x^4, y^4) \in X \times Y$ *is an Anti-Berge equilibrium(with respect to player 1) if*

$$\begin{cases} f_1(x^4, y^4) \leq f_1(x^4, y), & \forall y \in Y, \\ f_2(x^4, y^4) \geq f_2(x, y^4), & \forall x \in X. \end{cases}$$

In Nash equilibrium both of players maximizes their payoff functions simultaneously. In Berge equilibrium both of players mutually supports each other to maximize their payoffs while in the Anti-Berge equilibrium one of them minimizes other's payoff function. In other words, one of them behaves unpleasantly and is antagonistic to other.

Before we introduce Anti-Berge equilibrium for 3-person game, it is worth mentioning Berge equilibrium [10] for the game.

Definition 5. *A triple strategy* $(x^*, y^*, z^*) \in X \times Y \times Z$ *is a Berge equilibrium if*

$$\begin{cases} \hat{f}_1(x^*, y^*, z^*) \geq \hat{f}_1(x^*, y, z), & \forall (y, z) \in Y \times Z, \\ \hat{f}_2(x^*, y^*, z^*) \geq \hat{f}_2(x, y^*, z), & \forall (x, z) \in X \times Z, \\ \hat{f}_3(x^*, y^*, z^*) \geq \hat{f}_3(x, y, z^*), & \forall (x, y) \in X \times Y, \end{cases}$$

where the functions $\hat{f}_i(x, y, z), i = 1, 2, 3$ defined on a set $X \times Y \times Z$ of strategies are payoff functions of the players.

Now we introduce Anti-Berge equilibrium in the following.

Definition 6. *A triple strategy* $(x^*, y^*, z^*) \in X \times Y \times Z$ *is an Anti-Berge equilibrium (with respect to player 3) if*

$$\begin{cases} \hat{f}_1(x^*, y^*, z^*) \geq \hat{f}_1(x^*, y, z), & \forall (y, z) \in Y \times Z, \\ \hat{f}_2(x^*, y^*, z^*) \geq \hat{f}_2(x, y^*, z), & \forall (x, z) \in X \times Z, \\ \hat{f}_3(x^*, y^*, z^*) \leq \hat{f}_3(x, y, z^*), & \forall (x, y) \in X \times Y. \end{cases}$$

An existence of Anti-Berge equilibrium for a bimatrix game is given by the following proposition.

Theorem 1. *There exists an Anti-Berge equilibrium in a bimatrix game for mixed strategies.*

Proof. We follow up similarly the proof done for Berge equilibrium in [14]. Define the sets $S_1(x)$ and $S_2(y)$ as follows:

$$S_1(\bar{x}) = \left\{ \bar{y} \in Y \middle| f_1(\bar{x}, \bar{y}) = \max_{y \in Y} f_1(\bar{x}, y) \right\},$$

$$S_2(\bar{y}) = \left\{ \bar{x} \in X \middle| f_2(\bar{x}, \bar{y}) = \min_{x \in X} f_2(x, \bar{y}) \right\}.$$

Since the functions f_1 and f_2 are continuous and the sets X, Y are compact then there exist $\max_{y \in Y} f_1(\bar{x}, y)$, $\min_{x \in X} f_2(x, \bar{y})$. Thus $S_1(x) \neq \emptyset$ and $S_2(y) \neq \emptyset$.

Introduce the mapping $\mathcal{K}$ in the following:

$$\mathcal{K} \colon X \times Y \to S_1 \times S_2.$$

□

It is clear that if (x^*, y^*) is Anti-Berge equilibrium then $(x^*, y^*) \in \mathcal{K}(x^*, y^*)$. We show that $\mathcal{K}$ is convex compact. Indeed, for any $(\tilde{x}, \tilde{y}) \in \mathcal{K}(\bar{x}, \bar{y})$ and $(\hat{x}, \hat{y}) \in \mathcal{K}(\bar{x}, \bar{y})$ we have

$$f_1(\bar{x}, \tilde{y}) = \max_{y \in Y} f_1(\bar{x}, y),$$

$$f_2(\tilde{x}, \bar{y}) = \min_{x \in X} f_2(x, \bar{y}),$$

$$f_1(\bar{x}, \hat{y}) = \max_{y \in Y} f_1(\bar{x}, y),$$

$$f_2(\hat{x}, \bar{y}) = \min_{x \in X} f_2(x, \bar{y}).$$

Since f_1 and f_2 are bilinear functions, for $\alpha \in [0, 1]$ these equalities imply that

$$f_1(\bar{x}, \alpha \tilde{y} + (1 - \alpha) \hat{y}) = \max_{y \in Y} f_1(\bar{x}, y),$$

$$f_2(\alpha\tilde{x} + (1-\alpha)\hat{y}, \bar{y}) = \min_{x\in X} f_2(x, \bar{y}),$$

which means that

$$(\alpha\tilde{x} + (1-\alpha)\hat{y}, \alpha\tilde{y} + (1-\alpha)\hat{y}) \in \mathcal{K}(\bar{x}, \bar{y}).$$

Thus $\mathcal{K}$ is convex.

On the other hand, $\max_{y\in Y} f_1(\bar{x}, y)$ and $\min_{x\in X} f_2(x, \bar{y})$ are continuous functions on $X \times Y$, then $\mathcal{K}$ is continuous mapping. Since X and Y are compact then by Tikhonov theorem [8] $\mathcal{K}$ is also compact.

Therefore, conditions of fixed point theorem [4] are satisfied.

Hence, there exists (x^*, y^*) such that

$$(x^*, y^*) \in \mathcal{K}(x^*, y^*)$$

with $x^* \in S_2(y^*)$ and $y^* \in S_1(x^*)$.
This means that

$$f_1(x^*, y^*) = \max_{y\in Y} f_1(x^*, y) \geq f_1(x^*, y),\ \forall y \in Y,$$

$$f_2(x^*, y^*) = \min_{x\in X} f_2(x, y^*) \leq f_2(x, y^*),\ \forall x \in X$$

which proves the assertion.
For further purpose, it is useful to formulate the following theorem.

Theorem 2. *A pair strategy (x^*, y^*) is an Anti-Berge equilibrium if and only if*

$$f_1(x^*, y^*) \geq \left[{x^*}^T A\right]_j,\ j = 1, 2, \ldots, n, \tag{2.1}$$

$$f_2(x^*, y^*) \leq [By^*]_i,\ i = 1, 2, \ldots, m. \tag{2.2}$$

Proof. **Necessity**. Assume that (x^*, y^*) is an Anti-Berge equilibrium. Then by Definition 3, we have

$$f_1(x^*, y^*) \geq {x^*}^T Ay,\ \forall y \in Y, \tag{2.3}$$

$$f_2(x^*, y^*) \leq x^T By^*,\ \forall x \in X. \tag{2.4}$$

In the first inequality (2.3), successively choose $y = (0, 0, \ldots, 1, \ldots, 0)$ with 1 in each of the n spots, in (2.4) choose $x = (0, 0, \ldots, 1, \ldots, 0)$ with 1 in each of the m spots. We can easily see that

$$f_1(x^*, y^*) \geq \left[{x^*}^T A\right]_j,\ j = 1, \ldots, n,$$

$$f_2(x^*, y^*) \leq [By^*]_i,\ i = 1, \ldots, m.$$

Sufficiency. Suppose that for a pair $(x^*, y^*) \in X \times Y$, conditions (3.11) and (3.12) are satisfied. We choose $x \in X$, $y \in Y$ and multiply (3.11) by y_j and (3.12) by x_i respectively. We obtain

$$y_j f_1(x^*, y^*) \geq \left[{x^*}^T A\right]_j y_j,\ j = 1, 2, \ldots, n.$$

Summing up these inequalities and taking into account that $\sum_{j=1}^{n} y_j = 1$, we get

$$f_1(x^*, y^*) = \left(\sum_{j=1}^{n} y_j\right) f_1(x^*, y^*) \geq \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} x_i^* y_j = {x^*}^T A y.$$

By analogy, we also have

$$f_2(x^*, y^*) = \left(\sum_{i=1}^{m} x_i\right) f_2(x^*, y^*) \leq \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} x_i y_j^* = x^T B y^*.$$

Thus, we arrive at

$$f_1(x^*, y^*) \geq f_1(x^*, y), \quad \forall y \in Y,$$
$$f_2(x^*, y^*) \leq f_2(x, y^*), \quad \forall x \in X,$$

concluding that (x^*, y^*) is an Anti-Berge equilibrium. The proof is complete. □

3. Quadratic Programming Formulation of Anti-Berge Equilibrium

Theorem 3. *A pair strategy (x^*, y^*) is an Anti-Berge equilibrium (with respect to player 2) for the bimatrix game if and only if there exist scalars (p^*, q^*) such that (x^*, y^*, p^*, q^*) is a solution to the following quadratic programming problem :*

$$\max_{(x,y,p,q)} F(x, y, p, q) = x^T (A - B) y - p + q \tag{3.1}$$

subject to :

$$\left[x^T A\right]_j \leq p,\ j = 1, \ldots, n, \tag{3.2}$$

$$\left[By\right]_i \geq q,\ i = 1, \ldots, m, \tag{3.3}$$

$$\sum_{i=1}^{m} x_i = 1,\ x_i \geq 0,\ i = 1, \ldots, m, \tag{3.4}$$

$$\sum_{j=1}^{n} y_j = 1,\ y_j \geq 0,\ j = 1, \ldots, n. \tag{3.5}$$

Proof can be done similarly to the theorem in [14] proven for a Berge equilibrium.

Proof. **Necessity**. Suppose that (x^*, y^*) is an Anti-Berge equilibrium. Choose scalars p^* and q^* such that $p^* = f_1(x^*, y^*)$, $q^* = f_2(x^*, y^*)$.

We show that (x^*, y^*, p^*, q^*) is a solution to problem (3.1)–(3.5). First, we show that (x^*, y^*, p^*, q^*) is a feasible point for problem (3.1)–(3.5).

By Theorem 2, the equivalent characterization of an Anti-Berge equilibrium point, we have

$$p^* = f_1(x^*, y^*) \geq \left[{x^*}^T A\right]_j, \quad j = 1, \ldots, n,$$

$$q^* = f_2(x^*, y^*) \leq \left[By^*\right]_i, \quad i = 1, \ldots, m.$$

The rest of the constraints are satisfied because of $x^* \in X$ and $y^* \in Y$. It means that (x^*, y^*, p^*, q^*) is a feasible point.

Choose any $x \in X$ and $y \in Y$. Multiply (3.2)-(3.3) by y_j and x_i, respectively. If we sum up these inequalities, we obtain

$$f_1(x, y) = x^T Ay \leq p,$$

$$f_2(x, y) = x^T By \geq q.$$

Hence, we get

$$F(x, y, p, q) = x^T(A - B)y - p + q \leq 0$$

for all $x \in X$, $y \in Y$. But with $p^* = f_1(x^*, y^*)$ and $q^* = f_2(x^*, y^*)$, we have $F(x^*, y^*, p^*, q^*) = 0$. Hence, the point (x^*, y^*, p^*, q^*) is a solution to problem (3.1)–(3.5).

Sufficiency.Let $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$ be a solution to problem (3.1)–(3.5).

We show that $(\bar{x}, \bar{y})$ is an Anti-Berge equilibrium of the game. Since $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$ is a feasible point, the following constraints are satisfied:

$$\left[\bar{x}^T A\right]_j \leq \bar{p},\ j = 1, \ldots, n,\ \sum_{i=1}^{m} \bar{x}_i = 1,\ \bar{x}_i \geq 0,\ i = 1, \ldots, m, \tag{3.6}$$

$$\left[B\bar{y}\right]_i \geq \bar{q},\ i = 1, \ldots, m,\ \sum_{j=1}^{n} \bar{y}_j = 1,\ \bar{y}_j \geq 0,\ j = 1, \ldots, n, \tag{3.7}$$

Hence, we have

$$f_1(\bar{x}, \bar{y}) = \bar{x}^T A\bar{y} \leq \bar{p} \sum_{j=1}^{n} \bar{y}_j = \bar{p}, \tag{3.8}$$

$$f_2(\bar{x}, \bar{y}) = \bar{x}^T B\bar{y} \geq \bar{q} \sum_{i=1}^{m} \bar{x}_i = \bar{q}. \tag{3.9}$$

Summing up these inequalities, we obtain

$$F(\bar{x},\bar{y},\bar{p},\bar{q}) = \bar{x}^T(A-B)\bar{y} - \bar{p} + \bar{q} \le 0. \tag{3.10}$$

Taking into account (3.8) and (3.9), we conclude that the function $F(x,y,p,q)$ reaches its maximum at zero:

$$F(\bar{x},\bar{y},\bar{p},\bar{q}) = \left(\bar{x}^T A\bar{y} - \bar{p}\right) + \left(\bar{x}^T B\bar{y} - \bar{q}\right) = 0 \tag{3.11}$$

with

$$\bar{x}^T A\bar{y} = \bar{p}, \tag{3.12}$$

$$\bar{x}^T B\bar{y} = \bar{q}. \tag{3.13}$$

From (3.12)-(3.13) and (6)-(7) we have

$$\bar{p} = f_1(\bar{x},\bar{y}) = \bar{x}^T A\bar{y} \ge \left[\bar{x}^T A\right]_j \quad j = 1,\dots,n,$$

$$\bar{q} = f_2(\bar{x},\bar{y}) = \bar{x}^T B\bar{y} \le [B\bar{y}]_i \quad i = 1,\dots,m.$$

Now by Theorem 2, $(\bar{x},\bar{y})$ is an Anti-Berge equilibrium which completes the proof. □

Note that the condition

$$F(x^*,y^*,p^*,q^*) = 0$$

is necessary and sufficient for a (x^*,y^*) to be an Anti-Berge equilibrium.

We can also formulate the following assertion for Anti-Berge equilibrium (with respect to player 1).

Theorem 4. *A pair strategy $(\hat{x}^*,\hat{y}^*)$ is an Anti-Berge equilibrium (with respect to player 1) for the bimatrix game if and only if there exist scalars $(\hat{p}^*,\hat{q}^*)$ such that $(\hat{x}^*,\hat{y}^*,\hat{p}^*,\hat{q}^*)$ is a solution to the following quadratic programming problem :*

$$\max_{(x,y,p,q)} F(x,y,p,q) = x^T(B-A)y + p - q$$

subject to :

$$\left[x^T A\right]_j \ge p,\ j = 1,\dots,n,$$

$$[By]_i \le q,\ i = 1,\dots,m,$$

$$\sum_{i=1}^{m} x_i = 1,\ x_i \ge 0,\ i = 1,\dots,m,$$

$$\sum_{j=1}^{n} y_j = 1,\ y_j \ge 0,\ j = 1,\dots,n.$$

As an example, consider the following bimatrix game with matrices A and B :

$$A = \begin{pmatrix} 9 & 11 & 6 & 20 \\ 7 & 4 & 10 & 21 \\ 2 & 16 & 15 & 9 \\ 5 & 9 & 9 & 17 \\ 4 & 3 & 5 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 15 & 10 & 5 & 19 \\ 13 & 18 & 1 & 16 \\ 11 & 17 & 18 & 12 \\ 6 & 11 & 3 & 10 \\ 8 & 12 & 8 & 7 \end{pmatrix}$$

Problem (3.1)–(3.5) for finding Anti-Berge equilibrium (with respect to player 2) is formulated as:

$max\ F(x,y,p,q) = -6x_1y_1 + x_1y_2 + x_1y_3 + x_1y_4 - 6x_2y_1 - 14x_2y_2 + 9x_2y_3 + 5x_2y_4 - 9x_3y_1 - x_3y_2 - 3x_3y_3 - 3x_3y_4 - x_4y_1 - 2x_4y_2 + 6x_4y_3 + 7x_4y_4 - 4x_5y_1 - 9x_5y_2 - 3x_5y_3 - 5x_5y_4 + p - q$

$$\begin{cases} 9x_1 + 7x_2 + 2x_3 + 5x_4 + 4x_5 - p & \leq 0 \\ 11x_1 + 4x_2 + 16x_3 + 9x_4 + 3x_5 - p & \leq 0 \\ 6x_1 + 10x_2 + 15x_3 + 9x_4 + 5x_5 - p & \leq 0 \\ 20x_1 + 21x_2 + 9x_3 + 17x_4 + 2x_5 - p & \leq 0 \\ x_1 + x_2 + x_3 + x_4 + x_5 = 1 & \\ 15y_1 + 10y_2 + 5y_3 + 19y_4 - q & \geq 0 \\ 13y_1 + 18y_2 + y_3 + 16y_4 - q & \geq 0 \\ 11y_1 + 17y_2 + 18y_3 + 12y_4 - q & \geq 0 \\ 6y_1 + 11y_2 + 3y_3 + 10y_4 - q & \geq 0 \\ 8y_1 + 12y_2 + 8y_3 + 7y_4 - q & \geq 0 \\ y_1 + y_2 + y_3 + y_4 = 1 & \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0, & \\ y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ y_4 \geq 0,\ y_5 \geq 0. & \end{cases}$$

We can easily check that $F(x^*, y^*, p^*, q^*)=0$ with $x^*=(0,0,0,0.273,0.727)^T$, $y^* = (0,0,0.375,0.625)^T$, $p^* = 6.09$, $q^* = 7.375$ and $F^* = 0$. It means that (x^*, y^*) is an Anti-Berge equilibrium(with respect to player2) for the bimatrix game.

On the other hand, the game has also Anti-Berge equilibrium (with respect to player 1) in pure strategies: $x^* = (0,1,0,0,0)^T$, $y^* = (0,1,0,0)^T$. But there are two another Anti-Berge equilibria:

$$x^1 = (0.8125, 0, 0.1875, 0, 0)^T, y^1 = (0.764706, 0.235294, 0, 0)^T,$$

$$x^2=(0.532895, 0.447368, 0.019737, 0, 0)^T, y^2=(0.6875, 0.21875, 0.09375, 0)^T.$$

Conclusion

We examined so-called Anti-Berge equilibrium in a bimatrix game. By analogy of Nash and Berge equilibriums, we proved the existence of Anti-Berge equilibrium in the game. Finding an Anti-Berge equilibrium in the game has been reduced to a quadratic programming problem with an indefinite matrix. An example has been considered. We introduced also Anti-Berge equilibrium, a new concept of equilibria, for 3-person game. Computational aspects of Anti-Berge equilibria will be discussed in a next paper.

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Равновесие анти-Бержа для биматричных игр

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Аннотация. Рассматривается новая биматричная игра на основе равновесий Нэша и Бержа. Решение данной игры будем называть равновесием анти-Бержа. С помощью теоремы Милса [9] задача нахождения равновесия анти-Бержа сводится к задаче квадратичного программирования с линейными ограничениями. Новое понятие равновесия анти-Бержа иллюстрируется на численном примере.

Ключевые слова: равновесие Бержа, оптимизация, биматричная игра, равновесие анти-Бержа.

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